Lecture 3

Jan 31^{st}

Plan for the class

- A quick intro. to topology

2/3
- → definition
- → metric topology
- → cts maps, homeo.
- → mflds in $\mathbb{R}^n$. postponed
- → homotopy
- → intuitive idea of homology

Simplicial Structure.

- → simplex
- → simplicial cplx.
  (geometric & abstract
- → Examples.

Simplicial homology $\boxed{\text{? next class}}$

A quick intro. to topology

Def: A topology on a set X is a family $\mathcal{T}$ of subspaces of X that satisfies the following 3 conditions

(i) $\emptyset, X \in \mathcal{T}$

(ii) any arbitrary union of elts of $\mathcal{T}$ is an elt of $\mathcal{T}$.

(iii) any finite intersection of elts of $\mathcal{T}$ is an elt of $\mathcal{T}$.

The pair $(X, \underline{\mathcal{T}})$ is a topological space.

Elements of $\mathcal{T}$ are called open sets

A $Y \subset X$ is closed if $X \setminus Y$ is open.

Induced topology.

Let $Y \subset X$ then

$$\mathcal{T}_Y := \{ T \cap Y \mid \forall\, T \in \mathcal{T} \}$$

is a topology on Y.

Let $(X, \mathcal{T}_1)$ & $(Y, \mathcal{T}_2)$ be two top. spaces then a map

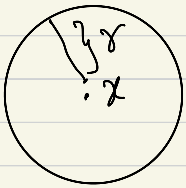
$$f : X \longrightarrow Y$$

is $\boxed{\text{Continuous}}$ if for every open set $T \in \mathcal{T}_2$ its inverse image $f^{-1}(T) \in \mathcal{T}_1$

Let (X, d) be a metric space. The smallest topology containing all the open balls (i.e., $B(x; r) := \{ y \in X \mid d(x, y) < r \}$).

is called <u>metric topology</u>.

<u>Example</u> $X = \mathbb{R}^n$, d is the Euclidean dist.



open ball of radius r centered at x.

<u>Continuity of functions on $\mathbb{R}^n$ is the usual ε-δ definition</u>

<u>Assumption</u>: A topological space is a subset of $\mathbb{R}^n$ equipped with induced topology.

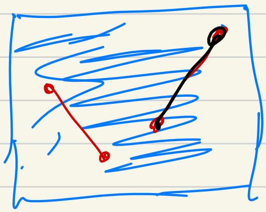
A space X is connected if for any two points $x \neq y \in X$ $\exists$ a cts. function $f : [0,1] \longrightarrow X$ s.t.

$f(0) = x$ & $f(1) = y$.

$[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

A c/s map $f : [0,1] \longrightarrow X$ is called a path in X.

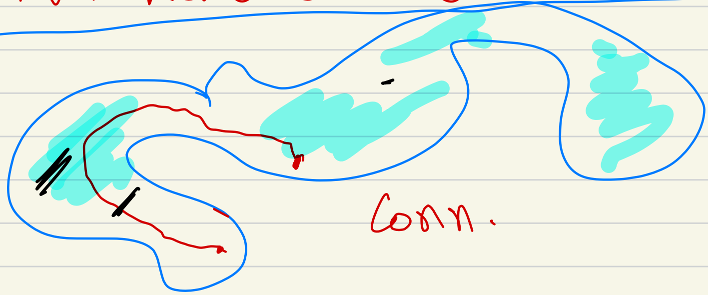
Examples



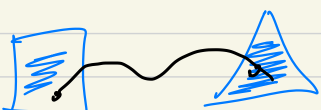
filled square in $\mathbb{R}^2$

convex $\Rightarrow$ connected

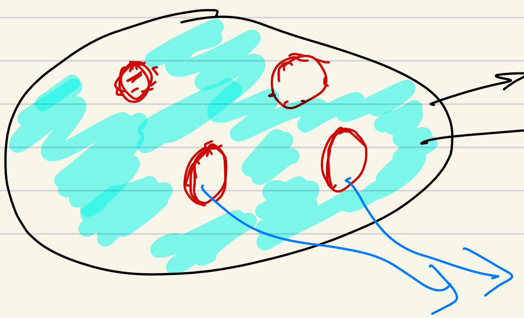
Convex hence connected



inf. is included



dis conn.



disc connected

holes

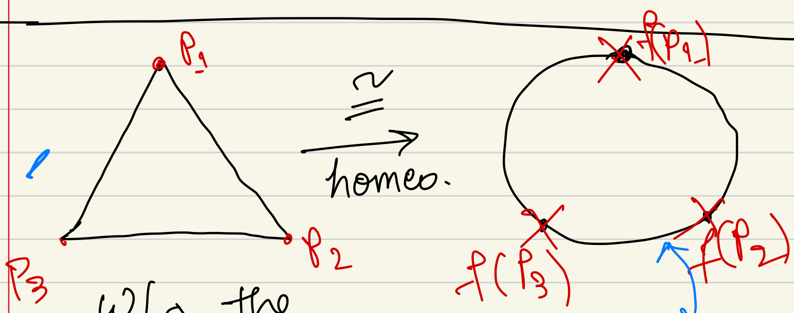
Connectivity is preserved under cts maps.

$$f : X \xrightarrow{\text{Cont.}} Y$$

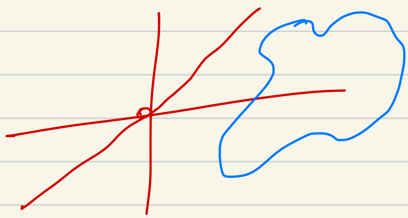
If X is conn. then $f(X)$ is conn.

Def A cts map $f : X \to Y$ is a homeomorphism if f is a bijection & f^{-1} is also cts.

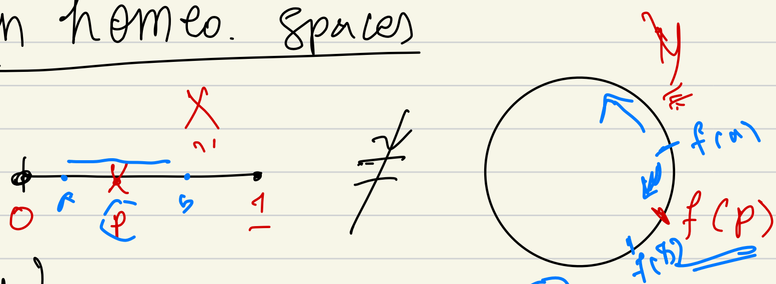
The two space X & Y are said to be homeomorphic.



w/o the interior



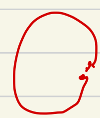
Non homeo. spaces

① 

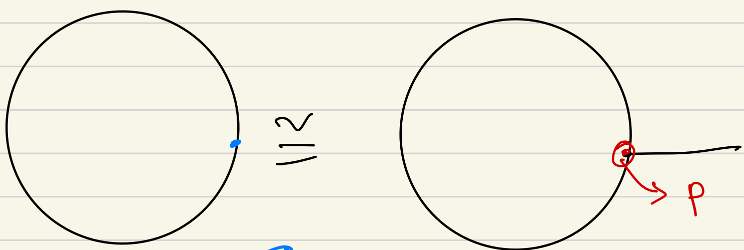
why?

Suppose $X \cong Y$

$f : X \setminus \{p\} \xrightarrow{\cong} Y \setminus \{f(p)\}$

$[0, p) \sqcup (p, 1] \underset{f^{-1}}{\overset{f}{\rightleftarrows}}$ 

disconn   conn.

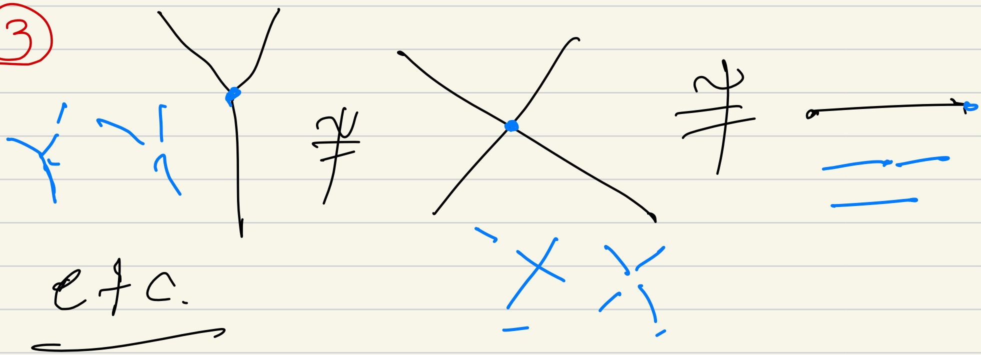
② 

X   C conn.   Y $Y - p =$ C— disconn

$Y \setminus \{p\}$ has 2 conn. components

$X \setminus \{f(p)\}$ is connected.

3



etc.

Fun H.W.

$0 \not\cong 9$

1. Consider the digits 0 to 9 as subsets of $\mathbb{R}^2$. Classify them up to homeo.

2. Do the same for A, B, . . . , Z

Homotopy

Definition

Given two space X, Y, two maps $f_0, f_1 : X \to Y$ are homotopic if $\exists$ a cts map

$$H : X \times \underbrace{[0,1]}_{\text{time parameter}} \to Y$$

such that $\boxed{f_0 \sim f_1}$

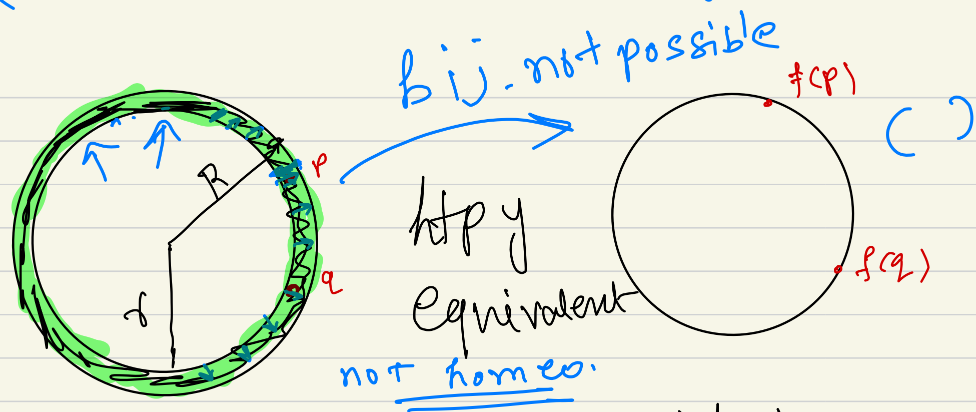
$\forall\, x \in X$

$$H(0, x) \Longrightarrow f_0(x)$$
$$H(1, x) = f_1(x)$$

Def^n Two spaces X & Y have the same homotopy type (or are homotopy equivalent) if $\exists$ two maps $f : X \to Y$ & $g : Y \to X$ s.t.

$\boxed{g \circ f \simeq 1_X}$ $\quad$ $\boxed{f \circ g \simeq 1_Y}$

$(\text{Homeo} \Rightarrow g \circ f = 1_X \text{ \& } f \circ g = 1_Y)$



X is a thick ring. Y is a circle.

X & Y are not homeo.

But X & Y are homotopic.

Part 2 of the fun HW

- Classify following shapes up to homotopy
- O to 9
- A to Z

O & 9 are not homeo. but O & 9 are htpy equiv.

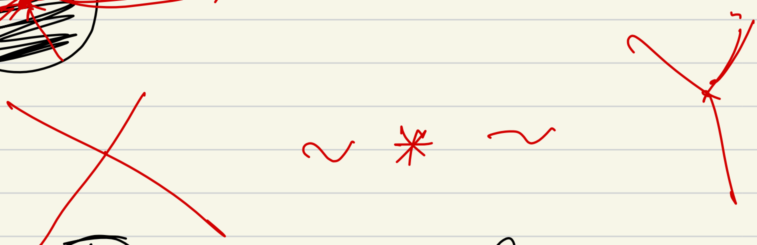
If X & Y are homeo. $\Longrightarrow$ X & Y are homotopic. $9 \to 9$

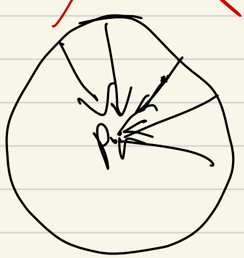
But $\Longleftarrow$ is false

Hint: 0 & 9 are htpic but not homeo.

& so on.

$\boxed{\text{Def}}$ A top. space X is contractible if X is htpy equivalent to a single point.





$f(x) = p$

$\{9 \xrightarrow{id} 9$ $\quad 9 \quad 0$

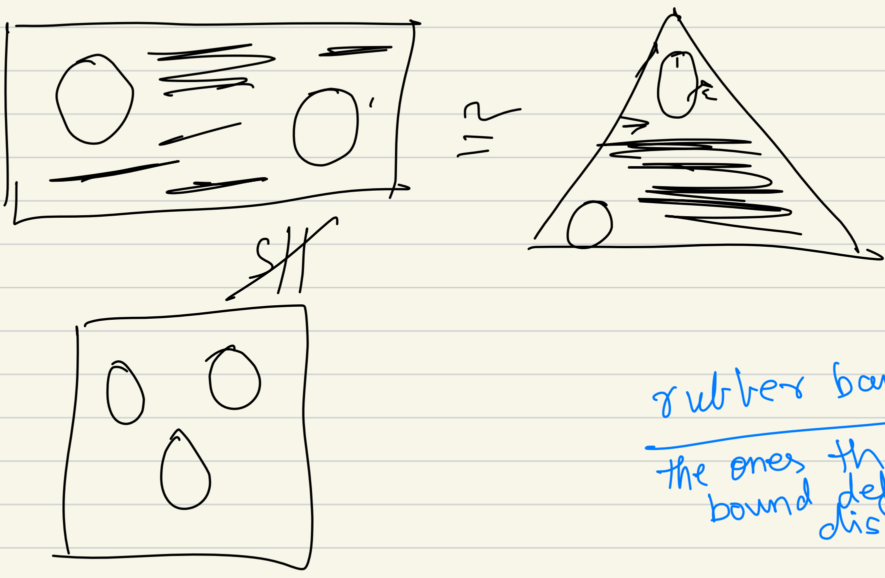
$\{I \xrightarrow{\text{const.}}$

$X \times [0, 1] \qquad 0 \times I$

Homology counts holes

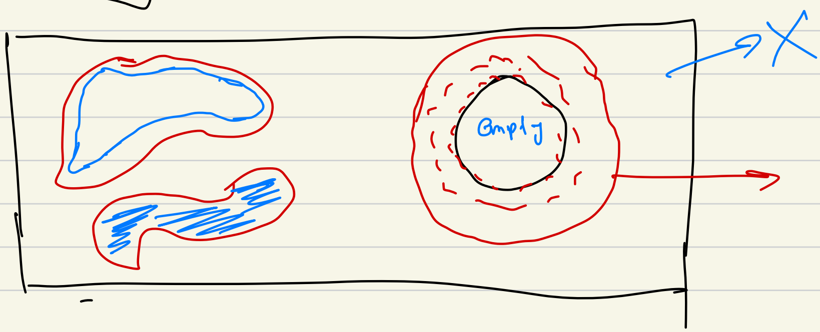
Trust me on this

$\boxed{\text{having holes is a topolgical property}}$



rubber bands

the ones that bound deformed disc.



Simplicial Complexes

A generalization of simple graphs

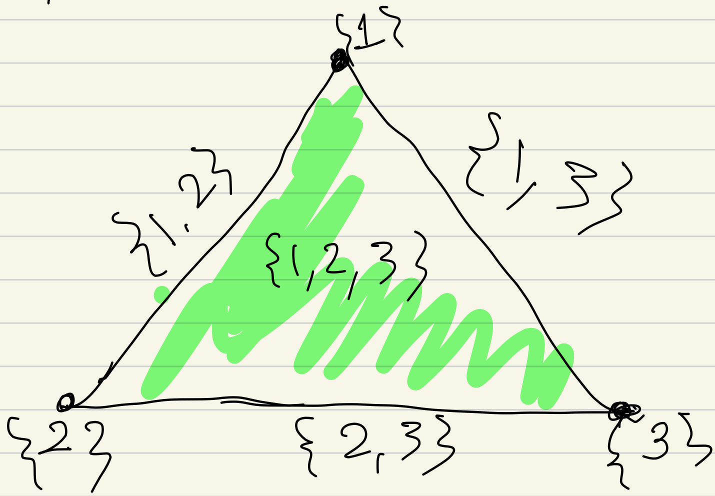
Def^n

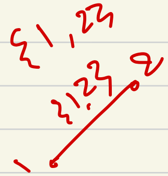
Abstract SC

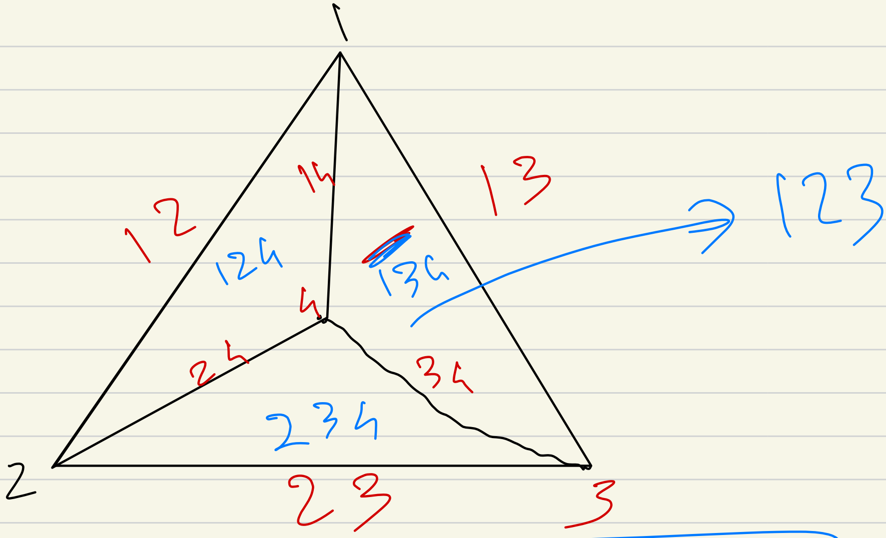
Geometric SC

The power set of $\{1, 2, 3\}$

$\emptyset$, 1, 2, 3, 12, 13, 23, 123







Power set = simplex