Lecture 4

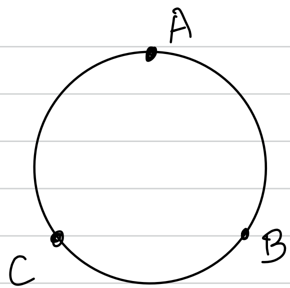
Simplicial Complexes

Feb. 2nd

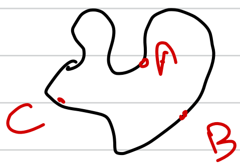
Our interest is in those topological spaces which can be combinatorially represented.

" All topological spaces are (homeomorphic to) simplicial complexes".

Suppose you want to represent a circle on a computer.



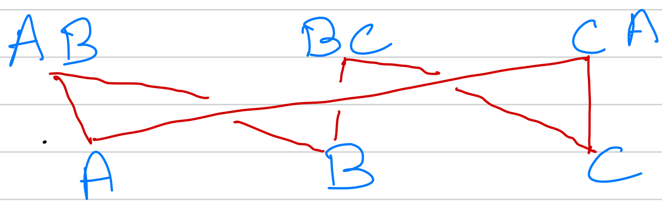
$$\text{Circle} = \{A\} \cup \{B\} \cup \{C\} \rightarrow \text{vertices}$$
$$\underbrace{\cup \{AB\} \cup \{BC\} \cup \{CA\}}_{\text{arcs.}}$$



more information vertex-arc containment

$A \subset AB$, $B \subset AB$ etc.

arcs



vertices

this info. represents a circle.

- This rep. of ○'le' is 'compact'
- This —||— is co-ordinate free.

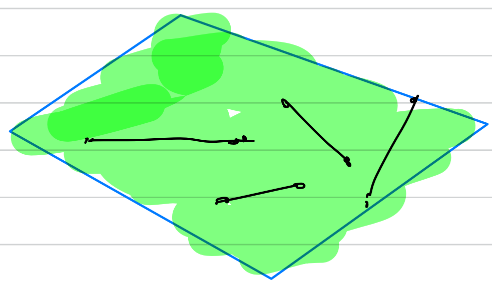
Partial order

$a \leq a$

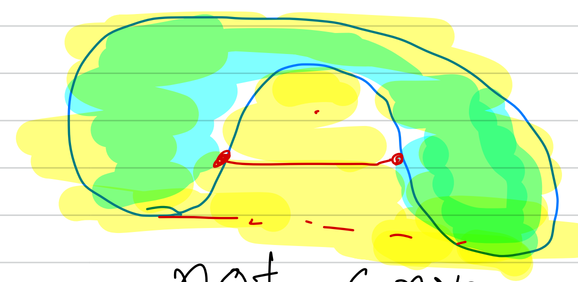
$a \leq b$, $b \leq c$

$a \leq c$

Def^n A subset S of $\mathbb{R}^k$ is **Convex** if for any pts $x, y \in S$, each point $\underbrace{(1-t)x + ty}_{\text{convex lin. combi.}}$, $t \in [0,1]$ is also contained in S.



Convex



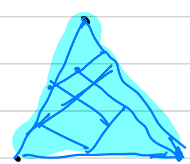
not convex

$$\text{Convex hull of } (S) = \bigcap \{ C \mid S \subset C \subset \mathbb{R}^k \ \& \ C \text{ is convex} \}$$

Suppose S is a finite subset,

say $S = \{ v_1, \ldots, v_n \} \subseteq \mathbb{R}^k$.

$$\operatorname{cvx}(S) = \left\{ \sum_{j=1}^{n} t_j v_j \;\middle|\; t_j \in [0,1] \ \& \ \sum t_j = 1 \right\}.$$

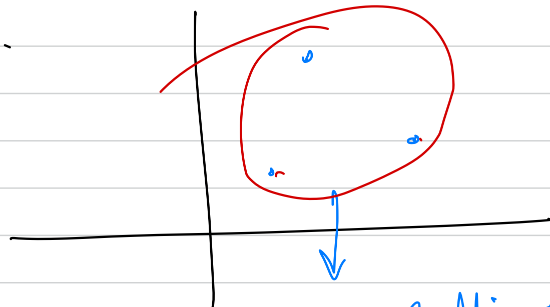


A subset $S = \{ x_0, \ldots, x_n \}$ is in general position in $\mathbb{R}^k$ if the pts of S are not contained

subset of S of card. at most $k+1$

in any affine (proper) subspace of of dimension less than k

e.g.



non collinear pts
$=$
general position

$\rightarrow$ not in general position.

Let A be any subset of $\mathbb{R}^d$. A is in general (linear) position if every k subset of A is affinely ind. for all $k \leq d+1$

Let $S \subset \mathbb{R}^k$ be a finite set.

The simplex associated to S is the set

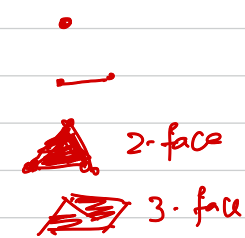
convex hull of S $\mathrm{CVX}(S)$. The elts of S are called vertices of S. In general, if $T \subseteq S$ is a proper subset then $\mathrm{CVX}(T) \subseteq \mathrm{CVX}(S)$ is called the proper face of $\mathrm{CVX}(S)$.

$|T| = 1$ $\quad \mathrm{CVX}(T)$ is a vertex

$|T| = 2$ $\quad \mathrm{CVX}(T)$ is an edge

$\vdots$

$|T| = |S| - 1$ $\quad \mathrm{CVX}(T)$ is a facet.

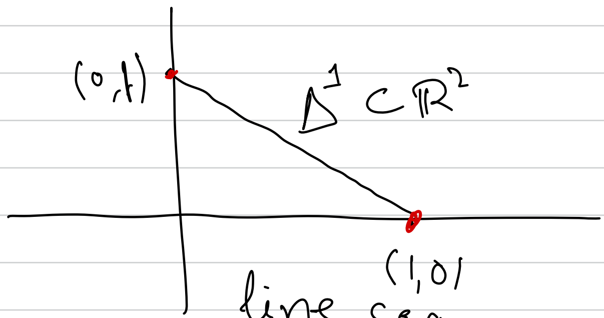


Def. The standard n-simplex Δ^n

Let $e_i = (0, \dots, 0, \underset{i^{th}\text{ pos}^n}{1}, 0, \dots, 0) \in \mathbb{R}^{n+1}$

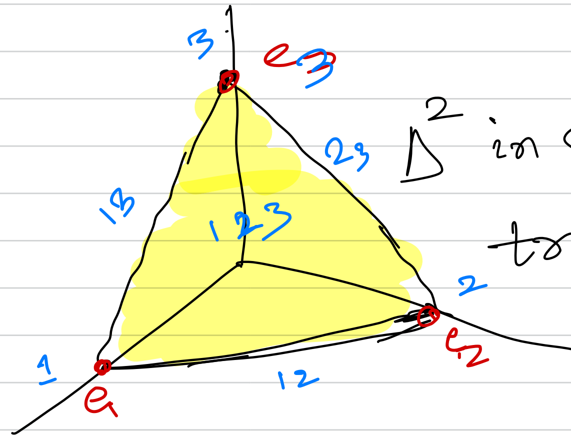
The convex hull of $\{e_1, \dots, e_{n+1}\}$ in $\mathbb{R}^{n+1}$ is the stdard n-simplex in $\mathbb{R}^{n+1}$.

$$\therefore \quad \Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; 0 \le x_i \le 1 \ \& \ \sum x_i = 1 \right\}$$



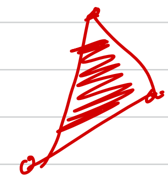
$\Delta^1 \subset \mathbb{R}^2$

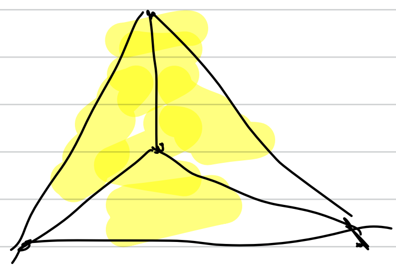
line seg.



Δ^2 in $\mathbb{R}^3$

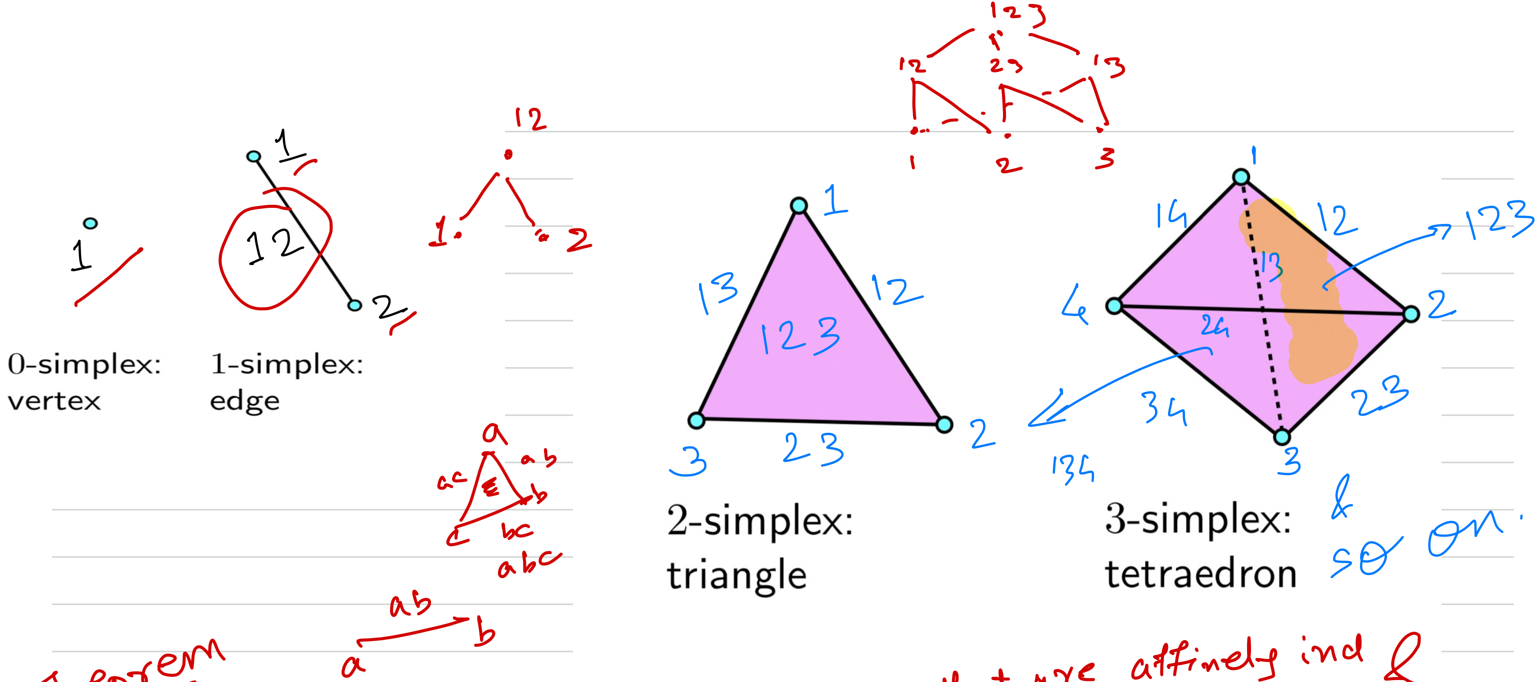
triangle





Δ^3 in $\mathbb{R}^4$

tetrahedron.



Theorem

Suppose S, S' are two subsets that are affinely ind & of $\mathbb{R}^k$ s.t. $|S| = |S'|$.
$cvx(S)$ & $cvx(S')$ are combinatorially equivalent. Why? $\rightarrow$ A bijection betn S & S' extends to $\mathcal{P}(S)$ & $\mathcal{P}(S')$.

We can intuitively imagine Convex(S) as a "visualization" of $\mathcal{P}(S)$.

affine ind.

"Simplex in a power set"

Every face of a simplex can be labeled by a subset of its vertices.

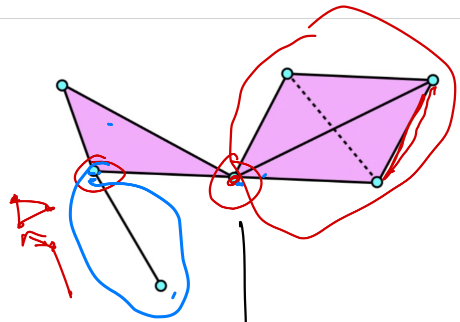
Consider $T = \{1, \dots, n+1\}$ send $i \longmapsto e_i \in \mathbb{R}^{n+1}$

call $T' = \{e_1, \dots, e_{n+1}\}$ $\text{cvx}(T')$ is labeled by subsets of T.

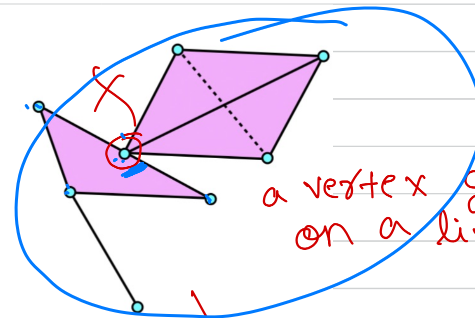
If $T \subset S$ then the dimension of the face $\mathrm{cvx}(T) \subset \mathrm{cvx}(S)$ is $|T|-1$.

<u>Def</u> A geometric simplicial complex is a collection of simplices $\mathcal{X}$ in $\mathbb{R}^n$ satisfying:

1. for any simplex $\sigma \in \mathcal{X}$, all faces of σ are also in $\mathcal{X}$
2. for any two $\sigma, \tau \in \mathcal{X}$, the intersection $\tau \cap \sigma$ is also a simplex & which is a common face.

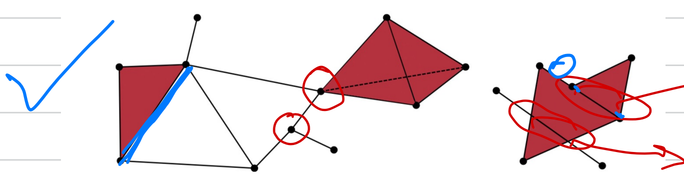


a simplicial cplx.



a vertex glued on a line.

not a simplicial cplx.



Valid

Invalid

not allowed.

not allowed

common face.

$A \in \Sigma$ $\quad A \subset V$

$\Rightarrow B \subset A$ is also $B \in \Sigma$

An abstract simplicial complex is a pair $K = (V, \Sigma')$ where V is a finite set & 'Σ' is a collection of subsets of V that is closed under containment.

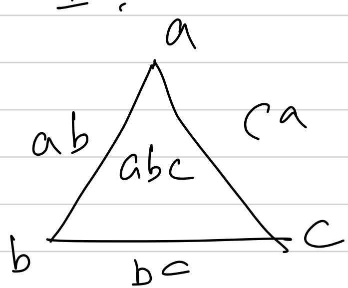
A j-face of K is a $j+1$ cardinality subset in Σ'. In particular 0-face is a vertex 1-face is an edge.

Example. $\quad V = \{a, b, c\}$

$$K = \{V, (\phi, a, b, c, ab)\}$$

$\mathcal{K} = \{ V, (a, b, c, ab, bc, ca) \}$

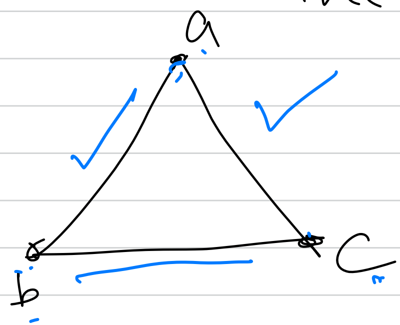
Example. If $\Sigma' = \mathcal{P}(V)$ then $\mathcal{K}$ is recovers labeling of a simplex of dimension $|V| - 1$.



Given a geometric simplicial cplx $\mathcal{X}$

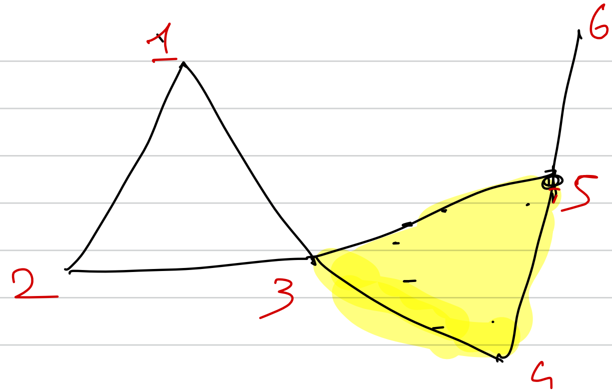
let V = the set of all vertices of $\mathcal{X}$

Σ' consists of those subsets for which the corr. vertices span a simplex.



$$\Sigma' = \{a, b, c, \{a,b\}, \{b,c\}, \{c,a\}\}$$

$$V = \{a, b, c\}$$



$S' = \{$ vertices : 1 to 6

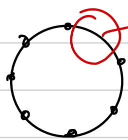
edges : 12, 23, 13, 35, 45,

34, 56,

2-faces : 345 $\}$

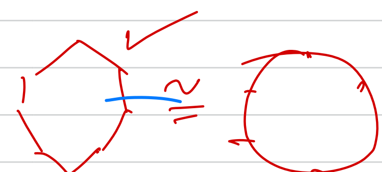
A geo. S.C. is a topological space.

We'd like to focus on spaces that are homeo. to GSC.

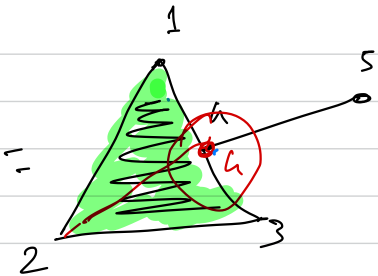


not a simplex

is homeo to an n-gon.

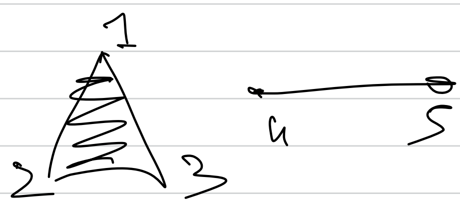


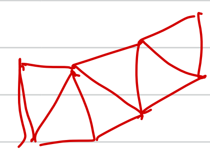
$| \cong C$

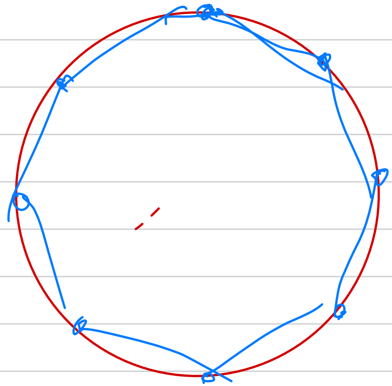
$\mathcal{K} =$ 

$\underline{\Delta(123)} \cap \underline{\Delta(45)}$ is not a simplex in the collection

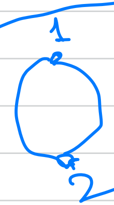
$$\mathcal{K} = \{1, 2, 3, 4, 5, \; 12, 23, 13, 45, \underline{123}\}$$







n-gon for $n \geq 3$ gives a simplicial cplx structure

$\{1\ 2\}$ 

$\times$ not a simplicial cplx