MULTIPLE INTEGRAL

* Double Integrals

* Change of order of Integration

* Double integrals in Polar Coordinates

* Area enclosed by plane curves

* Triple Integrals

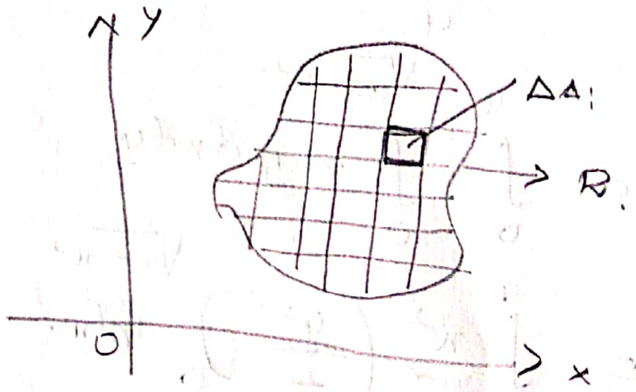
* volume of solids

* Change of Variables in double and Triple Integrals

Double integrals occur in many practical problems in science and engineering. It is used in problems involving area, volume, mass, centre of mass.

Double integrals in Cartesian coordinate.

Double integral is defined as the limit of a sum. Let $f(x,y)$ be a continuous fn of two independent variables x and y, defined in a simple closed region R. Subdivide R into element areas $\Delta A_1, \Delta A_2, \ldots \Delta A_n$ by drawing lines parallel to coordinate axes.



If $\lim_{\substack{n \to \infty \\ \Delta A_i \to 0}} \sum f(x_i, y_i) \Delta A_i$ exists then it is called the double integral, denoted by $\iint_R f(x,y)\,dx\,dy$.

1. Evaluate $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$

$$= \int_0^1 x\left(xy + \frac{y^2}{2}\right)_1^2 dx$$

$$= \int_0^1 \left(x + \frac{3}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{3x^2}{2.2}\right)_0^1$$

$$= \frac{13}{12}$$

2. Evaluate $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$$= \left(\sin^{-1} x\right)_0^1 \left(\sin^{-1} y\right)_0^1$$

$$= \frac{\pi^2}{4}$$

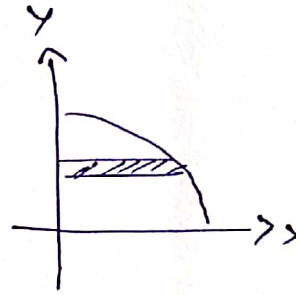
3. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\, dx\, dy$

$$= \int_0^a x^2 \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx.$$

$$= \frac{1}{2}\int_0^a x^2 (a^2-x^2)\, dx$$

$$= \frac{a^5}{15}$$

4. Evaluate $\iint_R xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$



$$\iint_R xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a (a^2y - y^3)\,dy$$

$$= \frac{a^4}{8}$$

5. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy$

$$= \int_2^3 \frac{1}{y}\left[\log x\right]_1^2 dy$$

$$= \int_2^3 \frac{1}{y}\log 2\,dy$$

$$= \log 2\,(\log y)_2^3$$

$$= \log 2 \log(3/2)$$

* Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

$$= \int_0^5 \int_0^{x^2} (x^3 + xy^2)\,dy\,dx \quad \text{(correct form)}$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx$$

$$= \left(\frac{x^6}{6} + \frac{x^8}{24}\right)_0^5$$

$$= 5^6 \left(\frac{29}{24}\right)$$

* Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\,dy\,dx \quad \text{(correct form)}$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \frac{\pi}{4}\left(\sinh^{-1} x\right)_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$

$$= \int_0^4 \left(\frac{e^{y/x}}{1/x} \right)_0^{x^2} dx$$

$$= \int_0^4 (xe^x - x)\, dx$$

$$= \int_0^4 x(e^x - 1)\, dx$$

$$= \left[x(e^x - x) - \left(e^x - \frac{x^2}{2}\right) \right]_0^4$$

$$= 3e^4 - 7$$

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$

$$= \int_0^a \left[\frac{y}{2}\sqrt{(a^2-x^2)-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{a^2-x^2}{2}\left(\frac{\pi}{2}\right) dx$$

$$= \frac{\pi}{2}\frac{1}{2}\left[a^2x - \frac{x^3}{3} \right]_0^a$$

$$= \frac{\pi a^3}{6}$$

Double Integration in Polar Coordinates.

1. Evaluate $$\int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\, dr\, d\theta .$$

$$= \int_0^{\pi/2}\int_0^{\infty} \frac{\frac{1}{2}\, d(r^2)}{(r^2+a^2)^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{-1}{r^2+a^2}\right)_0^{\infty} d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2} \frac{-1}{a^2}\, d\theta$$

$$= \frac{1}{2a^2}\left(\theta\right)_0^{\pi/2}$$

$$= \frac{\pi}{4a^2}$$

2. Evaluate $$\int_0^{\pi/2}\int_0^{\sin\theta} r\, d\theta\, dr$$

$$= \int_0^{\pi/2}\int_0^{\sin\theta} r\, dr\, d\theta \quad \text{Correct form}$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

* Evaluate $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr\, d\theta$

$$= \int_0^{\pi}\left(\sin\theta \ \frac{r^2}{2}\right)_0^{a\cos\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\cos^2\theta\sin\theta \, d\theta$$

$$= -\frac{a^2}{2}\int_0^{\pi}\cos^2\theta \, d(\cos\theta)$$

$$= -\frac{a^2}{2}\left(\frac{\cos^3\theta}{3}\right)_0^{\pi}$$

$$= \frac{a^2}{3}$$

* Evaluate $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 d\theta\, dr$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 dr\, d\theta \quad \text{(correct form)}$$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\frac{8\cos^3\theta}{3}\, d\theta$$

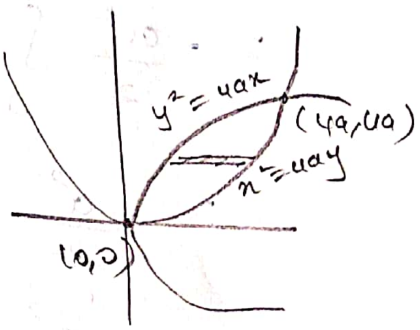
$$= 8/3\,(2)\int_0^{\pi/2}\cos^3\theta\, d\theta = \frac{32}{9}$$

Change of Order of Integration.

The process of changing a given double Integral with order of integration changed is called change of order of integration.

Evaluate by changing the order of integration

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$$



$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

$y : \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$

$\Rightarrow x^2 = 4ay$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax.$

Points of Intersection.

Solving $x^2 = 4ay$ & $y^2 = 4ax$

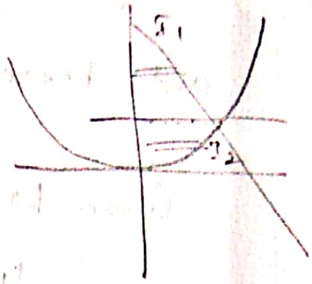
we get $(0,0)$ and $(4a, 4a)$.

Changing the order

$$\therefore \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{a}\sqrt{y}} dx\,dy = \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{a}\sqrt{y}} dy$$

$$= \frac{16a^2}{3}.$$

Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$



Given limit

$y = x^2$ to $y = 2-x$

$x = 0$ to $x = 1$

Divide the region into I_1 & I_2

$I = I_1 + I_2$

$$= \int_0^1 \int_{x^2}^{1} xy \, dy \, dx + \int_0^1 \int_1^{2-x} xy \, dy \, dx.$$

<u>In I_1</u> $x: 0$ to $x = 1$

$y = x^2$ to $y = 1$

Changing the order. $x = 0$ to $x = \sqrt{y}$

$y = 0$ to 1

$$\therefore I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy$$

$$= \frac{1}{2} \int_0^1 y^2 \, dy = \frac{1}{6}$$

<u>In I_2</u> $x = 0$ to $x = 1$, $y = 1$ to $y = 2-x$

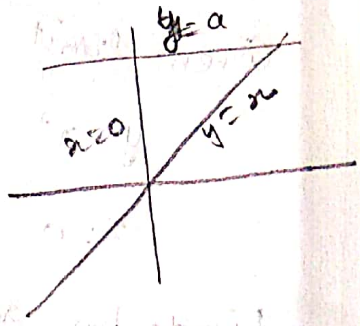
Changing $x = 0$ to $x = 2-y$ & $y = 1$ to $y = 2$

$$I_2 = \int_1^2 \int_0^{2-y} xy \, dx \, dy = \int_1^2 \left(\frac{x^2 y}{2} \right)_0^{2-y} dy$$

$$= 5/24.$$

$\therefore I = I_1 + I_2 = 3/8$

* Change the order of integration in $\int_0^a \int_x^a (x^2+y^2)\,dy\,dx$ and hence evaluate it



Given Limit:

$y = x$ to $y = a$

$x = 0$ to $x = a$

Change Limit

$x = 0$ to $x = y$

$y = 0$ to $y = a$

$$= \int_0^a \int_0^y (x^2+y^2)\,dx\,dy$$

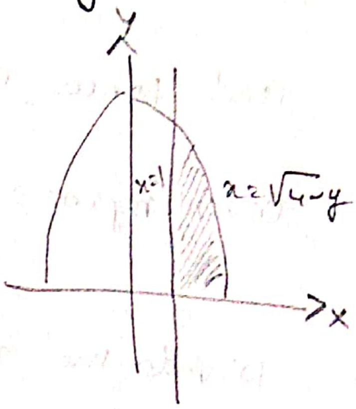
$$= \int_0^a \left(\frac{x^3}{3} + y^2 x\right)_0^y dy$$

$$= \int_0^a \frac{4}{3} y^3\,dy$$

$$= \frac{4}{3}\left(\frac{y^4}{4}\right)_0^a$$

$$= \frac{a^4}{4}$$

Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ by changing the order of integration



Given : $x = 1$ to $x = \sqrt{4-y}$

$y = 0$ to $y = 3$.

Changing the order we get

$x = 1$ to $x = 2$

$y = 0$ to $y = 4 - x^2$

$$\therefore I = \int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx$$

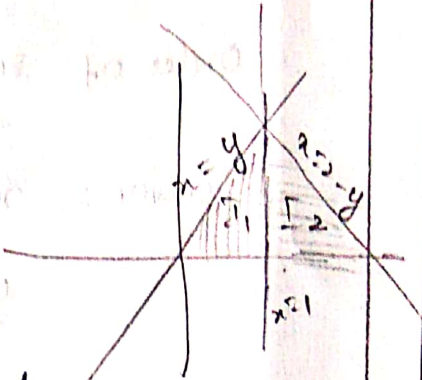
$$= \int_1^2 \left(xy + \frac{y^2}{2}\right)_0^{4-x^2} dx$$

$$= \frac{1}{2}\int_1^2 \left(x^4 - 2x^3 - 8x^2 + 8x + 16\right) dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - 2\frac{x^4}{4} - 8\frac{x^3}{3} + 8\frac{x^2}{2} + 16x\right]_1^2$$

$$= \frac{241}{60}.$$

* Change the order of integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ and hence evaluate it



Given region: $y=0$ to $y=1$

$x=y$ to $x=2-y$

Divide the region into $I = I_1 + I_2$

$$= \int_0^1 \int_y^1 xy\,dx\,dy + \int_0^1 \int_1^{2-y} xy\,dx\,dy$$

<u>In the region I_1</u>

$y=0$ to $y=1$ & $x=y$ to $x=1$

Changing the order $y=0$ to $y=x$, $x=0$ to $x=1$

$$\therefore I_1 = \int_0^1 \int_0^x xy\,dy\,dx = \frac{1}{8}.$$

<u>In the region I_2</u>

$y=0$ to $y=1$ & $x=1$ to $x=2-y$

Changing the order: $y=0$ to $y=2-x$, $x=1$ to $x=2$

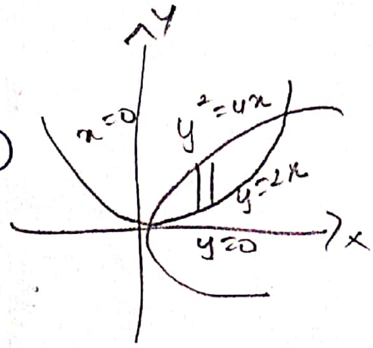
$$\therefore I_2 = \int_1^2 \int_0^{2-x} xy\,dy\,dx = \frac{5}{24}.$$

$$\therefore I = I_1 + I_2 = \frac{1}{3}.$$

Area as a Double Integral (cartesian)

* find the area enclosed by the curves

$y = 2x^2$ and $y^2 = 4x$



$y = 2x^2$ —— (1) $\quad y^2 = 4x$ —— (2)

sub (1) in (2)

Pt. of intersection is (0,0) & (1,2

x varies from $x=0$ to $x=1$

y varies from $y=2x^2$ to $y=2\sqrt{x}$

$\therefore$ the required area $= \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\,dx$

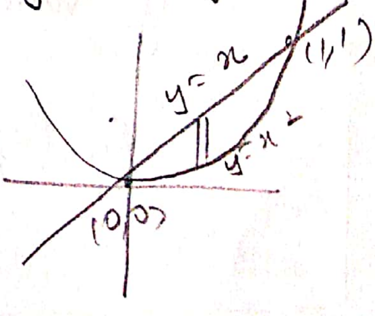
$$= \int_0^1 (y)_{2x^2}^{2\sqrt{x}}\, dx$$

$$= \int_0^1 (2\sqrt{x} - 2x^2)\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 2/3.$$

* find the area bounded by $y=x$ and $y=x^2$



$y = x$ —— (1) $\quad y = x^2$ —— (2)

Sub (1) in (2)

Pt. of intersection is (0,0) & (1,1)

Required area $= \int_0^1 \int_{x^2}^{x} dy\,dx$

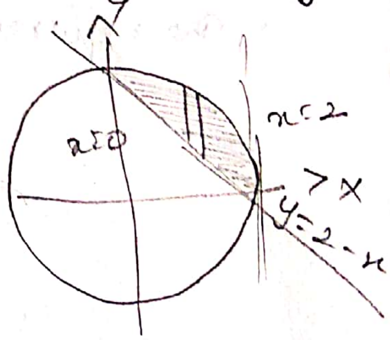
$= \int_0^1 (y)_{x^2}^{x}\,dx$

$= \int_0^1 (x-x^2)\,dx$

$= \left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^1$

$= \frac{1}{6}.$

Q: Find the smaller of the areas bounded by $y=2-x$ and $x^2+y^2=4$



$y = 2-x$ — (1)

$x^2+y^2=4$ — (2)

sub (1) in (2)

pt of intersection is $(0,2)$ and $(2,0)$

x : varies from $x=0$ to $x=2$

y varies from $y=2-x$ to $y=\sqrt{4-x^2}$

$\therefore$ Required Area $= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx.$

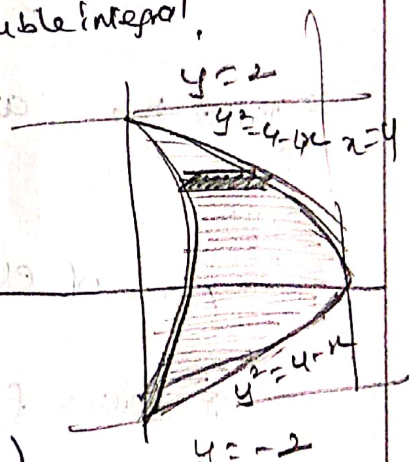
$$= \int_0^2 (y)_{2-x}^{\sqrt{4-x^2}}\, dx$$

$$= \int_0^2 \left(\sqrt{4-x^2} - (2-x)\right) dx$$

$$= \left(\frac{4}{2}\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}\right)_0^2 - \left(2x - \frac{x^2}{2}\right)_0^2$$

$= \pi - 2$ square units.

Q Find the area bounded by the parabolas $y^2 = 4-x$ and $y^2 = 4-4x$ as a double integral.



Soln: Given $y^2 = 4-x$ —(1)

$y^2 = 4-4x$ —(2)

Sub (1) in (2)

Pt of intersection is $(0,2)$ & $(0,-2)$

$\therefore$ x varies from $x = \frac{1}{4}(4-y^2)$

to $x = 4-y^2$

y varies from $y=-2$ to $y=2$

$$\therefore \text{ Required Area } = \int_{-2}^{2} \int_{\frac{1}{4}(4-y^2)}^{4-y^2} dx\, dy$$

$$= \int_{-2}^{2} (x)_{\frac{1}{4}(4-y^2)}^{4-y^2} dy$$

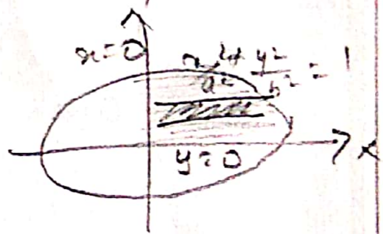
$$= \int_{-2}^{2} (4-y^2) - \frac{1}{4}(4-y^2) dy$$

$$= \int_{-2}^{2} \left(3 - \frac{3}{4}y^2\right) dy$$

$$= \left(3y - \frac{3}{4}\frac{y^3}{3}\right)_{-2}^{2}$$

$$= 8$$

* Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Soln

Area of ellipse = 4 × area of quadrant

x varies from x=0 to $x = \frac{a}{b}\sqrt{b^2-y^2}$

y varies from y=0 to y=b

$$\therefore \text{Required Area} = 4\int_{0}^{b}\int_{0}^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$$

$$= 4\int_{0}^{b} (x)_{0}^{a/b\sqrt{b^2-y^2}} dy$$

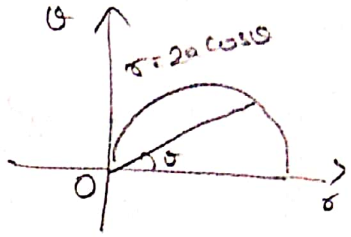
$$= \frac{4a}{b}\int_{0}^{b} \sqrt{b^2-y^2}\, dy$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{y}{b} + \frac{y}{2}\sqrt{b^2-y^2}\right]_{0}^{b}$$

$$= \pi ab \text{ sq. units.}$$

Area as Double Integral (Polar coordinates)

Evaluate $\iint r^2 \sin\theta \, dr \, d\theta$ where R is the semicircle $r = 2a\cos\theta$ about the initial line.



$$I = \int_0^{\pi/2} \int_0^{2a\cos\theta} (r^2 \sin\theta) \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3} \sin\theta \right)_{r=0}^{r=2a\cos\theta} d\theta$$

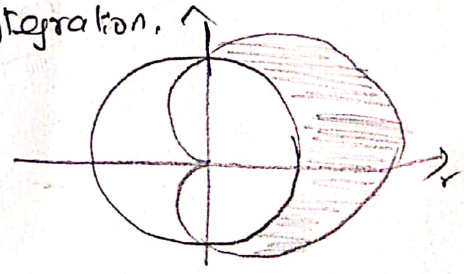
$$= \frac{8a^3}{3} \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \cos^3\theta \, d(\cos\theta)$$

$$= \frac{8a^3}{3} \left(\frac{\cos^4\theta}{4} \right)_0^{\pi/2}$$

$$= \frac{2a^3}{3}$$

Q Find the area that lies inside the cardiod $r = a(1+\cos\theta)$ and outside the circle $r = a$ by double integration.



Area of the enclosed region

$$= 2 \int_0^{\pi/2} \int_a^{a(1+\cos\theta)} r \, dr \, d\theta.$$

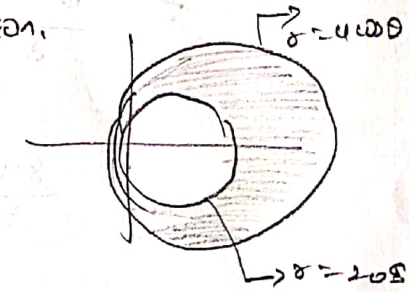
$$= 2\int_0^{\pi/2} \left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta.$$

$$= a^2 \int_0^{\pi/2} \left((1+\cos\theta)^2 - 1\right) d\theta$$

$$= a^2 \left(2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)_0^{\pi/2}$$

$$= \frac{a^2}{4}(\pi + 8)$$

Q Find the area of the region outside the inner circle $r = 2\cos\theta$ and inside the outercircle $r = 4\cos\theta$ by double integration.



$$\text{Area} = \iint r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{2\cos\theta}^{4\cos\theta} d\theta.$$

$$= 12\int_0^{\pi/2} (\cos^2\theta)\, d\theta$$

$$= 12 \times \frac{1}{2} \times \frac{\pi}{2} = 3\pi \text{ sq. units.}$$

Scanned by CamScanner

Triple Integration

* Evaluate $\int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z \, dz \, dy \, dx$

$$= \int_0^1 \int_0^x \left(\frac{z^2}{2}\right)_0^{\sqrt{x+y}} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^x (x+y) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left(xy + \frac{y^2}{2}\right)_0^x dx$$

$$= \frac{1}{2} \int_0^1 \frac{3}{2} x^2 \, dx$$

$$= \frac{3}{4} \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{1}{4}$$

* Evalte $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} \, dz \, dy \, dx$

$$= \int_0^{\log 2} \int_0^x \left(e^x e^y e^z\right)_0^{x+\log y} dy \, dx$$

$$= \int_0^{\log 2} \int_0^x \left(e^x e^y e^x e^{\log y} - e^x e^y\right) dy \, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left(e^{2x} e^{y} y - e^{x} e^{y}\right) dy\, dx$$

$$= \int_0^{\log 2} e^{2x} \left(ye^{y} - e^{y}\right) - e^{x} e^{y} \Big]_0^{x} dx$$

$$= \int_0^{\log 2} \left(e^{3x}(x-1) + e^{x}\right) dx$$

$$= \left[\frac{(x-1)e^{3x}}{3} - \frac{e^{3x}}{9} + e^{x}\right]_0^{\log 2}$$

$$= \left((\log 2 - 1)\frac{8}{3} - \frac{8}{9} + 2\right) - \left(\frac{-1}{3} - \frac{1}{9} + 1\right)$$

$$= \frac{8}{3}\log 2 - \frac{14}{9} - \frac{5}{9}$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}.$$

* Evaluate $\int_1^{3} \int_{1/x}^{1} \int_0^{\sqrt{xy}} xy\, dz\, dy\, dx$

$$= \int_1^{3} \int_{1/x}^{1} \left(xy \sqrt{xy}\right) dy\, dx.$$

$$= \int_1^{3} \int_{1/x}^{1} x^{3/2} y^{3/2}\, dy\, dx$$

$$= \int_1^3 \left[x^{3/2} \frac{y^{5/2}}{5/2} \right]_{1/x}^{1} dx$$

$$= \frac{2}{5} \int_1^3 \left(x^{3/2} - \frac{1}{x} \right) dx$$

$$= \frac{2}{5} \left[\frac{x^{5/2}}{5/2} - \log x \right]_1^3$$

$$= \frac{4}{25} 3^{5/2} - \frac{2}{5} - \frac{2}{5} \log 3$$

* Evaluate $\int_0^{\log a} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dz\, dy\, dx$

$$= \int_0^{\log a} \int_0^{x} \left(e^x e^y e^z \right)_0^{x+y} dy\, dx$$

$$= \int_0^{\log a} \int_0^{x} \left(e^{2x} e^{2y} - e^x e^y \right) dy\, dx$$

$$= \int_0^{\log a} \left(\frac{e^{2x} e^{2y}}{2} - e^x e^y \right)_0^x dx.$$

$$= \int_0^{\log a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right) dx$$

$$= \left(\frac{e^{4x}}{8} - \frac{3}{2} \frac{e^{2x}}{2} + e^x \right)_0^{\log a}$$

$$= \frac{1}{8} \left[a^4 + 8a - 6a^2 - 3 \right]..$$

Volume of solids.

* Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation

Volume = 8 × Volume in an octant

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a

$$\therefore V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\, dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2-y^2}\right) dy\,dx.$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2}\sqrt{a^2-x^2-y^2}\right]_0^{\sqrt{a^2-x^2}} dx.$$

$$= 8\int_0^a \frac{a^2-x^2}{2}\cdot\frac{\pi}{2}\Big)\, dx.$$

$$= 8\times\frac{\pi}{2}\left[a^2x-\frac{x^3}{3}\right]_0^a.$$

$$= \frac{4}{3}\pi a^3.$$

Scanned by CamScanner

* Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant

$$\text{Volume} = \iiint dz\,dy\,dx.$$

z varies from $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from $y=0$ to $y=b\sqrt{1-x^2/a^2}$

x varies from $x=0$ to $x=a$

$$\therefore \text{Volume} = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx.$$

$$= c\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx.$$

$$= \frac{c}{b}\int_0^a \left[\frac{y\sqrt{b^2(1-x^2/a^2)-y^2}}{2} + \frac{b^2(1-x^2/a^2)}{2}\sin^{-1}\left[\frac{y}{b\sqrt{1-x^2/a^2}}\right]\right]_0^{\sqrt{1-x^2/a^2}} dx.$$

$$= \frac{c}{b}\int_0^a \frac{b^2(1-x^2/a^2)}{2}\left(\pi/2\right) dx.$$

$$= \frac{\pi bc}{4} \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx.$$

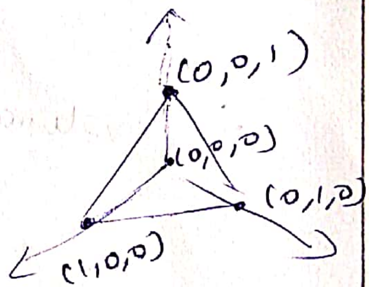
$$= \frac{\pi bc}{4} \left(x - \frac{x^3}{a^2 3}\right)_0^a$$

$$= \frac{\pi abc}{6}$$

$\therefore$ Volume of the ellipsoid $= 8 \times \frac{\pi abc}{6}$

$= \frac{4}{3}\pi abc.$

* Evaluate $\iiint_V dx\,dy\,dz$ where V is the volume of the tetrahedron whose vertices are $(0,0,0)$ $(0,1,0)$ $(1,0,0)$ and $(0,0,1)$



x varies from 0 to $1-y-z$

y varies from 0 to $1-z$

z varies from 0 to 1

$\therefore$ Volume $= \iiint dx\,dy\,dz$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} (x)_0^{1-y-z}\, dy\,dz.$$

$$= \int_0^1 \int_0^{1-z} (1-y-z)\,dy\,dz$$

$$= \int_0^1 \left(y - \frac{y^2}{2} - yz\right)_0^{1-z} dz$$

$$= \int_0^1 \left[(1-z) - \frac{(1-z)^2}{2} - z(1-z)\right] dz$$

$$= \int_0^1 (1-z)\frac{(1-z)}{2}\,dz$$

$$= \frac{1}{2}\int_0^1 (1-z)^2\,dz$$

$$= \frac{1}{2}\left[\frac{(1-z)^3}{3}(-1)\right]_0^1$$

$$= -\frac{1}{6}\left[(1-z)^3\right]_0^1$$

$$= \frac{1}{6}.$$

Change of Variables

* By changing into polar coordinates ST

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy = \frac{\pi}{4}. \quad \text{Hence evaluate } \int_0^\infty e^{-t^2} dt.$$

Take $x = r\cos\theta \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta.$

$$\therefore I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \frac{1}{2} d(r^2)\,d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[-e^{-r^2}\right]_0^\infty d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left(e^{-r^2}\right)_0^\infty d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} -d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta$$

$$= \frac{1}{2} \left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2}\right] = \frac{\pi}{4}$$

* By changing into polar Coordinates evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$$

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx.$$

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\therefore I = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{(r\cos\theta)^2+(r\sin\theta)^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r^2\cos\theta}{r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\cos\theta\right)_{r=0}^{r=2\cos\theta} d\theta.$$

$$= \int_0^{\pi/2} (2\cos^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}$$

* Evaluate by changing to polar $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy$

$x = r\cos\theta \quad y = r\sin\theta \quad dx dy = r d\theta dr$

$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} \, r \, dr d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^3\cos^2\theta}{\sqrt{r^2}} dr d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} r^2\cos^2\theta \, dr d\theta.$$

$$= \int_0^{\pi/4} \left(\cos^2\theta \frac{r^3}{3}\right)_0^{a\sec\theta} d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/4} \cos^2\theta \cdot \frac{1}{\cos^3\theta} d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/4} \sec\theta \, d\theta$$

$$= \frac{a^3}{3} \Big[\log(\sec\theta + \tan\theta)\Big]_0^{\pi/4}$$

$$= \frac{a^3}{3} \left[\log(\sqrt{2}+1) - \log 1\right]$$

$$= \frac{a^3}{3} \log(\sqrt{2}+1)$$

Scanned by CamScanner

Application of ~~Integral~~ Multiple Integration

The most important application of integrals involves finding areas bounded by a curve and x-axis. It includes finding solutions to the problems of work and energy. Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and x-axis, the double integral of a positive function of two variables represents the volume of the region defined between the surface defined by the function and the plane which contains its domain. If there are more variables, a multiple integral will yield hypervolumes of multidimensional functions. It is used for computing arc beneath a curve, the arc-length of a curve, for finding centre of mass, moment of inertia etc. A cable hung between two poles forms a curve called catenary. Calculus allows us to determine the exact length of the cable.

Scanned by CamScanner

Thus to determine the exact length of power cable required for connecting two substations, we need integral calculus. To determine the amount of materials necessary to construct a curved dome an architecht makes use of integral calculus.

While designing a utility vehicle, an engineer needs to know the centre of mass and moment of inertia of the vehicle for consideration of its safety features.

* Find the mass and moment of inertia of a sphere of radius 'a' with respect to a diameter if the density proportional to the distance from the centre.

$$\text{Mass} = \int_{0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} k\rho \, \rho^2 \sin\theta \, d\theta \, d\phi \, d\rho$$

$$= 2\pi k \left[-\cos\theta\right]_0^{\pi} \left(\frac{\rho^4}{4}\right)_0^{a}$$

$$= k\pi a^4$$

$$MI = \int_0^a \int_0^\pi \int_{\phi=0}^{2\pi} k\rho \rho^2 \sin\theta \, \rho^2 d\theta \, d\phi \, d\rho$$

$$= 2\pi K \left[-\cos\theta\right]_0^\pi \left[\frac{\rho^6}{6}\right]_0^a$$

$$= \frac{2}{3} K \pi a^4$$

* Find the mass, coordinates of the centre of mass and moments of inertia relative to the x-axis, y-axis and origin of the rectangle $0 \le x \le 4$, $0 \le y \le 2$ having mass density kxy

Mass $$M = P \int_0^4 \int_0^2 kxy \, dx \, dy$$

$$= K \left[\frac{x^2}{2}\right]_0^4 \left[\frac{y^2}{2}\right]_0^2$$

$$= 16k$$

$$\bar{x} = \frac{1}{M} \int_0^4 \int_0^2 kx^2 y \, dx dy$$

$$= \frac{K}{M} \left(\frac{x^3}{3}\right)_0^4 \left(\frac{y^2}{2}\right)_0^2 = \frac{16}{3}$$

$$\bar{y} = \frac{1}{M} \int_0^4 \int_0^2 kxy^2 \, dx\, dy$$

$$= \frac{K}{M} \int_0^4 x\, dx \int_0^2 y^2 dy$$

$$= \frac{k}{M} \left(\frac{x^2}{2}\right)_0^4 \left(\frac{y^3}{3}\right)_0^2$$

$$= \frac{8}{3}.$$

$$MI_x = \int_0^4 \int_0^2 kxyy^2 \, dx\, dy = k \left(\frac{x^2}{2}\right)_0^4 \left(\frac{y^4}{4}\right)_0^2$$

$$= 32k$$

$$MI_y = \int_0^4 \int_0^2 kxy\, x^2 \, dx\, dy = k \left(\frac{x^4}{4}\right)_0^4 \left(\frac{y^2}{2}\right)_0^2$$

$$= 128k.$$

Moment of inertia about the origin is

$$\int_0^4 \int_0^2 kxy\,(x^2+y^2)\, dx\, dy = k \int_0^4 x^3 dx + \int_0^2 y\, dy$$

$$+ k \int_0^4 x\, dx + \int_0^2 y^3 dy$$

$$= 128k + 32k$$

$$= 160k \;//.$$

Question Bank

Multiple Integral

PART A

1. Evaluate $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$

2. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy$

3. Sketch the region of integration $\int_0^5 \int_0^2 (x+y)\, dx\, dy$

4. Sketch the region of integration $\int_0^a \int_{x^2/a}^{x} \frac{x\, dy\, dx}{x^2+y^2}$

5. Change the order of integration for $\int_0^a \int_x^a f(x,y)\, dy\, dx$.

6. Change the order of integration for $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$.

7. Change the order of integration for $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

8. Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz\, dz\, dx\, dy$

9. Express $\int_0^\infty \int_0^\infty f(x,y)\, dx\, dy$ in polar coordinates

10. Evaluate $\int_0^1 \int_0^y \int_0^{x+y} dx\, dy\, dz$

11. Evaluate $\int_0^1 \int_0^{a^2} (x^2+y^2)\, dy\, dx$

12. Evaluate $\int_0^{\pi/2} \int_0^{a} r^2\, dr\, d\theta$

13. Evaluate $\int_0^{\pi/2} \int_0^{a\sqrt{2}} r\, dr\, d\theta$

14. Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} r\, dr\, d\theta$

15. Evaluate $\int_0^{2} \int_0^{\pi} r \sin^2\theta\, d\theta\, dr$.

16. Find the area bounded by the lines $x=0$, $y=1$ and $y=x$ using double integration

17. Sketch the contour integral $\int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} f(r,\theta)\, dr\, d\theta$.

18. Express the area of the cardiod $r=a(1+\cos\theta)$ in double integration

19. Express the area lying between the parabola $y=4x-x^2$ and the line $y=x$ in double integration

20. Express the volume of the sphere $x^2+y^2+z^2=a^2$ in triple integration.

PART B

1. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

2. Evaluate $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, d\theta\, dr.$

3. Change the order of integration and hence evaluate $\int_0^1 \int_y^{2-y} xy\, dx\, dy.$

4. Change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$ and hence evaluate it

5. Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and hence evaluate it

6. Using double integral find the area bounded by the parabolas $y^2=4ax$ and $x^2=4ay$

7. Using double integral find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

8. Evaluate by changing into polar coordinates

$$\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy$$

9. By changing to polar coordinates evaluate

$$\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

10. Find the area which is inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$

11. Find the area of the cardioids $r = a(1+\cos\theta)$

12. Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

13. Find the volume of the sphere $x^2+y^2+z^2 = a^2$

14. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by using triple integration

15. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dy\, dx\, dz}{\sqrt{1-x^2-y^2-z^2}}$

Numerical Integration by trapezoidal and simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rules, Romberg's method:

Formula: 1) Newton-cotes formula

$$\int_{x_0}^{x+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \cdots\right]$$

This is known as Newton-cotes quadrature formula.

2) **Trapezoidal rule:**

$$\text{Trapezoidal rule} = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{sum of the first and the last ordinates}) + 2(\text{sum of the remaining ordinates})\right]$$

3) Simpson's one third rule. ($\frac{1}{3}$ rule)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

$= \frac{h}{3}$[(sum of the first and last ordinates)+ 4(sum of the odd ordinates)+2(sum of the remaining even ordinates)]

4) Simpsons $\frac{3}{8}$ rule [Simpsons three eighth rule]

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

This is known as simpson's three-eighth rule.

Note:

Applying simpsons three-eighth rule, the number of sub intervals should be taken multiple of 3.

Formula:

Rule	Degree of y(x)	No of intervals	error	order
Trapezoidal rule	one	any	$\lvert E\rvert < \frac{(b-a)h^2}{12}M$	h^2
simpson's 1/3 rule	Two	even	$\lvert E\rvert < \frac{(b-a)h^4 M}{180}$	h^4
simpson's $\frac{3}{8}$ rule	three	multiple of 3	$\lvert E\rvert = \frac{3}{8}h^5$	.

Part-B:

1) Evaluate $I = \int_0^6 \frac{1}{1+x^2}dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) simpsons one-third rule (iv) simpson's three-eighth rule.

Soln:

(i) Direct integration

$$I = \int_0^6 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^6 = \tan^{-1}6 = 1.40564765.$$

(ii) By Trapezoidal rule,

$$I = \int_0^6 \frac{dx}{1+x^2} =$$

Here $b-a = 6-0 = 6$. Divide into 6 equal parts

$$h = \frac{6}{6} = 1.$$

x	0	1	2	3	4	5	6
$f(x) = \frac{1}{1+x^2}$	1.00	0.500	0.200	0.100	0.5884	0.038462	0.27027

There are 7 ordinates ($n=6$)

$$I = \int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\left[(1+0.027027)+2(0.5+0.2+0.1+0.058824+0.038462)\right]$$

$$= 1.41079950.$$

(iii) By simpson's one-third rule,

$$I = \frac{1}{3}\left[(1+0.027027)+2(0.2+0.058824)+4(0.500+0.100+0.038462)\right]$$

$$= \frac{1}{3}\left[1.027027+0.517648+2.553848\right]$$

$$= 1.36617433$$

(iv) By simpson's three-eights rule,

$$I = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+\cdots+y_{n-3})\right]$$

$$= \frac{3h}{8}\left[(1+0.027027)+3(0.5+0.2+0.058824+0.038462)+2(0.1)\right]$$

$$= 1.35708188$$

Concolution:

Here the value by Trapezoidal rule is closer to the actual valle than the value by simpsons rule.

2) By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\, dx$ by Trapezoidal and simpson's rule, verify your answer with integration.

Sol: Range $= \pi - 0 = \pi$.

$$\therefore h = \frac{\pi}{10}$$

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y=\sin x$	0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.8090	0.5878	0.3090	0

(i) By Trapezoidal rule

$$I = \frac{h}{2}\left[(y_0+y_{10}) + 2(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{\pi}{20}[(0+0)+2(0.3090+0.5878+0.8090+0.9511+1.0+0.9511+0.8090+0.5878+0.3090)]$$

$$= 1.9843.$$

(ii) By simpsons one thirds rule (∵ there are 11 ordinates)

$$I = \frac{h}{3}\left[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{\pi}{30}[(0+0)+2(0.5878+0.9511+0.9511+0.5878) + 4[0.3090+0.8090+1+0.8090+0.3090]]$$

$$= 2.00091.$$

Note: we cannot use simpson's three eighth's rule here

(iii) By actual Integration

$$I = \int_0^{\pi} \sin x\, dx = (-\cos x)_0^{\pi} = [+1-(-1)] = 2.$$

Hence simpsons rule is more accurate than the Trapezoidal rule.

3) The velocity v of a particle at a distance 's' from a point on its path is given by the table below

s in metre	0	10	20	30	40	50	60
v m/sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by using simpson's one third rule compare your answer with simpson's $\frac{3}{8}$ rule.

Soln:

$$v = \frac{ds}{dt}, \quad t = \int_0^{60} \frac{1}{v}\, ds, \quad \text{take } y = \frac{1}{v}.$$

Dividing the range into 10 equal parts find the value of $\int_0^{\pi/2} \sin x \, dx$ by (i) Trapezoidal rule (ii) Simpson's rule

Soln:

$$h = \frac{\frac{\pi}{2} - 0}{10} = \frac{\pi}{20}$$

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
$y = \sin x$	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9871	1

(i) By Trapezoidal rule

$$\int_0^{\pi/2} \sin x = \frac{h}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + \cdots + y_9)\right]$$

$$= \frac{\pi}{40}\left[0 + 1 + 2(5.8531)\right]$$

$$= 0.9980$$

(ii) By Simpson's $\frac{1}{3}$ rule

$$\int_0^{\pi/2} \sin x \, dx = \frac{h}{3}\left[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{\pi}{60}\left[1 + 12.7848 + 5.3138\right]$$

$$= \frac{\pi}{60}\left[19.0986\right] = 1.000$$

The table below gives the velocity v of a moving particle at time t seconds. Find the distance covered by the particle in 12 seconds and also the acceleration at $t = 2$ seconds using simpson rule:

t: (seconds)	0	2	4	6	8	10	12
v: (m/sec)	4	6	16	34	60	94	136

Soln:

we know that $v = \frac{ds}{dt}$, $\frac{dv}{dt} = a$

where $v =$ velocity, $a =$ acceleration, $s =$ distance

$\therefore ds = v\,dt \Rightarrow s = \int v\,dt$

$\therefore$ the distance moved by the particle in 12 seconds

$$= \int_0^{12} v\,dt$$

By simpson 1/3 rule

$$\int_0^{12} v\,dt = \frac{h}{3}\left[(v_0+v_6)+4(v_1+v_3+v_5)+2(v_4+v_2)\right]$$

Here $h=2$

$$= \frac{2}{3}\left[(4+136)+4(6+34+94)+2(16+60)\right]$$

$$= 552 \text{ meters.}$$

acceleration $= \left(\frac{dv}{dt}\right)_{t=2}$.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	4			
		2		
			8	
2 t_0	6 v_0			
		10 Δv_0		0
			8 $\Delta^2 v_0$	
4	16			
		18		0 $\Delta^3 v_0$
			8	
6	34			
		26		0
			8	
8	60			
		34		0
			8	
10	94			
		42		
12	136			

$$\left(\frac{dv}{dt}\right)_{t=2} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \cdots\right]$$

$$= \frac{1}{2}\left[10 - 8/2\right] = 3 \text{ m/sec}^2.$$