CS 101 End-semester Examination
Part B
Time: 90 minutes, Total: 50 points
Instructor: Apurva Mudgal

Instructions.

1. There are 5 questions with 10 points per question. You have to answer **all five** questions.

2. Each question has two parts. In the first part (5 points), you have to select one correct option. There is no negative marking. In the second part (5 points), you have to support the answer picked in first part by proofs.

   For the second part, if required, in addition to proving why the option selected by you is correct, you also need to prove why the options not selected by you are incorrect.

3. **Grading.** If the option selected in first part is correct, you get 5/5 marks. If it is incorrect, you get 0/5 marks.

   After this, *the second part will be graded only if the option selected by you in first part is correct.* There is partial marking in second part, based on the steps taken towards the answer.

Questions.

1. (10 points) Let A_1, A_2, A_3, A_4, A_5 be five finite sets which satisfy the following two properties:

   - $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 0$.
   - Let S_1 and S_2 be any two distinct, non-empty subsets of $\{1, 2, 3, 4, 5\}$ of equal cardinality. Then,

   $$| \cap_{i \in S_1} A_i| = | \cap_{i \in S_2} A_i|$$

   (a) (5 points) Is the following true? Answer 'Yes' or 'No'.
   The cardinality $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$ *is always divisible by* 5.

   (b) (5 points) Prove that the answer you have selected in part (a) above is correct.

2. (10 points) Let $G(V, E)$ be a graph on n ($n \geq 3$) vertices, such that G satisfies the following two properties:

   - G has *at most* $n - k$ edges for some non-negative integer k ($0 \leq k \leq n - 3$), and
   - G has a unique cycle.

(a) (5 points) What is the minimum possible number of connected components in G? (Pick one option)

i. $k-2$
ii. $k-1$
iii. k
iv. $k+1$
v. $k+2$

(b) (5 points) Prove that the answer you have selected in part (a) above is correct.

3. (10 points) Let $G(V, E)$ be a directed acyclic graph (DAG) on $3n$ vertices, with exactly n sources and exactly n sinks. Further, G has no vertex which is both a source and a sink i.e., has both indegree and outdegree equal to zero.

(a) (5 points) What is the maximum possible number of edges in G? (Pick one option)

i. $\frac{3n(3n-1)}{2}$
ii. $3.5n^2 - \frac{n}{2}$
iii. $3.5n^2 + \frac{n}{2}$
iv. $3.5n^2$
v. $3n^2$

(b) (5 points) Prove that the answer you have selected in part (a) above is correct.

4. (10 points) Consider the propositional logic formula:

$$\phi = (((((x_1 \to x_2) \to x_3) \to x_4) \to x_5) \to x_6) \to x_7$$

(a) (5 points) What is the total number of worlds over propositional variables $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ for which ϕ evaluates to true? (Pick one option)

i. 82
ii. 83
iii. 84
iv. 85
v. 86

(b) (5 points) Prove that the answer you have selected in part (a) above is correct.

5. (10 points) Let $G(V, E)$ be an undirected graph. We define a relation $\sim$ on vertices of V as follows:

- $u \sim u$ for every vertex $u \in V$.
- Let u, v be two distinct vertices of V. Then $u \sim v$ if and only if there are at least two distinct paths from u to v in G.

Note. P_1 and P_2 are distinct paths if and only if there is an edge e which belongs to exactly one of P_1 or P_2.

(a) (5 points) Which of the following is true? (Pick one option)

  i. $\sim$ is an equivalence relation.

  ii. $\sim$ is symmetric.

  iii. $\sim$ is not transitive.

(b) (5 points) Prove that the answer you have selected in part (a) above is correct.