

CS101 Part C (Graph Theory)
Tutorial 2

Note. A directed graph G is strongly connected if and only if it has exactly one strongly connected component.

1. Let $G(V,E)$ be a connected, undirected graph with exactly $2k$ ($k \geq 1$) vertices of odd degree.

Show that there exist z open walks W_1, W_2, \dots, W_z in G such that:

- (i) $1 \leq z \leq k$, and
- (ii) every edge in E occurs exactly once in these z open walks.

2. Let $G(V,E)$ be a *directed* graph such that for every vertex v in V , indegree of v equals the outdegree of v .

Show that, for some natural number l , there exist cycles C_1, C_2, \dots, C_l in G such that every edge of G occurs exactly once in these cycles.

3. Prove or disprove:

Let $G(V,E)$ be an undirected graph. Suppose G has an Eulerian cycle. Then, for any two distinct edges $(v, w_1), (v, w_2)$ sharing an endpoint, there exists a Eulerian cycle of G where these two edges are consecutive.

4. Let $G(V,E)$ be a directed graph. Define a relation R on vertices of G as follows: uRv if and only if *exactly one* of the following two conditions is true:

- i. there is a path from u to v in G
- ii. there is a path from v to u in G

Is R an equivalence relation? Prove your answer.

5. Let $G(V,E)$ be the following directed acyclic graph:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$E = \{(1,2), (1,3), (2,4), (3,4), (4,5), (4,6), (5,7), (6,7), (7,8), (7,9), (8,10), (9,10)\}$$

How many topological sorts does G have?

6. Let $G(V,E)$ be a directed graph on n vertices with a single strongly connected component. Suppose further that for every vertex v in V , removal of v increases the number of strongly connected components of G . What is the minimum possible number of edges in G ?

7. Let $G(V,E)$ be a directed graph. We call G *cycle-connected* if and only if it satisfies the following property:

For any two distinct edges e_1, e_2 in E , there is a cycle in G containing both e_1 and e_2 .

Prove or disprove:

G is cycle-connected if and only if G is strongly connected.

8. What is the maximum possible number of edges in a directed acyclic graph with $2n$ vertices?
9. Let K_n be the complete graph on n vertices. Show that the chromatic number of K_n is n .
10. Let $G(V,E)$ be the hypercube in d dimensions:

V = set of all bit vectors of length d

$E = (v_1, v_2)$ is an edge if and only if vectors v_1 and v_2 differ in exactly one bit.

What is the chromatic number of G ?