Lecture 21: 21 June, 2021

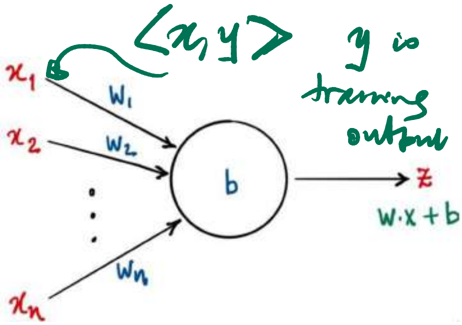
Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Data Mining and Machine Learning
April–July 2021

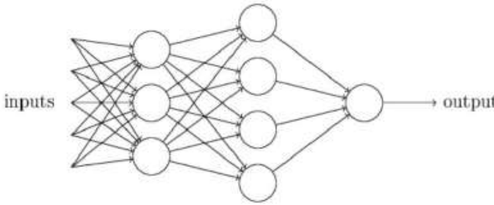
Linear separators and Perceptrons

- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)



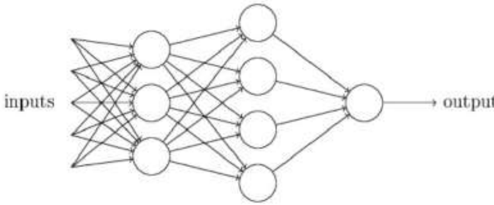
Linear separators and Perceptrons

- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)
- What if we cascade perceptrons?

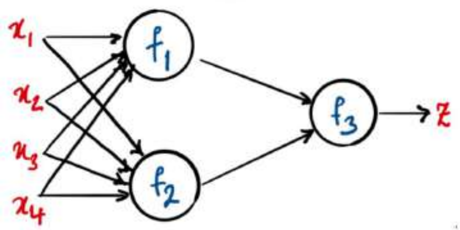


Linear separators and Perceptrons

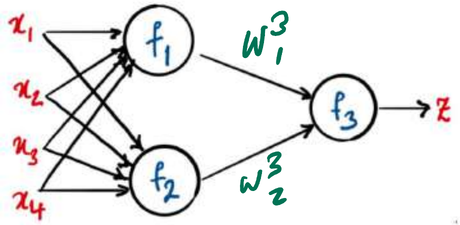
- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)
- What if we cascade perceptrons?
- Result is still a linear separator



- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)
- What if we cascade perceptrons?
- Result is still a linear separator
  - $f_1 = w_1 \cdot x + b_1, f_2 = w_2 \cdot x + b_2$

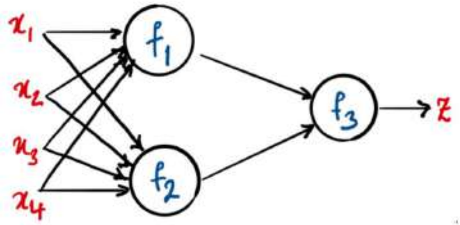


- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)
- What if we cascade perceptrons?
- Result is still a linear separator
  - $f_1 = w_1 \cdot x + b_1, f_2 = w_2 \cdot x + b_2$
  - $f_3 = w_3 \cdot \langle f_1, f_2 \rangle + b_3$

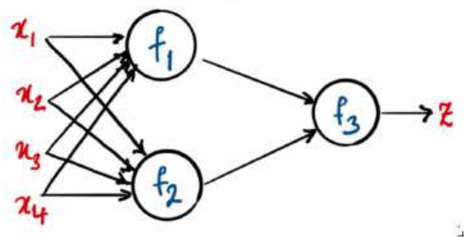


Linear separators and Perceptrons

- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)
- What if we cascade perceptrons?
- Result is still a linear separator
  - $f_1 = w_1 \cdot x + b_1, f_2 = w_2 \cdot x + b_2$
  - $f_3 = w_3 \cdot \langle f_1, f_2 \rangle + b_3$
  - $f_3 = w_3 \cdot \langle w_1 \cdot x + b_1, w_2 \cdot x + b_2 \rangle + b_3$

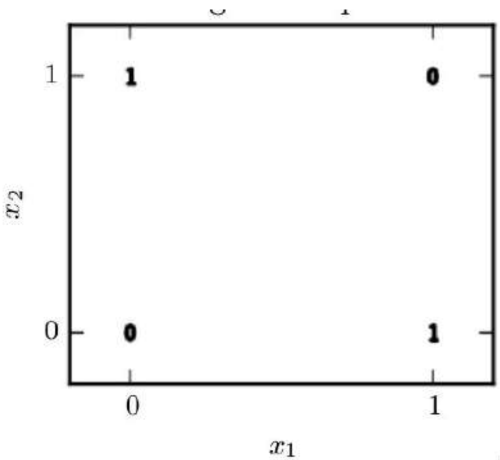


- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)
- What if we cascade perceptrons?
- Result is still a linear separator
  - $f_1 = w_1 \cdot x + b_1, f_2 = w_2 \cdot x + b_2$
  - $f_3 = w_3 \cdot \langle f_1, f_2 \rangle + b_3$
  - $f_3 = w_3 \cdot \langle w_1 \cdot x + b_1, w_2 \cdot x + b_2 \rangle + b_3$
  - $f_3 = \sum_{i=1}^{4} (w_{3_1} w_{1_i} + w_{3_2} w_{2_i}) \cdot x_i$
    $+(w_{3_1} b_1 + w_{3_2} b_2 + b_3)$

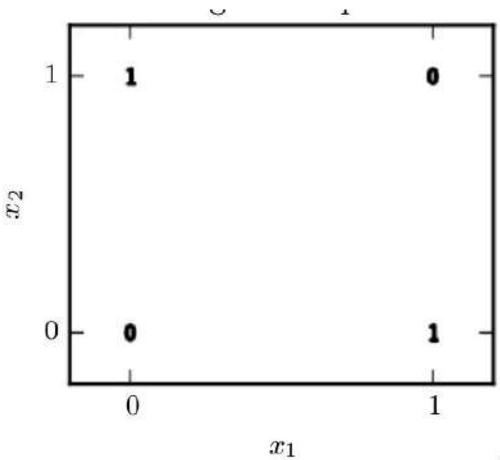


Limits of linearity

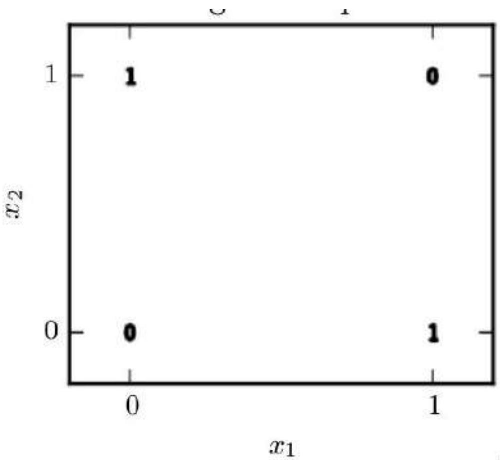
- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of x_1, x_2 is true (not both)



- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of x_1, x_2 is true (not both)
- Suppose $XOR(x_1, x_2) = ux_1 + vx_2 + b$

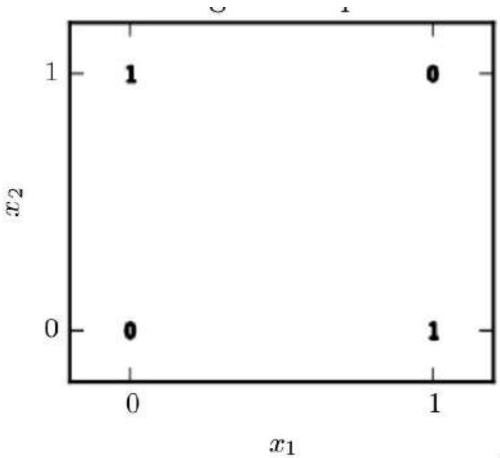


- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of x_1, x_2 is true (not both)
- Suppose $XOR(x_1, x_2) = ux_1 + vx_2 + b$
- $x_2 = 0$: As x_1 goes from 0 to 1, output goes from 0 to 1, so $u > 0$

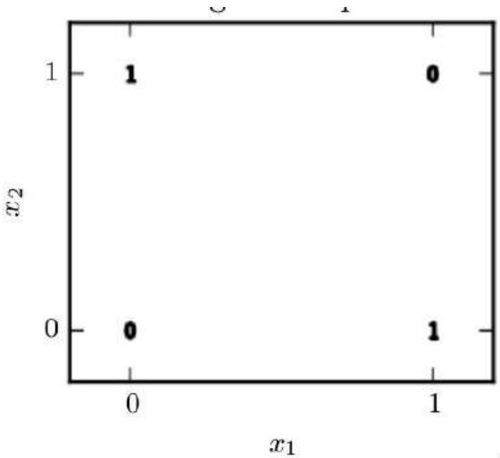


Limits of linearity

- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of x_1, x_2 is true (not both)
- Suppose $XOR(x_1, x_2) = ux_1 + vx_2 + b$
- $x_2 = 0$: As x_1 goes from 0 to 1, output goes from 0 to 1, so $u > 0$
- $x_2 = 1$: As x_1 goes from 0 to 1, output goes from 1 to 0, so $u < 0$

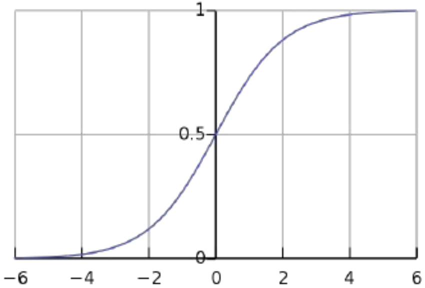


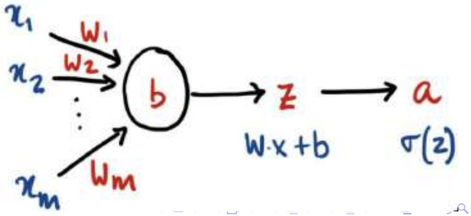
- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of x_1, x_2 is true (not both)
- Suppose $XOR(x_1, x_2) = ux_1 + vx_2 + b$
- $x_2 = 0$: As x_1 goes from 0 to 1, output goes from 0 to 1, so $u > 0$
- $x_2 = 1$: As x_1 goes from 0 to 1, output goes from 1 to 0, so $u < 0$
- Observed by Minsky and Papert, 1969, first "AI Winter"



Non-linear activation

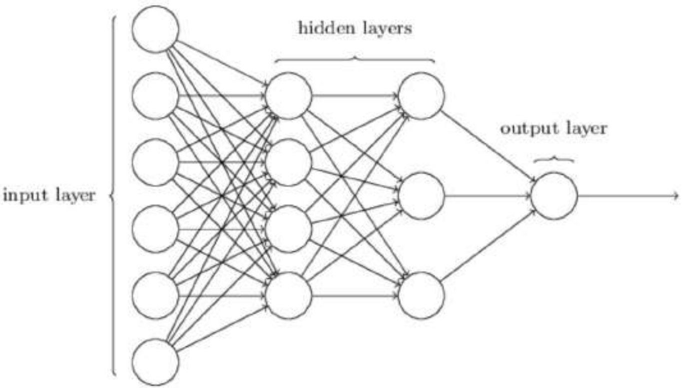
- Transform linear output z through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$





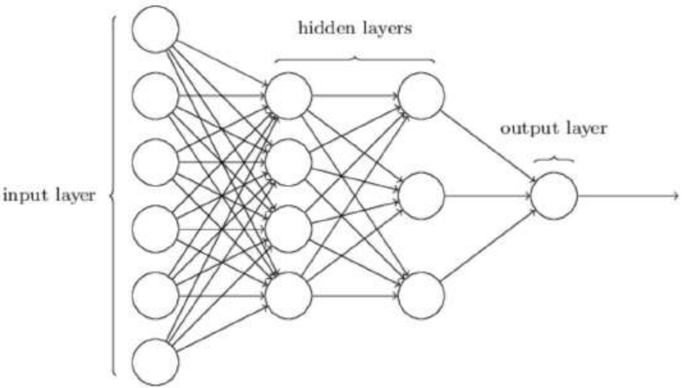
Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer



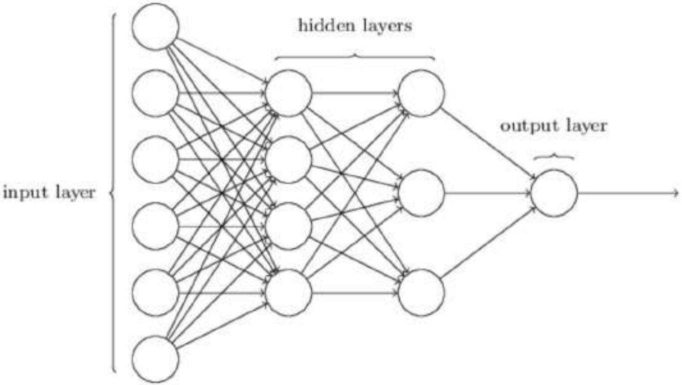
Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer
- Assumptions



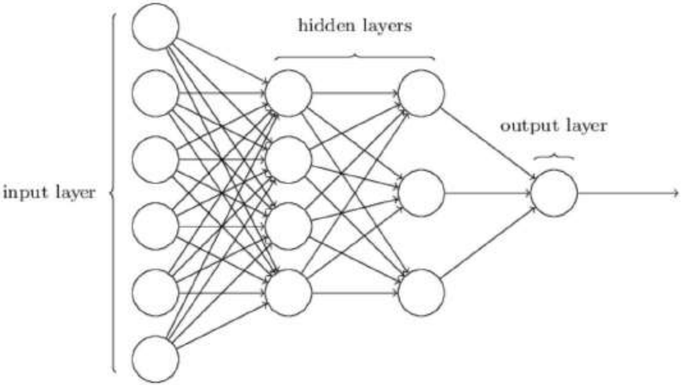
Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer
- Assumptions
  - Hidden neurons are arranged in layers



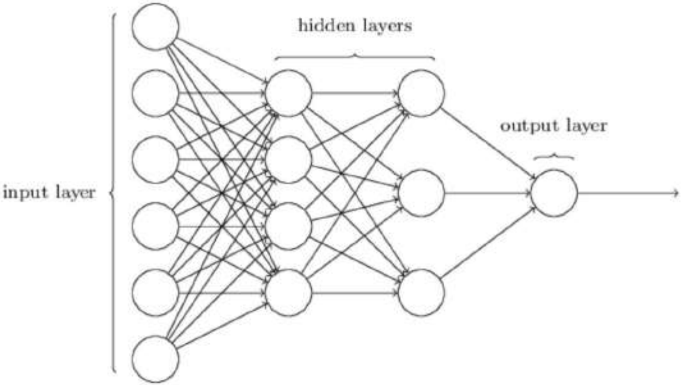
Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer
- Assumptions
  - Hidden neurons are arranged in layers
  - Each layer is fully connected to the next



Structure of a neural network

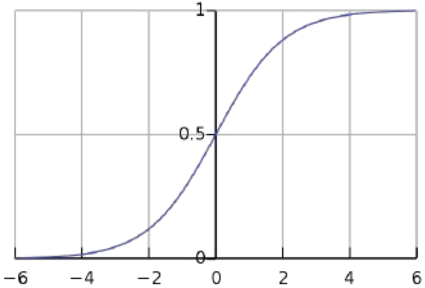
- Acyclic
- Input layer, hidden layers, output layer
- Assumptions
  - Hidden neurons are arranged in layers
  - Each layer is fully connected to the next
  - Set weight to zero to remove an edge

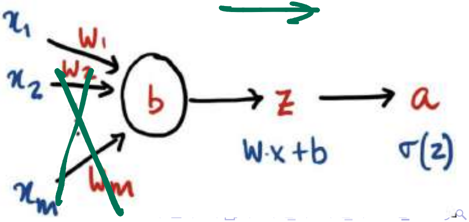


Non-linear activation

- Transform linear output z through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$
- Step is at $z = 0$
  - $z = wx + b$, so step is at $x = -\cancel{w/b}$ $-b/w$
  - Increasing w makes step steeper

Adjust b to shift the step





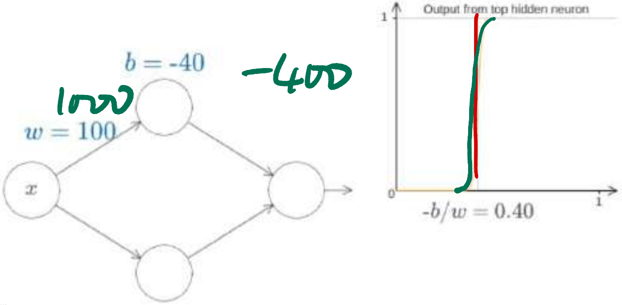
$X = \langle x_1 \rangle$

$Z = \overset{w_1}{W} \cdot X + b$

$w_1 x_1 + b$

Universality

- Create a step at $x = -\cancel{w/b}$ b/w

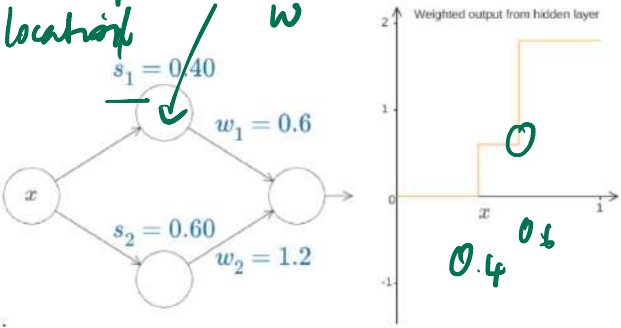


Universality

- Create a step at $x = -w/b$
- Cascade steps

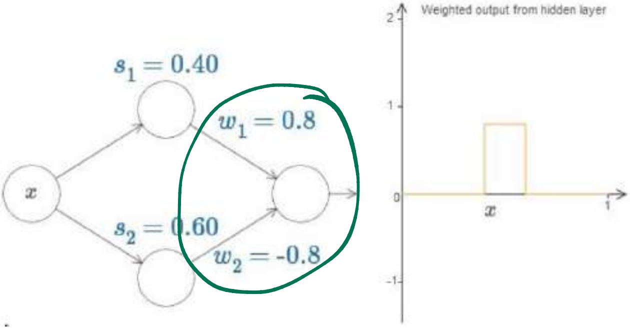
Step location

$-\frac{b}{w}$



Universality

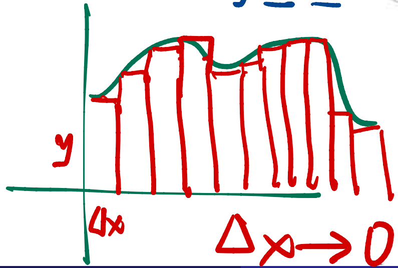
- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box

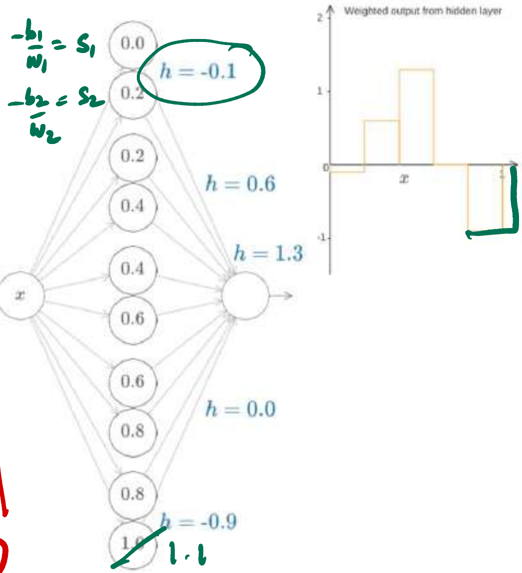


Universality

- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box
- Create many boxes

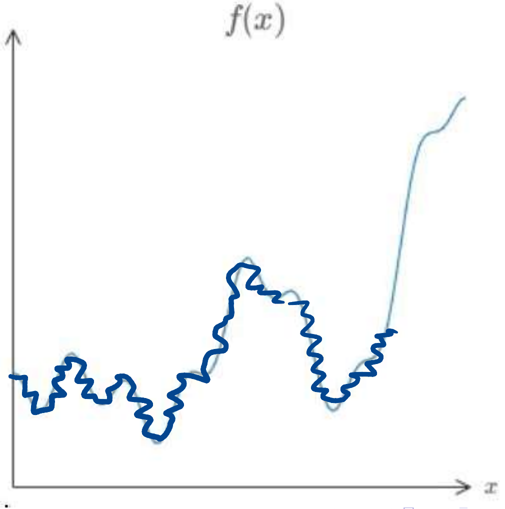
$\int f(x)\,dx$





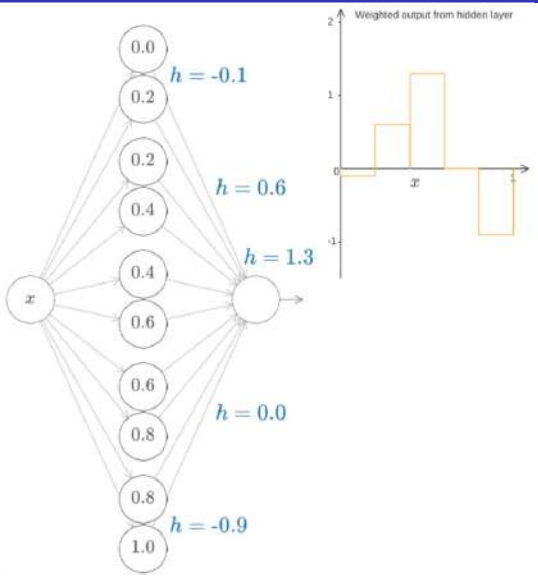
Universality

- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box
- Create many boxes
- Approximate any function



Universality

- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box
- Create many boxes
- Approximate any function
- Need only one hidden layer!

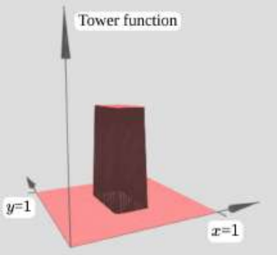


Non-linear activation

- With non-linear activation, network of neurons can approximate any function

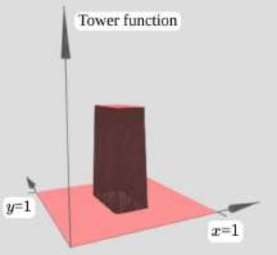
Non-linear activation

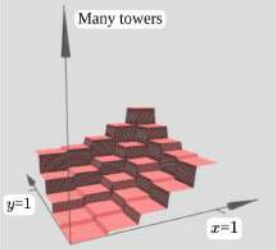
- With non-linear activation, network of neurons can approximate any function
  - Can build "rectangular" blocks



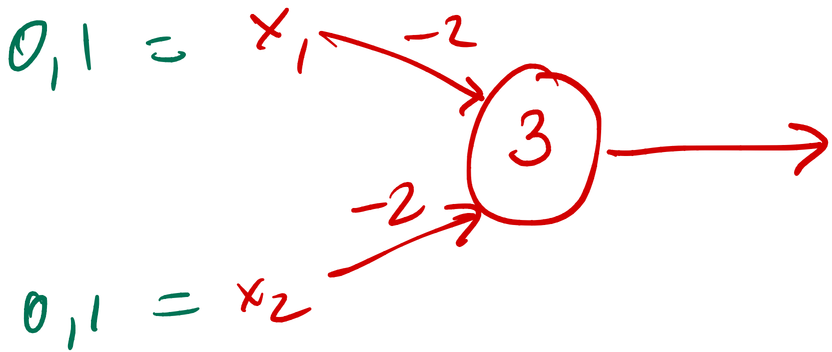
Non-linear activation

- With non-linear activation, network of neurons can approximate any function
  - Can build “rectangular” blocks
  - Combine blocks to capture any classification boundary





Related observation



x_1	x_2	z	NAND	AND
0	0	3	1	0
0	1	1	1	0
1	0	1	1	0
1	1	-1	0	1

$$z = -2 \cdot x_1 - 2x_2 + 3$$

True : > 0

False : < 0

Universality

NAND alone is universal

$f(x_1, x_2)$ – boolean function

4 combinations – 2 choices per combination

2^4 boolean functions $\quad 2 \times 2 \times 2 \times 2$

0,0 0,1 1,0 1,1

AND, NOT
OR, NOT → can define any boolean fn] UNIVERSAL

AND, OR is not universal

Example: Recognizing handwritten digits

- MNIST data set

Example: Recognizing handwritten digits

- MNIST data set
- 1000 samples of 10 handwritten digits
  - Assume input has been segmented

Example: Recognizing handwritten digits

- MNIST data set
- 1000 samples of 10 handwritten digits
  - Assume input has been segmented
- Each digit is 28×28 pixels
  - Grayscale value, 0 to 1
  - 784 pixels

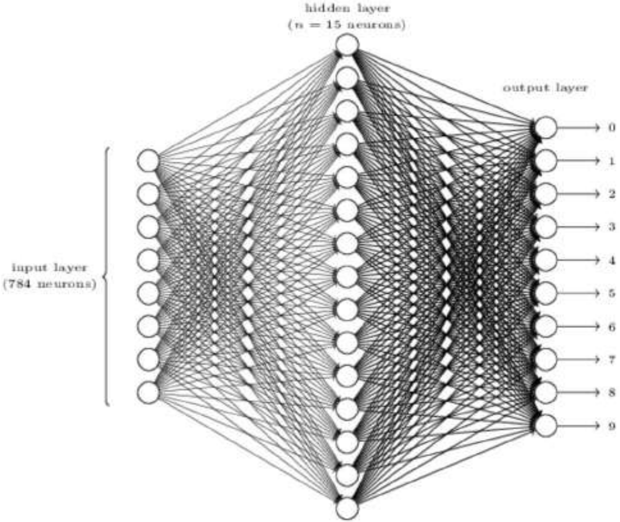
Example: Recognizing handwritten digits

- MNIST data set
- 1000 samples of 10 handwritten digits
  - Assume input has been segmented
- Each digit is 28×28 pixels
  - Grayscale value, 0 to 1
  - 784 pixels
- Input $x = (x_1, x_2, \ldots, x_{784})$

1 2 3 4 5

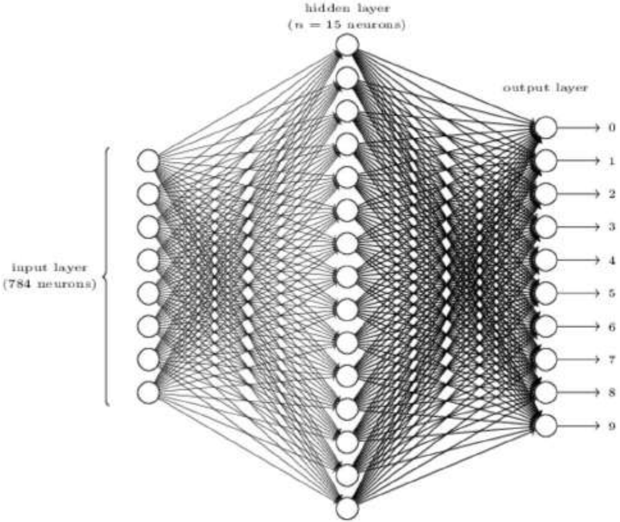
Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$



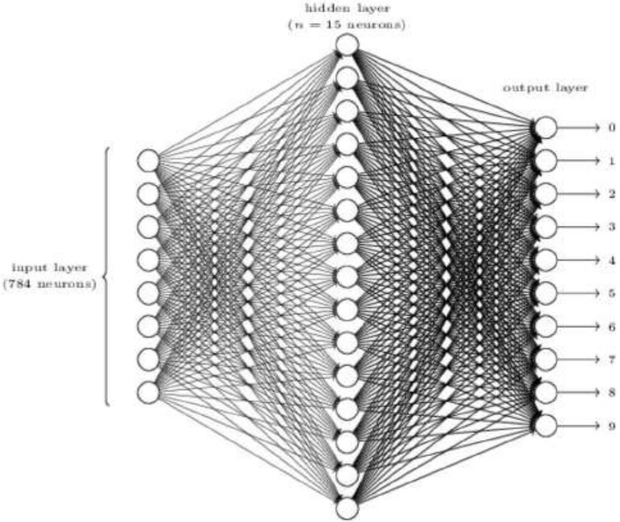
Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$
- Single hidden layer, 15 nodes



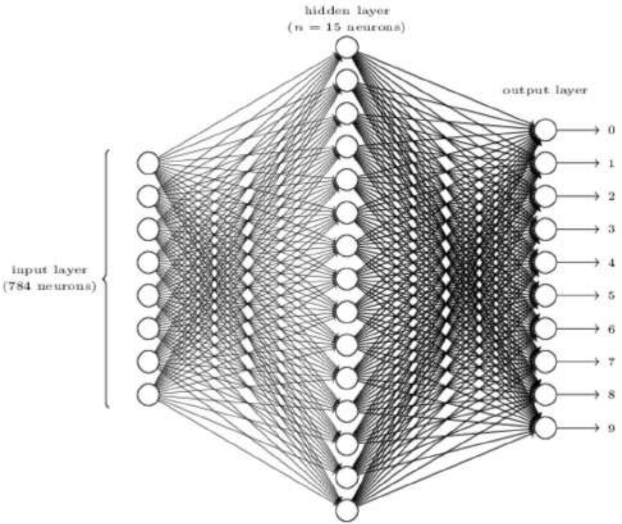
Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$
- Single hidden layer, 15 nodes
- Output layer, 10 nodes
  - Decision a_j for each digit $j \in \{0, 1, \ldots, 9\}$



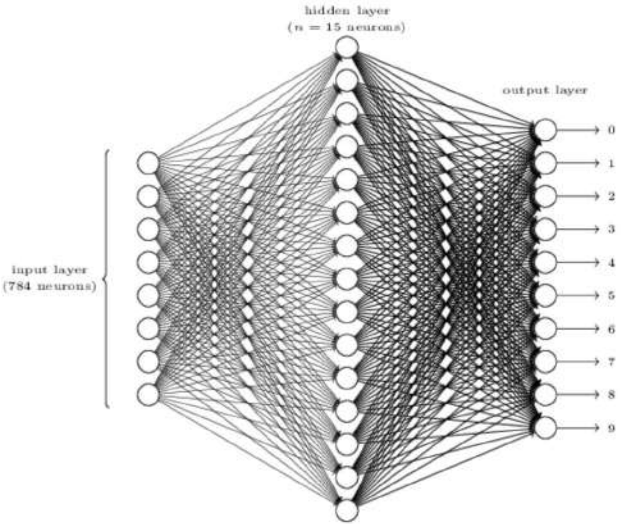
Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$
- Single hidden layer, 15 nodes
- Output layer, 10 nodes
  - Decision a_j for each digit $j \in \{0, 1, \ldots, 9\}$
- Final output is best a_j



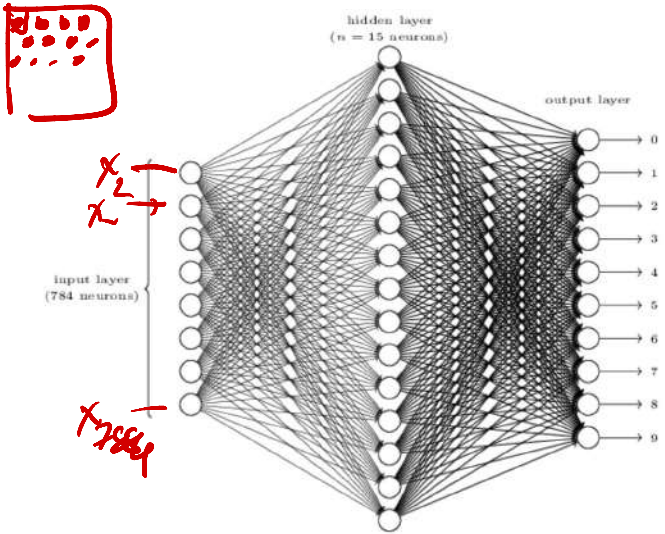
Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$
- Single hidden layer, 15 nodes
- Output layer, 10 nodes
  - Decision a_j for each digit $j \in \{0, 1, \ldots, 9\}$
- Final output is best a_j
  - Naïvely, $\arg\max_j a_j$



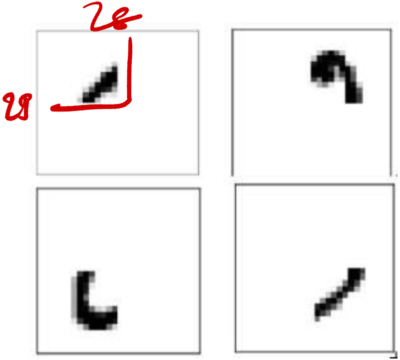
Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$
- Single hidden layer, 15 nodes
- Output layer, 10 nodes
  - Decision a_j for each digit $j \in \{0, 1, \ldots, 9\}$
- Final output is best a_j
  - Naïvely, $\arg\max_j a_j$
  - Softmax, $\arg\max_j \dfrac{e^{a_j}}{\sum_j e^{a_j}}$
    - "Smooth" version of $\arg\max$



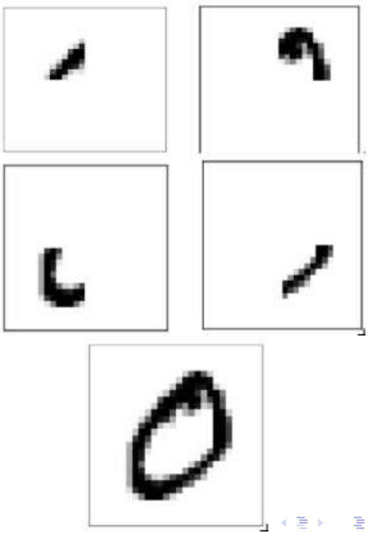
Example: Extracting features

- Hidden layers extract features
  - For instance, patterns in different quadrants



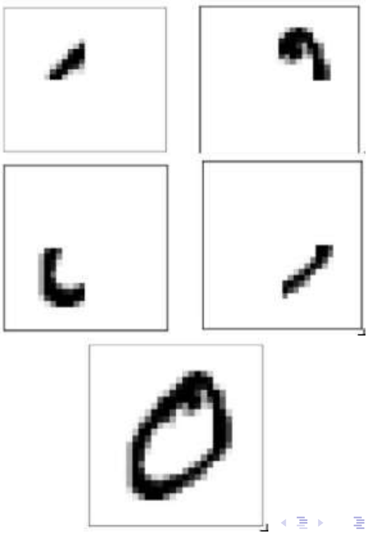
Example: Extracting features

- Hidden layers extract features
  - For instance, patterns in different quadrants
- Combination of features determines output



Example: Extracting features

- Hidden layers extract features
  - For instance, patterns in different quadrants
- Combination of features determines output
- Claim: Automatic identification of features is strength of the model



Example: Extracting features

- Hidden layers extract features
  - For instance, patterns in different quadrants
- Combination of features determines output
- Claim: Automatic identification of features is strength of the model
- Counter argument: implicitly extracted features are impossible to interpret
  - Explainability