TAM 212. Final Practice. Apr 29, 2013. (V3)

- There are 20 questions, each worth 1 point.
- This is a 3 hour exam.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- One two-sided sheet of hand-written notes is permitted.
- There are several different versions of this exam.
- Do not turn this page until instructed to do so.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Circle your discussion section:

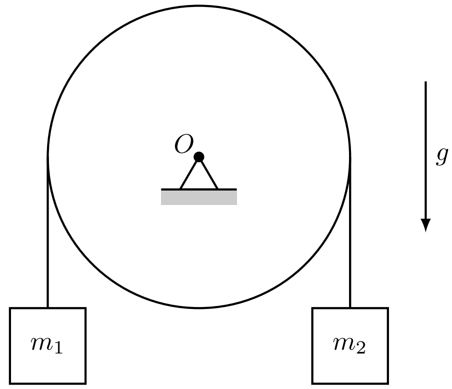
	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

95. D

96. C

1. (1 point) A rigid wheel with radius r and moment of inertia I_O is pinned at point O. An inextensible massless rope connects two masses m_1 and m_2, and moves without slipping on the wheel. Gravity g acts downwards.



At the instant shown, all bodies are stationary and we have:

$$
\begin{aligned}
r &= 2 \text{ m} \\
I_O &= 16 \text{ kg m}^2 \\
m_1 &= 2 \text{ kg} \\
m_2 &= 4 \text{ kg} \\
g &= 10 \text{ m/s}^2
\end{aligned}
$$

What is the magnitude of the angular acceleration $\vec{\alpha}$ of the wheel?

(A) $2 \text{ rad/s}^2 \leq \alpha < 3 \text{ rad/s}^2$

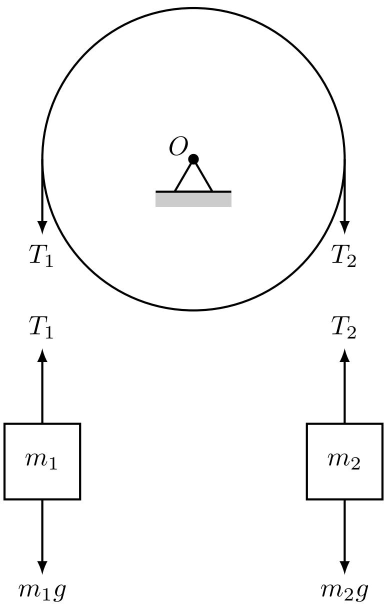
(B) $\alpha = 0 \text{ rad/s}^2$

(C) ★ $1 \text{ rad/s}^2 \leq \alpha < 2 \text{ rad/s}^2$

(D) $0 \text{ rad/s}^2 < \alpha < 1 \text{ rad/s}^2$

(E) $3 \text{ rad/s}^2 \leq \alpha$

Solution. Taking $\vec{\alpha} = \alpha\hat{k}$, we have that the acceleration of mass m_1 is $\vec{a}_1 = -r\alpha\hat{\jmath}$ and that of mass m_2 is $\vec{a}_2 = r\alpha\hat{\jmath}$. The free body diagram is:



Newton's equations for each mass and Euler's equations for the wheel give:

$$\left.\begin{aligned} T_1\hat{\jmath} - m_1 g\hat{\jmath} &= m_1\vec{a}_1 = -m_1 r\alpha\hat{\jmath} \\ T_2\hat{\jmath} - m_2 g\hat{\jmath} &= m_2\vec{a}_2 = m_2 r\alpha\hat{\jmath} \\ T_1 r\hat{k} - T_2 r\hat{k} &= I_O\vec{\alpha} = I_O\alpha\hat{k} \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \alpha &= -1 \text{ rad/s}^2 \\ T_1 &= 24 \text{ N} \\ T_2 &= 32 \text{ N} \end{aligned}\right.$$

The magnitude of the acceleration is thus $\alpha = 1$ rad/s^2.

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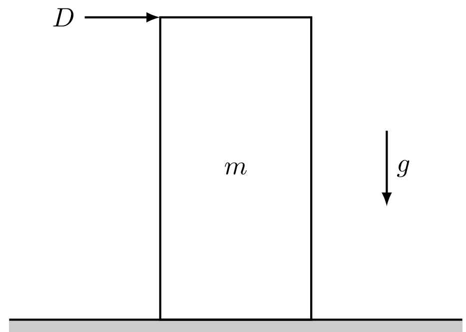
2. (1 point) A uniform rigid rectangular body of mass $m = 6$ kg, width 2 m, and height 4 m sits on a horizontal ground as shown, with gravity $g = 10$ m/s^2 acting vertically. A horizontal force D is applied and it is observed that the body begins to rotate without slipping at an angular acceleration of $\vec{\alpha} = -\hat{k}$ rad/s^2.



What is the minimum value μ of the coefficient of friction between the body and the ground that is consistent with the observed dynamics?

(A) $\frac{5}{6} \leq \mu$

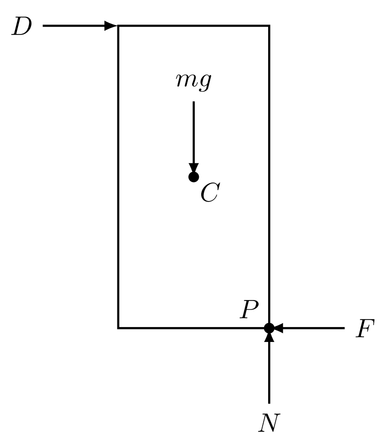
(B) $\frac{2}{6} \leq \mu < \frac{3}{6}$

(C) $\frac{4}{6} \leq \mu < \frac{5}{6}$

(D) $\frac{3}{6} \leq \mu < \frac{4}{6}$

(E) ★ $\mu < \frac{2}{6}$

Solution. The body is pivoting about the lower-right corner P, so the free body diagram with normal force N and friction F is:



The moment of inertia about P is:

$$I_P = I_C + m r_{PC}^2 = \frac{1}{12} 6(2^2 + 4^2) + 6(1^2 + 2^2) = 40 \text{ kg m}^2.$$

The acceleration of the center of mass is:

$$\vec{a}_C = \vec{a}_P + \vec{\alpha} \times \vec{r}_{PC} - \omega^2 \vec{r}_{PC} = 0 - \hat{k} \times (-\hat{\imath} + 2\hat{\jmath}) - 0 = 2\hat{\imath} + \hat{\jmath} \text{ m/s}^2.$$

The force and moment equations are:

$$\left.\begin{aligned} -4D\hat{k} + mg\hat{k} &= I_P\vec{\alpha} \\ D\hat{\imath} - mg\hat{\jmath} + N\hat{\jmath} - F\hat{\imath} &= m\vec{a}_C \end{aligned}\right\} \Longrightarrow \begin{cases} D = 25 \text{ N} \\ N = 66 \text{ N} \\ F = 13 \text{ N} \end{cases}$$

The minimum coefficient of friction is $\mu = \frac{F}{N} = \frac{13}{66}$.

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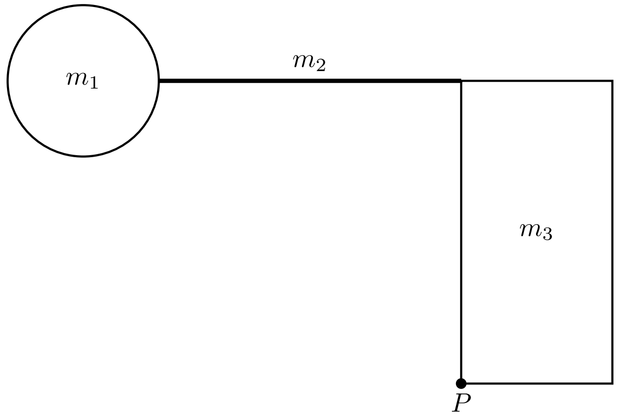
3. (1 point) A rigid body consists of four bodies joined together, as shown below (drawn to scale).



The component bodies are:

i. a uniform disk of radius 1 m and mass $m_1 = 1$ kg

ii. a uniform rod of length 4 m and mass $m_2 = 2$ kg

iii. a uniform rectangle of width 2 m, height 4 m, and mass $m_3 = 9$ kg

iv. a point mass at P with mass $m_4 = 2$ kg

What is the distance r_{PC} from point P to the center of mass C of the entire body?

(A) 2.5 m $\leq r_{PC}$

(B) 1.0 m $\leq r_{PC} <$ 1.5 m

(C) 1.5 m $\leq r_{PC} <$ 2.0 m

(D) $r_{PC} <$ 1.0 m

(E) ★ 2.0 m $\leq r_{PC} <$ 2.5 m

Solution. Total mass is $m = 14$ kg. Relative to P, the center of mass is at:

$$\begin{aligned}\vec{r}_{PC} &= \frac{1}{m}\left(m_1(-5\hat{\imath} + 4\hat{\jmath}) + m_2(-2\hat{\imath} + 4\hat{\jmath}) + m_3(\hat{\imath} + 2\hat{\jmath})\right) \\ &= \frac{15}{7}\hat{\jmath} \text{ m} \\ r_{PC} &= \frac{15}{7} \\ &\approx 2.14 \text{ m.}\end{aligned}$$

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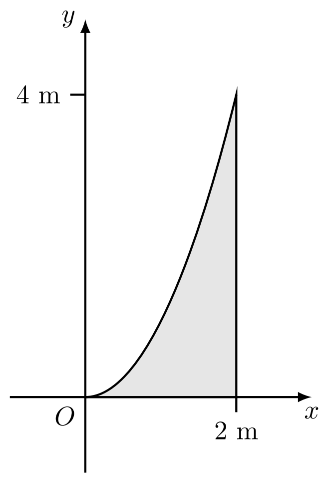
4. (1 point) A body has uniform thickness in the z direction and uniform density, and its shape in the x–y plane is bounded by the curves $y = x^2/\text{m}$, $y = 0$ m, and $x = 2$ m, as shown below.



What is the x coordinate C_x of the center of mass C of the body?

(A) $1.8\ \text{m} \le C_x$

(B) $1.6\ \text{m} \le C_x < 1.7\ \text{m}$

(C) ★ $1.5\ \text{m} \le C_x < 1.6\ \text{m}$

(D) $1.7\ \text{m} \le C_x < 1.8\ \text{m}$

(E) $C_x < 1.5\ \text{m}$

Solution. For thickness h and density ρ, the total mass is

$$\begin{aligned} m &= \int_{0\ \text{m}}^{2\ \text{m}} \rho h (x^2/\text{m})\, dx \\ &= \frac{8}{3}\rho h\ \text{m}^2. \end{aligned}$$

The x coordinate of the center of mass is then:

$$\begin{aligned} C_x &= \frac{1}{m}\int_0^2 \rho h x (x^2/\text{m})\, dx \\ &= \frac{1}{\frac{8}{3}\rho h\ \text{m}^2} 4\rho h\ \text{m}^3 \\ &= 1.5\ \text{m}. \end{aligned}$$

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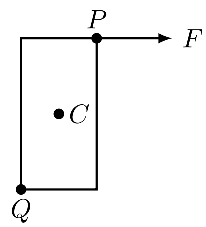
5. (1 point) A rigid 2D body has mass m, moment of inertia I_C and center of mass C, and is acted upon by a force $\vec{F}$ at point P as shown.



At the instant shown, the body is stationary and we have:

$$
\begin{aligned}
m &= 3 \text{ kg} \\
I_C &= 6 \text{ kg m}^2 \\
\vec{F} &= 6\hat{\imath} \text{ N} \\
\vec{r}_{CP} &= \hat{\imath} + 2\hat{\jmath} \text{ m} \\
\vec{r}_{CQ} &= -\hat{\imath} - 2\hat{\jmath} \text{ m.}
\end{aligned}
$$

What is the magnitude of the acceleration $\vec{a}_Q$ of point Q?

(A) $a_Q = 0 \text{ m/s}^2$

(B) ★ $2 \text{ m/s}^2 \le a_Q < 4 \text{ m/s}^2$

(C) $0 \text{ m/s}^2 < a_Q < 2 \text{ m/s}^2$

(D) $6 \text{ m/s}^2 \le a_Q$

(E) $4 \text{ m/s}^2 \le a_Q < 6 \text{ m/s}^2$

Solution.

$$
\begin{aligned}
\vec{a}_C &= \frac{1}{m}\vec{F} = 2\hat{\imath} \text{ m/s}^2 \\
\vec{\alpha} &= \frac{1}{I_C}\vec{M} = \frac{1}{I_C}\vec{r}_{CP} \times \vec{F} = -2\hat{k} \text{ rad/s}^2 \\
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) = 2\hat{\imath} - 2\hat{k} \times (-\hat{\imath} - 2\hat{\jmath}) = -2\hat{\imath} + 2\hat{\jmath} \\
a_Q &= 2\sqrt{2} \approx 2.83 \text{ m/s}^2
\end{aligned}
$$

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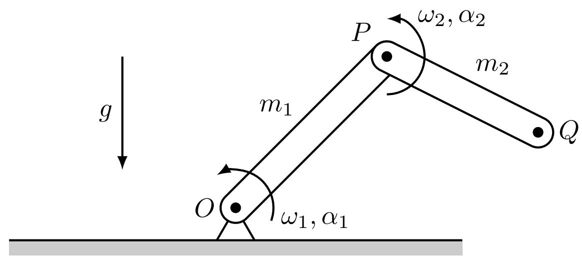
6. (1 point) Two thin uniform rods are connected with pin joints at O, P, and Q as shown, with masses $m_1 = 1$ kg and $m_2 = 2$ kg. The rods are being driven by pure moments applied at pins O and P, resulting in the angular accelerations given below. Gravity $g = 10\ \mathrm{m/s^2}$ acts vertically.



The positions and angular velocities of the rods at the current instant are:

$$\vec{r}_{OP} = 2\hat{\imath} + 2\hat{\jmath}\ \mathrm{m} \qquad \vec{r}_{PQ} = 2\hat{\imath} - \hat{\jmath}\ \mathrm{m}$$
$$\vec{\omega}_1 = \hat{k}\ \mathrm{rad/s} \qquad \vec{\omega}_2 = -2\hat{k}\ \mathrm{rad/s}$$
$$\vec{\alpha}_1 = 0 \qquad \vec{\alpha}_2 = 2\hat{k}\ \mathrm{rad/s^2}$$

What is the $\hat{\jmath}$ component R_y of the reaction force $\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath}$ on the rod at point O?

(A) $R_y = 27$ N

(B) $R_y = 30$ N

(C) $R_y = 25$ N

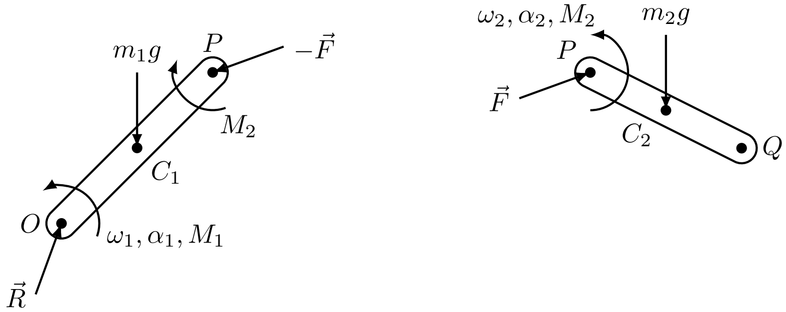
(D) ★ $R_y = 33$ N

(E) $R_y = 35$ N

Solution. Starting from the fixed point O and taking C_1 and C_2 to be the centers of the two rods, we have:

$$\vec{a}_{C_1} = \vec{a}_O + \vec{\alpha}_1 \times \frac{1}{2}\vec{r}_{OP} - \omega_1^2 \frac{1}{2}\vec{r}_{OP} = -\hat{\imath} - \hat{\jmath}$$
$$\vec{a}_P = \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OP} - \omega_1^2 \vec{r}_{OP} = -2\hat{\imath} - 2\hat{\jmath}$$
$$\vec{a}_{C_2} = \vec{a}_P + \vec{\alpha}_2 \times \frac{1}{2}\vec{r}_{PQ} - \omega_2^2 \frac{1}{2}\vec{r}_{PQ} = -5\hat{\imath} + 2\hat{\jmath}$$

Now the free body diagram is:



Taking Newton's equations for each rod gives:

$$\left.\begin{aligned} \vec{F} - m_2 g\hat{\jmath} &= m_2 \vec{a}_{C_2} \\ -\vec{F} + \vec{R} - m_1 g\hat{\jmath} &= m_1 \vec{a}_{C_1} \end{aligned}\right\} \Longrightarrow \begin{cases} \vec{F} = -10\hat{\imath} + 24\hat{\jmath} \text{ N} \\ \vec{R} = -11\hat{\imath} + 33\hat{\jmath} \text{ N} \end{cases}$$

Thus $R_y = 33$ N.

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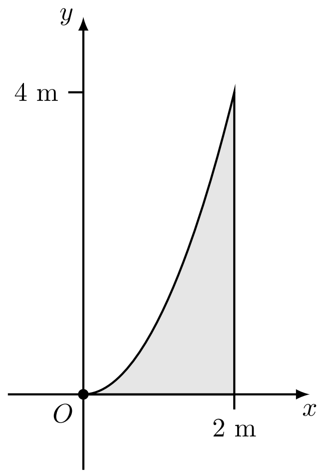
7. (1 point) A body has uniform thickness in the z direction and uniform density, and its shape in the x–y plane is bounded by the curves $y = x^2/\text{m}$, $y = 0$ m, and $x = 2$ m, as shown below. The total mass of the body is m.



What is the moment of inertia $I_{O,\hat{k}}$ about the $\hat{k}$ axis through the origin O?

(A) ★ $4m\ \text{m}^2 \leq I_{O,\hat{k}} < 6m\ \text{m}^2$

(B) $0\ \text{m}^2 \leq I_{O,\hat{k}} < 2m\ \text{m}^2$

(C) $6m\ \text{m}^2 \leq I_{O,\hat{k}} < 8m\ \text{m}^2$

(D) $2m\ \text{m}^2 \leq I_{O,\hat{k}} < 4m\ \text{m}^2$

(E) $8m\ \text{m}^2 \leq I_{O,\hat{k}}$

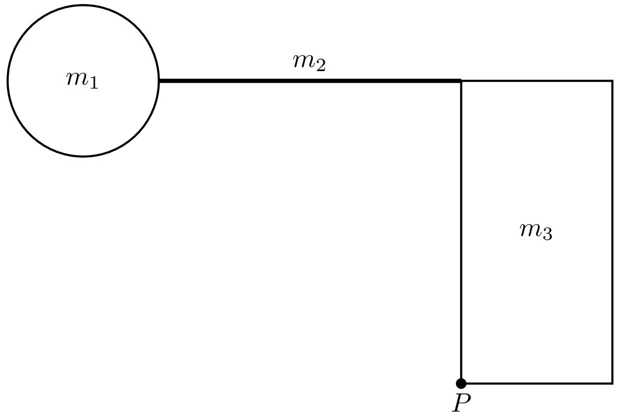
Solution. The area of the body is

$$A = \int_{0\ \text{m}}^{2\ \text{m}} \frac{x^2}{\text{m}}\,dx = \frac{8}{3}\ \text{m}^2.$$

For thickness h the density of the body is thus $\rho = m/(Ah)$. Consider now that a point with coordinates x, y has distance r to O, where $r^2 = x^2 + y^2$. Then the moment of inertia is:

$$\begin{aligned}
I_{O,\hat{k}} &= \int_{0\ \text{m}}^{2\ \text{m}} \int_{0\ \text{m}}^{x^2/\text{m}} h\rho(x^2 + y^2)\,dy\,dx \\
&= \frac{m}{A} \int_{0\ \text{m}}^{2\ \text{m}} \left[x^2 y + \frac{1}{3}y^3\right]_{0\ \text{m}}^{x^2/\text{m}} dx \\
&= \frac{m}{A} \int_{0\ \text{m}}^{2\ \text{m}} \left(x^4/\text{m} + \frac{1}{3}x^6/\text{m}^3\right) dx \\
&= \frac{m}{A} \left[\frac{1}{5}x^5/\text{m} + \frac{1}{21}x^7/\text{m}^3\right]_{0\ \text{m}}^{2\ \text{m}} \\
&= \frac{m}{A} \left(\frac{1}{5}32\ \text{m}^4 + \frac{1}{21}128\ \text{m}^4\right) \\
&= 4\frac{24}{35}m\ \text{m}^2 \\
&\approx 4.69m\ \text{m}^2
\end{aligned}$$

8. (1 point) A rigid body consists of four bodies joined together, as shown below (drawn to scale).



The component bodies are:

i. a uniform disk of radius 1 m and mass $m_1 = 1$ kg

ii. a uniform rod of length 4 m and mass $m_2 = 2$ kg

iii. a uniform rectangle of width 2 m, height 4 m, and mass $m_3 = 9$ kg

iv. a point mass at P with mass $m_4 = 2$ kg

What is the moment of inertia $I_{P,\hat{k}}$ about the $\hat{k}$ axis through the point P?

(A) $I_{P,\hat{k}} < 100$ kg m^2

(B) 200 kg m$^2 \leq I_{P,\hat{k}} < 300$ kg m^2

(C) 300 kg m$^2 \leq I_{P,\hat{k}} < 400$ kg m^2

(D) 400 kg m$^2 \leq I_{P,\hat{k}}$

(E) ★ 100 kg m$^2 \leq I_{P,\hat{k}} < 200$ kg m^2

Solution.

$$\begin{aligned} I_{P,\hat{k}} &= \frac{1}{2} m_1 1^2 + m_1(5^2 + 4^2) \\ &\quad + \frac{1}{12} m_2 4^2 + m_2(2^2 + 4^2) \\ &\quad + \frac{1}{12} m_3(2^2 + 4^2) + m_3(1^2 + 2^2) \\ &= 144\frac{1}{6} \text{ kg m}^2. \end{aligned}$$

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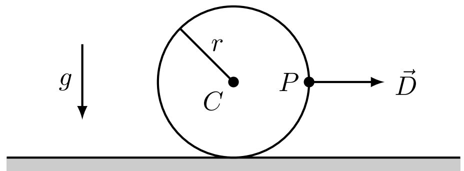
9. (1 point) A circular rigid body with center of mass C, mass $m = 2$ kg, moment of inertia $I_C = 1$ kg m^2, and radius $r = 1$ m is sitting on the ground as shown. The coefficient of friction between the body and the ground is $\mu = 0.1$. A driving force $\vec{D} = 3\hat{\imath}$ N acts at point P, and gravity $g = 10$ m/s^2 acts vertically.



What is the magnitude of the friction force $\vec{F}$?

(A) $F = 0$ N

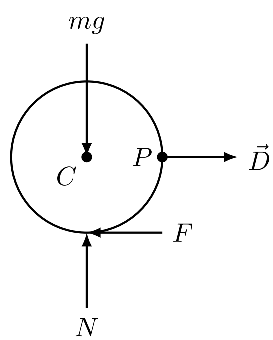
(B) ★ $F = 1$ N

(C) $F = 2$ N

(D) $F = 4$ N

(E) $F = 3$ N

Solution. With friction $\vec{F} = -F\hat{\imath}$ and normal force $\vec{N} = N\hat{\jmath}$, the free body diagram is:



Assuming sticking and taking $\vec{a} = a\hat{\imath}$ and $\vec{\alpha} = \alpha\hat{k}$, we have:

$$\left.\begin{aligned} \vec{D} - F\hat{\imath} + N\hat{\jmath} - mg\hat{\jmath} &= m\vec{a} = ma\hat{\imath} \\ -Fr\hat{k} &= I_C\vec{\alpha} = I_C\alpha\hat{k} \\ a &= -r\alpha \end{aligned}\right\} \implies \begin{cases} N = 20 \text{ N} \\ F = 1 \text{ N} \\ a = 1 \text{ m/s}^2 \\ \alpha = -1 \text{ rad/s}^2 \end{cases}$$

Checking the Coulomb friction condition gives:

$$\begin{aligned} |F| &\overset{?}{\leq} \mu|N| \\ 1 &\overset{?}{\leq} 0.1 \times 20 \\ 1 &\leq 2 \end{aligned}$$

Thus it is sticking and $F = 1$ N.

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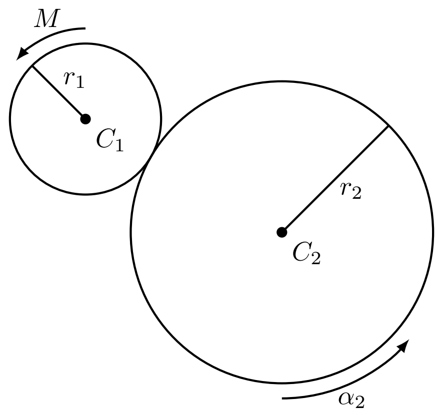
10. (1 point) Two meshed gears rotate about fixed centers as shown. The radii are $r_1 = 2$ m and $r_2 = 4$ m and the gears have moments of inertia $I_{C_1} = 1$ kg m^2 and $I_{C_2} = 4$ kg m^2, respectively. A pure moment $\vec{M} = 2\hat{k}$ N m is applied to the first gear, and this produces an angular acceleration of $\vec{\alpha}_2 = \alpha_2 \hat{k}$ for the second gear.



What is α_2?

(A) $1 \text{ rad/s}^2 \le \alpha_2$

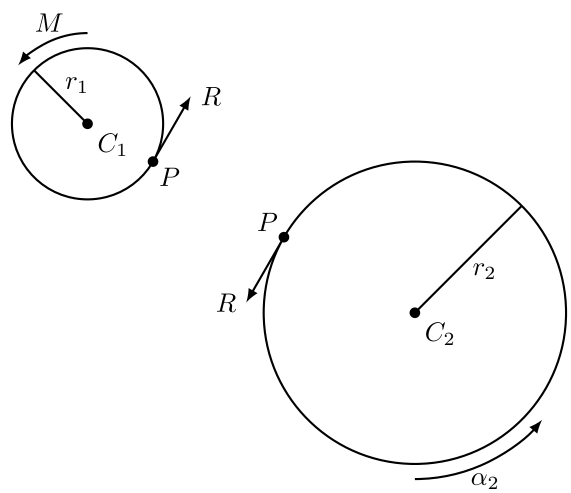
(B) ★ $-1 \text{ rad/s}^2 \le \alpha_2 < 0 \text{ rad/s}^2$

(C) $\alpha_2 = 0 \text{ rad/s}^2$

(D) $0 \text{ rad/s}^2 < \alpha_2 < 1 \text{ rad/s}^2$

(E) $\alpha_2 < -1 \text{ rad/s}^2$

Solution. Taking R to be the reaction force at the contact point P, the free body diagram is:



Taking angular accelerations $\vec{\alpha}_1 = \alpha_1 \hat{k}$ and $\vec{\alpha}_2 = \alpha_2 \hat{k}$, matching tangential accelerations at P gives $r_1 \alpha_1 = -r_2 \alpha_2$. Euler's equations are now:

$$\left.\begin{aligned} M\hat{k} + R r_1 \hat{k} &= I_{C_1} \alpha_1 \hat{k} \\ R r_2 \hat{k} &= I_{C_2} \alpha_2 \hat{k} \\ r_1 \alpha_1 &= -r_2 \alpha_2 \end{aligned}\right\} \implies \left\{\begin{aligned} \alpha_1 &= 1 \text{ rad/s}^2 \\ \alpha_2 &= -0.5 \text{ rad/s}^2 \\ R &= -0.5 \text{ N} \end{aligned}\right.$$