TAM 212. Midterm 1. Feb 21, 2013.

- There are 20 questions, each worth 5 points.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Circle your discussion section:

	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

94. A

95. D

96. C

1. (5 points) Points P and Q are moving in circular paths around the origin O with angular velocities ω_P and ω_Q and speeds v_P and v_Q, respectively.

The two particles are moving with the same angular velocity, so $\omega_P = \omega_Q$. Which statement is true?

(A) $2v_Q < v_P$

(B) $\frac{1}{2}v_Q < v_P \leq v_Q$

(C) ★ $v_P \leq \frac{1}{2}v_Q$

(D) $v_Q < v_P \leq 2v_Q$

Solution. If the particles both have the same angular velocity then we can observe from the diagram that r_P is much smaller than r_Q, so v_P is much smaller than ω_Q.
More precisely:

$$\omega = \frac{v_P}{r_P}$$
$$\omega = \frac{v_Q}{r_Q}$$
$$v_P = \frac{r_P}{r_Q}v_Q.$$

From the diagram, $\frac{r_P}{r_Q} \approx \frac{1}{4}$, so $v_P \approx \frac{1}{4}v_Q$, which is less than $\frac{1}{2}v_Q$.

2. (5 points) A particle P has position, velocity, and acceleration vectors given by:

$$\vec{r} = -2\hat{\imath} + 3\hat{\jmath} \text{ m}$$
$$\vec{v} = -4\hat{\imath} - \hat{\jmath} \text{ m/s}$$
$$\vec{a} = 2\hat{\imath} + 6\hat{\jmath} \text{ m/s}^2.$$

Consider the following statements:

(i) The particle is moving closer to the origin.

(ii) The particle is moving further from the origin.

(iii) The particle is speeding up.

(iv) The particle is slowing down.

Which statements are true?

(A) none of the other options

(B) ★ (ii) and (iv)

(C) (ii) and (iii)

(D) (i) and (iii)

(E) (i) and (iv)

Solution. $\vec{v} \cdot \vec{r} = 5 > 0$ so $\dot{r} > 0$ (moving further from the origin), and $\vec{a} \cdot \vec{v} = -14 < 0$ so $\dot{v} < 0$ (slowing down).

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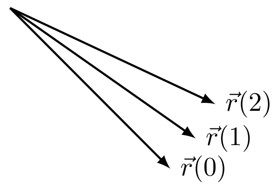
3. (5 points) The position vector $\vec{r}(t)$ of a point is shown below at $t = 0$ s, $t = 1$ s, and $t = 2$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

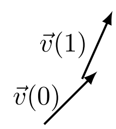
(A) ★ $\nwarrow$

(B) $\swarrow$

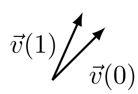
(C) $\nearrow$

(D) $\searrow$

Solution. This closely resembles uniform circular motion, so the acceleration is inwards towards the center. Alternatively, the velocity directions are roughly:



Re-drawing these with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly to the left and up.

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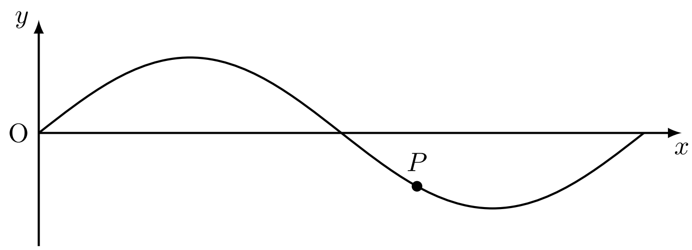
4. (5 points) A particle is moving to the left along a variable-height ground with ground height given by $y(x) = \sin(x/20)$ m. The particle's horizontal velocity component is a constant $v_x = -4$ m/s. What is the vertical component of velocity v_y when $x = 25\pi$ m?



(A) $v_y < -1$ m/s

(B) ★ $0 \text{ m/s} \leq v_y < 1$ m/s

(C) $v_y = 0$ m/s

(D) $-1 \text{ m/s} \leq v_y < 0$ m/s

(E) $1 \text{ m/s} \leq v_y$

Solution. Using $\dot{x} = v_x = -4$ m/s, we have:

$$\begin{aligned} y &= \sin(x/20) \\ v_y = \dot{y} &= \cos(x/20)\frac{\dot{x}}{20} \\ &= \cos(25\pi/20)\frac{-4}{20} \\ &= \cos(5\pi/4)\frac{-1}{5} \\ &= -\frac{1}{\sqrt{2}}\frac{-1}{5} \\ &\approx 0.14 \text{ m/s}. \end{aligned}$$

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5. (5 points) A car driving down the road travels a distance $s = \frac{3}{2}t^2$ m from its starting point. At $t = 1$ s the car is driving around a curve and the magnitude of its acceleration is $a = 5\ \mathrm{m/s^2}$. What is the radius of curvature ρ of the curve?

(A) $1\ \mathrm{m} \le \rho < 2\ \mathrm{m}$

(B) $3\ \mathrm{m} \le \rho < 4\ \mathrm{m}$

(C) ★ $2\ \mathrm{m} \le \rho < 3\ \mathrm{m}$

(D) $4\ \mathrm{m} \le \rho$

(E) $0\ \mathrm{m} \le \rho < 1\ \mathrm{m}$

Solution. We have $\dot{s}(t) = 3t$ and $\ddot{s} = 3$, so $\dot{s}(1) = 3$. The acceleration is:

$$\begin{aligned}
\vec{a} &= \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\,\hat{e}_n \\
&= 3\,\hat{e}_t + \frac{3^2}{\rho}\,\hat{e}_n \\
5^2 &= 3^2 + \frac{9^2}{\rho^2} \\
\rho^2 &= \frac{9^2}{4^2} \\
\rho &= \frac{9}{4} = 2.25\ \mathrm{m}.
\end{aligned}$$

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6. (5 points) The particle P has polar coordinates $r = 4$ m, $\theta = -135°$ and velocity $\vec{v} = \hat{\jmath}$. Which statement is true?

(A) $\dot{r} < 0$ and $\dot{\theta} \geq 0$

(B) $\dot{r} \geq 0$ and $\dot{\theta} \geq 0$

(C) ★ $\dot{r} < 0$ and $\dot{\theta} < 0$

(D) $\dot{r} \geq 0$ and $\dot{\theta} < 0$

Solution. The fastest way to solve this question is to draw a diagram and directly see that this velocity is causing r to decrease and θ to decrease.
Alternatively, we can convert $\vec{v}$ to polar coordinates to find:

$$\begin{aligned}\vec{v} &= \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta \\ &= -\frac{1}{\sqrt{2}}\,\hat{e}_r - \frac{1}{\sqrt{2}}\,\hat{e}_\theta.\end{aligned}$$

Thus we have a negative velocity component in the $\hat{e}_r$ direction (decreasing r), and a negative velocity component in the $\hat{e}_\theta$ direction (decreasing θ).

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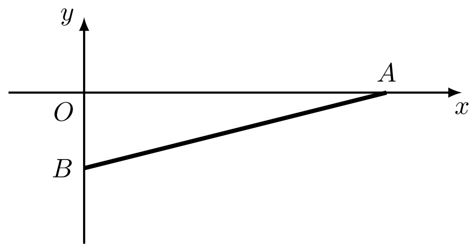
7. (5 points) A rod with fixed length is positioned as shown, with x the horizontal coordinate of end A and y the vertical coordinate of end B. End A can only move horizontally, while end B can only move vertically.



At an instant of time, we have $x = 4$ m and $y = -1$ m, and the rod is moving so that $\dot{y} = -2$ m/s. What is $\dot{x}$?

(A) $\dot{x} < -1$ m/s

(B) ★ -1 m/s $\leq \dot{x} < 0$ m/s

(C) 0 m/s $< \dot{x} < 1$ m/s

(D) 1 m/s $\leq \dot{x}$

(E) $\dot{x} = 0$ m/s

Solution. If the fixed length of the rod is ℓ, then we have:

$$\begin{aligned}
\ell^2 &= x^2 + y^2 \\
0 &= 2x\dot{x} + 2y\dot{y} \\
\dot{x} &= -\frac{y}{x}\dot{y} \\
&= -\frac{-1}{4}(-2) \\
&= -0.5 \text{ m/s.}
\end{aligned}$$

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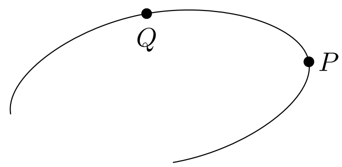
8. (5 points) A point is moving around the curve shown below with varying speed.



The radius of curvature and normal acceleration at P and Q are given by:

$$\rho_P = 2 \text{ m} \qquad\qquad \rho_Q = 4 \text{ m}$$
$$a_{P,n} = 4 \text{ m/s}^2 \qquad\qquad a_{Q,n} = 3 \text{ m/s}^2.$$

Which of the following is true about the velocities v_P at P and v_Q at Q?

(A) ★ $v_P < v_Q$

(B) $v_P = v_Q$

(C) $v_P > v_Q$

Solution. Using $a_n = v^2/\rho$ we find that $v_P^2 = 8$ and $v_Q^2 = 12$, so v_P is smaller.

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9. (5 points) A point is currently at position $x = -2$ m, $y = 1$ m, $z = 4$ m and is rotating in the x–y plane about the origin with angular velocity $\vec{\omega} = -3\hat{k}$ rad/s. The velocity $\vec{v}$ of the point is:

(A) $\vec{v} = -3\hat{\imath} - 6\hat{\jmath}$ m/s

(B) $\vec{v} = 3\hat{\imath} - 6\hat{\jmath}$ m/s

(C) $\vec{v} = -3\hat{\imath} + 6\hat{\jmath}$ m/s

(D) ★ $\vec{v} = 3\hat{\imath} + 6\hat{\jmath}$ m/s

Solution. The position vector is $\vec{r} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ m, so:

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r} \\ &= -3\hat{k} \times (-2\hat{\imath} + \hat{\jmath} + 4\hat{k}) \\ &= 3\hat{\imath} + 6\hat{\jmath} \text{ m/s.}\end{aligned}$$

---

10. (5 points) A particle moves so that its position in polar coordinates is given by

$$r = (3 + \sin t) \text{ m}$$
$$\theta = t \text{ rad.}$$

What is the $\hat{\jmath}$ component of velocity v_y at $t = \pi$ s?

(A) $0 \text{ m/s} < v_y < 2 \text{ m/s}$

(B) $-2 \text{ m/s} \le v_y < 0 \text{ m/s}$

(C) ★ $v_y < -2 \text{ m/s}$

(D) $2 \text{ m/s} \le v_y$

(E) $v_y = 0 \text{ m/s}$

Solution. We have:

$$r(t) = 3 + \sin t \qquad \theta(t) = t$$
$$\dot{r}(t) = \cos t \qquad \dot{\theta}(t) = 1.$$

At $t = 2$ this is:

$$r(\pi) = 3 \qquad \theta(\pi) = \pi$$
$$\dot{r}(\pi) = -1 \qquad \dot{\theta}(\pi) = 1.$$

The velocity is:

$$\begin{aligned} \vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \\ &= -\hat{e}_r + 3\,\hat{e}_\theta. \end{aligned}$$

Using $\theta = \pi$ we see that $\hat{e}_\theta = -\hat{\jmath}$, so the $\hat{\jmath}$ component is:

$$v_y = -3 \text{ m/s}.$$

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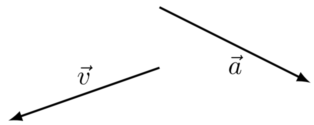
11. (5 points) The velocity $\vec{v}$ and acceleration $\vec{a}$ for a single particle P are shown below at a particular instant.



Which statement is true at this instant?

(A) ★ the particle is slowing down

(B) the particle's speed is not changing

(C) the particle is speeding up

Solution. The component of $\vec{a}$ in the direction of $\vec{v}$ is negative, so $\dot{v} < 0$.

---

12. (5 points) A particle starts at the origin at time $t = 0$ s and its velocity is given by

$$\vec{v} = 4t^3\,\hat{\imath} - t^2\,\hat{\jmath} \text{ m/s}.$$

At time $t = 1$ s, what is the particle's distance r from the origin?

(A) $2 \text{ m} \leq r < 4 \text{ m}$

(B) $6 \text{ m} \leq r < 8 \text{ m}$

(C) $4 \text{ m} \leq r < 6 \text{ m}$

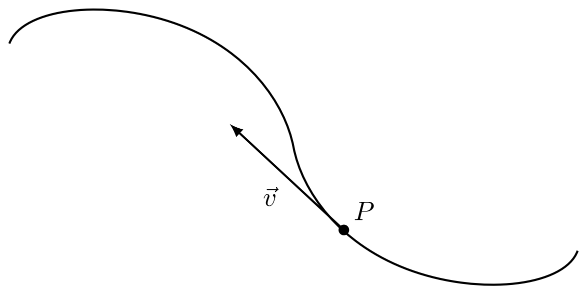
(D) ★ $0 \text{ m} \leq r < 2 \text{ m}$

(E) $8 \text{ m} \leq r$

Solution. $\vec{r} = \vec{r}(0) + \int_0^t \vec{v}(\tau)\,d\tau = t^4\,\hat{\imath} - \frac{1}{3}t^3\,\hat{\jmath}$, so $\vec{r}(1) = \hat{\imath} - \frac{1}{3}\hat{\jmath}$ m. Then $r(1)$ is more than 1 but less than 2 (in fact it's $r(1) \approx 1.05$ m).

---

13. (5 points) A car is driving on a curved track with the top view shown below. At a given instant the car is at point P with velocity $\vec{v}$ and its speed is decreasing, such that the tangential and normal components of its acceleration are equal in magnitude.



Which direction is the closest to the direction of the acceleration $\vec{a}$ at the instant shown?

(A) ★ $\rightarrow$

(B) $\downarrow$

(C) $\leftarrow$

(D) $\uparrow$

Solution. $\hat{e}_t$ is up-left, so $-\hat{e}_t$ is down-right and $\hat{e}_n$ is up-right, so equal components in these two directions give a total rightwards acceleration.

---

14. (5 points) A car is observed moving in the plane with velocity $\vec{v} = -2\hat{\imath} + 4\hat{\jmath}$ m/s and acceleration $\vec{a} = -\hat{\imath} - 3\hat{\jmath}$ m/s^2. At this instant, is it:

(A) stationary

(B) driving around a curve clockwise

(C) driving in a straight line

(D) ★ driving around a curve counterclockwise

Solution. Drawing a diagram shows it is curving counterclockwise.
Alternatively, $\vec{v} \times \vec{a} = 10\hat{k}$ which is in the $+\hat{k}$ direction, so it is curving in the positive direction (counterclockwise).

---

15. (5 points) If a particle has position vector $\vec{r}(t) = 2\cos(2t)\,\hat{\imath} + \sin(3t)\,\hat{\jmath} + (t^3 - 2t)\,\hat{k}$ m, what is its speed $v(0)$ at time $t = 0$ s?

(A) $v(0) = 0$ m/s

(B) 1 m/s $\leq v(0) < 3$ m/s

(C) 0 m/s $< v(0) < 1$ m/s

(D) ★ 3 m/s $\leq v(0) < 5$ m/s

(E) 5 m/s $\leq v(0)$

Solution. The velocity is $\vec{v} = \dot{\vec{r}} = -4\sin(2t)\,\hat{\imath} + 3\cos(3t)\,\hat{\jmath} + (3t^2 - 2)\,\hat{k}$, so $\vec{v}(0) = 3\hat{\jmath} - 2\hat{k}$. The magnitude $v(0)$ is more than 3 but less than $3 + 2 = 5$.

---

16. (5 points) A point is moving with position vector

$$\vec{r} = (t^2 - 2t)\,\hat{\imath} + (e^{2t} - 2t)\,\hat{\jmath} \text{ m}.$$

What is the radius of curvature ρ at $t = 0$ s?

(A) $\frac{1}{2}$ m $\leq \rho < 1$ m

(B) $\frac{3}{2}$ m $\leq \rho < 2$ m

(C) 2 m $\leq \rho$

(D) ★ 1 m $\leq \rho < \frac{3}{2}$ m

(E) 0 m $\leq \rho < \frac{1}{2}$ m

Solution. The position and velocity are:

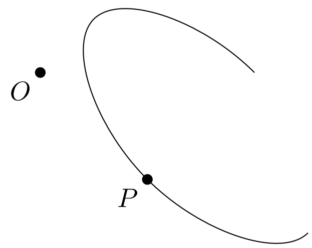
$$\begin{aligned}
\vec{r} &= (t^2 - 2t)\,\hat{\imath} + (e^{2t} - 2t)\,\hat{\jmath} = \hat{\jmath} \\
\vec{v} &= (2t - 2)\,\hat{\imath} + (2e^{2t} - 2)\,\hat{\jmath} = -2\hat{\imath} \\
\vec{a} &= 2\hat{\imath} + 4e^{2t}\,\hat{\jmath} = 2\hat{\imath} + 4\hat{\jmath}.
\end{aligned}$$

Thus $\hat{e}_t = -\hat{\imath}$ and $\hat{e}_n = \hat{\jmath}$, so $a_n = 4$. Also $v = 2$, so:

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
4 &= \frac{2^2}{\rho} \\
\rho &= 1 \text{ m}.
\end{aligned}$$

---

17. (5 points) A point is moving around the curve shown and is currently at position P. Consider a polar basis $\hat{e}_r, \hat{e}_\theta$ at P from the origin O and a tangential/normal basis $\hat{e}_t, \hat{e}_n$ at P.



Which of the following is true?

(A) $\hat{e}_n = -\hat{e}_r$

(B) $\hat{e}_n = -\hat{e}_\theta$

(C) $\hat{e}_n = \hat{e}_r$

(D) ★ $\hat{e}_n = \hat{e}_\theta$

Solution. Here $\hat{e}_r$ is to the down-right and $\hat{e}_\theta$ is up-right. Also $\hat{e}_n$ is inwards to the curve, so is up-right, which is $\hat{e}_\theta$.

---

18. (5 points) A position P has an associated polar basis with

$$\hat{e}_\theta = -\frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath}.$$

What is θ?

(A) $0 \leq \theta < \frac{1}{2}\pi$

(B) $\frac{3}{2}\pi \leq \theta < 2\pi$

(C) $\pi \leq \theta < \frac{3}{2}\pi$

(D) ★ $\frac{1}{2}\pi \leq \theta < \pi$

Solution. The fastest way to solve this question is to draw a diagram and immediately see that θ must be in the second quadrant.
Alternatively, we can use the formula $\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ to see that $-\sin\theta = -\frac{1}{2}$, so $\theta = 30°$ or $\theta = 150°$. Also $\cos\theta = -\frac{\sqrt{3}}{2}$ so $\theta = \pm 150°$. The only common solution is $\theta = 150° = \frac{5}{6}\pi$.

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19. (5 points) A particle is moving in the plane so that at a particular instant its polar coordinates have $\dot{r} = 3$ m/s and $\dot{\theta} = -4$ rad/s. The speed is $v = 5$ m/s. What is r?

(A) $0 \text{ m} \leq r < 1 \text{ m}$

(B) $3 \text{ m} \leq r < 4 \text{ m}$

(C) ★ $1 \text{ m} \leq r < 2 \text{ m}$

(D) $4 \text{ m} \leq r$

(E) $2 \text{ m} \leq r < 3 \text{ m}$

Solution. Using $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ we have:

$$\begin{aligned}
v &= \sqrt{v_r^2 + v_\theta^2} \\
5 &= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \\
25 &= 3^2 + r^2(-4)^2 \\
16r^2 &= 16 \\
r &= 1 \text{ m}.
\end{aligned}$$

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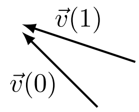
20. (5 points) The velocity $\vec{v}(t)$ of a point is shown below at $t = 0$ s and $t = 1$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

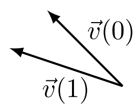
(A) $\nwarrow$

(B) ★ $\swarrow$

(C) $\searrow$

(D) $\nearrow$

Solution. Re-drawing the velocities with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly left-down.

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3. Fill in the following answers on the Scantron form:

94. B

95. E

96. D

1. (5 points) A particle is moving in the plane so that at a particular instant its polar coordinates have $\dot{r} = 3$ m/s and $\dot{\theta} = -1$ rad/s. The speed is $v = 5$ m/s. What is r?

(A) $3\text{ m} \le r < 4\text{ m}$

(B) $1\text{ m} \le r < 2\text{ m}$

(C) $2\text{ m} \le r < 3\text{ m}$

(D) ★ $4\text{ m} \le r$

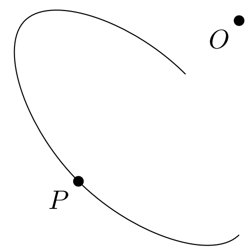
(E) $0\text{ m} \le r < 1\text{ m}$

Solution. Using $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ we have:

$$\begin{aligned}
v &= \sqrt{v_r^2 + v_\theta^2} \\
5 &= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \\
25 &= 3^2 + r^2(-1)^2 \\
r^2 &= 16 \\
r &= 4\text{ m.}
\end{aligned}$$

---

2. (5 points) A point is moving around the curve shown and is currently at position P. Consider a polar basis $\hat{e}_r, \hat{e}_\theta$ at P from the origin O and a tangential/normal basis $\hat{e}_t, \hat{e}_n$ at P.



Which of the following is true?

(A) $\hat{e}_n = \hat{e}_\theta$

(B) $\hat{e}_n = \hat{e}_r$

(C) $\hat{e}_n = -\hat{e}_\theta$

(D) ★ $\hat{e}_n = -\hat{e}_r$

Solution. Here $\hat{e}_r$ is to the down-left and $\hat{e}_\theta$ is down-right. Also $\hat{e}_n$ is inwards to the curve, so is up-right, which is opposite to $\hat{e}_r$.

---

3. (5 points) A particle P has position, velocity, and acceleration vectors given by:

$$\begin{aligned}\vec{r} &= \hat{\imath} - 3\hat{\jmath} \text{ m} \\ \vec{v} &= -2\hat{\imath} - 5\hat{\jmath} \text{ m/s} \\ \vec{a} &= 2\hat{\imath} - 2\hat{\jmath} \text{ m/s}^2.\end{aligned}$$

Consider the following statements:

(i) The particle is moving closer to the origin.

(ii) The particle is moving further from the origin.

(iii) The particle is speeding up.

(iv) The particle is slowing down.

Which statements are true?

(A) ★ (ii) and (iii)

(B) (i) and (iii)

(C) (ii) and (iv)

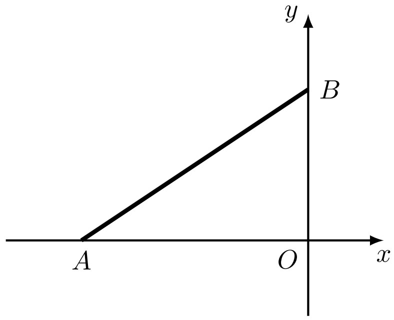
(D) none of the other options

(E) (i) and (iv)

Solution. $\vec{v} \cdot \vec{r} = 13 > 0$ so $\dot{r} > 0$ (moving further from the origin), and $\vec{a} \cdot \vec{v} = 6 > 0$ so $\dot{v} > 0$ (speeding up).

---

4. (5 points) A rod with fixed length is positioned as shown, with x the horizontal coordinate of end A and y the vertical coordinate of end B. End A can only move horizontally, while end B can only move vertically.



At an instant of time, we have $x = -3$ m and $y = 2$ m, and the rod is moving so that $\dot{y} = 2$ m/s. What is $\dot{x}$?

(A) ★ $1 \text{ m/s} \le \dot{x}$

(B) $\dot{x} = 0$ m/s

(C) $-1 \text{ m/s} \le \dot{x} < 0$ m/s

(D) $\dot{x} < -1$ m/s

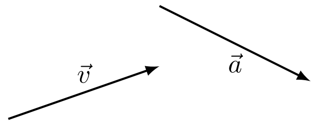
(E) $0 \text{ m/s} < \dot{x} < 1$ m/s

Solution. If the fixed length of the rod is ℓ, then we have:

$$\begin{aligned} \ell^2 &= x^2 + y^2 \\ 0 &= 2x\dot{x} + 2y\dot{y} \\ \dot{x} &= -\frac{y}{x}\dot{y} \\ &= -\frac{2}{-3}2 \\ &\approx 1.33 \text{ m/s.} \end{aligned}$$

---

5. (5 points) The velocity $\vec{v}$ and acceleration $\vec{a}$ for a single particle P are shown below at a particular instant.



Which statement is true at this instant?

(A) the particle's speed is not changing

(B) the particle is slowing down

(C) ★ the particle is speeding up

Solution. The component of $\vec{a}$ in the direction of $\vec{v}$ is positive, so $\dot{v} > 0$.

---

6. (5 points) A particle starts at the origin at time $t = 0$ s and its velocity is given by

$$\vec{v} = -t\,\hat{\imath} + 3t^2\,\hat{\jmath} \text{ m/s}.$$

At time $t = 2$ s, what is the particle's distance r from the origin?

(A) $6 \text{ m} \leq r < 8 \text{ m}$

(B) ★ $8 \text{ m} \leq r$

(C) $4 \text{ m} \leq r < 6 \text{ m}$

(D) $2 \text{ m} \leq r < 4 \text{ m}$

(E) $0 \text{ m} \leq r < 2 \text{ m}$

Solution. $\vec{r} = \vec{r}(0) + \int_0^t \vec{v}(\tau)\,d\tau = -\frac{1}{2}t^2\,\hat{\imath} + t^3\,\hat{\jmath}$, so $\vec{r}(2) = -2\,\hat{\imath} + 8\,\hat{\jmath}$ m. Then $r(2)$ is more than 8 but less than $8 + 2 = 10$ (in fact it's $r(2) \approx 8.25$ m).

---

7. (5 points) A particle moves so that its position in polar coordinates is given by

$$
\begin{aligned}
r &= (3 + \cos t) \text{ m} \\
\theta &= t \text{ rad.}
\end{aligned}
$$

What is the $\hat{\jmath}$ component of velocity v_y at $t = \pi$ s?

(A) $v_y < -2$ m/s

(B) $v_y = 0$ m/s

(C) $2 \text{ m/s} \leq v_y$

(D) $0 \text{ m/s} < v_y < 2$ m/s

(E) ★ $-2 \text{ m/s} \leq v_y < 0$ m/s

Solution. We have:

$$
\begin{aligned}
r(t) &= 3 + \cos t & \theta(t) &= t \\
\dot{r}(t) &= -\sin t & \dot{\theta}(t) &= 1.
\end{aligned}
$$

At $t = 2$ this is:

$$
\begin{aligned}
r(\pi) &= 2 & \theta(\pi) &= \pi \\
\dot{r}(\pi) &= 0 & \dot{\theta}(\pi) &= 1.
\end{aligned}
$$

The velocity is:

$$
\begin{aligned}
\vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \\
&= 2\,\hat{e}_\theta.
\end{aligned}
$$

Using $\theta = \pi$ we see that $\hat{e}_\theta = -\hat{\jmath}$, so the $\hat{\jmath}$ component is:

$$v_y = -2 \text{ m/s.}$$

---

8. (5 points) A point is currently at position $x = -2$ m, $y = -1$ m, $z = 4$ m and is rotating in the x–y plane about the origin with angular velocity $\vec{\omega} = -3\hat{k}$ rad/s. The velocity $\vec{v}$ of the point is:

(A) $\vec{v} = 3\hat{\imath} + 6\hat{\jmath}$ m/s

(B) ★ $\vec{v} = -3\hat{\imath} + 6\hat{\jmath}$ m/s

(C) $\vec{v} = -3\hat{\imath} - 6\hat{\jmath}$ m/s

(D) $\vec{v} = 3\hat{\imath} - 6\hat{\jmath}$ m/s

Solution. The position vector is $\vec{r} = -2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ m, so:

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r} \\ &= -3\hat{k} \times (-2\hat{\imath} - \hat{\jmath} + 4\hat{k}) \\ &= -3\hat{\imath} + 6\hat{\jmath} \text{ m/s.} \end{aligned}$$

---

9. (5 points) A point is moving with position vector

$$\vec{r} = (t^2 - 6t)\,\hat{\imath} + (e^{3t} - 3t)\,\hat{\jmath}\text{ m}.$$

What is the radius of curvature ρ at $t = 0$ s?

(A) $1\text{ m} \le \rho < \frac{3}{2}\text{ m}$

(B) $\frac{3}{2}\text{ m} \le \rho < 2\text{ m}$

(C) ★ $2\text{ m} \le \rho$

(D) $0\text{ m} \le \rho < \frac{1}{2}\text{ m}$

(E) $\frac{1}{2}\text{ m} \le \rho < 1\text{ m}$

Solution. The position and velocity are:

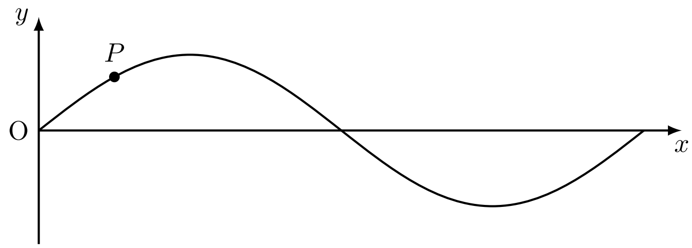
$$\begin{aligned}
\vec{r} &= (t^2 - 6t)\,\hat{\imath} + (e^{3t} - 3t)\,\hat{\jmath} = \hat{\jmath} \\
\vec{v} &= (2t - 6)\,\hat{\imath} + (3e^{3t} - 3)\,\hat{\jmath} = -6\hat{\imath} \\
\vec{a} &= 2\hat{\imath} + 9e^{3t}\,\hat{\jmath} = 2\hat{\imath} + 9\hat{\jmath}.
\end{aligned}$$

Thus $\hat{e}_t = -\hat{\imath}$ and $\hat{e}_n = \hat{\jmath}$, so $a_n = 9$. Also $v = 6$, so:

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
9 &= \frac{6^2}{\rho} \\
\rho &= 4\text{ m}.
\end{aligned}$$

---

10. (5 points) A particle is moving to the left along a variable-height ground with ground height given by $y(x) = \sin(x/20)$ m. The particle's horizontal velocity component is a constant $v_x = -4$ m/s. What is the vertical component of velocity v_y when $x = 5\pi$ m?



(A) $v_y = 0$ m/s

(B) ★ -1 m/s $\leq v_y < 0$ m/s

(C) 0 m/s $\leq v_y < 1$ m/s

(D) $v_y < -1$ m/s

(E) 1 m/s $\leq v_y$

Solution. Using $\dot{x} = v_x = -4$ m/s, we have:

$$\begin{aligned}
y &= \sin(x/20) \\
v_y = \dot{y} &= \cos(x/20)\frac{\dot{x}}{20} \\
&= \cos(5\pi/20)\frac{-4}{20} \\
&= \cos(\pi/4)\frac{-1}{5} \\
&= \frac{1}{\sqrt{2}}\frac{-1}{5} \\
&\approx -0.14 \text{ m/s.}
\end{aligned}$$

---

11. (5 points) A position P has an associated polar basis with

$$\hat{e}_\theta = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}.$$

What is θ?

(A) $\pi \leq \theta < \frac{3}{2}\pi$

(B) $0 \leq \theta < \frac{1}{2}\pi$

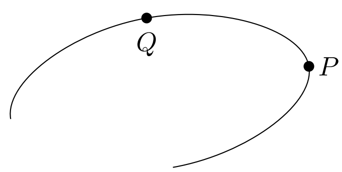
(C) $\frac{1}{2}\pi \leq \theta < \pi$

(D) ★ $\frac{3}{2}\pi \leq \theta < 2\pi$

Solution. The fastest way to solve this question is to draw a diagram and immediately see that θ must be in the fourth quadrant.
Alternatively, we can use the formula $\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ to see that $-\sin\theta = \frac{1}{2}$, so $\theta = -30°$ or $\theta = -150°$. Also $\cos\theta = \frac{\sqrt{3}}{2}$ so $\theta = \pm 30°$. The only common solution is $\theta = -30° = -\frac{1}{6}\pi$, or $\theta = \frac{11}{6}\pi$.

---

12. (5 points) A point is moving around the curve shown below with varying speed.



The radius of curvature and normal acceleration at P and Q are given by:

$$\rho_P = 2 \text{ m} \qquad \rho_Q = 4 \text{ m}$$
$$a_{P,n} = 4 \text{ m/s}^2 \qquad a_{Q,n} = 2 \text{ m/s}^2.$$

Which of the following is true about the velocities v_P at P and v_Q at Q?

(A) ★ $v_P = v_Q$

(B) $v_P < v_Q$

(C) $v_P > v_Q$

Solution. Using $a_n = v^2/\rho$ we find that $v_P^2 = 8$ and $v_Q^2 = 8$, so they are equal.

---

13. (5 points) If a particle has position vector $\vec{r}(t) = (2t - t^2)\,\hat{\imath} - 3\cos(t)\,\hat{\jmath} - \sin(4t)\,\hat{k}$ m, what is its speed $v(0)$ at time $t = 0$ s?

(A) $v(0) = 0$ m/s

(B) 6 m/s $\leq v(0)$

(C) ★ 4 m/s $\leq v(0) < 6$ m/s

(D) 0 m/s $< v(0) < 2$ m/s

(E) 2 m/s $\leq v(0) < 4$ m/s

Solution. The velocity is $\vec{v} = \dot{\vec{r}} = (2 - 2t)\,\hat{\imath} + 3\sin(t)\,\hat{\jmath} - 4\cos(4t)\,\hat{k}$, so $\vec{v}(0) = 2\hat{\imath} - 4\hat{k}$. The magnitude $v(0)$ is more than 4 but less than $4 + 2 = 6$.

---

14. (5 points) A car is observed moving in the plane with velocity $\vec{v} = 2\hat{\imath} - 4\hat{\jmath}$ m/s and acceleration $\vec{a} = -\hat{\imath} - 3\hat{\jmath}$ m/s^2. At this instant, is it:

(A) stationary

(B) ★ driving around a curve clockwise

(C) driving around a curve counterclockwise

(D) driving in a straight line

Solution. Drawing a diagram shows it is curving clockwise.
Alternatively, $\vec{v} \times \vec{a} = -10\hat{k}$ which is in the $-\hat{k}$ direction, so it is curving in the negative direction (clockwise).

---

15. (5 points) The particle P has polar coordinates $r = 4$ m, $\theta = -45°$ and velocity $\vec{v} = -\hat{\jmath}$. Which statement is true?

(A) $\dot{r} \geq 0$ and $\dot{\theta} \geq 0$

(B) $\dot{r} < 0$ and $\dot{\theta} < 0$

(C) $\dot{r} < 0$ and $\dot{\theta} \geq 0$

(D) ★ $\dot{r} \geq 0$ and $\dot{\theta} < 0$

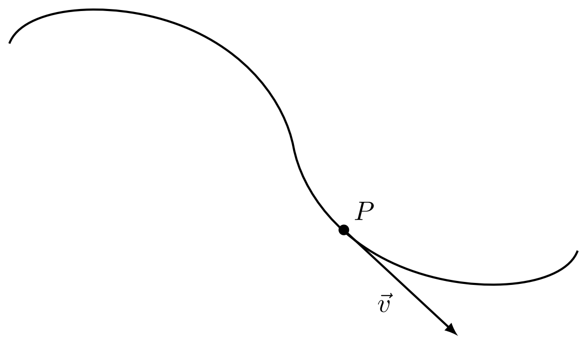
Solution. The fastest way to solve this question is to draw a diagram and directly see that this velocity is causing r to increase and θ to decrease.
Alternatively, we can convert $\vec{v}$ to polar coordinates to find:

$$\begin{aligned}\vec{v} &= -(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) \\ &= \frac{1}{\sqrt{2}}\,\hat{e}_r - \frac{1}{\sqrt{2}}\,\hat{e}_\theta.\end{aligned}$$

Thus we have a positive velocity component in the $\hat{e}_r$ direction (increasing r), and a negative velocity component in the $\hat{e}_\theta$ direction (decreasing θ).

---

16. (5 points) A car is driving on a curved track with the top view shown below. At a given instant the car is at point P with velocity $\vec{v}$ and its speed is decreasing, such that the tangential and normal components of its acceleration are equal in magnitude.



Which direction is the closest to the direction of the acceleration $\vec{a}$ at the instant shown?

(A) $\rightarrow$

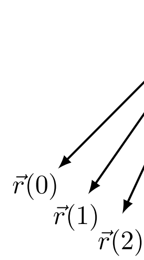
(B) ★ $\uparrow$

(C) $\downarrow$

(D) $\leftarrow$

Solution. $\hat{e}_t$ is down-right, so $-\hat{e}_t$ is up-left and $\hat{e}_n$ is up-right, so equal components in these two directions give a total upwards acceleration.

17. (5 points) The position vector $\vec{r}(t)$ of a point is shown below at $t = 0$ s, $t = 1$ s, and $t = 2$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

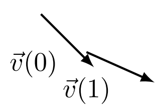
(A) ★ $\nearrow$

(B) $\nwarrow$

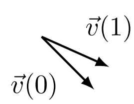
(C) $\searrow$

(D) $\swarrow$

Solution. This closely resembles uniform circular motion, so the acceleration is inwards towards the center. Alternatively, the velocity directions are roughly:



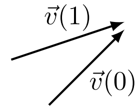
Re-drawing these with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly to the right and up.

---

18. (5 points) The velocity $\vec{v}(t)$ of a point is shown below at $t = 0$ s and $t = 1$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

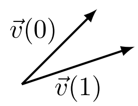
(A) ★ $\searrow$

(B) $\swarrow$

(C) $\nwarrow$

(D) $\nearrow$

Solution. Re-drawing the velocities with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly right-down.

---

19. (5 points) A car driving down the road travels a distance $s = (\frac{1}{2}t^3 + \frac{1}{2}t)$ m from its starting point. At $t = 1$ s the car is driving around a curve and the magnitude of its acceleration is $a = 5$ m/s^2. What is the radius of curvature ρ of the curve?

(A) $0 \text{ m} \leq \rho < 1 \text{ m}$

(B) ★ $1 \text{ m} \leq \rho < 2 \text{ m}$

(C) $3 \text{ m} \leq \rho < 4 \text{ m}$

(D) $2 \text{ m} \leq \rho < 3 \text{ m}$

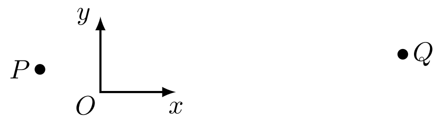
(E) $4 \text{ m} \leq \rho$

Solution. We have $\dot{s}(t) = \frac{3}{2}t^2 + \frac{1}{2}$ and $\ddot{s}(t) = 3t$, so $\dot{s}(1) = 2$ and $\ddot{s}(1) = 3$. The acceleration is:

$$\begin{aligned}
\vec{a} &= \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\,\hat{e}_n \\
&= 3\,\hat{e}_t + \frac{2^2}{\rho}\,\hat{e}_n \\
5^2 &= 3^2 + \frac{4^2}{\rho^2} \\
\rho^2 &= \frac{16}{16} \\
\rho &= 1 \text{ m}.
\end{aligned}$$

---

20. (5 points) Points P and Q are moving in circular paths around the origin O with angular velocities ω_P and ω_Q and speeds v_P and v_Q, respectively.



The two particles are moving with the same angular velocity, so $\omega_P = \omega_Q$. Which statement is true?

(A) $2v_Q < v_P$

(B) ★ $v_P \leq \frac{1}{2}v_Q$

(C) $\frac{1}{2}v_Q < v_P \leq v_Q$

(D) $v_Q < v_P \leq 2v_Q$

Solution. If the particles both have the same angular velocity then we can observe from the diagram that r_P is much smaller than r_Q, so v_P is much smaller than ω_Q.
More precisely:

$$\omega = \frac{v_P}{r_P}$$
$$\omega = \frac{v_Q}{r_Q}$$
$$v_P = \frac{r_P}{r_Q}v_Q.$$

From the diagram, $\frac{r_P}{r_Q} \approx \frac{1}{4}$, so $v_P \approx \frac{1}{4}v_Q$, which is less than $\frac{1}{2}v_Q$.

TAM 212. Midterm 1. Feb 21, 2013.

- There are 20 questions, each worth 5 points.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

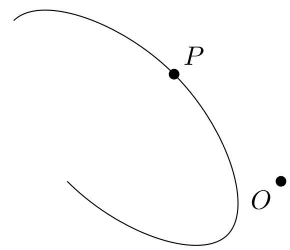
2. Circle your discussion section:

	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

94. C
95. A
96. E

1. (5 points) A point is moving around the curve shown and is currently at position P. Consider a polar basis $\hat{e}_r, \hat{e}_\theta$ at P from the origin O and a tangential/normal basis $\hat{e}_t, \hat{e}_n$ at P.



Which of the following is true?

(A) ★ $\hat{e}_n = \hat{e}_\theta$

(B) $\hat{e}_n = -\hat{e}_r$

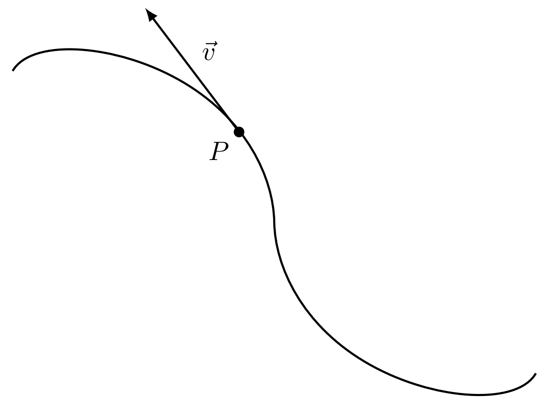
(C) $\hat{e}_n = \hat{e}_r$

(D) $\hat{e}_n = -\hat{e}_\theta$

Solution. Here $\hat{e}_r$ is to the up-left and $\hat{e}_\theta$ is down-left. Also $\hat{e}_n$ is inwards to the curve, so is down-left, which is $\hat{e}_\theta$.

---

2. (5 points) A car is driving on a curved track with the top view shown below. At a given instant the car is at point P with velocity $\vec{v}$ and its speed is decreasing, such that the tangential and normal components of its acceleration are equal in magnitude.



Which direction is the closest to the direction of the acceleration $\vec{a}$ at the instant shown?

(A) $\uparrow$

(B) $\leftarrow$

(C) ★ $\downarrow$

(D) $\rightarrow$

Solution. $\hat{e}_t$ is up-left, so $-\hat{e}_t$ is down-right and $\hat{e}_n$ is down-left, so equal components in these two directions give a total downwards acceleration.

3. (5 points) A car driving down the road travels a distance $s = 2t^2$ m from its starting point. At $t = 1$ s the car is driving around a curve and the magnitude of its acceleration is $a = 5\ \text{m/s}^2$. What is the radius of curvature ρ of the curve?

(A) $3\ \text{m} \leq \rho < 4\ \text{m}$

(B) $0\ \text{m} \leq \rho < 1\ \text{m}$

(C) ★ $4\ \text{m} \leq \rho$

(D) $1\ \text{m} \leq \rho < 2\ \text{m}$

(E) $2\ \text{m} \leq \rho < 3\ \text{m}$

Solution. We have $\dot{s}(t) = 4t$ and $\ddot{s} = 4$, so $\dot{s}(1) = 4$. The acceleration is:

$$\begin{aligned}
\vec{a} &= \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\,\hat{e}_n \\
&= 4\,\hat{e}_t + \frac{4^2}{\rho}\,\hat{e}_n \\
5^2 &= 4^2 + \frac{16^2}{\rho^2} \\
\rho^2 &= \frac{16^2}{3^2} \\
\rho &= \frac{16}{3} \approx 5.33\ \text{m}.
\end{aligned}$$

---

4. (5 points) If a particle has position vector $\vec{r}(t) = -\sin(2t)\,\hat{\imath} + (t + t^3)\,\hat{\jmath} + 3\cos(2t)\,\hat{k}$ m, what is its speed $v(0)$ at time $t = 0$ s?

(A) $1\ \text{m/s} \leq v(0) < 2\ \text{m/s}$

(B) $0\ \text{m/s} < v(0) < 1\ \text{m/s}$

(C) ★ $2\ \text{m/s} \leq v(0) < 3\ \text{m/s}$

(D) $3\ \text{m/s} \leq v(0)$

(E) $v(0) = 0\ \text{m/s}$

Solution. The velocity is $\vec{v} = \dot{\vec{r}} = -2\cos(2t)\,\hat{\imath} + (1 + 3t^2)\,\hat{\jmath} - 6\sin(2t)\,\hat{k}$, so $\vec{v}(0) = -2\hat{\imath} + \hat{\jmath}$. The magnitude $v(0)$ is more than 2 but less than $2 + 1 = 3$.

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5. (5 points) A particle is moving in the plane so that at a particular instant its polar coordinates have $\dot{r} = 3$ m/s and $\dot{\theta} = -2$ rad/s. The speed is $v = 5$ m/s. What is r?

(A) $1 \text{ m} \leq r < 2 \text{ m}$

(B) $3 \text{ m} \leq r < 4 \text{ m}$

(C) $4 \text{ m} \leq r$

(D) ★ $2 \text{ m} \leq r < 3 \text{ m}$

(E) $0 \text{ m} \leq r < 1 \text{ m}$

Solution. Using $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ we have:

$$\begin{aligned} v &= \sqrt{v_r^2 + v_\theta^2} \\ 5 &= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \\ 25 &= 3^2 + r^2(-2)^2 \\ 4r^2 &= 16 \\ r &= 2 \text{ m}. \end{aligned}$$

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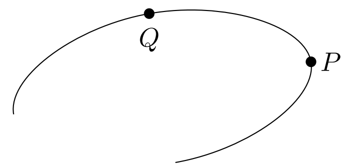
6. (5 points) A point is moving around the curve shown below with varying speed.



The radius of curvature and normal acceleration at P and Q are given by:

$$\rho_P = 2 \text{ m} \qquad \rho_Q = 4 \text{ m}$$
$$a_{P,n} = 6 \text{ m/s}^2 \qquad a_{Q,n} = 2 \text{ m/s}^2.$$

Which of the following is true about the velocities v_P at P and v_Q at Q?

(A) $v_P < v_Q$

(B) ★ $v_P > v_Q$

(C) $v_P = v_Q$

Solution. Using $a_n = v^2/\rho$ we find that $v_P^2 = 12$ and $v_Q^2 = 8$, so v_P is larger.

---

7. (5 points) A particle P has vector, velocity, and acceleration vectors given by:

$$\vec{r} = -3\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{v} = \hat{\imath} + 4\hat{\jmath} \text{ m/s}$$
$$\vec{a} = -9\hat{\imath} + 2\hat{\jmath} \text{ m/s}^2.$$

Consider the following statements:

(i) The particle is moving closer to the origin.

(ii) The particle is moving further from the origin.

(iii) The particle is speeding up.

(iv) The particle is slowing down.

Which statements are true?

(A) (ii) and (iii)

(B) (ii) and (iv)

(C) none of the other options

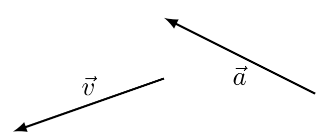
(D) ★ (i) and (iv)

(E) (i) and (iii)

Solution. $\vec{v} \cdot \vec{r} = -11 < 0$ so $\dot{r} < 0$ (moving closer to the origin), and $\vec{a} \cdot \vec{v} = -1 < 0$ so $\dot{v} < 0$ (slowing down).

---

8. (5 points) The velocity $\vec{v}$ and acceleration $\vec{a}$ for a single particle P are shown below at a particular instant.



Which statement is true at this instant?

(A) the particle's speed is not changing

(B) ★ the particle is speeding up

(C) the particle is slowing down

Solution. The component of $\vec{a}$ in the direction of $\vec{v}$ is positive, so $\dot{v} > 0$.

---

9. (5 points) The particle P has polar coordinates $r = 4$ m, $\theta = -135°$ and velocity $\vec{v} = -\hat{\jmath}$. Which statement is true?

(A) ★ $\dot{r} \geq 0$ and $\dot{\theta} \geq 0$

(B) $\dot{r} < 0$ and $\dot{\theta} \geq 0$

(C) $\dot{r} < 0$ and $\dot{\theta} < 0$

(D) $\dot{r} \geq 0$ and $\dot{\theta} < 0$

Solution. The fastest way to solve this question is to draw a diagram and directly see that this velocity is causing r to increase and θ to increase.
Alternatively, we can convert $\vec{v}$ to polar coordinates to find:

$$\begin{aligned}\vec{v} &= -(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) \\ &= \frac{1}{\sqrt{2}}\,\hat{e}_r + \frac{1}{\sqrt{2}}\,\hat{e}_\theta.\end{aligned}$$

Thus we have a positive velocity component in the $\hat{e}_r$ direction (increasing r), and a positive velocity component in the $\hat{e}_\theta$ direction (increasing θ).

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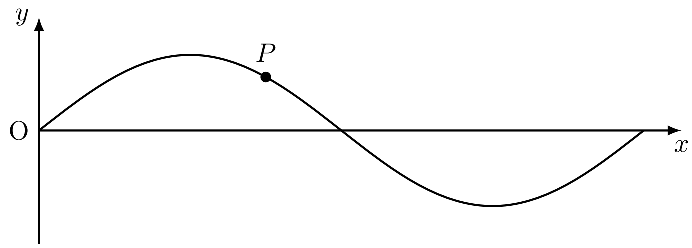
10. (5 points) A particle is moving to the left along a variable-height ground with ground height given by $y(x) = \sin(x/20)$ m. The particle's horizontal velocity component is a constant $v_x = -4$ m/s. What is the vertical component of velocity v_y when $x = 15\pi$ m?



(A) ★ $0 \text{ m/s} \le v_y < 1 \text{ m/s}$

(B) $v_y < -1 \text{ m/s}$

(C) $-1 \text{ m/s} \le v_y < 0 \text{ m/s}$

(D) $v_y = 0 \text{ m/s}$

(E) $1 \text{ m/s} \le v_y$

Solution. Using $\dot{x} = v_x = -4$ m/s, we have:

$$\begin{aligned} y &= \sin(x/20) \\ v_y = \dot{y} &= \cos(x/20)\frac{\dot{x}}{20} \\ &= \cos(15\pi/20)\frac{-4}{20} \\ &= \cos(3\pi/4)\frac{-1}{5} \\ &= -\frac{1}{\sqrt{2}}\frac{-1}{5} \\ &\approx 0.14 \text{ m/s.} \end{aligned}$$

---

11. (5 points) A particle moves so that its position in polar coordinates is given by

$$r = (3 + \cos t) \text{ m}$$
$$\theta = -t \text{ rad.}$$

What is the $\hat{\jmath}$ component of velocity v_y at $t = \pi$ s?

(A) ★ $2 \text{ m/s} \le v_y$

(B) $v_y < -2 \text{ m/s}$

(C) $v_y = 0 \text{ m/s}$

(D) $-2 \text{ m/s} \le v_y < 0 \text{ m/s}$

(E) $0 \text{ m/s} < v_y < 2 \text{ m/s}$

Solution. We have:

$$r(t) = 3 + \cos t \qquad \theta(t) = -t$$
$$\dot{r}(t) = -\sin t \qquad \dot{\theta}(t) = -1.$$

At $t = 2$ this is:

$$r(\pi) = 2 \qquad \theta(\pi) = -\pi$$
$$\dot{r}(\pi) = 0 \qquad \dot{\theta}(\pi) = -1.$$

The velocity is:

$$\begin{aligned} \vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \\ &= -2\,\hat{e}_\theta. \end{aligned}$$

Using $\theta = -\pi$ we see that $\hat{e}_\theta = -\hat{\jmath}$, so the $\hat{\jmath}$ component is:

$$v_y = 2 \text{ m/s.}$$

---

12. (5 points) A point is currently at position $x = 2$ m, $y = -1$ m, $z = 4$ m and is rotating in the x–y plane about the origin with angular velocity $\vec{\omega} = -3\hat{k}$ rad/s. The velocity $\vec{v}$ of the point is:

(A) $\vec{v} = 3\hat{\imath} - 6\hat{\jmath}$ m/s

(B) $\vec{v} = 3\hat{\imath} + 6\hat{\jmath}$ m/s

(C) ★ $\vec{v} = -3\hat{\imath} - 6\hat{\jmath}$ m/s

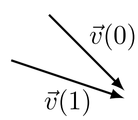
(D) $\vec{v} = -3\hat{\imath} + 6\hat{\jmath}$ m/s

Solution. The position vector is $\vec{r} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ m, so:

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r} \\ &= -3\hat{k} \times (2\hat{\imath} - \hat{\jmath} + 4\hat{k}) \\ &= -3\hat{\imath} - 6\hat{\jmath} \text{ m/s.}\end{aligned}$$

---

13. (5 points) The velocity $\vec{v}(t)$ of a point is shown below at $t = 0$ s and $t = 1$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

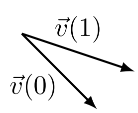
(A) $\nwarrow$

(B) $\searrow$

(C) ★ $\nearrow$

(D) $\swarrow$

Solution. Re-drawing the velocities with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly right-up.

---

14. (5 points) A point is moving with position vector

$$\vec{r} = (t^2 - 2t)\,\hat{\imath} + (8t - 8e^t)\,\hat{\jmath} \text{ m}.$$

What is the radius of curvature ρ at $t = 0$ s?

(A) ★ $\frac{1}{2}$ m $\leq \rho < 1$ m

(B) 1 m $\leq \rho < \frac{3}{2}$ m

(C) 0 m $\leq \rho < \frac{1}{2}$ m

(D) $\frac{3}{2}$ m $\leq \rho < 2$ m

(E) 2 m $\leq \rho$

Solution. The position and velocity are:

$$\begin{aligned}
\vec{r} &= (t^2 - 2t)\,\hat{\imath} + (8t - 8e^t)\,\hat{\jmath} = \hat{\jmath} \\
\vec{v} &= (2t - 2)\,\hat{\imath} + (8 - 8e^t)\,\hat{\jmath} = -2\hat{\imath} \\
\vec{a} &= 2\hat{\imath} - 8e^t\,\hat{\jmath} = 2\hat{\imath} - 8\hat{\jmath}.
\end{aligned}$$

Thus $\hat{e}_t = -\hat{\imath}$ and $\hat{e}_n = -\hat{\jmath}$, so $a_n = 8$. Also $v = 2$, so:

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
8 &= \frac{2^2}{\rho} \\
\rho &= \frac{1}{2} \text{ m}.
\end{aligned}$$

---

15. (5 points) A car is observed moving in the plane with velocity $\vec{v} = -2\hat{\imath} + 4\hat{\jmath}$ m/s and acceleration $\vec{a} = \hat{\imath} + 3\hat{\jmath}$ m/s^2. At this instant, is it:

(A) driving in a straight line

(B) driving around a curve counterclockwise

(C) ★ driving around a curve clockwise

(D) stationary

Solution. Drawing a diagram shows it is curving clockwise.
Alternatively, $\vec{v} \times \vec{a} = -10\hat{k}$ which is in the $-\hat{k}$ direction, so it is curving in the negative direction (clockwise).

---

16. (5 points) A position P has an associated polar basis with

$$\hat{e}_\theta = -\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}.$$

What is θ?

(A) ★ $0 \leq \theta < \frac{1}{2}\pi$

(B) $\frac{1}{2}\pi \leq \theta < \pi$

(C) $\frac{3}{2}\pi \leq \theta < 2\pi$

(D) $\pi \leq \theta < \frac{3}{2}\pi$

Solution. The fastest way to solve this question is to draw a diagram and immediately see that θ must be in the first quadrant.
Alternatively, we can use the formula $\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ to see that $-\sin\theta = -\frac{1}{2}$, so $\theta = 30^\circ$ or $\theta = 150^\circ$. Also $\cos\theta = \frac{\sqrt{3}}{2}$ so $\theta = \pm 30^\circ$. The only common solution is $\theta = 30^\circ = \frac{1}{6}\pi$.

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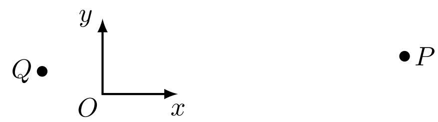
17. (5 points) Points P and Q are moving in circular paths around the origin O with angular velocities ω_P and ω_Q and speeds v_P and v_Q, respectively.



The two particles are moving with the same angular velocity, so $\omega_P = \omega_Q$. Which statement is true?

(A) $\frac{1}{2}v_Q < v_P \leq v_Q$

(B) ★ $2v_Q < v_P$

(C) $v_P \leq \frac{1}{2}v_Q$

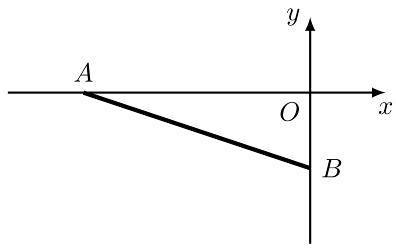
(D) $v_Q < v_P \leq 2v_Q$

Solution. If the particles both have the same angular velocity then we can observe from the diagram that r_P is much higher than r_Q, so v_P is much higher than ω_Q.
More precisely:

$$\omega = \frac{v_P}{r_P}$$
$$\omega = \frac{v_Q}{r_Q}$$
$$v_P = \frac{r_P}{r_Q} v_Q.$$

From the diagram, $\frac{r_P}{r_Q} \approx 4$, so $v_P \approx 4v_Q$, which is more than $2v_Q$.

18. (5 points) A rod with fixed length is positioned as shown, with x the horizontal coordinate of end A and y the vertical coordinate of end B. End A can only move horizontally, while end B can only move vertically.



At an instant of time, we have $x = -3$ m and $y = -1$ m, and the rod is moving so that $\dot{y} = -2$ m/s. What is $\dot{x}$?

(A) $\dot{x} = 0$ m/s

(B) -1 m/s $\leq \dot{x} < 0$ m/s

(C) ★ 0 m/s $< \dot{x} < 1$ m/s

(D) 1 m/s $\leq \dot{x}$

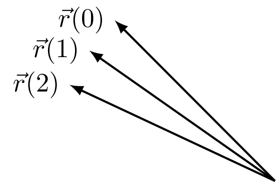
(E) $\dot{x} < -1$ m/s

Solution. If the fixed length of the rod is ℓ, then we have:

$$\begin{aligned} \ell^2 &= x^2 + y^2 \\ 0 &= 2x\dot{x} + 2y\dot{y} \\ \dot{x} &= -\frac{y}{x}\dot{y} \\ &= -\frac{-1}{-3}(-2) \\ &\approx 0.67 \text{ m/s.} \end{aligned}$$

---

19. (5 points) The position vector $\vec{r}(t)$ of a point is shown below at $t = 0$ s, $t = 1$ s, and $t = 2$ s.



Which direction is the closest to the direction of the acceleration $\vec{a}(0)$ at time $t = 0$ s?

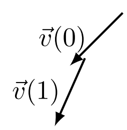
(A) $\swarrow$

(B) ★ $\searrow$

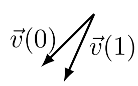
(C) $\nearrow$

(D) $\nwarrow$

Solution. This closely resembles uniform circular motion, so the acceleration is inwards towards the center. Alternatively, the velocity directions are roughly:



Re-drawing these with a common base gives:



so $\vec{a} = \dot{\vec{v}}$ is mainly to the right and down.

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20. (5 points) A particle starts at the origin at time $t = 0$ s and its velocity is given by

$$\vec{v} = -t^2\,\hat{\imath} + 4t\,\hat{\jmath}\ \text{m/s}.$$

At time $t = 1$ s, what is the particle's distance r from the origin?

(A) $0\ \text{m} \leq r < 2\ \text{m}$

(B) ★ $2\ \text{m} \leq r < 4\ \text{m}$

(C) $4\ \text{m} \leq r < 6\ \text{m}$

(D) $8\ \text{m} \leq r$

(E) $6\ \text{m} \leq r < 8\ \text{m}$

Solution. $\vec{r} = \vec{r}(0) + \int_0^t \vec{v}(\tau)\,d\tau = -\frac{1}{3}t^3\,\hat{\imath} + 2t^2\,\hat{\jmath}$, so $\vec{r}(1) = -\frac{1}{3}\,\hat{\imath} + 2\,\hat{\jmath}$ m. Then $r(1)$ is more than 2 but less than 3 (in fact it's $r(1) \approx 2.03$ m).
