Symmetric and Skew-Symmetric Matrices:

A square matrix $A=[a_{ij}]$ is said to be symmetric, if $a_{ij}=a_{ji}$ for all i,j, viz, if the $(i-j)^{th}$ element is the same as the $(j-i)^{th}$ element.
In otherwords, in a symmetric matrix all the elements are symmetrically placed w.r.t. its diagonal and are eq in magnitude and sign.

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & -4 \\ 3 & 1 & 5 \\ -4 & 5 & -1 \end{bmatrix}$ are symmetric matrices.

Property: 1. The total no of independent elements in a symm matrix of order $n = \frac{n(n+1)}{2}$.

2. If A is a symmetric matrix, its transpose $A^T=A$, since the matrix is not altered, if we interchange the rows and columns.

3. If A is a square matrix, then AA^T is a symmetric matrix.
For $(AA^T)^T=(A^T)^T A^T$ by the reversal law
$= AA^T$.

Skew-Symmetric matrix:

A square matrix $A=[a_{ij}]$ is said to be skew-symm if $a_{ij}=-a_{ji}$ for all i and j. viz, if the $(i-j)^{th}$ element is the negative of the $(j-i)^{th}$ ele

..., $a_{ii} = -a_{ii}$, $\forall i$, $a_{ii} = 0$, $\forall i$, in a skew symmetric matrix all diagonal elements are zero and elements symmetrically placed w.r.t. the diagonal are equal in magnitude and opposite in sign.

$$\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 4 & -5 \\ -4 & 0 & 2 \\ 5 & -2 & 0 \end{bmatrix}$$ are skew-symmetric matrices.

<u>Properties</u>:

If A is a skew-symmetric matrix, $A^T = -A$, since the interchange of rows and columns of A results in the negative of A.

If A & B are 2 symmetric matrices of the same order, then $A \pm B$ are also symmetric.
For $(A \pm B)^T = A^T \pm B^T = A \pm B$.

If A & B are two skew symmetric matrices of the same order, then $(A \pm B)$ are also skew-symmetric.
For $(A \pm B)^T = A^T \pm B^T = -(A \pm B)$

If A is a square matrix, then $(A + A^T)$ is symmetric and $(A - A^T)$ is skew-symmetric.
~~For $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$~~
and $(A - A^T)^T = A^T - A = -(A - A^T)$

Every square matrix A can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

A real square matrix A is called orthogonal if $AA^T = I =$

Properties:

1. If A is an orthogonal matrix, then $|A| = \pm 1$.

Proof: Since the value of a determinant is not altered by interchanging of rows & columns, $|A^T| = |A|$ — (1)

Since A is orthogonal, $AA^T = I$

$$|A||A^T| = |I|$$

$$|A|^2 = 1. \text{ by (1)}$$

$$\therefore \boxed{|A| = \pm 1}$$

2. If A is an orthogonal matrix, then A^{-1} exists & equals

Proof: $|A| = \pm 1 \neq 0$

$\therefore A^{-1}$ exists

By defn, $AA^T = I$

$$A^{-1}(AA^T) = A^{-1} I$$

$$(A^{-1}A)A^T = A^{-1}$$

$$IA^T = A^{-1}$$

$$\therefore \boxed{A^T = A^{-1}}$$

3. The inverse of an orthogonal matrix A is also orthogona

Proof: Since A is orthogonal, $A^T = A^{-1}$ by property 2.

Let $B = A^T = A^{-1}$

$$B^T = (A^{-1})^T$$

$$= (A^T)^{-1}$$

$$= B^{-1}$$

ie B or A^{-1} is also orthogonal.

Note: Property 3 also means that A^T is also orthogonal.

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... of 2 orthogonal matrices is also ...

...gonal matrix.

Let A & B be 2 orthogonal matrices of the same order.

Then by defn, $AA^T = A^TA = I$ ——— (1)

$BB^T = B^TB = I$ ——— (2)

$(AB)(AB)^T = ABB^TA^T$ by the reversal law for the transpose

$= AIA^T$ by (2)

$= AA^T = I$ by (1)

$(AB)^T AB = B^TA^TAB$

$= B^TIB$ by (1)

$= B^TB$

$= I$ by (2)

Thus $(AB)(AB)^T = (AB)^TAB = I$

$\therefore$ The product AB is also orthogonal.

Problems:

1) Find the symmetric and skew-symmetric parts of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Sol) $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\therefore$ $A + A^T = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$

Symmetric part $= \begin{bmatrix} a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & d \end{bmatrix}$

Skew-symmetric part $= \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-b) & 0 \end{bmatrix}$

(2) " " $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$, verify that AA' is a symmetric matrix.

Sol: $AA^T = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 3 \\ 8 & 14 & 8 \\ 3 & 8 & 13 \end{bmatrix}$

In AA^T, $a_{ij} = a_{ji}$ for all i & j.

$\therefore AA^T$ is symmetric.

(3) Express $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrices.

Sol: $A^T = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & -2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$

Symmetric part $= \frac{1}{2}(A + A^T) = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & -1/2 \\ 7/2 & -3/2 & -1/2 & 0 \end{bmatrix}$

Skew-symmetric part $= \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & 3/2 \\ 1/2 & -5/2 & -3/2 & 0 \end{bmatrix}$

(4) Find the conditions so that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ may be an orthogonal matrix.

Sol: If A is to be orthogonal, $AA^T = I$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ The required conditions are $a^2+b^2 = 1$, $c^2+d^2 = 1$ and $ac+bd = 0$

Prove That The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is orthogonal and hence find A^{-1}.

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Sol: $AA^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore$ A is orthogonal.

$A^{-1} = A^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(6) Find the values of a, b, c if $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal.

Sol: Since A is orthogonal, $AA^T = I$

$\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a^2+5 & ab+4 & ac-2 \\ ab+4 & b^2+5 & bc+2 \\ ac-2 & bc+2 & c^2+8 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

Equating like terms $a^2+5=9$, $b^2+5=9$, $c^2+5=9$

$\therefore \boxed{a=\pm 2;\ b=\pm 2;\ c=\pm 1}$

$ab+4=0$

When $a=2$, $b=-2$ and when $a=-2$, $b=2$.

$ac-2=0$

$\therefore$ when $a=2$, $c=1$ and when $a=-2$, $c=-1$

$\therefore$ The values of a, b, c are $(2,-2,1)$ or $(-2,2,-1)$.

and hence find A^{-1}. $\begin{bmatrix} \cdot & \cdot & \cdot \\ -1 & 2 & -2 \end{bmatrix}$ is orthogonal

Sol: $AA^T = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ A is orthogonal.

$\therefore$ By defn, $A^{-1} = A^T = \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$

Exercise:

Part-A

1. Define symmetric and skew-symmetric matrices.
2. Express a square matrix as the sum of a symmetric and a skew-symmetric matrices.
3. Define orthogonal matrix.
4. If A is an orthogonal matrix, P.T. $A^{-1} = A^T$.
5. If A is an orthogonal matrix, P.T. $|A| = \pm 1$.

Part-B

6. A and B are square matrices of the same order and A is symmetric. S.T. B^TAB is also symmetric.
7. Express $\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices.

Express $\begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & 1 & 4 & 6 \\ -3 & -2 & 2 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices.

9. Show that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal and hence find A^{-1}.

10. S.T. the matrix $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal and hence find A^{-1}.

11. Verify that $\frac{1}{7}\begin{bmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ -3 & 6 & 2 \end{bmatrix}$ is orthogonal and hence find A^{-1}.

12. Verify that the matrix $A = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ is orthogonal and hence find A^{-1}.

Solutions

7. $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 4 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 & -1 \\ 2 & 0 & 3 & 2 \\ -3 & -3 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}$

9. $A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

10. $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = A^{-1}$

11. $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ -6 & 3 & 2 \end{bmatrix}$

12. $\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$

MATRICES

Introduction:

The term 'matrix' was first introduced by Sylvester in 1850. He defined a matrix to be an arrangement of terms. In 1858 Cayley outlined a matrix algebra defining addition, multiplication, scalar multiplication and inverses. knowledge of matrix is very useful and important as it has a wider application in almost every field of Mathematics.

Economics are using matrices ~~for~~ for Social accounting, input - output tables and in the study of inter-industry economics. Matrices are also used in the study of Communication theory, network analysis in electrical engineering.

The purpose of matrices is to provide a kind of mathematical shorthand to help the study of problems represented by the entries. The matrices may represent transformations of co-ordinates spaces or systems of simultaneous linear equations.

The system of linear equations arises naturally in many areas of sciences, engineering economics and commerce. The analysis of electronic circuits, determination of the output of a chemical plant, finding the cost of chemical reaction are some of the problems which depend on the solutions of simultaneous linear equations. So, finding methods of solving such equations acquire considerable importance. In this connection methods using matrices play an important role.

Matrices

A matrix A of order $m \times n$ is an rectangular array of mn elements a_{ij} where $1 \le i \le m$, $1 \le j \le n$, arranged in m rows and n columns.

ie) $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

It is denoted as $A = (a_{ij})$, $i = 1, 2, \ldots, m$ & $j = 1, 2, \ldots n$

Square matrix

For any matrix if the number of rows is equal to the number of columns then the matrix is called square matrix.

Transpose of the matrix

If $A = (a_{ij})$ is an $m \times n$ matrix. then the $n \times m$ matrix $B = (b_{ij})$, where $b_{ij} = a_{ji}$ is called the transpose of the matrix A and it is denoted as A^T.

* Special Types of Matrices

Let A be a square matrix of order n,

1. $A = A^T$, then it is called a symmetric matrix i) $a_{ij} = a_{ji}$ for all i,j.
2. If the square matrix $A = -A^T$ then it is called skew-Symmetric matrix" i) $a_{ij} = -a_{ji}$ for all i,j.
3. An $m \times 1$ matrix is called a Column matrix.
4. An $1 \times n$ matrix is called a row matrix.
5. In the square matrix $A = (a_{ij})$, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the diagonal elements of A, and the diagonal elements constitute the principal diagonal of the matrix.
6. A square matrix is called a Diagonal matrix, if all the elements except the leading principal diagonal are zero.
7. A square matrix $A = (a_{ij})$ is called an Upper triangular matrix if all the entries below the leading diagonal are zero.

8. A square matrix $A = (a_{ij})$ is called an lower triangular matrix if all the entries above the leading diagonal are zero.

9. A square matrix A is said to be singular if $|A| = 0$, otherwise it is a non-singular matrix.

Eigenvalues and Eigen Vectors

Let $A = [a_{ij}]$ be an $n \times n$ matrix. If there exist a non-zero column vector X and a scalar λ, such that $AX = \lambda X$, then λ is called an eigen value and X is called an eigen vector corresponding to eigen value λ.

$$AX = \lambda X \quad \text{—} \quad (1)$$

$$(A - \lambda I)X = 0 \quad \text{———} \quad (2)$$

ie)
$$\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots\cdots & a_{mn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The above Eqns are linear homogeneous with n unknowns $x_1, x_2, \ldots, x_n$.

The solution of the above system should be a non-trivial solution only if the determinant of the coefficient matrix is zero.

i.e)
$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

i.e) $|A-\lambda I| = 0$ —— (3)

$|A-\lambda I|$ is a polynomial of degree n in λ and is called characteristic polynomial. It posses n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ and are called eigen values. The Equation (3) is called the characteristic equation.

For each eigen value of the system of Eqns (2) posses a non-zero vector X, called eigen vector of the matrix A to the eigen value λ.

In general, the characteristic equation of n^{th} order matrix can be written as

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \dots + (-1)^n D_n = 0$$

where D_i is the sum of all i^{th} order minors of A whose leading diagonals lie along the leading diagonal of A.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a given matrix of order 3, then the characteristic equation of A is

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

where

D_1 = Sum of leading diagonals $= a_{11} + a_{22} + a_{33}$

D_2 = Sum of the minors of leading diagonals

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$D_3 = |A|$

* Properties of Eigen values

1. The eigen values of A and A^T are same.

Let A be the square matrix of order n.

Then the characteristic polynomial of A is

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \quad \text{—— (1)}$$

The characteristic polynomial of A^T is

$$|A^T-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn}-\lambda \end{vmatrix} \quad \text{—— (2)}$$

The determinant (2) can be obtained by changing the rows into columns of determinant (1)

$$\therefore |A-\lambda I| = |A^T-\lambda I|$$

$\therefore$ The characteristic Eqns are identical

Thus the eigen values of A and A^T are Same.

2. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of the matrix A, then

(i) $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are the eigen values of the matrix A^k, where k is the positive integer.

(ii) $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigen values of the matrix kA, where k is a non-zero scalar.

(iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of the inverse matrix A^{-1}, provided A is non-singular

(i) Let λ be an eigen value and X be an eigen vector of A, then $AX = \lambda X$

$$A^2X = (AA)X = A(AX)$$
$$= A(\lambda X)$$
$$= \lambda(AX)$$
$$A^2X = \lambda^2 X$$

$\therefore \lambda^2$ is an eigen value of A^2

In the same way, we can prove that λ^k is an eigen value of A^k.

(ii) Let $AX = \lambda X$

Now, $(kA)X = k(AX) = k(\lambda X) = (k\lambda)X$

$\therefore k\lambda$ is an eigen value of kA.

(iii) Let $AX = \lambda X$

pre-multiplying both sides by A^{-1}

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda(A^{-1}X)$$

$$\therefore A^{-1}X = \frac{X}{\lambda}$$

Hence, $\frac{1}{\lambda}$ is an eigen value of A^{-1}

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3. Sum of the eigen values is equal to the sum of the principal diagonal elts of A.
4. The product of the eigen values is $|A|$.
5. A matrix A is singular iff zero is a characteristic root of A.
6. The eigen values of a diagonal matrix are the diagonal elts.
7. The eigen values of an upper or lower triangular matrix are the diagonal ~~matrix~~ elts.
8. The eigen values of real symmetric matrix are all real.

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* Properties of Eigen Vectors

1. If the eigen values of a matrix A are distinct, then the corresponding eigen vectors are linearly independent.

2. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

3. An eigen vector need not be unique (ie) Any scalar multiple of an eigen vector is also an eigen vector.

4. If two eigen values are equal, then the correspon-ding eigen vectors are either linearly dependent or linearly independent.

Problems

1. Find the sum and product of the eigen values of $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 2 & 4 \\ 1 & 2 & 7 \end{pmatrix}$

Sum of the eigen value of the matrix = Sum of the leading diagonal elts

$= 1+2+7 = 10$

Product of the eigen value $= |A|$

$$= \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 4 \\ 1 & 2 & 7 \end{vmatrix}$$

$$= 1(14-8) - 2(14-4) + 5(4-2)$$

$$= \cancel{4}\, 6 - 20 + 10$$

$$= -4$$

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2. If two of the eigen values of the matrix $\begin{pmatrix} 2 & -2 & 2 \\ -2 & -1 & -1 \\ 2 & -1 & -1 \end{pmatrix}$ are equal. find the remaining eigenvalue of the matrix.

Let the two eigenvalues of the matrix be 'x' (equal) and the third one as 'y'

$x + x + y =$ Sum of the leading diagonal elts

$$2x + y = 2 - 1 - 1 = 0$$

$$2x + y = 0 \quad \text{—— } (1)$$

Product of the eigenvalue $= |A|$

$$x^2 y = 16 \quad \text{—— } (2)$$

From (1)

$$\boxed{x = -y/2}$$

sub in (2)

$$\frac{y^3}{4} = 16$$

$$y^3 = 4^3$$

$$\boxed{\therefore y = 4}$$

from $x = -y/2$, we have $x = -2$

Hence the eigen values of the matrix are -2, -2 and 4.

3. find the eigen values of A^3, if $A = \begin{pmatrix} 3 & 1 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

Since A is the upper triangular matrix, its eigen values are its leading diagonals.

$\therefore$ Eigen values of A are $3, 4, -1$

$\therefore$ The eigen values of A^3 are $3^3, 4^3, (-1)^3$

i) $27, 64, -1$

4. The product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigen value.

Since A is 3×3 matrix, A has 3 eigen values.

Let a, b, c be the eigen values of A.

then $abc = |A|$

$abc = 32$ —— (1)

Given $ab = 16$, sub in (1)

$16c = 32$

$\boxed{c = 2}$

$a + b + c = 12$

$a + b = 12 - 2 = 10$

$\boxed{a + b = 10}$ ← (2)

$a = 16/b$, sub in (2)

$\frac{16}{b} + b = 10$

$16 + b^2 = 10b$

$b^2 - 10b + 16 = 0$

$b = 2 \; / \; b = 8$

16

-8 -2

$\therefore a = 16/b$, if $b = 2$, $a = 8$

(or) if $b = 8$, $a = 2$

Thus the eigen values of A

are 2, 8, 2 (or) 8, 2, 2.

5. Two of the eigen values of $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 and 6. find the eigen values of A^{-1}.

Let $\lambda_1 = 3$ and $\lambda_2 = 6$, λ_3 be the eigen values of A.

$$\lambda_1 + \lambda_2 + \lambda_3 = 3+5+3$$

$$9 + \lambda_3 = 11$$

$$\lambda_3 = 11 - 9 = 2$$

$$\boxed{\therefore \lambda_3 = 2}$$

Hence the eigen values of A^{-1} are $1/2$, $1/3$ & $1/6$.

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6. If $-1, 2, 3$ are the eigen values of A, write down the eigen values of $4A$, A^2, A^{-1} and $A+2I$.

Eigen values of A are $-1, 2, 3$

$\therefore$ Eigen values of $4A$ are $-4, 8, 12$

$\therefore$ Eigen values of A^2 are $1, 4, 9$

" A^{-1} are $1, 1/4, 1/9$

" $A+2I$ are $-1+2, 2+2, 3+2$

ie) $1, 4, 5$.

7. find the eigen values of $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ corresponding to the eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We know that $AX = \lambda X$

$$X(A - \lambda I) = 0$$

$$\left[\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 3 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2 - \lambda = 0$$

$$\boxed{\lambda = 2}$$

Eigen value corresponding to $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is 2

8. If $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ find the eigen values of $2A^2$.

characteristic Eqn of $A \Rightarrow \lambda^2 - D_1\lambda + D_2 = 0$

$D_1 = 4 + 2 = 6$

$D_2 = |A| = 8 - 3 = 5$

ii) $\lambda^2 - 6\lambda + 5 = 0$

$\therefore \lambda = 1, 5$

$\therefore$ Eigen values of A^2 are $1, 25$

$\therefore$ Eigen values of $2A^2$ are $2, 50$.

Problems :

1. Determine the eigen values and eigen vectors of $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$.

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(-3-\lambda)-2\right]-2\left[2(-3-\lambda)+7\right]+0=0$$

$$\Rightarrow \lambda^3-13\lambda+12=0$$

$\lambda=1$ is a root.

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda^2+\lambda-12)=0$$

$$(\lambda-1)(\lambda+4)(\lambda-3)=0$$

$\therefore$ The eigen values of A are $1, 3, -4$.

The eigen vector X is given by $(A-\lambda I)X=0$,

where $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

ii) $$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 &= 0 \\ 2x_1 + (1-\lambda)x_2 + x_3 &= 0 \\ -7x_1 + 2x_2 + (-3-\lambda)x_3 &= 0 \end{aligned}\right] \text{——— } (1)$$

Case (i) $\lambda = 1$, Eqns (1) becomes

$$\left.\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + x_3 &= 0 \end{aligned}\right\}$$
$$-7x_1 + 2x_2 - 4x_3 = 0$$

first two Eqns by cross multiplication

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-1} = \frac{x_3}{0-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4}$$

$\therefore$ The eigen vector corresponding to $\lambda = -1$

is $X_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$

Case (ii) $\lambda = -4$, Eqns (1) becomes

$$6x_1 + 2x_2 = 0$$
$$\left.\begin{aligned} 2x_1 + 5x_2 + x_3 &= 0 \\ -7x_1 + 2x_2 + x_3 &= 0 \end{aligned}\right\}$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ 5 & 1 & 2 & 5 \\ 2 & 1 & -7 & 2 \end{array}$$

$$\frac{x_1}{3} = \frac{x_2}{-9} = \frac{x_3}{39}$$

$$\frac{x_1}{1} = \frac{x_2}{-3} = \frac{x_3}{13}$$

$\therefore$ The eigen vector corresponding to $\lambda = 4$ is $X_2 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$

<u>Case (iii)</u> $\lambda = 3$, Eqns (1) becomes

$$\left.\begin{aligned} -x_1 + 2x_2 &= 0 \\ 2x_1 - 2x_2 + x_3 &= 0 \\ -7x_1 + 2x_2 - 6x_3 &= 0 \end{aligned}\right\}$$

$$\frac{x_1}{12-2} = \frac{x_2}{-7+12} = \frac{x_3}{4-14}$$

$$\frac{x_1}{10} = \frac{x_2}{5} = \frac{x_3}{-10}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ The eigen vector corresponding to $\lambda = 3$ is $X_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Hence the eigenvectors of matrix A is $\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$ & $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ corresponding to $\lambda = 1, -4$ & 3.

(2) Find the eigen values and eigen vectors of

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

char. Eqn of A is $|A-\lambda I|=0$

i.e)
$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\left[(3-\lambda)^2-1\right]+2\left[-2(3-\lambda)+2\right]+2\left[2-2(3-\lambda)\right]=0$$

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

$$(\lambda-2)(\lambda^2-10\lambda+16)=0$$

$$\begin{array}{c|cccc} & 1 & -12 & 36 & -32 \\ 2 & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$\lambda = 2, 2, 8$

Eigen vector X is given by $(A-\lambda I)X=0$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e)
$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

i.e) $$\left.\begin{aligned}(6-\lambda)x_1 - 2x_2 + 2x_3 &= 0\\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0\\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0\end{aligned}\right\} \text{——— (1)}$$

Case (i) $\lambda = 8$, Eqns (1) becomes

$$-2x_1 - 5x_2 - x_3 = 0$$

$$\left.\begin{aligned}-2x_1 - 5x_2 - x_3 &= 0\\ 2x_1 - x_2 - 5x_3 &= 0\end{aligned}\right\}$$

$$\frac{x_1}{24} = \frac{x_2}{-12} = \frac{x_3}{12}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$ The eigen vector corresponding to $\lambda = 8$ is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Case (ii) $\lambda = 2$, Eqns (1) becomes

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

All the Eqns are same. i.e) $2x_1 - x_2 + x_3 = 0$

There is one equation in three unknowns.

By giving arbitrary values to any two of the unknown x_1, x_2 & x_3, we can find the eigen vectors of A.

put $x_1 = 1$ & $x_2 = 0$, we get $x_3 = -2$

put $x_1 = 0$ & $x_2 = 1$, we get $x_3 = 1$

$\therefore$ The eigen vectors corresponding to $\lambda = 2$

are $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ +1 \end{pmatrix}$.

---

(3) Verify that the eigen vectors of the real Symmetric matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ are orthogonal in pairs.

char. Eqns of A is $|A - \lambda I| = 0$

ie) $\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$(\lambda - 1)(\lambda^2 - 3\lambda - 4) = 0$

$$(\lambda-1)(\lambda-4)(\lambda+1)=0$$

$\therefore$ The eigen values of A are $1, 4, -1$

The eigen vector X is given by $(A-\lambda I)X = 0$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i) $$\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{—} \quad (1)$$

Case (1) when $\lambda = 1$, Eqn (1) becomes

$$x_1 + x_2 - x_3 = 0$$

$$\left.\begin{aligned} x_1 - 2x_3 &= 0 \\ -x_1 - 2x_2 &= 0 \end{aligned}\right\}$$

$$\frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$\therefore \; X_1 = \begin{pmatrix} -2 \\ 1 \\ \end{pmatrix}$

Case (ii), $\lambda = 4$, Eqns (1) becomes

$$\left.\begin{aligned} -2x_1 + x_2 - x_3 &= 0 \\ x_1 - 3x_2 - 2x_3 &= 0 \end{aligned}\right\}$$

$$-x_1 - 2x_2 - 3x_3 = 0$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

(i) $\dfrac{x_1}{-1} = \dfrac{x_2}{-1} = \dfrac{x_3}{1}$.

$$\therefore X_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) $\lambda = -1$, Eqns (1) becomes

$$\left.\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}\right\}$$

$$-x_1 - 2x_2 + 2x_3 = 0$$

solving, $\dfrac{x_1}{0} = \dfrac{x_2}{5} = \dfrac{x_3}{5}$

(i) $\dfrac{x_1}{0} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$

$$\therefore X_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Now, $X_1^T X_2 = \begin{pmatrix} -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0$

$X_2^T X_3 = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$

$X_3^T X_1 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 + 1 - 1 = 0$

Since the inner product of the vectors are zero, the eigen vectors are orthogonal in pairs.

(4) find the eigen values and eigen vectors of (adj A), when $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

char. Eqn of matrix A is $|A - \lambda I| = 0$

i.e.) $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$(\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 4) = 0$

$\therefore$ Eigen values of A are 1, 1, 4

then, Eigen values of A^{-1} are 1, 1, 1/4

Since $(\text{adj } A) = |A| \cdot A^{-1}$

Now $|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 4$

$\therefore$ Eigen values of $(\text{adj } A) = 4 \times$ eigen values of A^{-1}

i.e) $4, 4, 1$

Eigen vector of A is given by $(A - \lambda I) X = 0$

i.e) $\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ —— (1)

Case (i) $\lambda = 4$, Eqn (1) becomes
$$\begin{cases} -2x_1 - x_2 + x_3 = 0 \\ -x_1 - 2x_2 + x_3 = 0 \\ x_1 - x_2 - 2x_3 = 0 \end{cases}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$\therefore$ Eigen vector corresponding to $\lambda = 4$

is $X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Case (ii) $\lambda = 1$ Eqn (1) reduces to $x_1 - x_2 + x_3 = 0$

By giving arbitrary values to any two unknown we find eigen vectors.

i.e) put $x_1 = 0$, $x_2 = 1$, then $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

put $x_1 = 1$, $x_3 = 0$, then $X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Thus the eigenvalues of A^{-1} and $(\text{adj } A)$ are $1, 1, 1/4$ and $4, 4, 1$ respectively and the corresponding Eigen vectors are X_2, X_3 & X_1.

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

i.e) If. $C_0\lambda^n + C_1\lambda^{n-1} + \dots + C_{n-1}\lambda + C_n = 0$ is the characteristic equation of $n \times n$ square matrix A, then $C_0 A^n + C_1 A^{n-1} + \dots + C_{n-1} A + C_n I = 0$, where I is the unit matrix of order n and RHS is a null matrix of order n.

Applications: 1. The inverse of a non-singular matrix can be obtained by using the Cayley-Hamilton theorem.

2. By using Cayley-Hamilton theorem, we can find the higher positive integral powers of given matrix A.

Problems

① Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$$

Let $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$

The char. Eqn is given by $|A - \lambda I| = 0$

i.e.) $\begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(3-\lambda) - 3 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0$$

we have to show that $A^2 - 8A + 12I = 0$

$$A^2 = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} = \begin{bmatrix} 28 & 24 \\ 8 & 12 \end{bmatrix}$$

$$A^2 - 8A + 12I = \begin{bmatrix} 28 & 24 \\ 8 & 12 \end{bmatrix} - 8\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 24 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 40 & 24 \\ 8 & 24 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the cayley - Hamilton theorem is verified.

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(2) Verify Cayley-Hamilton theorem for the matrix A and find its inverse where $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

Given $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

Char. Eqn is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

D_1 = Sum of diagonal elts

$D_1 = 2+2+2 = 6$

D_2 = Sum of the minors of leading diagonals

$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= 3+3+3$

$D_2 = 9$

$D_3 = |A| = 4$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To show that,

$A^3 - 6A^2 + 9A - 4 = 0$

$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$

$A^3 = A^2 \cdot A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$

$$\therefore \; A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$+ 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Cayley-Hamilton thm is verified.

To find A^{-1},

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$A^{-1} = \frac{-1}{4}(-A^2 + 6A - 9I)$$

$$\therefore \; A^{-1} = (A^2 - 6A + 9I) \cdot \frac{1}{4}$$

$$A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right]$$

$$\therefore \; A^{-1} = \frac{1}{4}\begin{pmatrix} 19 & -15 & 15 \\ -15 & 19 & -15 \\ 15 & -15 & 19 \end{pmatrix}$$

(3) Verify Cayley-Hamilton theorem and find A^4 for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$

Char. Eqn of A is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = -1$

$D_2 = \begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

$= -18$

$D_3 = |A| = 40$

$\therefore \lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

To verify cayley Hamilton thm, we show that

$A^3 + A^2 - 18A - 40I = 0$

Now, $A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{pmatrix}$

$A^3 = \begin{pmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{pmatrix}$

$A^3 + A^2 - 18A - 40I = \begin{pmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{pmatrix} + \begin{pmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{pmatrix}$

$-18 \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore$ Cayley-Hamilton theorem is verified.

Thus, $A^3 + A^2 - 18A - 40I = 0$ — ①

Multiply by A in ①

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$A^4 = -A^3 + 18A^2 + 40A$$

$$= 40\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix} + 18\begin{pmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{pmatrix} - \begin{pmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 248 & 100 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{pmatrix}$$

---

④ Use Cayley-Hamilton theorem to find the value of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ where $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

char Eqn of matrix A is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$$D_1 = 2 + 1 + 2 = 5$$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 7$$

$$D_3 = |A| = 3$$

$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton thm, we have

$$A^3 - 5A^2 + 7A - 3I = 0.$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5 \underbrace{(A^3 - 5A^2 + 7A - 3I)}_{0} + A(A^3 - 5A^2 + 8A - 2) + I$$

$$= A \underbrace{(A^3 - 5A^2 + 7A - 3I)}_{0} + A^2 + A + I$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

---

5. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$. Hence find A^4.

Let $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$

Char Eqn of A is $\lambda^2 - D_1\lambda + D_2 = 0$

$D_1 = 8$ & $D_2 = 12$

$\therefore \lambda^2 - 8\lambda + 12 = 0$

$\lambda = 2, 6$

when λ^n is divided by $\lambda^2 - 8\lambda + 12$, let the quotient be $Q(\lambda)$ and the remainder be $a\lambda + b$

$$\lambda^n = (\lambda^2 - 8\lambda + 12)\,Q(\lambda) + (a\lambda + b) \quad \text{---} \ (*)$$

put $\lambda = 2$,

$2^n = 0 + 2a + b$

$\therefore 2a + b = 2^n$ — (1)

put $\lambda = 6$,

$6^n = 0 + 6a + b$

$\therefore 6a + b = 6^n$ — (2)

solve (1) & (2)

$a = \dfrac{6^n - 2^n}{4}$ and $b = \dfrac{3(2^n) - 6^n}{2}$

Replace λ by A in (*)

$$A^n = (A^2 - 8A + 12I)\,Q(A) + (Aa + bI)$$

$\therefore \; A^n = aA + bI$ [$\because$ Cayley-Hamilton thm says $A^2 - 8A + 12I = 0$].

put $n=4$,

$$A^4 = \left(\frac{64-24}{4}\right)\begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} + \left[\frac{3(24)-64}{2}\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 320\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} - 624\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1600 & 960 \\ 320 & 960 \end{bmatrix} - \begin{bmatrix} 624 & 0 \\ 0 & 624 \end{bmatrix} = \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$$

$$\therefore \; A^4 = \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$$

Exercise

1. Verify Cayley-Hamilton thm for the matrix A and find A^4

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (iii) $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

2. Verify Cayley-Hamilton thm. and find A^{-1}

(i) $\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

3. find the eigen values and eigen vectors of a matrixes

(i) $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

4. find the eigen values and eigen vectors of a matrixes

(i) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Similar matrices:

Two matrices A and B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1}AP$.

Note: Two similar matrices have the same eigen values.

Diagonalisation of a matrix:

Let A be a square matrix of order n. Then A is said to be diagonalisable if there exists another non-singular square matrix M of order n such that $M^{-1}AM$ is a diagonal matrix. $M^{-1}AM$ is called the diagonalisation of A by similarity transform. It is denoted as D.

ie) $D = M^{-1}AM$

Diagonalisation by orthogonal transformation

If A is real symmetric matrix, then we can diagonalise the matrix by orthogonal transformation.

Procedure to diagonalise the matrix by orthogonal transformation.

1. Find the eigen values of the matrix A.
2. Find the eigen vectors corresponding to the eigen values of A.
3. Normalise each eigen vector X_r, for this, divide each element of the eigen vector X_r by the square root of the sum of the square of all the elts of X_r.
4. Form the normalised Modal Matrix N by using the normalised eigen vectors of A. Then,
5. $D = N^T A N$

Transforming the matrix A into D by means of the transformation $N^T A N = D$ is known as orthogonal transformation.

Orthogonal matrices

A square matrix A is said to be orthogonal if $AA^T = A^T A = I$.

If A is orthogonal then $\boxed{A^T = A^{-1}}$

Problems :

(1) Diagonalise the matrix $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ by orthogonal transformation.

Let $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

char. Eqns is given by $|A-\lambda I| = 0$

i.e) $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$(\lambda-1)(\lambda^2-5\lambda+4) = 0$$

$$(\lambda-1)(\lambda-1)(\lambda-4) = 0$$

$\therefore$ The eigen values of A are 1, 1, 4

The eigen vectors are given by $(A-\lambda I)X = 0$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e) $\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ ——— (1)

Case (i) $\lambda = 4$, Eqn (1) becomes

$$\left.\begin{array}{l} -2x_1 - x_2 + x_3 = 0 \\ -x_1 - 2x_2 - x_3 = 0 \\ x_1 - x_2 - 2x_3 = 0 \end{array}\right\}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$\therefore$ The eigen vector corresponding to $\lambda=4$ is $X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Case (ii) $\lambda = 1$, Eqn (1) becomes $x_1 - x_2 + x_3 = 0$

$\therefore$ By giving arbitrary values to any two unknowns we find the eigen vectors.

Taking $x_1 = 0$ and $x_2 = 1$ we get $x_3 = 1$

$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Now, To find the eigen vector X_3 orthogonal to X_2 and satisfying $x_1 - x_2 + x_3 = 0$ — (*)

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\therefore X_3^T X_2 = 0 + b + c = 0$

$b + c = 0$ — (1)

X_3 satisfying the Eqn (*) $a - b + c = 0$ — (2)

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from 1 & 2

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Hence the modal matrix $M = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

The normalised model matrix

$$N = \begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$$

Now,

$$N^T = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

$\therefore$ required orthogonal transformation is $D = N^T A N$.

$$D = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Diagonalise the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ by an orthogonal transformation.

Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Char. Eqn of A is $|A-\lambda I|=0$

i.e) $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$$(6-\lambda)\left[(3-\lambda)^2-1\right] + 2\left[-2(3-\lambda)+2\right] + 2\left[2-2(3-\lambda)\right] = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$(\lambda-2)(\lambda^2-10\lambda+16)=0$$

$$(\lambda-2)(\lambda-2)(\lambda-8)=0$$

$$\lambda = 2, 2, 8$$

The eigen vectors of A are given by $(A-\lambda I)X=0$, where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e) $\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ ——— (1)

Case (i) $\lambda = 8$, Eqn (1) becomes,

$$\left.\begin{aligned} -2x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 - 5x_2 - x_3 &= 0 \end{aligned}\right\}$$

$$2x_1 - x_2 - 5x_3 = 0$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) $\lambda = 2$, Eqn (1) becomes, $2x_1 - x_2 + x_3 = 0$ —— (*)

Take $x_1 = x_2 = 1$, then $x_3 = -1$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Now, let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the eigenvector which is orthogonal to X_2 and satisfying the Eqn (*)

$$X_3^T X_2 = a + b - c = 0 \quad \text{—— (1)}$$

from (*), we have $2a - b + c = 0$ —— (2)

from (1) & (2), $\dfrac{a}{0} = \dfrac{b}{-3} = \dfrac{c}{-3}$

$$\therefore X_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Modal matrix is $M = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

normalized modal matrix is

$$N = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

$$N^T = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

required orthogonal transformation is $D = N^TAN$

$$= \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Quadratic Forms

*Defn:- A homogeneous polynomial of degree two in any number of variables is called a quadratic form.

The general form of a quadratic form in n variables is

$$Q = c_{11}x_1^2 + c_{12}x_1x_2 + \dots + c_{1n}x_1x_n$$
$$+ c_{21}x_2x_1 + c_{22}x_2^2 + \dots + c_{2n}x_2x_n$$
$$+ \dots + c_{n1}x_nx_1 + c_{n2}x_nx_2 + \dots + c_{nn}x_n^2$$

i) $$Q = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\, x_i x_j \quad \text{--- (1)}$$

In general, the coefficients $c_{ij} \neq c_{ji}$. The coefficient of $x_{ij} = c_{ij} + c_{ji}$. If we define $a_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$. we obtain $a_{ij} = a_{ji}$ and $a_{ii} = c_{ii}$

Hence the matrix $A = [a_{ij}]$ is a symmetric matrix.

In matrix form, (1) becomes

$$Q = X^T A X, \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and $A = [a_{ij}]$ is a square matrix of order n, is called the matrix of the quadratic form.

<u>Linear transformation</u>

Let $Q = X^T A X$ be a quadratic form in n variables, where A is the matrix of the quadratic form.

Consider the transformation $X = PY$, where P is non-singular square matrix of order n.

Thus, $Q = X^T A X$ is transformed to

i.e) $Q = (PY)^T A (PY)$

$= Y^T P^T A P Y$

$= Y^T (P^T A P) Y$

$= Y^T B Y$, where $B = P^T A P$

Since A is symmetric i.e) $\boxed{A^T = A}$

then $B^T = (P^T A P)^T = P^T A^T P$

$B^T = P^T A P = B$

$\boxed{B^T = B}$

i.e) B is also symmetric matrix

$\therefore$ $Q = Y^T B Y$ is also a quadratic form in n-variables $(y_1, y_2, \ldots, y_n)$.

ie) $$Q = [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

is known as canonical form (or) sum of the squares form of Q.

Orthogonal Reduction of a Quadratic form

Let $X^T A X$ be a given quadratic form and N be normalised modal matrix (ie) N is orthogonal matrix) of A, then $X = NY$ will reduce $X^T A X$ to $Y^T D Y$, where D is the diagonal matrix whose elements are eigen values of A.

Nature of Quadratic form

When the quadratic form $X^T A X$ is reduced to a canonical form it will contain only r terms if rank of A is r.

* The number of positive terms in the canonical form is called the index of the quadratic form and it is denoted as p.

* The number of non-positive terms is $r-p$

* The difference of number of positive square terms and non-positive terms is called the signature of the quadratic form, and it is denoted as 's'.

i.e) Signature, $s = p-(r-p) = 2p-r$.

The quadratic form $Q = X^T A X$ is said to be

(i) Positive definite : if $r=n$ and $p=n$ (or) if all the eigen values of A are positive.

(ii) Negative definite: if $r=n$ and $p=0$ (or) if all the eigen values of A are negative.

(iii) Positive semi definite, if $r<n$ and $p=r$ or if atleast eigen value of A is zero and others are positive.

(iv) Negative semi definite, if $r<n$ and $p=0$ or if atleast one eigen value of A is zero and others are negative.

(v) Indefinite in all other cases, or the eigen values of A are positive as well as negative.

Alter Method: The nature of quadratic form can also be obtained from principal minors of the matrix.

Let $A=(a_{ij})$ be the matrix of the quadratic form of order n. Then, the principal minors

$$S_1 = |a_{11}| \; ; \; S_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$S_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \; \dots \; S_n = |A|.$$

The quadratic form $Q = X^T A X$ is said to be

(i) Positive definite if all $S_i > 0$, $i = 1, 2, \ldots, n$

(ii) Negative definite if all $S_i < 0$, $i = 1, 2, \ldots, n$

(iii) Positive semi definite if $S_i \geq 0$ and atleast one $S_i = 0$.

(iv) Negative semi definite if $S_i \leq 0$ and atleast one $S_i = 0$.

(v) Indefinite in all other cases.

Problems

① write the matrix of the quadratic form

(i) $x_1^2 + 2x_2^2 + 2x_1x_2$

(ii) $x_1^2 + 4x_2^2 - 3x_1x_2$

(i) The matrix of the quadratic form $Q: x_1^2 + 2x_2^2 + 2x_1x_2$ is a matrix A of order 2×2.

(i) $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

since $a_{ii} = c_{ii}$

$a_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$

(ii) The matrix of the quadratic form $Q: x_1^2 + 4x_2^2 - 3x_1x_2$ is a matrix A of order 2×2.

(i) $A = \begin{pmatrix} 1 & -3/2 \\ -3/2 & 4 \end{pmatrix}$

② write the following matrices in quadratic form.

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

we know that $Q = X^T A X$

(i) $Q = (x_1 \ x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= (x_1 \ x_2) \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = x_1 x_2 + x_2 x_1$

$Q = 2x_1 x_2$

(ii) $Q = (x_1 \ x_2 \ x_3) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= (x_1 \ x_2 \ x_3) \begin{pmatrix} x_1 + x_3 \\ x_2 - x_3 \\ x_1 - x_2 \end{pmatrix}$

$= x_1^2 + x_3 x_1 + x_2^2 - x_2 x_3 + x_1 x_3 - x_2 x_3$

$= x_1^2 + x_2^2 + 2x_1 x_3 - 2x_2 x_3$

(3) Reduce $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4xz$ to Canonical form through orthogonal reduction.

we know that $Q = X^T A X$

where A is the matrix of Q and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

The char Eqn of A is given by $|A-\lambda I| = 0$

i.e) $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3 \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)\left[3(7-\lambda) - 16\right] + 6\left[-18+8\right] + 2\left[24 - 2(7-\lambda)\right] = 0$

i.e) $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0, 3, 15$ are the eigen values of A.

Consider the Equations $(A-\lambda I)X = 0$

i.e) $\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ ——— (1)

<u>Case (1)</u> $\lambda = 0$, Eqns (1) becomes

$8x - 6y + 2z = 0$

$\left.\begin{aligned} -6x + 7y - 4z &= 0 \\ 2x - 4y + 3z &= 0 \end{aligned}\right\}$

$x/5 = y/10 = z/10$

$\therefore$ Eigen vector corresponding to $\lambda=0$ is

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii) $\lambda=3$, Eqn ① becomes

$$\left.\begin{aligned} 5x - 6y + 2z &= 0 \\ -6x + 4y - 4z &= 0 \\ 2x - 4y &= 0 \end{aligned}\right\}$$

on multiplication rule, we get

$$\frac{x}{-16} = \frac{y}{-8} = \frac{z}{16}$$

$\therefore$ Eigen vector corresponding to $\lambda=3$ is

$$X_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

Case (iii) $\lambda=15$, Eqn ① becomes

$$\left.\begin{aligned} -7x - 6y + 2z &= 0 \\ -6x - 8y - 4z &= 0 \\ 2x - 4y - 12z &= 0 \end{aligned}\right\}$$

$$\Rightarrow \frac{x}{80} = \frac{y}{-80} = \frac{z}{40}$$

$\therefore$ The Eigen vector corresponding to $\lambda=15$ is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

$$X_1^T X_2 = (1\ 2\ 2)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -2-2+4=0$$

$$X_1^T X_3 = (1\ 2\ 2)\begin{pmatrix}2\\-2\\1\end{pmatrix} = 2-4+2=0$$

$$X_2^T X_3 = (-2\ -1\ 2)\begin{pmatrix}2\\-2\\1\end{pmatrix} = -4+2+2=0$$

$\therefore$ X_1, X_2 & X_3 are pairwise orthogonal.

$\therefore$ Modal matrix, $M = \begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}$

Normalised modal matrix, $N = \begin{pmatrix}1/3 & -2/3 & 2/3\\ 2/3 & -1/3 & -2/3\\ 2/3 & 2/3 & 1/3\end{pmatrix}$

$$N^T = \begin{pmatrix}1/3 & 2/3 & 2/3\\ -2/3 & -1/3 & 2/3\\ 2/3 & -2/3 & 1/3\end{pmatrix}$$

Since N is orthogonal, $\boxed{N^T = N^{-1}}$

Hence, $N^T A N = D$

$$= \begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{pmatrix}$$ [by diagonalization]

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Consider the orthogonal transformation

put $X = N Y$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Thus $Q = X^T A X$

$= Y^T D Y$

$= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} (y_1 \ y_2 \ y_3)$

$\therefore$ The req. Canonical form is

$3y_2^2 + 15y_3^2$.

Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation. find the nature.

$Q = X^T A X$, where A is the matrix of Q.

Thus, $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The char Eqn of A is given by $|A - \lambda I| = 0$

(i) $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right]+1\left[-1(1-\lambda)-0\right]=0$

i.e) $\lambda^3-4\lambda^2+3\lambda=0$

$\lambda(\lambda-1)(\lambda-3)=0$

$\therefore \lambda=0,1,3$ are the eigen values of A.

The eigenvector of A are given by the Eqn $(A-\lambda I)X=0$, where $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e) $\left.\begin{array}{l} (1-\lambda)x_1-x_2=0 \\ -x_1+(2-\lambda)x_2+x_3=0 \\ x_2+(1-\lambda)x_3=0 \end{array}\right\} \longleftarrow (1)$

Case (i) $\lambda=0$, Eqn (1) becomes

$x_1-x_2=0$

$-x_1+2x_2+x_3=0$

$x_2+x_3=0$

we have $\dfrac{x_1}{-1}=\dfrac{x_2}{-1}=\dfrac{x_3}{1}$

$\therefore X_1=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Case (ii) $\lambda = 1$, Eqn (I) becomes

$$\left.\begin{array}{l} -x_2 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{array}\right]$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Case (iii) $\lambda = 3$, Eqns (I) becomes

$$\left.\begin{array}{l} -2x_1 - x_2 = 0 \\ -x_1 - x_2 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array}\right]$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -1 & 0 & -2 & -1 \\ -1 & 1 & -1 & -1 \end{array}$$

$$\frac{x_1}{-1+0} = \frac{x_2}{0+2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Note that X_1, X_2 & X_3 are pairwise orthogonal.

$$\therefore M = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

Normalised modal matrix is $N = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0/\sqrt{2} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$

Since N is orthogonal matrix,
we have $N^T = N^{-1}$

Thus $N^T A N = D$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Consider the orthogonal transformation
$X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Q = X^T A X = Y^T D Y$$

$$= (y_1 \quad y_2 \quad y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$Q = y_2^2 + 3y_3^2$$

Rank, $r = 2$

Index, $p = 2$

Signature $= 2p - r = 4 - 2 = 2$

Nature of quadratic form is positive semi definite

(5) Reduce the quadratic form $2xy + 2yz + 2zx$ into canonical form by means of orthogonal transformation. Find its nature.

Quadratic form $Q = X^T A X$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and A is the matrix of quadratic form, i.e) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Char Eqn of A is given by $|A - \lambda I| = 0$

i.e) $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\lambda^3 - 3\lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1)^2 = 0$

$\therefore \lambda = -1, -1, 2$ are the eigen values of A.

Consider the Eqn $(A - \lambda I)X = 0$

i.e) $\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ ——— (1)

Case (i) $\lambda = 2$, Eqn (1) becomes

$$\left.\begin{aligned} -2x + y + z &= 0 \\ x - 2y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}\right\}$$

By cross multiplication of last two Eqns $\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$

$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case (ii) $\lambda = -1$, Eqn (1) becomes

$x + y + z = 0$

By giving arbitrary values to any of the two unknowns x, y, z we can find the eigen vectors.

put $x = 1$ & $y = 0$ we get $z = -1$

i.e) $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be eigen vector which is orthogonal to X_2 and satisfying the Eqn $x+y+z=0$

$$X_2 X_3^T = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} (a \quad b \quad c)$$

$$\boxed{0 = a - c} \quad \text{--- (2)}$$

we have $\boxed{a+b+c=0}$ --- (3)

from (2) & (3), we have $\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$

(ii) $X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore$ Modal matrix, $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$.

Normalised modal matrix is

$$N = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

Then

$$N^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

Since N is orthogonal, $N^T = N^{-1}$

Thus, $N^T A N = D$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Consider the orthogonal transformation is

$X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

Thus, $Q = X^T A X = Y^T D Y$ (by diagonalisation)

$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$Q = 2y_1^2 - y_2^2 - y_3^2$ is the canonical form

and the nature of QF is indefinite

$\therefore$ Orthogonal tranformation is $X = NY$

reduces to $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

ie) $x = 1/\sqrt{3}\, y_1 + 1/\sqrt{2}\, y_2 + 1/\sqrt{6}\, y_3$

$y = 1/\sqrt{3}\, y_1 - 2/\sqrt{6}\, y_3$

$z = 1/\sqrt{3}\, y_1 - 1/\sqrt{2}\, y_2 + 1/\sqrt{6}\, y_3$.

Exercise

① Reduce the Quadratic form $x_1^2+5x_2^2+x_3^2+2x_1x_2+2x_2x_3+6x_1x_3$ to Canonical form through an orthogonal transformation.

② Reduce the QF into canonical form by an orthogonal transformation and find the nature.

(i) $10x_1^2+2x_2^2+5x_3^2+6x_2x_3-10x_3x_1-4x_1x_2$.

(ii) $x_1^2+3x_2^2+3x_3^2-2x_2x_3$.

(iii) $6x^2+3y^2+3z^2-4yz+4zx-2xy$.

③ Diagonalise the following matrices by orthogonal transformation

(i) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$

Question Bank

Matrices

Part - A

1. If 3 and 6 are two eigen values of $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, find the eigen values of A^{-1}.

2. Find the eigen values of $\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$.

3. Find the sum of the eigen values of A^{-1}, if $A = \begin{pmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{pmatrix}$.

4. If the sum of two eigen values and trace of a 3×3 matrix A are equal, find the value of $|A|$.

5. Discuss the nature of the quadratic form $Q = 2xy + 2yz + 2zx$.

6. What is the matrix associated to the quadratic form $Q = 2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_2$.

7. Compute the eigen values of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

8. Find the constant a and b such that the matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and -2 as its eigen values.

9. If 1 and 2 are the eigen values of a 2×2 matrix A, what are the eigen values of A^2 and A^{-1}?

10. State Cayley - Hamilton theorem.

11. If -1 is an eigen value of the matrix $A=\begin{pmatrix}1 & -2\\ -3 & 2\end{pmatrix}$ find the eigen values of A^4 using properties.

12. Use Cayley - Hamilton theorem to find $A^4 - 8A^3 - 12A^2$ when $A=\begin{bmatrix}5 & 3\\ 1 & 3\end{bmatrix}$.

13. Find the eigen values of $A=\begin{pmatrix}0 & 0 & 2\\ 0 & -1 & 4\\ 3 & 1 & -5\end{pmatrix}$.

Also find the eigenvalues of $-3A$.

14. Find the nature of the Quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$.

15. The product of two eigenvalues of the matrix $A = \begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}$ is 16. Find the third eigenvalues of A.

16. Find the eigen values of A^{-1}, where $A=\begin{pmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{pmatrix}$

17. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$.

18. Write down the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$.

19. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{pmatrix}$.

20. Write down the quadratic form whose associated matrix is $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Part - B

1. Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

2. Using Cayley-Hamilton theorem, find A^{-1}, when $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

3. Reduce the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ to canonical form by orthogonal reduction and state its nature.

4. Using Cayley-Hamilton theorem, find the inverse of the matrix $A = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & 7 \\ -3 & 0 & 8 \end{bmatrix}$.

5. Reduce the QF $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form by an orthogonal transformation and find its nature.

6. find the eigen values and eigen vectors of the matrix A^{-1} given $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

7. Using cayley Hamilton theorem evaluate the matrix $A^4 + A^3 - 18A^2 - 39A + 2I$, given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$.

8. Diagonalise the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by mean of an orthogonal transformation.

9. Reduce the QF $x^2+y^2+z^2-2xy-2yz-2zx$ into a canonical form through an orthogonal transformation.

10. Find the eigen values and Eigen vectors of the matrix $A=\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$.

11. Using cayley-Hamilton theorem, find the inverse of $A=\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$.

12. Find A^n using cayley-Hamilton theorem, Taking $A=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Also, find A^8.

13. Verify cayley-Hamilton theorem for the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ and hence find A^{-1} and A^4.

14. find the eigen values and eigen vectors of $A=\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

15. Reduce the QF $3x^2+5y^2+3z^2-2xy-2yz+2zx$ into canonical form.

EIGENVALUE OF A MATRIX BY POWER METHOD

① Find the dominant eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ Find also the Least Latent root and hence the third eigen value.

Soln: Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an approximate eigen vector

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7 X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.5714\, X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\, X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706 X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix}$$

$$= 4.0072\, X_7$$

$$AX_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix}$$

$$= 3.9982\, X_8$$

$$AX_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4\, X_9$$

$$AX_9 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

Domiant eigen value $= 4$, corresponding eigen vector is $(1, 0.5, 0)^T$.

To find the least eigen value, Let $B = A - 4I$ since $\lambda_1 = 4$.

$$B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

we will find the dominant eigen value of B

Let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector

$$BY_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3Y_2$$

$$BY_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5Y_3$$

$$BY_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigen value of B is -5.

Adding 4, smallest eigen value of $A = -5+4 = -1$

Sum of eigen values = Trace of A

$$4 + (-1) + \lambda_3 = 1+2+3$$

$$3 + \lambda_3 = 6$$

$$\lambda_3 = 6-3$$

$$\lambda_3 = 3$$

$\therefore$ All the three eigen values are $4, 3, -1$.

2) Find by power method the Largest eigen value and the corresponding eigenvector of a matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial vector $(1\ 1\ 1)^T$

Solution:

Let the initial eigenvector $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix} = 13 \begin{bmatrix} 0.231 \\ 0.692 \\ 1 \end{bmatrix} = 13 X_2$$

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 2 & 10 \end{bmatrix} \begin{bmatrix} 0.231 \\ 0.692 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.307 \\ 6.077 \\ 12.537 \end{bmatrix} = 12.537 \begin{bmatrix} 1.104 \\ 0.485 \\ 1 \end{bmatrix} = 12.537 X_3$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 2 & 10 \end{bmatrix} \begin{bmatrix} 1.104 \\ 0.485 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.559 \\ 5.282 \\ 11.836 \end{bmatrix} = 11.836 \begin{bmatrix} 0.047 \\ 0.446 \\ 1 \end{bmatrix} = 11.836 X_4$$

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 2 & 10 \end{bmatrix} \begin{bmatrix} 0.047 \\ 0.446 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.385 \\ 5.033 \\ 11.737 \end{bmatrix} = 11.737 \begin{bmatrix} 0.032 \\ 0.429 \\ 1 \end{bmatrix} = 11.737 X_5$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 2 & 10 \end{bmatrix} \begin{bmatrix} 0.032 \\ 0.429 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.319 \\ 4.954 \\ 11.684 \end{bmatrix} = 11.684 \begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix} = 11.684 X_6$$

$$AX_6 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 2 & 10 \end{bmatrix} \begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.296 \\ 4.927 \\ 11.665 \end{bmatrix} = 11.665 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix} = 11.665 X_7$$

$$AX_7 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 2 & 10 \end{bmatrix} \begin{bmatrix} 0.025 \\ 0.421 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.291 \\ 4.919 \\ 11.663 \end{bmatrix} = 11.663 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix} = 11.663 X_8$$

Domiant eigen vector is $= \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$

And Domiant eigen value $= 11.663$.

Part-A

1. Determine the Largest eigenvalue and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ correct to two decimal places using power method.

Soln: $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be the initial vector

$$Ax_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2X_2$$

$$AX_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2X_3$$

∴ This show that the Largest eigen value = 2 and corresponding eigen vector $[1 \ \ 1]^T$.

2) If the eigen values of A are −3, 3, 1 then the domiant eigenvalue of A is no domiant eigenvalue

3) If the eigen value of A are 1, 3, 4 then the domiant eigen value = 4.

4) If the eigenvalues of A are −4, 3, 1 then the domiant eigenvalue of A is = −4.

5) Write down the procedure to find the numerically smallest eigen value of A is matrix by power method.

(or)

How will you find the smallest eigenvalue of a square matrix A?

Sol:

By power method the Largest eigen value of A^{-1} can be found. Then smallest eigenvalue of A is the reciprocal of Largest eigenvalue of A^{-1}.

6) What type of eigenvalue can be obtained using Power method?

Soln

We can obtain dominant eigenvalue of the matrix.