Position, Velocity, Acceleration

1) cartesian

2) polar

pattern

geometry

(A) need to write

$x(t), y(t)$ — cartesian

OR

$r(t), \theta(t)$ — polar

vector algebra

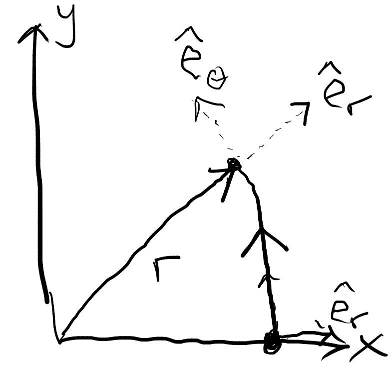
(B) take derivative wrt t

$\Rightarrow v, a$

$$
\begin{tikzcd}
\text{cartesian} & \text{polar} \\
x(t), y(t) \arrow[r, leftrightarrow] \arrow[d, "\text{derivatives}"] \arrow[dr] & r(t), \theta(t) \arrow[d, "\text{derivatives}"] \arrow[d, bend right] \\
v_x(t), v_y(t) \arrow[r, leftrightarrow] \arrow[d, "\text{derivatives}"] & v_r(t), v_\theta(t) \arrow[d, "\text{derivatives}"] \\
a_x(t), a_y(t) \arrow[r, leftrightarrow] & a_r(t), a_\theta(t)
\end{tikzcd}
$$

polar coordinates:

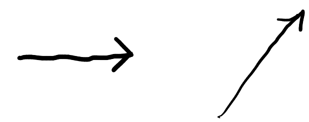
$$\vec{r} = r\hat{e}_r$$



$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r)$$

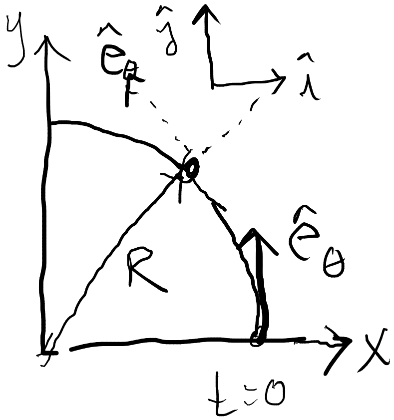
$$= \left(\frac{dr}{dt}\right)\hat{e}_r + r\frac{d}{dt}(\hat{e}_r)$$

$t=0$ $t=\Delta t$



$$= \underbrace{\dot{r}\hat{e}_r}_{\text{changes of magnitude of } \vec{r}} + r\dot{\theta}\hat{e}_\theta$$

changes in the direction of $\vec{r}$



particle moves on a circular path, at a constant angular velocity $\dot{\theta}$

$\theta = \dot{\theta} t$

geometry

cartesian: $x(t) = R\cos\theta(t)$

$y(t) = R\sin\theta(t)$

algebra

$V_x = \frac{dx}{dt} = -R(\sin\theta)(\dot{\theta})$

$V_y = \frac{dy}{dt} = R(\cos\theta)\dot{\theta}$

$\vec{V} = V_x\hat{i} + V_y\hat{j}$

$= -\dot{\theta}R\,\underline{\sin\theta\,\hat{i}} + \dot{\theta}R\,\underline{\cos\theta\,\hat{j}}$

$= (\dot{\theta})(R)\hat{e}_\theta$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

geometry

$$r(t) = R$$

$$\theta(t) = \dot{\theta} t$$

$$\vec{v} = \cancelto{0}{\dot{r}}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$= R\dot{\theta}\,\hat{e}_\theta$$

Key Concept:

for a circular trajectory, the velocity of the particle is parallel to the $\hat{e}_\theta$ direction

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\underline{\underline{\dot{r}\hat{e}_r}} + r\dot{\theta}\hat{e}_\theta\right)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \frac{d}{dt}(r\dot{\theta})\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$

$$= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta) + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$