Quiz 2 Solutions

02 April 2021 16:15

Question 4

(a) 14

(b) $A = \exists x \forall y \, P(x,y)$

$B = \forall x \exists y \, P(x,y)$

Consider the 2X2 matrix:

$P(1,1)$	$P(2,1)$
$P(1,2)$	$P(2,2)$

A is true if one column of matrix has both entries T.

B is true if each column of matrix has at least one entry T.

$A \to B$ is false iff $\left(\begin{array}{c} A \text{ is true} \\ \text{and} \\ B \text{ is false} \end{array} \right)$

A is true and B is false

IFF

One column has both entries T and the other column has both entries F

T	F
T	F

F	T
F	T

Quiz 2 solutions

02 April 2021 16:25

Question 4 (b) (contd.)

Thus,

\# of worlds over $D = \{1, 2\}$ where $A \to B$ is true

$= 16 - (\#$ of worlds where $A \to B$ is false$)$

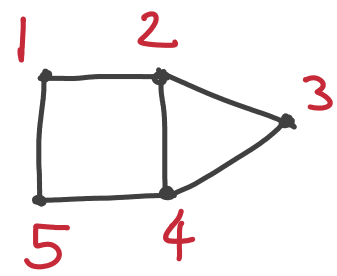
$= 16 - 2$

$= 14$

Quiz 2 Solutions

02 April 2021 10:31

Question 7. $V = \{1,2,3,4,5\}$



$$E = \{\{1,2\},\{2,3\},\{2,4\},\{3,4\},\{4,5\},\{1,5\}\}$$

(a)

At least one vertex from edge $\{u,v\}$ is picked in W

$$\|$$

$$x_u \vee x_v$$

Thus,

At least one vertex from each edge is picked in W

$$\|$$

$$\bigwedge_{\{u,v\} \in E} (x_u \vee x_v)$$

or, $(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4) \wedge (x_3 \vee x_4) \wedge (x_4 \vee x_5) \wedge (x_1 \vee x_5)$

Quiz 2 solutions

02 April 2021 15:42

Question 7.

(b) $\phi_1 \rightarrow \phi_2$, where

ϕ_1 : W has size at most 2

= No three distinct variables x_u, x_v, x_w are simultaneously true

$$= \bigwedge_{\{u,v,w\} \subseteq \{1,2,\ldots,5\}} \left(\neg (x_u \wedge x_v \wedge x_w) \right)$$

ϕ_2 : there exists an edge $\{u,v\}$ such that neither u nor v is picked in W

$$= \bigvee_{\{u,v\} \in E} \left(\neg (x_u \vee x_v) \right)$$

Marking scheme. There may be other propositional formulas equivalent to above two statements. You will get full 5 points, if your formula is correct.

Quiz 2 solutions

02 April 2021 15:54

Question 8.

(a) Option (iii) – for every Quiz2-tree T,

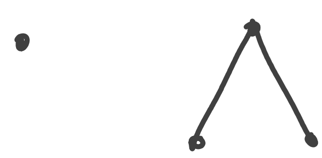
$$i(T) \leq \ell(T) - 1$$

(b)

For every Quiz2-tree, $i(T) \leq \ell(T) - 2$

Not true. Take

(1 point)



etc.

$i(T) = 0$, $\ell(T) = 1$ $\qquad$ $i(T) = 1$, $\ell(T) = 2$

For every Quiz2-tree, $i(T) \leq \ell(T) - 1$

Proceed as in Interactive Session 23.

P(n): $i(T) \leq \ell(T) - 1$ for all Quiz2-trees T with $N(T) = n$.

(1 point)

$N(\bullet) = 1$

N  $= 1 + \sum_{j=1}^{z} N(T_j)$

We prove $\forall n\, P(n)$ using induction.

Quiz 2 solutions

02 April 2021 16:00

Question 8 (b) (contd.)

Base case

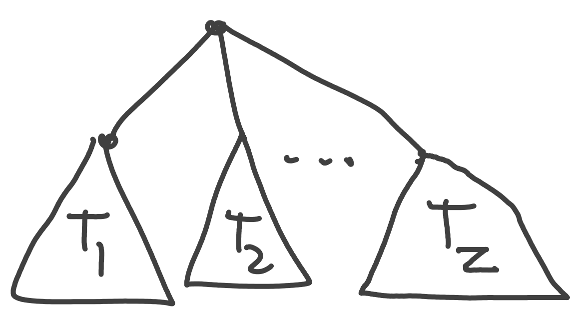
$P(1)$ • $i(T) = 0$, $\ell(T) = 1$ $\quad i(T) \leq \ell(T) - 1$

Inductive step

Suppose $P(i)$ is true for all $i \leq n$ $(n \geq 1)$

Consider a Quiz 2-tree T with $N(T) = n+1$

Let $T =$



$z \geq 2$

(3 points)

$\because$ $N(T) = 1 + \sum_{j=1}^{z} N(T_j)$, we conclude that

$$N(T_j) \leq n \text{ for each } 1 \leq j \leq z.$$

By inductive hypothesis:

$$i(T_j) \leq \ell(T_j) - 1 \quad \text{for } 1 \leq j \leq z$$

By definition:

$$\begin{cases} i(T) = 1 + \sum_{j=1}^{z} i(T_j) \\ \ell(T) = \sum_{j=1}^{z} \ell(T_j) \end{cases}$$

Quiz 2 solutions

02 April 2021 16:03

Question 8 (b) (contd.)

Thus,

$$
\begin{aligned}
i(T) &= 1 + \sum_{j=1}^{z} i(T_j) \\
&\geqslant 1 + \sum_{j=1}^{z} \left(\ell(T_j) - 1\right) \\
&= \sum_{j=1}^{z} \ell(T_j) + (1 - z) \\
&= \ell(T) + 1 - z \\
&\geq \ell(T) - 1 \qquad (\because z \geq 2)
\end{aligned}
$$

We conclude that $P(n+1)$ is also true.