TAM 212. Midterm 2 Practice. Mar 25, 2013. (V2)

- There are 50 questions, each worth 1 point.
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.
- The notation $\vec{r}_{PQ}$ denotes the position vector from P to Q.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

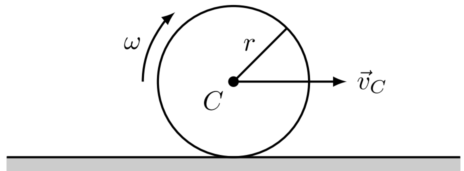
2. Circle your discussion section:

	Monday	**Tuesday**	**Wednesday**	**Thursday**
8–9		ADI (260) Karthik		
9–10		ADC (260) Venanzio		ADK (260) Aaron
10–11		ADD (256) Aaron ADQ (344) Jan	ADS (252) Ray	ADT (243) Aaron ADU (344) Jan
11–12		ADE (252) Jan		ADL (256) Kumar
12–1	ADA (243) Ray ADP (135) Seung	ADF (335) Seung ADG (336) Kumar	ADJ (256) Ray ADR (252) Lin	ADN (260) Kumar
1–2				
2–3				
3–4				
4–5	ADV (252) Karthik		ADO (260) Mazhar ADW (252) Lin	
5–6	ADB (260) Mazhar	ADH (260) Karthik	ADM (243) Mazhar	

3. Fill in the following answers on the Scantron form:

91. A
92. A
93. A
94. A
95. D
96. C

1. (1 point) A circular rigid body with radius $r = 2$ m is rolling without slipping on a flat surface in 2D as shown. The speed of the center is $v_C = 6$ m/s.



What is the angular velocity ω?

(A) 0 rad/s $\leq \omega <$ 1 rad/s

(B) 1 rad/s $\leq \omega <$ 2 rad/s

(C) 2 rad/s $\leq \omega <$ 3 rad/s

(D) 4 rad/s $\leq \omega$

(E) ★ 3 rad/s $\leq \omega <$ 4 rad/s

Solution. $v_c = r\omega$ so $\omega = v_c/r = 6/2 = 3$ rad/s.

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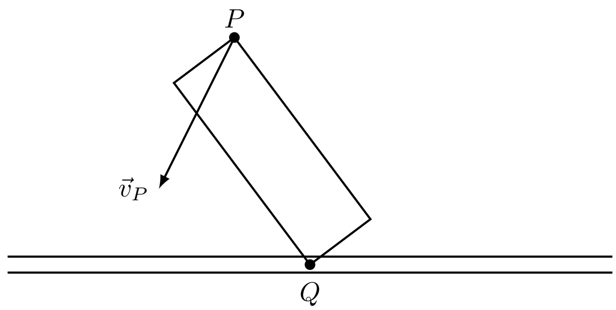
2. (1 point) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$. A pin at point Q constrains that point to move in a horizontal slot.



Point P on the body has:

$$\vec{r}_{PQ} = \hat{\imath} - 3\hat{\jmath} \text{ m}$$
$$\vec{v}_P = -\hat{\imath} - 2\hat{\jmath} \text{ m/s.}$$

What is the speed v_Q of point Q?

(A) ★ $4 \text{ m/s} \le v_Q$

(B) $1 \text{ m/s} \le v_Q < 2 \text{ m/s}$

(C) $2 \text{ m/s} \le v_Q < 3 \text{ m/s}$

(D) $3 \text{ m/s} \le v_Q < 4 \text{ m/s}$

(E) $0 \text{ m/s} \le v_Q < 1 \text{ m/s}$

Solution. Taking $\vec{v}_Q = v_Q\,\hat{\imath}$ with unknown speed v_Q, we have:

$$\begin{aligned}
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
v_Q\,\hat{\imath} &= -\hat{\imath} - 2\hat{\jmath} + \omega\hat{k} \times (\hat{\imath} - 3\hat{\jmath}) \\
&= -\hat{\imath} - 2\hat{\jmath} + 3\omega\,\hat{\imath} + \omega\,\hat{\jmath} \\
&= (-1 + 3\omega)\,\hat{\imath} + (-2 + \omega)\,\hat{\jmath}
\end{aligned}$$

Equating components and solving gives:

$$\begin{aligned}
0 &= -2 + \omega \\
\implies \omega &= 2 \text{ rad/s} \\
v_Q &= -1 + 3\omega \\
&= 5 \text{ m/s.}
\end{aligned}$$

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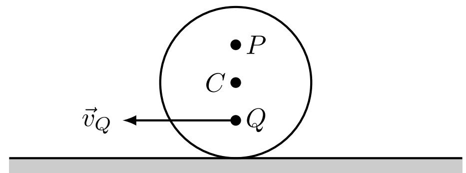
3. (1 point) A circular rigid body is rolling without slipping on a flat surface in 2D as shown. The velocity of point Q is $\vec{v}_Q = -\hat{\imath}$ m/s.



What is the $\hat{\imath}$ component v_{Px} of point P?

(A) $2 \text{ m/s} \le v_{Px}$

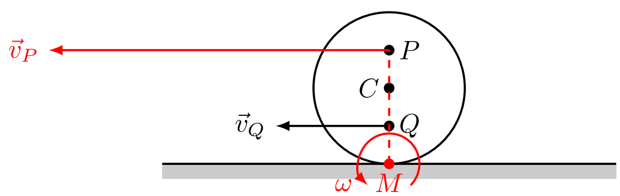
(B) $-2 \text{ m/s} \le v_{Px} < 0 \text{ m/s}$

(C) $v_{Px} = 0$ m/s

(D) ★ $v_{Px} < -2$ m/s

(E) $0 \text{ m/s} < v_{Px} < 2 \text{ m/s}$

Solution. Consider the instantaneous center M at the contact point. Now $r_{MP} \approx 3r_{MQ}$, so $v_P \approx 3v_Q \approx 3$ m/s, and $\vec{v}_P$ has the same direction as $\vec{v}_Q$ (the body is rolling counterclockwise), so $\vec{v}_P \approx -3\hat{\imath}$ m/s.



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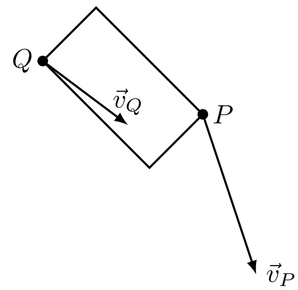
4. (1 point) A rigid body is moving in 2D as shown below.

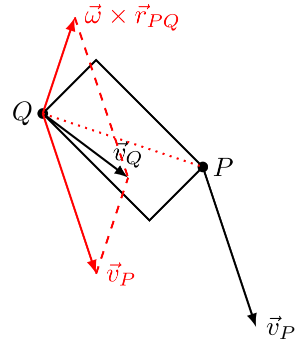


What is the direction of the angular velocity of the body?

(A) ★ ↻ (clockwise)

(B) ↺ (counterclockwise)

Solution. Considering the required term $\vec{\omega} \times \vec{r}_{PQ}$ shows that it is up-right, so considering rotation about P we see that ω is clockwise.



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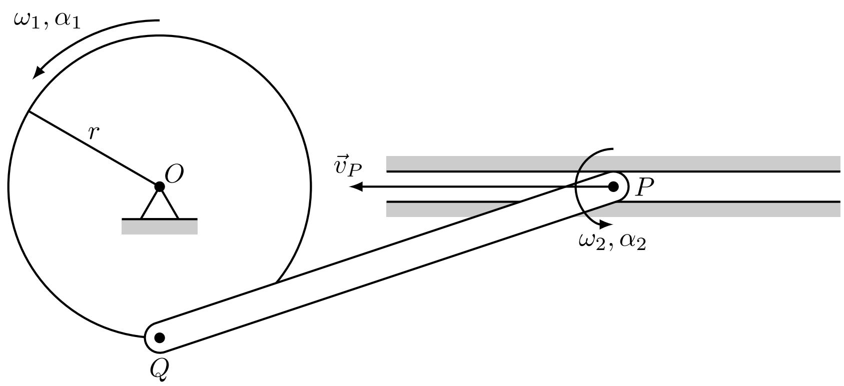
5. (1 point) A circular rigid body with radius $r = 2$ m rotates about the fixed center O as shown. A rigid rod connects pins P and Q, and point P is constrained to only move horizontally. Point P has velocity $\vec{v}_P = -4\hat{\imath}$ m/s and acceleration $\vec{a}_P = 0$. The angular velocity and angular acceleration of the circular body are $\vec{\omega}_1 = \omega_1\hat{k}$ and $\vec{\alpha}_1 = \alpha_1\hat{k}$, while those of the rod are $\vec{\omega}_2 = \omega_2\hat{k}$ and $\vec{\alpha}_2 = \alpha_2\hat{k}$.



The position vectors are:

$$\vec{r}_{OQ} = -2\hat{\jmath} \text{ m}$$
$$\vec{r}_{PQ} = -6\hat{\imath} - 2\hat{\jmath} \text{ m}.$$

What is α_1?

(A) $-1 \text{ rad/s}^2 \leq \alpha_1 < 0 \text{ rad/s}^2$

(B) $0 \text{ rad/s}^2 < \alpha_1 < 1 \text{ rad/s}^2$

(C) $\alpha_1 = 0 \text{ rad/s}^2$

(D) ★ $\alpha_1 < -1 \text{ rad/s}^2$

(E) $1 \text{ rad/s}^2 \leq \alpha_1$

Solution. We first use the fact that $\vec{v}_Q = -v_Q\hat{\imath}$ to find:

$$\begin{aligned} \vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\ -v_Q\hat{\imath} &= -4\hat{\imath} + \omega_2\hat{k} \times (-6\hat{\imath} - 2\hat{\jmath}) \\ (4 - v_Q)\hat{\imath} &= 2\omega_2\,\hat{\imath} - 6\omega_2\,\hat{\jmath}. \end{aligned}$$

Comparing $\hat{\jmath}$ components shows that $\omega_2 = 0$ and $v_Q = 4$ m/s, so $\omega_1 = -4/2 = -2$ rad/s. The fact that $\vec{v}_P = \vec{v}_Q$ could have also been realized from the fact that $\vec{v}_P$ and $\vec{v}_Q$ are parallel and offset, so there is no instantaneous center and thus no rotation.

Now we have:

$$\begin{aligned} \vec{a}_Q &= \vec{a}_Q \\ \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OQ} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OQ}) &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_2 \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\ 0 + \alpha_1\hat{k} \times (-2\hat{\jmath}) - \omega_1^2(-2\hat{\jmath}) &= 0 + \alpha_2\hat{k} \times (-6\hat{\imath} - 2\hat{\jmath}) + 0 \\ 2\alpha_1\,\hat{\imath} + 2\omega_1^2\,\hat{\jmath} &= 2\alpha_2\,\hat{\imath} - 6\alpha_2\,\hat{\jmath}. \end{aligned}$$

Equating components shows that $\alpha_1 = \alpha_2 = -\omega_1^2/3 = -4/3 \approx -1.33 \text{ rad/s}^2$.

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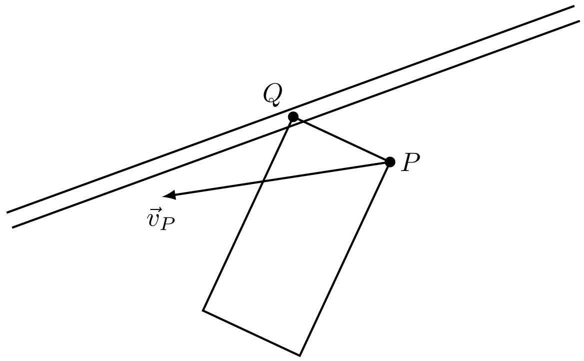
6. (1 point) A rigid body is moving in 2D as shown below, with point Q constrained to move in the angled slot.

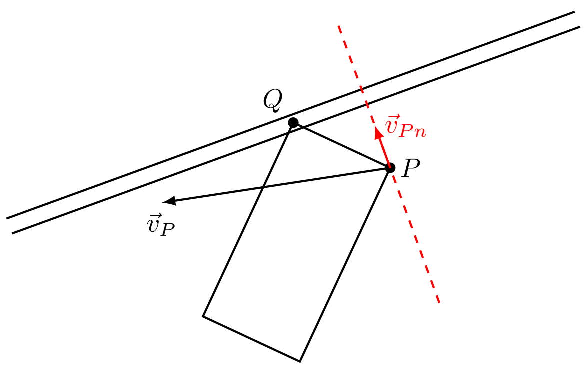


What is the direction of the angular velocity of the body?

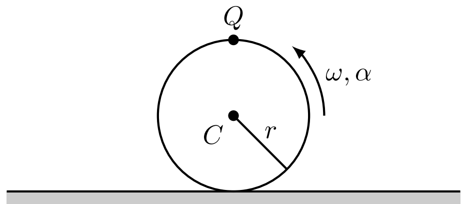
(A) ↻ (clockwise)

(B) ★ ↺ (counterclockwise)

Solution. The normal component of $\vec{v}_P$ is towards the slot, so P is moving closer to the slot. This means the line PQ, and hence the entire body, must be rotating counterclockwise.



7. (1 point) A circular rigid body with radius $r = 1$ m is rolling without slipping with angular velocity $\vec{\omega} = 2\hat{k}$ on a flat surface in 2D as shown. The body is speeding up and has angular acceleration $\vec{\alpha} = \alpha\hat{k}$. Point Q is at the top of the body and has acceleration magnitude $a_Q = 5$ m/s^2.



What is α?

(A) 1 rad/s^2 $\leq \alpha <$ 1.5 rad/s^2

(B) ★ 1.5 rad/s^2 $\leq \alpha <$ 2 rad/s^2

(C) 0.5 rad/s^2 $\leq \alpha <$ 1 rad/s^2

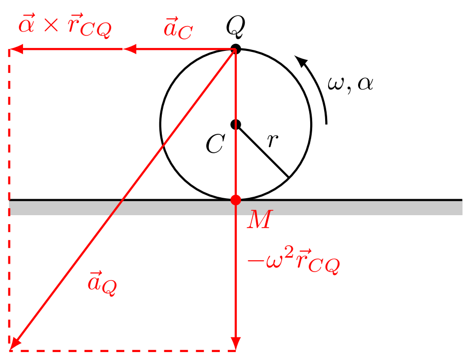
(D) 0 rad/s^2 $\leq \alpha <$ 0.5 rad/s^2

(E) 2 rad/s^2 $\leq \alpha$

Solution. The acceleration of Q is:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= -r\alpha\,\hat{\imath} + \alpha\hat{k} \times r\hat{\jmath} + \omega\hat{k} \times (\omega\hat{k} \times r\hat{\jmath}) \\
&= -r\alpha\,\hat{\imath} - r\alpha\,\hat{\imath} - r\omega^2\,\hat{\jmath} \\
&= -2r\alpha\,\hat{\imath} - r\omega^2\,\hat{\jmath} \\
&= -2\alpha\,\hat{\imath} - 4\,\hat{\jmath}.
\end{aligned}$$

To have $a_Q = 5$ we need $5^2 = (2\alpha)^2 + 4^2$, so $\alpha = 1.5$ rad/s^2.



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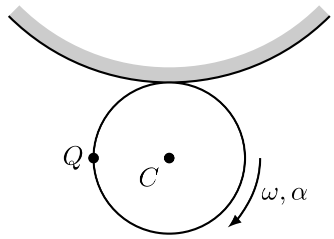
8. (1 point) A circular rigid body is rolling without slipping on a curved surface in 2D as shown. At the current instant the body is rotating clockwise and rate of rotation is increasing (ω and α are both positive in the direction shown), such that $r\alpha = r\omega^2$.



What is the direction of the acceleration $\vec{a}_Q$ of point Q?

(A) $\leftarrow$

(B) $\rightarrow$

(C) ★ $\uparrow$

(D) $\nearrow$

(E) $\nwarrow$

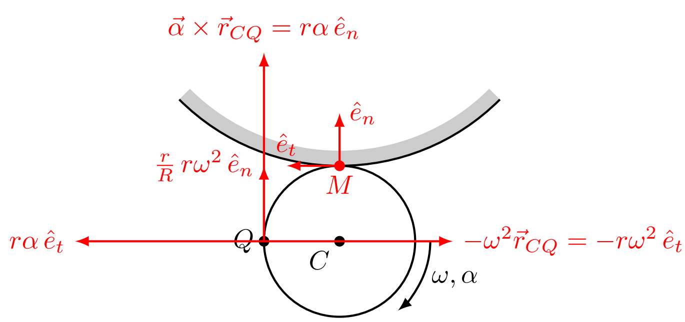
Solution. The acceleration of point C is

$$\vec{a}_C = r\alpha\,\hat{e}_t + \frac{(r\omega)^2}{R}\,\hat{e}_n$$

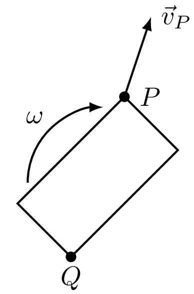
and using this we find the acceleration components at Q are:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= r\alpha\,\hat{e}_t + \frac{r}{R} r\omega^2\,\hat{e}_n + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= r\alpha\,\hat{e}_t + \frac{r}{R} r\omega^2\,\hat{e}_n + r\alpha\,\hat{e}_n - r\omega^2\,\hat{e}_t \\
&= \left(1 + \frac{r}{R}\right) r\alpha\,\hat{e}_n.
\end{aligned}$$

Because $r\alpha = r\omega^2$, the horizontal components cancel and we are left with a purely vertical acceleration upwards at Q.



9. (1 point) A rigid body is moving in 2D as shown, with a clockwise rotation (ω is positive in the direction indicated). The angular velocity ω, distance r_{PQ}, and speed v_P satisfy $\omega r_{PQ} = 2v_P$.



What is the direction of $\vec{v}_Q$?

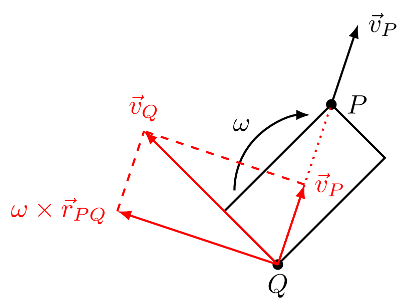
(A) $\nearrow$

(B) $\searrow$

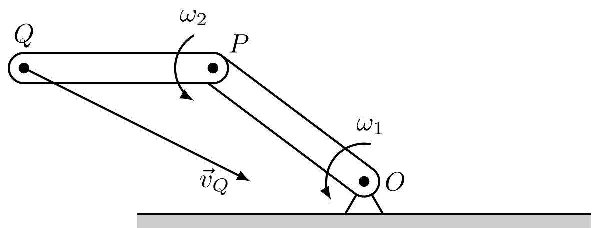
(C) ★ $\nwarrow$

(D) $\swarrow$

Solution.



10. (1 point) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = \omega_1 \hat{k}$ and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The positions and velocities at the current instant are:

$$\begin{aligned}
\vec{r}_{OP} &= -4\hat{\imath} + 3\hat{\jmath} \text{ m} \\
\vec{r}_{PQ} &= -5\hat{\imath} \text{ m} \\
\vec{v}_Q &= 6\hat{\imath} - 3\hat{\jmath} \text{ m/s.}
\end{aligned}$$

What is ω_1?

(A) $\omega_1 = 0$ rad/s

(B) $1 \text{ rad/s} \le \omega_1$

(C) $0 \text{ rad/s} < \omega_1 < 1 \text{ rad/s}$

(D) $-1 \text{ rad/s} \le \omega_1 < 0 \text{ rad/s}$

(E) ★ $\omega_1 < -1$ rad/s

Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned}
\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\
&= -3\omega_1 \,\hat{\imath} - 4\omega_1 \,\hat{\jmath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\
6\hat{\imath} - 3\hat{\jmath} &= -3\omega_1 \,\hat{\imath} + (-4\omega_1 - 5\omega_2)\,\hat{\jmath}.
\end{aligned}$$

Comparing $\hat{\imath}$ components gives $\omega_1 = -2$ rad/s.

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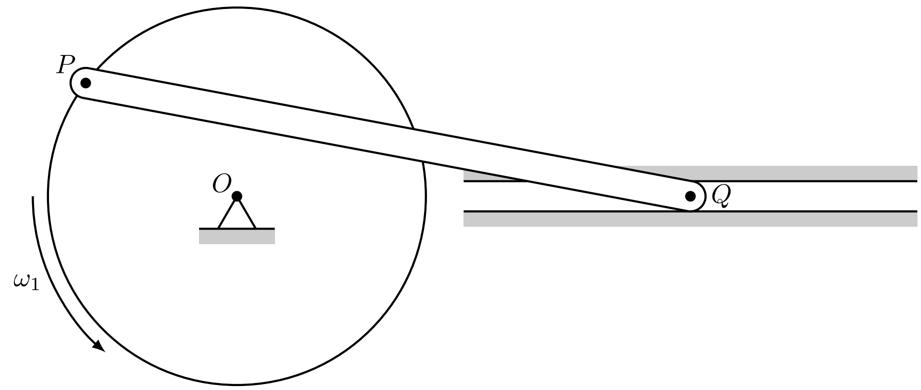
11. (1 point) A circular rigid body rotates about the fixed center O with angular velocity $\vec{\omega}_1 = 2\hat{k}$ rad/s as shown. A rigid rod connects pins P and Q, and point Q is constrained to only move horizontally.



At the current instant the positions are:

$$\begin{aligned}\vec{r}_{OP} &= -4\hat{\imath} + 3\hat{\jmath} \text{ m} \\ \vec{r}_{PQ} &= 16\hat{\imath} - 3\hat{\jmath} \text{ m.}\end{aligned}$$

What is the speed v_Q of point Q?

(A) $0 \text{ m/s} \le v_Q < 3 \text{ m/s}$

(B) $9 \text{ m/s} \le v_Q < 12 \text{ m/s}$

(C) $6 \text{ m/s} \le v_Q < 9 \text{ m/s}$

(D) ★ $3 \text{ m/s} \le v_Q < 6 \text{ m/s}$

(E) $12 \text{ m/s} \le v_Q$

Solution. Let the unknown angular velocity of the rod be $\vec{\omega}_2 = \omega_2\hat{k}$. Starting from the fixed point O, we have:

$$\begin{aligned}\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\ &= 0 + 2\hat{k} \times (-4\hat{\imath} + 3\hat{\jmath}) \\ &= -6\hat{\imath} - 8\hat{\jmath} \\ \vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\ &= (-6\hat{\imath} - 8\hat{\jmath}) + \omega_2\hat{k} \times (16\hat{\imath} - 3\hat{\jmath}) \\ &= (-6 + 3\omega_2)\,\hat{\imath} + (-8 + 16\omega_2)\,\hat{\jmath}.\end{aligned}$$

Zero vertical velocity for Q gives $-8 + 16\omega_2 = 0$, so $\omega_2 = 0.5$ rad/s and $\vec{v}_Q = -4.5\hat{\imath}$ m/s, giving a speed of $v_Q = 4.5$ m/s.

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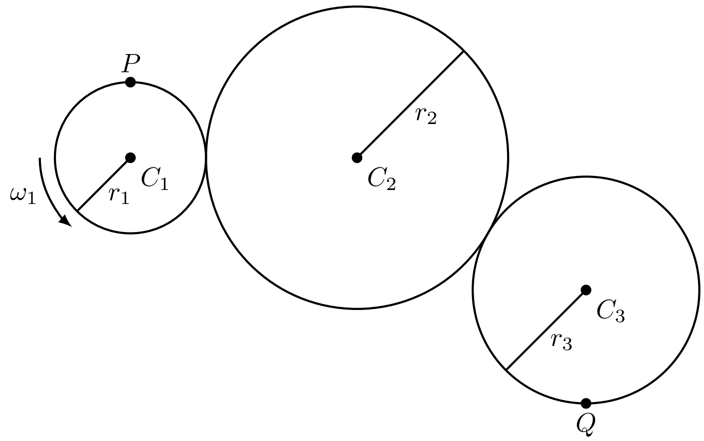
12. (1 point) Three meshed gears rotate about fixed centers as shown. The radii are $r_1 = 2$ m, $r_2 = 4$ m, and $r_3 = 3$ m and the gear at C_1 is rotating counterclockwise as shown.



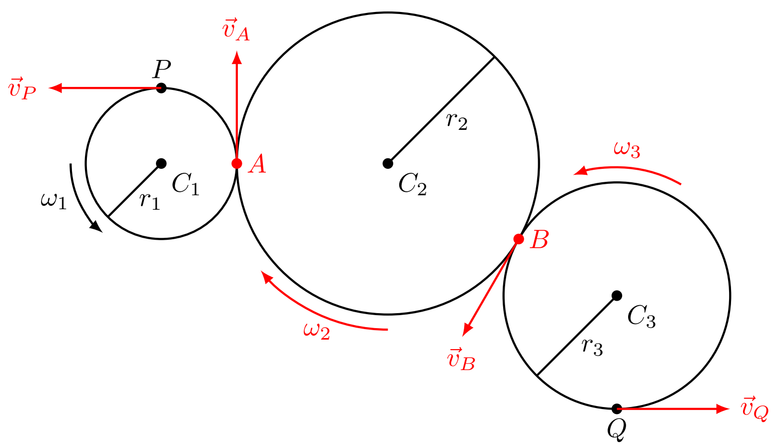
What is the relationship between the speeds v_P and v_Q of points P and Q?

(A) $v_p > v_Q$

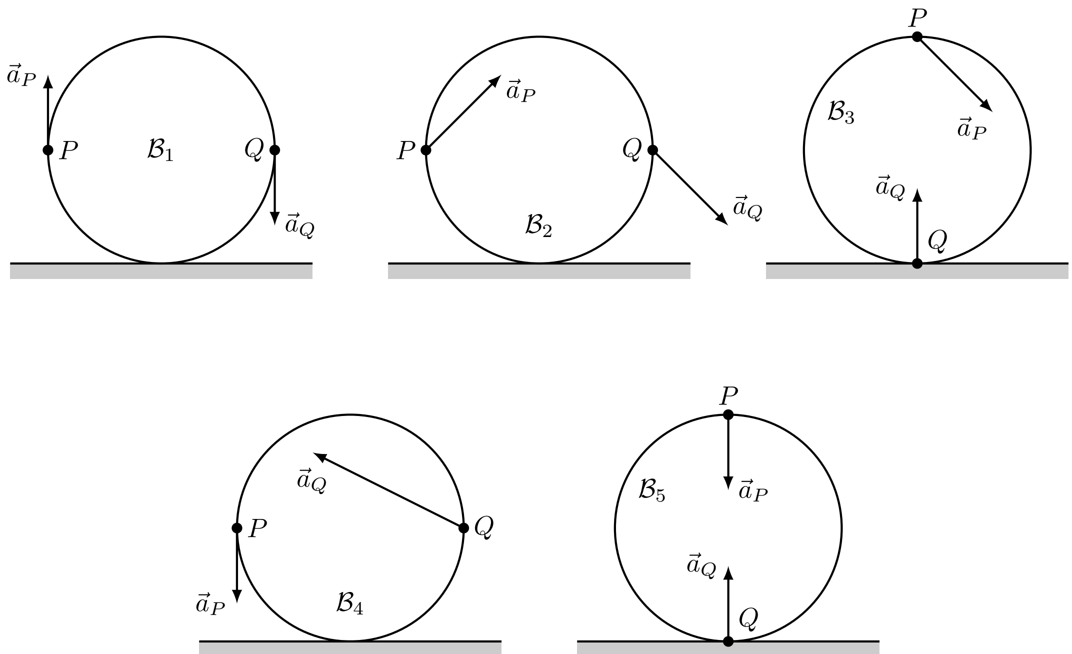
(B) $v_p < v_Q$

(C) ★ $v_p = v_Q$

Solution. Each gear is a circular rigid body with fixed center, so all points on the edge have speed $v = r\omega$ with directions as shown below. Thus $v_P = v_A = v_B = v_Q$.



13. (1 point) Five circular rigid bodies are rolling without slipping as shown, with the accelerations of points P and Q on the bodies as drawn.



Which body does *not* have physically possible accelerations for points P and Q?

(A) ★ $\mathcal{B}_1$

(B) $\mathcal{B}_4$

(C) $\mathcal{B}_2$

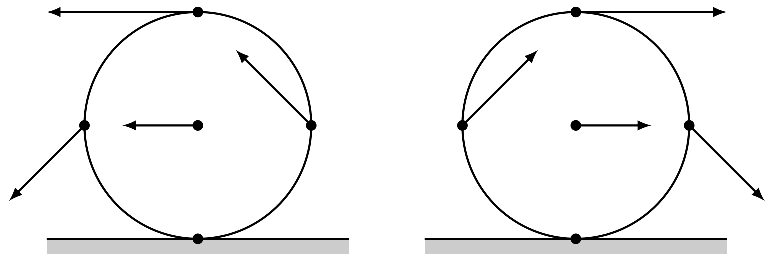
(D) $\mathcal{B}_5$

(E) $\mathcal{B}_3$

Solution. The acceleration of a rolling body is determined by the angular velocity $\vec{\omega} = \omega\hat{k}$ and the angular acceleration $\vec{\alpha} = \alpha\hat{k}$. Taking C to be the center and P to be the contact point, any point Q on the body has acceleration:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= \vec{\alpha} \times \vec{r}_{PC} + \vec{\alpha} \times \vec{r}_{CQ} - \omega^2 \vec{r}_{CQ} \\
&= \vec{\alpha} \times \vec{r}_{PQ} - \omega^2 \vec{r}_{CQ} \\
&= r\alpha \hat{r}_{PQ}^{\perp} - r\omega^2 \hat{r}_{CQ}.
\end{aligned}$$

Thus α causes rotational acceleration about the contact point, while ω causes centripetal acceleration towards the center, as shown below.

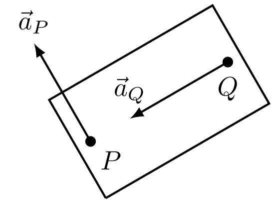


$\omega \neq 0$, $\alpha = 0$ $\qquad$ $\omega = 0$, $\alpha > 0$ $\qquad$ $\omega = 0$, $\alpha < 0$

Combining these, we see that:

- Body $\mathcal{B}_1$ is not possible.
- Body $\mathcal{B}_2$ has $\omega = 0$ and $\alpha < 0$, and it is possible.
- Body $\mathcal{B}_3$ has $\omega \neq 0$ and $\alpha < 0$ so that $2r|\alpha| = r\omega^2$, and it is possible.
- Body $\mathcal{B}_4$ has $\omega \neq 0$ and $\alpha > 0$ so that $r\alpha = r\omega^2$, and it is possible.
- Body $\mathcal{B}_5$ has $\omega \neq 0$ and $\alpha = 0$, and it is possible.

14. (1 point) A rigid body is moving in 2D as shown below. The accelerations of points P and Q on the body are as shown.



What is the direction of the angular acceleration α of the body?

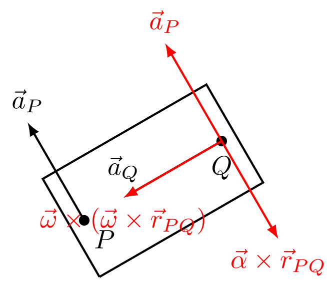
(A) ★ ↻ (clockwise)

(B) ↺ (counterclockwise)

Solution. Starting from P, the acceleration of Q is:

$$\vec{a}_Q = \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}).$$

Drawing the directions of the acceleration terms at Q shows that $\vec{\alpha} \times \vec{r}_{PQ}$ must be down-right, so α must be clockwise.



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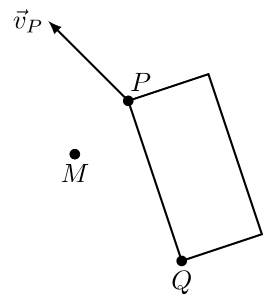
15. (1 point) A rigid body is moving in 2D as shown below with points P and Q attached to the body. The instantaneous center of the body is at point M.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

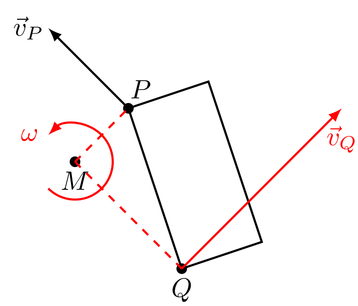
(A) ★ $\nearrow$

(B) $\swarrow$

(C) $\searrow$

(D) $\nwarrow$

Solution. The direction of $\vec{v}_P$ shows that ω is counterclockwise. Then $\vec{v}_Q$ is orthogonal to $\vec{r}_{MQ}$ in a counterclockwise direction, so it is up-right.



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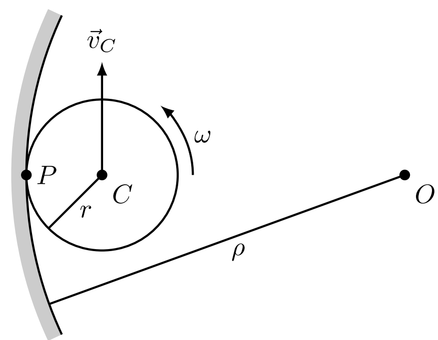
16. (1 point) A circular rigid body with radius $r = 2$ m is rolling without slipping on a curved surface with radius of curvature ρ in 2D as shown. The angular velocity of the body is a constant $\vec{\omega} = 2\hat{k}$ rad/s. Point P is fixed to the edge of the body and, at the instant shown, is the contact point. The magnitude of acceleration of P is $a_P = 10$ rad/s^2.



What is the radius of curvature ρ of the surface?

(A) ★ $9 \text{ m} \leq \rho < 12 \text{ m}$

(B) $12 \text{ m} \leq \rho$

(C) $3 \text{ m} \leq \rho < 6 \text{ m}$

(D) $0 \text{ m} \leq \rho < 3 \text{ m}$

(E) $6 \text{ m} \leq \rho < 9 \text{ m}$

Solution. Because we are rolling on the inside, $R = \rho - r$, giving:

$$\begin{aligned} a_P &= \frac{\rho}{R} r\omega^2 \\ 10 &= \frac{\rho}{\rho - 2} 2 \times 2^2 \\ \rho &= 10 \text{ m}. \end{aligned}$$

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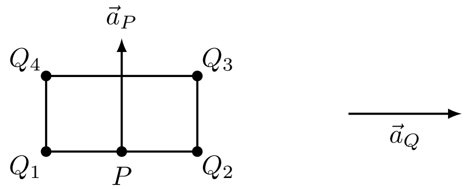
17. (1 point) A rigid body is moving in 2D as shown below with zero angular acceleration and some angular velocity. The acceleration of point P is shown, as is the acceleration $\vec{a}_Q$ of one of the Q_i points.



Which point Q_i has the given acceleration $\vec{a}_Q$?

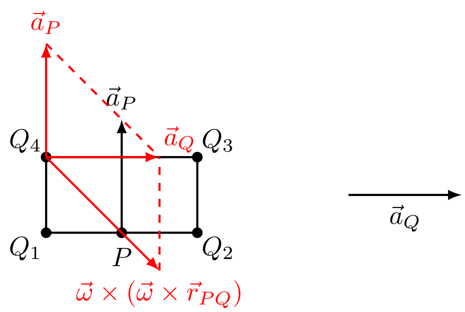
(A) ★ Q_4

(B) Q_1

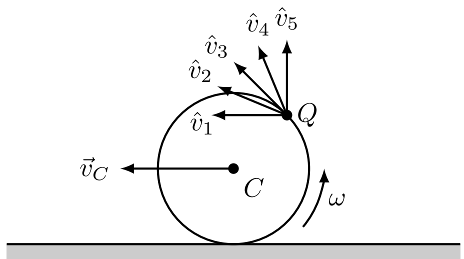
(C) Q_2

(D) Q_3

Solution. Only at Q_4 will the $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ})$ term have both negative $\hat{\jmath}$ component (to cancel out $\vec{a}_P$) and positive $\hat{\imath}$ component (to give $\vec{a}_Q$).



18. (1 point) A circular rigid body is rolling without slipping on a flat surface in 2D in a counterclockwise direction as shown.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

(A) $\hat{v}_1$

(B) $\hat{v}_4$

(C) ★ $\hat{v}_2$

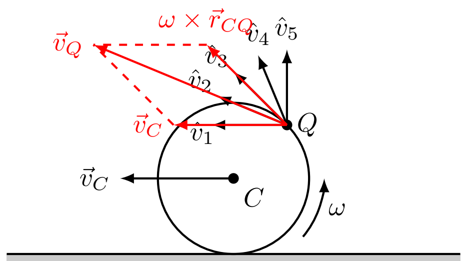
(D) $\hat{v}_5$

(E) $\hat{v}_3$

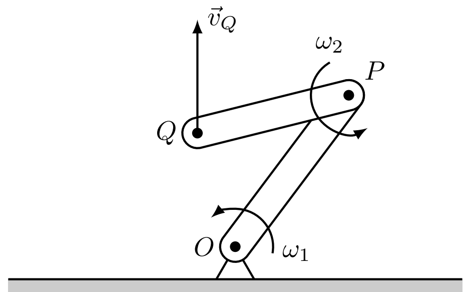
Solution. Starting from point C, the velocity at point Q is given by:

$$\vec{v}_Q = \vec{v}_C + \vec{\omega} \times \vec{r}_{CQ}.$$

The two terms on the right hand side above both have magnitude ωr, where r is the radius of the body. The $\vec{\omega} \times \vec{r}_{CQ}$ term is orthogonal to $\vec{r}_{CP}$ (in the $\hat{v}_3$ direction), while the $\vec{v}_C$ term is in the $\hat{v}_1$ direction. Adding the two terms thus gives a resultant $\vec{v}_Q$ exactly half way between $\hat{v}_1$ and $\hat{v}_3$, which is the $\hat{v}_2$ direction.



19. (1 point) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = -\hat{k}$ rad/s and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The velocity $\vec{v}_Q$ of point Q is directly upwards and the positions of the rods are:

$$\vec{r}_{OP} = 3\hat{\imath} + 4\hat{\jmath} \text{ m}$$
$$\vec{r}_{PQ} = -4\hat{\imath} - \hat{\jmath} \text{ m}$$

What is the speed v_Q of point Q?

(A) 9 m/s $\leq v_Q <$ 12 m/s

(B) 3 m/s $\leq v_Q <$ 6 m/s

(C) 6 m/s $\leq v_Q <$ 9 m/s

(D) ★ 12 m/s $\leq v_Q$

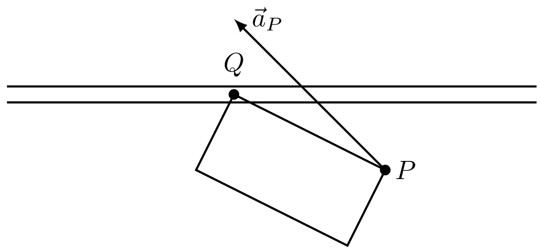
(E) 0 m/s $\leq v_Q <$ 3 m/s

Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned}
\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\
&= 4\hat{\imath} - 3\hat{\jmath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\
&= (4 + \omega_2)\,\hat{\imath} + (-3 - 4\omega_2)\,\hat{\jmath} \text{ m/s.}
\end{aligned}$$

Zero horizontal component implies $4 + \omega_2 = 0$, so $\omega_2 = -4$ rad/s. Then $\vec{v}_Q = 13\hat{\jmath}$ m/s, giving $v_Q = 13$ m/s.

20. (1 point) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega \hat{k}$ and zero angular acceleration. A pin at point Q constrains that point to move in a horizontal slot.



Point P on the body has:

$$\begin{aligned} \vec{r}_{PQ} &= -2\hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{a}_P &= -2\hat{\imath} + 2\hat{\jmath} \text{ m/s.} \end{aligned}$$

What is the magnitude a_Q of the acceleration $\vec{a}_Q$ of point Q?

(A) $1 \text{ m/s}^2 \leq a_Q < 2 \text{ m/s}^2$

(B) $3 \text{ m/s}^2 \leq a_Q < 4 \text{ m/s}^2$

(C) $0 \text{ m/s}^2 \leq a_Q < 1 \text{ m/s}^2$

(D) $4 \text{ m/s}^2 \leq a_Q$

(E) ★ $2 \text{ m/s}^2 \leq a_Q < 3 \text{ m/s}^2$

Solution. Taking $\vec{a}_Q = a_Q \, \hat{\imath}$, we have:

$$\begin{aligned} \vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ a_Q \, \hat{\imath} &= -2\hat{\imath} + 2\hat{\jmath} + \omega\hat{k} \times (\omega\hat{k} \times (-2\hat{\imath} + \hat{\jmath})) \\ &= -2\hat{\imath} + 2\hat{\jmath} + 2\omega^2 \, \hat{\imath} - \omega^2 \, \hat{\jmath} \\ &= (-2 + 2\omega^2) \, \hat{\imath} + (2 - \omega^2) \, \hat{\jmath}. \end{aligned}$$

Equating components and solving gives:

$$\begin{aligned} 0 &= 2 - \omega^2 \\ \Longrightarrow \omega^2 &= 2 \\ a_Q &= -2 + 2\omega^2 \\ &= 2 \text{ m/s}^2. \end{aligned}$$

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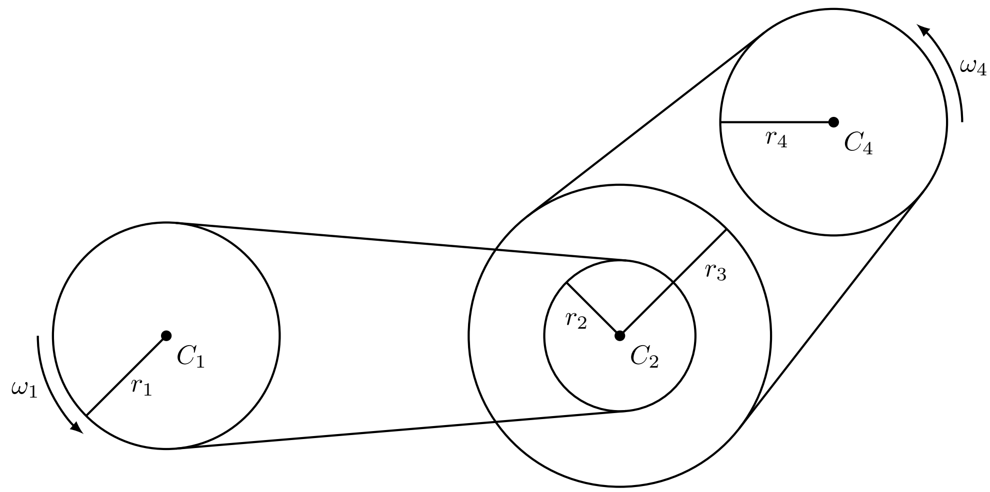
21. (1 point) Four gears are positioned with fixed centers and two chains connect pairs of gears as shown. The gears have radii $r_1 = 3$ m, $r_2 = 2$ m, $r_3 = 4$ m, and $r_4 = 3$ m, and the two gears centered at C_2 are locked together so they have the same angular velocity. The gear centered at C_1 has angular velocity $\vec{\omega}_1 = 3\hat{k}$ rad/s, while the gear at C_4 has angular velocity $\vec{\omega}_4 = \omega_4\hat{k}$.



What is ω_4?

(A) ★ 4 rad/s $\leq \omega_4$

(B) 3 rad/s $\leq \omega_4 <$ 4 rad/s

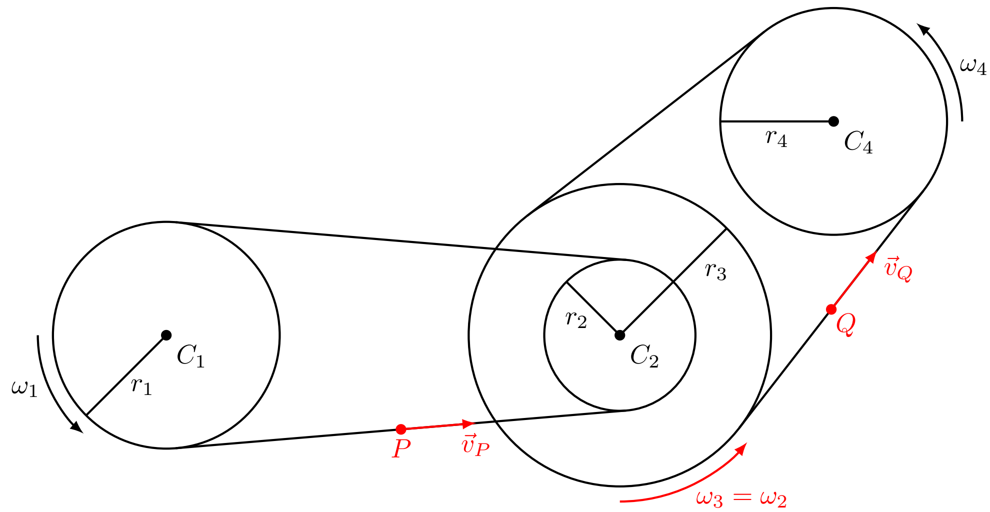
(C) 0 rad/s $\leq \omega_4 <$ 1 rad/s

(D) 2 rad/s $\leq \omega_4 <$ 3 rad/s

(E) 1 rad/s $\leq \omega_4 <$ 2 rad/s

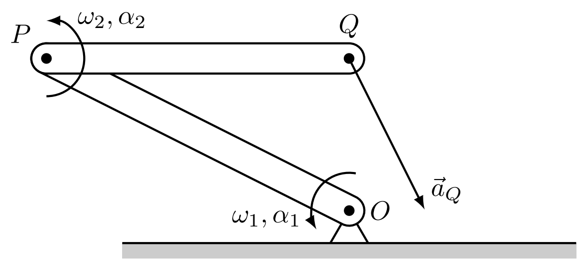
Solution. Each chain has the same speed for all points on it, and the two gears centered at C_2 have the same angular velocity $\omega_2 = \omega_3$. This gives:

$$\begin{aligned} r_1\omega_1 = v_P = r_2\omega_2 &\Longrightarrow \omega_2 = 4.5 \\ \omega_3 = \omega_2 &= 4.5 \\ r_3\omega_3 = v_Q = r_4\omega_4 &\Longrightarrow \omega_4 = 6 \text{ rad/s}. \end{aligned}$$



ω_4
r_4
C_4
r_3
$\vec{v}_Q$
r_2
Q
C_1
C_2
ω_1
r_1
P
$\vec{v}_P$
$\omega_3 = \omega_2$

22. (1 point) Two rods are connected with pin joints at O, P, and Q as shown. The rods have angular velocities and angular accelerations as indicated.



At a particular instant we observe:

$$\vec{r}_{OP} = -2\hat{\imath} + \hat{\jmath} \text{ m} \qquad \vec{r}_{PQ} = 2\hat{\imath} \text{ m}$$
$$\vec{\omega}_1 = \hat{k} \text{ rad/s} \qquad \vec{\omega}_2 = -\hat{k} \text{ rad/s}$$
$$\vec{\alpha}_1 = \alpha_1 \hat{k} \qquad \vec{\alpha}_2 = \alpha_2 \hat{k}$$
$$\vec{a}_Q = \hat{\imath} - 2\hat{\jmath} \text{ m/s}^2.$$

What is α_2?

(A) $-1 \text{ rad/s}^2 \leq \alpha_2 < 0 \text{ rad/s}^2$

(B) $1 \text{ rad/s}^2 \leq \alpha_2$

(C) $0 \text{ rad/s}^2 < \alpha_2 < 1 \text{ rad/s}^2$

(D) $\alpha_2 = 0 \text{ rad/s}^2$

(E) ★ $\alpha_2 < -1 \text{ rad/s}^2$

Solution. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{a}_P &= \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OP} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OP}) \\
&= 0 + \alpha_1 \hat{k} \times (-2\hat{\imath} + \hat{\jmath}) + \hat{k} \times (\hat{k} \times (-2\hat{\imath} + \hat{\jmath})) \\
&= (-\alpha_1 \,\hat{\imath} - 2\alpha_1 \,\hat{\jmath}) + (2\hat{\imath} - \hat{\jmath}) \\
&= (2 - \alpha_1)\,\hat{\imath} + (-1 - 2\alpha_1)\,\hat{\jmath} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_w \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\
\hat{\imath} - 2\hat{\jmath} &= (2 - \alpha_1)\,\hat{\imath} + (-1 - 2\alpha_1)\,\hat{\jmath} + \alpha_2 \hat{k} \times 2\hat{\imath} + (-\hat{k}) \times ((-\hat{k}) \times 2\hat{\imath}) \\
&= -\alpha_1 \,\hat{\imath} + (-1 - 2\alpha_1 + 2\alpha_2)\,\hat{\jmath} \text{ m/s}^2.
\end{aligned}$$

Equating components gives:

$$\begin{aligned}
1 &= -\alpha_1 \implies \alpha_1 = -1 \text{ rad/s}^2 \\
-2 &= -1 - 2\alpha_1 + 2\alpha_2 = 1 + 2\alpha_2 \implies \alpha_2 = -1.5 \text{ rad/s}^2.
\end{aligned}$$

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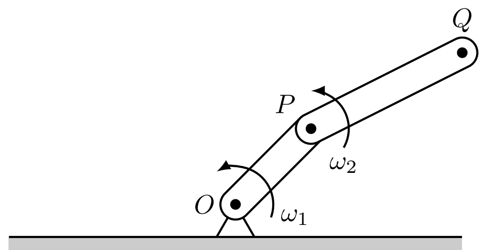
23. (1 point) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = \omega_1 \hat{k}$ and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The positions and angular velocities of the rods at the current instant are:

$$\vec{r}_{OP} = \hat{\imath} + \hat{\jmath} \text{ m} \qquad \vec{r}_{PQ} = 2\hat{\imath} + \hat{\jmath} \text{ m}$$
$$\vec{\omega}_1 = 2\hat{k} \text{ rad/s} \qquad \vec{\omega}_2 = -\hat{k} \text{ rad/s.}$$

What is the $\hat{\jmath}$ component v_{Qy} of the velocity $\vec{v}_Q$ of point Q?

(A) $-2 \text{ m/s} \leq v_{Qy} < 0 \text{ m/s}$

(B) ★ $v_{Qy} = 0 \text{ m/s}$

(C) $0 \text{ m/s} < v_{Qy} < 2 \text{ m/s}$

(D) $v_{Qy} < -2 \text{ m/s}$

(E) $2 \text{ m/s} \leq v_{Qy}$

Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned}
\vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\
&= -2\hat{\imath} + 2\hat{\jmath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\
&= (-2\hat{\imath} + 2\hat{\jmath}) + (\hat{\imath} - 2\hat{\jmath}) \\
&= -\hat{\imath} \\
v_{Qy} &= 0 \text{ m/s.}
\end{aligned}$$

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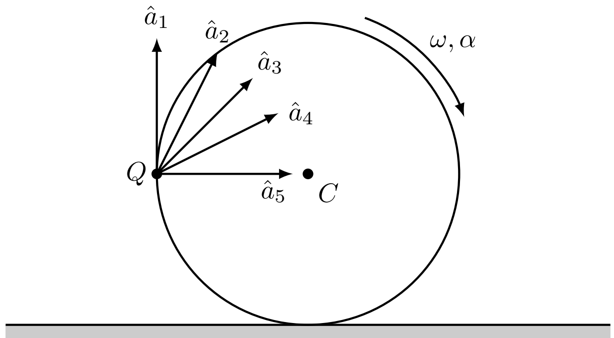
24. (1 point) A circular rigid body is rolling without slipping on a flat surface in 2D as shown. The angular velocity $\vec{\omega} = -\omega\hat{k}$ and angular acceleration $\vec{\alpha} = -\alpha\hat{k}$ satisfy $\alpha = \omega^2$.



What is the direction of the acceleration $\vec{a}_Q$ of point Q?

(A) $\hat{a}_5$

(B) $\hat{a}_3$

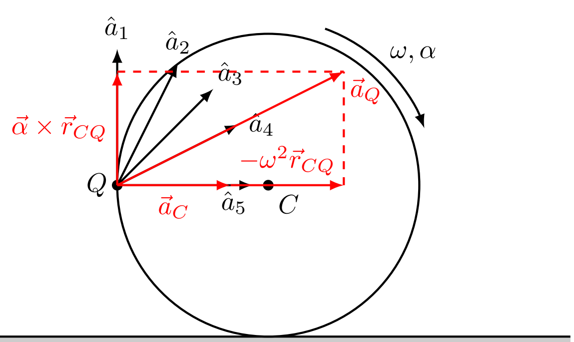
(C) ★ $\hat{a}_4$

(D) $\hat{a}_2$

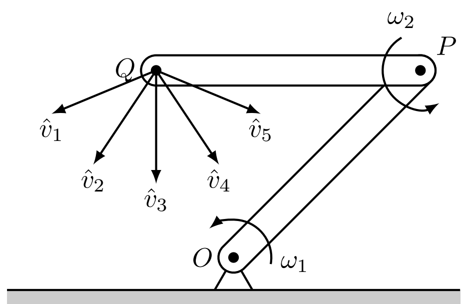
(E) $\hat{a}_1$

Solution. The acceleration of Q is given by:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= r\alpha\,\hat{\imath} + (-\alpha\hat{k}) \times (-r\hat{\imath}) + (-\omega\hat{k}) \times (-\omega\hat{k} \times (-r\hat{\imath})) \\
&= r\alpha\,\hat{\imath} + r\alpha\,\hat{\jmath} + r\omega^2\,\hat{\imath} \\
&= r\alpha\,\hat{\imath} + r\alpha\,\hat{\jmath} + r\alpha\,\hat{\imath} \\
&= 2r\alpha\,\hat{\imath} + r\alpha\,\hat{\jmath}.
\end{aligned}$$



25. (1 point) Two equal-length rods are connected with pin joints at O, P, and Q as shown, so that the line OP is at 45° from horizontal. Both rods are rotating counterclockwise with the same angular velocity.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

(A) $\hat{v}_4$

(B) $\hat{v}_3$

(C) $\hat{v}_2$

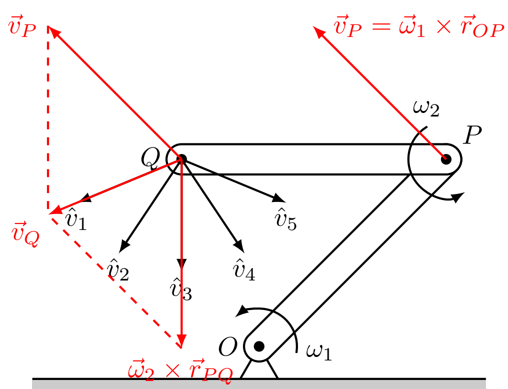
(D) ★ $\hat{v}_1$

(E) $\hat{v}_5$

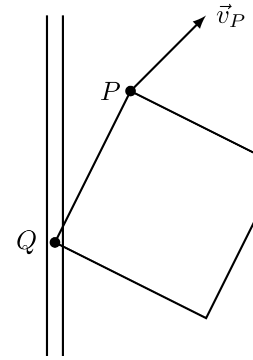
Solution. For rod lengths r and angular velocity ω, the velocity $\vec{v}_P$ is up-left at 45° with magnitude $r\omega$. Then:

$$\vec{v}_Q = \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ}.$$

The term $\vec{\omega}_2 \times \vec{r}_{PQ}$ is straight down, also with magnitude $r\omega$. Adding this term to $\vec{v}_P$ results in $\vec{v}_Q$ to the down-left at an angle of 22.5° below the horizontal.



26. (1 point) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$. A pin at point Q constrains that point to move in a vertical slot.



Point P on the body has:

$$\vec{r}_{PQ} = -\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{v}_P = \hat{\imath} + \hat{\jmath} \text{ m/s}.$$

What is ω?

(A) 1 rad/s $\leq \omega$

(B) ★ -1 rad/s $\leq \omega < 0$ rad/s

(C) 0 rad/s $< \omega < 1$ rad/s

(D) $\omega < -1$ rad/s

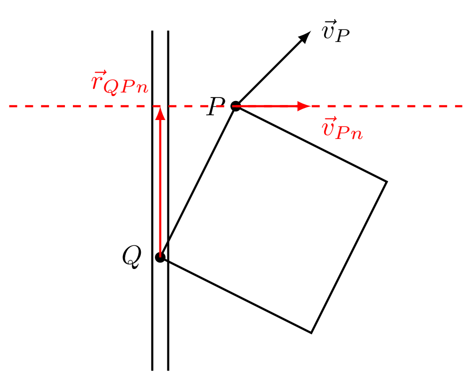
(E) $\omega = 0$ rad/s

Solution. Taking $\vec{v}_Q = v_Q\,\hat{\jmath}$ with unknown speed v_Q, we have:

$$\begin{aligned}
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
v_Q\,\hat{\jmath} &= \hat{\imath} + \hat{\jmath} + \omega\hat{k} \times (-\hat{\imath} - 2\hat{\jmath}) \\
&= \hat{\imath} + \hat{\jmath} + 2\omega\,\hat{\imath} - \omega\,\hat{\jmath} \\
&= (1 + 2\omega)\,\hat{\imath} + (1 - \omega)\,\hat{\jmath}
\end{aligned}$$

Equating $\hat{\imath}$ components gives $0 = 1 + 2\omega$, so $\omega = -0.5$ rad/s.

Alternatively, a fast solution is to observe that the rotation is caused by the normal velocity component $v_{Pn} = 1$ m/s acting at a distance $r_{PQt} = 2$ m, so the angular velocity is 1/2 rad/s clockwise.



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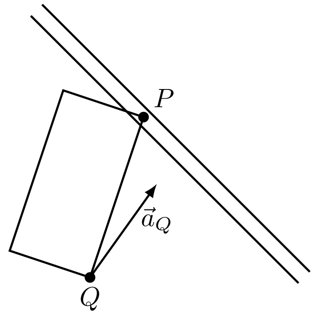
27. (1 point) A rigid body is moving in 2D as shown below with zero angular acceleration. A pin at point P constrains that point to move in an angled slot.



In which direction is the acceleration $\vec{a}_P$ of point P?

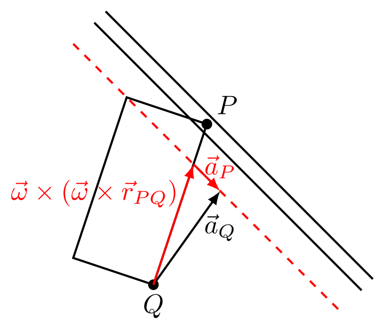
(A) $\swarrow$

(B) $\nwarrow$

(C) $\nearrow$

(D) ★ $\searrow$

Solution. $\vec{a}_Q$ must consist of the centripetal term $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ})$ plus the acceleration $\vec{a}_P$ in the slot direction, as shown below, so $\vec{a}_P$ must be down-right.



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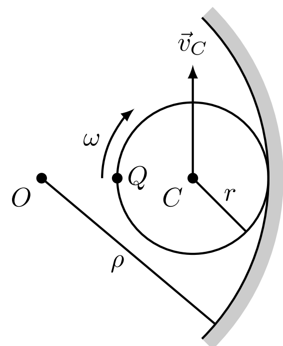
28. (1 point) A circular rigid body with radius $r = 1$ m is rolling without slipping on a curved surface with radius of curvature $\rho = 3$ m in 2D as shown. The speed v_C of the center is a constant $v_C = 2$ m/s. Point Q is fixed to the edge of the body and, at the instant shown, the points O-Q-C form a horizontal line.



What is the $\hat{\imath}$ component a_{Qx} of the acceleration $\vec{a}_Q$ of point Q?

(A) $-2 \text{ m/s}^2 \leq a_{Qx} < 0 \text{ m/s}^2$

(B) $a_{Qx} < -2 \text{ m/s}^2$

(C) $0 \text{ m/s}^2 < a_{Qx} < 2 \text{ m/s}^2$

(D) $a_{Qx} = 0 \text{ m/s}^2$

(E) ★ $2 \text{ m/s}^2 \leq a_{Qx}$

Solution. The angular velocity is $\omega = v_C/r = 2$ rad/s. Because the speed v_C is constant, the tangential acceleration $r\alpha\,\hat{e}_t$ is zero and so $\alpha = 0 \text{ rad/s}^2$. The radius of curvature of the center is $R = \rho - r = 2$ m, and the local basis vectors are $\hat{e}_t = \hat{\jmath}$ and $\hat{e}_n = -\hat{\imath}$. Thus:

$$\begin{aligned}
\vec{a}_C &= r\alpha\,\hat{e}_t + \frac{v_C^2}{R}\,\hat{e}_n \\
&= 2\hat{e}_n \\
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= 2\hat{e}_n + r\omega^2\,\hat{\imath} \\
&= -2\hat{\imath} + 4\hat{\imath} \\
&= 2\hat{\imath} \\
a_{Qx} &= 2 \text{ m/s}^2
\end{aligned}$$

---

29. (1 point) A rigid body is moving in 2D with angular velocity $\vec{\omega} = \omega\hat{k}$. Two points P and Q are fixed to the body and have:

$$\begin{aligned}
\vec{r}_{PQ} &= 2\hat{\imath} - 2\hat{\jmath} \text{ m} \\
\vec{v}_P &= 3\hat{\jmath} \text{ m/s} \\
\vec{v}_Q &= -2\hat{\imath} + \hat{\jmath} \text{ m/s.}
\end{aligned}$$

What is ω?

(A) $2 \text{ rad/s} \leq \omega$

(B) $\omega = 0 \text{ rad/s}$

(C) $\omega < -2 \text{ rad/s}$

(D) $0 \text{ rad/s} < \omega < 2 \text{ rad/s}$

(E) ★ $-2 \text{ rad/s} \leq \omega < 0 \text{ rad/s}$

Solution.

$$\begin{aligned}
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
-2\hat{\imath} + \hat{\jmath} &= 3\hat{\jmath} + \omega\hat{k} \times (2\hat{\imath} - 2\hat{\jmath}) \\
-2\hat{\imath} - 2\hat{\jmath} &= 2\omega\,\hat{\imath} - 2\omega\,\hat{\jmath} \\
\implies \omega &= -1 \text{ rad/s.}
\end{aligned}$$

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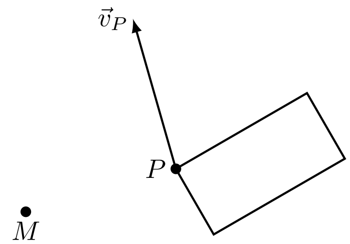
30. (1 point) A rigid body is moving in 2D as shown below. The instantaneous center of the body is at point M.

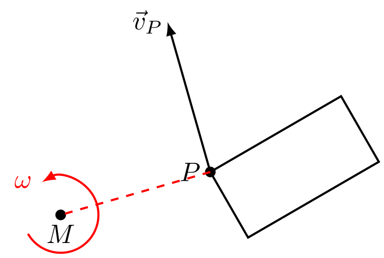


What is the direction of the angular velocity of the body?

(A) ↻ (clockwise)

(B) ★ ↺ (counterclockwise)

Solution. The velocity $\vec{v}_P$ is caused by rotation of $\vec{r}_{MP}$ about M, so this rotation must be counterclockwise.



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31. (1 point) A rigid body is moving in 2D with points P and Q attached to it. We have:

$$\vec{r}_P = -\hat{\imath} - 2\hat{\jmath} \text{ m} \qquad \vec{r}_Q = \hat{\imath} - \hat{\jmath} \text{ m}$$
$$\vec{v}_P = 2\hat{\imath} + \hat{\jmath} \text{ m/s} \qquad \vec{v}_Q = 3\hat{\imath} - \hat{\jmath} \text{ m/s.}$$

What is the x coordinate M_x of the instantaneous center M of the body?

(A) ★ $M_x = 0$ m

(B) $-2 \text{ m} \leq M_x < 0$ m

(C) $0 \text{ m} < M_x < 2$ m

(D) $M_x < -2$ m

(E) $2 \text{ m} \leq M_x$

Solution. We first determine the angular velocity $\vec{\omega} = \omega\hat{k}$:

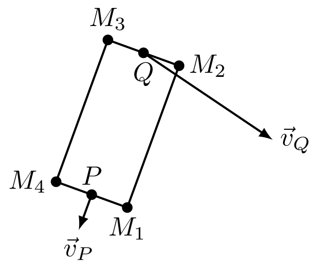
$$\begin{aligned}
\vec{r}_{PQ} &= \vec{r}_Q - \vec{r}_P \\
&= 2\hat{\imath} + \hat{\jmath} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
3\hat{\imath} - \hat{\jmath} &= 2\hat{\imath} + \hat{\jmath} + \omega\hat{k} \times (2\hat{\imath} + \hat{\jmath}) \\
&= (2 - \omega)\,\hat{\imath} + (1 + 2\omega)\,\hat{\jmath}.
\end{aligned}$$

Equating terms and solving gives $\omega = -1$ rad/s (both components must give the same ω solution). Now we find the instantaneous center:

$$\begin{aligned}
\vec{r}_{PM} &= \frac{1}{\omega}\vec{v}_P^{\perp} \\
&= \frac{1}{-1}(-\hat{\imath} + 2\hat{\jmath}) \\
&= \hat{\imath} - 2\hat{\jmath} \\
\vec{r}_M &= \vec{r}_P + \vec{r}_{PM} \\
&= -4\hat{\jmath} \\
M_x &= 0 \text{ m.}
\end{aligned}$$

---

32. (1 point) A rigid body is moving in 2D as shown below.



Which point M_i is the instantaneous center?

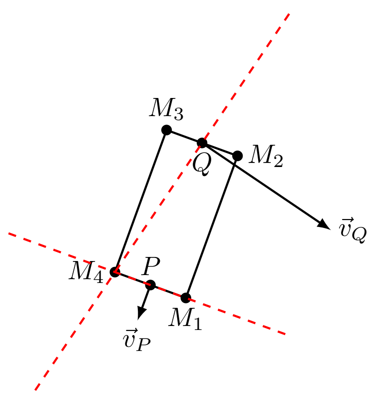
(A) M_1

(B) M_3

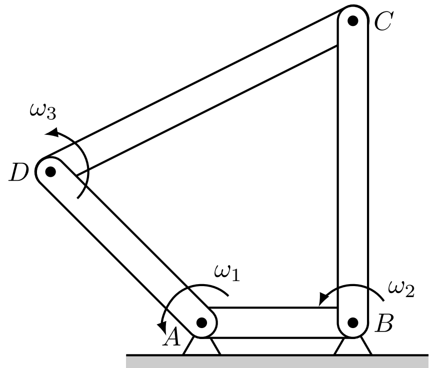
(C) M_2

(D) ★ M_4

Solution. The lines through P and Q perpendicular to $\vec{v}_P$ and $\vec{v}_Q$ intersect at M_4, so that is the instantaneous center.



33. (1 point) A four-bar linkage has rigid rods connecting pins at A, B, C, and D, as shown. The angular velocities are $\vec{\omega}_1 = 2\hat{k}$ for rod AD, $\vec{\omega}_2 = \omega_2\hat{k}$ for rod BC, and $\vec{\omega}_3 = \omega_3\hat{k}$ for rod DC.



At the current instant the positions are:

$$\begin{aligned}\vec{r}_{AB} &= \hat{\imath} \text{ m} \\ \vec{r}_{BC} &= 2\hat{\jmath} \text{ m} \\ \vec{r}_{AD} &= -\hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{r}_{DC} &= 2\hat{\imath} + \hat{\jmath} \text{ m.}\end{aligned}$$

What is ω_2?

(A) 0 rad/s $\leq \omega_2 <$ 0.5 rad/s

(B) 2 rad/s $\leq \omega_2$

(C) ★ 1.5 rad/s $\leq \omega_2 <$ 2 rad/s

(D) 1 rad/s $\leq \omega_2 <$ 1.5 rad/s

(E) 0.5 rad/s $\leq \omega_2 <$ 1 rad/s

Solution. Starting from the fixed point A we have:

$$\begin{aligned}\vec{v}_D &= \vec{v}_A + \vec{\omega}_1 \times \vec{r}_{AD} \\ &= 0 + 2\hat{k} \times (-\hat{\imath} + \hat{\jmath}) \\ &= -2\hat{\imath} - 2\hat{\jmath} \\ \vec{v}_C &= \vec{v}_D + \vec{\omega}_3 \times \vec{r}_{DC} \\ &= (-2\hat{\imath} - 2\hat{\jmath}) + \omega_3\hat{k} \times (2\hat{\imath} + \hat{\jmath}) \\ &= (-2 - \omega_3)\,\hat{\imath} + (-2 + 2\omega_3)\,\hat{\jmath}.\end{aligned}$$

We can also get to C from the fixed point B:

$$\begin{aligned}\vec{v}_C &= \vec{v}_B + \vec{\omega}_2 \times \vec{r}_{BC} \\ &= 0 + \omega_2\hat{k} \times 2\hat{\jmath} \\ &= -2\omega_2\,\hat{\imath}.\end{aligned}$$

Equating the two expressions for $\vec{v}_C$ gives:

$$(-2 - \omega_3)\,\hat{\imath} + (-2 + 2\omega_3)\,\hat{\jmath} = -2\omega_2\,\hat{\imath}.$$

The $\hat{\jmath}$ components gives $\omega_3 = 1$, and the $\hat{\imath}$ components give $\omega_2 = 1 + \omega_3/2 = 1.5$ rad/s.

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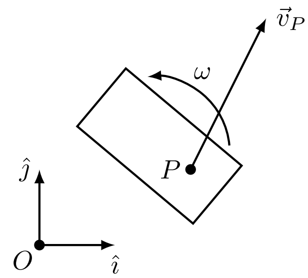
34. (1 point) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = 2\hat{k}$ rad/s.



Relative to the origin O, the point P has:

$$\begin{aligned}\vec{r}_P &= 2\hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{v}_P &= \hat{\imath} + 2\hat{\jmath} \text{ m/s.}\end{aligned}$$

What is the y coordinate M_y of the instantaneous center M of the body?

(A) $4 \text{ m} \leq M_y < 5 \text{ m}$

(B) $3 \text{ m} \leq M_y < 4 \text{ m}$

(C) ★ $M_y < 2 \text{ m}$

(D) $5 \text{ m} \leq M_y$

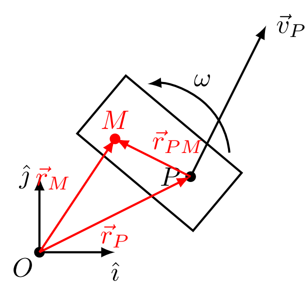
(E) $2 \text{ m} \leq M_y < 3 \text{ m}$

Solution. The position of M relative to P is:

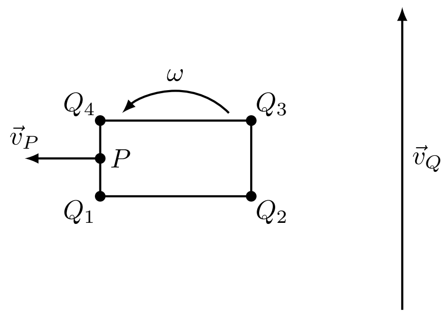
$$\begin{aligned}\vec{r}_{PM} &= \frac{1}{\omega^2}\,\omega\hat{k} \times \vec{v}_P \\ &= \frac{1}{2^2}\,2\hat{k} \times (\hat{\imath} + 2\hat{\jmath}) \\ &= \frac{1}{4}(-4\hat{\imath} + 2\hat{\jmath}) \\ &= -\hat{\imath} + 0.5\hat{\jmath} \text{ m.}\end{aligned}$$

Then the position of M is:

$$\begin{aligned}\vec{r}_M &= \vec{r}_P + \vec{r}_{PM} \\ &= (2\hat{\imath} + \hat{\jmath}) + (-\hat{\imath} + 0.5\hat{\jmath}) \\ &= \hat{\imath} + 1.5\hat{\jmath} \text{ m} \\ M_y &= 1.5 \text{ m.}\end{aligned}$$



35. (1 point) A rigid body is moving in 2D as shown below with counterclockwise angular velocity. The velocity of point P is shown, as is the velocity $\vec{v}_Q$ of one of the Q_i points.



Which point Q_i has the given velocity $\vec{v}_Q$?

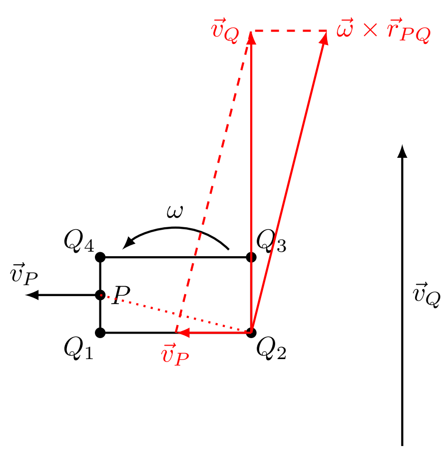
(A) Q_4

(B) Q_1

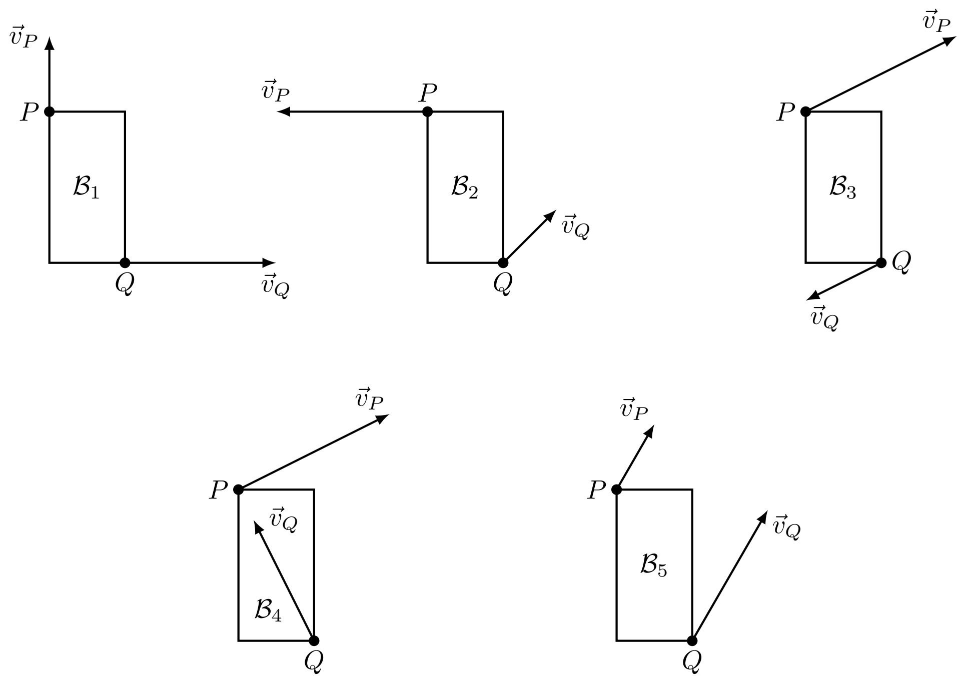
(C) Q_3

(D) ★ Q_2

Solution. Only at Q_2 will the $\vec{\omega} \times \vec{r}_{PQ}$ term have positive $\hat{\imath}$ component (to cancel out $\vec{v}_P$) and positive $\hat{\jmath}$ component (to give $\vec{v}_Q$).



36. (1 point) Five bodies moving in 2D are shown below with the velocities of points P and Q on the bodies as drawn.



Which body has physically possible velocities for point P and Q?

(A) $\mathcal{B}_1$

(B) $\mathcal{B}_2$

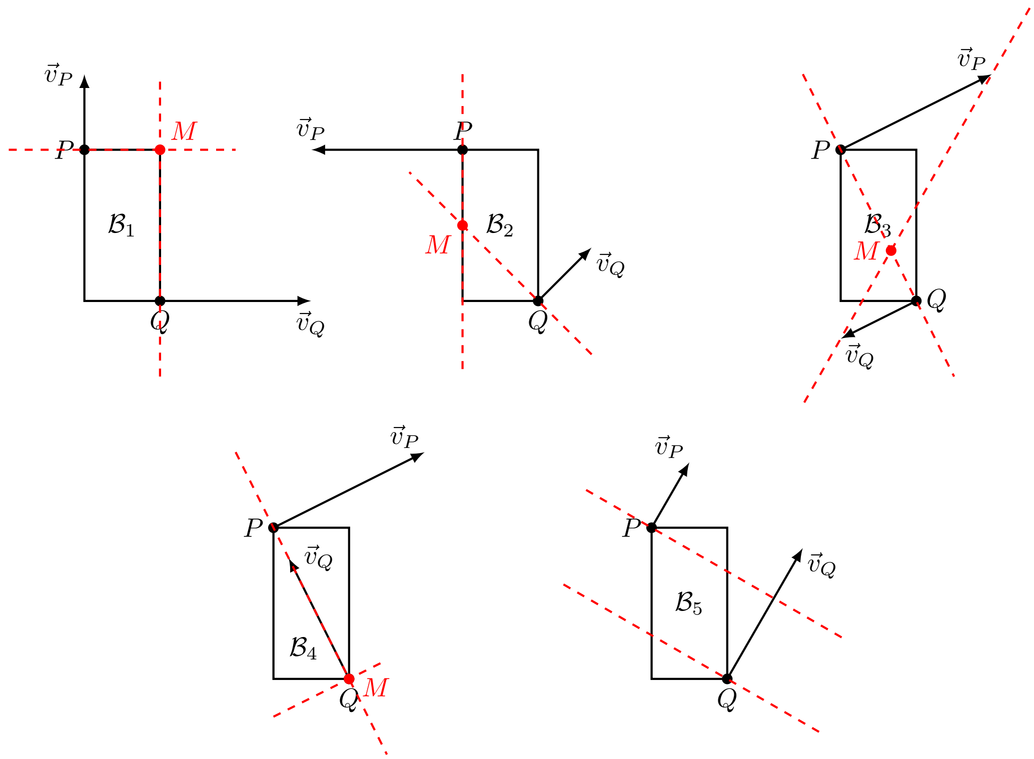
(C) ★ $\mathcal{B}_3$

(D) $\mathcal{B}_5$

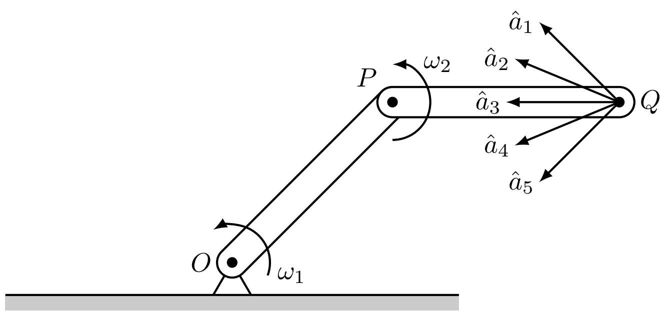
(E) $\mathcal{B}_4$

Solution.

- Body $\mathcal{B}_1$ has P rotating clockwise and Q rotating counterclockwise about M, so it is impossible.
- Body $\mathcal{B}_2$ has P closer to M than Q, but v_P is larger than v_Q, so it is impossible.
- Body $\mathcal{B}_3$ is possible.
- Body $\mathcal{B}_4$ has M at Q, but $\vec{v}_Q \neq 0$, so it is impossible.
- Body $\mathcal{B}_5$ does not have an instantaneous center, and the velocities of P and Q are not equal, so it is impossible.


$\vec{v}_P$
P
M
$\mathcal{B}_1$
Q
$\vec{v}_Q$
$\vec{v}_P$
P
M
$\mathcal{B}_2$
$\vec{v}_Q$
Q
$\vec{v}_P$
P
$\mathcal{B}_3$
M
Q
$\vec{v}_Q$
$\vec{v}_P$
P
$\vec{v}_Q$
$\mathcal{B}_4$
Q
M
$\vec{v}_P$
P
$\vec{v}_Q$
$\mathcal{B}_5$
Q

37. (1 point) Two equal-length rods are connected with pin joints at O, P, and Q as shown, so that the line OP is at 45° from horizontal. Both rods are rotating counterclockwise with the same constant angular velocity.



What is the direction of the acceleration $\vec{a}_Q$ of point Q?

(A) $\hat{a}_2$

(B) ★ $\hat{a}_4$

(C) $\hat{a}_5$

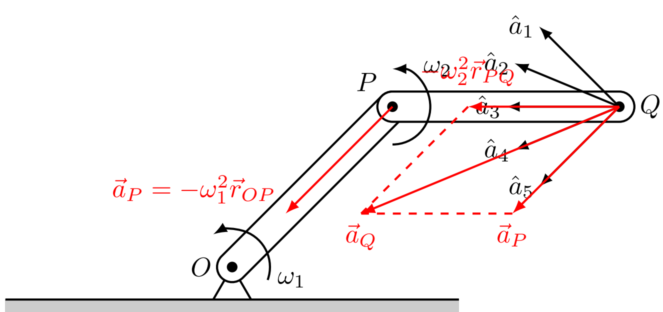
(D) $\hat{a}_3$

(E) $\hat{a}_1$

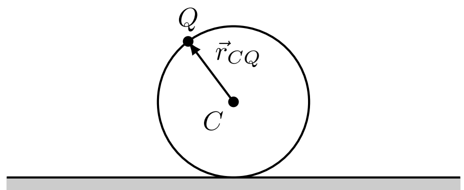
Solution. The angular accelerations are both zero, so the acceleration of P is $\vec{a}_P = \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OP}) = -\omega_1^2 \vec{r}_{OP}$. Then:

$$\begin{aligned}\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ &= -\omega_1^2 \vec{r}_{OP} - \omega_2^2 \vec{r}_{PQ}.\end{aligned}$$

Because $\omega_1 = \omega_2$ and $r_{OP} = r_{PQ}$, the acceleration $\vec{a}_Q$ is exactly midway between the two components, at an angle of 22.5° below the horizontal.



38. (1 point) A circular rigid body is rolling without slipping with angular velocity $\vec{\omega} = -\hat{k}$ and angular acceleration $\vec{\alpha} = 2\hat{k}$ on a flat surface in 2D as shown. Point Q is offset from the center C by $\vec{r}_{CQ} = -3\hat{\imath}+4\hat{\jmath}$ m.



What is the $\hat{\jmath}$ component a_{Qy} of the acceleration $\vec{a}_Q$ of point Q?

(A) ★ $a_{Qy} < -8 \text{ m/s}^2$

(B) $0 \text{ m/s}^2 < a_{Qy} < 8 \text{ m/s}^2$

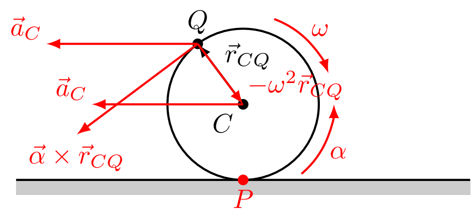
(C) $8 \text{ m/s}^2 \leq a_{Qy}$

(D) $-8 \text{ m/s}^2 \leq a_{Qy} < 0 \text{ m/s}^2$

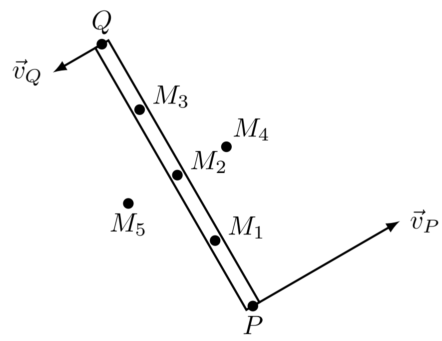
(E) $a_{Qy} = 0 \text{ m/s}^2$

Solution. First note that the radius is $r = 5$ m. Now the acceleration of Q is:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= -r\alpha\,\hat{\imath} + 2\hat{k} \times (-3\hat{\imath} + 4\hat{\jmath}) + (-\hat{k}) \times ((-\hat{k}) \times (-3\hat{\imath} + 4\hat{\jmath})) \\
&= -10\hat{\imath} - 8\hat{\imath} - 6\hat{\jmath} + 3\hat{\imath} - 4\hat{\jmath} \\
&= -15\hat{\imath} - 10\hat{\jmath} \\
a_{Qy} &= -10 \text{ m/s}^2.
\end{aligned}$$



39. (1 point) A rigid rod is moving in 2D as shown below.



Which point M_i is the instantaneous center?

(A) M_1

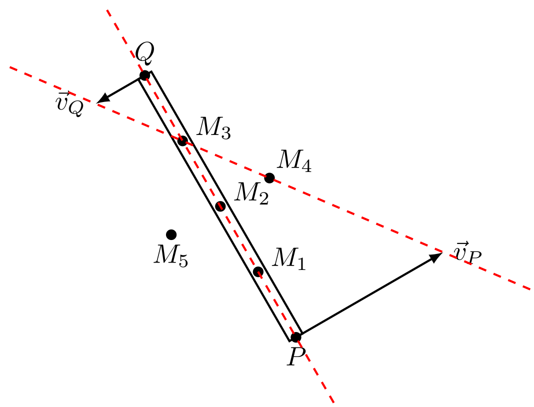
(B) ★ M_3

(C) M_5

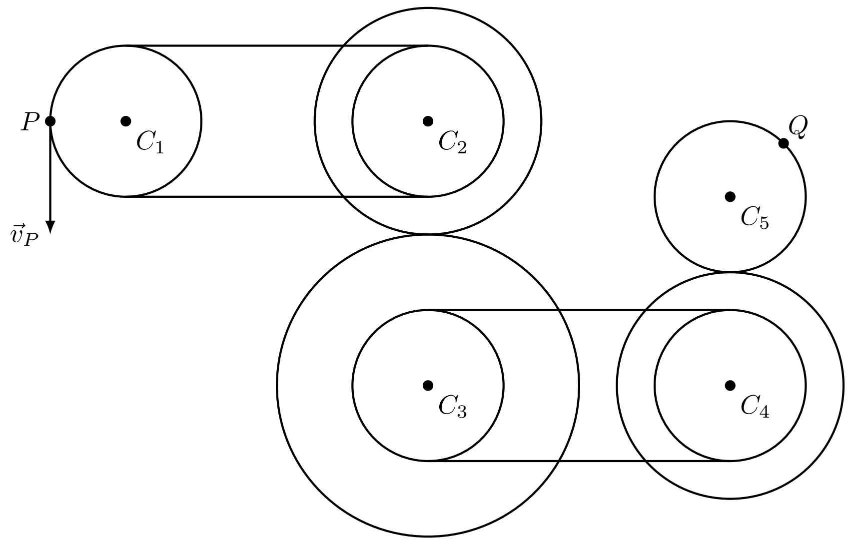
(D) M_4

(E) M_2

Solution. The lines through P and Q perpendicular to $\vec{v}_P$ and $\vec{v}_Q$ are the same, so we additionally intersect with the line through the ends of the velocity vectors.



40. (1 point) A set of gears and chains are positioned as shown with fixed centers C_1 through C_5. All gear pairs that have the same center are fixed together to rotate at the same angular velocity. All gears and chains are meshed so there is no slip. Point P has velocity $\vec{v}_P$ in the direction indicated.



What is the direction of the velocity $\vec{v}_Q$ of point Q?

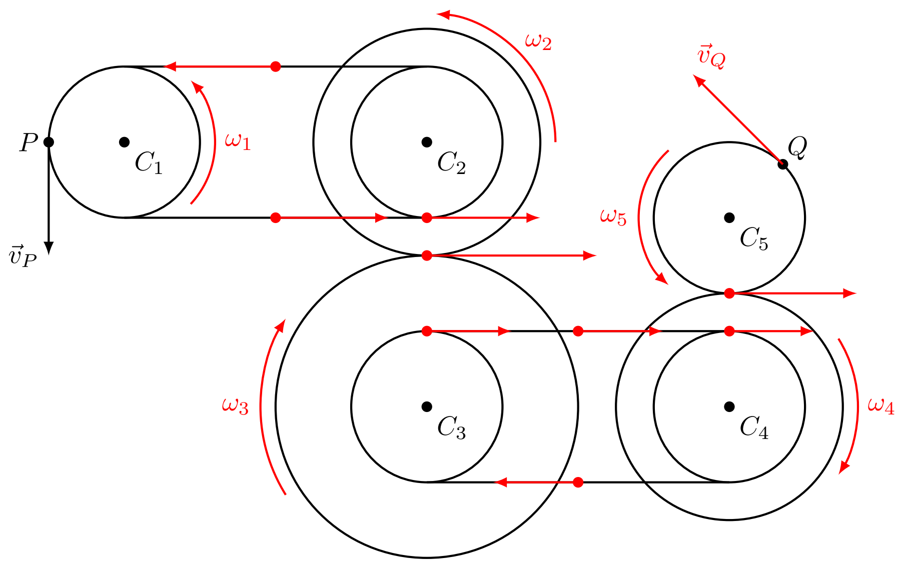
(A) $\searrow$

(B) $\nearrow$

(C) $\swarrow$

(D) ★ $\nwarrow$

Solution.



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41. (1 point) A rigid body is moving in 2D with angular velocity $\vec{\omega} = -\hat{k}$ rad/s and angular acceleration $\vec{\alpha} = 2\hat{k}$ rad/s^2. Points P and Q on the body have:

$$\begin{aligned}\vec{r}_{PQ} &= -2\hat{\imath} + \hat{\jmath} \text{ m} \\ \vec{a}_P &= \hat{\imath} - \hat{\jmath} \text{ m/s}^2.\end{aligned}$$

What is the $\hat{\imath}$ component a_{Qx} of the acceleration $\vec{a}_Q$ of point Q?

(A) $a_{Qx} < -3$ m/s^2

(B) -3 m/s$^2 \leq a_{Qx} < 0$ m/s^2

(C) $a_{Qx} = 0$ m/s^2

(D) 3 m/s$^2 \leq a_{Qx}$

(E) ★ 0 m/s$^2 < a_{Qx} < 3$ m/s^2

Solution.

$$\begin{aligned}\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ &= \hat{\imath} - \hat{\jmath} + 2\hat{k} \times (-2\hat{\imath} + \hat{\jmath}) - \hat{k} \times (-\hat{k} \times (-2\hat{\imath} + \hat{\jmath})) \\ &= \hat{\imath} - \hat{\jmath} - 2\hat{\imath} - 4\hat{\jmath} + 2\hat{\imath} - \hat{\jmath} \\ &= \hat{\imath} - 6\hat{\jmath} \\ a_{Qx} &= 1 \text{ m/s}^2.\end{aligned}$$

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42. (1 point) A rigid body is moving in 2D with angular velocity $\vec{\omega} = \omega\hat{k}$, for a non-negative value of ω, and zero angular acceleration. Points P and Q on the body are a distance $r_{PQ} = 0.2$ m apart and it is observed that:

$$\begin{aligned}\vec{a}_P &= 2\hat{\imath} - \hat{\jmath} \text{ m/s}^2 \\ \vec{a}_Q &= -\hat{\imath} + 3\hat{\jmath} \text{ m/s}^2.\end{aligned}$$

What is ω?

(A) ★ 5 rad/s $\leq \omega <$ 10 rad/s

(B) 10 rad/s $\leq \omega <$ 15 rad/s

(C) 0 rad/s $\leq \omega <$ 5 rad/s

(D) 20 rad/s $\leq \omega$

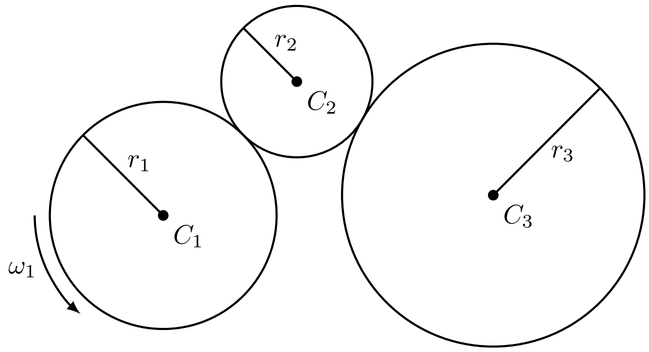
(E) 15 rad/s $\leq \omega <$ 20 rad/s

Solution. The fast solution is to note that the relative acceleration magnitude is $\|(2\hat{\imath} - \hat{\jmath}) - (-\hat{\imath} + 3\hat{\jmath})\| = 5 = \omega^2 r_{PQ}$, so $\omega = \sqrt{5/r_{PQ}} = 5$ rad/s.

Alternatively,

$$\begin{aligned}\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}) \\ -\hat{\imath} + 3\hat{\jmath} &= 2\hat{\imath} - \hat{\jmath} + \omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{PQ}) \\ -3\hat{\imath} + 4\hat{\jmath} &= -\omega^2 \vec{r}_{PQ} \\ \|-3\hat{\imath} + 4\hat{\jmath}\| &= \|-\omega^2 \vec{r}_{PQ}\| \\ 5 &= \omega^2 r_{PQ} \\ 5 &= \omega^2 \frac{1}{5} \\ \omega^2 &= 25 \\ \omega &= 5 \text{ rad/s.}\end{aligned}$$

43. (1 point) Three meshed gears rotate about fixed centers as shown. The radii are $r_1 = 3$ m, $r_2 = 2$ m, and $r_3 = 4$ m and the corresponding angular velocities are $\vec{\omega}_1 = 2\hat{k}$ rad/s, $\vec{\omega}_2 = \omega_2\hat{k}$, and $\vec{\omega}_3 = \omega_3\hat{k}$.



What is ω_3?

(A) $\omega_3 < -2$ rad/s

(B) 2 rad/s $\leq \omega_3$

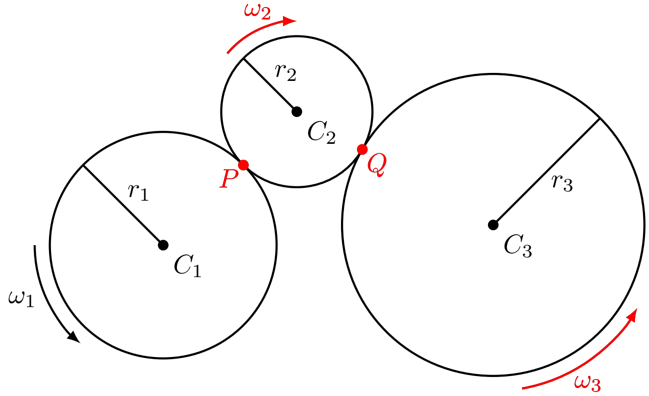
(C) ★ 0 rad/s $< \omega_3 < 2$ rad/s

(D) -2 rad/s $\leq \omega_3 < 0$ rad/s

(E) $\omega_3 = 0$ rad/s

Solution. Matching the velocities at points P and Q shows that the gear at C_2 rotates clockwise and the gear at C_3 rotates counterclockwise. Also:

$$\begin{aligned} r_1\omega_1 = v_P = r_2\omega_2 = v_Q &= r_3\omega_3 \\ \omega_3 &= \frac{r_1}{r_3}\omega_1 \\ &= \frac{3}{4}2 \\ &= 1.5 \text{ rad/s}. \end{aligned}$$



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44. (1 point) A rigid body is moving in 2D with angular velocity $\vec{\omega} = 3\hat{k}$ rad/s. Points P and Q on the body have:

$$\begin{aligned}\vec{r}_{PQ} &= \hat{\imath} + 4\hat{\jmath} \text{ m} \\ \vec{v}_P &= -3\hat{\imath} - 2\hat{\jmath} \text{ m/s.}\end{aligned}$$

What is the $\hat{\jmath}$ component v_{Qy} of the velocity $\vec{v}_Q$ of point Q?

(A) $-2 \text{ m/s} \leq v_{Qy} < 0 \text{ m/s}$

(B) $v_{Qy} = 0 \text{ m/s}$

(C) ★ $0 \text{ m/s} < v_{Qy} < 2 \text{ m/s}$

(D) $2 \text{ m/s} \leq v_{Qy}$

(E) $v_{Qy} < -2 \text{ m/s}$

Solution.

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ &= -3\hat{\imath} - 2\hat{\jmath} + 3\hat{k} \times (\hat{\imath} + 4\hat{\jmath}) \\ &= -3\hat{\imath} - 2\hat{\jmath} - 12\hat{\imath} + 3\hat{\jmath} \\ &= -15\hat{\imath} + \hat{\jmath} \\ v_{Qy} &= 1 \text{ m/s}\end{aligned}$$

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45. (1 point) A rigid body is moving in 2D with angular velocity $\vec{\omega} = 2\hat{k}$ rad/s. Two points P and Q are fixed to the body and the offset between them is in the direction $\hat{r}_{PQ} = \frac{1}{5}(-3\hat{\imath} + 4\hat{\jmath})$ (note that this is the unit vector in the direction of the offset vector $\vec{r}_{PQ}$, not the actual offset vector $\vec{r}_{PQ}$). The velocities are:

$$\begin{aligned}
\vec{v}_P &= 2\hat{\imath} + 3\hat{\jmath} \text{ m/s} \\
\vec{v}_Q &= -2\hat{\imath} \text{ m/s.}
\end{aligned}$$

What is the distance r_{PQ} between P and Q?

(A) ★ 2 m $\leq r_{PQ} <$ 3 m

(B) 3 m $\leq r_{PQ} <$ 4 m

(C) 1 m $\leq r_{PQ} <$ 2 m

(D) 0 m $\leq r_{PQ} <$ 1 m

(E) 4 m $\leq r_{PQ}$

Solution.

$$\begin{aligned}
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
-2\hat{\imath} &= 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k} \times r_{PQ}\frac{1}{5}(-3\hat{\imath} + 4\hat{\jmath}) \\
-4\hat{\imath} - 3\hat{\jmath} &= -\frac{8}{5}r_{PQ}\,\hat{\imath} - \frac{6}{5}r_{PQ}\,\hat{\jmath} \\
\Longrightarrow r_{PQ} &= \frac{5}{2} = 2.5 \text{ m.}
\end{aligned}$$

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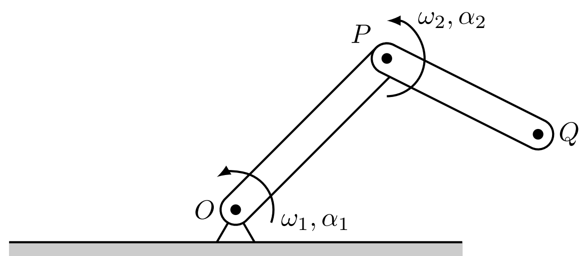
46. (1 point) Two rods are connected with pin joints at O, P, and Q as shown. The rods have angular velocities and angular accelerations as indicated.



The positions and angular velocities of the rods at the current instant are:

$$\vec{r}_{OP} = 2\hat{\imath} + 2\hat{\jmath} \text{ m} \qquad \vec{r}_{PQ} = 2\hat{\imath} - \hat{\jmath} \text{ m}$$
$$\vec{\omega}_1 = \hat{k} \text{ rad/s} \qquad \vec{\omega}_2 = -\hat{k} \text{ rad/s}$$
$$\vec{\alpha}_1 = 0 \qquad \vec{\alpha}_2 = \hat{k} \text{ rad/s}^2$$

What is the $\hat{\imath}$ component a_{Qx} of the acceleration $\vec{a}_Q$ of point Q?

(A) $0 \text{ m/s}^2 < a_{Qx} < 2 \text{ m/s}^2$

(B) ★ $a_{Qx} < -2 \text{ m/s}^2$

(C) $2 \text{ m/s}^2 \leq a_{Qx}$

(D) $a_{Qx} = 0 \text{ m/s}^2$

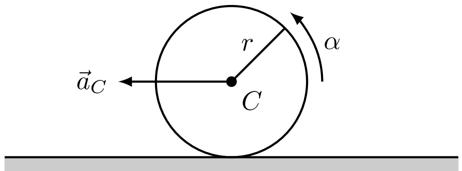
(E) $-2 \text{ m/s}^2 \leq a_{Qx} < 0 \text{ m/s}^2$

Solution. Starting from the fixed point O, we have:

$$\begin{aligned}
\vec{a}_P &= \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OP} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OP}) \\
&= 0 + 0 + \hat{k} \times (\hat{k} \times (2\hat{\imath} + 2\hat{\jmath})) \\
&= -2\hat{\imath} - 2\hat{\jmath} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_w \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\
&= (-2\hat{\imath} - 2\hat{\jmath}) + \hat{k} \times (2\hat{\imath} - \hat{\jmath}) + (-\hat{k}) \times ((-\hat{k}) \times (2\hat{\imath} - \hat{\jmath})) \\
&= (-2\hat{\imath} - 2\hat{\jmath}) + (\hat{\imath} + 2\hat{\jmath}) + (-2\hat{\imath} + \hat{\jmath}) \\
&= -3\hat{\imath} + \hat{\jmath} \\
a_{Qx} &= -3 \text{ m/s}^2.
\end{aligned}$$

---

47. (1 point) A circular rigid body is rolling without slipping with angular acceleration $\vec{\alpha} = 3\hat{k}$ on a flat surface in 2D as shown. The acceleration of the center is $\vec{a}_C = -2$ m/s.



What is the radius r of the rigid body?

(A) $3 \text{ m} \leq r < 4 \text{ m}$

(B) $2 \text{ m} \leq r < 3 \text{ m}$

(C) $4 \text{ m} \leq r$

(D) ★ $0 \text{ m} \leq r < 1 \text{ m}$

(E) $1 \text{ m} \leq r < 2 \text{ m}$

Solution. $a_c = r\alpha$ so $r = a_c/\alpha = 2/3 \approx 0.67$ m.

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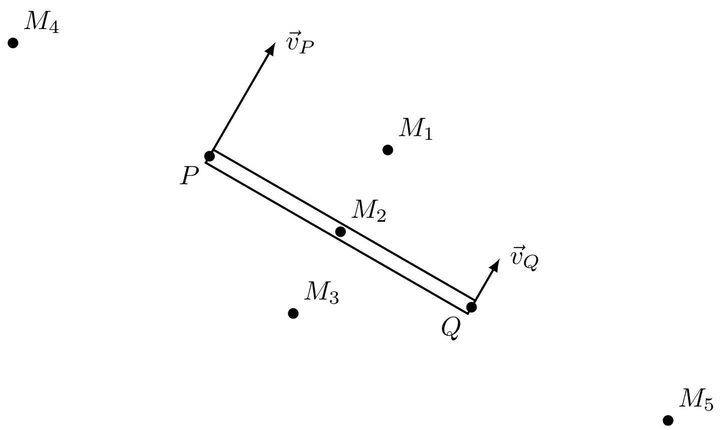
48. (1 point) A rigid rod is moving in 2D as shown below.



Which point M_i is the instantaneous center?

(A) M_4

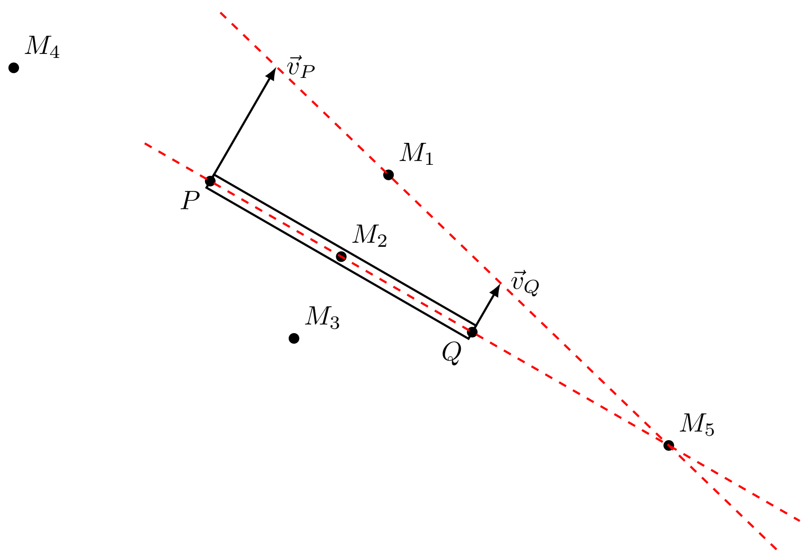
(B) M_3

(C) M_2

(D) ★ M_5

(E) M_1

Solution. The lines through P and Q perpendicular to $\vec{v}_P$ and $\vec{v}_Q$ are the same, so we additionally intersect with the line through the ends of the velocity vectors.



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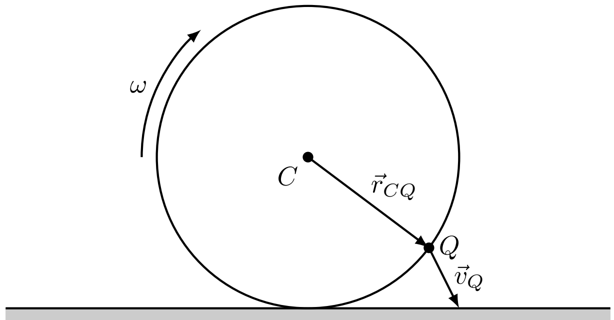
49. (1 point) A circular rigid body with is rolling without slipping on a flat surface in 2D as shown, with angular velocity $\vec{\omega} = -\omega\hat{k}$.



Point Q has:

$$\vec{r}_{CQ} = 4\hat{\imath} - 3\hat{\jmath} \text{ m}$$
$$\vec{v}_Q = \hat{\imath} - 2\hat{\jmath} \text{ m/s}.$$

What is the angular velocity ω?

(A) 2 rad/s $\leq \omega <$ 3 rad/s

(B) ★ 0 rad/s $\leq \omega <$ 1 rad/s

(C) 4 rad/s $\leq \omega$

(D) 3 rad/s $\leq \omega <$ 4 rad/s

(E) 1 rad/s $\leq \omega <$ 2 rad/s

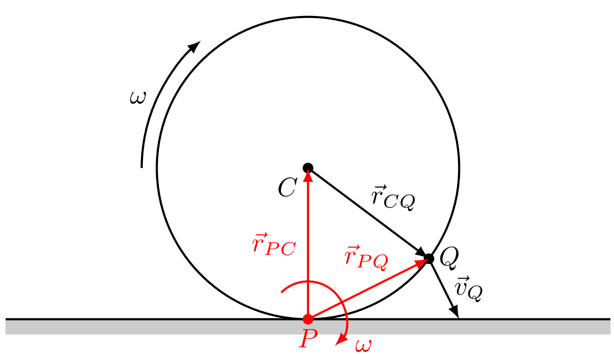
Solution. Observe that the radius of the rigid body is $r = r_{CQ} = 5$ m. Taking P to be the contact point, we have:

$$\begin{aligned}\vec{r}_{PQ} &= \vec{r}_{PC} + \vec{r}_{CQ} \\ &= 5\hat{\jmath} + (4\hat{\imath} - 3\hat{\jmath}) \\ &= 4\hat{\imath} + 2\hat{\jmath}.\end{aligned}$$

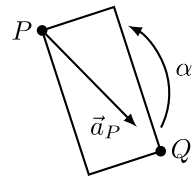
Now taking $\vec{\omega} = -\omega\hat{k}$, the velocity of Q is:

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\ \hat{\imath} - 2\hat{\jmath} &= 0 - \omega\hat{k} \times (4\hat{\imath} + 2\hat{\jmath}) \\ &= 2\omega\,\hat{\jmath} - 4\omega\,\hat{\jmath}.\end{aligned}$$

Equating components and solving gives $\omega = 0.5$ rad/s.



50. (1 point) A rigid body is moving in 2D as shown below, with a counterclockwise angular acceleration and points P and Q on the body. We know that $a_P = 2\omega^2 r_{PQ}$ and $\alpha = \omega^2$.



What is the direction of the acceleration $\vec{a}_Q$?

(A) ★ $\rightarrow$

(B) $\uparrow$

(C) $\leftarrow$

(D) $\downarrow$

Solution. Consider the acceleration equation for point Q:

$$\vec{a}_Q = \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}).$$

We know that the last two terms on the right hand side have the same magnitude ($\alpha r_{PQ} = \omega^2 r_{PQ}$) and the first term on the right hand side has twice this magnitude ($a_P = 2\omega^2 r_{PQ}$). Drawing the right-hand-side terms shows that the resulting direction for $\vec{a}_Q$ is to the right: