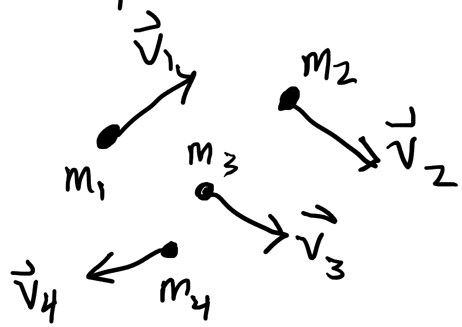
TAM 212

Principle of Work & Kinetic Energy

Kinetic Energy of point masses

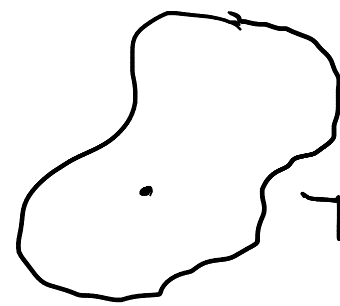


$$T = \frac{1}{2}\sum m_i v_i^2$$

$$= \frac{1}{2}\sum m_i \vec{v}_i \cdot \vec{v}_i$$

$\Longrightarrow$

Kinetic Energy of a Rigid Body



$$T = \frac{1}{2}\int \rho(\vec{r})\, v^2(\vec{r})\, dV$$

$$T = \frac{1}{2}\int \rho\, (\vec{v}\cdot\vec{v})\, dV$$

Since the velocities of different points are related to each other by the angular velocity, the K.E. can be expressed

$$T = \frac{1}{2} m v_{\underline{c}}^2 + \frac{1}{2} I_{\underline{c},\hat{k}}\, \omega^2$$

CENTER OF MASS

"translational" motion of mass center

"rotational" motion of rigid body relative to mass center

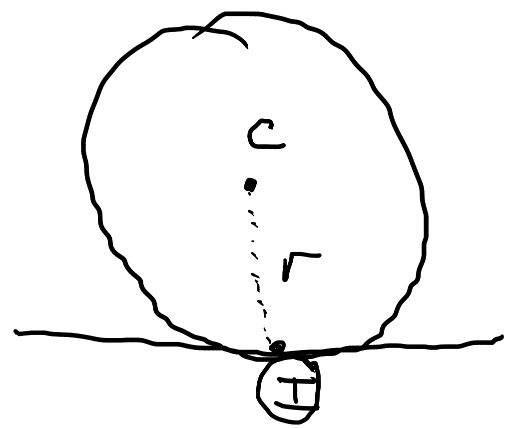
also: we can write T w/ respect to the instaneous center I

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_{c,\hat{k}} \omega^2$$

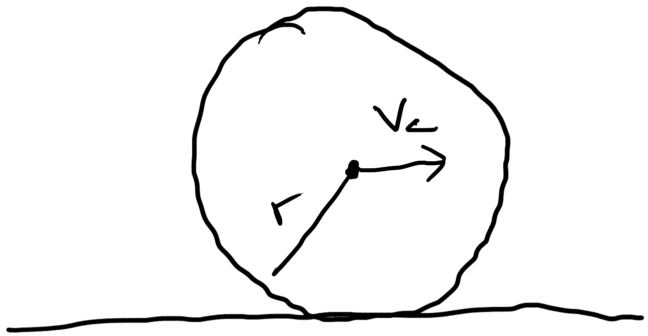
$$= \frac{1}{2} m (r\omega)^2 + \frac{1}{2} I_{c,\hat{k}} \omega^2$$

$$= \frac{1}{2} \underbrace{\left(I_{c,\hat{k}} + m r^2 \right)} \omega^2$$

$$= \frac{1}{2} I_{I,\hat{k}} \omega^2$$



Example Rolling cyllinder, solid $\qquad I_{c,\hat{k}} = \frac{1}{2} m r^2$



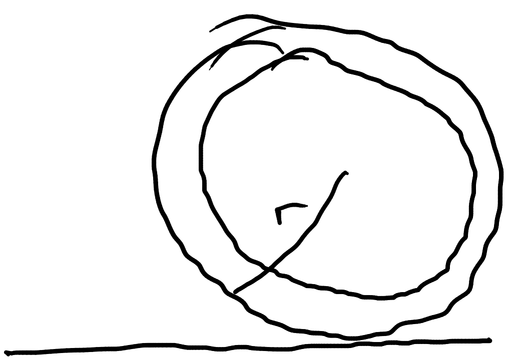
$$T = T_v + T_\omega$$

$$= \frac{1}{2} m v_c^2 + \frac{1}{2} I_{c,\hat{k}} \omega^2$$

$$= \frac{1}{2} m (r\omega)^2 + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\omega^2 = \frac{3}{4} m (r\omega)^2$$

2/3 of energy is translational,
1/3 of energy is rotational

Example: Hollow rolling cylinder, same mass $\qquad I_{c,\hat{k}} = m r^2$



$$T = T_v + T_\omega$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} I_{c,\hat{k}} \omega^2$$

$$= \frac{1}{2} m (r\omega)^2 + \frac{1}{2}\left(m r^2\right)\omega^2 = m (r\omega)^2$$

$\Rightarrow$ 1/2 ; 1/2 trans, rotational

PRINCIPLE OF WORK & KINETIC ENERGY

$W = \Delta T$

work done by external forces/moments

change in KE

For a rigid body

$$T = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} I_{c,\hat{k}} (\omega \hat{k} \cdot \omega \hat{k})$$

$$\frac{dT}{dt} = \frac{1}{2} m (\vec{a}_c \cdot \vec{v}_c + \vec{v}_c \cdot \vec{a}_c) + \frac{1}{2} I_{c,\hat{k}} (\alpha \hat{k} \cdot \omega \hat{k} + \omega \hat{k} \cdot \alpha \hat{k})$$

$$= \underbrace{m\vec{a}_c}\cdot \vec{v}_c + \underbrace{I_{c,\hat{k}}(\alpha \hat{k}}\cdot \omega \hat{k})$$

$$= \underbrace{\sum \vec{F} \cdot \vec{v}_c}_{\text{work of ext. forces}} + \underbrace{\sum \vec{M}_c \cdot \omega \hat{k}}_{\text{work done by ext. moments}}$$

CENTER OF MASS

Note: $\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots$

$\sum \vec{M}_c = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots\ldots$

$(\vec{a} \times \vec{b}) \cdot \vec{c} =$
$(\vec{c} \times \vec{a}) \cdot \vec{b}$

$$\frac{dT}{dt} = (\vec{F}_1 + \vec{F}_2 + \ldots\ldots) \cdot \vec{V}_c + (\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots\ldots) \cdot \omega \hat{k}$$

$$= (\vec{F}_1 + \vec{F}_2 + \ldots\ldots) \cdot \vec{V}_c + \underbrace{(\omega \hat{k} \times \vec{r}_1)}_{a} \cdot \vec{F}_1 + \underbrace{(\omega \hat{k} \times \vec{r}_2)} \cdot \vec{F}_2 + \ldots\ldots$$

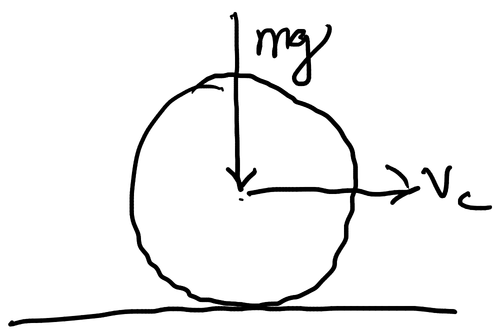
$\vec{V}_1 = \vec{V}_c + \vec{\omega} \times \vec{r}_{c1}$

$$= (\underbrace{\vec{V}_c + \omega \hat{k} \times \vec{r}_1}) \cdot \vec{F}_1 + (\vec{V}_c + \omega \hat{k} \times \vec{r}_2) \cdot \vec{F}_2 + \ldots\ldots$$

$$= \vec{V}_1 \cdot \vec{F}_1 + \vec{V}_2 \cdot \vec{F}_2 + \ldots\ldots$$

$$= \sum \vec{F}_i \cdot \vec{V}_i$$

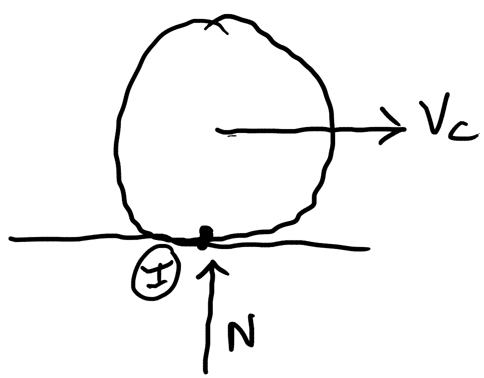
Example) Cyllinder rolls w/o slip on flat ground with initial velocity $\vec{V}_c = V_c\hat{\imath} = r\omega\hat{\imath}$.



work done by gravity? $\vec{F}_g = -mg\,\hat{\jmath}$

$$\vec{V}_c = r\omega\,\hat{\imath}$$

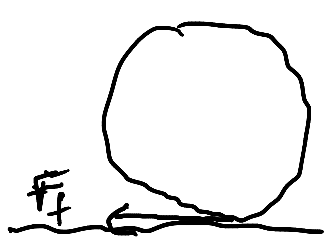
$$\vec{F}_g \cdot \vec{V}_c = 0$$



work done by N? $\vec{N} = mg\,\hat{\jmath}$

$$\vec{V}_{I} = 0$$

$$\vec{N} \cdot \vec{V}_{I} = 0$$



work done by friction $\vec{F}_f$? $\vec{F}_f = F_f\,\hat{\imath}$

$$\vec{V}_I = 0$$

$$\vec{F}_f \cdot \vec{V}_I = 0$$

Note: if the wheel were slipping, then friction would do non-zero (negative) work

$$\left.\begin{array}{l}\vec{v}_c \neq 0 \\ \vec{F} \text{ opposes } \vec{v}_c\end{array}\right\} = \vec{F}\cdot\vec{v}_c < 0$$

contact point.