Unit I: Induction and Recursion

Ref. Kenneth H. Rosen, “Discrete Mathematics and Its Applications”

By

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Mathematical Induction

Definition:

- Mathematical Induction is a powerful method for showing a property is true for all nonnegative integers.
- It plays a central role in discrete mathematics and computer science.
- It is used to write proofs of claims on non negative integers.

Example

The Principle of Mathematical Induction.

Let P be a ***predicate*** on nonnegative integers, If

- P (0) is true,

  and
- P (n) -> P(n +1), for all nonnegative integers, n

  then
- P (m) is true for all nonnegative integers, m.

Mathematical Induction can be used to prove

1. Summation formulae
2. Inequalities
3. Identities for combinations of sets
4. Divisibility results
5. Theorems about algorithms etc.

1. Summation formulae

Ex. 1 Show that if n is a positive integer then $$1+2+\ldots+n=\frac{n(n+1)}{2}$$

Proof: The induction hypothesis, P (n), will be $$1+2+\ldots+n=\frac{n(n+1)}{2} \tag{1}$$

Base case: P(0) is true, because both sides of equation (1) equal zero when n = 0.

Inductive step: Assume that P(n) is true, where n is any nonnegative integer.

Then, Lets prove,

$$1+2+\cdots+n+(n+1)=\frac{n(n+1)}{2}+(n+1)$$

i.e. $$1+2+\cdots+n+(n+1)=\frac{(n+1)(n+2)}{2}$$

Prove: $1+2+\cdots+n+(n+1)=\frac{(n+1)(n+2)}{2}$

Assumption: $1+2+\ldots+n=\frac{n(n+1)}{2}$

L.H.S. = $1+2+\cdots+n+(n+1)$	**R.H.S** $=\frac{n(n+1)}{2}$
$= \underline{1+2+\cdots+n}+(n+1)$	Let us replace n by $n+1$
$= \frac{n(n+1)}{2}+(n+1)$	$= \frac{(n+1)(n+1+1)}{2}$
$= \frac{n(n+1)+2(n+1)}{2}$	$= \frac{(n+1)(n+2)}{2}$
$= \frac{(n+1)(n+2)}{2}$	

which proves P(n + 1). So it follows by induction that P (n) is true for all nonnegative n.

A Template for Induction Proofs

1. State that the proof uses induction.
2. Define an appropriate predicate P (n)
3. Prove that P (0) is true.
4. Prove that P (n) -> P (n +1) for every nonnegative integer n.
5. Invoke induction.

Ex. 2 Conjecture a formula for the sum of first n positive odd integers. Then prove Conjecture using mathematical induction.

Solution:

1. The sums of the first n positive odd integers for n=1,2,3,4,5 are
   - for n=1, first one odd number, $1=1$
   - for n=2, first two odd numbers, $1+3=4$
   - for n=3, first three odd numbers, $1+3+5= 9$
   - for n=4, first four odd numbers, $1+3+5+7=16$
   - for n=5, first five odd numbers, $1+3+5+7+9=25$

   …

   - for n=n, first n odd numbers, $1+3+5+7+9+\ldots+(2n-1)=?$

   The sum of the first n positive odd integers is n^2,

   that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

   We need a method to prove that this conjecture is correct, if in fact it is.

2. Let P(n) denote the proposition that the sum of the first n odd positive integers is n^2.
Our conjecture is that P(n) is true for all positive integers n.
We use mathematical induction to prove this conjecture.

BASIS STEP: P(1) states that the sum of the first one odd positive integer is 1^2.
This is true because the sum of the first odd positive integer is 1.
i.e. $1 = 1^2$

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true for an arbitrary positive integer k (k terms), that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.
Let us prove it is true for a positive integer (k+1) (k+1 terms)
that is $1 + 3 + 5 + \cdots + (2k - 1) + (2k+1) = (k+1)^2$.

$2k-1$ — k^{th} odd int
$k+1^{th}$ odd
odd

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = [1 + 3 + \cdots + (2k - 1)] + (2k + 1)$$
$$= k^2 + (2k + 1)$$

$\to 2k-1 \quad 2k \cancel{-1} + \cancel{1} \quad 2k+1$

$$= k^2 + 2k + 1$$
$$= (k + 1)^2.$$

$k^2 + 2k + 1 = k^2 + k + k + 1$
$= k(k+1) + 1(k+1)$
$= (k+1)(k+1)$
$= (k+1)^2$

This shows that P(k + 1) follows from P(k).

Ex 3: Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Solution: Let P(n) be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: P(0) is true because $2^0 = 1 = 2^1 - 1$.
This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, assume that P(k) is true for an arbitrary nonnegative integer k.
i.e., $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$
we must show that when we assume that P(k) is true, then P(k + 1) is also true.
i.e., $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$ (assuming the inductive hypothesis P(k).)

Under the assumption of P(k),
we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

We have completed the inductive step.

Because the basis step and the inductive step have completed; by mathematical induction we have shown that P(n) is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

EXAMPLE 4 **Sums of Geometric Progressions** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

Let P(n) be
$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1, \quad \text{— ①}$$

where n is a nonnegative integer.

$$ar^0 + ar^1 + ar^2 + \cdots + ar^n$$
$$a + ar + ar^2 + \cdots + ar^n = \sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r-1}$$

$$p(0) = ar^0 = a = \frac{ar^{0+1} - a}{r-1}$$
$$= \frac{ar - a}{r-1} = \frac{a\cancel{(r-1)}}{\cancel{(r-1)}} = a$$

Solution: To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n + 1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r - 1)}{r - 1} = a.$$

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+1}}{1}.$$

$$= \frac{ar^{k+1} - a + (r-1)(ar^{k+1})}{r-1}$$

Rewriting the right-hand side of this equation shows that

$$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$$

$$\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1}$$

$$= \frac{ar^{k+2} - a}{r - 1}.$$

$$= \frac{ar^{k+2} - a}{r-1}$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

H.W. ① Find a formula for $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = ?$

② Find a formula for $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = ?$

Find the formula for $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$ by examining the values of this expression for small values of n.
Prove the formula you conjectured.

$P(n) \Rightarrow \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} = 1-\frac{1}{2^n}$ Rutuja

$= \frac{2^n-1}{2^n}$ correct

$S_1 = \frac{1}{2}$,

$S_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$,

$S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$,

$S_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16}, \ldots$

$S_n = \frac{2^n-1}{2^n}$

$n = 1, \quad \frac{1}{2} = \frac{1}{2} \quad \frac{2^1-1}{2^1}$

$n = 2, \quad \frac{1}{2}+\frac{1}{4} = \frac{3}{4} \quad \frac{2^2-1}{2^2}$

$n = 3, \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} \quad \frac{2^3-1}{2^3}$

$n = 4, \quad \underbrace{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}+\frac{1}{16} = \frac{15}{16} \quad \frac{2^4-1}{2^4}$

$\frac{2^n-1}{2^n} \qquad \frac{7}{8}+\frac{1}{16} = \frac{14+1}{16} = \frac{15}{16}$

Find a formula for $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}$ by examining the values of this expression for small values of n. Prove the formula you conjectured above.

$\frac{n}{n+1}$ ← Pallavi correct

$S_1 = \frac{1}{2}$,

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$,

$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}, \ldots$

$S_n = \frac{n}{n+1}$

$n = 1, \quad \frac{1}{2} \Rightarrow \frac{n}{n+1}$

$n = 2, \quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$n = 3, \quad \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

$\vdots$

$n = n \Rightarrow \frac{n}{n+1}$ ✓