TAM 212. Midterm 1 Practice. Feb 11, 2013. (V3)

- There are 50 questions worth points as shown in each question.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Circle your discussion section:

	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

91. E

92. E

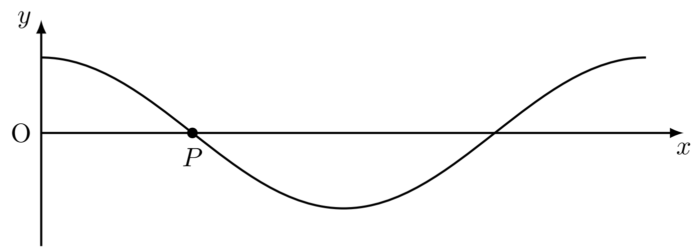
93. D

94. A

95. E

96. D

1. (1 point) A particle is moving to the right along a variable-height ground with ground height given by $y(x) = \cos(x/10)$ m. The particle's horizontal velocity component is a constant $v_x = 20$ m/s. What is the vertical component of velocity v_y when $x = 5\pi$ m?



(A) $v_y = 0$ m/s

(B) $v_y < -3$ m/s

(C) $3 \text{ m/s} \leq v_y$

(D) $0 \text{ m/s} \leq v_y < 3$ m/s

(E) ★ $-3 \text{ m/s} \leq v_y < 0$ m/s

Solution. Using $\dot{x} = v_x = 20$ m/s, we have:

$$\begin{aligned} y &= \cos(x/10) \\ v_y = \dot{y} &= -\sin(x/10)\frac{\dot{x}}{10} \\ &= -\sin(5\pi/10)\frac{20}{10} \\ &= -2 \text{ m/s}. \end{aligned}$$

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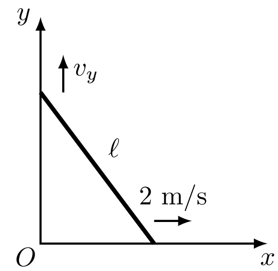
2. (1 point) A ladder leaning against the wall has a fixed length of $\ell = 5$ m. The bottom of the ladder is 3 m from the wall and is moving along the ground away from the wall at a speed of 2 m/s. What is the vertical component of the velocity v_y of the top of the ladder, assuming it remains in contact with the wall?



(A) 0 m/s $< v_y <$ 1 m/s

(B) 1 m/s $< v_y$

(C) ★ $v_y < -1$ m/s

(D) -1 m/s $\leq v_y < 0$ m/s

(E) $v_y = 0$ m/s

Solution. Taking x to be the horizontal coordinate of the bottom of the ladder, and y the vertical coordinate of the top of the ladder, we have $x = 3$ and $y = 4$ at the instant shown. Then:

$$\begin{aligned}
\ell^2 &= x^2 + y^2 \\
0 &= 2x\dot{x} + 2y\dot{y} \\
\dot{y} &= -\frac{x}{y}\dot{x} \\
v_y &= -\frac{3}{4}2 \\
&= -1.5 \text{ m/s.}
\end{aligned}$$

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3. (1 point) Given a polar basis $\hat{e}_r, \hat{e}_\theta$, the time derivative of the angular basis vector satisfies

$$\dot{\hat{e}}_\theta = -\hat{e}_r.$$

(A) True

(B) ★ False

Solution. False. Actually $\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$.

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4. (1 point) The particle P has polar coordinates $r = 3$ m, $\theta = 60°$ and velocity $\vec{v} = -\hat{\imath}$. Which statement is true?

(A) $\dot{r} \geq 0$ and $\dot{\theta} \geq 0$

(B) ★ $\dot{r} < 0$ and $\dot{\theta} \geq 0$

(C) $\dot{r} < 0$ and $\dot{\theta} < 0$

(D) $\dot{r} \geq 0$ and $\dot{\theta} < 0$

Solution. The fastest way to solve this question is to draw a diagram and directly see that this velocity is causing r to decrease and θ to increase.
Alternatively, we can convert $\vec{v}$ to polar coordinates to find:

$$\begin{aligned}\vec{v} &= -(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) \\ &= -\frac{1}{2}\,\hat{e}_r + \frac{\sqrt{3}}{2}\,\hat{e}_\theta.\end{aligned}$$

Thus we have a negative velocity component in the $\hat{e}_r$ direction (decreasing r), and a positive velocity component in the $\hat{e}_\theta$ direction (increasing θ).

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5. (1 point) A point currently has position vector $\vec{r} = -2\hat{\imath} + 4\hat{\jmath}$ m and is rotating about the origin in the x–y plane with angular velocity $\omega = -3$ rad/s. What is the $\hat{\jmath}$ component of the velocity v_y of the point?

(A) $-5 \text{ m/s} \leq v_y < 0 \text{ m/s}$

(B) $0 \text{ m/s} < v_y < 5 \text{ m/s}$

(C) ★ $5 \text{ m/s} \leq v_y$

(D) $v_y = 0 \text{ m/s}$

(E) $v_y < -5 \text{ m/s}$

Solution. The angular velocity vector is $\vec{\omega} = -3\hat{k}$ rad/s, so:

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r} \\ &= -3\hat{k} \times (-2\hat{\imath} + 4\hat{\jmath}) \\ &= 12\hat{\imath} + 6\hat{\jmath} \text{ m/s} \\ v_y &= 6 \text{ m/s}. \end{aligned}$$

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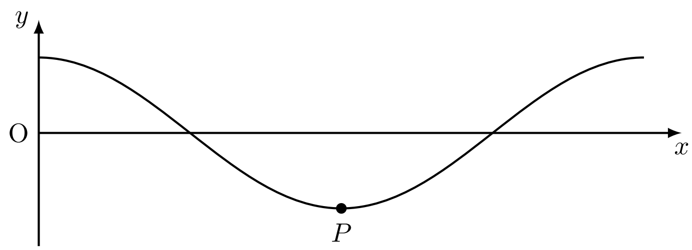
6. (1 point) A particle is moving to the right along a variable-height ground at a constant speed of $v = 20$ m/s. The ground height is given by $y(x) = \cos(x/10)$ m. When the particle is at the lowest point on the ground, what is the vertical acceleration a_y?



(A) ★ $3 \text{ m/s}^2 \leq a_y$

(B) $0 \text{ m/s}^2 < a_y < 3 \text{ m/s}^2$

(C) $a_y < -3 \text{ m/s}^2$

(D) $-2 \text{ m/s}^2 \leq a_y < 0 \text{ m/s}^2$

(E) $a_y = 0 \text{ m/s}^2$

Solution. Note that at this point $\cos(x/10) = -1$ m, $\hat{e}_t = \hat{\imath}$, and $\hat{e}_n = \hat{\jmath}$. The velocity is purely horizontal, so $\dot{x} = v = 20$ m/s. Constant speed means that $\ddot{s} = 0$ and the acceleration is purely vertical, so $\ddot{x} = 0$. Then:

$$\begin{aligned}
\dot{y} &= -\sin(x/10)\frac{\dot{x}}{10} \\
a_y = \ddot{y} &= -\cos(x/10)\frac{\dot{x}^2}{100} - \sin(x/10)\frac{\ddot{x}}{10} \\
&= \frac{20^2}{100} \\
&= 4 \text{ m/s}^2.
\end{aligned}$$

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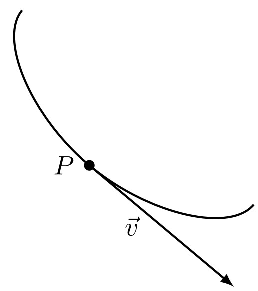
7. (1 point) A point P is moving around a curve and at a given instant has position and velocity $\vec{v}$ as shown.



Which direction is the closest to the direction of the normal basis vector $\hat{e}_n$ at the instant shown?

(A) ★ $\nearrow$

(B) $\searrow$

(C) $\nwarrow$

(D) $\swarrow$

Solution. $\hat{e}_n$ is inwards to the curve.

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8. (1 point) A car driving down the road is at a distance $s = t^2$ m from its starting point. At $t = 2$ s the car is driving around a curve with radius of curvature $\rho = 8$ m. What is the magnitude of the acceleration a at this time?

(A) $a < 0\ \mathrm{m/s^2}$

(B) $1\ \mathrm{m/s^2} \leq a < 2\ \mathrm{m/s^2}$

(C) $a = 0\ \mathrm{m/s^2}$

(D) $0\ \mathrm{m/s^2} \leq a < 1\ \mathrm{m/s^2}$

(E) ★ $2\ \mathrm{m/s^2} \leq a$

Solution. We have $\dot{s}(t) = 2t$ and $\ddot{s} = 2$, so $\dot{s}(2) = 4$. The acceleration is:

$$\begin{aligned}\vec{a} &= \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\,\hat{e}_n \\ &= 2\,\hat{e}_t + \frac{4^2}{8}\,\hat{e}_n \\ &= 2\,\hat{e}_t + 2\,\hat{e}_n\ \mathrm{m/s^2}.\end{aligned}$$

The magnitude of $\vec{a}$ is thus more than $2\ \mathrm{m/s^2}$.

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9. (1 point) If a particle has position vector $\vec{r}(t) = 5e^t\,\hat{\imath} + \sin(2t)\,\hat{\jmath} + (1 - 3t)\,\hat{k}$ m, what is its speed $v(0)$ at time $t = 0$ s?

(A) $15\ \mathrm{m/s} \leq v(0)$

(B) $v(0) = 0\ \mathrm{m/s}$

(C) $10\ \mathrm{m/s} \leq v(0) < 15\ \mathrm{m/s}$

(D) ★ $5\ \mathrm{m/s} \leq v(0) < 10\ \mathrm{m/s}$

(E) $0\ \mathrm{m/s} < v(0) < 5\ \mathrm{m/s}$

Solution. The velocity is $\vec{v} = \dot{\vec{r}} = 5e^t\,\imath + 2\cos(2t)\,\hat{\jmath} - 3\,\hat{k}$ m/s, so $\vec{v}(0) = 5\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ m/s. The magnitude of $\vec{v}(0)$ is more than 5, but less than $5 + 2 + 3 = 10$.

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10. (1 point) Given a polar basis $\hat{e}_r, \hat{e}_\theta$, the time derivative of the radial basis vector satisfies

$$\dot{\hat{e}}_r \cdot \hat{e}_r = 0.$$

(A) False

(B) ★ True

Solution. True. This hold for any unit vector.

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11. (1 point) A car is observed moving in the plane with velocity $\vec{v} = 2\hat{\imath} - 4\hat{\jmath}$ and acceleration $\vec{a} = 2\hat{\imath} + \hat{\jmath}$. At this instant, is it:

(A) slowing down

(B) ★ keeping its speed constant

(C) speeding up

Solution. $\vec{a} \cdot \vec{b} = 0$ so $\dot{v} = 0$ and the speed is being instantaneously kept constant.

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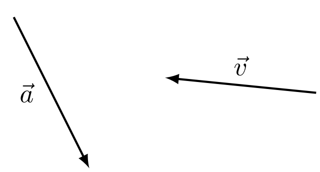
12. (1 point) The velocity $\vec{v}$ and acceleration $\vec{a}$ for a single particle P are shown below at a particular instant.



Which statement is true at this instant?

(A) the particle's speed is not changing

(B) ★ the particle slowing down

(C) the particle is speeding up

Solution. The component of $\vec{a}$ in the direction of $\vec{v}$ is negative, so $\dot{v} < 0$.

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13. (1 point) At a certain instant, particles P and Q have position vectors and velocities given by:

$$\vec{r}_P = 3\hat{\imath} + 4\hat{\jmath} \text{ m} \qquad \vec{r}_Q = -2\hat{\imath} + 6\hat{\jmath} \text{ m}$$
$$\vec{v}_P = -\hat{\imath} - 2\hat{\jmath} \text{ m/s} \qquad \vec{v}_Q = 2\hat{\imath} + 5\hat{\jmath} \text{ m/s}$$

Which statement is true at this instant?

(A) the two particles are moving further apart

(B) ★ the two particles are moving closer together

(C) the two particles are staying at the same distance from each other

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Solution. Take $\vec{R} = \vec{r}_P - \vec{r}_Q$, so R is the distance between the particles. Then:

$$\begin{aligned}
\vec{R} &= (3\hat{\imath} + 4\hat{\jmath} \text{ m}) - (-2\hat{\imath} + 6\hat{\jmath} \text{ m}) \\
&= 5\hat{\imath} - 2\hat{\jmath} \text{ m} \\
\dot{\vec{R}} &= \dot{\vec{r}}_P - \dot{\vec{r}}_Q \\
&= \vec{v}_P - \vec{v}_Q \\
&= (-\hat{\imath} - 2\hat{\jmath} \text{ m/s}) - (2\hat{\imath} + 5\hat{\jmath} \text{ m/s}) \\
&= -3\hat{\imath} - 7\hat{\jmath} \text{ m/s} \\
\dot{\vec{R}} \cdot \vec{R} &= -1 \\
&< 0.
\end{aligned}$$

So $\dot{R} < 0$.

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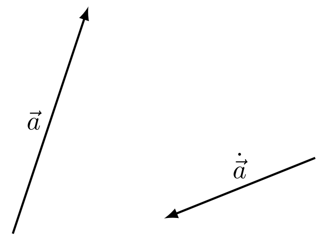
14. (1 point) The vector $\vec{a}$ and its derivative $\dot{\vec{a}}$ are shown below.



Which statement is true?

(A) the length of $\vec{a}$ is increasing

(B) ★ the length of $\vec{a}$ is decreasing

(C) the length of $\vec{a}$ is staying the same

Solution. The derivative $\dot{\vec{a}}$ has a negative component in the direction of $\vec{a}$.

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15. (1 point) A particle is moving in the plane with changing radius so that at a particular instant we have $r = 1$ m and $\dot{r} = -4$ m/s. The speed is $v = 5$ m/s. What is the magnitude of $\dot{\theta}$?

(A) $6 \text{ rad/s} \leq |\dot{\theta}| < 8 \text{ rad/s}$

(B) $8 \text{ rad/s} \leq |\dot{\theta}|$

(C) ★ $2 \text{ rad/s} \leq |\dot{\theta}| < 4 \text{ rad/s}$

(D) $4 \text{ rad/s} \leq |\dot{\theta}| < 6 \text{ rad/s}$

(E) $0 \text{ rad/s} \leq |\dot{\theta}| < 2 \text{ rad/s}$

Solution. Using $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ we have:

$$\begin{aligned}
v &= \sqrt{v_r^2 + v_\theta^2} \\
5 &= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \\
25 &= (-4)^2 + \dot{\theta}^2 \\
\dot{\theta}^2 &= 9 \\
|\dot{\theta}| &= 3.
\end{aligned}$$

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16. (1 point) A particle starts at the origin at time $t = 0$ s and its velocity is given by $\vec{v} = t^3\,\hat{\imath} - t\,\hat{\jmath}$ m. At time $t = 2$ s, what is the particle's distance r from the origin?

(A) 12 m $\leq r <$ 16 m

(B) 0 m $\leq r <$ 4 m

(C) ★ 4 m $\leq r <$ 8 m

(D) 16 m $\leq r$

(E) 8 m $\leq r <$ 12 m

Solution. $\vec{r} = \vec{r}(0) + \int_0^t \vec{v}(\tau)\,d\tau = \frac{1}{4}t^4\,\hat{\imath} - \frac{1}{2}t^2\,\hat{\jmath}$ m, so $\vec{r}(2) = 4\hat{\imath} - 2\hat{\jmath}$ m. Then $r(2)$ is more than 4 m but less than $4 + 2 = 6$ m.

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17. (1 point) A particle moves so that its position vector in the Cartesian basis is given by

$$\vec{r} = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}\ \text{m}.$$

Using cylindrical coordinates, what is the angular component of velocity v_θ at $t = \pi/4$ s?

(A) $0\ \text{m/s} \le v_\theta < 1\ \text{m/s}$

(B) $-1\ \text{m/s} \le v_\theta < 0\ \text{m/s}$

(C) $v_\theta = 0\ \text{m/s}$

(D) $v_\theta < -1\ \text{m/s}$

(E) ★ $1\ \text{m/s} \le v_\theta$

Solution. If we realize that the $\hat{\imath}, \hat{\jmath}$ motion is uniform circular motion with constant $R = 1$ m and $\dot{\theta} = 1$ rad/s, then we known that $v_\theta = R\dot{\theta} = 1$ m/s.
Alternatively, $\vec{r} = \frac{1}{\sqrt{2}}\,\hat{\imath} + \frac{1}{\sqrt{2}}\,\hat{\jmath} + \frac{\pi}{4}\,\hat{k}$, so $\hat{e}_r = \frac{1}{\sqrt{2}}\,\hat{\imath} + \frac{1}{\sqrt{2}}\,\hat{\jmath}$ and $\hat{e}_\theta = -\frac{1}{\sqrt{2}}\,\hat{\imath} + \frac{1}{\sqrt{2}}\,\hat{\jmath}$. Then:

$$\begin{aligned}
\vec{r}(t) &= \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k} \\
\vec{v}(t) &= -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k} \\
\vec{v}(\pi/4) &= -\frac{1}{\sqrt{2}}\,\hat{\imath} + \frac{1}{\sqrt{2}}\,\hat{\jmath} + \hat{k} \\
v_\theta &= \vec{v} \cdot \hat{e}_\theta \\
&= \left(-\frac{1}{\sqrt{2}}\,\hat{\imath} + \frac{1}{\sqrt{2}}\,\hat{\jmath}\right) \cdot \left(-\frac{1}{\sqrt{2}}\,\hat{\imath} + \frac{1}{\sqrt{2}}\,\hat{\jmath} + \hat{k}\right) \\
&= 1\text{m/s}.
\end{aligned}$$

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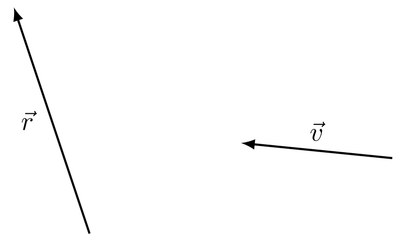
18. (1 point) The position vector $\vec{r}$ and velocity $\vec{v}$ for a single particle P are shown below at a particular instant.



Which statement is true at this instant?

(A) the distance of P from the origin is staying the same

(B) the distance of P from the origin is decreasing

(C) ★ the distance of P from the origin is increasing

Solution. The component of $\vec{v}$ in the direction of $\vec{r}$ is positive, so $\dot{r} > 0$.

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19. (1 point) A point is rotating about the origin in the x–y plane with angular velocity $\omega = 2$ rad/s and velocity $\vec{v} = 4\hat{\imath} - 2\hat{\jmath}$ m/s. What is the x coordinate of the point?

(A) $2 \text{ m} \leq x$

(B) ★ $-2 \text{ m} \leq x < 0 \text{ m}$

(C) $x < -2 \text{ m}$

(D) $0 \text{ m} < x < 2 \text{ m}$

(E) $x = 0 \text{ m}$

Solution. Taking $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$ and $\vec{\omega} = 2\hat{k}$ rad/s, the velocity is:

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r} \\ 4\hat{\imath} - 2\hat{\jmath} &= 2\hat{k} \times (x\,\hat{\imath} + y\,\hat{\jmath}) \\ &= -2y\,\hat{\imath} + 2x\,\hat{\jmath}. \end{aligned}$$

Comparing the $\hat{\jmath}$ components gives $-2 = 2x$, so $x = -1$ m.

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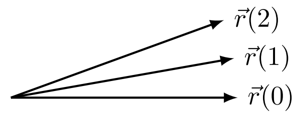
20. (1 point) The position vector $\vec{r}(t)$ of a point is shown below at $t = 0$ s, $t = 1$ s, and $t = 2$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

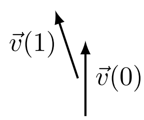
(A) ★ $\leftarrow$

(B) $\rightarrow$

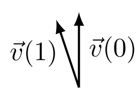
(C) $\uparrow$

(D) $\downarrow$

Solution. The velocity directions are roughly:



Re-drawing these with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly to the left.

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21. (1 point) If $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ and the derivative is $\dot{\vec{a}} = 2\hat{\imath} - \hat{\jmath}$, what can we say about the rate of change of the length of $\vec{a}$ at this instant?

(A) ★ the length of $\vec{a}$ is increasing

(B) the length of $\vec{a}$ is staying the same

(C) the length of $\vec{a}$ is decreasing

Solution. $\dot{\vec{a}} \cdot \vec{a} = 2 > 0$, so $\dot{a} = \dot{\vec{a}} \cdot \hat{a}$, so $\dot{a} > 0$.

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22. (1 point) A point is moving with position vector

$$\vec{r} = (t^2 - 2t)\,\hat{\imath} + t^2\,\hat{\jmath}.$$

What is the radius of curvature ρ at $t = 1$ s?

(A) ρ is infinite

(B) $\rho = 0$ m

(C) $0 \text{ m} < \rho < \frac{1}{2}$ m

(D) $\frac{1}{2} \text{ m} \leq \rho < 1$ m

(E) ★ $1 \text{ m} \leq \rho < \infty$

Solution. The position and velocity are:

$$\begin{aligned}
\vec{r} &= (t^2 - 2t)\,\hat{\imath} + t^2\,\hat{\jmath} = -\hat{\imath} + \hat{\jmath} \\
\vec{v} &= (2t - 2)\,\hat{\imath} + 2t\,\hat{\jmath} = 2\hat{\jmath} \\
\vec{a} &= 2\hat{\imath} + 2\hat{\jmath}.
\end{aligned}$$

Thus $\hat{e}_t = \hat{\jmath}$ and $\hat{e}_n = \hat{\imath}$, so $a_n = 2$. Also $v = 2$, so:

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
2 &= \frac{2^2}{\rho} \\
\rho &= 2 \text{ m}.
\end{aligned}$$

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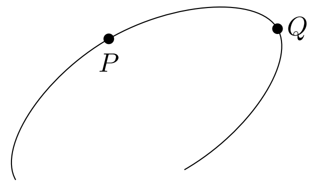
23. (1 point) A car is driving on a curved race track at constant speed, with the top view shown below.



How does the magnitude of the car's acceleration a_P at point P compare to the value a_Q at point Q?

(A) $a_P > a_Q$

(B) $a_P = a_Q$

(C) ★ $a_P < a_Q$

Solution. The radius of curvature is higher at P, so the normal acceleration is lower, and there is no tangential component.

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24. (1 point) A particle is moving with position vector given by $\vec{r} = t\,\hat{\imath} + t^2\,\hat{\jmath}$ m. At time $t = 1$ s, what is the vertical component of the normal vector $e_{n,y}$?

(A) $\frac{1}{2} \leq e_{n,y}$

(B) $-\frac{1}{2} \leq e_{n,y} < 0$

(C) $e_{n,y} = 0$

(D) $e_{n,y} < -\frac{1}{2}$

(E) ★ $0 \leq e_{n,y} < \frac{1}{2}$

Solution. Computing the velocity and acceleration gives:

$$\begin{aligned} \vec{r} &= t\,\hat{\imath} + t^2\,\hat{\jmath} \\ \vec{v} &= \hat{\imath} + 2t\,\hat{\jmath} \\ \vec{a} &= 2\hat{\jmath}. \end{aligned}$$

At $t = 1$ s this gives:

$$\begin{aligned} \vec{v} &= \hat{\imath} + 2\hat{\jmath} \\ \vec{a} &= 2\hat{\jmath}. \end{aligned}$$

Now the normal vector must point up and left, so:

$$\begin{aligned} \hat{e}_t &= \hat{v} = \frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{\jmath} \\ \hat{e}_n &= -\frac{2}{\sqrt{5}}\hat{\imath} + \frac{1}{\sqrt{5}}\hat{\jmath}. \end{aligned}$$

Thus $e_{n,y} = \frac{1}{\sqrt{5}} < \frac{1}{2}$.

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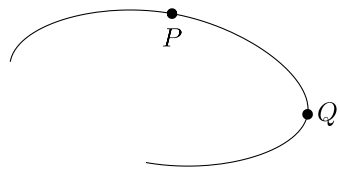
25. (1 point) A point is moving around the curve shown below with varying speed.



The radius of curvature and speed at P and Q are given by:

$$\rho_P = 4 \text{ m} \qquad \rho_Q = 2 \text{ m}$$
$$v_P = 4 \text{ m/s} \qquad v_Q = 2 \text{ m/s}.$$

Which of the following is true about the normal accelerations $a_{P,n}$ at P and $a_{Q,n}$ at Q?

(A) ★ $a_{P,n} > a_{Q,n}$

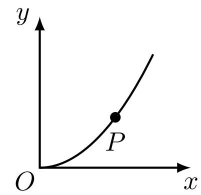
(B) $a_{P,n} = a_{Q,n}$

(C) $a_{P,n} < a_{Q,n}$

Solution. $a_{P,n} = v_{P,n}^2/\rho_P = 4^2/4 = 4$ m and $a_{Q,n} = v_{Q,n}^2/\rho_Q = 2^2/2 = 2$ m, so $a_{P,n}$ is larger.

---

26. (1 point) A particle is moving to the right along the curve $y = \frac{1}{3}x^2$ m at a constant speed of $v = \frac{5}{3}$ m/s. What is the horizontal component of velocity v_x when $x = 2$ m?



(A) ★ $0 \text{ m/s} \le v_x < 2 \text{ m/s}$

(B) $2 \text{ m/s} \le v_x$

(C) $-2 \text{ m/s} \le v_x < 0 \text{ m/s}$

(D) $v_x = 0$ m/s

(E) $v_x < -2$ m/s

Solution. The position vector of P is:

$$\begin{aligned}
\vec{r} &= x\,\hat{\imath} + \frac{1}{3}x^2\,\hat{\jmath} \\
\vec{v} &= \dot{x}\,\hat{\imath} + \frac{2}{3}x\dot{x}\,\hat{\jmath} \\
&= \dot{x}\,\hat{\imath} + \frac{4}{3}\dot{x}\,\hat{\jmath} \\
v &= \sqrt{\dot{x}^2 + \left(\frac{4}{3}\dot{x}\right)^2} \\
v^2 &= \left(1 + \frac{16}{9}\right)\dot{x}^2 \\
\frac{25}{9} &= \frac{25}{9}\dot{x}^2 \\
v_x = \dot{x} &= 1 \text{ m/s.}
\end{aligned}$$

Here we used the fact that the particle is moving to the right to choose the positive solution.

---

27. (1 point) For a certain position P the distance from the origin is $r = 4$ m and the polar basis vectors are:

$$\hat{e}_r = -\frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath}$$
$$\hat{e}_\theta = \frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}$$

What is the horizontal coordinate x?

(A) ★ $-2 \text{ m} \leq x < 0 \text{ m}$

(B) $2 \text{ m} < x$

(C) $x < -2 \text{ m}$

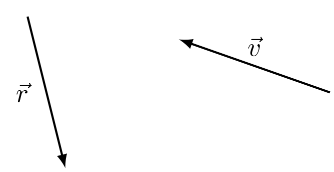
(D) $0 \text{ m} < x < 2 \text{ m}$

(E) $x = 0 \text{ m}$

Solution. $\vec{r} = r\,\hat{e}_r = 4\left(-\frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath}\right) = -2\hat{\imath} - 2\sqrt{3}\hat{\jmath}$, so $x = -2$ m.

---

28. (1 point) The position vector $\vec{r}$ and velocity $\vec{v}$ for a single particle P are shown below at a particular instant.



Which statement about the polar angle derivative $\dot{\theta}$ is true at this instant?

(A) ★ $\dot{\theta} < 0$

(B) $\dot{\theta} = 0$

(C) $\dot{\theta} > 0$

Solution. From the diagram it is clear that $\vec{r}$ is rotating clockwise, so $\dot{\theta} < 0$.

---

29. (1 point) The vector $\vec{a} = 3\hat{\imath} - 4\hat{\jmath}$ has derivative $\dot{\vec{a}} = -2\hat{\imath} - \hat{\jmath}$. What is the rate of change $\dot{a}$ of the length?

(A) $\dot{a} < -1$

(B) $0 < \dot{a} < 1$

(C) ★ $-1 \leq \dot{a} < 0$

(D) $1 \leq \dot{a}$

(E) $\dot{a} = 0$

Solution. $a = 5$ so $\hat{a} = 0.6\hat{\imath} - 0.8\hat{\jmath}$. Then $\dot{a} = \dot{\vec{a}} \cdot \hat{a} = -2(0.6) - (-0.8) = -0.4$.

---

30. (1 point) A particle moves so that its position in polar coordinates is given by

$$r = \frac{1}{2}t^2 \text{ m}$$
$$\theta = 2t \text{ rad.}$$

What is the radial component of acceleration a_r at $t = 2$ s?

(A) $0 \text{ m/s}^2 \leq a_r < 7 \text{ m/s}^2$

(B) $7 \text{ m/s}^2 \leq a_r$

(C) $a_r < -7 \text{ m/s}^2$

(D) $a_r = 0 \text{ m/s}^2$

(E) ★ $-7 \text{ m/s}^2 \leq a_r < 0 \text{ m/s}^2$

Solution. We have:

$$\begin{aligned} r(t) &= \frac{1}{2}t^2 & \theta(t) &= 2t \\ \dot{r}(t) &= t & \dot{\theta}(t) &= 2 \\ \ddot{r}(t) &= 1 & \ddot{\theta}(t) &= 0. \end{aligned}$$

Evaluating these at $t = 2$ gives:

$$\begin{aligned} r(2) &= 2 & \theta(2) &= 4 \\ \dot{r}(2) &= 2 & \dot{\theta}(2) &= 2 \\ \ddot{r}(2) &= 1 & \ddot{\theta}(2) &= 0. \end{aligned}$$

The radial acceleration is:

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= 1 - 2 \cdot 2^2 \\ &= -7 \text{ m/s}^2. \end{aligned}$$

---

31. (1 point) A position P has an associated polar basis with

$$\hat{e}_\theta = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}.$$

What is θ?

(A) $\frac{3}{2}\pi \le \theta < 2\pi$

(B) ★ $\pi \le \theta < \frac{3}{2}\pi$

(C) $0 \le \theta < \frac{1}{2}\pi$

(D) $\frac{1}{2}\pi \le \theta < \pi$

Solution. The fastest way to solve this question is to draw a diagram and immediately see that θ must be in the third quadrant.
Alternatively, we can use the formula $\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ to see that $-\sin\theta = \frac{1}{\sqrt{2}}$, so $\theta = -45^\circ$ or $\theta = -135^\circ$. Also $\cos\theta = -\frac{1}{\sqrt{2}}$ so $\theta = \pm 135^\circ$. The only common solution is $\theta = -135^\circ = \frac{5}{4}\pi$.

---

32. (1 point) The particle P has polar coordinates $r = 2$ m, $\theta = -\frac{3}{4}\pi$ rad and rates of change $\dot{r} > 0$ and $\dot{\theta} < 0$. Which statement about the velocity $\vec{v}$ of P must be true?

(A) $v_y < 0$

(B) $v_y \geq 0$

(C) $v_x \geq 0$

(D) ★ $v_x < 0$

Solution. The fastest way to solve this question is to draw a diagram with P in the third quadrant, and to see that $\dot{r} > 0$ implies down-left movement, while $\dot{\theta} < 0$ implies up-left moment, so a combination of these may be up or down (depending on the relative magnitudes), but must be left. This is $v_x < 0$.
Alternatively, we can write the velocity of P as:

$$\begin{aligned}
\vec{v}_P &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \\
&= \dot{r}\left(-\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}\right) + r\dot{\theta}\left(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}\right) \\
&= \frac{1}{\sqrt{2}}(-\dot{r} + r\dot{\theta})\,\hat{\imath} + \frac{1}{\sqrt{2}}(-\dot{r} - r\dot{\theta})\,\hat{\jmath}.
\end{aligned}$$

Now $\dot{r} > 0$ so $-\dot{r} < 0$. Also $r > 0$ and $\dot{\theta} < 0$ so $r\dot{\theta} < 0$ and $-r\dot{\theta} > 0$. This means that $-\dot{r} + r\dot{\theta} < 0$, but $-\dot{r} - r\dot{\theta}$ may be positive, negative, or zero. Thus we see that $v_x < 0$ is the only statement that is guaranteed to always be true.

---

33. (1 point) A position P has an associated polar basis with

$$\hat{e}_r = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}.$$

What is $\hat{e}_\theta$?

(A) $\hat{e}_\theta = \frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}$

(B) ★ $\hat{e}_\theta = -\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$

(C) $\hat{e}_\theta = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$

(D) $\hat{e}_\theta = -\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}$

Solution. The fastest way to solve this question is to draw a diagram and realize that $\hat{e}_\theta$ is always a 90° rotation of $\hat{e}_r$, so we can immediately deduce the answer.
Alternatively, this basis vector is $\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$, so $\cos\theta = \frac{1}{2}$ and thus $\theta = \pm 60°$. But $\sin\theta = \frac{\sqrt{3}}{2}$, so $\theta = 60°$ or $\theta = 120°$. The only common solution is $\theta = 60°$. Then $\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ to get the answer.

---

34. (1 point) A particle P has position vector and velocity:

$$\vec{r} = 5\hat{\imath} + 2\hat{\jmath} \text{ m}$$
$$\vec{v} = 2\hat{\imath} - 4\hat{\jmath} \text{ m/s}.$$

Is the distance from P to the origin:

(A) ★ increasing

(B) staying the same

(C) decreasing

Solution. $\vec{v} \cdot \vec{r} = 2 > 0$ so $\dot{r} = \vec{v} \cdot \hat{r} > 0$.

---

35. (1 point) A particle P has position vector, velocity, and acceleration given by:

$$\vec{r} = 4\hat{\imath} + 2\hat{\jmath} \text{ m}$$
$$\vec{v} = 2\hat{\imath} - 7\hat{\jmath} \text{ m/s}$$
$$\vec{a} = -3\hat{\imath} - 2\hat{\jmath} \text{ m/s}^2$$

Consider the following statements:

(i) The particle is moving closer to the origin.

(ii) The particle is moving further from the origin.

(iii) The particle is speeding up.

(iv) The particle is slowing down.

Which statements are true?

(A) ★ (i) and (iii)

(B) (ii) and (iii)

(C) (i) and (iv)

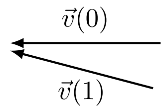
(D) (ii) and (iv)

(E) none of the other options

Solution. $\vec{v} \cdot \vec{r} = -6 < 0$ so $\dot{r} < 0$ (moving closer to the origin), and $\vec{a} \cdot \vec{v} = 8 > 0$ so $\dot{v} > 0$ (speeding up).

---

36. (1 point) The velocity $\vec{v}(t)$ of a point is shown below at $t = 0$ s and $t = 1$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

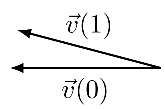
(A) ★ $\uparrow$

(B) $\downarrow$

(C) $\leftarrow$

(D) $\rightarrow$

Solution. Re-drawing the velocities with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly upwards.

---

37. (1 point) A vector $\vec{a}$ and its derivative $\dot{\vec{a}}$ have $\|\vec{a}\| = 3$, $\|\dot{\vec{a}}\| = 4$, and $\vec{a} \cdot \dot{\vec{a}} = -6$. What can we say about the rate of change of the length of $\vec{a}$ at this instant?

(A) the length of $\vec{a}$ is increasing

(B) ★ the length of $\vec{a}$ is decreasing

(C) the length of $\vec{a}$ is staying the same

Solution. $\dot{a} = \dot{\vec{a}} \cdot \hat{a}$, so $\dot{a}$ is negative.

---

38. (1 point) A point has Cartesian coordinates $x = -3$ m, $y = -2$ m. What is its polar coordinate angle θ?

(A) ★ π rad $\leq \theta < \frac{3}{2}\pi$ rad

(B) $\frac{1}{2}\pi$ rad $\leq \theta < \pi$ rad

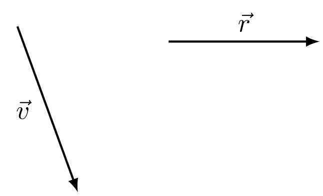
(C) 0 rad $\leq \theta < \frac{1}{2}\pi$ rad

(D) $\frac{3}{2}\pi$ rad $\leq \theta < 2\pi$ rad

Solution. The point lies in the third quadrant.

---

39. (1 point) The position vector $\vec{r}$ and velocity $\vec{v}$ for a single particle P are shown below at a particular instant.



Which statement about $\dot{r}$ is true at this instant?

(A) $\dot{r} = 0$

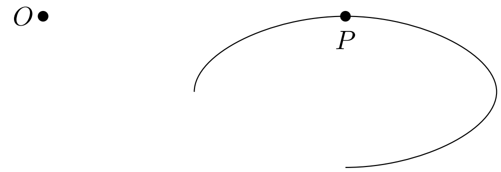
(B) ★ $\dot{r} > 0$

(C) $\dot{r} < 0$

Solution. $\vec{v} \cdot \vec{r} > 0$ so $\vec{v}$ has a positive component in the $\vec{r}$ direction, meaning that $\vec{r}$ is getting longer and $\dot{r} > 0$.

---

40. (1 point) A point is moving around the curve shown and is currently at position P. Consider a polar basis $\hat{e}_r, \hat{e}_\theta$ at P from the origin O and a tangential/normal basis $\hat{e}_t, \hat{e}_n$ at P.



Which of the following is true?

(A) $\hat{e}_n = -\hat{e}_r$

(B) ★ $\hat{e}_n = -\hat{e}_\theta$

(C) $\hat{e}_n = \hat{e}_r$

(D) $\hat{e}_n = \hat{e}_\theta$

Solution. Here $\hat{e}_r$ is to the right and $\hat{e}_\theta$ is upwards. Also $\hat{e}_n$ is inwards to the curve, so is downwards, which is opposite to $\hat{e}_\theta$.

---

41. (1 point) A position P has an associated polar basis so that

$$\hat{\imath} = \frac{1}{\sqrt{2}}\hat{e}_r - \frac{1}{\sqrt{2}}\hat{e}_\theta.$$

What is θ?

(A) $\pi \le \theta < \frac{3}{2}\pi$

(B) $\frac{3}{2}\pi \le \theta < 2\pi$

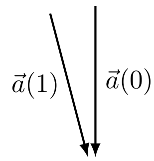
(C) $\frac{1}{2}\pi \le \theta < \pi$

(D) ★ $0 \le \theta < \frac{1}{2}\pi$

Solution. The fastest way to solve this question is to draw a diagram and see that θ must be in the first quadrant.
Alternatively, using the formula $\hat{\imath} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$, we have $\cos\theta = \frac{1}{\sqrt{2}}$ so $\theta = \pm 45°$. Also $-\sin\theta = -\frac{1}{\sqrt{2}}$ so $\theta = 45°$ or $\theta = 135°$. The only common solution is $\theta = 45°$.

---

42. (1 point) The vector $\vec{a}(t)$ is pictured below at $t = 0$ s and $t = 1$ s.



Which direction is the closest to the direction of $\dot{\vec{a}}(0)$?

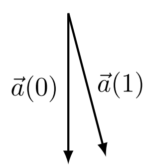
(A) $\leftarrow$

(B) $\downarrow$

(C) $\uparrow$

(D) ★ $\rightarrow$

Solution. Drawing the vectors with the same base gives



so the derivative is mainly to the right.

---

43. (1 point) A point is currently at position $x = 3$ m, $y = 2$ m, $z = 0$ m and is rotating in the x–y plane about the origin with angular velocity $\vec{\omega} = 2\hat{k}$. The velocity $\vec{v}$ of the point is:

(A) ★ $\vec{v} = -4\hat{\imath} + 6\hat{\jmath}$ m/s

(B) $\vec{v} = -4\hat{\imath} - 6\hat{\jmath}$ m/s

(C) $\vec{v} = 4\hat{\imath} - 6\hat{\jmath}$ m/s

(D) $\vec{v} = 4\hat{\imath} + 6\hat{\jmath}$ m/s

Solution. The position vector is $\vec{r} = 3\hat{\imath} + 2\hat{\jmath}$ m, so:

$$\begin{aligned} \vec{v} &= \omega \times \vec{r} \\ &= 2\hat{k} \times (3\hat{\imath} + 2\hat{\jmath}) \\ &= -4\hat{\imath} + 6\hat{\jmath} \text{ m/s}. \end{aligned}$$

---

44. (1 point) A particle moves so that its position in polar coordinates is given by

$$r = 2t \text{ m}$$
$$\theta = -\frac{\pi}{8}t^2 \text{ rad.}$$

What is the $\hat{\imath}$ component of velocity v_x at $t = 2$ s?

(A) ★ $v_x < -6$ m/s

(B) -6 m/s $\leq v_x < 0$ m/s

(C) $v_x = 0$ m/s

(D) 6 m/s $\leq v_x$

(E) 0 m/s $\leq v_x < 6$ m/s

Solution. We have:

$$r(t) = 2t \qquad \theta(t) = -\frac{\pi}{8}t^2$$
$$\dot{r}(t) = 2 \qquad \dot{\theta}(t) = -\frac{\pi}{4}t.$$

At $t = 2$ this is:

$$r(2) = 4 \qquad \theta(2) = -\frac{\pi}{2}$$
$$\dot{r}(2) = 2 \qquad \dot{\theta}(2) = -\frac{\pi}{2}.$$

The velocity is:

$$\begin{aligned}\vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \\ &= 2\,\hat{e}_r - 2\pi\,\hat{e}_\theta.\end{aligned}$$

Using $\theta = -\frac{\pi}{2}$ we see that $\hat{e}_\theta = \hat{\imath}$, so the $\hat{\imath}$ component is:

$$v_x = -2\pi \approx -6.28 \text{ m/s}.$$

---

45. (1 point) The position vector of a particle is given by

$$\vec{r} = (4t^2 - 2)\,\hat{\imath} + (4t^2 - t^3)\,\hat{\jmath} \text{ m}.$$

Which statement is true at time $t = 0$?

(A) the $\hat{\jmath}$ component of the particle's velocity is decreasing

(B) the $\hat{\jmath}$ component of the particle's velocity is staying the same

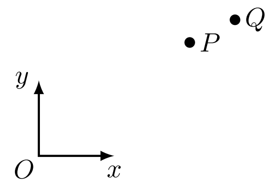
(C) ★ the $\hat{\jmath}$ component of the particle's velocity is increasing

---

Solution. $\vec{v}(t) = 8t\,\hat{\imath} + (8t - 3t^2)\,\hat{\jmath}$ m/s and $\vec{a}(t) = 8\hat{\imath} + (8 - 6t)\,\hat{\jmath}$ m/s^2, so $\vec{a}(0) = 8\hat{\imath} + 8\hat{\jmath}$ m/s^2, so $\dot{v}_y = a_y > 0$.

---

46. (1 point) Points P and Q are moving in circular paths around the origin O with angular velocities ω_P and ω_Q.



The two particles are moving with the same speed. Which statement is true?

(A) $\frac{1}{2}|\omega_Q| < |\omega_P| \leq |\omega_Q|$

(B) $|\omega_P| \leq \frac{1}{2}|\omega_Q|$

(C) ★ $|\omega_Q| < |\omega_P| \leq 2|\omega_Q|$

(D) $2|\omega_Q| < |\omega_P|$

Solution. If the particles both have speed v then we can observe from the diagram that ω_P is a little bit higher than ω_Q.
More precisely, taking absolute values to ignore the direction, we have:

$$|\omega_P| = \frac{v}{r_P}$$
$$|\omega_Q| = \frac{v}{r_Q}$$
$$|\omega_P| = \frac{r_Q}{r_P}|\omega_Q|.$$

From the diagram, $\frac{r_Q}{r_P} \approx 1.25$, so ω_P is larger than $|\omega_Q|$, but smaller than $2|\omega_Q|$.

---

47. (1 point) A spaceship is moving with velocity and acceleration given in a polar basis by

$$\vec{v} = 3\,\hat{e}_r + 4\,\hat{e}_\theta \text{ m/s}$$
$$\vec{a} = -7\,\hat{e}_r - \hat{e}_\theta \text{ m/s}^2.$$

What is the radius of curvature of the spaceship's path?

(A) $0 \text{ m} \le \rho < 2 \text{ m}$

(B) $\rho = 0 \text{ m}$

(C) $6 \text{ m} \le \rho$

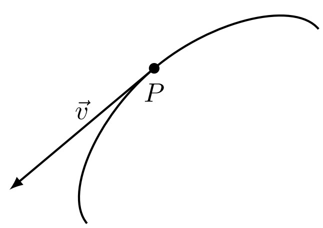
(D) $2 \text{ m} \le \rho < 4 \text{ m}$

(E) ★ $4 \text{ m} \le \rho < 6 \text{ m}$

Solution. $v = 5$ and $\|\vec{v} \times \vec{a}\| = 25$, so $\rho = v^3/\|\vec{v} \times \vec{a}\| = 5$ m.
Alternatively, $\hat{e}_t = \hat{v} = 0.6\hat{\imath} + 0.8\hat{\jmath}$ so $a_t = \vec{a} \cdot \hat{e}_t = -5$. Now $a^2 = 50 = a_t^2 + a_n^2 = 25 + a_n^2$, so $a_n = 5$, and $a_n = v^2/\rho$ gives $\rho = 5$ m.

---

48. (1 point) A car is driving on a curved track with the top view shown below. At a given instant the car is at point P with velocity $\vec{v}$ and its speed is increasing, such that the tangential and normal components of its acceleration are equal in magnitude.



Which direction is the closest to the direction of the acceleration $\vec{a}$ at the instant shown?

(A) $\rightarrow$

(B) $\leftarrow$

(C) $\uparrow$

(D) ★ $\downarrow$

Solution. $\hat{e}_t$ is down-left and $\hat{e}_n$ is down-right, so equal components in these two directions give a total downwards acceleration.

---

49. (1 point) A point has polar coordinates $r = 4$ m, $\theta = -120°$. What is its horizontal coordinate x?

(A) $x < -2$ m

(B) $2 \text{ m} < x$

(C) $0 \text{ m} < x < 2$ m

(D) $x = 0$ m

(E) ★ $-2 \text{ m} \leq x < 0$ m

Solution. $x = r\cos\theta = 4\cos(-120°) = 4(-1/2) = -2$ m.

---

50. (1 point) A car is observed moving in the plane with velocity $\vec{v} = 3\hat{\imath} + 2\hat{\jmath}$ and acceleration $\vec{a} = -2\hat{\imath} + 4\hat{\jmath}$. At this instant, is it:

(A) stationary

(B) ★ driving around a curve counterclockwise

(C) driving around a curve clockwise

(D) driving in a straight line

Solution. Drawing a diagram shows it is curving counterclockwise.
Alternatively, $\vec{v} \times \vec{a} = 16\hat{k}$ which is in the $+\hat{k}$ direction, so it is curving in the positive direction (counterclockwise).
