Chapter 4 – Training Linear Models

This notebook contains all the sample code and solutions to the exercises in chapter 4.

[Run in Google Colab (https://colab.research.google.com/github/ageron/handson-ml2/blob/master/04_training_linear_models.ipynb)](https://colab.research.google.com/github/ageron/handson-ml2/blob/master/04_training_linear_models.ipynb)

Setup

First, let's import a few common modules, ensure MatplotLib plots figures inline and prepare a function to save the figures. We also check that Python 3.5 or later is installed (although Python 2.x may work, it is deprecated so we strongly recommend you use Python 3 instead), as well as Scikit-Learn ≥0.20.

In [1]:

```
# Python ≥3.5 is required
import sys
assert sys.version_info >= (3, 5)

# Scikit-Learn ≥0.20 is required
import sklearn
assert sklearn.__version__ >= "0.20"

# Common imports
import numpy as np
import os

# to make this notebook's output stable across runs
np.random.seed(42)

# To plot pretty figures
%matplotlib inline
import matplotlib as mpl
import matplotlib.pyplot as plt
mpl.rc('axes', labelsize=14)
mpl.rc('xtick', labelsize=12)
mpl.rc('ytick', labelsize=12)

# Where to save the figures
PROJECT_ROOT_DIR = "."
CHAPTER_ID = "training_linear_models"
IMAGES_PATH = os.path.join(PROJECT_ROOT_DIR, "images", CHAPTER_ID)
os.makedirs(IMAGES_PATH, exist_ok=True)

def save_fig(fig_id, tight_layout=True, fig_extension="png", resolution=
    path = os.path.join(IMAGES_PATH, fig_id + "." + fig_extension)
    print("Saving figure", fig_id)
    if tight_layout:
        plt.tight_layout()
    plt.savefig(path, format=fig_extension, dpi=resolution)

# Ignore useless warnings (see SciPy issue #5998)
import warnings
warnings.filterwarnings(action="ignore", message="^internal gelsd")
```

Linear regression using the Normal Equation

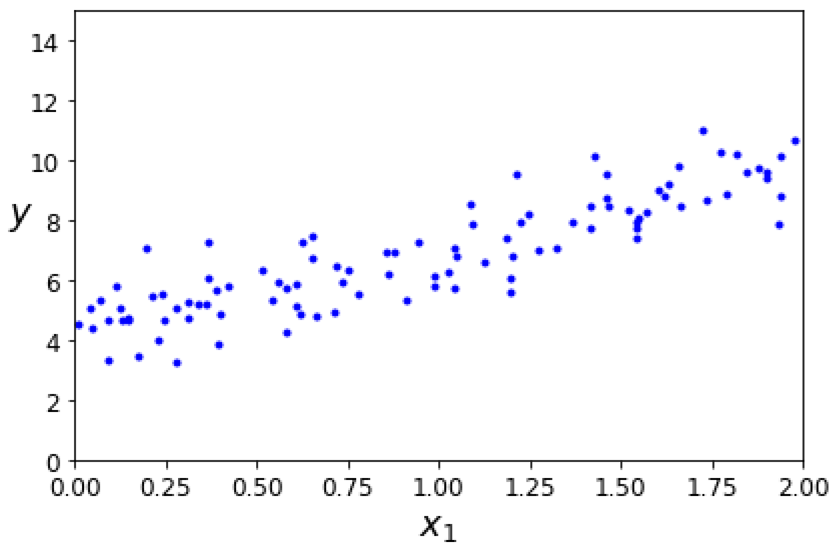
In [2]:

```
import numpy as np

X = 2 * np.random.rand(100, 1)
y = 4 + 3 * X + np.random.randn(100, 1)
```

In [3]:

```
plt.plot(X, y, "b.")
plt.xlabel("$x_1$", fontsize=18)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.axis([0, 2, 0, 15])
save_fig("generated_data_plot")
plt.show()
```

Saving figure generated_data_plot



In [4]:

```
X_b = np.c_[np.ones((100, 1)), X]  # add x0 = 1 to each instance
theta_best = np.linalg.inv(X_b.T.dot(X_b)).dot(X_b.T).dot(y)
```

In [5]:

```
theta_best
```

Out[5]:

```
array([[4.21509616],
       [2.77011339]])
```

In [6]:

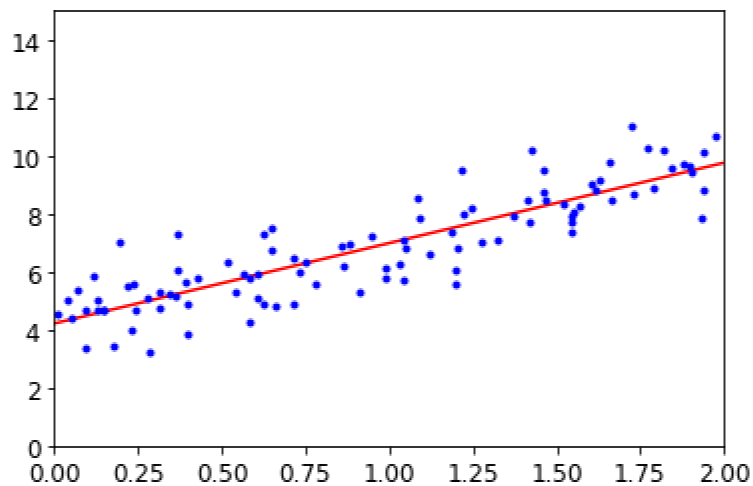
```
X_new = np.array([[0], [2]])
X_new_b = np.c_[np.ones((2, 1)), X_new]  # add x0 = 1 to each instance
y_predict = X_new_b.dot(theta_best)
y_predict
```

Out[6]:

```
array([[4.21509616],
       [9.75532293]])
```

In [7]:

```
plt.plot(X_new, y_predict, "r-")
plt.plot(X, y, "b.")
plt.axis([0, 2, 0, 15])
plt.show()
```

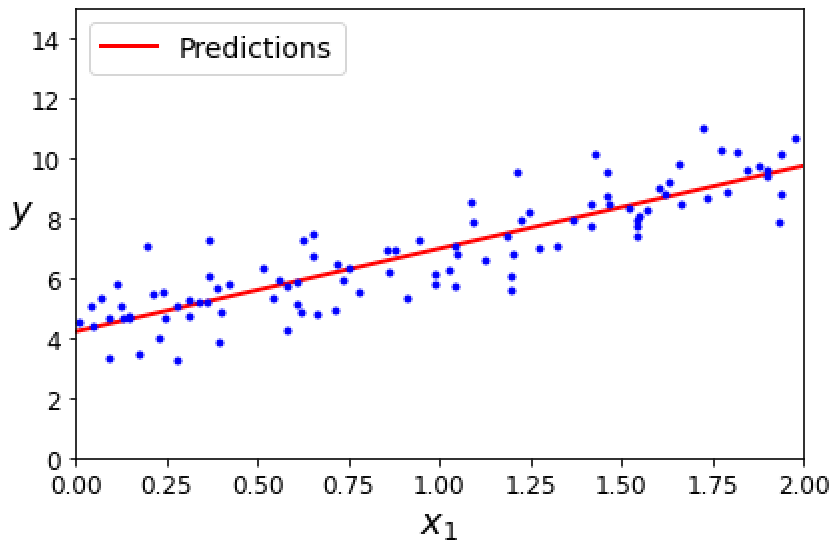


The figure in the book actually corresponds to the following code, with a legend and axis labels:

In [8]:

```
plt.plot(X_new, y_predict, "r-", linewidth=2, label="Predictions")
plt.plot(X, y, "b.")
plt.xlabel("$x_1$", fontsize=18)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.legend(loc="upper left", fontsize=14)
plt.axis([0, 2, 0, 15])
save_fig("linear_model_predictions_plot")
plt.show()
```

```
Saving figure linear_model_predictions_plot
```



In [9]:

```
from sklearn.linear_model import LinearRegression

lin_reg = LinearRegression()
lin_reg.fit(X, y)
lin_reg.intercept_, lin_reg.coef_
```

Out[9]:

```
(array([4.21509616]), array([[2.77011339]]))
```

In [10]:

```
lin_reg.predict(X_new)
```

Out[10]:

```
array([[4.21509616],
       [9.75532293]])
```

The `LinearRegression` class is based on the `scipy.linalg.lstsq()` function (the name stands for "least squares"), which you could call directly:

In [11]:

```
theta_best_svd, residuals, rank, s = np.linalg.lstsq(X_b, y, rcond=1e-6)
theta_best_svd
```

Out[11]:

```
array([[4.21509616],
       [2.77011339]])
```

This function computes $\mathbf{X}^{+}\mathbf{y}$, where $\mathbf{X}^{+}$ is the *pseudoinverse* of $\mathbf{X}$ (specifically the Moore-Penrose inverse). You can use `np.linalg.pinv()` to compute the pseudoinverse directly:

In [12]:

```
np.linalg.pinv(X_b).dot(y)
```

Out[12]:

```
array([[4.21509616],
       [2.77011339]])
```

Linear regression using batch gradient descent

In [13]:

```
eta = 0.1  # learning rate
n_iterations = 1000
m = 100

theta = np.random.randn(2,1)  # random initialization

for iteration in range(n_iterations):
    gradients = 2/m * X_b.T.dot(X_b.dot(theta) - y)
    theta = theta - eta * gradients
```

In [14]:

```
theta
```

Out[14]:

```
array([[4.21509616],
       [2.77011339]])
```

In [15]:

```
X_new_b.dot(theta)
```

Out[15]:

```
array([[4.21509616],
       [9.75532293]])
```

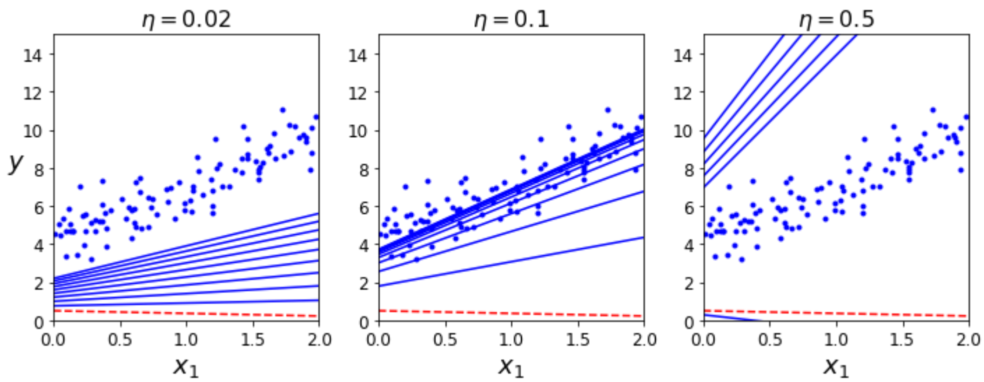
In [16]:

```
theta_path_bgd = []

def plot_gradient_descent(theta, eta, theta_path=None):
    m = len(X_b)
    plt.plot(X, y, "b.")
    n_iterations = 1000
    for iteration in range(n_iterations):
        if iteration < 10:
            y_predict = X_new_b.dot(theta)
            style = "b-" if iteration > 0 else "r--"
            plt.plot(X_new, y_predict, style)
        gradients = 2/m * X_b.T.dot(X_b.dot(theta) - y)
        theta = theta - eta * gradients
        if theta_path is not None:
            theta_path.append(theta)
    plt.xlabel("$x_1$", fontsize=18)
    plt.axis([0, 2, 0, 15])
    plt.title(r"$\eta = {}$".format(eta), fontsize=16)
```

In [17]:

```
np.random.seed(42)
theta = np.random.randn(2,1)  # random initialization

plt.figure(figsize=(10,4))
plt.subplot(131); plot_gradient_descent(theta, eta=0.02)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.subplot(132); plot_gradient_descent(theta, eta=0.1, theta_path=theta
plt.subplot(133); plot_gradient_descent(theta, eta=0.5)

save_fig("gradient_descent_plot")
plt.show()
```

```
Saving figure gradient_descent_plot
```



Stochastic Gradient Descent

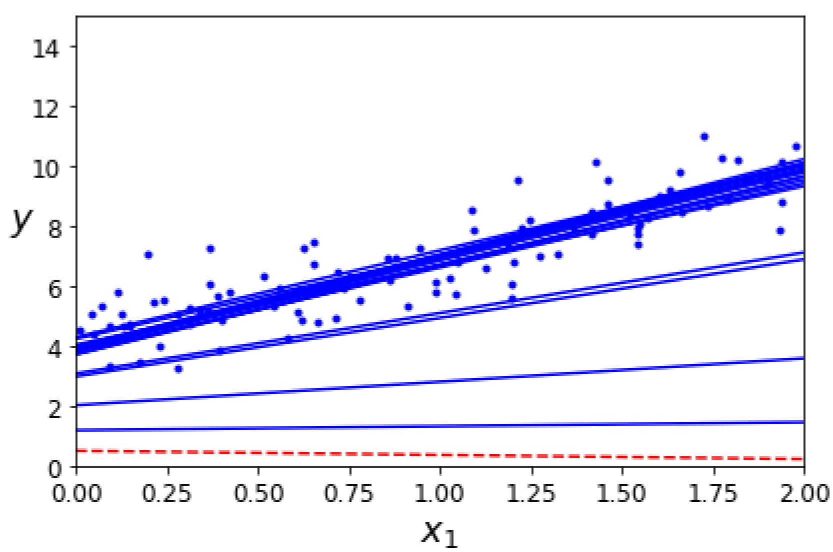
In [18]:

```
theta_path_sgd = []
m = len(X_b)
np.random.seed(42)
```

In [19]:

```
n_epochs = 50
t0, t1 = 5, 50  # learning schedule hyperparameters

def learning_schedule(t):
    return t0 / (t + t1)

theta = np.random.randn(2,1)  # random initialization

for epoch in range(n_epochs):
    for i in range(m):
        if epoch == 0 and i < 20:                    # not shown in the
            y_predict = X_new_b.dot(theta)           # not shown
            style = "b-" if i > 0 else "r--"         # not shown
            plt.plot(X_new, y_predict, style)        # not shown
        random_index = np.random.randint(m)
        xi = X_b[random_index:random_index+1]
        yi = y[random_index:random_index+1]
        gradients = 2 * xi.T.dot(xi.dot(theta) - yi)
        eta = learning_schedule(epoch * m + i)
        theta = theta - eta * gradients
        theta_path_sgd.append(theta)                 # not shown

plt.plot(X, y, "b.")                                 # not shown
plt.xlabel("$x_1$", fontsize=18)                     # not shown
plt.ylabel("$y$", rotation=0, fontsize=18)           # not shown
plt.axis([0, 2, 0, 15])                              # not shown
save_fig("sgd_plot")                                 # not shown
plt.show()                                           # not shown
```

```
Saving figure sgd_plot
```



In [20]:

```
theta
```

Out[20]:

```
array([[4.21076011],
       [2.74856079]])
```

In [21]:

```
from sklearn.linear_model import SGDRegressor

sgd_reg = SGDRegressor(max_iter=1000, tol=1e-3, penalty=None, eta0=0.1,
sgd_reg.fit(X, y.ravel())
```

Out[21]:

```
SGDRegressor(eta0=0.1, penalty=None, random_state=42)
```

In [22]:

```
sgd_reg.intercept_, sgd_reg.coef_
```

Out[22]:

```
(array([4.24365286]), array([2.8250878]))
```

Mini-batch gradient descent

In [23]:

```
theta_path_mgd = []

n_iterations = 50
minibatch_size = 20

np.random.seed(42)
theta = np.random.randn(2,1)  # random initialization

t0, t1 = 200, 1000
def learning_schedule(t):
    return t0 / (t + t1)

t = 0
for epoch in range(n_iterations):
    shuffled_indices = np.random.permutation(m)
    X_b_shuffled = X_b[shuffled_indices]
    y_shuffled = y[shuffled_indices]
    for i in range(0, m, minibatch_size):
        t += 1
        xi = X_b_shuffled[i:i+minibatch_size]
        yi = y_shuffled[i:i+minibatch_size]
        gradients = 2/minibatch_size * xi.T.dot(xi.dot(theta) - yi)
        eta = learning_schedule(t)
        theta = theta - eta * gradients
        theta_path_mgd.append(theta)
```

In [24]:

```
theta
```

Out[24]:

```
array([[4.25214635],
       [2.7896408 ]])
```

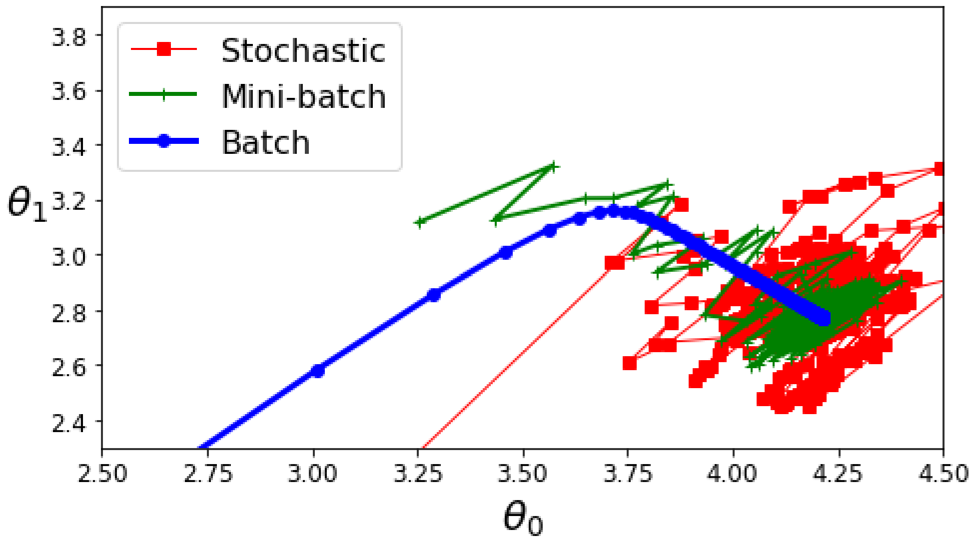
In [25]:

```
theta_path_bgd = np.array(theta_path_bgd)
theta_path_sgd = np.array(theta_path_sgd)
theta_path_mgd = np.array(theta_path_mgd)
```

In [26]:

```
plt.figure(figsize=(7,4))
plt.plot(theta_path_sgd[:, 0], theta_path_sgd[:, 1], "r-s", linewidth=1,
plt.plot(theta_path_mgd[:, 0], theta_path_mgd[:, 1], "g-+", linewidth=2,
plt.plot(theta_path_bgd[:, 0], theta_path_bgd[:, 1], "b-o", linewidth=3,
plt.legend(loc="upper left", fontsize=16)
plt.xlabel(r"$\theta_0$", fontsize=20)
plt.ylabel(r"$\theta_1$   ", fontsize=20, rotation=0)
plt.axis([2.5, 4.5, 2.3, 3.9])
save_fig("gradient_descent_paths_plot")
plt.show()
```

```
Saving figure gradient_descent_paths_plot
```



Polynomial regression

In [27]:

```
import numpy as np
import numpy.random as rnd

np.random.seed(42)
```

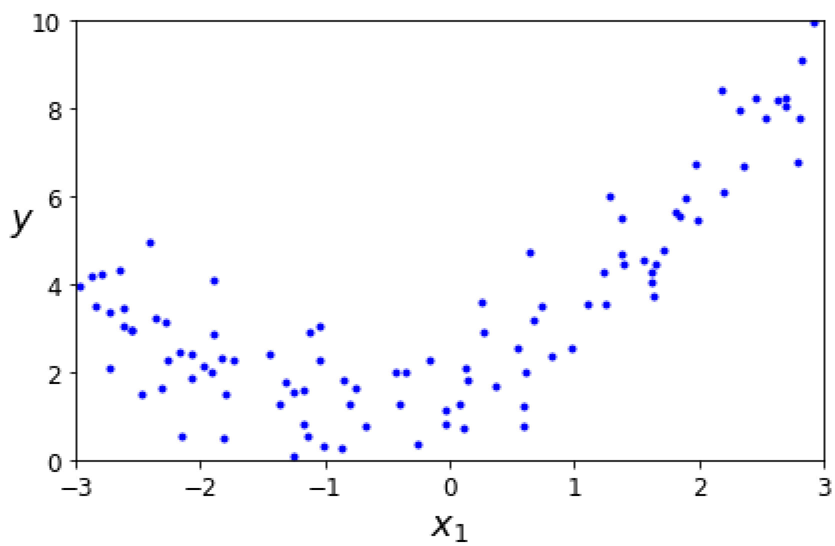
In [28]:

```
m = 100
X = 6 * np.random.rand(m, 1) - 3
y = 0.5 * X**2 + X + 2 + np.random.randn(m, 1)
```

In [29]:

```
plt.plot(X, y, "b.")
plt.xlabel("$x_1$", fontsize=18)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.axis([-3, 3, 0, 10])
save_fig("quadratic_data_plot")
plt.show()
```

Saving figure quadratic_data_plot



In [30]:

```
from sklearn.preprocessing import PolynomialFeatures
poly_features = PolynomialFeatures(degree=2, include_bias=False)
X_poly = poly_features.fit_transform(X)
X[0]
```

Out[30]:

```
array([-0.75275929])
```

In [31]:

```
X_poly[0]
```

Out[31]:

```
array([-0.75275929,  0.56664654])
```

In [32]:

```
lin_reg = LinearRegression()
lin_reg.fit(X_poly, y)
lin_reg.intercept_, lin_reg.coef_
```

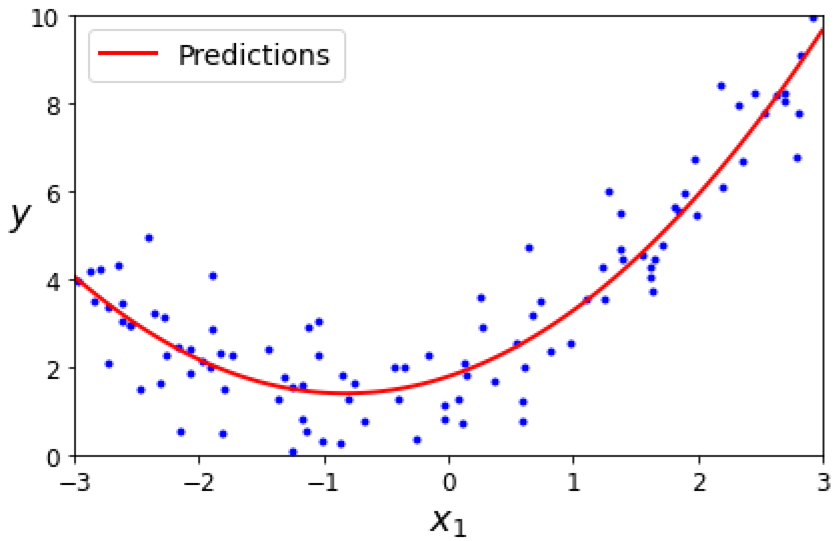
Out[32]:

```
(array([1.78134581]), array([[0.93366893, 0.56456263]]))
```

In [33]:

```
X_new=np.linspace(-3, 3, 100).reshape(100, 1)
X_new_poly = poly_features.transform(X_new)
y_new = lin_reg.predict(X_new_poly)
plt.plot(X, y, "b.")
plt.plot(X_new, y_new, "r-", linewidth=2, label="Predictions")
plt.xlabel("$x_1$", fontsize=18)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.legend(loc="upper left", fontsize=14)
plt.axis([-3, 3, 0, 10])
save_fig("quadratic_predictions_plot")
plt.show()
```

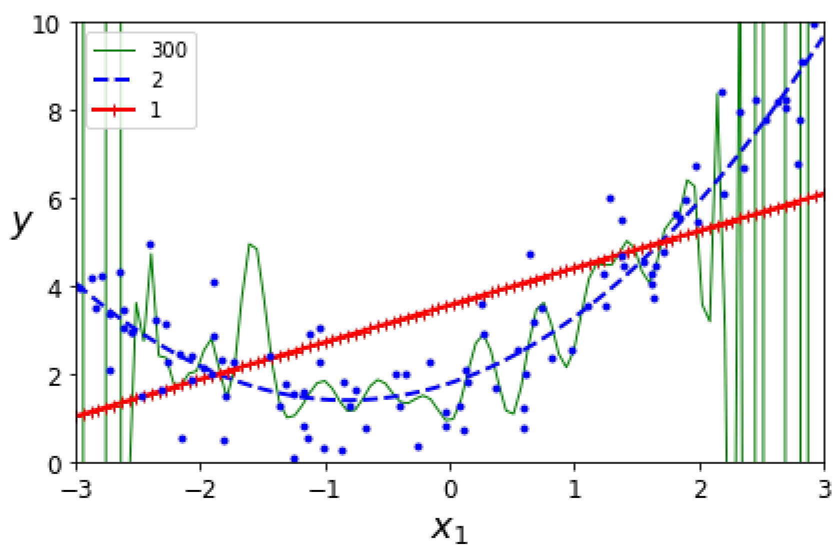
```
Saving figure quadratic_predictions_plot
```



In [34]:

```
from sklearn.preprocessing import StandardScaler
from sklearn.pipeline import Pipeline

for style, width, degree in (("g-", 1, 300), ("b--", 2, 2), ("r-+", 2, 1
    polybig_features = PolynomialFeatures(degree=degree, include_bias=Fa
    std_scaler = StandardScaler()
    lin_reg = LinearRegression()
    polynomial_regression = Pipeline([
            ("poly_features", polybig_features),
            ("std_scaler", std_scaler),
            ("lin_reg", lin_reg),
        ])
    polynomial_regression.fit(X, y)
    y_newbig = polynomial_regression.predict(X_new)
    plt.plot(X_new, y_newbig, style, label=str(degree), linewidth=width)

plt.plot(X, y, "b.", linewidth=3)
plt.legend(loc="upper left")
plt.xlabel("$x_1$", fontsize=18)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.axis([-3, 3, 0, 10])
save_fig("high_degree_polynomials_plot")
plt.show()
```

```
Saving figure high_degree_polynomials_plot
```



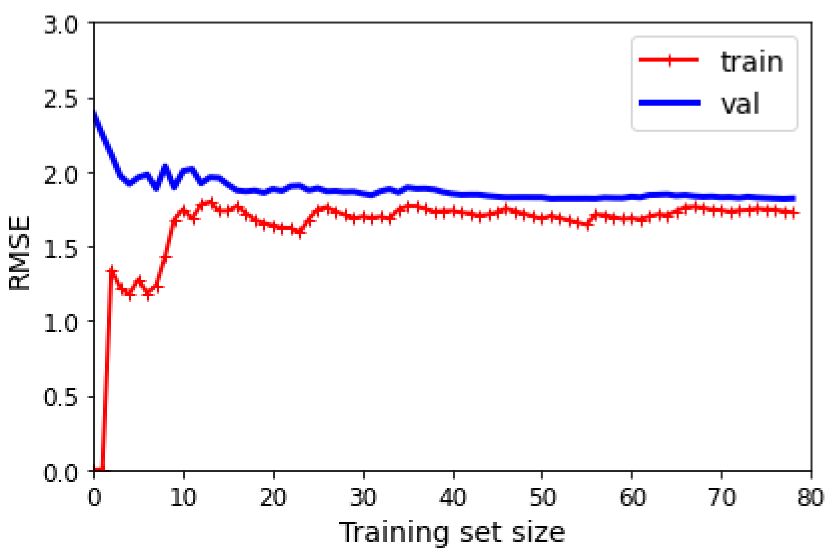
In [35]:

```
from sklearn.metrics import mean_squared_error
from sklearn.model_selection import train_test_split

def plot_learning_curves(model, X, y):
    X_train, X_val, y_train, y_val = train_test_split(X, y, test_size=0.
    train_errors, val_errors = [], []
    for m in range(1, len(X_train)):
        model.fit(X_train[:m], y_train[:m])
        y_train_predict = model.predict(X_train[:m])
        y_val_predict = model.predict(X_val)
        train_errors.append(mean_squared_error(y_train[:m], y_train_pred
        val_errors.append(mean_squared_error(y_val, y_val_predict))

    plt.plot(np.sqrt(train_errors), "r-+", linewidth=2, label="train")
    plt.plot(np.sqrt(val_errors), "b-", linewidth=3, label="val")
    plt.legend(loc="upper right", fontsize=14)   # not shown in the book
    plt.xlabel("Training set size", fontsize=14) # not shown
    plt.ylabel("RMSE", fontsize=14)              # not shown
```

In [36]:

```
lin_reg = LinearRegression()
plot_learning_curves(lin_reg, X, y)
plt.axis([0, 80, 0, 3])                         # not shown in the book
save_fig("underfitting_learning_curves_plot")   # not shown
plt.show()                                      # not shown
```

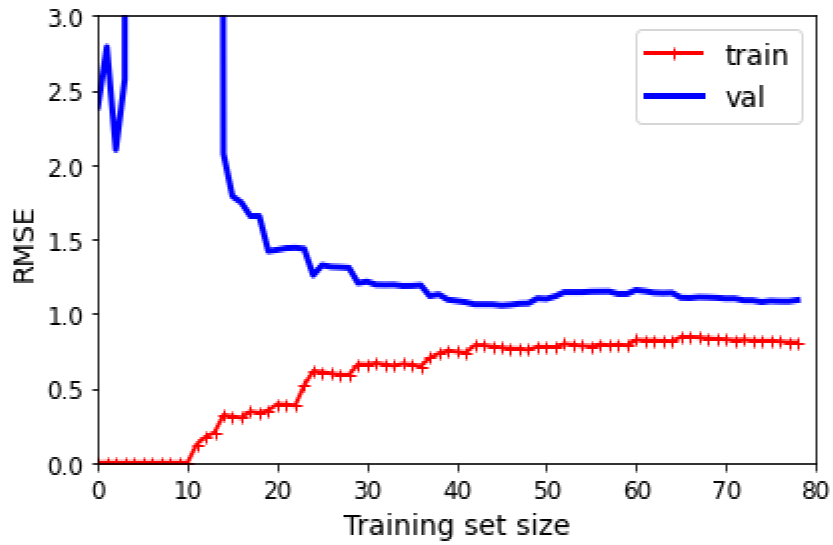
Saving figure underfitting_learning_curves_plot



In [37]:

```
from sklearn.pipeline import Pipeline

polynomial_regression = Pipeline([
        ("poly_features", PolynomialFeatures(degree=10, include_bias=Fal
        ("lin_reg", LinearRegression()),
    ])

plot_learning_curves(polynomial_regression, X, y)
plt.axis([0, 80, 0, 3])           # not shown
save_fig("learning_curves_plot")  # not shown
plt.show()                        # not shown
```

Saving figure learning_curves_plot



Regularized models

In [38]:

```
np.random.seed(42)
m = 20
X = 3 * np.random.rand(m, 1)
y = 1 + 0.5 * X + np.random.randn(m, 1) / 1.5
X_new = np.linspace(0, 3, 100).reshape(100, 1)
```

In [39]:

```
from sklearn.linear_model import Ridge
ridge_reg = Ridge(alpha=1, solver="cholesky", random_state=42)
ridge_reg.fit(X, y)
ridge_reg.predict([[1.5]])
```

Out[39]:

```
array([[1.55071465]])
```

In [40]:

```
ridge_reg = Ridge(alpha=1, solver="sag", random_state=42)
ridge_reg.fit(X, y)
ridge_reg.predict([[1.5]])
```

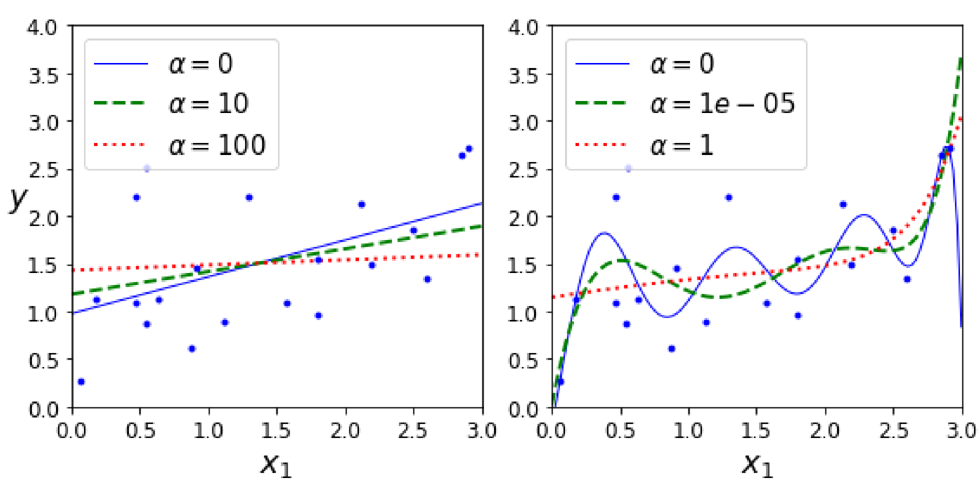
Out[40]:

```
array([[1.5507201]])
```

In [41]:

```
from sklearn.linear_model import Ridge

def plot_model(model_class, polynomial, alphas, **model_kargs):
    for alpha, style in zip(alphas, ("b-", "g--", "r:")):
        model = model_class(alpha, **model_kargs) if alpha > 0 else Line
        if polynomial:
            model = Pipeline([
                    ("poly_features", PolynomialFeatures(degree=10, incl
                    ("std_scaler", StandardScaler()),
                    ("regul_reg", model),
                ])
        model.fit(X, y)
        y_new_regul = model.predict(X_new)
        lw = 2 if alpha > 0 else 1
        plt.plot(X_new, y_new_regul, style, linewidth=lw, label=r"$\alph
    plt.plot(X, y, "b.", linewidth=3)
    plt.legend(loc="upper left", fontsize=15)
    plt.xlabel("$x_1$", fontsize=18)
    plt.axis([0, 3, 0, 4])

plt.figure(figsize=(8,4))
plt.subplot(121)
plot_model(Ridge, polynomial=False, alphas=(0, 10, 100), random_state=42
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.subplot(122)
plot_model(Ridge, polynomial=True, alphas=(0, 10**-5, 1), random_state=4

save_fig("ridge_regression_plot")
plt.show()
```

Saving figure ridge_regression_plot



Note: to be future-proof, we set `max_iter=1000` and `tol=1e-3` because these will be the default values in Scikit-Learn 0.21.

In [42]:

```
sgd_reg = SGDRegressor(penalty="l2", max_iter=1000, tol=1e-3, random_sta
sgd_reg.fit(X, y.ravel())
sgd_reg.predict([[1.5]])
```

Out[42]:

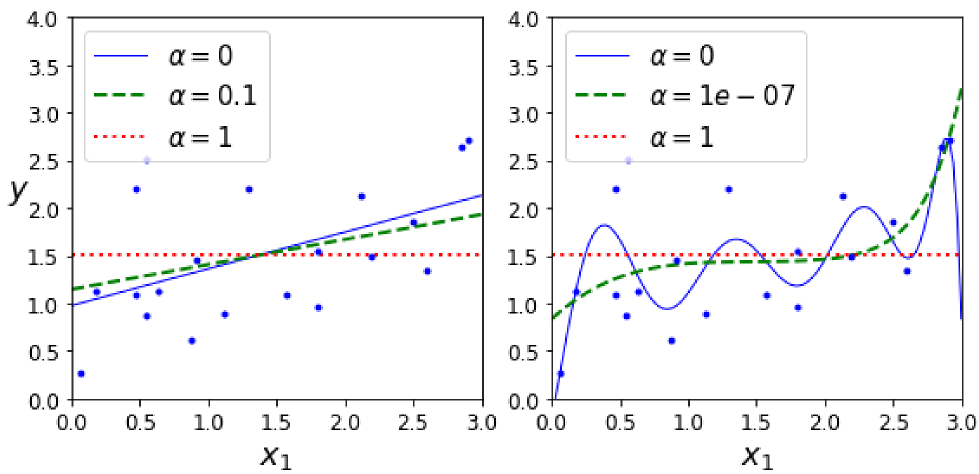
```
array([1.47012588])
```

In [43]:

```
from sklearn.linear_model import Lasso

plt.figure(figsize=(8,4))
plt.subplot(121)
plot_model(Lasso, polynomial=False, alphas=(0, 0.1, 1), random_state=42)
plt.ylabel("$y$", rotation=0, fontsize=18)
plt.subplot(122)
plot_model(Lasso, polynomial=True, alphas=(0, 10**-7, 1), random_state=4

save_fig("lasso_regression_plot")
plt.show()
```

```
/home/madhavan/miniconda3/lib/python3.7/site-packages/skle
arn/linear_model/_coordinate_descent.py:532: ConvergenceWa
rning: Objective did not converge. You might want to incre
ase the number of iterations. Duality gap: 2.8028677038274
32, tolerance: 0.0009294783355207351
  positive)
```

```
Saving figure lasso_regression_plot
```



In [44]:

```
from sklearn.linear_model import Lasso
lasso_reg = Lasso(alpha=0.1)
lasso_reg.fit(X, y)
lasso_reg.predict([[1.5]])
```

Out[44]:

```
array([1.53788174])
```

In [45]:

```
from sklearn.linear_model import ElasticNet
elastic_net = ElasticNet(alpha=0.1, l1_ratio=0.5, random_state=42)
elastic_net.fit(X, y)
elastic_net.predict([[1.5]])
```

Out[45]:

```
array([1.54333232])
```

In [46]:

```
np.random.seed(42)
m = 100
X = 6 * np.random.rand(m, 1) - 3
y = 2 + X + 0.5 * X**2 + np.random.randn(m, 1)

X_train, X_val, y_train, y_val = train_test_split(X[:50], y[:50].ravel()
```

Early stopping example:

In [47]:

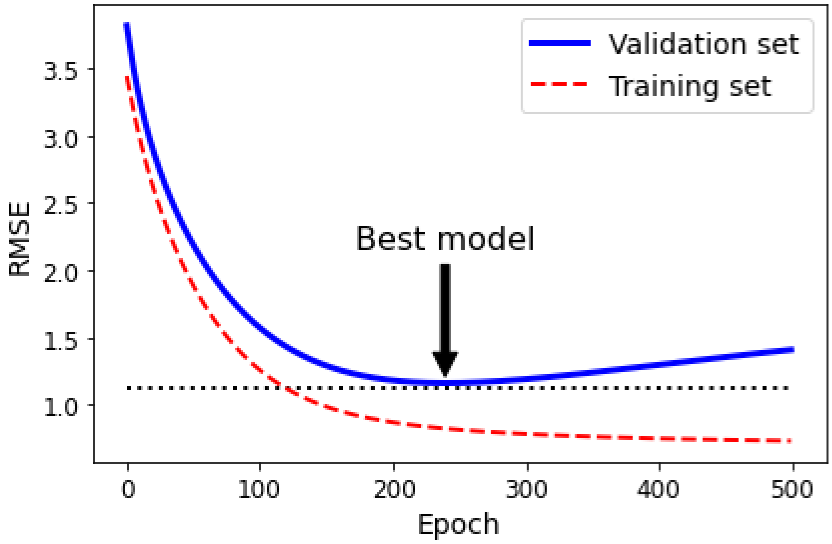
```
from sklearn.base import clone

poly_scaler = Pipeline([
        ("poly_features", PolynomialFeatures(degree=90, include_bias=Fal
        ("std_scaler", StandardScaler())
    ])

X_train_poly_scaled = poly_scaler.fit_transform(X_train)
X_val_poly_scaled = poly_scaler.transform(X_val)

sgd_reg = SGDRegressor(max_iter=1, tol=-np.infty, warm_start=True,
                       penalty=None, learning_rate="constant", eta0=0.00

minimum_val_error = float("inf")
best_epoch = None
best_model = None
for epoch in range(1000):
    sgd_reg.fit(X_train_poly_scaled, y_train)  # continues where it left
    y_val_predict = sgd_reg.predict(X_val_poly_scaled)
    val_error = mean_squared_error(y_val, y_val_predict)
    if val_error < minimum_val_error:
        minimum_val_error = val_error
        best_epoch = epoch
        best_model = clone(sgd_reg)
```

Create the graph:

In [48]:

```
sgd_reg = SGDRegressor(max_iter=1, tol=-np.infty, warm_start=True,
                       penalty=None, learning_rate="constant", eta0=0.00

n_epochs = 500
train_errors, val_errors = [], []
for epoch in range(n_epochs):
    sgd_reg.fit(X_train_poly_scaled, y_train)
    y_train_predict = sgd_reg.predict(X_train_poly_scaled)
    y_val_predict = sgd_reg.predict(X_val_poly_scaled)
    train_errors.append(mean_squared_error(y_train, y_train_predict))
    val_errors.append(mean_squared_error(y_val, y_val_predict))

best_epoch = np.argmin(val_errors)
best_val_rmse = np.sqrt(val_errors[best_epoch])

plt.annotate('Best model',
             xy=(best_epoch, best_val_rmse),
             xytext=(best_epoch, best_val_rmse + 1),
             ha="center",
             arrowprops=dict(facecolor='black', shrink=0.05),
             fontsize=16,
            )

best_val_rmse -= 0.03  # just to make the graph look better
plt.plot([0, n_epochs], [best_val_rmse, best_val_rmse], "k:", linewidth=
plt.plot(np.sqrt(val_errors), "b-", linewidth=3, label="Validation set")
plt.plot(np.sqrt(train_errors), "r--", linewidth=2, label="Training set"
plt.legend(loc="upper right", fontsize=14)
plt.xlabel("Epoch", fontsize=14)
plt.ylabel("RMSE", fontsize=14)
save_fig("early_stopping_plot")
plt.show()
```

```
Saving figure early_stopping_plot
```



In [49]:

```
best_epoch, best_model
```

Out[49]:

```
(239,
 SGDRegressor(eta0=0.0005, learning_rate='constant', max_i
ter=1, penalty=None,
              random_state=42, tol=-inf, warm_start=True))
```

In [50]:

```
%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np
```

In [51]:

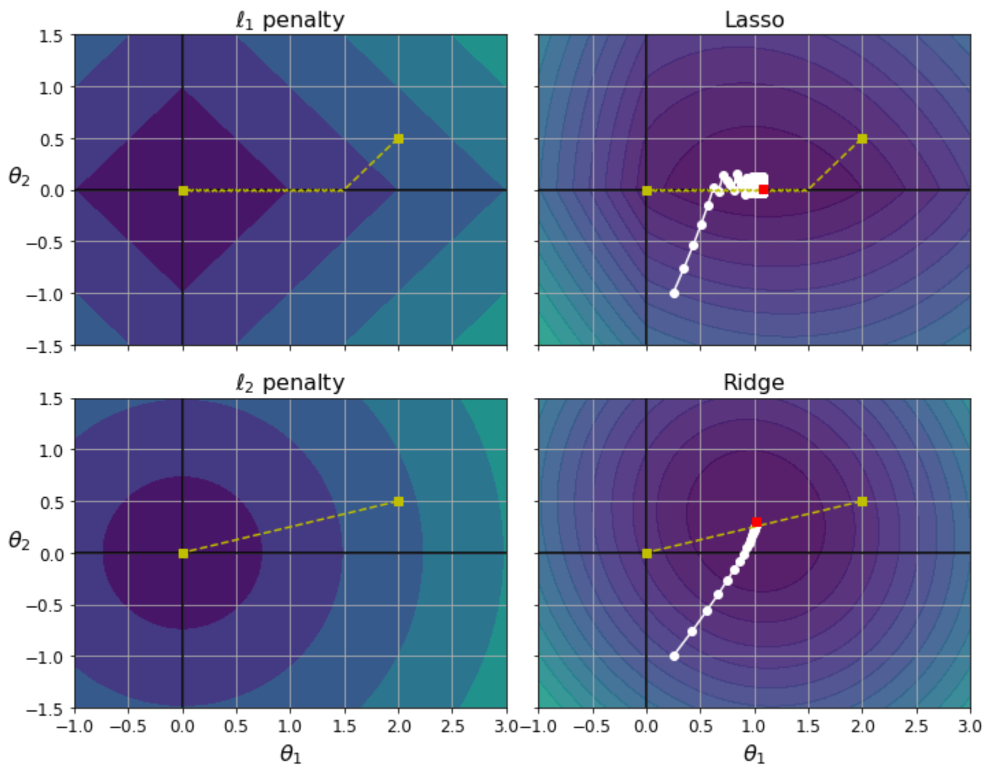
```
t1a, t1b, t2a, t2b = -1, 3, -1.5, 1.5

t1s = np.linspace(t1a, t1b, 500)
t2s = np.linspace(t2a, t2b, 500)
t1, t2 = np.meshgrid(t1s, t2s)
T = np.c_[t1.ravel(), t2.ravel()]
Xr = np.array([[1, 1], [1, -1], [1, 0.5]])
yr = 2 * Xr[:, :1] + 0.5 * Xr[:, 1:]

J = (1/len(Xr) * np.sum((T.dot(Xr.T) - yr.T)**2, axis=1)).reshape(t1.sha

N1 = np.linalg.norm(T, ord=1, axis=1).reshape(t1.shape)
N2 = np.linalg.norm(T, ord=2, axis=1).reshape(t1.shape)

t_min_idx = np.unravel_index(np.argmin(J), J.shape)
t1_min, t2_min = t1[t_min_idx], t2[t_min_idx]

t_init = np.array([[0.25], [-1]])
```

In [52]:

```
def bgd_path(theta, X, y, l1, l2, core = 1, eta = 0.05, n_iterations = 2
    path = [theta]
    for iteration in range(n_iterations):
        gradients = core * 2/len(X) * X.T.dot(X.dot(theta) - y) + l1 * n
        theta = theta - eta * gradients
        path.append(theta)
    return np.array(path)

fig, axes = plt.subplots(2, 2, sharex=True, sharey=True, figsize=(10.1,
for i, N, l1, l2, title in ((0, N1, 2., 0, "Lasso"), (1, N2, 0,  2., "Ri
    JR = J + l1 * N1 + l2 * 0.5 * N2**2

    tr_min_idx = np.unravel_index(np.argmin(JR), JR.shape)
    t1r_min, t2r_min = t1[tr_min_idx], t2[tr_min_idx]

    levelsJ=(np.exp(np.linspace(0, 1, 20)) - 1) * (np.max(J) - np.min(J)
    levelsJR=(np.exp(np.linspace(0, 1, 20)) - 1) * (np.max(JR) - np.min(
    levelsN=np.linspace(0, np.max(N), 10)

    path_J = bgd_path(t_init, Xr, yr, l1=0, l2=0)
    path_JR = bgd_path(t_init, Xr, yr, l1, l2)
    path_N = bgd_path(np.array([[2.0], [0.5]]), Xr, yr, np.sign(l1)/3, n

    ax = axes[i, 0]
    ax.grid(True)
    ax.axhline(y=0, color='k')
    ax.axvline(x=0, color='k')
    ax.contourf(t1, t2, N / 2., levels=levelsN)
    ax.plot(path_N[:, 0], path_N[:, 1], "y--")
    ax.plot(0, 0, "ys")
    ax.plot(t1_min, t2_min, "ys")
    ax.set_title(r"$\ell_{}$ penalty".format(i + 1), fontsize=16)
    ax.axis([t1a, t1b, t2a, t2b])
    if i == 1:
        ax.set_xlabel(r"$\theta_1$", fontsize=16)
    ax.set_ylabel(r"$\theta_2$", fontsize=16, rotation=0)

    ax = axes[i, 1]
    ax.grid(True)
    ax.axhline(y=0, color='k')
    ax.axvline(x=0, color='k')
    ax.contourf(t1, t2, JR, levels=levelsJR, alpha=0.9)
    ax.plot(path_JR[:, 0], path_JR[:, 1], "w-o")
    ax.plot(path_N[:, 0], path_N[:, 1], "y--")
    ax.plot(0, 0, "ys")
    ax.plot(t1_min, t2_min, "ys")
    ax.plot(t1r_min, t2r_min, "rs")
    ax.set_title(title, fontsize=16)
```

```
    ax.axis([t1a, t1b, t2a, t2b])
    if i == 1:
        ax.set_xlabel(r"$\theta_1$", fontsize=16)

save_fig("lasso_vs_ridge_plot")
plt.show()
```

```
Saving figure lasso_vs_ridge_plot
```

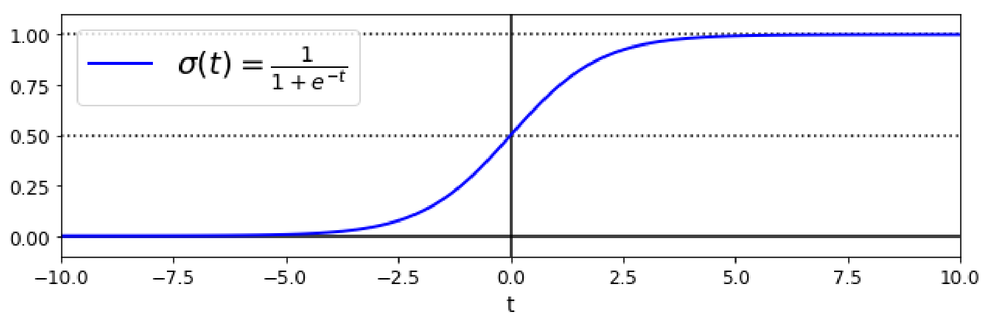


Logistic regression

In [53]:

```
t = np.linspace(-10, 10, 100)
sig = 1 / (1 + np.exp(-t))
plt.figure(figsize=(9, 3))
plt.plot([-10, 10], [0, 0], "k-")
plt.plot([-10, 10], [0.5, 0.5], "k:")
plt.plot([-10, 10], [1, 1], "k:")
plt.plot([0, 0], [-1.1, 1.1], "k-")
plt.plot(t, sig, "b-", linewidth=2, label=r"$\sigma(t) = \frac{1}{1 + e^
plt.xlabel("t")
plt.legend(loc="upper left", fontsize=20)
plt.axis([-10, 10, -0.1, 1.1])
save_fig("logistic_function_plot")
plt.show()
```

Saving figure logistic_function_plot



In [54]:

```
from sklearn import datasets
iris = datasets.load_iris()
list(iris.keys())
```

Out[54]:

```
['data',
 'target',
 'frame',
 'target_names',
 'DESCR',
 'feature_names',
 'filename']
```

In [55]:

```python
print(iris.DESCR)
```

```
.. _iris_dataset:

Iris plants dataset
--------------------

**Data Set Characteristics:**

    :Number of Instances: 150 (50 in each of three classe
s)
    :Number of Attributes: 4 numeric, predictive attribute
s and the class
    :Attribute Information:
        - sepal length in cm
        - sepal width in cm
        - petal length in cm
        - petal width in cm
        - class:
                - Iris-Setosa
                - Iris-Versicolour
                - Iris-Virginica

    :Summary Statistics:

    ============== ==== ==== ======= ===== ===============
=====
                    Min  Max   Mean    SD   Class Correlat
ion
    ============== ==== ==== ======= ===== ===============
=====
    sepal length:   4.3  7.9   5.84   0.83    0.7826
    sepal width:    2.0  4.4   3.05   0.43   -0.4194
    petal length:   1.0  6.9   3.76   1.76    0.9490  (hig
h!)
    petal width:    0.1  2.5   1.20   0.76    0.9565  (hig
h!)
    ============== ==== ==== ======= ===== ===============
=====

    :Missing Attribute Values: None
    :Class Distribution: 33.3% for each of 3 classes.
    :Creator: R.A. Fisher
    :Donor: Michael Marshall (MARSHALL%PLU@io.arc.nasa.go
v)
    :Date: July, 1988

The famous Iris database, first used by Sir R.A. Fisher. T
```

```
he dataset is taken
from Fisher's paper. Note that it's the same as in R, but
not as in the UCI
Machine Learning Repository, which has two wrong data poin
ts.

This is perhaps the best known database to be found in the
pattern recognition literature.  Fisher's paper is a class
ic in the field and
is referenced frequently to this day.  (See Duda & Hart, f
or example.)  The
data set contains 3 classes of 50 instances each, where ea
ch class refers to a
type of iris plant.  One class is linearly separable from
the other 2; the
latter are NOT linearly separable from each other.

.. topic:: References

   - Fisher, R.A. "The use of multiple measurements in tax
onomic problems"
     Annual Eugenics, 7, Part II, 179-188 (1936); also in
"Contributions to
     Mathematical Statistics" (John Wiley, NY, 1950).
   - Duda, R.O., & Hart, P.E. (1973) Pattern Classificatio
n and Scene Analysis.
     (Q327.D83) John Wiley & Sons.  ISBN 0-471-22361-1.  S
ee page 218.
   - Dasarathy, B.V. (1980) "Nosing Around the Neighborhoo
d: A New System
     Structure and Classification Rule for Recognition in
Partially Exposed
     Environments".  IEEE Transactions on Pattern Analysis
and Machine
     Intelligence, Vol. PAMI-2, No. 1, 67-71.
   - Gates, G.W. (1972) "The Reduced Nearest Neighbor Rul
e".  IEEE Transactions
     on Information Theory, May 1972, 431-433.
   - See also: 1988 MLC Proceedings, 54-64.  Cheeseman et
al"s AUTOCLASS II
     conceptual clustering system finds 3 classes in the d
ata.
   - Many, many more ...
```

In [56]:

```
X = iris["data"][:, 3:]  # petal width
y = (iris["target"] == 2).astype(np.int)  # 1 if Iris virginica, else 0
```

```
/home/madhavan/miniconda3/lib/python3.7/site-packages/ipyk
ernel_launcher.py:2: DeprecationWarning: `np.int` is a dep
recated alias for the builtin `int`. To silence this warni
ng, use `int` by itself. Doing this will not modify any be
havior and is safe. When replacing `np.int`, you may wish
to use e.g. `np.int64` or `np.int32` to specify the precis
ion. If you wish to review your current use, check the rel
ease note link for additional information.
Deprecated in NumPy 1.20; for more details and guidance: h
ttps://numpy.org/devdocs/release/1.20.0-notes.html#depreca
tions (https://numpy.org/devdocs/release/1.20.0-notes.html
#deprecations)
```

Note: To be future-proof we set `solver="lbfgs"` since this will be the default value in Scikit-Learn 0.22.

In [57]:

```
from sklearn.linear_model import LogisticRegression
log_reg = LogisticRegression(solver="lbfgs", random_state=42)
log_reg.fit(X, y)
```

Out[57]:

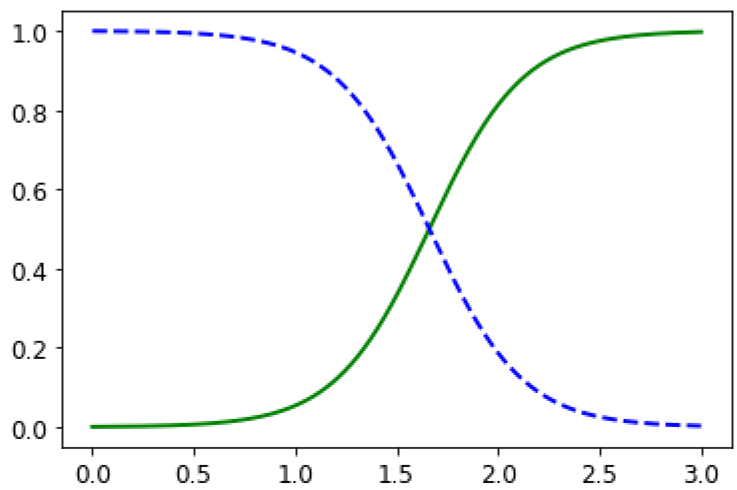
```
LogisticRegression(random_state=42)
```

In [58]:

```
X_new = np.linspace(0, 3, 1000).reshape(-1, 1)
y_proba = log_reg.predict_proba(X_new)

plt.plot(X_new, y_proba[:, 1], "g-", linewidth=2, label="Iris virginica"
plt.plot(X_new, y_proba[:, 0], "b--", linewidth=2, label="Not Iris virgi
```

Out[58]:

```
[<matplotlib.lines.Line2D at 0x7fc0630d3910>]
```

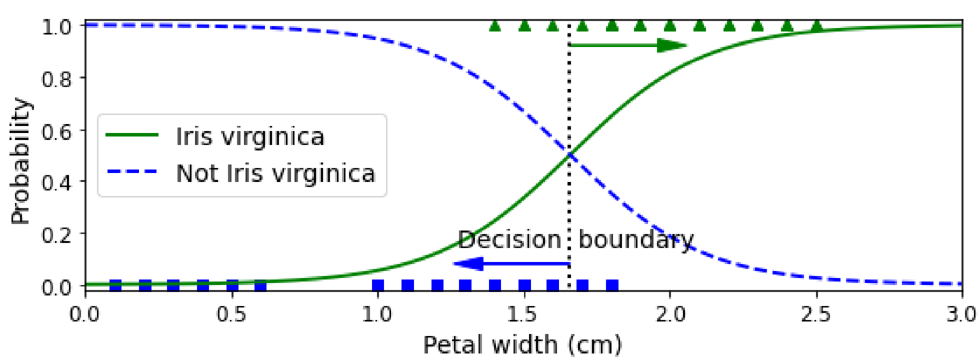


The figure in the book actually is actually a bit fancier:

In [59]:

```
X_new = np.linspace(0, 3, 1000).reshape(-1, 1)
y_proba = log_reg.predict_proba(X_new)
decision_boundary = X_new[y_proba[:, 1] >= 0.5][0]

plt.figure(figsize=(8, 3))
plt.plot(X[y==0], y[y==0], "bs")
plt.plot(X[y==1], y[y==1], "g^")
plt.plot([decision_boundary, decision_boundary], [-1, 2], "k:", linewidt
plt.plot(X_new, y_proba[:, 1], "g-", linewidth=2, label="Iris virginica"
plt.plot(X_new, y_proba[:, 0], "b--", linewidth=2, label="Not Iris virgi
plt.text(decision_boundary+0.02, 0.15, "Decision  boundary", fontsize=14
plt.arrow(decision_boundary, 0.08, -0.3, 0, head_width=0.05, head_length
plt.arrow(decision_boundary, 0.92, 0.3, 0, head_width=0.05, head_length=
plt.xlabel("Petal width (cm)", fontsize=14)
plt.ylabel("Probability", fontsize=14)
plt.legend(loc="center left", fontsize=14)
plt.axis([0, 3, -0.02, 1.02])
save_fig("logistic_regression_plot")
plt.show()
```

```
Saving figure logistic_regression_plot

/home/madhavan/miniconda3/lib/python3.7/site-packages/matp
lotlib/patches.py:1338: VisibleDeprecationWarning: Creatin
g an ndarray from ragged nested sequences (which is a list
-or-tuple of lists-or-tuples-or ndarrays with different le
ngths or shapes) is deprecated. If you meant to do this, y
ou must specify 'dtype=object' when creating the ndarray.
  verts = np.dot(coords, M) + (x + dx, y + dy)
```



In [60]:

```
decision_boundary
```

Out[60]:

```
array([1.66066066])
```

In [61]:

```
log_reg.predict([[1.7], [1.5]])
```

Out[61]:

```
array([1, 0])
```

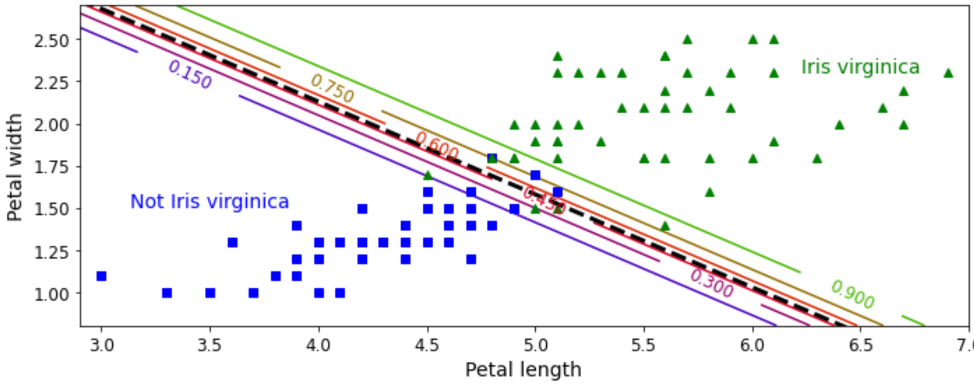
In [62]:

```
from sklearn.linear_model import LogisticRegression

X = iris["data"][:, (2, 3)]  # petal length, petal width
y = (iris["target"] == 2).astype(np.int)

log_reg = LogisticRegression(solver="lbfgs", C=10**10, random_state=42)
log_reg.fit(X, y)

x0, x1 = np.meshgrid(
        np.linspace(2.9, 7, 500).reshape(-1, 1),
        np.linspace(0.8, 2.7, 200).reshape(-1, 1),
    )
X_new = np.c_[x0.ravel(), x1.ravel()]

y_proba = log_reg.predict_proba(X_new)

plt.figure(figsize=(10, 4))
plt.plot(X[y==0, 0], X[y==0, 1], "bs")
plt.plot(X[y==1, 0], X[y==1, 1], "g^")

zz = y_proba[:, 1].reshape(x0.shape)
contour = plt.contour(x0, x1, zz, cmap=plt.cm.brg)


left_right = np.array([2.9, 7])
boundary = -(log_reg.coef_[0][0] * left_right + log_reg.intercept_[0]) /

plt.clabel(contour, inline=1, fontsize=12)
plt.plot(left_right, boundary, "k--", linewidth=3)
plt.text(3.5, 1.5, "Not Iris virginica", fontsize=14, color="b", ha="cen
plt.text(6.5, 2.3, "Iris virginica", fontsize=14, color="g", ha="center"
plt.xlabel("Petal length", fontsize=14)
plt.ylabel("Petal width", fontsize=14)
plt.axis([2.9, 7, 0.8, 2.7])
save_fig("logistic_regression_contour_plot")
plt.show()
```

```
/home/madhavan/miniconda3/lib/python3.7/site-packages/ip
ykernel_launcher.py:4: DeprecationWarning: `np.int` is a
deprecated alias for the builtin `int`. To silence this
warning, use `int` by itself. Doing this will not modify
any behavior and is safe. When replacing `np.int`, you m
ay wish to use e.g. `np.int64` or `np.int32` to specify
the precision. If you wish to review your current use, c
heck the release note link for additional information.
Deprecated in NumPy 1.20; for more details and guidance:
https://numpy.org/devdocs/release/1.20.0-notes.html#depr
```

```
ecations (https://numpy.org/devdocs/release/1.20.0-note
s.html#deprecations)
  after removing the cwd from sys.path.
```

```
Saving figure logistic_regression_contour_plot
```



In [63]:

```
X = iris["data"][:, (2, 3)]  # petal length, petal width
y = iris["target"]

softmax_reg = LogisticRegression(multi_class="multinomial",solver="lbfgs
softmax_reg.fit(X, y)
```

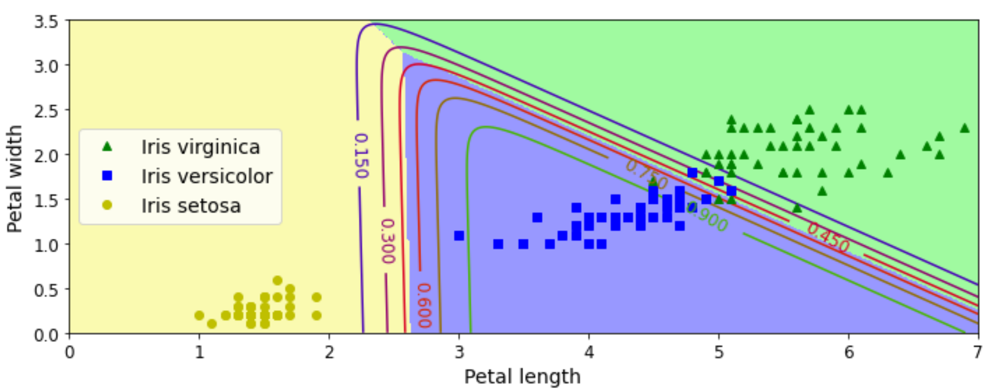
Out[63]:

```
LogisticRegression(C=10, multi_class='multinomial', random
_state=42)
```

In [64]:

```
x0, x1 = np.meshgrid(
        np.linspace(0, 8, 500).reshape(-1, 1),
        np.linspace(0, 3.5, 200).reshape(-1, 1),
    )
X_new = np.c_[x0.ravel(), x1.ravel()]


y_proba = softmax_reg.predict_proba(X_new)
y_predict = softmax_reg.predict(X_new)

zz1 = y_proba[:, 1].reshape(x0.shape)
zz = y_predict.reshape(x0.shape)

plt.figure(figsize=(10, 4))
plt.plot(X[y==2, 0], X[y==2, 1], "g^", label="Iris virginica")
plt.plot(X[y==1, 0], X[y==1, 1], "bs", label="Iris versicolor")
plt.plot(X[y==0, 0], X[y==0, 1], "yo", label="Iris setosa")

from matplotlib.colors import ListedColormap
custom_cmap = ListedColormap(['#fafab0','#9898ff','#a0faa0'])

plt.contourf(x0, x1, zz, cmap=custom_cmap)
contour = plt.contour(x0, x1, zz1, cmap=plt.cm.brg)
plt.clabel(contour, inline=1, fontsize=12)
plt.xlabel("Petal length", fontsize=14)
plt.ylabel("Petal width", fontsize=14)
plt.legend(loc="center left", fontsize=14)
plt.axis([0, 7, 0, 3.5])
save_fig("softmax_regression_contour_plot")
plt.show()
```

Saving figure softmax_regression_contour_plot



In [65]:

```
softmax_reg.predict([[5, 2]])
```

Out[65]:

```
array([2])
```

In [66]:

```
softmax_reg.predict_proba([[5, 2]])
```

Out[66]:

```
array([[6.38014896e-07, 5.74929995e-02, 9.42506362e-01]])
```