TAM 212. Midterm 2. Apr 4, 2013.

- There are 20 questions, each worth 5 points.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.
- The notation $\vec{r}_{PQ}$ denotes the position vector from P to Q.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

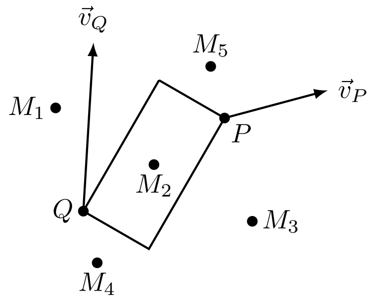
2. Circle your discussion section:

	Monday	Tuesday	Wednesday	Thursday
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

94. A
95. D
96. C

1. (5 points) A rigid body is moving in 2D as shown below.



Which point M_i is the instantaneous center?

(A) M_5

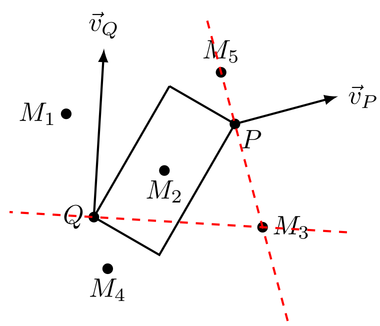
(B) M_2

(C) ★ M_3

(D) M_1

(E) M_4

Solution. The lines through P and Q perpendicular to $\vec{v}_P$ and $\vec{v}_Q$ intersect at M_3, so that is the instantaneous center.



---

2. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -2\hat{k}$ rad/s. Two points P and Q are fixed to the body and the offset between them is in the direction $\hat{r}_{PQ} = \frac{1}{5}(3\hat{\imath} - 4\hat{\jmath})$ (note that this is the unit vector in the direction of the offset vector $\vec{r}_{PQ}$, not the actual offset vector $\vec{r}_{PQ}$). The velocities are:

$$\begin{aligned}\vec{v}_P &= 3\hat{\imath} + 4\hat{\jmath} \text{ m/s} \\ \vec{v}_Q &= -5\hat{\imath} - 2\hat{\jmath} \text{ m/s.}\end{aligned}$$

What is the distance r_{PQ} between P and Q?

(A) $8 \text{ m} \le r_{PQ}$

(B) ★ $4 \text{ m} \le r_{PQ} < 6 \text{ m}$

(C) $2 \text{ m} \le r_{PQ} < 4 \text{ m}$

(D) $6 \text{ m} \le r_{PQ} < 8 \text{ m}$

(E) $0 \text{ m} \le r_{PQ} < 2 \text{ m}$

Solution.

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ -5\hat{\imath} - 2\hat{\jmath} &= 3\hat{\imath} + 4\hat{\jmath} - 2\hat{k} \times r_{PQ}\frac{1}{5}(3\hat{\imath} - 4\hat{\jmath}) \\ -8\hat{\imath} - 6\hat{\jmath} &= -\frac{8}{5}r_{PQ}\,\hat{\imath} - \frac{6}{5}r_{PQ}\,\hat{\jmath} \\ \implies r_{PQ} &= 5 \text{ m.}\end{aligned}$$

---

3. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -2\hat{k}$ rad/s and angular acceleration $\vec{\alpha} = -\hat{k}$ rad/s^2. Points P and Q on the body have:

$$\vec{r}_{PQ} = 2\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{a}_P = 3\hat{\imath} - 3\hat{\jmath} \text{ m/s}^2.$$

What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) $a_{Qy} < -3$ m/s^2

(B) $a_{Qy} = 0$ m/s^2

(C) -3 m/s$^2 \leq a_{Qy} < 0$ m/s^2

(D) ★ 3 m/s$^2 \leq a_{Qy}$

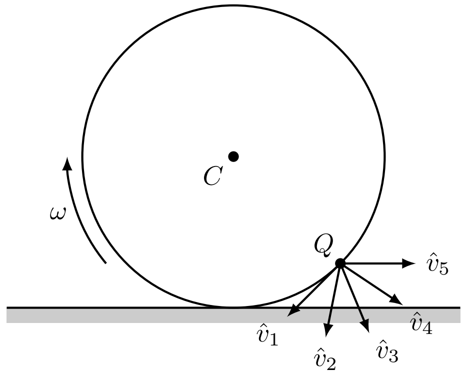
(E) 0 m/s$^2 < a_{Qy} < 3$ m/s^2

Solution.

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\
&= 3\hat{\imath} - 3\hat{\jmath} - \hat{k} \times (2\hat{\imath} - 2\hat{\jmath}) - 2\hat{k} \times (-2\hat{k} \times (2\hat{\imath} - 2\hat{\jmath})) \\
&= (3\hat{\imath} - 3\hat{\jmath}) + (-2\hat{\imath} - 2\hat{\jmath}) + (-8\hat{\imath} + 8\hat{\jmath}) \\
&= -7\hat{\imath} + 3\hat{\jmath} \\
a_{Qy} &= 3 \text{ m/s}^2.
\end{aligned}$$

---

4. (5 points) A circular rigid body is rolling without slipping on a flat surface in 2D in a clockwise direction as shown.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

(A) $\hat{v}_2$

(B) ★ $\hat{v}_3$

(C) $\hat{v}_1$

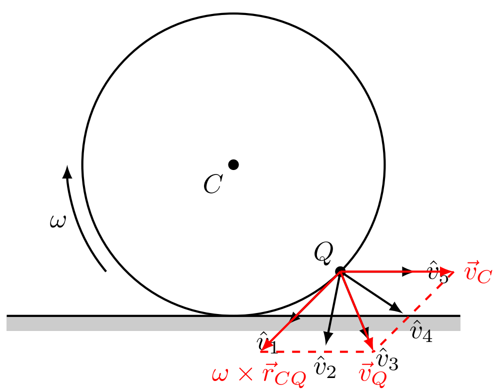
(D) $\hat{v}_4$

(E) $\hat{v}_5$

Solution. Starting from point C, the velocity at point Q is given by:

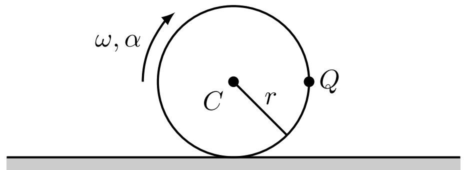
$$\vec{v}_Q = \vec{v}_C + \vec{\omega} \times \vec{r}_{CQ}.$$

The two terms on the right hand side above both have magnitude ωr, where r is the radius of the body. The $\vec{\omega} \times \vec{r}_{CQ}$ term is orthogonal to $\vec{r}_{CP}$, while the $\vec{v}_C$ term is horizontal. Adding the two terms gives a resultant $\vec{v}_Q$ exactly half way between the two component vectors.



---

5. (5 points) A circular rigid body with radius $r = 2$ m is rolling without slipping with angular velocity $\vec{\omega} = -\hat{k}$ rad/s on a flat surface in 2D as shown. The body is speeding up and has angular acceleration $\vec{\alpha} = -\alpha\hat{k}$. Point Q is at the right edge of the body and has acceleration $\vec{a}_Q = 3\hat{\imath} - 5\hat{\jmath}$ m/s^2.



What is α?

(A) 1 rad/s$^2 \leq \alpha <$ 1.5 rad/s^2

(B) 0 rad/s$^2 \leq \alpha <$ 0.5 rad/s^2

(C) 1.5 rad/s$^2 \leq \alpha <$ 2 rad/s^2

(D) 0.5 rad/s$^2 \leq \alpha <$ 1 rad/s^2

(E) ★ 2 rad/s$^2 \leq \alpha$

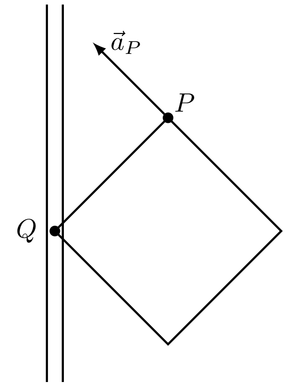
Solution. The acceleration of Q is:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= (-\alpha\hat{k}) \times r\hat{\jmath} + (-\alpha\hat{k}) \times r\hat{\imath} + (-\omega\hat{k}) \times (-\omega\hat{k} \times r\hat{\imath}) \\
&= r\alpha\,\hat{\imath} - r\alpha\,\hat{\jmath} - r\omega^2\,\hat{\imath} \\
&= r(\alpha - \omega^2)\,\hat{\imath} - r\alpha\,\hat{\jmath} \\
3\hat{\imath} - 5\hat{\jmath} &= 2(\alpha - 1)\,\hat{\imath} - 2\alpha\,\hat{\jmath}
\end{aligned}$$

So $\alpha = 2.5$ rad/s^2.

---

6. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$ and zero angular acceleration. A pin at point Q constrains that point to move in a vertical slot.



Point P on the body has:

$$\vec{r}_{PQ} = -\hat{\imath} - \hat{\jmath} \text{ m}$$
$$\vec{a}_P = -\hat{\imath} + \hat{\jmath} \text{ m/s}^2.$$

What is the magnitude a_Q of the acceleration $\vec{a}_Q$ of point Q?

(A) $0 \text{ m/s}^2 \le a_Q < 1 \text{ m/s}^2$

(B) ★ $2 \text{ m/s}^2 \le a_Q < 3 \text{ m/s}^2$

(C) $3 \text{ m/s}^2 \le a_Q < 4 \text{ m/s}^2$

(D) $4 \text{ m/s}^2 \le a_Q$

(E) $1 \text{ m/s}^2 \le a_Q < 2 \text{ m/s}^2$

Solution. Taking $\vec{a}_Q = a_Q\,\hat{\jmath}$, we have:

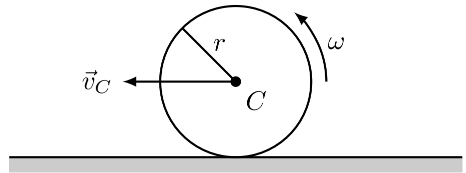
$$\begin{aligned}
\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\
a_Q\,\hat{\jmath} &= -\hat{\imath} + \hat{\jmath} + \omega\hat{k} \times (\omega\hat{k} \times (-\hat{\imath} - \hat{\jmath})) \\
\hat{\imath} + (a_Q - 1)\,\hat{\jmath} &= \omega^2\hat{\imath} + \omega^2\hat{\jmath}.
\end{aligned}$$

Equating components and solving gives:

$$\begin{aligned}
\omega^2 &= 1 \\
a_Q &= \omega^2 + 1 \\
&= 2 \text{ m/s}^2.
\end{aligned}$$

---

7. (5 points) A circular rigid body with radius $r = 2$ m is rolling without slipping on a flat surface in 2D as shown. The speed of the center is $v_C = 9$ m/s.



What is the angular velocity ω?

(A) 0 rad/s $\leq \omega <$ 1 rad/s

(B) 1 rad/s $\leq \omega <$ 2 rad/s

(C) 2 rad/s $\leq \omega <$ 3 rad/s

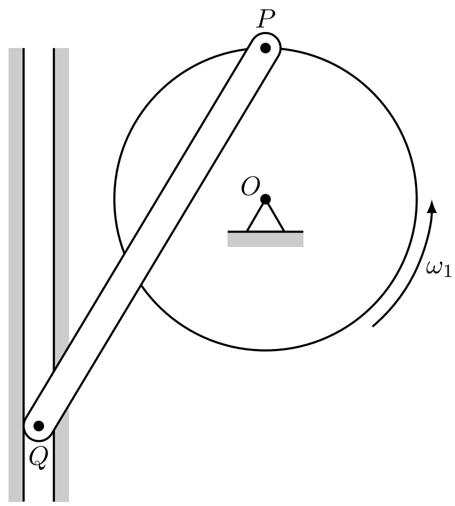
(D) ★ 4 rad/s $\leq \omega$

(E) 3 rad/s $\leq \omega <$ 4 rad/s

Solution. $v_c = r\omega$ so $\omega = v_c/r = 9/2 = 4.5$ rad/s.

---

8. (5 points) A circular rigid body rotates about the fixed center O with angular velocity $\vec{\omega}_1 = 5\hat{k}$ rad/s as shown. A rigid rod connects pins P and Q, and point Q is constrained to only move vertically.



At the current instant the positions are:

$$\vec{r}_{OP} = 2\hat{\jmath} \text{ m}$$
$$\vec{r}_{PQ} = -3\hat{\imath} - 5\hat{\jmath} \text{ m.}$$

What is the speed v_Q of point Q?

(A) 8 m/s $\leq v_Q$

(B) 2 m/s $\leq v_Q <$ 4 m/s

(C) 4 m/s $\leq v_Q <$ 6 m/s

(D) ★ 6 m/s $\leq v_Q <$ 8 m/s

(E) 0 m/s $\leq v_Q <$ 2 m/s

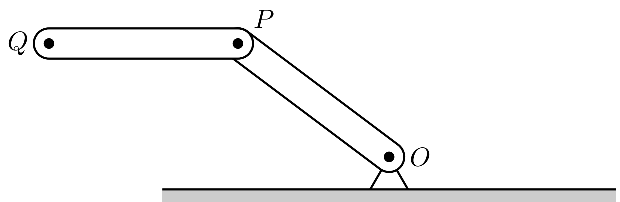
Solution. Let the unknown angular velocity of the rod be $\vec{\omega}_2 = \omega_2\hat{k}$ and the unknown velocity of Q be $\vec{v}_Q = v_Q\,\hat{\jmath}$. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\
&= 0 + 5\hat{k} \times (2\hat{\jmath}) \\
&= -10\hat{\imath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\
&= -10\hat{\imath} + \omega_2\hat{k} \times (-3\hat{\imath} - 5\hat{\jmath}) \\
v_Q\,\hat{\jmath} &= (-10 + 5\omega_2)\,\hat{\imath} - 3\omega_2\,\hat{\jmath}.
\end{aligned}$$

Solving this gives $\omega_2 = 2$ rad/s and $v_Q = -3\omega_2 = -6$, so the speed is 6 m/s.

---

9. (5 points) Two rods are connected with pin joints at O, P, and Q as shown. The angular velocity and acceleration for rod OP are $\vec{\omega}_1$ and $\vec{\alpha}_1$, while the angular velocity and acceleration for rod PQ are $\vec{\omega}_2$ and $\vec{\alpha}_2$.



The positions and angular velocities of the rods at the current instant are:

$$\vec{r}_{OP} = -4\hat{\imath} + 3\hat{\jmath} \text{ m} \qquad \vec{r}_{PQ} = -5\hat{\imath} \text{ m}$$
$$\vec{\omega}_1 = 0 \qquad \vec{\omega}_2 = 2\hat{k} \text{ rad/s}$$
$$\vec{\alpha}_1 = 1\hat{k} \text{ rad/s}^2 \qquad \vec{\alpha}_2 = -\hat{k} \text{ rad/s}^2.$$

What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) $2 \text{ m/s}^2 \le a_{Qy}$

(B) $-2 \text{ m/s}^2 \le a_{Qy} < 0 \text{ m/s}^2$

(C) $a_{Qy} = 0 \text{ m/s}^2$

(D) $a_{Qy} < -2 \text{ m/s}^2$

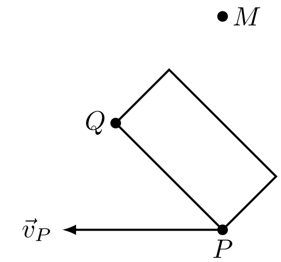
(E) ★ $0 \text{ m/s}^2 < a_{Qy} < 2 \text{ m/s}^2$

Solution. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{a}_P &= \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OP} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OP}) \\
&= 0 + \hat{k} \times (-4\hat{\imath} + 3\hat{\jmath}) + 0 \\
&= -3\hat{\imath} - 4\hat{\jmath} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_w \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\
&= (-3\hat{\imath} - 4\hat{\jmath}) - \hat{k} \times (-5\hat{\imath}) + 2\hat{k} \times (2\hat{k} \times (-5\hat{\imath})) \\
&= (-3\hat{\imath} - 4\hat{\jmath}) + 5\hat{\jmath} + 20\hat{\imath} \\
&= 17\hat{\imath} + \hat{\jmath} \\
a_{Qy} &= 1 \text{ m/s}^2.
\end{aligned}$$

---

10. (5 points) A rigid body is moving in 2D as shown below with points P and Q attached to the body. The instantaneous center of the body is at point M.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

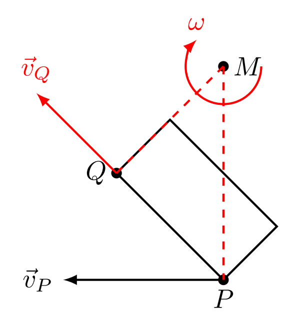
(A) $\searrow$

(B) $\swarrow$

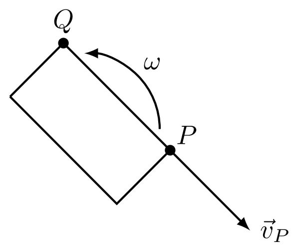
(C) $\nearrow$

(D) $\bigstar$ $\nwarrow$

Solution. The direction of $\vec{v}_P$ shows the direction of ω. Then $\vec{v}_Q$ is orthogonal to $\vec{r}_{MQ}$ in the ω direction.



11. (5 points) A rigid body is moving in 2D as shown, with a counterclockwise rotation (ω is positive in the direction indicated). The angular velocity ω, distance r_{PQ}, and speed v_P satisfy $\omega r_{PQ} = v_P$.



What is the direction of $\vec{v}_Q$?

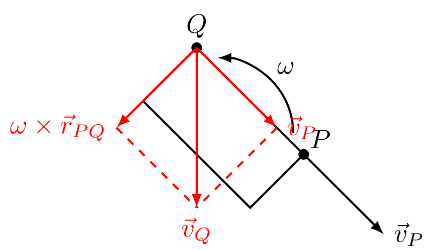
(A) $\leftarrow$

(B) $\uparrow$

(C) ★ $\downarrow$

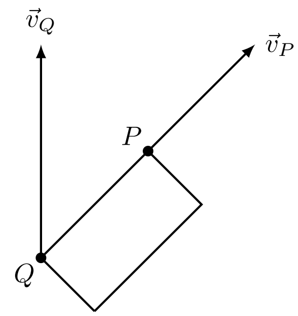
(D) $\rightarrow$

Solution.



---

12. (5 points) A rigid body is moving in 2D as shown below.

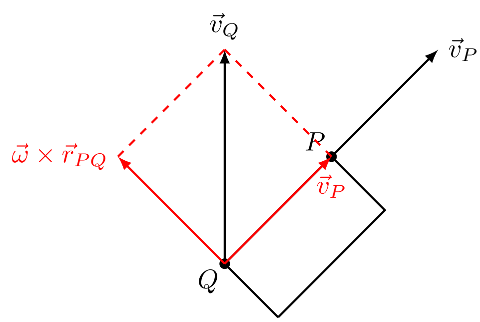


What is the direction of the angular velocity of the body?

(A) ↺ (counterclockwise)

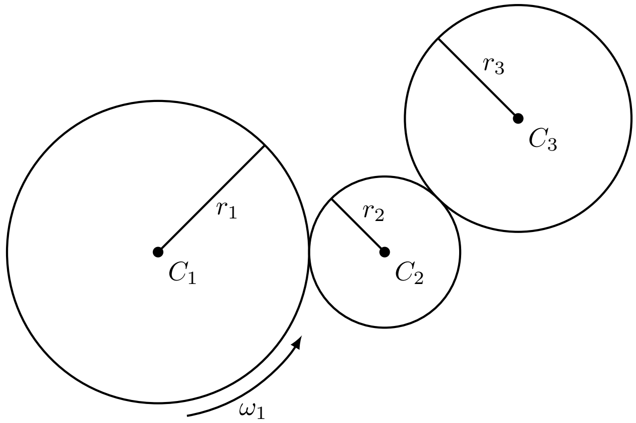
(B) ★ ↻ (clockwise)

Solution. Considering the required term $\vec{\omega} \times \vec{r}_{PQ}$ shows that it is up-left, so considering rotation about P we see that ω is clockwise.



---

13. (5 points) Three meshed gears rotate about fixed centers as shown. The radii are $r_1 = 4$ m, $r_2 = 2$ m, and $r_3 = 3$ m and the corresponding angular velocities are $\vec{\omega}_1 = 3\hat{k}$ rad/s, $\vec{\omega}_2 = \omega_2\hat{k}$, and $\vec{\omega}_3 = \omega_3\hat{k}$.



What is ω_3?

(A) 3 rad/s $\leq \omega_3 <$ 4 rad/s

(B) 1 rad/s $\leq \omega_3 <$ 2 rad/s

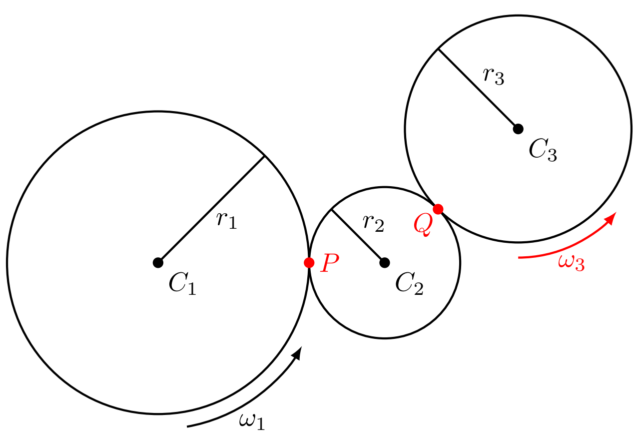
(C) ★ 4 rad/s $\leq \omega_3$

(D) 2 rad/s $\leq \omega_3 <$ 3 rad/s

(E) 0 rad/s $\leq \omega_3 <$ 1 rad/s

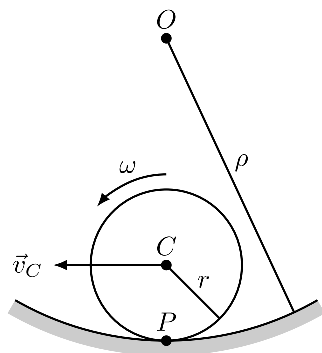
Solution. Matching the velocities at points P and Q shows that the gear at C_2 rotates clockwise and the gear at C_3 rotates counterclockwise. Also:

$$\begin{aligned} r_1\omega_1 &= v_P = v_Q = r_3\omega_3 \\ \omega_3 &= \frac{r_1}{r_3}\omega_1 \\ &= \frac{4}{3}3 \\ &= 4 \text{ rad/s.} \end{aligned}$$



---

14. (5 points) A circular rigid body with radius $r = 3$ m is rolling without slipping on a curved surface with radius of curvature ρ in 2D as shown. The angular velocity of the body is a constant $\vec{\omega} = \hat{k}$ rad/s. Point P is fixed to the edge of the body and, at the instant shown, is the contact point. The magnitude of acceleration of P is $a_P = 6$ m/s^2.



What is the radius of curvature ρ of the surface?

(A) $0 \text{ m} \leq \rho < 3 \text{ m}$

(B) ★ $6 \text{ m} \leq \rho < 9 \text{ m}$

(C) $3 \text{ m} \leq \rho < 6 \text{ m}$

(D) $9 \text{ m} \leq \rho < 12 \text{ m}$

(E) $12 \text{ m} \leq \rho$

Solution. Because we are rolling on the inside, $R = \rho - r$, giving:

$$\begin{aligned} a_P &= \frac{\rho}{R} r\omega^2 \\ 6 &= \frac{\rho}{\rho - 3} 3 \times 1^2 \\ \rho &= 6 \text{ m}. \end{aligned}$$

---

15. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -2\hat{k}$ rad/s. Points P and Q on the body have:

$$\begin{aligned}\vec{r}_{PQ} &= 2\hat{\imath} - \hat{\jmath} \text{ m} \\ \vec{v}_P &= -\hat{\imath} + \hat{\jmath} \text{ m/s.}\end{aligned}$$

What is the $\hat{\imath}$ component v_{Qx} of the velocity $\vec{v}_Q$ of point Q?

(A) $-2 \text{ m/s} \le v_{Qx} < 0 \text{ m/s}$

(B) $0 \text{ m/s} < v_{Qx} < 2 \text{ m/s}$

(C) $2 \text{ m/s} \le v_{Qx}$

(D) $v_{Qx} = 0 \text{ m/s}$

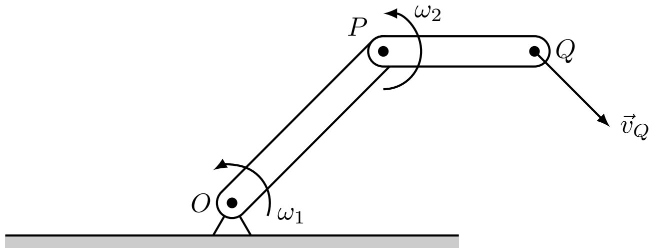
(E) ★ $v_{Qx} < -2 \text{ m/s}$

Solution.

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ &= -\hat{\imath} + \hat{\jmath} - 2\hat{k} \times (2\hat{\imath} - \hat{\jmath}) \\ &= -\hat{\imath} + \hat{\jmath} - 2\hat{\imath} - 4\hat{\jmath} \\ &= -3\hat{\imath} - 3\hat{\jmath} \\ v_{Qx} &= -3 \text{ m/s}\end{aligned}$$

---

16. (5 points) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = \omega_1 \hat{k}$ and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The positions and velocities at the current instant are:

$$\begin{aligned}
\vec{r}_{OP} &= 2\hat{\imath} + 2\hat{\jmath} \text{ m} \\
\vec{r}_{PQ} &= 2\hat{\imath} \text{ m} \\
\vec{v}_Q &= 2\hat{\imath} - 2\hat{\jmath} \text{ m/s}.
\end{aligned}$$

What is ω_2?

(A) ★ $\omega_2 = 0$ rad/s

(B) -1 rad/s $\leq \omega_2 < 0$ rad/s

(C) 1 rad/s $\leq \omega_2$

(D) $\omega_2 < -1$ rad/s

(E) 0 rad/s $< \omega_2 < 1$ rad/s

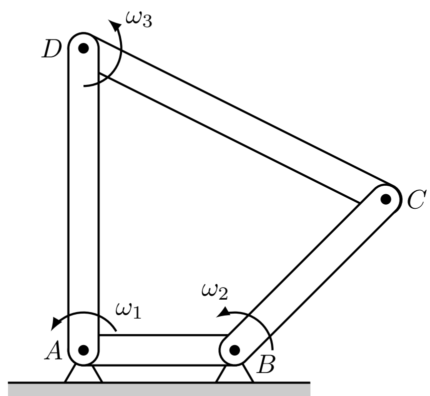
Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned}
\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\
&= -2\omega_1\,\hat{\imath} + 2\omega_1\,\hat{\jmath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\
2\hat{\imath} - 2\hat{\jmath} &= -2\omega_1\,\hat{\imath} + (2\omega_1 + 2\omega_2)\,\hat{\jmath}.
\end{aligned}$$

Comparing $\hat{\imath}$ components gives $\omega_1 = -1$ rad/s and $\omega_2 = 0$ rad/s.

---

17. (5 points) A four-bar linkage has rigid rods connecting pins at A, B, C, and D, as shown. The angular velocities are $\vec{\omega}_1 = \hat{k}$ for rod AD, $\vec{\omega}_2 = \omega_2 \hat{k}$ for rod BC, and $\vec{\omega}_3 = \omega_3 \hat{k}$ for rod DC.



At the current instant the positions are:

$$
\begin{aligned}
\vec{r}_{AB} &= \hat{\imath} \text{ m} \\
\vec{r}_{BC} &= \hat{\imath} + \hat{\jmath} \text{ m} \\
\vec{r}_{AD} &= 2\hat{\jmath} \text{ m} \\
\vec{r}_{DC} &= 2\hat{\imath} - \hat{\jmath} \text{ m.}
\end{aligned}
$$

What is ω_2?

(A) ★ 1 rad/s $\leq \omega_2 <$ 1.5 rad/s

(B) 0 rad/s $\leq \omega_2 <$ 0.5 rad/s

(C) 0.5 rad/s $\leq \omega_2 <$ 1 rad/s

(D) 1.5 rad/s $\leq \omega_2 <$ 2 rad/s

(E) 2 rad/s $\leq \omega_2$

Solution. Starting from the fixed point A we have:

$$
\begin{aligned}
\vec{v}_D &= \vec{v}_A + \vec{\omega}_1 \times \vec{r}_{AD} \\
&= 0 + \hat{k} \times 2\hat{\jmath} \\
&= -2\hat{\imath} \\
\vec{v}_C &= \vec{v}_D + \vec{\omega}_3 \times \vec{r}_{DC} \\
&= -2\hat{\imath} + \omega_3 \hat{k} \times (2\hat{\imath} - \hat{\jmath}) \\
&= (-2 + \omega_3)\,\hat{\imath} + 2\omega_3\,\hat{\jmath}.
\end{aligned}
$$

We can also get to C from the fixed point B:

$$
\begin{aligned}
\vec{v}_C &= \vec{v}_B + \vec{\omega}_2 \times \vec{r}_{BC} \\
&= 0 + \omega_2 \hat{k} \times (\hat{\imath} + \hat{\jmath}) \\
&= -\omega_2\,\hat{\imath} + \omega_2\,\hat{\jmath}.
\end{aligned}
$$

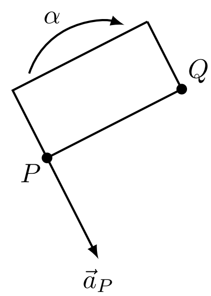
Equating the two expressions for $\vec{v}_C$ gives:

$$
\begin{aligned}
-2 + \omega_3 &= -\omega_2 \\
2\omega_3 &= \omega_2.
\end{aligned}
$$

Solving these equations gives $\omega_3 = \frac{2}{3} \approx 0.67$ rad/s and $\omega_2 = \frac{4}{3} \approx 1.33$ rad/s.

---

18. (5 points) A rigid body is moving in 2D as shown below, with a counterclockwise angular acceleration and points P and Q on the body (α is positive in the direction shown). We know that $a_P = \omega^2\, r_{PQ} = \alpha\, r_{PQ}$.



What is the direction of the acceleration $\vec{a}_Q$?

(A) $\uparrow$

(B) $\rightarrow$

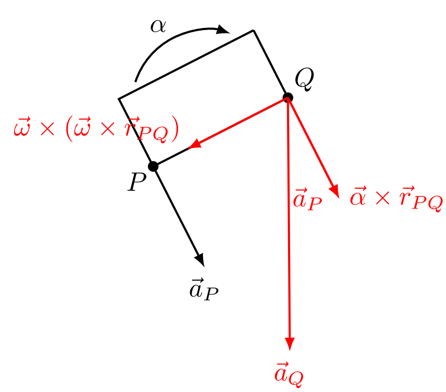
(C) ★ $\downarrow$

(D) $\leftarrow$

Solution. Consider the acceleration equation for point Q:

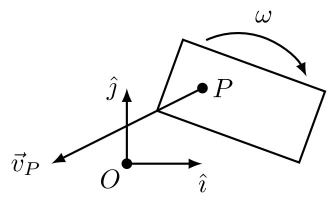
$$\vec{a}_Q = \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}).$$

All three terms on the right hand side have the same magnitude ($a_P = \omega^2\, r_{PQ} = \alpha\, r_{PQ}$). Drawing the right-hand-side terms shows that the resulting direction for $\vec{a}_Q$ is down:



---

19. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = -\hat{k}$ rad/s.



Relative to the origin O, the point P has:

$$\begin{aligned}\vec{r}_P &= \hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{v}_P &= -2\hat{\imath} - \hat{\jmath} \text{ m/s.}\end{aligned}$$

What is the x coordinate M_x of the instantaneous center M of the body?

(A) $M_x < -1$ m

(B) $0 \text{ m} < M_x < 1$ m

(C) $-1 \text{ m} \leq M_x < 0$ m

(D) ★ $M_x = 0$ m

(E) $1 \text{ m} \leq M_x$

Solution. The position of M relative to P is:

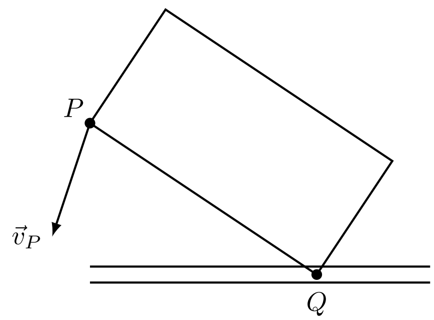
$$\begin{aligned}\vec{r}_{PM} &= \frac{1}{\omega^2}\,\omega\hat{k} \times \vec{v}_P \\ &= -\frac{1}{1^2}\,\hat{k} \times (-2\hat{\imath} - \hat{\jmath}) \\ &= -\hat{\imath} + 2\hat{\jmath} \text{ m.}\end{aligned}$$

Then the position of M is:

$$\begin{aligned}\vec{r}_M &= \vec{r}_P + \vec{r}_{PM} \\ &= (\hat{\imath} + \hat{\jmath}) + (-\hat{\imath} + 2\hat{\jmath}) \\ &= 3\hat{\jmath} \text{ m} \\ M_x &= 0 \text{ m.}\end{aligned}$$

---

20. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega \hat{k}$. A pin at point Q constrains that point to move in a horizontal slot.



Point P on the body has:

$$\vec{r}_{PQ} = 3\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{v}_P = -\hat{\imath} - 3\hat{\jmath} \text{ m/s.}$$

What is ω?

(A) $\omega < -2$ rad/s

(B) $\omega = 0$ rad/s

(C) 2 rad/s $\leq \omega$

(D) ★ 0 rad/s $< \omega <$ 2 rad/s

(E) -2 rad/s $\leq \omega <$ 0 rad/s

Solution. Taking $\vec{v}_Q = v_Q \,\hat{\imath}$ with unknown speed v_Q, we have:

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ v_Q \,\hat{\imath} &= -\hat{\imath} - 3\hat{\jmath} + \omega\hat{k} \times (3\hat{\imath} - 2\hat{\jmath}) \\ (v_Q + 1)\,\hat{\imath} + 3\hat{\jmath} &= 2\omega\,\hat{\imath} + 3\omega\,\hat{\jmath} \end{aligned}$$

Equating $\hat{\jmath}$ components gives $\omega = 1$ rad/s.

TAM 212. Midterm 2. Apr 4, 2013.

- There are 20 questions, each worth 5 points.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.
- The notation $\vec{r}_{PQ}$ denotes the position vector from P to Q.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Circle your discussion section:

	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

94. B

95. E

96. D

1. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -\hat{k}$ rad/s. Two points P and Q are fixed to the body and the offset between them is in the direction $\hat{r}_{PQ} = \frac{1}{5}(3\hat{\imath} + 4\hat{\jmath})$ (note that this is the unit vector in the direction of the offset vector $\vec{r}_{PQ}$, not the actual offset vector $\vec{r}_{PQ}$). The velocities are:

$$\begin{aligned} \vec{v}_P &= 2\hat{\jmath} \text{ m/s} \\ \vec{v}_Q &= 4\hat{\imath} - \hat{\jmath} \text{ m/s}. \end{aligned}$$

What is the distance r_{PQ} between P and Q?

(A) $6 \text{ m} \le r_{PQ} < 8 \text{ m}$

(B) ★ $4 \text{ m} \le r_{PQ} < 6 \text{ m}$

(C) $0 \text{ m} \le r_{PQ} < 2 \text{ m}$

(D) $8 \text{ m} \le r_{PQ}$

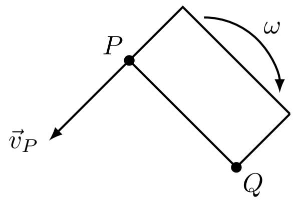
(E) $2 \text{ m} \le r_{PQ} < 4 \text{ m}$

Solution.

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ 4\hat{\imath} - \hat{\jmath} &= 2\hat{\jmath} - \hat{k} \times r_{PQ} \frac{1}{5}(3\hat{\imath} + 4\hat{\jmath}) \\ 4\hat{\imath} - 3\hat{\jmath} &= \frac{4}{5} r_{PQ}\, \hat{\imath} - \frac{3}{5} r_{PQ}\, \hat{\jmath} \\ \Longrightarrow r_{PQ} &= 5 \text{ m}. \end{aligned}$$

---

2. (5 points) A rigid body is moving in 2D as shown, with a clockwise rotation (ω is positive in the direction indicated). The angular velocity ω, distance r_{PQ}, and speed v_P satisfy $\omega r_{PQ} = v_P$.



What is the direction of $\vec{v}_Q$?

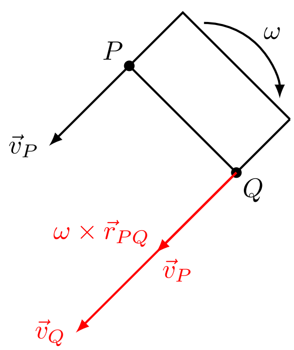
(A) $\nwarrow$

(B) $\nearrow$

(C) $\searrow$

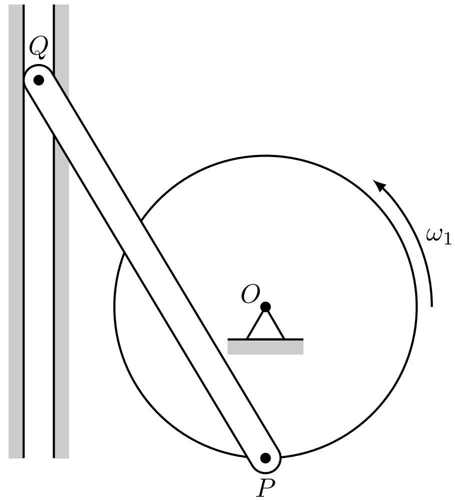
(D) $\bigstar$ $\swarrow$

Solution.



---

3. (5 points) A circular rigid body rotates about the fixed center O with angular velocity $\vec{\omega}_1 = 5\hat{k}$ rad/s as shown. A rigid rod connects pins P and Q, and point Q is constrained to only move vertically.



At the current instant the positions are:

$$\vec{r}_{OP} = -2\hat{\jmath} \text{ m}$$
$$\vec{r}_{PQ} = -3\hat{\imath} + 5\hat{\jmath} \text{ m.}$$

What is the speed v_Q of point Q?

(A) $2 \text{ m/s} \le v_Q < 4 \text{ m/s}$

(B) ★ $6 \text{ m/s} \le v_Q < 8 \text{ m/s}$

(C) $0 \text{ m/s} \le v_Q < 2 \text{ m/s}$

(D) $8 \text{ m/s} \le v_Q$

(E) $4 \text{ m/s} \le v_Q < 6 \text{ m/s}$

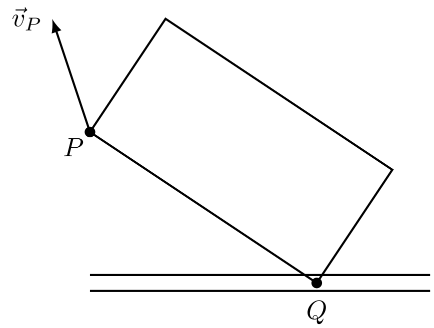
Solution. Let the unknown angular velocity of the rod be $\vec{\omega}_2 = \omega_2\hat{k}$ and the unknown velocity of Q be $\vec{v}_Q = v_Q\,\hat{\jmath}$. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\
&= 0 + 5\hat{k} \times (-2\hat{\jmath}) \\
&= 10\hat{\imath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\
&= 10\hat{\imath} + \omega_2\hat{k} \times (-3\hat{\imath} + 5\hat{\jmath}) \\
v_Q\,\hat{\jmath} &= (10 - 5\omega_2)\,\hat{\imath} - 3\omega_2\,\hat{\jmath}.
\end{aligned}$$

Solving this gives $\omega_2 = 2$ rad/s and $v_Q = -3\omega_2 = -6$, so the speed is 6 m/s.

---

4. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$. A pin at point Q constrains that point to move in a horizontal slot.



Point P on the body has:

$$\vec{r}_{PQ} = 3\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{v}_P = -\hat{\imath} + 3\hat{\jmath} \text{ m/s.}$$

What is ω?

(A) $0 \text{ rad/s} < \omega < 2 \text{ rad/s}$

(B) $\omega = 0 \text{ rad/s}$

(C) $2 \text{ rad/s} \leq \omega$

(D) $\omega < -2 \text{ rad/s}$

(E) ★ $-2 \text{ rad/s} \leq \omega < 0 \text{ rad/s}$

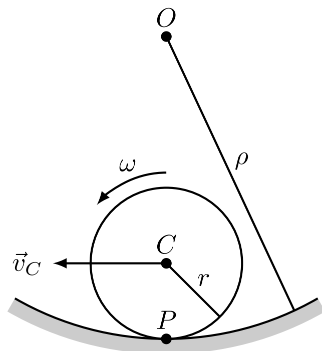
Solution. Taking $\vec{v}_Q = v_Q\,\hat{\imath}$ with unknown speed v_Q, we have:

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ v_Q\,\hat{\imath} &= -\hat{\imath} + 3\hat{\jmath} + \omega\hat{k} \times (3\hat{\imath} - 2\hat{\jmath}) \\ (v_Q + 1)\,\hat{\imath} - 3\hat{\jmath} &= 2\omega\,\hat{\imath} + 3\omega\,\hat{\jmath} \end{aligned}$$

Equating $\hat{\jmath}$ components gives $\omega = -1$ rad/s.

---

5. (5 points) A circular rigid body with radius $r = 2$ m is rolling without slipping on a curved surface with radius of curvature ρ in 2D as shown. The angular velocity of the body is a constant $\vec{\omega} = 2\hat{k}$ rad/s. Point P is fixed to the edge of the body and, at the instant shown, is the contact point. The magnitude of acceleration of P is $a_P = 16$ m/s^2.



What is the radius of curvature ρ of the surface?

(A) $9 \text{ m} \leq \rho < 12 \text{ m}$

(B) $0 \text{ m} \leq \rho < 3 \text{ m}$

(C) $6 \text{ m} \leq \rho < 9 \text{ m}$

(D) $12 \text{ m} \leq \rho$

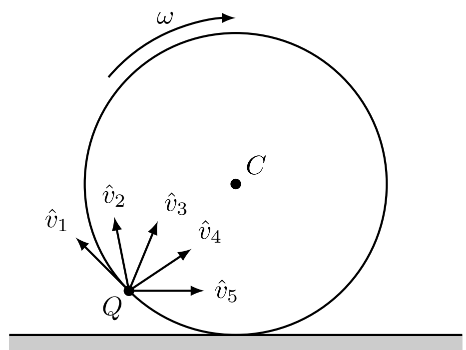
(E) ★ $3 \text{ m} \leq \rho < 6 \text{ m}$

Solution. Because we are rolling on the inside, $R = \rho - r$, giving:

$$
\begin{aligned}
a_P &= \frac{\rho}{R} r \omega^2 \\
16 &= \frac{\rho}{\rho - 2} 2 \times 2^2 \\
\rho &= 4 \text{ m}.
\end{aligned}
$$

---

6. (5 points) A circular rigid body is rolling without slipping on a flat surface in 2D in a clockwise direction as shown.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

(A) $\hat{v}_5$

(B) ★ $\hat{v}_3$

(C) $\hat{v}_4$

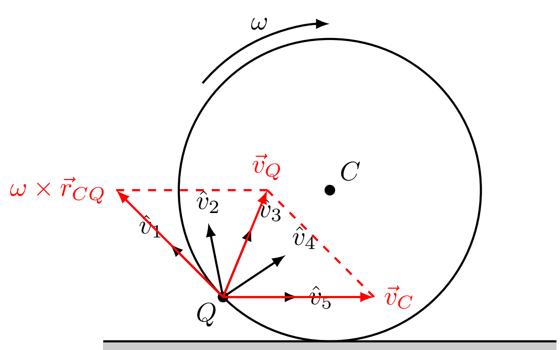
(D) $\hat{v}_2$

(E) $\hat{v}_1$

Solution. Starting from point C, the velocity at point Q is given by:

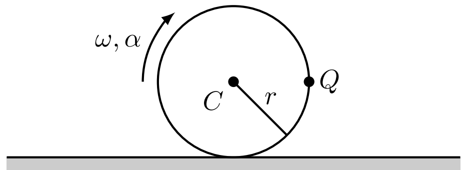
$$\vec{v}_Q = \vec{v}_C + \vec{\omega} \times \vec{r}_{CQ}.$$

The two terms on the right hand side above both have magnitude ωr, where r is the radius of the body. The $\vec{\omega} \times \vec{r}_{CQ}$ term is orthogonal to $\vec{r}_{CP}$, while the $\vec{v}_C$ term is horizontal. Adding the two terms gives a resultant $\vec{v}_Q$ exactly half way between the two component vectors.



---

7. (5 points) A circular rigid body with radius $r = 2$ m is rolling without slipping with angular velocity $\vec{\omega} = -2\hat{k}$ rad/s on a flat surface in 2D as shown. The body is speeding up and has angular acceleration $\vec{\alpha} = -\alpha\hat{k}$. Point Q is at the right edge of the body and has acceleration $\vec{a}_Q = -6\hat{\imath} - 2\hat{\jmath}$ m/s^2.



What is α?

(A) ★ 1 rad/s$^2 \leq \alpha < 1.5$ rad/s^2

(B) 1.5 rad/s$^2 \leq \alpha < 2$ rad/s^2

(C) 2 rad/s$^2 \leq \alpha$

(D) 0 rad/s$^2 \leq \alpha < 0.5$ rad/s^2

(E) 0.5 rad/s$^2 \leq \alpha < 1$ rad/s^2

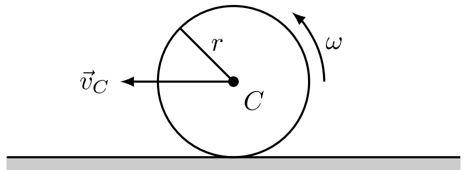
Solution. The acceleration of Q is:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= (-\alpha\hat{k}) \times r\hat{\jmath} + (-\alpha\hat{k}) \times r\hat{\imath} + (-\omega\hat{k}) \times (-\omega\hat{k} \times r\hat{\imath}) \\
&= r\alpha\,\hat{\imath} - r\alpha\,\hat{\jmath} - r\omega^2\,\hat{\imath} \\
&= r(\alpha - \omega^2)\,\hat{\imath} - r\alpha\,\hat{\jmath} \\
-6\hat{\imath} - 2\hat{\jmath} &= 2(\alpha - 4)\,\hat{\imath} - 2\alpha\,\hat{\jmath}
\end{aligned}$$

So $\alpha = 1$ rad/s^2.

---

8. (5 points) A circular rigid body with radius $r = 3$ m is rolling without slipping on a flat surface in 2D as shown. The speed of the center is $v_C = 7$ m/s.



What is the angular velocity ω?

(A) ★ 2 rad/s $\leq \omega <$ 3 rad/s

(B) 1 rad/s $\leq \omega <$ 2 rad/s

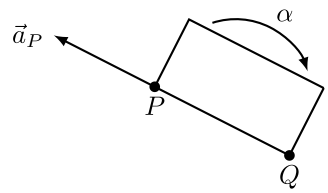
(C) 3 rad/s $\leq \omega <$ 4 rad/s

(D) 0 rad/s $\leq \omega <$ 1 rad/s

(E) 4 rad/s $\leq \omega$

Solution. $v_c = r\omega$ so $\omega = v_c/r = 7/3 \approx 2.33$ rad/s.

9. (5 points) A rigid body is moving in 2D as shown below, with a clockwise angular acceleration and points P and Q on the body (α is positive in the direction shown). We know that $a_P = \omega^2 r_{PQ} = \alpha r_{PQ}$.



What is the direction of the acceleration $\vec{a}_Q$?

(A) ★ ←

(B) ↑

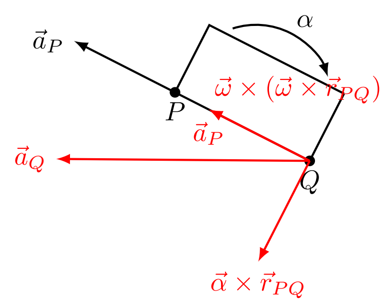
(C) ↓

(D) →

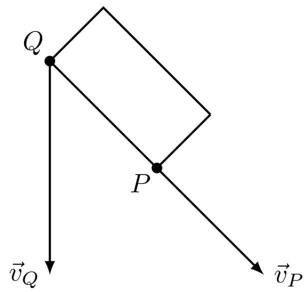
Solution. Consider the acceleration equation for point Q:

$$\vec{a}_Q = \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}).$$

All three terms on the right hand side have the same magnitude ($a_P = \omega^2 r_{PQ} = \alpha r_{PQ}$). Drawing the right-hand-side terms shows that the resulting direction for $\vec{a}_Q$ is left:



10. (5 points) A rigid body is moving in 2D as shown below.

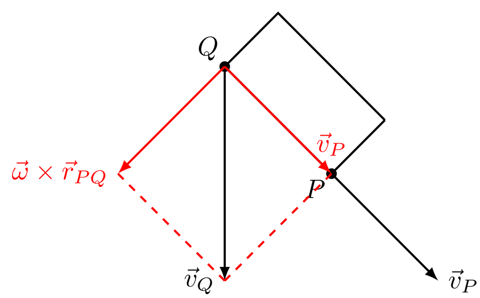


What is the direction of the angular velocity of the body?

(A) ↻ (clockwise)

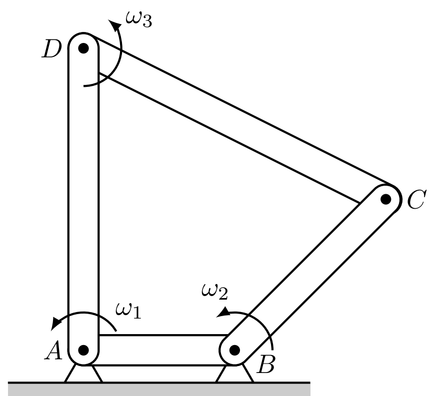
(B) ★ ↺ (counterclockwise)

Solution. Considering the required term $\vec{\omega} \times \vec{r}_{PQ}$ shows that it is down-left, so considering rotation about P we see that ω is counterclockwise.



---

11. (5 points) A four-bar linkage has rigid rods connecting pins at A, B, C, and D, as shown. The angular velocities are $\vec{\omega}_1 = 2\hat{k}$ for rod AD, $\vec{\omega}_2 = \omega_2\hat{k}$ for rod BC, and $\vec{\omega}_3 = \omega_3\hat{k}$ for rod DC.



At the current instant the positions are:

$$\begin{aligned}
\vec{r}_{AB} &= \hat{\imath} \text{ m} \\
\vec{r}_{BC} &= \hat{\imath} + \hat{\jmath} \text{ m} \\
\vec{r}_{AD} &= 2\hat{\jmath} \text{ m} \\
\vec{r}_{DC} &= 2\hat{\imath} - \hat{\jmath} \text{ m.}
\end{aligned}$$

What is ω_2?

(A) $1 \text{ rad/s} \le \omega_2 < 1.5 \text{ rad/s}$

(B) $0 \text{ rad/s} \le \omega_2 < 0.5 \text{ rad/s}$

(C) $1.5 \text{ rad/s} \le \omega_2 < 2 \text{ rad/s}$

(D) ★ $2 \text{ rad/s} \le \omega_2$

(E) $0.5 \text{ rad/s} \le \omega_2 < 1 \text{ rad/s}$

Solution. Starting from the fixed point A we have:

$$\begin{aligned}
\vec{v}_D &= \vec{v}_A + \vec{\omega}_1 \times \vec{r}_{AD} \\
&= 0 + 2\hat{k} \times 2\hat{\jmath} \\
&= -4\hat{\imath} \\
\vec{v}_C &= \vec{v}_D + \vec{\omega}_3 \times \vec{r}_{DC} \\
&= -4\hat{\imath} + \omega_3\hat{k} \times (2\hat{\imath} - \hat{\jmath}) \\
&= (-4 + \omega_3)\,\hat{\imath} + 2\omega_3\,\hat{\jmath}.
\end{aligned}$$

We can also get to C from the fixed point B:

$$\begin{aligned}
\vec{v}_C &= \vec{v}_B + \vec{\omega}_2 \times \vec{r}_{BC} \\
&= 0 + \omega_2\hat{k} \times (\hat{\imath} + \hat{\jmath}) \\
&= -\omega_2\,\hat{\imath} + \omega_2\,\hat{\jmath}.
\end{aligned}$$

Equating the two expressions for $\vec{v}_C$ gives:

$$\begin{aligned} -4 + \omega_3 &= -\omega_2 \\ 2\omega_3 &= \omega_2. \end{aligned}$$

Solving these equations gives $\omega_3 = \frac{4}{3} \approx 1.33$ rad/s and $\omega_2 = \frac{8}{3} \approx 2.67$ rad/s.

---

12. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = \hat{k}$ rad/s and angular acceleration $\vec{\alpha} = -\hat{k}$ rad/s^2. Points P and Q on the body have:

$$\vec{r}_{PQ} = \hat{\imath} - \hat{\jmath} \text{ m}$$
$$\vec{a}_P = 3\hat{\imath} \text{ m/s}^2.$$

What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) $0 \text{ m/s}^2 < a_{Qy} < 3 \text{ m/s}^2$

(B) $-3 \text{ m/s}^2 \leq a_{Qy} < 0 \text{ m/s}^2$

(C) ★ $a_{Qy} = 0 \text{ m/s}^2$

(D) $3 \text{ m/s}^2 \leq a_{Qy}$

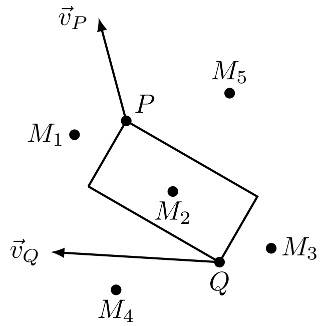
(E) $a_{Qy} < -3 \text{ m/s}^2$

Solution.

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\
&= 3\hat{\imath} - \hat{k} \times (\hat{\imath} - \hat{\jmath}) + \hat{k} \times (\hat{k} \times (\hat{\imath} - \hat{\jmath})) \\
&= (3\hat{\imath}) + (-\hat{\imath} - \hat{\jmath}) + (-\hat{\imath} + \hat{\jmath}) \\
&= \hat{\imath} \\
a_{Qy} &= 0 \text{ m/s}^2.
\end{aligned}$$

---

13. (5 points) A rigid body is moving in 2D as shown below.



Which point M_i is the instantaneous center?

(A) M_2

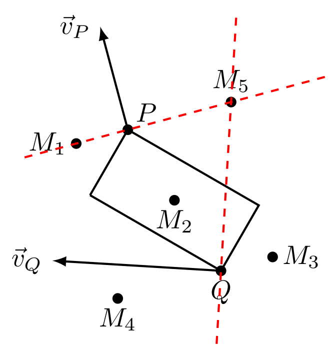
(B) M_1

(C) ★ M_5

(D) M_4

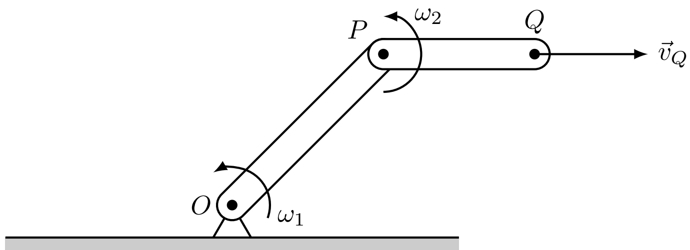
(E) M_3

Solution. The lines through P and Q perpendicular to $\vec{v}_P$ and $\vec{v}_Q$ intersect at M_5, so that is the instantaneous center.



---

14. (5 points) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = \omega_1 \hat{k}$ and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The positions and velocities at the current instant are:

$$\begin{aligned} \vec{r}_{OP} &= 2\hat{\imath} + 2\hat{\jmath} \text{ m} \\ \vec{r}_{PQ} &= 2\hat{\imath} \text{ m} \\ \vec{v}_Q &= 2\hat{\imath} \text{ m/s.} \end{aligned}$$

What is ω_2?

(A) 0 rad/s $< \omega_2 <$ 1 rad/s

(B) $\omega_2 < -1$ rad/s

(C) -1 rad/s $\leq \omega_2 <$ 0 rad/s

(D) $\omega_2 = 0$ rad/s

(E) ★ 1 rad/s $\leq \omega_2$

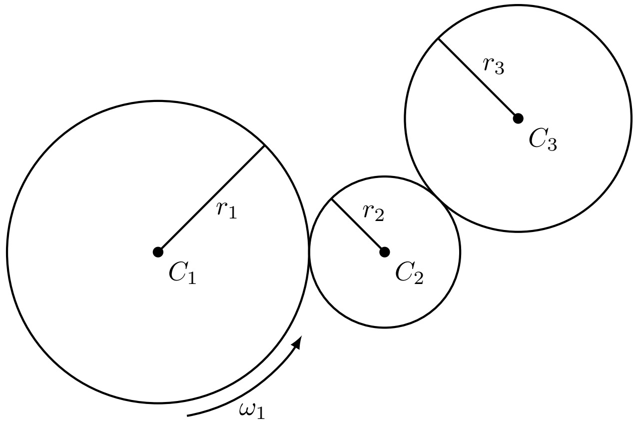
Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned} \vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\ &= -2\omega_1 \,\hat{\imath} + 2\omega_1 \,\hat{\jmath} \\ \vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\ 2\hat{\imath} &= -2\omega_1 \,\hat{\imath} + (2\omega_1 + 2\omega_2)\,\hat{\jmath}. \end{aligned}$$

Comparing $\hat{\imath}$ components gives $\omega_1 = -1$ rad/s and $\omega_2 = 1$ rad/s.

---

15. (5 points) Three meshed gears rotate about fixed centers as shown. The radii are $r_1 = 4$ m, $r_2 = 2$ m, and $r_3 = 3$ m and the corresponding angular velocities are $\vec{\omega}_1 = 2\hat{k}$ rad/s, $\vec{\omega}_2 = \omega_2\hat{k}$, and $\vec{\omega}_3 = \omega_3\hat{k}$.



What is ω_3?

(A) ★ 2 rad/s $\leq \omega_3 <$ 3 rad/s

(B) 3 rad/s $\leq \omega_3 <$ 4 rad/s

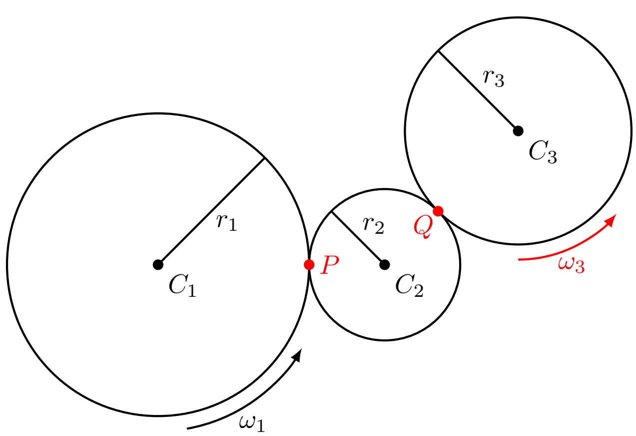
(C) 4 rad/s $\leq \omega_3$

(D) 0 rad/s $\leq \omega_3 <$ 1 rad/s

(E) 1 rad/s $\leq \omega_3 <$ 2 rad/s

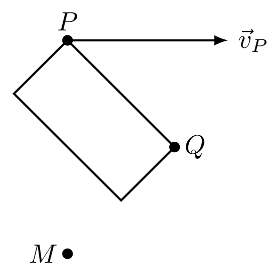
Solution. Matching the velocities at points P and Q shows that the gear at C_2 rotates clockwise and the gear at C_3 rotates counterclockwise. Also:

$$\begin{aligned} r_1\omega_1 &= v_P = v_Q = r_3\omega_3 \\ \omega_3 &= \frac{r_1}{r_3}\omega_1 \\ &= \frac{4}{3}2 \\ &\approx 2.67 \text{ rad/s}. \end{aligned}$$



---

16. (5 points) A rigid body is moving in 2D as shown below with points P and Q attached to the body. The instantaneous center of the body is at point M.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

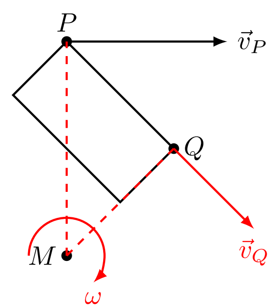
(A) ★ $\searrow$

(B) $\swarrow$

(C) $\nearrow$

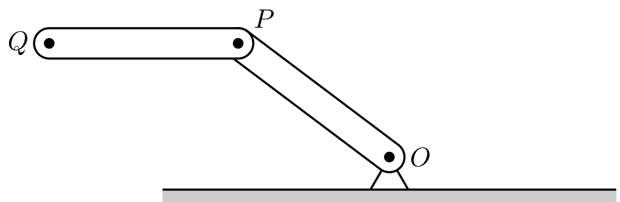
(D) $\nwarrow$

Solution. The direction of $\vec{v}_P$ shows the direction of ω. Then $\vec{v}_Q$ is orthogonal to $\vec{r}_{MQ}$ in the ω direction.



---

17. (5 points) Two rods are connected with pin joints at O, P, and Q as shown. The angular velocity and acceleration for rod OP are $\vec{\omega}_1$ and $\vec{\alpha}_1$, while the angular velocity and acceleration for rod PQ are $\vec{\omega}_2$ and $\vec{\alpha}_2$.



The positions and angular velocities of the rods at the current instant are:

$$\vec{r}_{OP} = -4\hat{\imath} + 3\hat{\jmath} \text{ m} \qquad \vec{r}_{PQ} = -5\hat{\imath} \text{ m}$$
$$\vec{\omega}_1 = 0 \qquad \vec{\omega}_2 = -\hat{k} \text{ rad/s}$$
$$\vec{\alpha}_1 = -2\hat{k} \text{ rad/s}^2 \qquad \vec{\alpha}_2 = 2\hat{k} \text{ rad/s}^2.$$

What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) $a_{Qy} = 0 \text{ m/s}^2$

(B) $a_{Qy} < -2 \text{ m/s}^2$

(C) ★ $-2 \text{ m/s}^2 \leq a_{Qy} < 0 \text{ m/s}^2$

(D) $0 \text{ m/s}^2 < a_{Qy} < 2 \text{ m/s}^2$

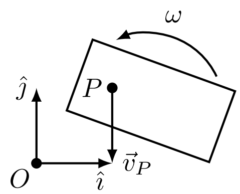
(E) $2 \text{ m/s}^2 \leq a_{Qy}$

Solution. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{a}_P &= \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OP} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OP}) \\
&= 0 - 2\hat{k} \times (-4\hat{\imath} + 3\hat{\jmath}) + 0 \\
&= 6\hat{\imath} + 8\hat{\jmath} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_w \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\
&= (6\hat{\imath} + 8\hat{\jmath}) + 2\hat{k} \times (-5\hat{\imath}) - \hat{k} \times (-\hat{k} \times (-5\hat{\imath})) \\
&= (6\hat{\imath} + 8\hat{\jmath}) - 10\hat{\jmath} + 5\hat{\imath} \\
&= 11\hat{\imath} - 2\hat{\jmath} \\
a_{Qy} &= -2 \text{ m/s}^2.
\end{aligned}$$

---

18. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \hat{k}$ rad/s.



Relative to the origin O, the point P has:

$$\vec{r}_P = \hat{\imath} + \hat{\jmath} \text{ m}$$
$$\vec{v}_P = -\hat{\jmath} \text{ m/s.}$$

What is the x coordinate M_x of the instantaneous center M of the body?

(A) ★ $1 \text{ m} \leq M_x$

(B) $M_x = 0$ m

(C) $0 \text{ m} < M_x < 1 \text{ m}$

(D) $-1 \text{ m} \leq M_x < 0 \text{ m}$

(E) $M_x < -1$ m

Solution. The position of M relative to P is:

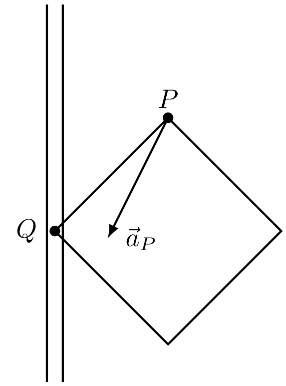
$$\begin{aligned}\vec{r}_{PM} &= \frac{1}{\omega^2}\,\omega\hat{k} \times \vec{v}_P \\ &= \frac{1}{1^2}\,\hat{k} \times (-\hat{\jmath}) \\ &= \hat{\imath} \text{ m.}\end{aligned}$$

Then the position of M is:

$$\begin{aligned}\vec{r}_M &= \vec{r}_P + \vec{r}_{PM} \\ &= (\hat{\imath} + \hat{\jmath}) + \hat{\imath} \\ &= 2\hat{\imath} + \hat{\jmath} \text{ m} \\ M_x &= 2 \text{ m.}\end{aligned}$$

---

19. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$ and zero angular acceleration. A pin at point Q constrains that point to move in a vertical slot.



Point P on the body has:

$$\begin{aligned} \vec{r}_{PQ} &= -\hat{\imath} - \hat{\jmath} \text{ m} \\ \vec{a}_P &= -\hat{\imath} - 2\hat{\jmath} \text{ m/s}^2. \end{aligned}$$

What is the magnitude a_Q of the acceleration $\vec{a}_Q$ of point Q?

(A) $3 \text{ m/s}^2 \leq a_Q < 4 \text{ m/s}^2$

(B) $0 \text{ m/s}^2 \leq a_Q < 1 \text{ m/s}^2$

(C) ★ $1 \text{ m/s}^2 \leq a_Q < 2 \text{ m/s}^2$

(D) $2 \text{ m/s}^2 \leq a_Q < 3 \text{ m/s}^2$

(E) $4 \text{ m/s}^2 \leq a_Q$

Solution. Taking $\vec{a}_Q = a_Q\,\hat{\jmath}$, we have:

$$\begin{aligned} \vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ a_Q\,\hat{\jmath} &= -\hat{\imath} - 2\hat{\jmath} + \omega\hat{k} \times (\omega\hat{k} \times (-\hat{\imath} - \hat{\jmath})) \\ \hat{\imath} + (a_Q + 2)\,\hat{\jmath} &= \omega^2\hat{\imath} + \omega^2\hat{\jmath}. \end{aligned}$$

Equating components and solving gives:

$$\begin{aligned} \omega^2 &= 1 \\ a_Q &= \omega^2 - 2 \\ &= -1 \text{ m/s}^2. \end{aligned}$$

The magnitude is thus 1 m/s^2.

---

20. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -\hat{k}$ rad/s. Points P and Q on the body have:

$$\begin{aligned} \vec{r}_{PQ} &= 3\hat{\imath} - 2\hat{\jmath} \text{ m} \\ \vec{v}_P &= \hat{\imath} + 2\hat{\jmath} \text{ m/s.} \end{aligned}$$

What is the $\hat{\imath}$ component v_{Qx} of the velocity $\vec{v}_Q$ of point Q?

(A) $2 \text{ m/s} \leq v_{Qx}$

(B) $v_{Qx} < -2 \text{ m/s}$

(C) ★ $-2 \text{ m/s} \leq v_{Qx} < 0 \text{ m/s}$

(D) $0 \text{ m/s} < v_{Qx} < 2 \text{ m/s}$

(E) $v_{Qx} = 0 \text{ m/s}$

Solution.

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ &= \hat{\imath} + 2\hat{\jmath} - \hat{k} \times (3\hat{\imath} - 2\hat{\jmath}) \\ &= \hat{\imath} + 2\hat{\jmath} - 2\hat{\imath} - 3\hat{\jmath} \\ &= -\hat{\imath} - \hat{\jmath} \\ v_{Qx} &= -1 \text{ m/s} \end{aligned}$$

TAM 212. Midterm 2. Apr 4, 2013.

- There are 20 questions, each worth 5 points.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.
- The notation $\vec{r}_{PQ}$ denotes the position vector from P to Q.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Circle your discussion section:

	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

94. C

95. A

96. E

1. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = \hat{k}$ rad/s. Two points P and Q are fixed to the body and the offset between them is in the direction $\hat{r}_{PQ} = \frac{1}{5}(-3\hat{\imath} - 4\hat{\jmath})$ (note that this is the unit vector in the direction of the offset vector $\vec{r}_{PQ}$, not the actual offset vector $\vec{r}_{PQ}$). The velocities are:

$$\begin{aligned}\vec{v}_P &= -3\hat{\imath} + 3\hat{\jmath} \text{ m/s}\\ \vec{v}_Q &= 5\hat{\imath} - 3\hat{\jmath} \text{ m/s.}\end{aligned}$$

What is the distance r_{PQ} between P and Q?

(A) $6 \text{ m} \le r_{PQ} < 8 \text{ m}$

(B) $2 \text{ m} \le r_{PQ} < 4 \text{ m}$

(C) $4 \text{ m} \le r_{PQ} < 6 \text{ m}$

(D) ★ $8 \text{ m} \le r_{PQ}$

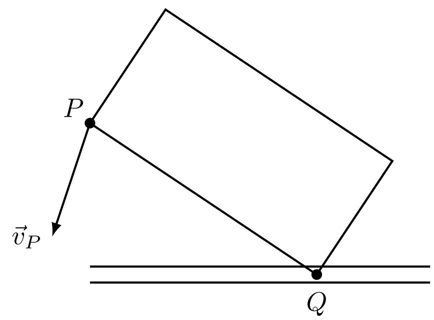
(E) $0 \text{ m} \le r_{PQ} < 2 \text{ m}$

Solution.

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ}\\ 5\hat{\imath} - 3\hat{\jmath} &= -3\hat{\imath} + 3\hat{\jmath} + \hat{k} \times r_{PQ}\frac{1}{5}(-3\hat{\imath} - 4\hat{\jmath})\\ 8\hat{\imath} - 6\hat{\jmath} &= \frac{4}{5}r_{PQ}\,\hat{\imath} - \frac{3}{5}r_{PQ}\,\hat{\jmath}\\ \implies r_{PQ} &= 10 \text{ m.}\end{aligned}$$

---

2. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$. A pin at point Q constrains that point to move in a horizontal slot.



Point P on the body has:

$$\vec{r}_{PQ} = 3\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{v}_P = -2\hat{\imath} - 6\hat{\jmath} \text{ m/s}.$$

What is ω?

(A) $0 \text{ rad/s} < \omega < 2 \text{ rad/s}$

(B) ★ $2 \text{ rad/s} \le \omega$

(C) $\omega < -2 \text{ rad/s}$

(D) $-2 \text{ rad/s} \le \omega < 0 \text{ rad/s}$

(E) $\omega = 0 \text{ rad/s}$

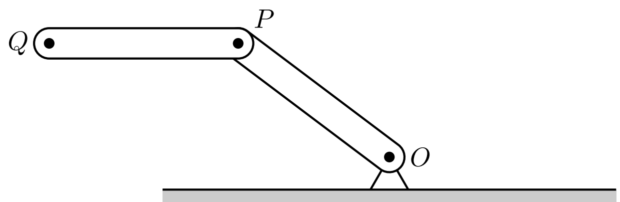
Solution. Taking $\vec{v}_Q = v_Q\,\hat{\imath}$ with unknown speed v_Q, we have:

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ v_Q\,\hat{\imath} &= -2\hat{\imath} - 6\hat{\jmath} + \omega\hat{k} \times (3\hat{\imath} - 2\hat{\jmath}) \\ (v_Q + 2)\,\hat{\imath} + 6\hat{\jmath} &= 2\omega\,\hat{\imath} + 3\omega\,\hat{\jmath} \end{aligned}$$

Equating $\hat{\jmath}$ components gives $\omega = 2$ rad/s.

---

3. (5 points) Two rods are connected with pin joints at O, P, and Q as shown. The angular velocity and acceleration for rod OP are $\vec{\omega}_1$ and $\vec{\alpha}_1$, while the angular velocity and acceleration for rod PQ are $\vec{\omega}_2$ and $\vec{\alpha}_2$.



The positions and angular velocities of the rods at the current instant are:

$$\vec{r}_{OP} = -4\hat{\imath} + 3\hat{\jmath} \text{ m} \qquad \vec{r}_{PQ} = -5\hat{\imath} \text{ m}$$
$$\vec{\omega}_1 = 0 \qquad \vec{\omega}_2 = \hat{k} \text{ rad/s}$$
$$\vec{\alpha}_1 = 2\hat{k} \text{ rad/s}^2 \qquad \vec{\alpha}_2 = -\hat{k} \text{ rad/s}^2.$$

What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) $0 \text{ m/s}^2 < a_{Qy} < 2 \text{ m/s}^2$

(B) $2 \text{ m/s}^2 \leq a_{Qy}$

(C) ★ $a_{Qy} < -2 \text{ m/s}^2$

(D) $-2 \text{ m/s}^2 \leq a_{Qy} < 0 \text{ m/s}^2$

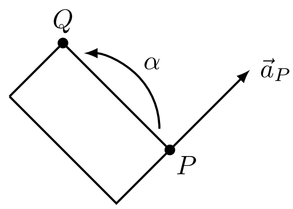
(E) $a_{Qy} = 0 \text{ m/s}^2$

Solution. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{a}_P &= \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OP} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OP}) \\
&= 0 + 2\hat{k} \times (-4\hat{\imath} + 3\hat{\jmath}) + 0 \\
&= -6\hat{\imath} - 8\hat{\jmath} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_w \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\
&= (-6\hat{\imath} - 8\hat{\jmath}) - \hat{k} \times (-5\hat{\imath}) + \hat{k} \times (\hat{k} \times (-5\hat{\imath})) \\
&= (-6\hat{\imath} - 8\hat{\jmath}) + 5\hat{\jmath} + 5\hat{\imath} \\
&= -\hat{\imath} - 3\hat{\jmath} \\
a_{Qy} &= -3 \text{ m/s}^2.
\end{aligned}$$

---

4. (5 points) A rigid body is moving in 2D as shown below, with a counterclockwise angular acceleration and points P and Q on the body (α is positive in the direction shown). We know that $a_P = \omega^2\, r_{PQ} = \alpha\, r_{PQ}$.



What is the direction of the acceleration $\vec{a}_Q$?

(A) $\swarrow$

(B) $\nwarrow$

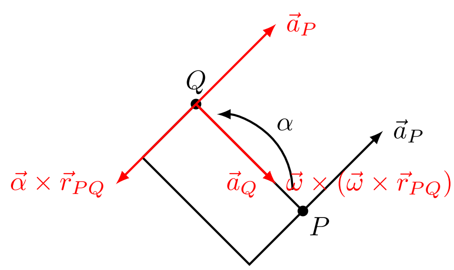
(C) ★ $\searrow$

(D) $\nearrow$

Solution. Consider the acceleration equation for point Q:

$$\vec{a}_Q = \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}).$$

All three terms on the right hand side have the same magnitude ($a_P = \omega^2\, r_{PQ} = \alpha\, r_{PQ}$). Drawing the right-hand-side terms shows that the resulting direction for $\vec{a}_Q$ is down-right:



---

5. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -\hat{k}$ rad/s and angular acceleration $\vec{\alpha} = -2\hat{k}$ rad/s^2. Points P and Q on the body have:

$$\begin{aligned}\vec{r}_{PQ} &= 2\hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{a}_P &= \hat{\imath} + 3\hat{\jmath} \text{ m/s}^2.\end{aligned}$$

What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) $3 \text{ m/s}^2 \le a_{Qy}$

(B) $a_{Qy} = 0 \text{ m/s}^2$

(C) $0 \text{ m/s}^2 < a_{Qy} < 3 \text{ m/s}^2$

(D) ★ $-3 \text{ m/s}^2 \le a_{Qy} < 0 \text{ m/s}^2$

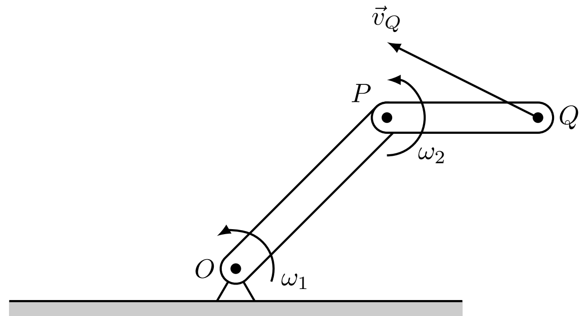
(E) $a_{Qy} < -3 \text{ m/s}^2$

Solution.

$$\begin{aligned}\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ &= \hat{\imath} + 3\hat{\jmath} - 2\hat{k} \times (2\hat{\imath} + \hat{\jmath}) - \hat{k} \times (-\hat{k} \times (2\hat{\imath} + \hat{\jmath})) \\ &= (\hat{\imath} + 3\hat{\jmath}) + (2\hat{\imath} - 4\hat{\jmath}) + (-2\hat{\imath} - \hat{\jmath}) \\ &= \hat{\imath} - 2\hat{\jmath} \\ a_{Qy} &= -2 \text{ m/s}^2.\end{aligned}$$

---

6. (5 points) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = \omega_1 \hat{k}$ and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The positions and velocities at the current instant are:

$$\begin{aligned}\vec{r}_{OP} &= 2\hat{\imath} + 2\hat{\jmath} \text{ m} \\ \vec{r}_{PQ} &= 2\hat{\imath} \text{ m} \\ \vec{v}_Q &= -4\hat{\imath} + 2\hat{\jmath} \text{ m/s.}\end{aligned}$$

What is ω_2?

(A) $0 \text{ rad/s} < \omega_2 < 1 \text{ rad/s}$

(B) $\omega_2 < -1 \text{ rad/s}$

(C) ★ $-1 \text{ rad/s} \leq \omega_2 < 0 \text{ rad/s}$

(D) $\omega_2 = 0 \text{ rad/s}$

(E) $1 \text{ rad/s} \leq \omega_2$

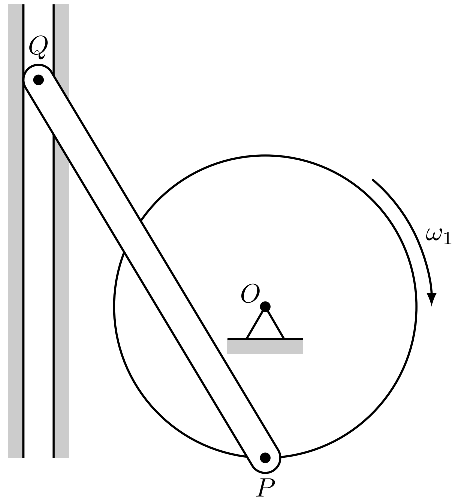
Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned}\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\ &= -2\omega_1 \,\hat{\imath} + 2\omega_1 \,\hat{\jmath} \\ \vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\ -4\hat{\imath} + 2\hat{\jmath} &= -2\omega_1 \,\hat{\imath} + (2\omega_1 + 2\omega_2)\,\hat{\jmath}.\end{aligned}$$

Comparing $\hat{\imath}$ components gives $\omega_1 = 2$ rad/s and $\omega_2 = -1$ rad/s.

---

7. (5 points) A circular rigid body rotates about the fixed center O with angular velocity $\vec{\omega}_1 = -10\hat{k}$ rad/s as shown. A rigid rod connects pins P and Q, and point Q is constrained to only move vertically.



At the current instant the positions are:

$$\begin{aligned}\vec{r}_{OP} &= -2\hat{\jmath} \text{ m}\\ \vec{r}_{PQ} &= -3\hat{\imath} + 5\hat{\jmath} \text{ m.}\end{aligned}$$

What is the speed v_Q of point Q?

(A) ★ $8 \text{ m/s} \le v_Q$

(B) $0 \text{ m/s} \le v_Q < 2 \text{ m/s}$

(C) $4 \text{ m/s} \le v_Q < 6 \text{ m/s}$

(D) $2 \text{ m/s} \le v_Q < 4 \text{ m/s}$

(E) $6 \text{ m/s} \le v_Q < 8 \text{ m/s}$

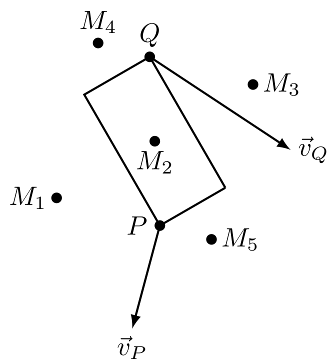
Solution. Let the unknown angular velocity of the rod be $\vec{\omega}_2 = \omega_2\hat{k}$ and the unknown velocity of Q be $\vec{v}_Q = v_Q\,\hat{\jmath}$. Starting from the fixed point O, we have:

$$\begin{aligned}\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP}\\ &= 0 - 10\hat{k} \times (-2\hat{\jmath})\\ &= -20\hat{\imath}\\ \vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ}\\ &= -20\hat{\imath} + \omega_2\hat{k} \times (-3\hat{\imath} + 5\hat{\jmath})\\ v_Q\,\hat{\jmath} &= (-20 - 5\omega_2)\,\hat{\imath} - 3\omega_2\,\hat{\jmath}.\end{aligned}$$

Solving this gives $\omega_2 = -4$ rad/s and $v_Q = -3\omega_2 = 12$ m/s.

---

8. (5 points) A rigid body is moving in 2D as shown below.



Which point M_i is the instantaneous center?

(A) M_2

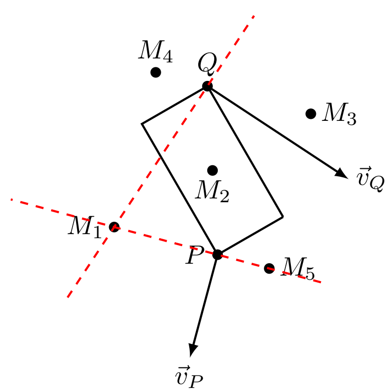
(B) M_4

(C) ★ M_1

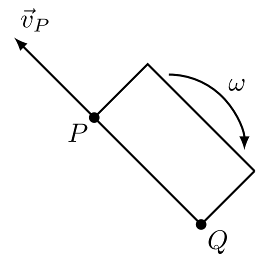
(D) M_5

(E) M_3

Solution. The lines through P and Q perpendicular to $\vec{v}_P$ and $\vec{v}_Q$ intersect at M_1, so that is the instantaneous center.



9. (5 points) A rigid body is moving in 2D as shown, with a clockwise rotation (ω is positive in the direction indicated). The angular velocity ω, distance r_{PQ}, and speed v_P satisfy $\omega r_{PQ} = v_P$.



What is the direction of $\vec{v}_Q$?

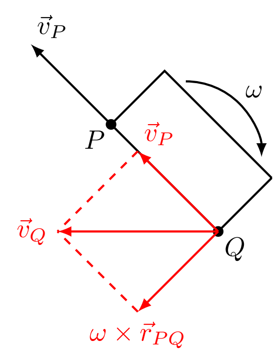
(A) $\downarrow$

(B) $\uparrow$

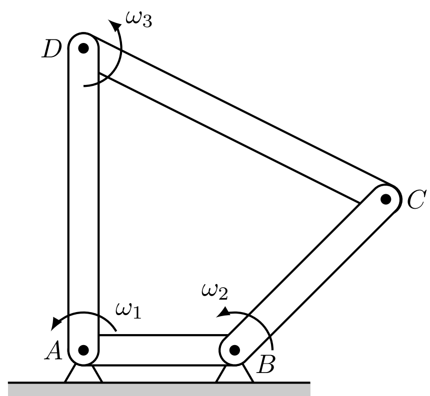
(C) ★ $\leftarrow$

(D) $\rightarrow$

Solution.



10. (5 points) A four-bar linkage has rigid rods connecting pins at A, B, C, and D, as shown. The angular velocities are $\vec{\omega}_1 = 3\hat{k}$ for rod AD, $\vec{\omega}_2 = \omega_2\hat{k}$ for rod BC, and $\vec{\omega}_3 = \omega_3\hat{k}$ for rod DC.



At the current instant the positions are:

$$\begin{aligned}\vec{r}_{AB} &= \hat{\imath} \text{ m}\\ \vec{r}_{BC} &= \hat{\imath} + \hat{\jmath} \text{ m}\\ \vec{r}_{AD} &= 2\hat{\jmath} \text{ m}\\ \vec{r}_{DC} &= 2\hat{\imath} - \hat{\jmath} \text{ m.}\end{aligned}$$

What is ω_2?

(A) 6 rad/s $\leq \omega_2 <$ 9 rad/s

(B) 0 rad/s $\leq \omega_2 <$ 3 rad/s

(C) 12 rad/s $\leq \omega_2$

(D) ★ 3 rad/s $\leq \omega_2 <$ 6 rad/s

(E) 9 rad/s $\leq \omega_2 <$ 12 rad/s

Solution. Starting from the fixed point A we have:

$$\begin{aligned}\vec{v}_D &= \vec{v}_A + \vec{\omega}_1 \times \vec{r}_{AD}\\ &= 0 + 3\hat{k} \times 2\hat{\jmath}\\ &= -6\hat{\imath}\\ \vec{v}_C &= \vec{v}_D + \vec{\omega}_3 \times \vec{r}_{DC}\\ &= -6\hat{\imath} + \omega_3\hat{k} \times (2\hat{\imath} - \hat{\jmath})\\ &= (-6 + \omega_3)\,\hat{\imath} + 2\omega_3\,\hat{\jmath}.\end{aligned}$$

We can also get to C from the fixed point B:

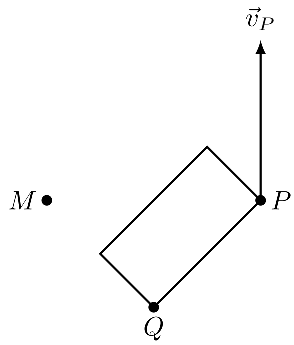
$$\begin{aligned}\vec{v}_C &= \vec{v}_B + \vec{\omega}_2 \times \vec{r}_{BC}\\ &= 0 + \omega_2\hat{k} \times (\hat{\imath} + \hat{\jmath})\\ &= -\omega_2\,\hat{\imath} + \omega_2\,\hat{\jmath}.\end{aligned}$$

Equating the two expressions for $\vec{v}_C$ gives:

$$\begin{aligned} -6 + \omega_3 &= -\omega_2 \\ 2\omega_3 &= \omega_2. \end{aligned}$$

Solving these equations gives $\omega_3 = 2$ rad/s and $\omega_2 = 4$ rad/s.

11. (5 points) A rigid body is moving in 2D as shown below with points P and Q attached to the body. The instantaneous center of the body is at point M.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

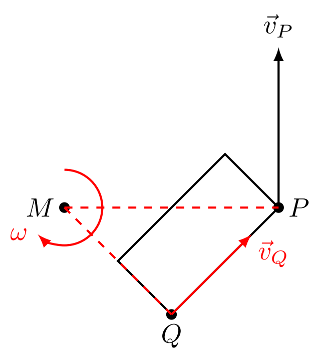
(A) $\nwarrow$

(B) $\swarrow$

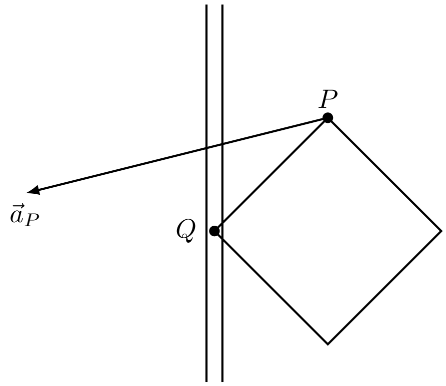
(C) $\searrow$

(D) ★ $\nearrow$

Solution. The direction of $\vec{v}_P$ shows the direction of ω. Then $\vec{v}_Q$ is orthogonal to $\vec{r}_{MQ}$ in the ω direction.



12. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$ and zero angular acceleration. A pin at point Q constrains that point to move in a vertical slot.



Point P on the body has:

$$\begin{aligned}\vec{r}_{PQ} &= -\hat{\imath} - \hat{\jmath} \text{ m} \\ \vec{a}_P &= -4\hat{\imath} - \hat{\jmath} \text{ m/s}^2.\end{aligned}$$

What is the magnitude a_Q of the acceleration $\vec{a}_Q$ of point Q?

(A) $4 \text{ m/s}^2 \le a_Q$

(B) $1 \text{ m/s}^2 \le a_Q < 2 \text{ m/s}^2$

(C) $0 \text{ m/s}^2 \le a_Q < 1 \text{ m/s}^2$

(D) $2 \text{ m/s}^2 \le a_Q < 3 \text{ m/s}^2$

(E) ★ $3 \text{ m/s}^2 \le a_Q < 4 \text{ m/s}^2$

Solution. Taking $\vec{a}_Q = a_Q\,\hat{\jmath}$, we have:

$$\begin{aligned}\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ a_Q\,\hat{\jmath} &= -4\hat{\imath} - \hat{\jmath} + \omega\hat{k} \times (\omega\hat{k} \times (-\hat{\imath} - \hat{\jmath})) \\ 4\hat{\imath} + (a_Q + 1)\,\hat{\jmath} &= \omega^2\hat{\imath} + \omega^2\hat{\jmath}.\end{aligned}$$

Equating components and solving gives:

$$\begin{aligned}\omega^2 &= 4 \\ a_Q &= \omega^2 - 1 \\ &= 3 \text{ m/s}^2.\end{aligned}$$

---

13. (5 points) A rigid body is moving in 2D with angular velocity $\vec{\omega} = 2\hat{k}$ rad/s. Points P and Q on the body have:

$$\begin{aligned} \vec{r}_{PQ} &= \hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{v}_P &= 2\hat{\imath} - \hat{\jmath} \text{ m/s.} \end{aligned}$$

What is the $\hat{\imath}$ component v_{Qx} of the velocity $\vec{v}_Q$ of point Q?

(A) $-2 \text{ m/s} \le v_{Qx} < 0 \text{ m/s}$

(B) ★ $v_{Qx} = 0 \text{ m/s}$

(C) $2 \text{ m/s} \le v_{Qx}$

(D) $0 \text{ m/s} < v_{Qx} < 2 \text{ m/s}$

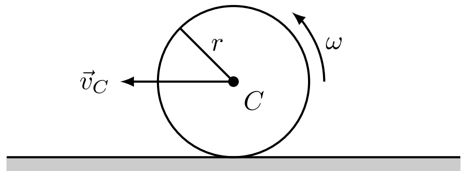
(E) $v_{Qx} < -2 \text{ m/s}$

Solution.

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ &= 2\hat{\imath} - \hat{\jmath} + 2\hat{k} \times (\hat{\imath} + \hat{\jmath}) \\ &= 2\hat{\imath} - \hat{\jmath} - 2\hat{\imath} + 2\hat{\jmath} \\ &= \hat{\jmath} \\ v_{Qx} &= 0 \text{ m/s} \end{aligned}$$

---

14. (5 points) A circular rigid body with radius $r = 6$ m is rolling without slipping on a flat surface in 2D as shown. The speed of the center is $v_C = 4$ m/s.



What is the angular velocity ω?

(A) 2 rad/s $\leq \omega <$ 3 rad/s

(B) 1 rad/s $\leq \omega <$ 2 rad/s

(C) 3 rad/s $\leq \omega <$ 4 rad/s

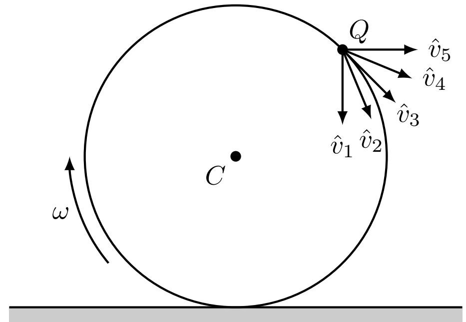
(D) ★ 0 rad/s $\leq \omega <$ 1 rad/s

(E) 4 rad/s $\leq \omega$

Solution. $v_c = r\omega$ so $\omega = v_c/r = 4/6 \approx 0.67$ rad/s.

---

15. (5 points) A circular rigid body is rolling without slipping on a flat surface in 2D in a clockwise direction as shown.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

(A) $\hat{v}_1$

(B) $\hat{v}_2$

(C) ★ $\hat{v}_4$

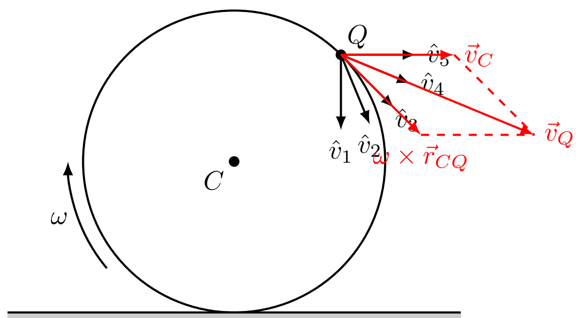
(D) $\hat{v}_3$

(E) $\hat{v}_5$

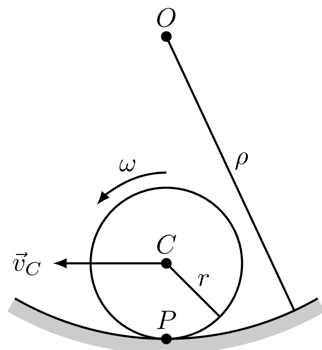
Solution. Starting from point C, the velocity at point Q is given by:

$$\vec{v}_Q = \vec{v}_C + \vec{\omega} \times \vec{r}_{CQ}.$$

The two terms on the right hand side above both have magnitude ωr, where r is the radius of the body. The $\vec{\omega} \times \vec{r}_{CQ}$ term is orthogonal to $\vec{r}_{CP}$, while the $\vec{v}_C$ term is horizontal. Adding the two terms gives a resultant $\vec{v}_Q$ exactly half way between the two component vectors.



16. (5 points) A circular rigid body with radius $r = 3$ m is rolling without slipping on a curved surface with radius of curvature ρ in 2D as shown. The angular velocity of the body is a constant $\vec{\omega} = 2\hat{k}$ rad/s. Point P is fixed to the edge of the body and, at the instant shown, is the contact point. The magnitude of acceleration of P is $a_P = 30$ m/s^2.



What is the radius of curvature ρ of the surface?

(A) 12 m $\leq \rho$

(B) ★ 3 m $\leq \rho <$ 6 m

(C) 6 m $\leq \rho <$ 9 m

(D) 0 m $\leq \rho <$ 3 m

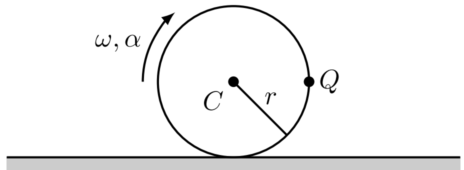
(E) 9 m $\leq \rho <$ 12 m

Solution. Because we are rolling on the inside, $R = \rho - r$, giving:

$$\begin{aligned} a_P &= \frac{\rho}{R} r\omega^2 \\ 30 &= \frac{\rho}{\rho - 3} 3 \times 2^2 \\ \rho &= 5 \text{ m.} \end{aligned}$$

---

17. (5 points) A circular rigid body with radius $r = 2$ m is rolling without slipping with angular velocity $\vec{\omega} = -\hat{k}$ rad/s on a flat surface in 2D as shown. The body is speeding up and has angular acceleration $\vec{\alpha} = -\alpha\hat{k}$. Point Q is at the right edge of the body and has acceleration $\vec{a}_Q = \hat{\imath} - 3\hat{\jmath}$ m/s^2.



What is α?

(A) $0.5 \text{ rad/s}^2 \leq \alpha < 1 \text{ rad/s}^2$

(B) $2 \text{ rad/s}^2 \leq \alpha$

(C) ★ $1.5 \text{ rad/s}^2 \leq \alpha < 2 \text{ rad/s}^2$

(D) $1 \text{ rad/s}^2 \leq \alpha < 1.5 \text{ rad/s}^2$

(E) $0 \text{ rad/s}^2 \leq \alpha < 0.5 \text{ rad/s}^2$

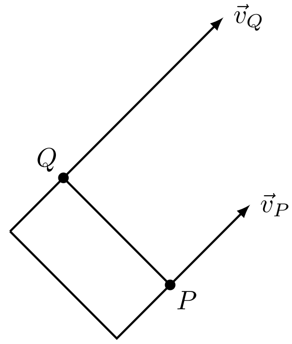
Solution. The acceleration of Q is:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= (-\alpha\hat{k}) \times r\hat{\jmath} + (-\alpha\hat{k}) \times r\hat{\imath} + (-\omega\hat{k}) \times (-\omega\hat{k} \times r\hat{\imath}) \\
&= r\alpha\,\hat{\imath} - r\alpha\,\hat{\jmath} - r\omega^2\,\hat{\imath} \\
&= r(\alpha - \omega^2)\,\hat{\imath} - r\alpha\,\hat{\jmath} \\
\hat{\imath} - 3\hat{\jmath} &= 2(\alpha - 1)\,\hat{\imath} - 2\alpha\,\hat{\jmath}
\end{aligned}$$

So $\alpha = 1.5$ rad/s^2.

---

18. (5 points) A rigid body is moving in 2D as shown below.

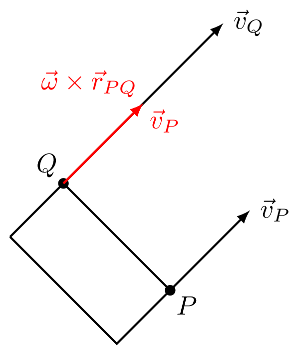


What is the direction of the angular velocity of the body?

(A) ↺ (counterclockwise)

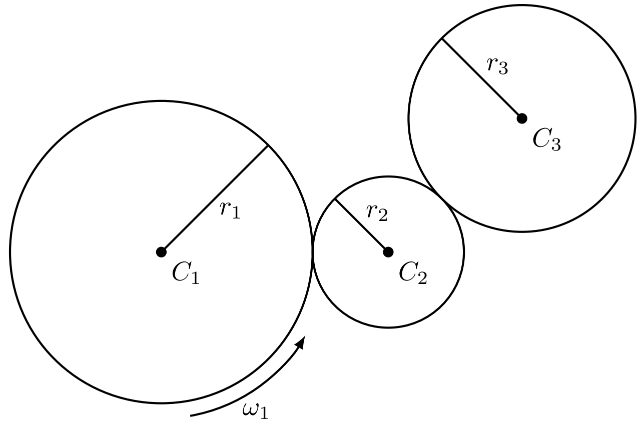
(B) ★ ↻ (clockwise)

Solution. Considering the required term $\vec{\omega} \times \vec{r}_{PQ}$ shows that it is up-right, so considering rotation about P we see that ω is clockwise.



---

19. (5 points) Three meshed gears rotate about fixed centers as shown. The radii are $r_1 = 4$ m, $r_2 = 2$ m, and $r_3 = 3$ m and the corresponding angular velocities are $\vec{\omega}_1 = \hat{k}$ rad/s, $\vec{\omega}_2 = \omega_2 \hat{k}$, and $\vec{\omega}_3 = \omega_3 \hat{k}$.



What is ω_3?

(A) 2 rad/s $\leq \omega_3 <$ 3 rad/s

(B) 4 rad/s $\leq \omega_3$

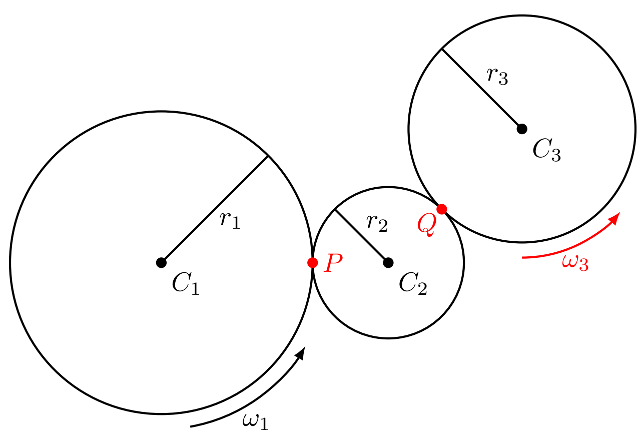
(C) 3 rad/s $\leq \omega_3 <$ 4 rad/s

(D) 0 rad/s $\leq \omega_3 <$ 1 rad/s

(E) ★ 1 rad/s $\leq \omega_3 <$ 2 rad/s

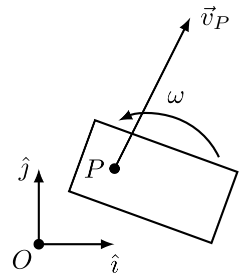
Solution. Matching the velocities at points P and Q shows that the gear at C_2 rotates clockwise and the gear at C_3 rotates counterclockwise. Also:

$$\begin{aligned} r_1 \omega_1 &= v_P = v_Q = r_3 \omega_3 \\ \omega_3 &= \frac{r_1}{r_3} \omega_1 \\ &= \frac{4}{3} 1 \\ &\approx 1.33 \text{ rad/s.} \end{aligned}$$



---

20. (5 points) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = 2\hat{k}$ rad/s.



Relative to the origin O, the point P has:

$$\vec{r}_P = \hat{\imath} + \hat{\jmath} \text{ m}$$
$$\vec{v}_P = 2\hat{\imath} + 4\hat{\jmath} \text{ m/s.}$$

What is the x coordinate M_x of the instantaneous center M of the body?

(A) $M_x = 0$ m

(B) $M_x < -1$ m

(C) $0 \text{ m} < M_x < 1$ m

(D) $1 \text{ m} \leq M_x$

(E) ★ $-1 \text{ m} \leq M_x < 0$ m

Solution. The position of M relative to P is:

$$\begin{aligned}
\vec{r}_{PM} &= \frac{1}{\omega^2}\,\omega\hat{k} \times \vec{v}_P \\
&= \frac{1}{2^2}\,2\hat{k} \times (2\hat{\imath} + 4\hat{\jmath}) \\
&= \frac{1}{4}(-8\hat{\imath} + 4\hat{\jmath}) \\
&= -2\hat{\imath} + \hat{\jmath} \text{ m.}
\end{aligned}$$

Then the position of M is:

$$\begin{aligned}
\vec{r}_M &= \vec{r}_P + \vec{r}_{PM} \\
&= (\hat{\imath} + \hat{\jmath}) + (-2\hat{\imath} + \hat{\jmath}) \\
&= -\hat{\imath} + 2\hat{\jmath} \text{ m} \\
M_x &= -1 \text{ m.}
\end{aligned}$$
