Functions of Several Variables

Function of two independent variables

A Symbol z which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write $z = f(x,y)$ or $\phi(x,y)$

When z is a function of three or more variables $x, y, t, \ldots$ we represent the relation by writing $z = f(x, y, t \ldots)$.

For such functions, no geometrical representation is possible. However, the concepts of a region and neighbourhood can easily be extended to functions of three or more variables.

In this chapter, we will see, Partial derivatives Euler's theorem for homogeneous functions. - Total derivatives - Differentiation of implicit functions. Jacobians - Taylor's expansion - and other applications of several variables like, maxima and minima - Method of Lagrangian multipliers.

Partial Derivatives

Let $u = f(x, y)$ be a function of two independent variables x and y. Differentiating u with respect to x, keeping 'y' as constant is known as the partial differential coefficient of u w.r. to x, It is denoted by $\frac{\partial u}{\partial x}$

Here $\frac{\partial u}{\partial x}$ will represent the first derivative of u w.r. to x, we can do w.r to y as $\frac{\partial u}{\partial y}$

Here both $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are again in general a function of x and y. Hence each partial derivatives may again be differentiated w.r. to 'x' and respectively. and is denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \quad , \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

Problems:-

① If $u = \frac{y}{z} + \frac{z}{x}$ find the value of

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

Soln: Given, $u = \frac{y}{z} + \frac{z}{x}$, $\frac{\partial u}{\partial x} = 0 - z/x^2$

$\frac{\partial u}{\partial y} = \frac{1}{z} + 0$

$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x}$

$x\frac{\partial u}{\partial x} = \frac{-z}{x}$; $y\frac{\partial u}{\partial y} = \frac{y}{z}$

$z\frac{\partial u}{\partial z} = -y/z + z/x$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(2) If $u = \sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$. Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$

Soln:

Given $u = \sin^{-1}\left(\frac{x^3-y^3}{x+y}\right) \Rightarrow \sin u = \frac{x^3-y^3}{x+y}$

Diff. Partially w.r to x we get

$$\cos u \cdot \frac{\partial u}{\partial x} = \frac{(x+y)\,3x^2 - (x^3-y^3)}{(x+y)^2}$$

Diff. Partially w.r to y we get.

$$\cos u \frac{\partial u}{\partial y} = \frac{(x+y)(-3y^2) - (x^3-y^3)(1)}{(x+y)^2}$$

$$x + \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{\cos u}\left[\frac{3(x+y)(x^3-y^3) - (x^3-y^3)(x+y)}{(x+y)^2}\right]$$

$$= \frac{2}{\cos u}\left[\frac{x^3-y^3}{x+y}\right]$$

$$= \frac{2\sin u}{\cos u} = 2\tan u.$$

(3) If $u = (x-y)(y-z)(z-x)$ show that.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Soln:

Given: $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$

$$= -(x-y)(y-z) + (y-z)(z-x)$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$\frac{\partial u}{\partial y} = (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (x-y)(y-z) - (x-y)(z-x)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

④ If $f(x,y) = \log\sqrt{x^2+y^2}$, show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Soln:

Given: $f = \log\sqrt{x^2+y^2}$

$$= \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

similarly $\frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

Now, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Euler's theorem on homogeneous function:

If u is a homogeneous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

① If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Solution: $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right) \Rightarrow \tan u = \frac{x^3+y^3}{x-y}$

Let $f(x,y) = \tan u = \frac{x^3+y^3}{x-y}$

$$f(tx,ty) = \frac{(tx)^3+(ty)^3}{tx-ty} = \frac{t^3(x^3+y^3)}{t(x-y)}$$

$$= t^2 f(x,y)$$

$\therefore f$ is a homogenous function of degree 2 in x and y.

By Euler's theorem we get $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$ —①

Here $f = \tan u$ $\quad \frac{\partial f}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$

$$\frac{\partial f}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

$\therefore$ ① $\Rightarrow$ $x\sec^2 u \frac{\partial u}{\partial x} + y\sec^2 u \frac{\partial u}{\partial y} = 2\tan u$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \frac{1}{\sec^2 u} = 2\sin u \cos u$$

$$= \sin 2u$$

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2. If u is a homogeneous function of degree n in x and y, then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Soln: By Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$ ... (1)

Differentiating (1) partially w.r.to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

$$\text{(i.e.,)}\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x} \quad \text{—— (2)}$$

Differentiating (1) partially w.r.to y, we get

$$x\frac{\partial^2 u}{\partial y \partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial y \partial x} = (n-1)\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

(2) $\times x$ + (3) $\times y$ $\Rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial^2 u}{\partial y \partial x} = (n-1)x\frac{\partial u}{\partial x}+(n-1)y\frac{\partial u}{\partial y}$$

$$= (n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2} = (n-1)nu$$

$$\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

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Total derivatives - change of Variables - Partial differentiation of Implicit functions

Total derivative:- If $u = f(x,y)$, where $x = \phi(t)$, $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x,y)$. Thus we can find the ordinary derivative $\frac{du}{dt}$, which is called Total derivative of u

ie, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ [without substitution of x & y]

Composite function of one variable:

If $u = f(x,y,z)$ where x, y, z are all functions of a variable t, then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Composite function of two variables:

If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$ then z is a function of u, v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Differentiation of Implicit functions

If $f(x,y) = c$ be a implicit relation between x and y which defines as a deferential function of x then, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

becomes $0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$

ie, $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$ $\quad [\because \frac{\partial f}{\partial y} \neq 0]$

Problems

① If $u = x\log(xy)$ where $x^3+y^3+3xy = 1$ then find $\frac{du}{dx}$?

Soln:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= x\left[\frac{1}{x}\right] + [\log x + \log y] + x\left(\frac{1}{y}\right)\frac{dy}{dx}$$

$$= 1 + \log x + \log y + \frac{x}{y}\frac{dy}{dx} \quad \text{—①}$$

Given: $x^3+y^3+3xy = 1$

differentiating w.r. to x, we get,

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left[y + x\frac{dy}{dx}\right] = 0$$

$$(x+y^2)\frac{dy}{dx} = -[y+x^2]$$

$$\frac{dy}{dx} = \frac{-[y+x^2]}{x+y^2}$$

$$① \Rightarrow \therefore \frac{du}{dx} = \log x + \log y + 1 - \frac{x(y+x^2)}{y(x+y^2)}$$

(2) Find $\frac{dy}{dx}$ when $y \sin x = x \cos y$

Soln: Let $f(x,y) = y \sin x - x \cos y$

$$\frac{\partial f}{\partial x} = y \cos x - \cos y$$

$$\frac{\partial f}{\partial y} = \sin x + x \sin y$$

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\left(\frac{y \cos x - \cos y}{\sin x + x \sin y}\right)$$

(3) If $z = f(y-z, z-x, x-y)$, show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = 0.$$

Soln: Let $u = y-z$, $v = z-x$, $w = x-y$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \quad -(1)$$

Similarly $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} \quad -(2)$

$$\frac{\partial z}{\partial z} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \quad -(3)$$

Adding (1) + (2) + (3) implies

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = 0$$

④ If $g(x,y) = \varphi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$ then prove that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right]$

Soln: $g(x,y) = \varphi(u,v)$

$u = x^2 - y^2$ $\qquad\qquad$ $v = 2xy$

$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = -2y$ $\qquad\qquad$ $\frac{\partial v}{\partial x} = 2y, \ \frac{\partial v}{\partial y} = 2x$

$$\frac{\partial g}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial \varphi}{\partial u}(2x) + \frac{\partial \varphi}{\partial v}(2y)$$

$$= 2x \frac{\partial \varphi}{\partial u} + 2y \frac{\partial \varphi}{\partial v}$$

$$\frac{\partial g}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial \varphi}{\partial u}(-2y) + \frac{\partial \varphi}{\partial v}(2x)$$

$$= -2y \frac{\partial \varphi}{\partial u} + 2x \frac{\partial \varphi}{\partial v}$$

$$\frac{\partial}{\partial y} = -2y \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial x} = 2x \frac{\partial}{\partial u} + 2y \frac{\partial}{\partial v}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \left(2x \frac{\partial}{\partial u} + 2y \frac{\partial}{\partial v}\right)\left(2x \frac{\partial \varphi}{\partial u} + 2y \frac{\partial \varphi}{\partial v}\right)$$

$$= 2x \frac{\partial}{\partial u}\left[2x \frac{\partial \varphi}{\partial u} + 2y \frac{\partial \varphi}{\partial v}\right] + 2y \frac{\partial}{\partial v}\left[2x \frac{\partial \varphi}{\partial u} + 2y \frac{\partial \varphi}{\partial v}\right]$$

$$= 4x^2 \frac{\partial^2 \varphi}{\partial u^2} + 4xy \frac{\partial^2 \varphi}{\partial u \partial v} + 4xy \frac{\partial^2 \varphi}{\partial v \partial u} + 4y^2 \frac{\partial^2 \varphi}{\partial v^2} \quad \text{—①}$$

Similarly

$$\frac{\partial^2 g}{\partial y^2} = 4y^2 \frac{\partial^2 \varphi}{\partial u^2} - 4xy \frac{\partial^2 \varphi}{\partial u \partial v} - 4xy \frac{\partial^2 \varphi}{\partial v \partial u} + 4x^2 \frac{\partial^2 \varphi}{\partial v^2} \quad \text{—②}$$

Adding ① and ② we get.

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right)$$

JACOBIANS

Properties of Jacobians.

Def. $\dfrac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$ and $\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$

(i) $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1$

(ii) $\dfrac{\partial(u,v)}{\partial x,y} \cdot \dfrac{\partial(x,y)}{\partial(r,s)} = \dfrac{\partial(u,v)}{\partial(r,s)}$

(iii) If u, v, w are functionally dependent functions of three independent variables x, y, z

then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Problems:

① If $x = r\cos\theta$, $y = r\sin\theta$ find (i) $\dfrac{\partial(x,y)}{\partial(r,\theta)}$, (ii) $\dfrac{\partial(r,\theta)}{\partial(x,y)}$

Soln: Given: $x = r\cos\theta$ $\quad y = r\sin\theta$

$\dfrac{\partial x}{\partial r} = \cos\theta$ $\qquad \dfrac{\partial y}{\partial r} = \sin\theta$

$\dfrac{\partial x}{\partial \theta} = -r\sin\theta$ $\qquad \dfrac{\partial y}{\partial \theta} = r\cos\theta$

(i) $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r[\cos^2\theta + \sin^2\theta] = r$

(ii) W.k.T. $\dfrac{\partial(x,y)}{\partial(r,\theta)} \cdot \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$

which implies $r \cdot \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$

$$\Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

(2) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3, if $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$

Soln:

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}.$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2}\left[\frac{x_1^2 x_2 x_3}{x_2^2 x_3^2} - \frac{x_1^2}{x_2 x_3}\right] - \left(\frac{x_3}{x_1}\right)\left[\frac{-x_1}{x_3} - \frac{x_1 x_2}{x_2 x_3}\right] + \frac{x_2}{x_1}\left[\frac{x_1}{x_2} + \frac{x_1}{x_2}\right]$$

$$= -1 + 1 + 1 + 1 + 1 + 1$$

$$= 4$$

③ Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Soln:

The Jacobian of Transformation $\Big\}$ $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$x = r\sin\theta\cos\phi$ $\qquad y = r\sin\theta\sin\phi$ $\qquad z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\qquad \frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ $\qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ $\qquad \frac{\partial z}{\partial \theta} = -r\sin\theta$

$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$ $\qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$ $\qquad \frac{\partial z}{\partial \phi} = 0$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \cos\theta\left[r^2\cos\theta\sin\theta\cos^2\phi + r^2\cos\theta\sin\theta\sin^2\phi\right]$
$\qquad + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$

$= r^2\sin\theta\left[\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin\theta[\sin^2\theta(\cos^2\phi + \sin^2\phi)]\right]$

$= r^2\sin\theta\left[\cos^2\theta + \sin^2\theta\right)$

$= r^2\sin\theta$

Taylor Series for a function of two variables.

By Taylor's theorem:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \cdots\cdots$$

Problems:

① Expand $e^x \cos y$ about $(0, \pi/2)$ upto the third term using Taylor's series.

Soln:

Given $a = 0$, $b = \pi/2$ $h = x - a = x$
$k = y - b = y - \pi/2$

By Taylor theorem we have the above formula,

Function	Value at $(0, \pi/2)$
$\therefore f(x,y) = e^x \cos y$	$f = 0$
$f_x = e^x \cos y$	$f_x = 0$
$f_y = -e^x \sin y$	$f_y = -1$
$f_{xx} = e^x \cos y$	$f_{xx} = 0$
$f_{xy} = -e^x \sin y$	$f_{xy} = -1$
$f_{yy} = -e^x \cos y$	$f_{yy} = 0$
$f_{xxx} = e^x \cos y$	$f_{xxx} = 0$
$f_{xxy} = -e^x \sin y$	$f_{xxy} = -1$
$f_{xyy} = -e^x \cos y$	$f_{xyy} = 0$
$f_{yyy} = e^x \sin y$	$f_{yyy} = 1$

$$\therefore f(x,y) = 0 + \left[x(0) + \left(y-\tfrac{\pi}{2}\right)(-1)\right] + \frac{1}{2!}\left[x^2(0) + 2x\left(y-\tfrac{\pi}{2}\right)(-1) + \left(y-\tfrac{\pi}{2}\right)^2(0)\right]$$

$$+ \frac{1}{3!}\left[x^3(0) + 3x^2\left(y-\tfrac{\pi}{2}\right)(-1) + 3x\left(y-\tfrac{\pi}{2}\right)^2(0) + \left(y-\tfrac{\pi}{2}\right)^3(1)\right] + \cdots$$

$$f(x,y) = -y + \tfrac{\pi}{2} + \frac{1}{2!}\left[-2xy + 2x\left(\tfrac{\pi}{2}\right)\right] + \frac{1}{3!}\left[-3x^2y + \tfrac{3\pi}{2}x^2 + \left(y-\tfrac{\pi}{2}\right)^3\right] + \cdots$$

(2) Expand $e^x \log(1+y)$ in powers of x and y upto terms of third degree.

Soln:

Function	Value at $(0,0)$
$f(x,y) = e^x \log(1+y)$	$f = 0$
$f_x = e^x \log(1+y)$	$f_x = 0$
$f_y = e^x (1+y)^{-1}$	$f_y = 1$
$f_{xx} = e^x \log(1+y)$	$f_{xx} = 0$
$f_{xy} = e^x (1+y)^{-1}$	$f_{xy} = 1$
$f_{yy} = -e^x (1+y)^{-2}$	$f_{yy} = -1$
$f_{xxx} = e^x \log(1+y)$	$f_{xxx} = 0$
$f_{xxy} = e^x (1+y)^{-1}$	$f_{xxy} = 1$
$f_{xyy} = -e^x (1+y)^{-2}$	$f_{xyy} = -1$
$f_{yyy} = 2e^x (1+y)^{-3}$	$f_{yyy} = 2$

8

By Taylor series expansion, for $h = x - a = x$
$k = y - b = y$

$$f(x,y) = e^x \log(1+y) = 0 + \frac{x(0) + y(1)}{1!} + \frac{x^2(0) + 2xy(1) + y^2(-1)}{2!}$$
$$+ \frac{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)}{3!} + \cdots$$

$$e^x \log(1+y) = y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}[3x^2y - 3xy^2 + 2y^3] + \cdots$$

Maxima and Minima for functions of Two Variables

* Neccessary Conditions for a maximum or a minimum

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

* Sufficient Conditions.

If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$

$f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$

(i) $f(a,b)$ is Maximum values if $AC - B^2 > 0$ and $A < 0$ (or $B<0$)

(ii) $f(a,b)$ is Minimum value if $AC - B^2 > 0$ and $A > 0$ (or $B>0$)

(iii) $f(a,b)$ is not an extremum (saddle) if $AC - B^2 < 0$

(iv) If $AC - B^2 = 0$, then the test is inconclusive.

Problems

① Find the extreme values of the function.

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Soln: Given $f(x,y) = x^3+y^3-3x-12y+20$

$f_x(x,y) = 3x^2-3$, $f_y(x,y) = 3y^2-12$

$A = f_{xx} = 6x$; $B = f_{xy} = 0$; $C = f_{yy} = 6y$

To find the stationary points.

$f_x = 0$ $f_y = 0$

$\therefore$ $3x^2-3=0$ $3y^2-12=0$

$x^2-1=0$ $y^2-4=0$

$x = \pm 1$ $y = \pm 2$

$\therefore$ the stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$

$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$ $6>0,\ A>0$ $B=0$ $C=6y$ $=12$ $AC-B^2 = 72>0$ min point	$A=6>0$ $B=0$ $C=-12$ $AC-B^2=-72<0$ saddle point	$A=-6<0$ $B=0$ $C=12$ $AC-B^2<0$ saddle point	$A=-6<0$ $B=0$ $C=-12$ $72>0$ max. point.

$\therefore$ Maximum value of $f(x,y)$ is

$$f(-1,-2) = (-1)^3+(-2)^3-3(-1)-12(-2)+20$$
$$= -1-8+3+24+20$$

Minimum value of $f(x,y)$ is

$$f(1,2) = 1^3 + (2)^3 - 3(1) - 12(2) + 20$$
$$= 2$$

(2) A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

Soln. $u = x^2 + 2y^2 - x$

$U_x = 2x - 1$

$U_y = 4y$

$U_x = 0$ $\qquad$ $U_y = 0$

$\Rightarrow 2x - 1 = 0$ $\qquad$ $4y = 0$

$x = 1/2$ $\qquad$ $y = 0$

$A = U_{xx} = 2$; $B = U_{xy} = 0$; $C = U_{yy} = 4$

$\Delta = AC - B^2 > 0$

U is minimum at $(1/2, 0)$ and its minimum value is $-1/4$

(3) Find the maxima and minima of $xy(a - x - y)$

Soln

Given: $f(x,y) = xy(a-x-y)$

i.e $f = axy - x^2y - xy^2$

$f_x = ay - 2xy - y^2$

$f_y = ax - x^2 - 2xy$

Let $A = \frac{\partial^2 f}{\partial x^2} = -2y$; $B = f_{xy} = a-2x-2y$

and $C = f_{yy} = -2x$

$AC - B^2 = 4xy - (a-2x-2y)^2$

Solving. $f_x = 0$ $\qquad$ $f_y = 0$

$ay - 2xy - y^2 = 0$ and $ax - x^2 - 2xy = 0$

$y(a-2x-y) = 0$ $\qquad$ $x(a-x-2y) = 0$

$y=0$, $a-2x-y=0$ $\qquad$ $x=0$, $a-x-2y=0$

Solving stationary points are $(0,0), (a,0) (0,a) \& (a/3, a/3)$

At $(0,0)$, $AC-B^2 =$ Negative value

At $(a,0)$, $AC-B^2 =$ Negative value

At $(0,a)$, $AC-B^2 =$ Negative value

$\therefore$ f does not have maxima or minima at these pts.

At $(\frac{a}{3}, a/3)$, $AC-B^2 =$ positive and $A<0$ if $a>0$

f is maximum at $(a/3, a/3)$ if $a>0$

f is minimum at this point $a<0$

Maximum (or) Minimum value $= f(a/3, a/3) = \frac{a^3}{27}$

Lagrange's Method of undetermined Multipliers.

To find the maxima and minimum values of $f(x,y,z)$ where (x,y,z) are subject to a constraint equation $g(x,y,z)=0$.

We define a function

$$F(x,y,z,\lambda) = f(x,y,z) + \lambda\, g(x,y,z) \quad \text{— ①}$$

where λ is called Lagrange Multiplier which is Independent of x, y, z

Neccessary conditions for a maximum or minimum are :

$$\frac{\partial F}{\partial x}=0 \text{ — ②}, \quad \frac{\partial F}{\partial y}=0 \text{ — ③} \quad \text{and} \quad \frac{\partial F}{\partial z}=0 \text{ — ④}$$

Problems

① A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction.

Soln: Let x, y, z be the length, breadth and height of the box,

Surface area $= xy + 2yz + 2zx$

volume $= xyz = 32$

Let the auxiliary function F be

$$F(x,y,z,\lambda) = (xy+2yz+2zx) + \lambda\,(xyz-32)$$

where λ is Lagrange multiplier,

$$F_x = \frac{\partial F}{\partial x} = y+2z+\lambda yz \qquad F_y = \frac{\partial F}{\partial y} = x+2z+\lambda zx \qquad F_z = \frac{\partial F}{\partial z} = 2x+2y+\lambda xy$$

when F is extremum,

$F_x = 0$ $\qquad$ $F_y = 0$ $\qquad$ $F_z = 0$

$y+2z+\lambda yz = 0$ $\qquad$ $x+2z+\lambda zx = 0$ $\qquad$ $2x+2y+\lambda xy = 0$

$\frac{1}{z}+\frac{2}{y} = -\lambda$ — ① $\qquad$ $\frac{1}{z}+\frac{2}{x} = -\lambda$ — ② $\qquad$ $\frac{2}{y}+\frac{2}{x} = -\lambda$ — ③

From ① & ② we get

$\frac{1}{z}+\frac{2}{y} = \frac{1}{z}+\frac{2}{x}$

$\Rightarrow x = y$ — ④

From ② & ③ we get

$\frac{1}{z}+\frac{2}{x} = \frac{2}{y}+\frac{2}{x}$

$\Rightarrow y = 2z$ — ⑤

From ④ & ⑤ we get $x = y = 2z$

Volume $= xyz = 32 \Rightarrow (2z)(2z)z = 32$

$\Rightarrow 4z^3 = 32$

$z = 2, \therefore x=4, y=4,$

Cost minimum when $x=4, y=4, z=2$

Thus the dimension of the box are, 4, 4, 2.

② The temperature $u(x,y,z)$ at any point in space is $u = 400xyz^2$. Find the highest temperature on surface of the sphere. $x^2+y^2+z^2=1$.

Soln:- Given $u = f = 400xyz^2$, $\phi = x^2+y^2+z^2-1=0$ —(*)

Let the auxiliary function F be

$$F(x,y,z,\lambda) = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$$

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$F_x = 400yz^2 + 2x\lambda$ $\quad F_y = 400xz^2 + 2y\lambda$ $\quad F_z = 800xyz + 2z\lambda$

For maximum or minimav.

$F_x = 0$ $\quad F_y = 0$ $\quad F_z = 0$

$400yz^2 + 2x\lambda = 0$ $\quad 400xz^2 + 2y\lambda = 0$ $\quad 800xyz + 2z\lambda = 0$

$\frac{200yz^2}{x} = -\lambda$ — ① $\quad \frac{200xz^2}{y} = -\lambda$ — ② $\quad 400xy = -\lambda$ — ③

From ① & ② we get

$$\frac{200yz^2}{x} = \frac{200xz^2}{y}$$

$$y^2 = x^2 \text{ — ④}$$

From ② & ③ we get

$$\frac{200xz^2}{y} = 400xy$$

$$z^2 = 2y^2 \text{ — ⑤}$$

From ④ & ⑤ we get

$$x^2 = y^2 = \frac{z^2}{2} \text{ — ⑥}$$

$(*) \Rightarrow \frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 - 1 = 0 \quad 2z^2 = 1 \quad z = \pm\frac{1}{\sqrt{2}}$

⑥ $\Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2};\ y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$\therefore U = 400xyz^2$; we select x, y, z to be positive

$$U = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$U = 50$$

$\therefore$ Maximum temperature is 50

Question Bank - UNIT-IV -

Functions of several variables.

PART-A - Questions

1) Given $u(x,y) = x^2 \tan^{-1}(y/x)$, find the value of $x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy}$

2) write the sufficient condition for $f(x,y)$ to have a maximum value at (a,b)

3.) If $u = f(x-y, y-z, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

4.) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

5.) If $u = xy$ and $v = x+y$ find $\frac{\partial(x,y)}{\partial(u,v)}$

6) Find the Taylor series expansion of $e^x \sin y$ near the point $(1, \pi/4)$ upto the second degree terms

7.) If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$

8.) If $u = x/y + y/z + z/x$ find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

9.) If $u = x^y$ show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

10.) If $x = u^2 - v^2$ and $y = 2uv$. find the Jacobian of x and y with respect to u and v

11.) If $u = y^2/2x$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

12.) If $u = 2xy$, $v = x^2 - y^2$ find $\frac{\partial(u,v)}{\partial(x,y)}$

13.) write down the the first two non-zero terms of the Taylor's series expansion of $e^x \cos y$ in powers of x and y.

14) If $r = \frac{yz}{x}$, $s = \frac{xy}{z}$, $t = \frac{xy}{z}$, find $\frac{\partial(r,s,t)}{\partial(x,y,z)}$

15) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ if $u = y^x$

16) If $f(x,y) = \log\sqrt{x^2+y^2}$, show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

17) State Neccessary Conditions for a maximum or a minimum.

18) Find the Stationary points for the function
$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

19) A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y) = x^2 + 2y^2 - x$
Find the coldest point on the plate.

20) If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

PART - B - Questions

1) If $u = f(x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$ prove that
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

2) Expand by Taylor's series the function $f(x,y) = x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ into the third powers.

3) Find the maximum and maximum values of $x^2 - xy + y^2 - 2x + y$

4) A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction.

5. Find the Taylor's expansion of $e^x \cos y$ in the neighborhood of the point $(1, \pi/4)$ upto third degree terms.

6. If $u = \log(x^2+y^2) + \tan^{-1}(y/x)$ prove that $U_{xx} + U_{yy} = 0$

7. Discuss the maxima and minima of the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

8. Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

9. If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$, $v = 2xy$ prove that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]$

10. Find the shortest and longest distances from point $(1,2,-1)$ to the sphere $x^2+y^2+z^2 = 24$, using the method of Lagrange multipliers.

11. Expand $e^{-x} \log y$ as a Taylor Series in powers of x and $(y-1)$ upto third degree terms.

12. If $u = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

13. If $u = x^2+y^2+z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$ Find $\frac{du}{dt}$.

14. If $x+y+z = u$, $y+z = uv$, $z = uvw$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

15. If F is a function of x and y and if $x = e^u \sin v$, $y = e^u \cos v$, prove that $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = e^{-2u}\left[\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}\right]$