Euler's Second Law

$$\begin{cases} (1) \quad \sum \vec{M}_c = 0 & \longrightarrow \text{pure translations} \\ (2) \quad \sum \vec{M}_c = I_{c,\hat{k}} \vec{\alpha} & \longrightarrow \text{always} \\ (3) \quad \sum \vec{M}_o = I_{o,\hat{k}} \vec{\alpha} & \longrightarrow \text{for } o = \text{fixed point} \end{cases}$$

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generalize to arbitrary points P

1. $\sum \vec{M}_p = I_{c,\hat{k}} \vec{\alpha} + (\vec{r}_{pc} \times m\vec{a}_c)$

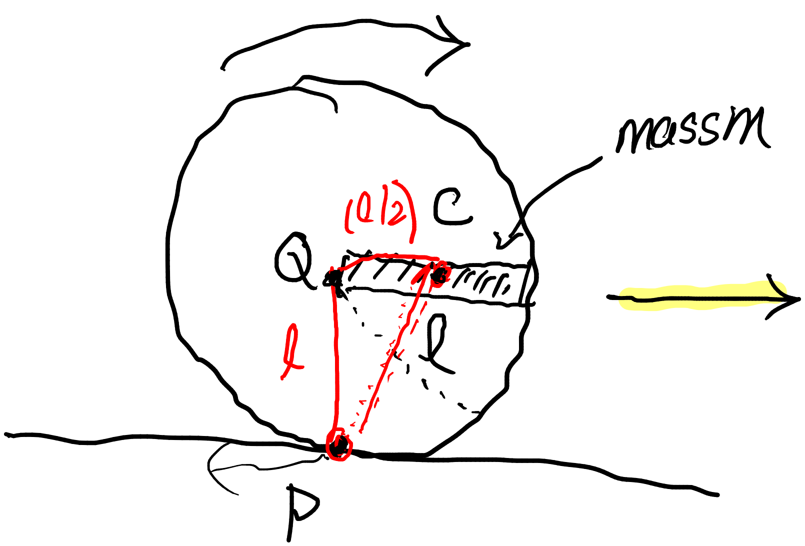
2. $\sum \vec{M}_p = I_{p,\hat{k}} \vec{\alpha} + (\vec{r}_{pc} \times m\vec{a}_p)$

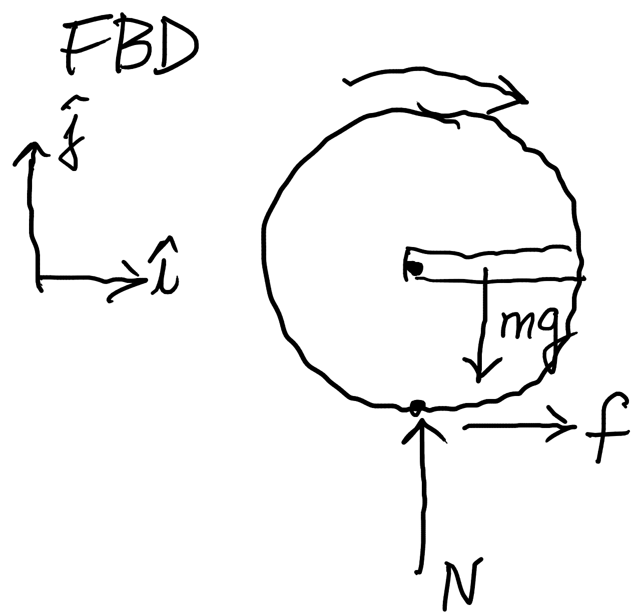
quick sanity check

1. $P \Rightarrow C \qquad \sum \vec{M}_c = I_{c,\hat{k}} \vec{\alpha} + (\cancel{\vec{r}_{cc} \times m\vec{a}_c}^{\,0}) = I_{c,\hat{k}} \vec{\alpha}$

2. $P \Rightarrow \text{fixed point } O \qquad \sum \vec{M}_o = I_{o,\hat{k}} \vec{\alpha} + (\cancel{\vec{r}_{oc} \times m\vec{a}_o}^{\,0}) = I_{o,\hat{k}} \vec{\alpha}$

Example) Heavy body of mass m welded to a hoop of negligible mass, We release it from rest, find initial angular acceleration $\vec{\alpha}$





$\vec{\alpha} = (-\alpha)\hat{k}$

if $\alpha > 0$

$$\boxed{\vec{a}_Q = \alpha l \hat{i}}$$

version 2: $\sum \vec{M}_P = I_{P,\hat{k}}\vec{\alpha} + (\vec{r}_{PC} \times m\vec{a}_P)$

$$\sum \vec{M}_P = \left(-\frac{mgl}{2}\right)\hat{k}$$

$$I_{P,\hat{k}} = I_{C,\hat{k}} + m d_{PC}^2$$ parallel axis theorem

$$= \frac{1}{12} m l^2 + m\left(l^2 + \left(\frac{l}{2}\right)^2\right)$$

$$= m l^2 \left(\frac{1}{12} + 1 + \frac{1}{4}\right)$$

$$= \frac{4}{3} m l^2$$

$$\vec{\alpha} = (-\alpha)\hat{k}$$

$$\vec{r}_{PC} = \left(\frac{l}{2}\hat{i} + l\hat{j}\right)$$

$$m\vec{a}_P = m\left(\vec{a}_Q + \vec{\alpha}\times\vec{r}_{QP} - \cancelto{\omega=0}{\omega^2\vec{r}_{QP}}\right)$$

$$= m\left[\left(l\alpha\hat{i} + (-\alpha\hat{k})\times(-l\hat{j})\right)\right]$$

$$= m\left[l\alpha\hat{i} + (-\alpha)(-l)(-\hat{i})\right]$$

$$= m\left[l\alpha\hat{i} - l\alpha\hat{i}\right]$$

$$= 0$$

So our equation becomes:

$$\sum \vec{M}_P = I_{P,\hat{k}}\,\vec{\alpha} + \cancelto{0}{\left(\vec{r}_{PC}\times m\vec{a}_P\right)}$$

$$\left(-\frac{mgl}{2}\right)\hat{k} = \frac{4}{3}ml^2(-\alpha\hat{k}) \implies$$

$$g/2 = \frac{4}{3}l\alpha$$

$$\alpha = \frac{3}{8}\frac{g}{l}\ \text{rad/s}^2$$

$$\vec{\alpha} = -\frac{3}{8}(g/l)\,\hat{k}\quad \text{CW}$$

version 1)

$$\sum \vec{M}_p = I_{c,\hat{k}}\,\vec{\alpha} + (\vec{r}_{pc} \times m\vec{a}_c)$$

$$\sum \vec{M}_p = \left(\frac{-mgl}{2}\right)\hat{k}$$

$$I_{c,\hat{k}} = \frac{1}{12} m l^2$$

$$\vec{\alpha} = (-\alpha)\hat{k}$$

$$\vec{r}_{pc} = \frac{l}{2}\hat{i} + l\hat{j}$$

$$m\vec{a}_c = m\left(\vec{a}_Q + \vec{\alpha} \times \vec{r}_{QC} - \overset{\omega=0}{\cancel{\omega^2}}\,\vec{r}_{QC}\right)$$

$$= m\left(\left[l\alpha\hat{i} + (-\alpha\hat{k}) \times \left(\frac{l}{2}\hat{i}\right)\right]\right) = m\left[l\alpha\hat{i} + \left(\frac{-\alpha l}{2}\right)\hat{j}\right]$$

$$= m\alpha l\hat{i} + \left(\frac{-m\alpha l}{2}\hat{j}\right)$$

$$\left(-\frac{mg\ell}{2}\right)\hat{k} = \frac{1}{12}m\ell^2(-\alpha)\hat{k} + \left[\left(\frac{\ell}{2}\hat{i} + \ell\hat{j}\right) \times m\left(\alpha\ell\hat{i} - \frac{m\alpha\ell}{2}\hat{j}\right)\right]$$

$$\alpha = \frac{3}{8}\left(\frac{g}{\ell}\right) \text{ rad/s}^2$$