15/03/21

[A (really) brief intro. to simplicial homology.]

- The dimension of a simpl. cplx is the max. dim. of simplices that it contains.

- A maximal simplex is the simplex which is maximal w.r.t. containment

$K =$ 

$\dim K = 3$

maximal simplices in K
$\{8\}$, $\{45\}$, $\{5,6,7\}$, $\{1,2,3,4\}$

The space of chains

Given a (G.) S.C. K, for every non-ve integer p the vector space of p-chains in K is defined as the free vector space (over $\mathbb{F}_2$) formed by p-simplices. Denote $C_p(K)$

$\mathbb{F}_2 = \{0, 1\}$ $\quad 1+0=1, \quad 1+1=0=0+1$

this is a field. $\quad 1 \cdot 1 = 1, \quad 1 \cdot 0 = 0$

Free vector on a finite set.

Let $X = \{x_1, \ldots, x_n\}$ be a finite set

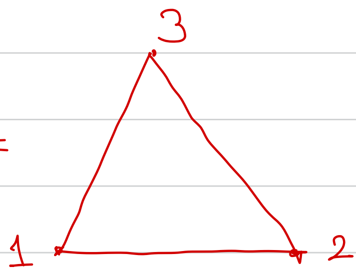
$Vect_{\mathbb{F}_2}(X)$ consists of all formal

$\mathbb{F}_2$-linear combination.

$$\sum_{i=1}^{n} \varepsilon_i x_i \in \mathrm{Vect}_{\mathbb{F}_2}(X) \qquad \varepsilon_i \in \mathbb{F}_2.$$

Let us take $n = 3$.

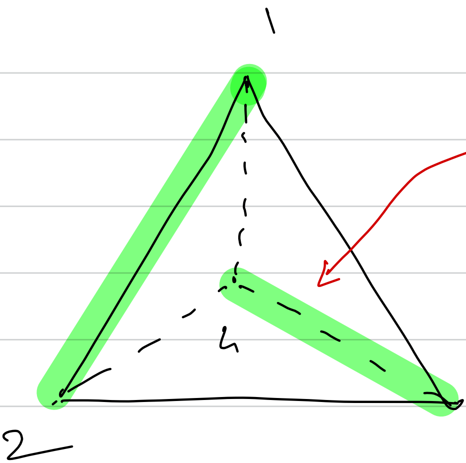
$$\mathrm{Vect}_{\mathbb{F}_2}(X) = \{0, x_1, x_2, x_3, x_1+x_2, x_1+x_3, x_2+x_3, x_1+x_2+x_3\}$$

$K =$ 

$$C_0(K) = \{0, 1, 2, 3, 1+2, 1+3, 2+3, 1+2+3\}$$

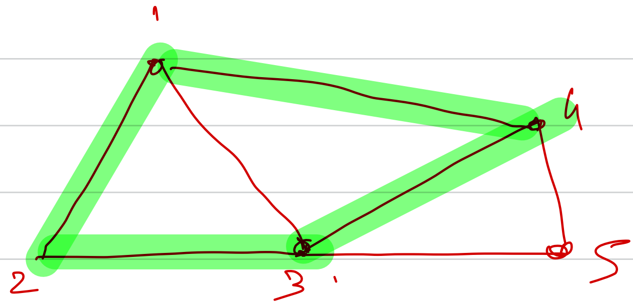
$$C_1(K) = \{0, 12, 23, 13, 12+23, 12+13, 13+23, 12+23+13\}$$

☆ $C_p(K)$ consists all possible subsets of p-simplices in K.



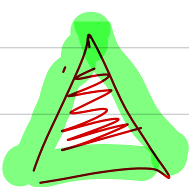
$12 + 34$

chains need not be conn.

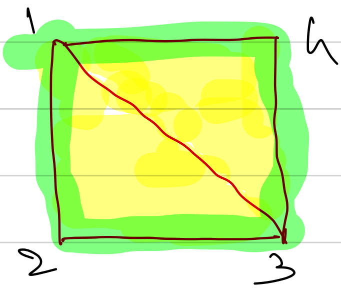


$12 + 23 + 13 + 13 + 34 + 14$?

Def^n The boundary of a p-simplex is the set of all its $(p-1)$-faces.



$\boxed{\text{Def}^n}$ The boundary of a p-chain is the mod-2 sum (i.e, the sum in $C_{p-1}(K)$) of the boundaries of its simplices.

K: 

$\sigma = 123 + 134$ is a 2-chain.

What is the boundary of σ?

$12 + 23 + \cancel{13} + \cancel{13} + 34 + 14$

$= 12 + 23 + 34 + 14 \in C_1(K)$.

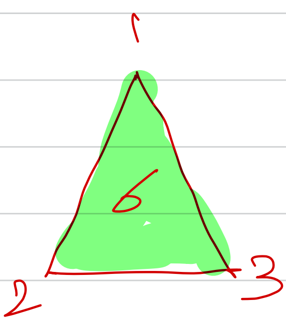
We get a linear map

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$

$$\text{chain} \longmapsto \text{bdry of that chain.}$$

Def: A p-cycle is a p-chain which is in the kernel of ∂_p.

$$\partial_p(\sigma) = 0 \Rightarrow \sigma \in C_p(K) \text{ is a cycle.}$$



$$\partial_2 \sigma = 12 + 23 + 13 =: \tau$$

τ is a 1-chain.

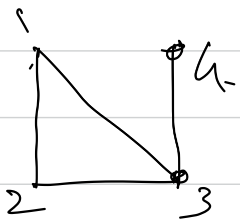
$$\partial_1(12 + 23 + 13)$$
$$= \partial_1(12) + \partial_1(23) + \partial_1(13)$$
$$= 1 + 2 + 2 + 3 + 1 + 3$$
$$= 0$$

In general,

<u>Theorem</u> The composition

$$\cdots \to C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} C_{p-2}(K)$$

is the zero linear map.



$\sigma = 12 + 13 + 23 + 34$

Not a cycle.

$$\partial_1(12 + 13 + 23 + 34) = \cancel{1} + \cancel{2} + \cancel{1} + \cancel{3} + \cancel{2} + \cancel{3} + 3 + 4$$

$$= 3 + 4$$

Denote by

$Z_p(K) :=$ kernel of

$\partial_p : C_p \to C_{p-1}$

$\boxed{\text{Th}^m}$ The boundary of any $(p+1)$- chain is always a p- cycle.

$$\xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} C_{p-2}(K)$$

$$\operatorname{Im}(\partial_{p+1}) \subset \operatorname{Ker}(\partial_p)$$

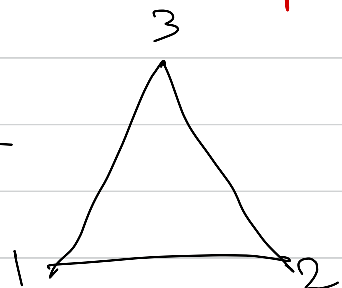
$$\therefore \; B_p := \operatorname{Im}(\partial_{p+1})$$

$B_p \subset Z_p$ both are vector spaces of $\mathbb{F}_2$

<u>Def:</u> The p-th simplicial homology of K is $H_p(K) := \dfrac{Z_p}{B_p}$ vector space.

The pth Betti number is

$$\beta_p \doteq \dim H_p(K).$$

$K =$ 

$$0 \to C_1(K) \xrightarrow{\partial_1} C_0(K) \to 0$$

$C_0(K) = \{0, 1, 2, 3, 1+2, 1+3, 2+3, 1+2+3\}$

$C_1(K) = \{0, 12, 23, 13, 12+23, 12+13, 23+13, 12+23+13\}$

$$\partial_1 : C_1(K) \to C_0(K)$$

what is Z_1?

$\partial_1(12+23+13) = 0$

Z_1 is $12+23+13$

B_1 is zero since B_1 is the image of the map $\underset{\substack{\| \\ 0}}{C_2} \to C_1$

$$\therefore \; H_1(K) = Z_1(K)/0$$

$$\cong Z_1(K) \cong \mathbb{Z}_2$$

$$\therefore \; \beta_1 = 1$$

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for H_0 $\qquad C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$

$$\therefore \; Z_0 = C_0$$

$$\therefore \; H_0 = C_0 / \mathrm{Im}\, \partial_1 \cong \mathbb{Z}_2$$

$$= C_0(K) / \langle \{1+2,\ 1+3,\ 2+3\} \rangle$$

$$\therefore \; \beta_0 = 1$$

Fun exc. Let K be the bdry of Δ^3 then

$$\boxed{\begin{aligned} \beta_0(K) &= 1 \\ \beta_1(K) &= 0 \\ \beta_2(K) &= 1 \end{aligned}}$$

Theorem $\beta_0(K)$ is the no. of connected components of K.