Topological Data Analysis

Name: Pragya Shukla

Roll No: MDS202027

Task

A fintie, simple, undirected graph can be considered as a finite metric space as follows: the elements of this metric space are the vertices (nodes) of the graph and the distance between to elements is the graph distance (the least number of edges needed to go from one vertex to the other). You may also assume that the graphs are connected, i.e., between any two vertices there is a path.

Write the following program - Choose any 2 of the following graph classes

1. trees on n vertices
2. hypercube graphs (1-skeleton of a hypercube, i.e., vertices have coordinates 0 or 1 and two such vertices are adjacent if their coordinates differ in exactly one place. This graph has 2^n vertices
3. cycle graphs on n vertices
4. a complete bipartatite graph on n vertices
5. a barbell graph on n vertices
6. lollipop graph on n vertices

For smaller number of vertices (say not more than 11/12 vertices) do the following. For the given input 'n' and the graph class prepare the distance matrix (rows and columns indexed by vertices and entries are the distance between them). Use this distance matrix to find the persistent homology. The aim is to find if there is any (homology) pattern among the VR complexes of a particular graph class (before we hit the diameter).

Import Libraries

In [53]:
```
import numpy as np
from numpy.random import default_rng
rng = default_rng(42)

from scipy.spatial.distance import pdist, squareform
from scipy.sparse import csr_matrix

from gtda.graphs import GraphGeodesicDistance
from gtda.homology import VietorisRipsPersistence, SparseRipsPersistence, FlagserPersistence

from igraph import Graph

from IPython.display import SVG, display
```

The two graphs we have chosen from above are **Hypercube graph** and **Cycle graph**.

First we will make adjacency matrix for both the graphs.

- For hypercube graph, it is obtained by putting 1 at location (i,j) if XOR of binary notation of i,j only has one set bit.
- For cycle graph, it is obtained by putting 1 at location (i,i+1) in the matrix.

Hypercube Graph

In [54]:
```
def hypercube_adjacency(n):
    n_vertices=2**n                              # Qn has 2^n vertices
    mat=np.zeros((n_vertices,n_vertices))
    for i in range(n_vertices):
        for j in range(n_vertices):
            if len([int(x) for x in bin(i^j).replace("0b","") if x=='1'])==1:
                mat[i,j]=1.
    return mat
```

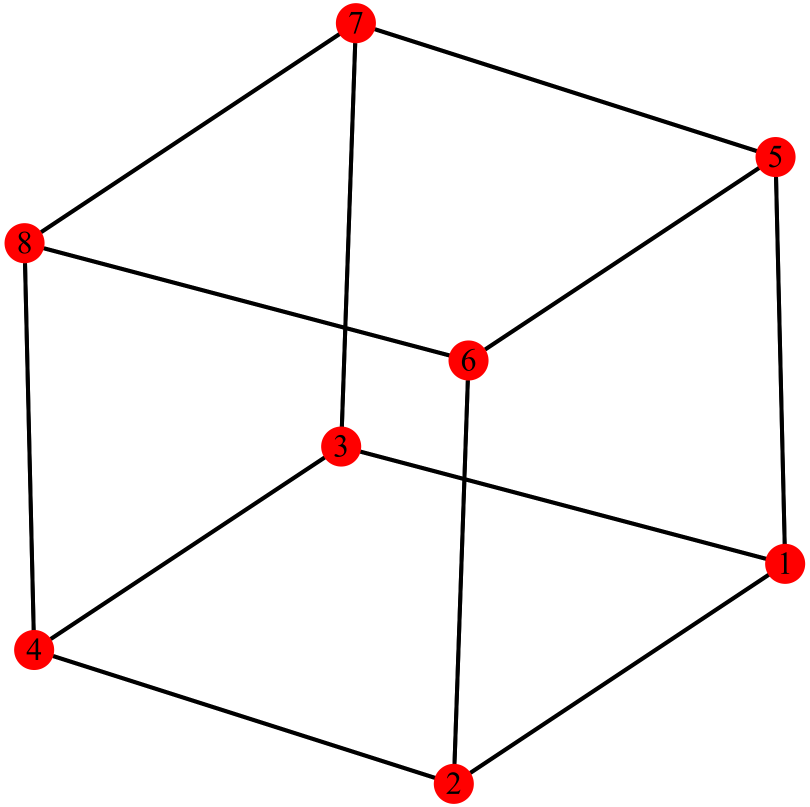
Cycle Graph

In [55]:
```
def circle_adjacency(n_vertices, directed=False):
    weights = np.ones(n_vertices)
    rows = np.arange(n_vertices)
    columns = np.arange(1, n_vertices + 1) % n_vertices
    directed_adjacency = csr_matrix((weights, (rows, columns)))
    if not directed:
        return directed_adjacency + directed_adjacency.T
    return directed_adjacency
```

1. Hypercube Graph Geodesic Persistence

n = 3

In [56]:
```
n_vertices = 3                  # We will have 8 vertices in the graph for n_vertices = 3 (since 2^n)
hy = hypercube_adjacency(n_vertices)
from igraph import plot
row, col = hy.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "hypercube.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```



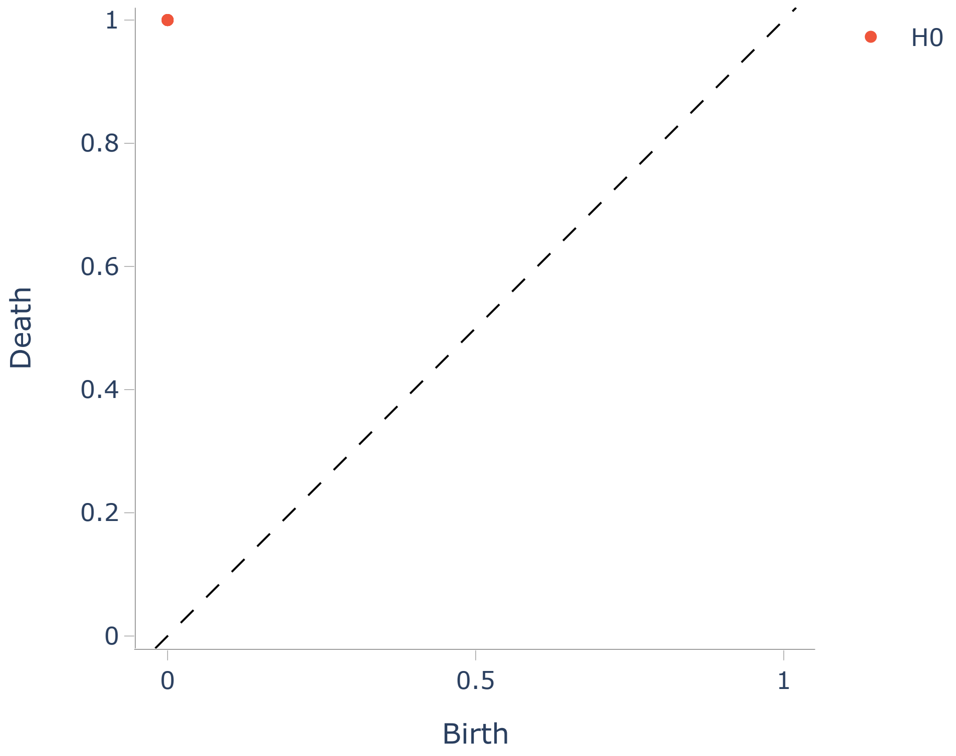
In [57]:

```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True,method='D').fit_transform([hy])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 1. 1. 1. 1. 1. 1.]
  [1. 0. 1. 1. 1. 1. 1. 1.]
  [1. 1. 0. 1. 1. 1. 1. 1.]
  [1. 1. 1. 0. 1. 1. 1. 1.]
  [1. 1. 1. 1. 0. 1. 1. 1.]
  [1. 1. 1. 1. 1. 0. 1. 1.]
  [1. 1. 1. 1. 1. 1. 0. 1.]
  [1. 1. 1. 1. 1. 1. 1. 0.]]]
```

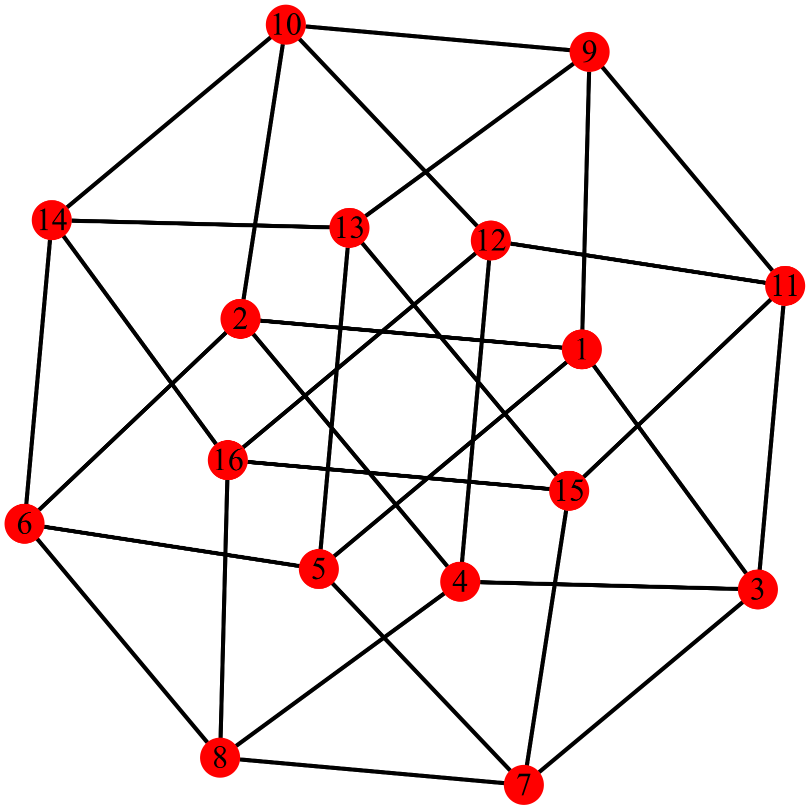
In [58]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```



n = 4

In [59]:
```
n_vertices = 4
hy = hypercube_adjacency(n_vertices)
from igraph import plot
row, col = hy.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "hypercube.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```



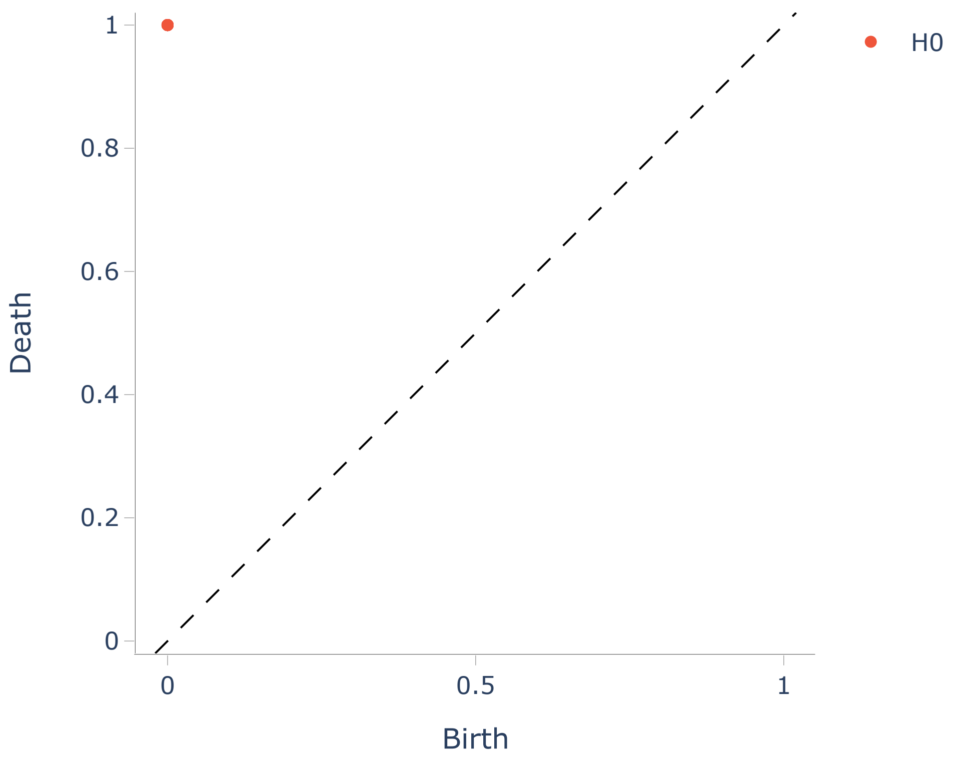
Above generated is a Hypercube Graph with 16 vertices

In [60]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True,method='D').fit_transform([hy])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 0. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 0. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 0. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 0. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 0. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 0. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 0. 1. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 0. 1. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 0. 1. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 0. 1. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 0. 1. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 0. 1. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 0. 1. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 0. 1.]
  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 0.]]]
```

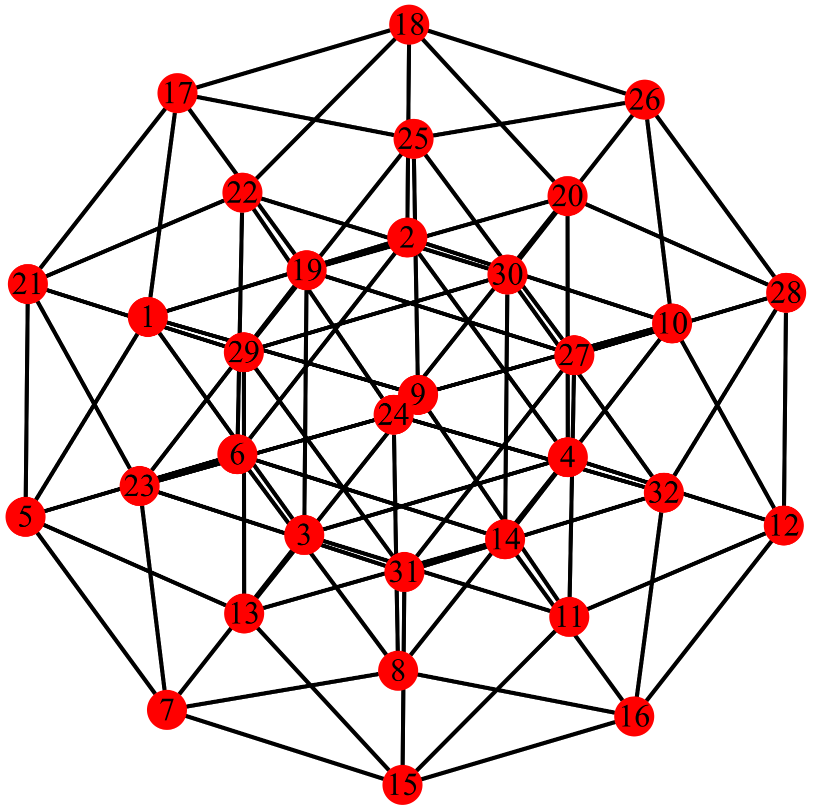
In [61]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```



n = 5

In [62]:
```
n_vertices = 5
hy = hypercube_adjacency(n_vertices)
from igraph import plot
row, col = hy.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "hypercube.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```

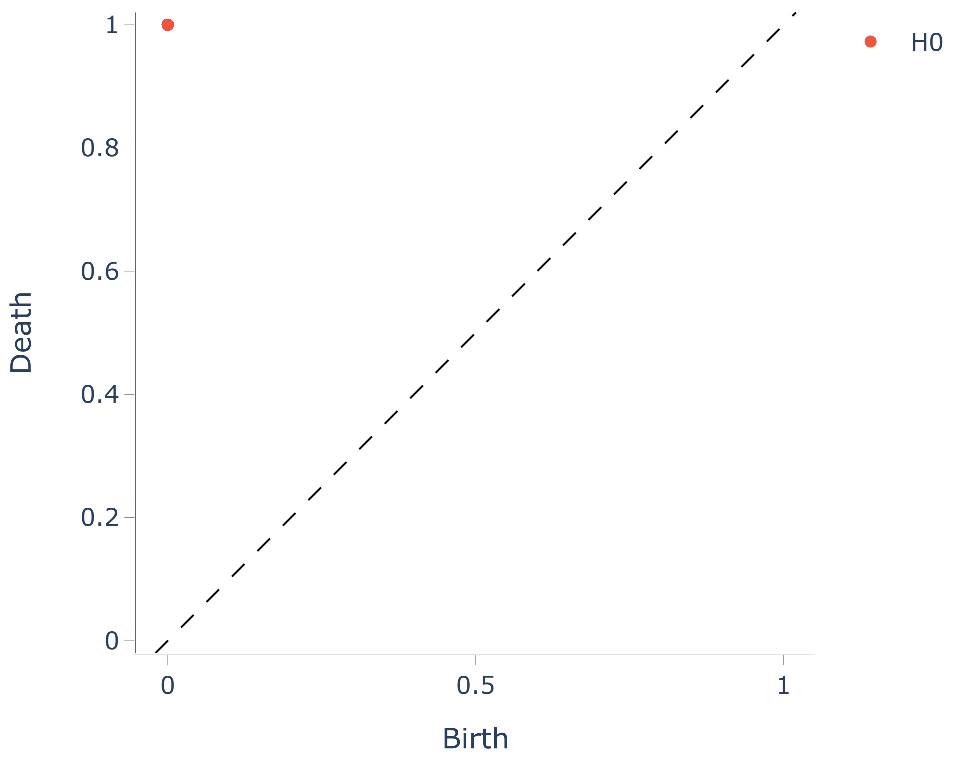


In [63]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True,method='D').fit_transform([hy])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 1. ... 1. 1. 1.]
  [1. 0. 1. ... 1. 1. 1.]
  [1. 1. 0. ... 1. 1. 1.]
  ...
  [1. 1. 1. ... 0. 1. 1.]
  [1. 1. 1. ... 1. 0. 1.]
  [1. 1. 1. ... 1. 1. 0.]]]
```

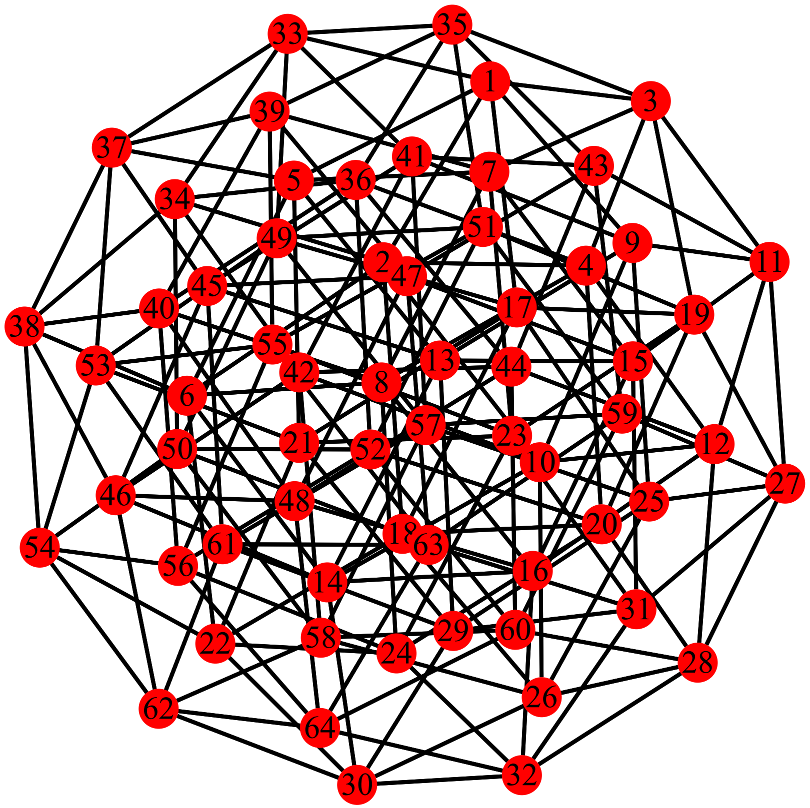
In [64]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```



n = 6

In [65]:
```
n_vertices = 6
hy = hypercube_adjacency(n_vertices)
from igraph import plot
row, col = hy.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "hypercube.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```

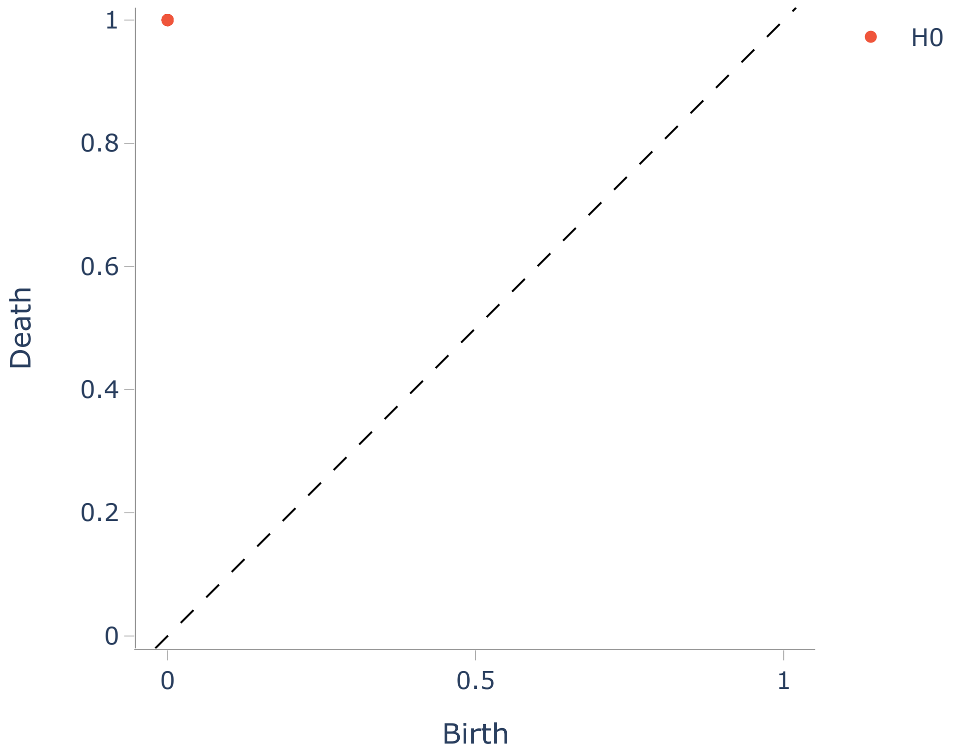


In [66]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True,method='D').fit_transform([hy])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 1. ... 1. 1. 1.]
  [1. 0. 1. ... 1. 1. 1.]
  [1. 1. 0. ... 1. 1. 1.]
  ...
  [1. 1. 1. ... 0. 1. 1.]
  [1. 1. 1. ... 1. 0. 1.]
  [1. 1. 1. ... 1. 1. 0.]]]
```

In [67]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```

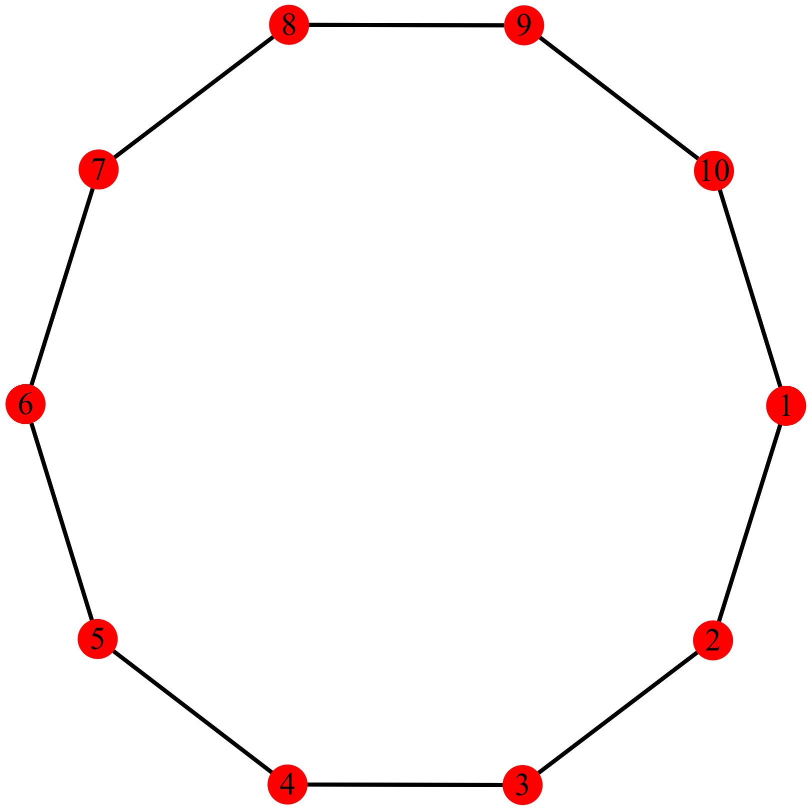


Conclusion: There is only one zero dimensional homology feature which takes birth at t=0 and dies at t=1 which means at t=1 every vertex is connected to every other vertex. We can see this in all the Persistence Diagrams above.

Cycle Graph Geodesic Persistence

n = 10

In [68]:
```
n_vertices = 10
undirected_circle = circle_adjacency(n_vertices)
from igraph import plot
row, col = undirected_circle.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "undirected_circle.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```

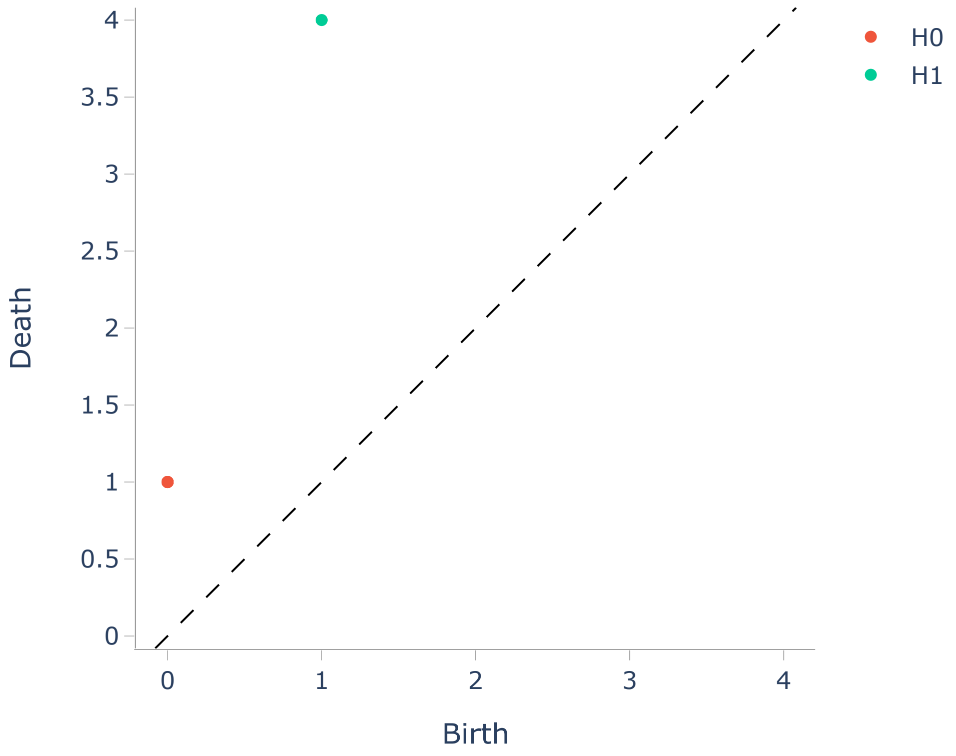


In [69]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True).fit_transform([undirected_circle])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 2. 3. 4. 5. 4. 3. 2. 1.]
  [1. 0. 1. 2. 3. 4. 5. 4. 3. 2.]
  [2. 1. 0. 1. 2. 3. 4. 5. 4. 3.]
  [3. 2. 1. 0. 1. 2. 3. 4. 5. 4.]
  [4. 3. 2. 1. 0. 1. 2. 3. 4. 5.]
  [5. 4. 3. 2. 1. 0. 1. 2. 3. 4.]
  [4. 5. 4. 3. 2. 1. 0. 1. 2. 3.]
  [3. 4. 5. 4. 3. 2. 1. 0. 1. 2.]
  [2. 3. 4. 5. 4. 3. 2. 1. 0. 1.]
  [1. 2. 3. 4. 5. 4. 3. 2. 1. 0.]]]
```

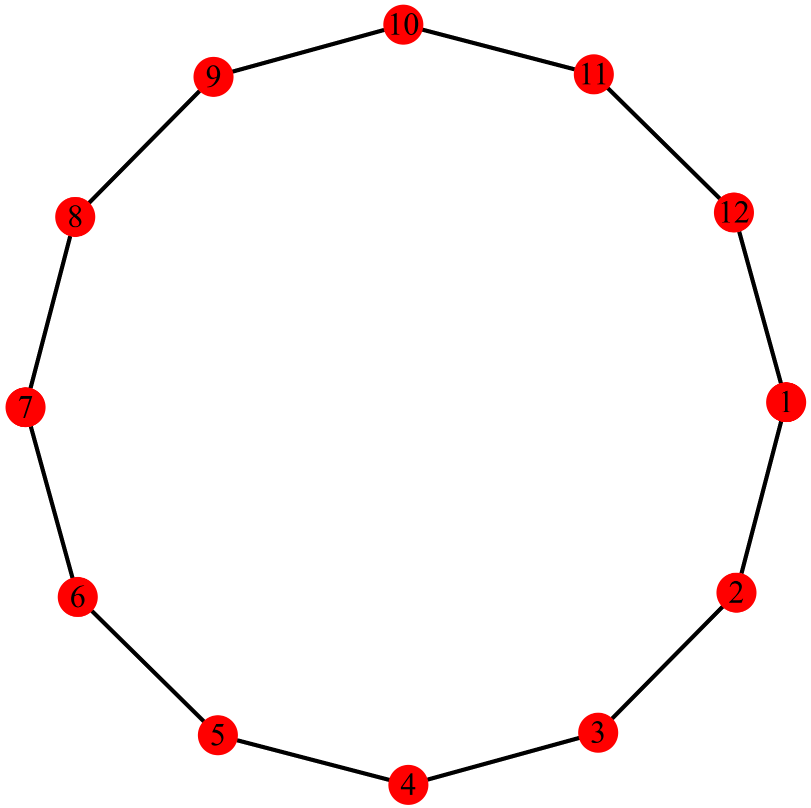
In [70]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```



n = 12

In [71]:
```
n_vertices = 12
undirected_circle = circle_adjacency(n_vertices)
from igraph import plot
row, col = undirected_circle.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "undirected_circle.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```

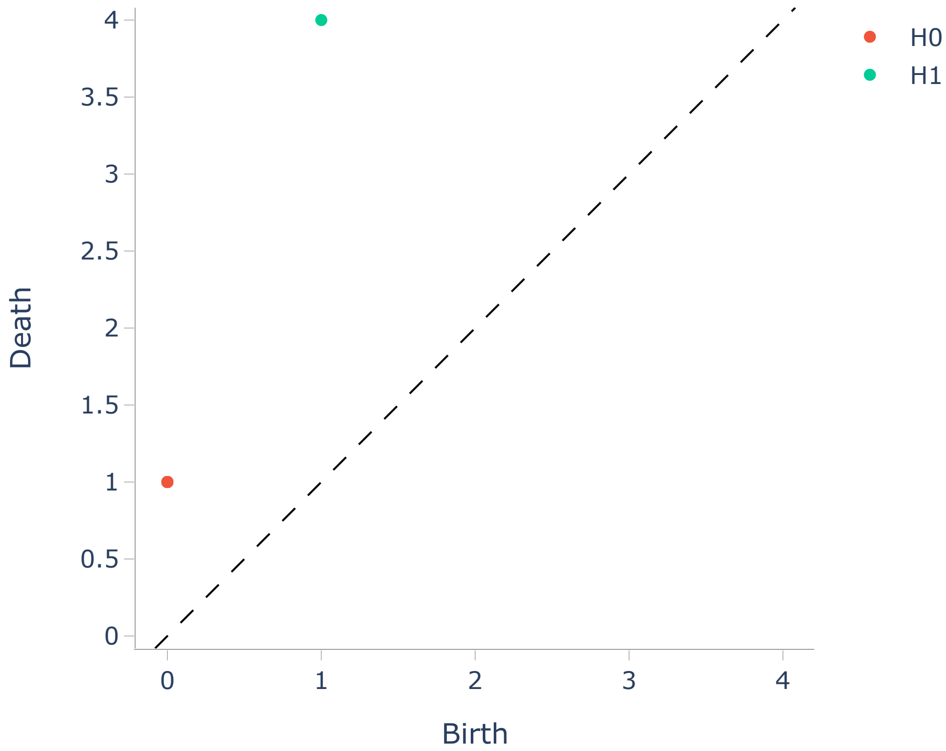


In [72]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True).fit_transform([undirected_circle])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 2. 3. 4. 5. 6. 5. 4. 3. 2. 1.]
  [1. 0. 1. 2. 3. 4. 5. 6. 5. 4. 3. 2.]
  [2. 1. 0. 1. 2. 3. 4. 5. 6. 5. 4. 3.]
  [3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 5. 4.]
  [4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 5.]
  [5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6.]
  [6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5.]
  [5. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4.]
  [4. 5. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3.]
  [3. 4. 5. 6. 5. 4. 3. 2. 1. 0. 1. 2.]
  [2. 3. 4. 5. 6. 5. 4. 3. 2. 1. 0. 1.]
  [1. 2. 3. 4. 5. 6. 5. 4. 3. 2. 1. 0.]]]
```

In [73]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

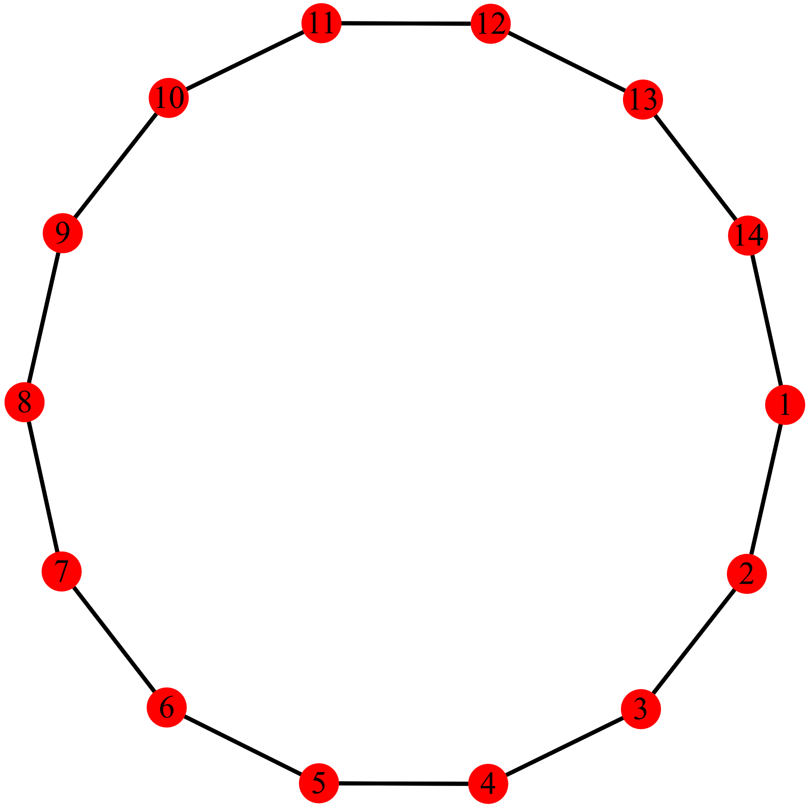
```
Persistence diagram is:
```



n = 14

In [76]:

```
n_vertices = 14
undirected_circle = circle_adjacency(n_vertices)
from igraph import plot
row, col = undirected_circle.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "undirected_circle.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```

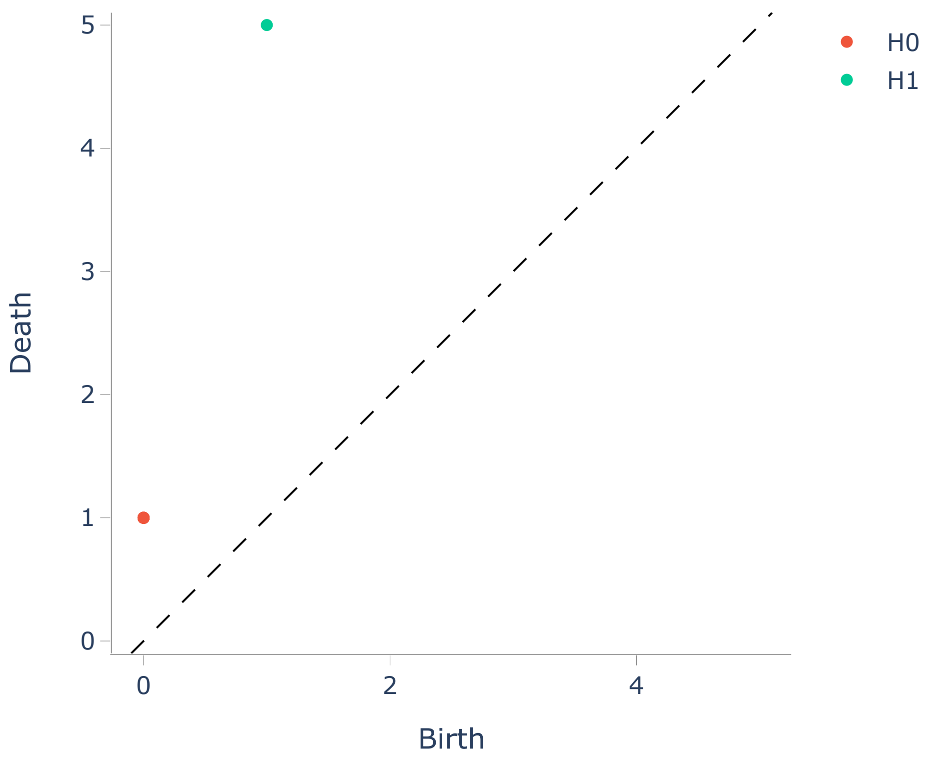


In [77]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True).fit_transform([undirected_circle])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 2. 3. 4. 5. 6. 7. 6. 5. 4. 3. 2. 1.]
  [1. 0. 1. 2. 3. 4. 5. 6. 7. 6. 5. 4. 3. 2.]
  [2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 6. 5. 4. 3.]
  [3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 6. 5. 4.]
  [4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 6. 5.]
  [5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 6.]
  [6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7.]
  [7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6.]
  [6. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5.]
  [5. 6. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4.]
  [4. 5. 6. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3.]
  [3. 4. 5. 6. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2.]
  [2. 3. 4. 5. 6. 7. 6. 5. 4. 3. 2. 1. 0. 1.]
  [1. 2. 3. 4. 5. 6. 7. 6. 5. 4. 3. 2. 1. 0.]]]
```

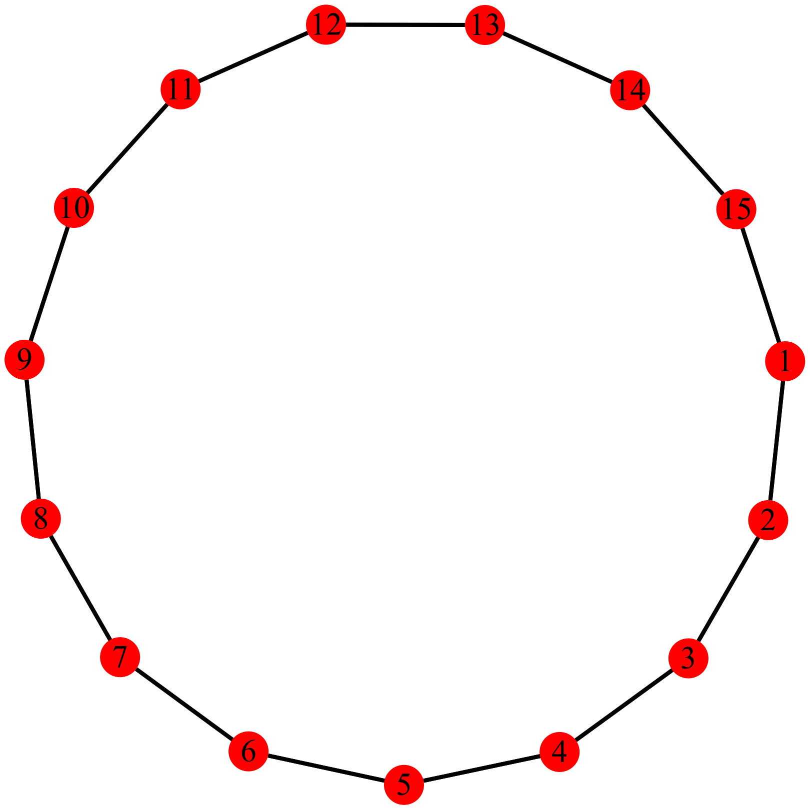
In [78]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```



n = 15

In [79]:
```
n_vertices = 15
undirected_circle = circle_adjacency(n_vertices)
from igraph import plot
row, col = undirected_circle.nonzero()
graph = Graph(n=n_vertices, edges=list(zip(row, col)), directed=False)
fname = "undirected_circle.svg"
graph.write_svg(fname)
display(SVG(filename=fname))
```

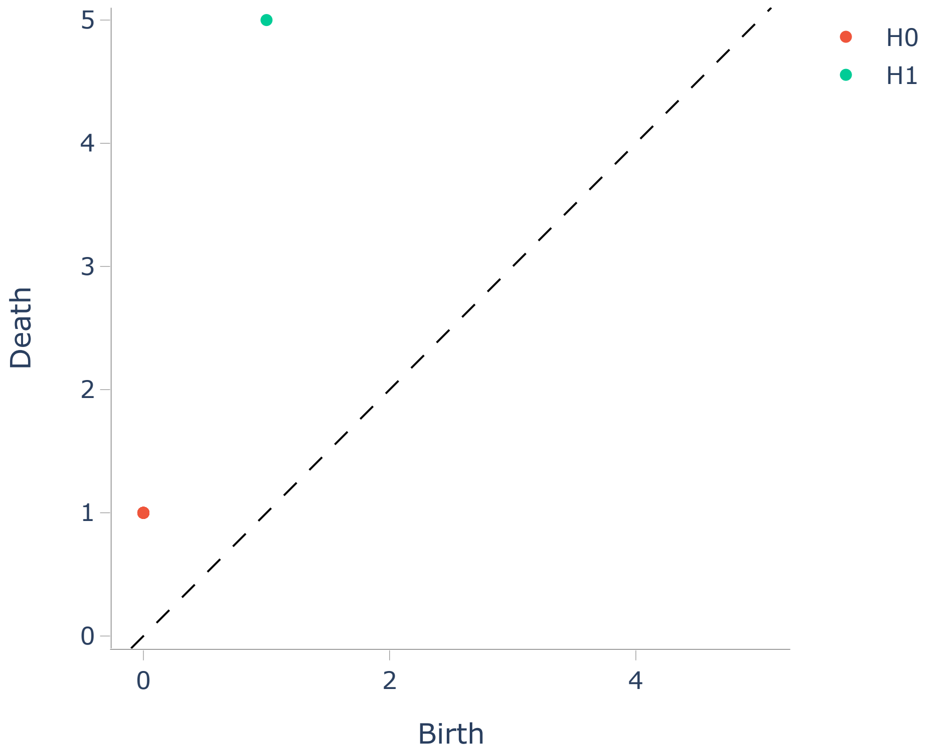


In [80]:
```
X_ggd = GraphGeodesicDistance(directed=False, unweighted=True).fit_transform([undirected_circle])
print(f"The distance matrix is:\n {X_ggd}")
```

```
The distance matrix is:
 [[[0. 1. 2. 3. 4. 5. 6. 7. 7. 6. 5. 4. 3. 2. 1.]
  [1. 0. 1. 2. 3. 4. 5. 6. 7. 7. 6. 5. 4. 3. 2.]
  [2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 7. 6. 5. 4. 3.]
  [3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 7. 6. 5. 4.]
  [4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 7. 6. 5.]
  [5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 7. 6.]
  [6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7. 7.]
  [7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6. 7.]
  [7. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5. 6.]
  [6. 7. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4. 5.]
  [5. 6. 7. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3. 4.]
  [4. 5. 6. 7. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2. 3.]
  [3. 4. 5. 6. 7. 7. 6. 5. 4. 3. 2. 1. 0. 1. 2.]
  [2. 3. 4. 5. 6. 7. 7. 6. 5. 4. 3. 2. 1. 0. 1.]
  [1. 2. 3. 4. 5. 6. 7. 7. 6. 5. 4. 3. 2. 1. 0.]]]
```

In [81]:
```
print(f"Persistence diagram is:\n ")
VietorisRipsPersistence(metric="precomputed").fit_transform_plot(X_ggd);
```

```
Persistence diagram is:
```



Conclusion: There is a topological feature in dimension 1 whose death value is finite. This is because, at some point, we have enough triangles to completely fill the 1D hole. This can be seen in all the Persistence Diagram for Cycle Graph above.