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Question 1.

(a) No.

(b) Counterexample:

Take $p = q = F$. Then,

$p \to q$ is T

$q \to p$ is T

$(p \to q) \to (q \to p)$ is T

But $p \wedge q$ is F.

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Question 2.

(a) 2 (Option (iii)).

(b) We construct the truth table:

p	q	$\overbrace{p \to q}^{A}$	$\overbrace{q \to (p \to q)}^{B}$	$A \to B$	$(A \to B) \to q$
F	F	T	T	T	F
F	T	T	T	T	T
T	F	F	T	T	F
T	T	T	T	T	T

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Question 3.

(a) Option (ii) – $A \vdash B$, but B does not derive A.

(b) $A \vdash B$:

(2.5 points)

1. $\exists y \forall x \forall z\, R(x,y,z)$ (Assumption)
2. $\forall x \forall z\, R(x, \alpha(), z)$ (EI, $y \to \alpha()$)
3. $\forall z\, R(a, \alpha(), z)$ (UI, $x \to a$)
4. $R(a, \alpha(), b)$ (UI, $z \to b$)
5. $\forall z\, R(a, \alpha(), z)$ (UG, $b \to z$ No special point depends on b)
6. $\exists y \forall z\, R(a, y, z)$ (EI, $\alpha() \to y$)
7. $\forall x \exists y \forall z\, R(x,y,z)$ (UG, $a \to x$ No special point depends on a)

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Question 2.

(b) (contd.)

(2.5 points)

B does not derive A

World W

Let $D = \{1, 2\}$

Define predicate $R : D \times D \times D \to \{T, F\}$ as follows:

$$R(1,1,1) = R(1,1,2) = R(2,2,1) = R(2,2,2) = T$$
$$R(2,1,1) = R(2,1,2) = R(1,2,1) = R(1,2,2) = F$$

In world W,

A is false, but B is true

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Question 5.

(a) Option (iii) — $|U| = |\mathbb{R}|$

(b) $\underline{|U| \geq |\mathbb{R}|.}$

(2.5 points) Let $U_1 = \{ f \mid f : \mathbb{N} \to \{0,1\} \}$

Then, $|U_1| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, and

$U_1 \subseteq U$ ($\because$ for $f \in U_1$, $\max_{i \in \mathbb{N}} f(i) \leq 1$)

Thus, $|U| \geq |U_1| = |\mathbb{R}|$.

$\underline{|\mathbb{R}| \geq |U|}$

(2.5 points) Let $f \in U$. Define the set T_f as follows:

$$T_f = \{ (i, f(i)) \mid i \in \mathbb{N} \}$$

Note that $T_f \subseteq \mathbb{N} \times \mathbb{N}$.

Further, if $f_1 \neq f_2$ then $T_{f_1} \neq T_{f_2}$.

Thus,

$$|U| = |\{T_f \mid f \in U\}| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})|$$
$$= |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|.$$

By Schröder-Bernstein theorem, we conclude that $|U| = |\mathbb{R}|$.

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Question 6.

(a) Option (ii) : $\sim$ is not symmetric

(b)

$\sim$ is reflexive

(1 point)

$(a_1, a_2, \ldots, a_m) \sim (a_1, a_2, \ldots, a_m)$ (Take $k_i = i$ for $1 \leq i \leq m$)

$\sim$ is not symmetric

(2 points)

$(2, 3) \sim (2, 3, 5)$, but

$(2, 3, 5) \not\sim (2, 3)$

$\sim$ is transitive

(2 points)

Let $(a_1, a_2, \ldots, a_m) \sim (b_1, b_2, \ldots, b_n)$, and

$(b_1, b_2, \ldots, b_n) \sim (c_1, c_2, \ldots, c_\ell)$

$\Rightarrow \exists \quad 1 \leq k_1 < k_2 < \ldots < k_m \leq n$

such that $a_i = b_{k_i}$

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and

$\exists \quad 1 \leq z_1 < z_2 < \ldots < z_n \leq l$

such that $b_i = c_{z_i}$

Then,

$$a_i = c_{z_{k_i}} \quad \text{for each } 1 \leq i \leq m$$

Hence,

$$(a_1, a_2, \ldots, a_m) \sim (c_1, c_2, \ldots, c_l)$$

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Question 9

(a) Yes

(b)

1. $A \land B$ (assumption)
2. $(\forall x\, Q(x)) \land (\exists y\, P(y))$ $\left(\frac{F \land G}{G}, \text{ formula } \#1 \right)$
3. $\forall x\, Q(x)$ $\left(\frac{F \land G}{F}, \text{ formula } \#2 \right)$
4. $Q(a)$ $\left(\underset{x \to a}{UI}, \text{ formula } \#3 \right)$
5. $\exists y\, P(y)$ $\left(\frac{F \land G}{G}, \text{ formula } \#2 \right)$
6. $P(\alpha())$ $\left(\underset{y \to \alpha()}{EI}, \text{ formula } \#5 \right)$
7. $P(\alpha()) \land Q(a)$ $\left(\frac{F \quad G}{F \land G}, \right.$ apply this rule where $F = P(\alpha())$ and G varies over formulas in family $Q(a)$, #4 $\left. \right)$

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Question 9

(b) (contd.)

8. $\exists y\,(P(y) \wedge Q(a))$ $\left(\begin{array}{c} EG \\ \alpha() \to y \end{array}\right)$

9. $\forall x\,(\exists y\,(P(y) \wedge Q(x)))$

$\left(\begin{array}{cc} UG & \text{No special} \\ x \to a & \text{point depends} \\ & \text{on } a \end{array}\right)$

10. $(\forall x\,(\exists y\,(P(y) \wedge Q(x)))) \to \forall y\, P(y)$

$\left(\dfrac{F \wedge G}{F},\ \#1\right)$

11. $\forall y\, P(y)$ $\left(\begin{array}{c} F \\ F \to G \\ \hline G \end{array},\ \text{formulas } \#9 \text{ and } \#10\right)$

12. $P(a)$ $\left(\begin{array}{l} UI,\ \text{formula } \#11 \\ y \to a \end{array}\right)$

13. $\forall x\, P(x)$ $\left(\begin{array}{cc} UG, & \text{No special} \\ a \to x & \text{point depends} \\ & \text{on } a \end{array}\right)$