CS 101 Quiz 2
Time: 2 hours, Total: 80 points
Instructor: Apurva Mudgal

Instructions.

1. There are 9 questions with 10 points per question. You have to answer **any eight** out of these nine questions.
2. Each question (except Question 7, which asks you to write two propositional formulas) has two parts. In the first part (5 points), you have to select one correct option. There is no negative marking. In the second part (5 points), you have to support the answer picked in first part by proofs.

   For the second part, if required, in addition to proving why the option selected by you is correct, you also need to prove why the options not selected by you are incorrect.
3. **Grading.** If the option selected in first part is correct, you get 5/5 marks. If it is incorrect, you get 0/5 marks.

   After this, *the second part will be graded only if the option selected by you in first part is correct.* There is partial marking in second part, based on the steps taken towards the answer.

Questions.

1. (10 points)
   (a) (5 points) Is the following true? Answer 'Yes' or 'No'.
   $$(p \to q) \to (q \to p) \vdash p \wedge q$$
   (b) (5 points) If your answer to part (a) is 'Yes', give a proof *using rules of inference.* If the answer to part (a) is 'No', give a counterexample.
2. (10 points) Consider the formula:
   $$((p \to q) \to (q \to (p \to q))) \to q$$
   (a) (5 points) Note that there are four possible truth assignments over variables $\{p, q\}$. What is the number of truth assignments over $\{p, q\}$ for which the above formula evaluates to true? (Pick one option.)
      i. 0
      ii. 1
      iii. 2
      iv. 3

v. 4

(b) (5 points) Prove that the answer you selected in part (a) is correct.

3. (10 points) Consider the following two predicate logic formulas:

$$A = \exists y \forall x \forall z R(x, y, z)$$
$$B = \forall x \exists y \forall z R(x, y, z)$$

(a) (5 points) Which of the following four possibilities is correct? (Pick one option)

i. $A \vdash B$ and $B \vdash A$.

ii. $A \vdash B$, but B does not derive A.

iii. A does not derive B, but $B \vdash A$.

iv. Neither A derives B, nor B derives A.

(b) (5 points) Prove that the answer you selected in part (a) is correct.

4. (10 points)

(a) (5 points) Suppose there is a single predicate $P(\cdot, \cdot)$ of arity 2. With this predicate, there are a total of $2^4 = 16$ worlds with domain of discourse $D = \{1, 2\}$.

In how many worlds over domain of discourse $\{1, 2\}$ is the following formula true?

$$(\exists x \forall y P(x, y)) \rightarrow (\forall x \exists y P(x, y))$$

Your answer should be a single integer in the range 0-16.

(b) (5 points) Prove that the answer given by you for part (a) is correct.

5. (10 points) Consider the following set U:

$$U = \left\{ f \mid f : \mathbb{N} \rightarrow \mathbb{N} \ and \ \max_{i \in \mathbb{N}} f(i) \ is \ a \ finite \ integer \right\}$$

(a) (5 points) Which of the following is true? (pick one option)

i. U is a finite set.

ii. $|U| = |\mathbb{N}|$ (same cardinality as natural numbers).

iii. $|U| = |\mathbb{R}|$ (same cardinality as real numbers).

iv. $|U| = |\mathcal{P}(\mathbb{R})|$ (same cardinality as power set of real numbers).

(b) (5 points) Prove that the answer given by you for part (a) is correct.

6. (10 points) Let U be the set of all finite sequences of natural numbers i.e.,

$$U = \Big\{(a_1, a_2, \dots, a_m) \mid m \in \mathbb{N} \ and\ a_i \in \mathbb{N}\ for\ each\ 1 \le i \le m\Big\}$$

Define a relation $\sim$ on U as follows. Let $(a_1, a_2, \dots, a_m)$ and $(b_1, b_2, \dots, b_n)$ belong to set U. Then, $(a_1, a_2, \dots, a_m) \sim (b_1, b_2, \dots, b_n)$ if and only if the first tuple is a subsequence of the second tuple. In other words, there exist indices $1 \le k_1 < k_2 < \dots < k_m \le n$ such that $a_i = b_{k_i}$ for each $1 \le i \le m$.

(a) (5 points) Which of the following is true? (pick one option)

i. $\sim$ is not reflexive.

ii. $\sim$ is not symmetric.

iii. $\sim$ is not transitive.

iv. $\sim$ is an equivalence relation.

(b) (5 points) Prove that the answer selected by you for part (a) is correct.

7. (10 points) (Fill in the blanks.) Consider the graph G with 5 vertices and 6 edges shown in Figure 1. We pick a subset W of vertices of G.

For $1 \le i \le 5$, let x_i be a propositional variable such that x_i is true if and only if vertex i is picked in set W.

Express the following two statements as propositional formulas over variables $\{x_1, x_2, \dots, x_5\}$:

(a) (5 points) "At least one vertex from each edge is picked in W".

(b) (5 points) "If W has size at most 2, then there exists an edge (u, v) such that neither u nor v is picked in W."

Note. This question is an exception. There is partial marking in both parts, based on the steps taken towards the answer.

8. (10 points) A *Quiz2-tree* is defined as follows:

(a) A single node $\bullet$ is a Quiz2-tree.

(b) If $T_1, T_2, \dots, T_l$ are Quiz2-trees, where $l \ge 2$, the tree shown in Figure 2 is also a Quiz2-tree.

Let $i(T)$ denote the number of interior nodes, and $l(T)$ denote the number of leaves in Quiz2-tree T.

(a) (5 points) Which of the following is true? (Pick one option.)

i. For every Quiz2-tree T, $i(T) \le l(T) - 2$.

ii. There exists a Quiz2-tree T for which $i(T) > l(T)$.

iii. For every Quiz2-tree T, $i(T) \le l(T) - 1$.

iv. There exists a Quiz2-tree T for which $i(T) = l(T)$.

(b) (5 points) Prove that the answer selected by you in part (a) is correct.

9. (10 points)

Let

$$A = (\forall x \exists y (P(y) \land Q(x))) \to \forall y P(y)$$
$$B = (\forall x Q(x)) \land (\exists y P(y))$$

(a) (5 points) Is the following true? Answer 'Yes' or 'No'.

$$A \land B \vdash \forall x P(x)$$

(b) (5 points) If your answer to part (a) is 'Yes', give a proof *using rules of inference*. If the answer to part (a) is 'No', give a counterexample.