CS101 Quiz 1
Time: 2 hours, Total: 90 points
Instructor: Apurva Mudgal

Instructions. Answer **any three** out of the following four questions. You can use results discussed in class without proofs. For full credit, you have to support your answers with rigorous proofs as given in lectures. There is partial credit, if you complete some of the steps towards the answer.

1. (30 points) Suppose you have n_1 *distinguishable* red balls and n_2 *indistinguishable* yellow balls. A red ball can be distinguished from a yellow ball.

   How many ways are there to distribute these $n_1 + n_2$ balls into m *distinguishable* urns so that there is no urn with only yellow balls?

   Prove your answer.

   Note. An empty urn is allowed. You do not need to simplify your answer.

   Solution: (9 points) Let U be the set of all ways to distribute these $n_1 + n_2$ balls into m distinguishable urns, so that no urn has only yellow balls.

   For $1 \leq i \leq m$, let A_i be the set of all ways to distribute these $n_1 + n_2$ balls into m distinguishable urns, so that (i) no urn has only yellow balls, and (ii) exactly i urns are non-empty. Observe that every non-empty urn must contain at least one red ball.

   Note that A_i's form a partition of U. Hence, by first version of sum rule:

$$|U| = \sum_{i=1}^{m} |A_i|$$

   (8 points) For $1 \leq i \leq m$, let B_i be the set of all ways to distribute n_1 distinguishable red balls into m urns so that exactly i urns are non-empty.

   We define a function $\psi_1 : A_i \to B_i$ as follows. Take a distribution $D \in A_i$. Let $D' \in B_i$ be the distribution obtained by removing all yellow balls from distribution D. We define $\psi_1(D) = D'$.

   Let K_i be the number of ways to distribute n_2 indistinguishable yellow balls into i distinguishable urns. From Lecture 3, it follows that:

$$K_i = \binom{n_2 + i - 1}{n_2}$$

   Prove yourself that:

   (a) ψ_1 is onto, and

   (b) for every distribution $D' \in B_i$, its pre-image $\{D \mid D \in A_i \textit{ and } \psi_1(D) = D'\}$ under ψ_1 has exactly K_i elements.

By mapping version of product rule, for each $1 \leq i \leq m$:

$$|A_i| = K_i \cdot |B_i| = \binom{n_2 + i - 1}{n_2} \cdot |B_i|$$

(8 points) Let C_i be the number of ways to distribute n_1 distinguishable red balls into i distinguishable urns. Note that $|C_i| = H_{n_1,i}$, where $H_{n_1,i}$ is the number of surjective functions from set $\{1, 2, \ldots, n_1\}$ to set $\{1, 2, \ldots, i\}$.

Define a function $\psi_2 : B_i \to C_i$ as follows. Take a distribution $D' \in B_i$. Suppose the non-empty urns under distribution D' have indices $1 \leq z_1 < z_2 < \ldots < z_i \leq m$. Construct a distribution $D'' \in C_i$, by putting all balls sent to urn z_l under D' into urn l under D'', for each $1 \leq l \leq i$.

Prove yourself that:

(a) ψ_2 is onto, and

(b) for every distribution $D'' \in C_i$, its pre-image $\{D' \mid D' \in B_i \textit{ and } \psi_2(D') = D''\}$ under ψ_2 has exactly $\binom{m}{i}$ elements.

By mapping version of product rule, we conclude that:

$$|B_i| = \binom{m}{i} \cdot |C_i| = \binom{m}{i} \cdot H_{n_1,i}$$

(5 points) Combining everything together, we obtain the answer:

$$|U| = \sum_{i=1}^{m} \left(\binom{n_2 + i - 1}{n_2} \cdot \binom{m}{i} \cdot H_{n_1,i} \right)$$

2. (30 points) There are $2n$ letters $L_1, L_2, \ldots, L_{2n}$ and $2n$ envelopes $E_1, E_2, \ldots, E_{2n}$. For each $1 \leq i \leq n$:

   (a) letter L_{2i-1} is posted correctly if and only if it goes to one of the two envelopes $\{E_{2i-1}, E_{2i}\}$, and

   (b) letter L_{2i} is posted correctly if and only if it goes to one of the two envelopes $\{E_{2i-1}, E_{2i}\}$.

   Let U be the set of all ways to post $2n$ letters in $2n$ envelopes such that no letter is posted correctly.

   Let $S \subseteq \{L_1, L_2, \ldots, L_{2n}\}$. We define $np(S)$, the *number of complete pairs* in S, as follows:

$$np(S) = \left| \left\{ i \;\middle|\; 1 \leq i \leq n \textit{ and both } L_{2i-1} \textit{ and } L_{2i} \textit{ are in } S \right\} \right|$$

(a) (5 points) Let K_S be the set of all ways to post $2n$ letters in $2n$ envelopes such that every letter in S is posted correctly.

Express $|U|$ in terms of the cardinalities $|K_S|$, where $S \subseteq \{L_1, L_2, \ldots, L_{2n}\}$. Justify your answer.

(b) (10 points) Suppose $|S| = k$ and $np(S) = l$. What is the cardinality of set K_S? Prove your answer.

(c) (10 points) Count the number of subsets S which satisfy $|S| = k$ and $np(S) = l$. Prove your answer.

(d) (5 points) Write the full expression for $|U|$ using parts (a), (b), and (c) above. You do not need to simplify your answer.

Note. Exactly one letter can be posted in any envelope.

3. (30 points) Consider the recurrence:

$$a_n = \sum_{i=0}^{n-2} a_i a_{n-i-2}$$

with base cases $a_0 = a_1 = 1$.

(a) (10 points) Show that the generating function $\sum_{i=0}^{\infty} a_i z^i$ has positive radius R of absolute convergence.

Solution: (2 points) Let b_n be the Catalan numbers i.e., $b_0 = 1$ and

$$b_n = \sum_{i=0}^{n-1} b_i b_{n-i-1}, \ n \geq 1$$

We prove that, $a_n \leq b_n$ for all $n \geq 0$.

(1 point) *Base Case.* $a_0 = 1 = b_0$ and $a_1 = 1 = (b_0)^2 = b_1$.

(5 points)*Inductive Step.* Suppose $a_i \leq b_i$ for all $1 \leq i \leq n$. Then,

$$a_{n+1} = \sum_{i=0}^{n-i-1} a_i a_{n-i-1}$$

By inductive hypothesis $a_i \leq b_i$ and $a_{n-i-1} \leq b_{n-i-1}$ and hence:

$$a_{n+1} \leq \sum_{i=0}^{n-i-1} b_i b_{n-i-1}$$

Since $b_0 \leq b_1 \leq b_2 \leq \cdots$ (i.e., Catalan numbers are a non-decreasing sequence), we conclude that $b_{n-i-1} \leq b_{n-i}$, and hence:

$$a_{n+1} \leq \sum_{i=0}^{n-i} b_i b_{n-i} = b_{n+1}$$

Thus, the claim is true.

(2 points) By Lecture 7 (part 2), $b_{n+1} \leq 4^{n+1}$. Hence, $a_n \leq 4^{n+1}$ and $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq 4$. Thus, radius of absolute convergence of $\sum_{i=0}^{\infty} a_i z^i$ is at least $\frac{1}{4}$.

(b) (20 points) Let $G(z) = \sum_{i=0}^{\infty} a_i z^i$ for $|z| < R$. Find a simple expression for $G(z)$. Prove your answer.

Solution: (3 points) We have that

$$G(z) = \sum_{i=0}^{\infty} a_i z^i, \ |z| < R$$

Multiplying both sides by z^2 and arranging terms (since z^2 is absolutely convergent in the whole complex plane), we obtain:

$$z^2 G(z) = a_0 z^2 + a_1 z^3 + a_2 z^4 + \cdots, \ |z| < R$$

(3 points) Further,

$$z^2 G(z) \cdot G(z) = \Big(\sum_{i=0}^{\infty} a_i z^{i+2}\Big) \cdot \Big(\sum_{i=0}^{\infty} a_i z^i\Big), \ |z| < R$$

(3 points) Due to absolute convergence, we can rearrange terms to get:

$$z^2 \cdot (G(z))^2 = (a_0)^2 z^2 + (a_0 a_1 + a_1 a_0) z^3 + (a_0 a_2 + a_1 a_1 + a_2 a_0) z^4$$
$$+ (a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0) z^5 + \cdots, \ |z| < R$$

(2 points) Since $a_0 = a_1 = 1$, and $a_n = \sum_{i=0}^{n-2} a_i a_{n-i-2}$ for $n \geq 2$, we get that:

$$z^2 \cdot (G(z))^2 = a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots, \ |z| < R$$

(2 points) Since $\sum_{i=0}^{\infty} a_i z^i$ is absolutely convergent for $|z| < R$, we can write:

$$z^2 \cdot (G(z))^2 = G(z) - a_0 - a_1 z = G(z) - 1 - z, \ |z| < R$$

Or,

$$z^2 \cdot (G(z))^2 - G(z) + (1 + z) = 0, \ |z| < R$$

(3 points) By solving this quadratic equation, we get that, for any z ($|z| < R$), the possible values of $G(z)$ are:

$$\frac{1 \pm \sqrt{1-4z^2(1+z)}}{2z^2}$$

(2 points) Color a point z ($z \neq 0$) in the region $|z| < R$ red if $G(z) = \frac{1-\sqrt{1-4z^2(1+z)}}{2z^2}$, and color it blue if $G(z) = \frac{1+\sqrt{1-4z^2(1+z)}}{2z^2}$. By an argument similar to that used to prove Claim 1 on Slide 3 of Lecture 7 (part 3), we conclude that *all points except origin are of the same color.*

(2 points) Now, note that $G(0) = 1$. Note that:

$$\lim_{z \to 0} \frac{1 + \sqrt{1-4z^2(1+z)}}{2z^2} = \infty$$

$$\lim_{z \to 0} \frac{1 - \sqrt{1-4z^2(1+z)}}{2z^2} = 1$$

Thus, all points in $|z| < R$ are colored red, and hence:

$$G(z) = \frac{1 - \sqrt{1-4z^2(1+z)}}{2z^2}, \ |z| < R$$

4. (30 points)

   (a) (20 points) Solve the recurrence:

$$a_n = 2a_{n-1} - a_{n-2} + n(n-1)(n-2)$$

with base cases $a_0 = a_1 = 1$.

Prove your answer.

Solution:

(6 points). Let $f(n) = n(n-1)(n-2)$. Prove by induction, that $f(n) \leq 4^n$ for $n \geq 2$. The recurrence is a special case of the general recurrence discussed in page 3 of Interactive Session 18. We conclude, from there, than $a_n \leq \gamma^{n+1}$ for some positive real number γ. Thus, $a_0 + a_1 z + a_2 z^2 + \cdots$ has positive radius $R > 0$ of (absolute) convergence.

(2 points) Let function $G(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in $|z| < R$. Multiply both sides by $1 - 2z + z^2$:

$$(1 - 2z + z^2)G(z) = (1 - 2z + z^2) \cdot \Big(\sum_{i=0}^{\infty} a_i z^i\Big), \ |z| < R$$

(2 points) Due to absolute convergence, we can rearrange terms. Thus:

$$(1-2z+z^2)G(z) = a_0+(a_1-2a_0)z+(a_2-2a_1+a_0)z^2+(a_3-2a_2+a_1)z^3+\cdots, \ |z| < R$$

(2 points) Since $a_0 = a_1 = 1$ and a_n satisfies the given recurrence for $n \geq 2$, we get that:

$$(1-2z+z^2)G(z) = 1 - z + \sum_{n=2}^{\infty} n(n-1)(n-2)z^n, \ |z| < R$$

(2 points) Using Binomial Theorem, conclude that:

$$\frac{6z^3}{(1-z)^4} = \sum_{n=2}^{\infty} n(n-1)(n-2)z^n, \ |z| < 1$$

(2 points) Substituting in the above equation we get that:

$$G(z) = \frac{1-z}{(1-z)^2} + \frac{6z^3}{(1-z)^6} = \frac{1}{1-z} + \frac{6z^3}{(1-z)^6}, \ |z| < \min(1, R)$$

(2 points) Thus, for $n \geq 0$,

$$a_n = 1 + 6.[coefficient\ of\ z^{n-3}\ in\ (1-z)^{-6}]$$

(2 points) Write the full expression for coefficient of z^{n-3} in $(1-z)^{-6}$, using Binomial Theorem.

(b) (10 points) Prove or disprove:
You are given $2n$ balls $B_1, B_2, \ldots, B_{2n}$. You color each ball with exactly one of two colors: red and yellow. Then, there exist two balls B_i and B_j $(i \neq j)$ of the same color such that greatest common divisor (gcd) of i and j is 1.

Solution:

(5 points) The claim is false.

(5 points) Suppose $n = 2$, and ball B_1 is colored red, whereas ball B_2 is colored yellow.