Unit 3. Inner product spaces

Definition:

An Inner product on a Vector space V is an operation that assigns to every pair of Vectors u and v in V a real number $\langle u,v\rangle$ such that the following properties hold for all Vectors u, v and w in V and all scalars c:

(i) $\langle u,v\rangle = \langle v,u\rangle$

(ii) $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

(iii) $\langle cu,v\rangle = c\langle u,v\rangle$

(iv) $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u=0$

A Vector space with an inner product is called an Inner product space.

Examples

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be two Vectors in $\mathbb{R}^2$. $\langle u,v\rangle = 2u_1v_1 + 3u_2v_2$ defines an inner product.

(1) Let $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{aligned}
\langle u, v+w\rangle &= 2u_1(v_1+w_1) + 3u_2(v_2+w_2) \\
&= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2 \\
&= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2) \\
&= \langle u,v\rangle + \langle u,w\rangle
\end{aligned}$$

If c is Scalar, then

$$\langle cu, v\rangle = 2(cu_1)v_1 + 3(cu_2)v_2$$
$$= c(2u_1v_1 + 3u_2v_2)$$
$$= c\langle u, v\rangle$$

Also $\langle u, u\rangle = 2u_1^2 + 3v_1^2 \geq 0$ and $\langle u, v\rangle = \langle v, u\rangle$

$\therefore \langle u, v\rangle$ as defined is an Inner product.

In General,

Let $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ are Vectors in R^n

and $w_1, w_2 \dots w_n$ are positive Scalars, Then

$\langle u, v\rangle = w_1u_1v_1 + w_2u_2v_2 + \dots + w_nu_nv_n$ defines an inner product on $\mathbb{R}^n$, Called weighted dot product. We Can write the weighted dot product as $\langle u, v\rangle = u^TWv$ where W is $n\times n$ diagonal matrix.

$$W = \begin{bmatrix} w_1 & \dots & & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & & \ddots & \\ \vdots & & & w_n \end{bmatrix}$$

The following result further generalizes this type of inner product.

Result: Let A be a symmetric, positive definite $n \times n$ matrix and let u and v be vectors in $\mathbb{R}^n$. Then $\langle u, v\rangle = u^T A v$ defines an inner product.

Examples:

1. Let $A = \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$, Then the matrix A is positive definite.

$$\therefore \langle u, v\rangle = u^T A v = (u_1, u_2)\begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$= 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 7u_2v_2$$

defines an inner product on $\mathbb{R}^2$.

2. Suppose $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\langle u, v\rangle = u_1v_1 + 2u_1v_2 + 2u_2v_1 + 5u_2v_2$ Then the symmetric matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Length, Distance, and Orthogonality:

Let u and v be vectors in inner product space V.

(i) The Length (or Norm) of v is

$$\|v\| = \sqrt{\langle v, v\rangle}$$

Note: We can take the square root of this non-negativity quantity. ($\because \langle v, v\rangle \geq 0$)

(ii) The distance between u & V is

$d(u,v) = \|u-v\|$.

(iii) u and V are orthogonal, if $\langle u,v \rangle = 0$

Example

1. Let $f(x) = x$, $g(x) = 3x-2$ in $C[0,1]$.

Consider the inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx \quad \text{Then}$$

(i) $\|f\| = \sqrt{\langle f, f \rangle}$

$$\langle f, f \rangle = \int_0^1 f(x)^2\, dx = \int_0^1 x^2\, dx = 1/3$$

$$\therefore \|f\| = \sqrt{1/3} = 1/\sqrt{3}$$

(ii) $d(f,g) = \|f-g\|$

$$\langle f-g, f-g \rangle = \int_0^1 [f(x) - g(x)]^2\, dx$$

$$= \int_0^1 (x - 3x + 2)^2\, dx = \int_0^1 [2(1-x)]^2\, dx$$

$$= 4/3$$

$$\therefore d(f,g) = \|f-g\| = \sqrt{\langle f-g, f-g \rangle} = \sqrt{4/3}$$

$$= 2/\sqrt{3}$$

(iii)

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx = \int_0^1 x(3x-2)\, dx$$

$$= \int_0^1 (3x^2 - 2x)\, dx = 0$$

Thus f & g are Orthogonal.

Cauchy - Schwarz Inequality:

Let u and v be Vectors in an inner product space V. Then

$$|\langle u, v \rangle| \leq \|u\|\, \|v\|$$

Proof: If $v = 0$, then the result is true. So assume that $v \neq 0$. Then for any $c \in F$, we have

$$\|u - cv\|^2 \geq 0$$

$$\langle u - cv, u - cv \rangle \geq 0$$

$$\langle u, u - cv \rangle - c \langle v, u - cv \rangle \geq 0$$

$$\langle u, u \rangle - \bar{c} \langle u, v \rangle - c \langle v, u \rangle + c\bar{c} \langle v, v \rangle \geq 0$$

Setting $c = \dfrac{\langle u, v \rangle}{\langle v, v \rangle}$

$$\langle u, u \rangle - \frac{\overline{\langle u, v \rangle}}{\langle v, v \rangle} \langle u, v \rangle - \frac{\langle u, v \rangle}{\langle v, v \rangle} \langle v, u \rangle + \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot \frac{\overline{\langle u, v \rangle}}{\langle v, v \rangle} \cdot \langle v, v \rangle \geq$$

$$\|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2} - \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \frac{|\langle u,v\rangle|^2}{\|v\|^2} \geq 0$$

$$\|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2} \geq 0$$

$$\Rightarrow \|u\|^2 . \|v\|^2 - |\langle u,v\rangle|^2 \geq 0$$

$$\Rightarrow \boxed{|\langle u,v\rangle| \leq \|u\| . \|v\|}$$

Triangle Inequality:

Let u and v be Vectors in an inner product space V. Then

$$\|u+v\| \leq \|u\| + \|v\|$$

Proof:

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + 2 \operatorname{Re}\langle u,v\rangle + \|v\|^2$$

$$\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

$$\therefore \|u+v\| \leq \|u\| + \|v\|$$

Here $\operatorname{Re}\langle u,v\rangle$ denotes the real part of the Complex number $\langle u,v\rangle$.

The Gram - Schmidt Orthogonalization Process:

Definition:

A Set of Vectors $\{v_1, v_2, \dots v_k\}$ in $\mathbb{R}^n$ is Called an Orthogonal Set if all pairs of distinct Vectors in the Set are Orthogonal. i.e $v_i \cdot v_j = 0$ whenever $i \neq j$, for $i, j = 1, 2 \dots k$.

An Orthogonal basis for a Subspace W of R^n is a basis of W that is an Orthogonal Set.

Let V be an inner product space. A Vector u in V is a Unit Vector if $\|u\| = 1$.

A Set of Vectors in $\mathbb{R}^n$ is an Orthonormal Set if it is an Orthogonal Set of Unit Vectors. An Orthonormal basis for a Subspace W of $\mathbb{R}^n$ is a basis of W that is an Orthonormal Set.

Theorem: (Gram - Schmidt Process)

Let V be an inner product space. Let $S = \{u_1, u_2 \dots u_n\}$ be a Linearly independent Subset of V.

Define $S' = \{v_1, v_2 \dots v_n\}$ where

$v_1 = u_1$ and

$$V_k = u_k - \sum_{j=1}^{k-1} \frac{\langle u_k, v_j \rangle}{\|v_j\|^2} v_j \quad \text{for } 2 \leq k \leq n.$$

Then S' is an Orthogonal basis.

1. Apply the Gram-Schmidt process to Construct an Orthonormal basis.

Let $V = R^3$

Let $u_1 = (1,1,0)$, $u_2 = (2,0,1)$, $u_3 = (2,2,1)$ Then $\{u_1, u_2, u_3\}$ is Linearly independent.

Take $V_1 = u_1 = (1,1,0)$

$$V_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (2,0,1) - \frac{2}{2}(1,1,0)$$

$$= (1,-1,1)$$

$$V_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= (2,2,1) - \frac{4}{2}(1,1,0) - \frac{1}{3}(1,-1,1)$$

$$= (-1/3, 1/3, 2/3) \quad \therefore \vartheta_3 = (-1,1,2)$$

$\therefore S' = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ is an Orthogonal basis.

for Each Vector divided by its Length, we will get an Orthonormal basis.

$$\beta = \left\{ \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), \left(\tfrac{1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{-1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}\right) \right\}$$

2. Apply the Gram-Schmidt process to Construct an Orthonormal basis for the Subspace $W = \text{Span}(u_1, u_2, u_3)$ of $\mathbb{R}^4$, where

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad u_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Clearly $S = \{u_1, u_2, u_3\}$ Linearly independent.

Take $v_1 = u_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

Scaling v_2,

$$v_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 .$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore$ $S' = \{v_1, v_2, v_3\}$ is an Orthogonal basis.

$$\beta = \left\{ \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{3}{\sqrt{20}}, \tfrac{3}{\sqrt{20}}, \tfrac{1}{\sqrt{20}}, \tfrac{1}{\sqrt{20}}\right), \left(-\tfrac{1}{\sqrt{6}}, 0, \tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}\right) \right\}$$

is an Orthonormal basis.

The QR factorization:

Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthogonal columns and R is an invertible upper triangular matrix.

Example:

Find a QR factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

Let $u_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ $u_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Using Gram-Schmidt process, we obtained the following orthonormal basis.

$q_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$, $q_2 = \begin{pmatrix} 3/\sqrt{20} \\ 3/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \end{pmatrix}$ $q_3 = \begin{pmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$

so $Q = \begin{bmatrix} q_1, & q_2 & q_3 \end{bmatrix}$

$$Q = \begin{bmatrix} 1/2 & 3/\sqrt{20} & -1/\sqrt{6} \\ -1/2 & 3/\sqrt{20} & 0 \\ -1/2 & 1/\sqrt{20} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{20} & 2/\sqrt{6} \end{bmatrix}$$

Now

$$A = QR$$

$$Q^T A = Q^T Q R$$

$$Q^T A = I R \quad (\because Q \text{ is orthonormal})$$

$$\therefore R = Q^T A$$

$$R = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ \frac{3}{\sqrt{20}} & 3/\sqrt{20} & 1/\sqrt{20} & 1/\sqrt{20} \\ -1/\sqrt{6} & 0 & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & [illegible] & 1/2 \\ 0 & 10/\sqrt{20} & 15/\sqrt{20} \\ 0 & 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

The Modified Gram-Schmidt Process

The Gram-Schmidt Orthogonalization Process may yield grossly inaccurate results due to roundoff error under finite-digit arithmetic. A modification of that algorithm exists which is more stable and which generates the same vectors in the absence of rounding. This modification also transforms a set of linearly independent vectors $\{X_1, X_2 .. X_n\}$ into a set of orthonormal vectors $\{Q_1, Q_2 ... Q_n\}$. Such that each vector Q_k $(k=1,2..n)$ is a linear combination of X_1 through X_{k-1}. The modified algorithm is iterative with the k^{th} iteration given by the following steps:

Step 1: Set $r_{kk} = \|x_k\|_2$ and $Q_k = (1/r_{kk}) x_k$

Step 2: for $j = k+1, k+2 ... n$,

set $r_{kj} = \langle X_j, Q_k \rangle$

Step 3: for $j = k+1, k+2 ... n$, replace X_j by $X_j - r_{kj} Q_k$.

1. Use the modified Gram - Schmidt process to Construct an Orthogonal Set of Vectors from the Linearly independent set $\{X_1, X_2, X_3\}$ when

$$X_1 = \begin{pmatrix} -4 \\ 3 \\ 6 \end{pmatrix} \quad X_2 = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \quad X_3 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

First Iteration :

Step1:

$$r_{11} = \|X_1\|_2 = \sqrt{61} = 7.810250$$

$$Q_1 = \frac{1}{r_{11}} X_1 = \begin{pmatrix} -4/\sqrt{61} \\ 3/\sqrt{61} \\ 6/\sqrt{61} \end{pmatrix}$$

Step2

$$r_{12} = \langle X_2, Q_1 \rangle = \frac{19}{\sqrt{61}}$$

$$r_{12} = 2.432701$$

$$r_{13} = \langle X_3, Q_1 \rangle = \frac{1}{\sqrt{61}} = 0.128037$$

Step3:

$$X_2 \leftarrow X_2 - r_{12} Q_1$$

$$X_2 = \begin{pmatrix} 198/61 \\ -240/61 \\ 252/61 \end{pmatrix}$$

$$X_3 \leftarrow X_3 - r_{13} Q_1 \qquad X_3 = \begin{pmatrix} 126/61 \\ 180/61 \\ -6/61 \end{pmatrix}$$

Second Iteration:

Step1:

$$r_{22} = \|X_2\|_2$$

$$= \sqrt{\left(\frac{198}{61}\right)^2 + \left(\frac{-240}{61}\right)^2 + \left(\frac{256}{61}\right)^2}$$

$$= 6.563686$$

~~Step~~

$$Q_2 = \frac{1}{r_{22}} X_2 = \begin{pmatrix} 0.494524 \\ -0.599423 \\ 0.629395 \end{pmatrix}$$

Step2:

$$r_{23} = \langle X_3, Q_2 \rangle = -0.809222$$

Step3:

$$X_3 \leftarrow X_3 - r_{23} Q_2$$

$$X_3 = \begin{pmatrix} 2.465753 \\ 2.465753 \\ 0.410959 \end{pmatrix}$$

Third Iteration

Step1:

$$r_{33} = \sqrt{(2.465753)^2 + (2.465753)^2 + (0.410959)^2}$$

$$= 3.511234$$

$$Q_3 = \frac{1}{r_{33}} X_3 = \begin{pmatrix} 0.702247 \\ 0.702247 \\ 0.117041 \end{pmatrix}$$

An Orthonormal set is $\{Q_1, Q_2, Q_3\}$

$$\therefore Q = \begin{bmatrix} -0.512148 & 0.494524 & 0.702247 \\ 0.384111 & -0.599423 & 0.702247 \\ 0.768221 & 0.629395 & 0.117041 \end{bmatrix}$$

$$R = \begin{bmatrix} 7.810250 & 2.432701 & 0.128037 \\ 0 & 6.563686 & -0.809222 \\ 0 & 0 & 3.511234 \end{bmatrix}$$

2. Use the modified Gram-Schmidt process to construct an orthogonal set of vectors from the linearly independent set $\{X_1, X_2, X_3, X_4\}$ when

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad X_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

First Iteration:

Step1:

$$r_{11} = \| x_1 \|_2 = \sqrt{3}$$

$$Q_1 = \frac{1}{r_{11}} x_1 = \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

Step2:

$$r_{12} = \langle x_2, Q_1 \rangle = 2/\sqrt{3}$$

$$r_{13} = \langle x_3, Q_1 \rangle = 2/\sqrt{3}$$

$$r_{14} = \langle x_4, Q_1 \rangle = 2/\sqrt{3}$$

Step3:

$$x_2 \leftarrow x_2 - r_{12} Q_1 \qquad x_2 = \begin{pmatrix} 1 \\ -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$x_3 \leftarrow x_3 - r_{13} Q_1$$

$$x_3 = \begin{pmatrix} 1 \\ 1/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$x_4 \leftarrow x_4 - r_{14} Q_1$$

$$x_4 = \begin{pmatrix} 1 \\ 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

Second Iteration:

Step1:

$$r_{22} = \|X_2\|_2 = \sqrt{15}/3$$

$$Q_2 = \frac{1}{r_{22}} X_2 = \begin{pmatrix} 3/\sqrt{15} \\ -2/\sqrt{15} \\ \frac{1}{\sqrt{15}} \\ 1/\sqrt{15} \end{pmatrix}$$

Step2:

$$r_{23} = \langle X_3, Q_2 \rangle = 2/\sqrt{15}$$

$$r_{24} = \langle X_4, Q_2 \rangle = 2/\sqrt{15}$$

Step3:

$$X_3 \leftarrow X_3 - r_{23} Q_2 = \begin{pmatrix} 3/5 \\ 3/5 \\ -4/5 \\ 1/5 \end{pmatrix}$$

$$X_4 \leftarrow X_4 - r_{24} Q_2 = \begin{pmatrix} 3/5 \\ 3/5 \\ 1/5 \\ -4/5 \end{pmatrix}$$

Third Iteration:

Step1:

$$r_{33} = \|X_3\|_2 = \sqrt{35}/5$$

$$Q_3 = \frac{1}{r_{33}} X_3$$

$$Q_3 = \begin{pmatrix} 3/\sqrt{35} \\ 3/\sqrt{35} \\ -4/\sqrt{35} \\ 1/\sqrt{35} \end{pmatrix}$$

step2:

$$r_{34} = \langle x_4, Q_3 \rangle = 2/\sqrt{35}$$

step3:

$$x_4 \leftarrow x_4 - r_{34} Q_3$$

$$x_4 = \begin{pmatrix} 3/7 \\ 3/7 \\ 3/7 \\ -6/7 \end{pmatrix}$$

Fourth Iteration:

$$r_{44} = \|x_4\|_2 = \sqrt{63}/7$$

$$Q_4 = \frac{1}{r_{44}} x_4$$

$$= \begin{pmatrix} 1/\sqrt{7} \\ 1/\sqrt{7} \\ 1/\sqrt{7} \\ -2/\sqrt{7} \end{pmatrix}$$

$$Q = \begin{bmatrix} 0 & 3/\sqrt{15} & 3/\sqrt{35} & 1/\sqrt{7} \\ 1/\sqrt{3} & -2/\sqrt{15} & 3/\sqrt{35} & 1/\sqrt{7} \\ 1/\sqrt{3} & 1/\sqrt{15} & -4/\sqrt{35} & 1/\sqrt{7} \\ 1/\sqrt{3} & 1/\sqrt{15} & 1/\sqrt{35} & -2/\sqrt{7} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 0 & \sqrt{15}/3 & 2/\sqrt{15} & 2/\sqrt{15} \\ 0 & 0 & \sqrt{35}/5 & 2/\sqrt{35} \\ 0 & 0 & 0 & \sqrt{63}/7 \end{bmatrix}$$

Singular Value DeComposition:

If A is an $m \times n$ matrix, the Singular Values of A are the Square roots of the eigen Values of $A^T A$ and are denoted by $\sigma_1, \sigma_2, \dots \sigma_n$. It is Conventional to arrange the singular Values so that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots \geq \sigma_n$$

Let A be an $m \times n$ matrix with singular Values $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} \dots = \sigma_n = 0$

Then there exist an $m \times m$ Orthogonal matrix U, and $n \times n$ Orthogonal matrix V and $m \times n$ matrix Σ Of the form $A = U \Sigma V^T$

A factorization of A is called an singular Value decomposition of A.

The column of U are called left singular values of A, and the columns of V are called right singular vectors of A. The matrices U and V are not uniquely determined by A, but Σ must contain the singular values of A.

1. Find a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

We compute $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find its eigen values are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$ with corresponding eigen vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Now we Normalize them to obtain

$$V_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad V_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

The singular values are

$\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{1} = 1$, $\sigma_3 = 0$ Thus

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$ and

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find u:

$$u_1 = \frac{1}{\sigma_1} A V_1$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A V_2$$

$$= \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$A = U \Sigma V^T$$

2. find a Singular Value decomposition

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

We Compute

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equations are

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, 3$$

$$\therefore \sigma_1 = \sqrt{3}, \quad \sigma_2 = 1$$

Corresponding eigen Vectors are

$$V_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

To find u :

$$u_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} AV_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Now, we need to extend $\{u_1, u_2\}$ to an Orthonormal basis for $\mathbb{R}^3$.

If e_3 is the third standard basis Vector in $\mathbb{R}^3$, it is Clear that $\{u_1, u_2, e_3\}$ is Linearly independent.

Applying Gram-Schmidt to $\{u_1, u_2, e_3\}$, we find

$$u_3 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

and we have the SVD

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Applications of Real Inner product spaces.

Introduction:

The defn of inner product space which are vector spaces with an inner product defined on them, which allow us to introduce the concept of length (norm) of a vector, orthogonality concept. further we discuss some of the applications of inner product spaces like least squares approximations, quadratic forms, fourier approximation etc.

Applications of Inner product space.

* find the linear or quadratic least squares approximation of a function
* find the n^{th} order approximation of a function
* Quadratic forms.

Least square approximations of functions

Many problems in the physical sciences and engineering involve an approximation of a function f by another function g. If f is in $C[a,b]$ (the inner product space of all continuous fns on $[a,b]$, the g is usually chosen from a subspace W of $C[a,b]$

In particular, to approximate the function
$f(x) = e^x$.

Theorem: least square approximation

Let f be continuous on $[a,b]$, and let W be a finite dimensional subspace of $C[a,b]$. The least squares approximating function f w.r.t W is given by

$$g = \langle f, w_1 \rangle w_1 + \langle f, w_2 \rangle w_2 + \cdots + \langle f, w_n \rangle w_n.$$

where $B = \{ w_1, w_2 \cdots w_n \}$ is an orthonormal basis for W.

Fourier Approximations:

A special type of least squares approximation is called Fourier approximation

Consider the function

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx.$$

In the subspace W of $C[-\pi, \pi]$ spanned by the basis $\{ 1, \cos x, \cdots \cos nx, \sin x \cdots \sin nx \}$

we can easily check that the set

$$B = \{ w_0, w_1 \cdots w_n, w_{n+1} \cdots w_{2n} \}$$
$$= \{ \tfrac{1}{\sqrt{2}}, \cos x \cdots \cos nx, \sin x \cdots \sin nx \}$$

forms an orthonormal basis in the inner product space $C[-\pi,\pi]$

Quadratic Forms:

Defn: A real quadratic form q in variables $x_1, \dots x_n$ is a polynomial such that every term has degree 2.

That is $q(x_1, x_2 \dots x_n) = \sum_i c_i x_i^2 + \sum_{i<j} d_{ij} x_i x_j$

where $c_i \in R$, $d_{ij} \in R$ for every $i, j = 1, 2 \dots n$

The quadratic form q defines a symmetric matrix $A = [a_{ij}]$ where $a_{ii} = c_i$ and $a_{ij} = a_{ji} = \frac{1}{2} d_{ij}$

If the matrix A of q is diagonal, then q has the diagonal representation

$$q(x) = X^T A X = a_{11} x^2 + a_{22} x^2 + \dots + a_{nn} x^2$$

That is, the quadratic polynomial representing q will contain no "coss product" terms

Defn: A quadratic form $X^T A X$ is said to be positive definite if $X^T A X > 0$ for every non zero $x \in R^n$.

Result: A quadratic form $X^T A X$ is positive definite iff if all the eigen values of the symmetric matrix A are positive.

Result: A quadratic form $X^T A X$ is positive if and only if all the eigen values of the symmetric matrix A are positive.

Gram_schmidt process:

The application of the Gram-schmidt process to the column vectors of a full column rank matrix yields the QR decomposition.

We define the projection operator by

$$proj_u (v) = \frac{\langle u, v \rangle}{\langle u, u \rangle} u,$$

where $\langle u, v \rangle$ denotes the inner product of the vectors u and v.

$\langle u, v \rangle = u^T v$ for vectors on $\mathbb{R}^n$. or $\langle u, v \rangle = u^* v$ for vectors on $\mathbb{C}^n$. This operator projects the vector v orthogonally onto the line spanned by vector u.

If $u = 0$, we define $proj_0 (v) := 0$

i.e the projection map $proj_0$ is the zero map, sending every vector to the zero vector.

PART A

1. what is a innerproduct space?

2. what is a norm?.

3. Describe the process of normalizing a vector.?

4. Give two examples of innerproduct spaces.

5. Normalize the vector $u = (1,3,-4,2)$.

6. Normalize the vector $v = (4,-2,2,1)$.

7. Normalize the vector $(5,-1,-2,6)$.

8. How is the angle between two vectors u and v defined in terms of innerproduct ?

9. Find the angle between the vectors $u=(2,3,5)$ and $v=(1,-4,3)$.

10. when are two vectors $u, v \in V$, a vectorspace said to be orthogonal.

11. Find a non-zero vector w that is orthogonal to $u_1=(1,2,1)$ and $u_2=(2,5,4)$.

12. what are orthogonal matrices.

13. what is positive definite matrix?

14. The vectors $u_1=(1,1,0)$, $u_2=(1,2,3)$ and $u_3=(1,3,5)$ form a basis S of $\mathbb{R}^3$. Find the matrix A that represents the innerproduct in $\mathbb{R}^3$.

15. what is a normed vector space.

16. Expand (a) $\langle 5u_1+8u_2, 6v_1-7v_2\rangle$.

(b) $\langle 3u+5v, 4u-6v\rangle$

(c) $\|2u-3v\|^2$.

17. Consider the vectors $u = (1, 2, 4)$, $v = (2, -3, 1)$, $w = (4, \ldots$ in $\mathbb{R}^3$. Find.

(a) $u \cdot v$ (d) $(u+v) \cdot w$

(b) $u \cdot w$ (e) $\|u\|$

(c) $v \cdot w$. (f) $\|v\|$.

18. Verify that $\langle u, v \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$.
where $u = (x_1, x_2)$ $v = (y_1, y_2)$ defines an innerproduct on $\mathbb{R}^2$.

19. Find $\cos\theta$ where θ is the angle between.
$u = (1, 3, -5, 4)$ and $v = (2, -3, 4, 1)$ in $\mathbb{R}^4$.

20. Find k so that $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ in $\mathbb{R}^4$
are orthogonal.

PART-B.

1. Apply Gramschmidt orthogonalisation process to find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by $v_1 = (1,1,1,1)$, $v_2 = (1,2,4,5)$, $v_3 = (1,-3,-4,-2)$.

2. Show that $S = \{(1,1,0,-1),(1,2,1,3),(1,1,-9,2),(16,-13,1,3)\}$. is an orthogonal basis of $\mathbb{R}^4$.

3. Consider the subspace V of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1,1,1,1)$, $v_2 = (1,1,2,4)$ and $v_3 = (1,2,-4,-3)$. Find an orthonormal basis of V.

4. verify that the following is an innerproduct on $\mathbb{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$: $f(u,v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$.

5. Show that $\langle u,v\rangle = x_1y_1 + x_2y_2$, where $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ is not an innerproduct.

6. Show that $\langle u,v\rangle = x_1y_2x_3 + y_1x_2y_3$, where $u=(x_1,x_2,x_3)$ and $v = (y_1, y_2, y_3)$ is not an innerproduct.

7. Using Gramschmidt process obtain a orthonormal basis from the set: $w_1 = (1,0,1,0)$, $w_2 = (1,1,1,1)$ and $w_3 = (0,1,2,1)$

8. use modified Gramschmidt process to construct an orthogonal set of vectors from the set $X_1 = [-4,3,6]$, $X_2 = [2,-3,6]$, $X_3 = [2,3,0]$

9. construct a Q,R decomposition for the matrix $A = \begin{bmatrix} -4 & 2 & 2 \\ 3 & -3 & 3 \\ 6 & 6 & 0 \end{bmatrix}$

10. use modified Gramschmidt process to construct an orthogonal set of vectors from $X_1 = [0,1,1,1]$, $X_2 = [1,0,1,1]$, $X_3 = [1,1,0,1]$ $X_4 = [1,1,1,0]$.

11. Find the QR decomposition of the matrix $A = \begin{bmatrix} 5 & 2 & 2 \\ 3 & 6 & 3 \\ 6 & 6 & 9 \end{bmatrix}$

12. Construct the QR decomposition of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

13. Construct QR decomposition of $B = \begin{bmatrix} -4 & 4 & 2 \\ 4 & -4 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

14. Construct QR decomposition of $C = \begin{bmatrix} 4 & -3 & 2 \\ 3 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$

15. Construct QR decomposition of $D = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 1 \\ 2 & 1 & 8 \end{bmatrix}$