Vector Space

Introduction

Many familiar physical notions, such as forces, velocities, and accelerations, involve both a magnitude and a direction. Any such entity involving both magnitude and direction is called a vector. Vectors are represented by arrows in which the length of the arrow denotes the magnitude of the vector and the direction of the arrow represents the direction of the vector. In most physical situations involving vectors, only the magnitude and direction of the vectors are significant; Consequently, we regard vectors with the same magnitude and direction as being equal irrespective of their positions.

Algebraic Description

The algebraic description of vector addition and scalar multiplication for vectors in a plane yield the following properties for arbitrary vectors x, y and z and arbitrary real numbers 'a' & 'b'.

1. $x+y = y+x$
2. $\{x+y\}+z = x+\{y+z\}$
3. There exist a vector denoted 0 such that $x+0 = x$ for each vector 'x'.

4. For each vector x there is a vector y such that $x+y=0$.

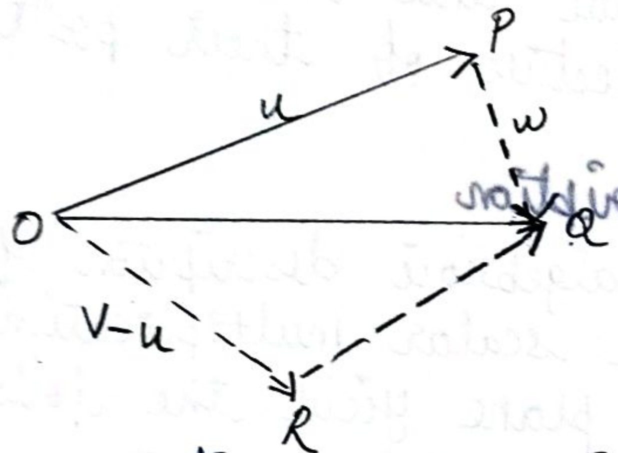
5. $1x = x$.

6. $(ab)x = a(bx)$.

7. $a(x+y) = ax+ay$.

8. $(a+b)x = ax+bx$

Consider the equation of a line in space that passes through two distinct points P and Q. Let 'O' denote the origin of a coordinate system in space, and let 'u' & 'v' denote the vectors beginning at P and ending at Q, then "tail to head" addition shows that $u+w=v$, and hence $w=v-u$, where $-u$ denotes the vector $(-1)u$.



Example:

The equation of the line through the points P and Q having coordinates $(-2,0,1)$ and $(4,5,3)$ respectively. The endpoint R of the vector emanating from the origin and having the same direction as the vector beginning at P and terminating at Q has coordinates $(4,5,3)-(-2,0,1)=(6,5,2)$. Hence the desired eqn. is $x=(-2,0,1)+t(6,5,2)$.

Vector Spaces:

Definition:

A Vector space (or linear space) V over a field F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y in V there is a unique element $x+y$ in V, and for each element a in 'F' and each element x in V there is a unique element ax in V, such that the following condition hold:

(VS1) For all x, y in V, $x+y = y+x$
(commutativity ~~and~~ addition)

(VS2) For all x, y, z in V, $(x+y)+z = x+(y+z)$
(associativity of addition)

(VS3) There exists an element in V denoted by '0' such that $x+0 = x$ for each in 'V'.

(VS4) For each element x in V there exists an element y in V such that $x+y=0$

(VS5) For each element x in V $1x = x$.

(VS6) For each pair a, b of elements in 'F' and each element x in V. $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in V

$$a(x+y) = ax + ay.$$

(VS 8) For each pair a, b of elements in F and each element x in V $(a+b)x = ax + bx$.

The elements '$x+y$' and 'ax' are called the sum of x and y and the product of 'a' and 'x', respectively.

Example:-

The set of all n-tuples with entries from a field F forms a vector space, which we denoted by F^n, under the operations of coordinatewise addition and multiplication: that is, if $x = (a_1 \dots a_n) \in F^n$, $y = (b_1 \dots b_n) \in F^n$ and $c \in F$, then

$$x+y = (a_1+b_1 \dots \dots, a_n+b_n) \text{ and}$$

$$cx = (ca_1 \dots ca_n)$$

For example, in R^4,

$(3, -2, 0, 5) + (-1, 1, 4, 2) = (2, -1, 4, 7)$

and $-5(1, -2, 0, 3) = (-5, 10, 0, -15)$

Eliments of F^n will often be written as column vectors: $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

rather than as row vector $(a_1 \dots a_n)$. Since a 1-tuple with entry from F may be regarded as an element of F, we will write F rather than F^1 for the vector space of 1-tuples from F.

Example:-

Let F be any field. A sequence in F is a function σ from the positive integers into F. As usual, the sequence σ such that $\sigma(n) = a_n$ will be denoted by $\{a_n\}$. Let V consist of all sequences $\{a_n\}$ in F that have only a finite number of nonzero terms a_n. If $\{a_n\}$ and $\{b_n\}$ are sequences in V and $t \in F$, then $\{a_n\}+\{b_n\}$ is that sequence $\{c_n\}$ in V such that $c_n = a_n + b_n$ $(n = 1, 2 \dots)$ and $t\{a_n\}$ is that sequence $\{d_n\}$ in V such that $d_n = ta_n$ $(n = 1, 2 \dots)$. Under these operations V is a vector space.

Example:

The set of all $m \times n$ matrices with entries from a field 'F' is a vector space, which we denote by $M_{m \times n}(F)$ under the following operations of addition and scalar multiplication: For $A, B \in M_{m \times n}(F)$ and $c \in F$;

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}.$$

For instance

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and

$$-3\begin{pmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{pmatrix} \text{ in } M_{2 \times 3}(R).$$

Example:

The set of all polynomials with coefficients from a field 'F' is a vector space, which we denote by $P(F)$, under the following operations. For

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

and $g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0$

in $P(F)$ and $c \in F$,

$(f+g)x = (a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \cdots + (a_0+b_0)$

and

$(cf)(x) = ca_nx^n + ca_{n-1}x^{n-1} + \cdots + ca_0$

Example:-

Let $S = \{(a_1,a_2) : a_1,a_2 \in R\}$, For (a_1,a_2), $(b_1,b_2) \in S$ and $c \in R$, define

$(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2)$ and

$c(a_1,a_2) = (ca_1, ca_2)$

since (VS1), (VS2) and (VS8) all fail to hold, S is not a vector space under the operation.

Example:

Let $S = \{(a_1,a_2) : a_1,a_2 \in R\}$ For (a_1,a_2), $(b_1,b_2) \in S$ and $c \in R$ define

$(a_1,a_2) + (b_1,b_2) = (a_1+b_1, 0)$ and

$c(a_1,a_2) = (ca_1, 0)$

The S under these operations is not a vector space because (VS3) [hence (VS4)] and (VS5) fail.

Problems:-

① Every vector space contains a zero vector.

Yes, Every vector space contains a zero vector. It's condition (VS3).

② A vector space may have more than one zero vector.

No, If x, y are both zero vectors, Then by condition (VS3) $x = x + y = y$.

③ Write the zero vector of $M_{3\times4}(F)$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ It's the 3×4 matrix with all entries 0.

④ If $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ what are M_{13}, M_{21} and M_{22}?

$M_{13} = 3$; $M_{21} = 4$; $M_{22} = 5$.

⑤ perform the operation Indicated

soln.,

(i) $$\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{pmatrix}$$

(ii) $$4\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 8 & 20 & -12 \\ 4 & 0 & 28 \end{pmatrix}$$

(iii) $(2x^4-7x^3+4x+3)+(8x^3+2x^2-6x+7)$

$2x^4+x^3+2x^2-2x+10.$

(iv) $5(2x^7-6x^4+8x^2-3x)$

$10x^7-30x^4+40x^2-15x.$

6. If any Vector space V, show that
$(a+b)(x+y)=ax+ay+bx+by$ for any $x,y \in V$
and any $a,b \in F$.

By (VS 7) and (VS 8), we have

$(a+b)(x+y)=a(x+y)+b(x+y)=ax+ay+bx+by.$

7. Let V denote the set of ordered pairs of real numbers. If (a_1,a_2) and (b_1,b_2) are elements of V and c is an element of F, define $(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2b_2)$ and
$c(a_1,a_2)=(ca_1,a_2).$
Is V a vector space under these operations? Justify your answer.

No, If it's not a vector space, we have $0(a_1,a_2)=(0,a_2)$ be the zero vector.
But since a_2 is arbitrary, this is a contradiction to the uniqueness of zero vector.

Subspace:

Definition:

A Subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F under the operations of addition and scalar multiplication defined on V.

Example:

Let V be any vector space. Then the set $\{0\}$ consisting of the zero vector alone, and also the entire space V are subspace of V.

Example:

Let M be an $n \times n$ matrix. The diagonal of M consist of the entries $M_{11}, M_{22} \ldots M_{nn}$. An $n \times n$ matrix D is called a diagonal matrix if each entry not on the diagonal of D is zero, that is, if $D_{ij} = 0$ whenever $i \neq j$. The set of all diagonal matrices in $M_{n \times n}(F)$ is a subspace of $M_{n \times n}(F)$.

Example:

Let V be the space of all square $n \times n$ matrices. Then the set W consisting of those matrices $A = (a_{ij})$ for which $a_{ij} = a_{ji}$, called symmetric matrices, is a subspace of V.

Example:

Let V be the space of polynomials. Then the set W consisting of polynomials with degree $= n$ for a fixed n, is a subspace of V.

Example:

Consider any homogeneous system of linear equations in 'n' unknowns with, say, real coefficients

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

Recall any particular solution of the system may be viewed as a point in R^n. The set W of all solutions of the homogeneous system is a subspace of R^n called the solution space. We comment that the solution set of a non homogeneous system of linear equation in 'n' unknown is not a subspace of R^n.

Theorem:-

* Let W_1 and W_2 be subspace of a vector space V then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

* If W is a Subspace of V and $x_1, x_2 \ldots x_n$ are elements of W, then $a_1x_1 + \ldots + a_nx_n$ is an element of W for any scalars $a_1, \ldots a_n$ in F.
* Any intersection of subspace of a Vector space V is a subspace of V.

Problems:

1) Determine the transpose of each of the following matrices. In addition, if the matrix is square, compute its trace.

(a) $\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 8 & -6 \\ 3 & 4 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

(d) $(1, -1, 3, 5)$

Soln..

(a) $\begin{pmatrix} -4 & 5 \\ 2 & -1 \end{pmatrix}$ with Trace $= -5$

(b) $\begin{pmatrix} 0 & 3 \\ 8 & 4 \\ -6 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{pmatrix}$ with Trace $= 12$

(d) $\begin{pmatrix} 1 \\ -1 \\ 3 \\ 5 \end{pmatrix}$

(2) Determine if the following sets are subspace of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answer

(a) $W_1 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

(b) $W_2 = \{(a_1, a_2, a_3) \in R^3 : a_1 = a_3 + 2\}$

(c) $W_3 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0\}$

soln:-

(a) Yes. It's a line $t(3, 1, -1)$

(b) No. It's contains no $(0, 0, 0)$

(c) Yes. It's a plane with normal vector $(2, -7, 1)$.

(3) Consider a homogeneous system of linear equations in 'n' unknowns $x_1, x_2 \dots x_n$ over a field F.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

S.T the solution set W is subspace of the vector space F^n.

Soln:- $0 = (0, 0 \ldots 0) \in W$ since, clearly

$$a_{11}0 + a_{12}0 + \ldots + a_{1n}0 = 0 \quad \text{for } i = 1 \ldots m$$

Suppose $u = (u_1, u_2 \ldots u_n)$ and $V = (V_1, V_2 \ldots V_n)$ belong to W i.e., for $i = 1, \ldots m$.

$$a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n = 0$$

$$a_{11}V_1 + a_{12}V_2 + \ldots + a_{1n}V_n = 0$$

Let 'a' and 'b' be scalars in F. Then

$$au + bV = (au_1 + bV_1, \; au_2 + bV_2 \ldots au_n + bV_n)$$

and, for $i = 1 \ldots m$.

$$a_{11}(au_1 + bV_1) + a_{12}(au_2 + bV_2) + \ldots + a_{1n}(au_n + bV_n)$$

$$= a(a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n) + b(a_{11}V_1 + a_{12}V_2 + \ldots + a_{1n}V_n)$$

$$= a0 + b0 = 0$$

Hence $au + bV$ is a solution of the system, i.e., belongs to W. Accordingly, W is a subspace of F^n.

4. P.T $tr(aA+bB) = a\,tr(A) + b\,tr(B)$ for any $A, B \in M_{n\times n}(F)$.

Soln: We know that

$$tr(aA+bB) = \sum_{i=1}^{n}(aA_{ii} + bB_{ii})$$

$$= a\sum_{i=1}^{n} A_{ii} + b\sum_{i=1}^{n} B_{ii}$$

$$tr(aA+bB) = a\,tr(A) + b\,tr(B)$$

Linear Combination and Systems of Linear Equations.

Definition:

Let V be a space and S a nonempty subset of V. A vector x in V is said to be in linear combination of elements of S if there exist a finite number of elements $y_1, \dots y_n$ in S and scalars $a_1 \dots a_n$ in F such that $x = a_1y_1 + \dots + a_ny_n$. In this situation it is also customary to say that 'x' is a linear combination of $y_1, \dots y_n$.

Example:

(1) We will S.T $2x^3-2x^2+12x-6$ is a linear combination of

x^3-2x^2-5x-3 and $3x^3-5x^2-4x-9$

in $P_3(R)$, but that

$3x^2-2x^2+7x+8$ is not such a linear combination. In the first case we find scalars 'a' & 'b' such that

$$2x^3-2x^2+12x-6 = a(x^3-2x^2-5x-3) + b(3x^3-5x^2-4x-9)$$

$$=(a+3b)x^3+(-2a-5b)x^2 + (-5a-4b)x+(-3a-9b)$$

Thus we are led to the following system of linear equations

$$a+3b=2 \quad \text{—}(1)$$
$$-2a-5b=-2 \quad \text{—}(2)$$
$$-5a-4b=12 \quad \text{—}(3)$$
$$-3a-9b=-6 \quad \text{—}(4)$$

Adding appropriate multiples of the first equation to the others in order to eliminate a, we find

$$a+3b=2$$

Solving (2) & (3) eqn.,

$$b=2$$

Solving ① & ③

$$11b = 22$$

Solving ① & ④

$$0 = 0$$

Now adding the appropriate multiples of the second eqn, to the others yields

$$a = -4$$
$$b = 2$$
$$0 = 0$$
$$0 = 0$$

Hence $2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9)$ ⑤

Hence,

In the second case we wish to show that there are no scalars 'a' and 'b' for which,

$$3x^3 - 2x^2 + 7x + 8 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

As above we obtain a system of linear eqns.,

$$a + 3b = 3$$
$$-2a - 5b = -2$$
$$-5a - 4b = 7$$
$$-3a - 9b = 8$$

Eliminating 'a' as before yields

$$a+3b=3$$
$$b=4$$
$$11b=22$$
$$0=17$$

But the presence of the inconsistent eqn., $0=17$ indicates that system has no solutions. Hence $3x^3-2x^2+7x+8$ is not a linear combination of

$$x^3-2x^2-5x-3$$

& $3x^3-5x^2-4x-9$.

② Write the vector $V=(1,-2,5)$ as a linear combination of the vectors $x=(1,1,1)$, $y=(1,2,3)$ and $z=(2,-1,1)$.

To express V as $V=ax+by+cz$ with a,b and c as yet unknown scalars. Thus we require

$$(1,-2,5) = a(1,1,1)+b(1,2,3)+c(2,-1,1)$$
$$= (a,a,a)+(b,2b,3b)+(2c,-c,c)$$
$$= (a+b+2c,\ a+2b-c,\ a+3b+c)$$

Form the equivalent system of equations by setting corresponding components equal to each other, and then reduce to echelon form

$$\begin{aligned} a+b+2c &= 1 \\ a+2b-c &= -2 \\ a+3b+c &= 5 \end{aligned} \quad \text{(or)} \quad \begin{aligned} a+b+2c &= 1 \\ b-3c &= -3 \\ 2b-c &= 4 \end{aligned} \quad \text{(or)} \quad \begin{aligned} a+b+2c &= 1 \\ b-3c &= -8 \\ 5c &= 10 \end{aligned}$$

Note that the above system is consistent and so has a solution. Solve for the unknowns to obtain $a=-6$; $b=3$; $c=2$

Hence $V = -6x+3y+2z$

3 Solve the system of linear equations

$$2x_1-2x_2-3x_3 = -2 \quad \text{—— (1)}$$

$$3x_1-3x_2-2x_3+5x_4 = 7 \quad \text{—— (2)}$$

$$x_1-x_2-2x_3-x_4 = -3 \quad \text{—— (3)}$$

This system of linear equation has 3 equations with 4 unknowns.

(2) $- 3\times$(3) gives

$$\begin{array}{r} 3x_1-3x_2-2x_3+5x_4=7 \\ \overset{(-)}{3x_1}\overset{(+)}{-3x_2}\overset{(+)}{-6x_3}\overset{(+)}{-3x_4}\overset{(+)}{=-9} \\ \hline 4x_3+8x_4=16 \end{array}$$

$$x_3+2x_4=4$$

Let $x_4 = k_1$

$$\therefore x_3 = 4-2k_1$$

$$\therefore (1) \Rightarrow 2x_1-2x_2-3(4-2k_1) = -2$$

$$2x_1-2x_2-12+6k_1 = -2$$

$$2x_1-2x_2 = 10-6k_1$$

$$x_1-x_2 = 5-3k_1$$

Let $x_2 = K_2$

$\therefore x_1 - K_2 = 5 - 3K_1$

$x_1 = 5 - 3K_1 + K_2$

$\therefore$ The solution set is $\{5-3K_1+K_2, K_2, 4-2K_1+K_1\}$

where $K_1, K_2 \in F$.

(2) Solve the system of linear equation

$x_1 + 2x_2 - x_3 + x_4 = 5$ — ①

$x_1 + 4x_2 - 3x_3 - 3x_4 = 6$ — ②

$2x_1 + 3x_2 - x_3 + 4x_4 = 8$ — ③

This system of linear equations has 3 equations with 4 unknowns.

① − ②

$x_1 + 2x_2 - x_3 + x_4 = 5$

$\overset{(-)}{x_1} \overset{(-)}{+4x_2} \overset{(+)}{-3x_3} \overset{(+)}{-3x_4} = \overset{-}{6}$

$-2x_2 + x_3 + 4x_4 = -1$ — ④

② × 2 − ③

$2x_1 + 8x_2 - 6x_3 - 6x_4 = 12$

$\overset{(-)}{2x_1} \overset{(-)}{+3x_2} \overset{(+)}{-x_3} \overset{(-)}{+4x_4} = \overset{(-)}{8}$

$5x_2 - 5x_3 - 10x_4 = 4$ — ⑤

⑤ × 2 + 5 × ④

$-10x_2 + 5x_3 + 20x_4 = -20$

$10x_2 - 10x_3 - 20x_4 = 8$

$-5x_3 = -12$

$x_3 = 12/5$ — ⑥

$\therefore$ ④ $\Rightarrow -2x_2 + \frac{12}{5} + 4x_4 = -1$

$-2x_2 + 4x_4 = \frac{-17}{5}$

$-10x_2 + 20x_4 = -17$ — ⑦

⑤ $\Rightarrow 5x_2 - 5 \cdot \frac{12}{5} - 10x_4 = 4$

$5x_2 - 10x_4 = 16$ — ⑧

⑦ + 2 × ⑧

$-10x_2 + 20x_4 = -17$

$10x_2 - 20x_4 = 16$

$\boxed{0 = -1}$

not possible

$\therefore$ no solution

$\therefore$ This system of linear equations has no solution.

Linear Dependence and linear Independence

Introduction:

The set $S = \{x_1, x_2, x_3, x_4\} \subseteq R^3$, where $x_1 = (2,-1,4)$, $x_2 = (1,-1,3)$, $x_3 = (1,1,-1)$ and $x_4 = (1,-2,-1)$. To see if S is linearly dependent, we must check whether or not there is a vector in S that is linear combination of the others. Now the vectors x_4 is a linear combination of x_1, x_2 and x_3 if and only if there are scalars a, b and 'c' such that

$$x_4 = ax_1 + bx_2 + cx_3$$

that is, if and only if

$$x_4 = (2a+b+c,\ -a-b+c,\ 4a+3b-c)$$

Thus x_4 is a linear combination of the x_1, x_2 and x_3 if and only if the system

$$2a+b+c = 1$$
$$-a-b+c = -2$$
$$4a+3b-c = -1$$

has a solution.

Definition:-

A subset S of a vector space V is said to be linearly dependent if there exists a finite number of distinct vectors $x_1, x_2 \ldots x_n$ in S and scalars $a_1, a_2 \ldots a_n$ not all zero, such that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$.

In this case we will also say that the elements of 'S' are linearly dependent.

Example:

In R^4 the set

$S = \{(1,3,-4,2), (2,2,-4,0), (1,-3,2,-4), (-1,0,1,0)\}$ is linearly dependent because

$$4(1,3,-4,2) - 3(2,2,-4,0) + 2(1,-3,2,-4) + 0(-1,0,1,0) = (0,0,0,0)$$

similarly in $M_{2\times3}(R)$ the set

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}$$

is linearly dependent since

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition:-

A subset S of a vector space that is not linearly dependent is said to be linearly independent. As before, we will also say that the elements of S are linearly independent.

Example:

We show that the vectors $x = (6,2,3,4)$, $y = (0,5,-3,1)$ and $z = (0,0,7,-2)$ are independent. For

Suppose $ax+by+cz=0$, where a, b and c are unknown scalars. Then

$$(0,0,0,0) = a(6,2,3,4)+b(0,5,-3,1)+c(0,0,7,-2)$$

and so, by the equality of the corresponding components,

$$6a = 0$$
$$2a+5b = 0$$
$$3a-3b+7c = 0$$
$$4a+b-2c = 0$$

The first equation yields $a=0$;

The second equation yields with $a=0$, $b=0$;

The third equation yields with $a=0$, $b=0$, $c=0$

Thus $ax+by+cz=0$ implies $a=0$, $b=0$, $c=0$

Accordingly x, y, z are independent

Problems:

① Show that the set $\{1, x, x^2 \ldots x^n\}$ is linearly independent in $P_n(F)$.

Solution

Let $S' = \{x^{m_1}, x^{m_2} \ldots x^{m_n}\}$ be any finite subset of having 'n' vectors, where $m_1, m_2 \ldots m_n$ are non-negative integers.

Let $a_0, a_1, a_2 \ldots a_n \in F$

$\therefore a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n = 0$

(i.e., Zero polynomial)

Then by definition of equality of two vectors we have $a_0 = a_1 = 0, a_2 = 0 \ldots a_n = 0$

$\Rightarrow$ Every finite subset of S in linearly Independent and hence S is linearly independent.

(2) Let V be the vector space of 2×3 matrices over R. Show that the vectors A, B, C form L.I set where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$$

Soln.,

Let $a, b, c \in R$ then for linearly independent.

$aA + bB + cC = 0 \Rightarrow a = 0;\ b = 0;\ c = 0;$

Now $aA + bB + cC = 0$

$$\Rightarrow a\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$$

$$+ c\begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a-2c & 4a+5b-3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow 2a+b+4c = 0,\quad a+b-c = 0,\quad -a-3b+2c = 0$

$3a-2b+c = 0,\quad -2a-2c = 0,\quad 4a+5b-3c = 0$

These equations have only solution *

$a = 0,\ b = 0,\ c = 0.$

*

c.11/02

Bases and Dimension:

Definition - Basis

A basis β for a vector space V is a linearly independent subset of V that generates V. (If β is a basis for V, we also say that the elements of β form a basis for V.)

Definition - Dimension:

A vector space V is called finite-dimensional if it has a basis consisting of a finite number of elements: the unique number of elements in each basis for V is called the dimension of V and is denoted $\dim(V)$. If a vector space is not finite-dimensional, then it is called infinite-dimensional.

Examples:

* The four vectors F^4 $(1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1)$ are linearly independent since they form a matrix in echelon form. Furthermore since $\dim F^4 = 4$ they form a basis of F^4.
* The four vectors in R^3 $(257,-132,58), (43,0,-17), (521,-317,94), (328,-512,-731)$ must be linearly dependent since they come from a vector space of dimension 3.

* The vector space $\{0\}$ has dimension Zero.

* The vector space F^n has dimension n.

* The vector space $M_{m\times n}(F)$ has dimension mn.

Theorem:-

* Let W be a Subspace of a finite-dimensional Vector space V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, the $W = V$.

* If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional, and $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

* Let W_1 and W_2 be finite dimensional subspace of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if

$$\dim(V) = \dim(W_1) + \dim(W_2).$$

Problems:

① Determine the basis for the set R^3

$\{(1,0,-1), (2,5,1), (0,-4,3)\}$

The vectors form a basis if and only if they are independent.

Let us take a, b, c as scalars

$\therefore$ the given set form the linear Combination as

$$a(1,0,-1) + b(2,5,1) + c(0,-4,3) = (0,0,0)$$

$$a + 2b = 0 \quad \text{—} ①$$

$$5b - 4c = 0 \quad \text{—} ②$$

$$-a + b + 3c = 0 \quad \text{—} ③$$

$① + ③ \Rightarrow 3b + 3c = 0 \quad \text{—} ④$

$④ \times 4 + ② \times 3 \Rightarrow$

$$\begin{array}{r} 12b + \cancel{12c} = 0 \\ 15b - \cancel{12c} = 0 \\ \hline b = 0 \end{array}$$

so $① \Rightarrow a = 0$

from this $③ \Rightarrow c = 0$

$a = 0, b = 0, c = 0$

$\therefore$ The given ~~set~~ are linearly independent in R^3

$\therefore \{(1,0,-1), (2,5,1), (0,-4,3)\}$ form a basis in R^3.

(2) Determine the basis for the set $\{2-x-x^2, -2+x+2x^2, 4-2x+x^2\}$ in $P_2(R)$.

Solution:

The vectors form a basis if and only if they are independent.

Let us take a, b, c as scalars

$\therefore$ the given set form the linear combination as

$$a(2,-1,-1) + b(-2,1,2) + c(4,-2,1) = (0,0,0)$$

$$2a - 2b + 4c = 0 \quad — (1)$$

$$-a + b - 2c = 0 \quad — (2)$$

$$-a + 2b + c = 0 \quad — (3)$$

(2) + (3)

$$-a + b - 2c = 0$$

$$\underline{\overset{(+)}{-}a \overset{(-)}{+} 2b \overset{(+)}{-} c = 0}$$

$$-b - 3c = 0$$

$$-b = 3c$$

$$\frac{b}{3} = \frac{c}{-1}$$

$\therefore$ (1) $\Rightarrow$ $2a - 2(3) + 4(-1) = 0$

$2a - 6 - 4 = 0$

$2a = 10$

$a = 5$

$\therefore a = 5, b = 3, c = -1$

The given set are linearly dependent in $P_2(R)$. $\therefore$ The given set do not form basis in $P_2(R)$.

③ Suppose that V is a vector space with (4)
a basis $\{x_1, x_2, x_3\}$. S.T $\{x_1+x_2+x_3, x_2+x_3, x_3\}$ is also a basis for V.

The vectors form a basis if and only if they are independent.

Let us take a, b, c as scalars

$\therefore$ the given set form the linear combination as

$$a(1,0,0) + b(1,1,0) + c(1,1,1) = (0,0,0)$$

$$a+b+c = 0 \quad \text{— ①}$$

$$b+c = 0 \quad \text{— ②}$$

$$c = 0 \quad \text{— ③}$$

from ③ $b=0$, ① $\Rightarrow a=0$.

$a=0, b=0, c=0$

$\therefore$ The given set are linearly independent

$\therefore \{x_1+x_2+x_3, x_2+x_3, x_3\}$ is a basis for V.

④ Let W_1 and W_2 be two subspaces of R^4 given by $W_1 = \{a,b,c,d\} : b-2c+d=0\}$

$W_2 = \{a,b,c,d\} : a=d, b=2c\}$

Find the basis and dimension of (i) W_1 (ii) W_2 (iii) $W_1 \cap W_2$ and hence find $\dim(W_1+W_2)$

Solution:

(i) Giv., $V=\{(a,b,c,d\}$; $b=2c-d$

Let $(a,b,c,d)\in W_1$ then

$(a,b,c,d) = (a, 2c-d, c, d)$

$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$.

$\therefore (a,b,c,d)$ = linear combination of Linearly Independent set

$\{(1,0,0,0),(0,2,1,0),(0,-1,0,1)\}$ which forms the basis of W_1 $\therefore \dim W_1 = 3$

(ii) Given $W_2 = \{(a,b,c,d) : a=d, b=2c\}$

Let $\alpha\in W_2 \Rightarrow \alpha=(a,b,c,d)$ where $a=d, b=2c$

$\therefore \alpha = (d,2c,c,d)$

$= d(1,0,0,1) + c(0,2,1,0)$.

$\Rightarrow \alpha$ = linear combination of Linearly independent set $\{(1,0,0,1),(0,2,1,0)\}$

$\therefore$ which forms a basis $\therefore \dim W_2 = 2$.

(iii) $W_1\cap W_2 = \{(a,b,c,d) / b-2c+d=0, a=d, b=2c\}$.

Now $b-2c+d=0$, $a=d$, $b=2c$ gv., $b=2c$,

$a=0, d=0$

$\therefore (a,b,c,d) = (0,2c,c,0) = c(0,2,1,0)$

$\Rightarrow (a,b,c,d)$ = multiple of the vector $(0,2,1,0)$

$\therefore$ Basis of $W_1\cap W_2 = (0,2,1,0)$

$\Rightarrow \dim(W_1\cap W_2) = \dim W_1 + \dim W_2 - \dim W_1\cap W_2$

(iv)

$= 3+2-1 = 4$

Linear Transformations and Matrices

Linear Transformations, Null spaces and Ranges.

Definition:

Let V and W be vector spaces (over F). A function $T: V \to W$ is called a linear transformation from V into W if for all $x, y \in V$ and $c \in F$ we have (a) $T(x+y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$.

We often simply call T linear.

A function $T: V \to W$

1. If T is linear, then $T(0) = 0$.
2. T is a linear if and only if $T(ax+y) = aT(x) + T(y)$ for all $x, y \in V$ and $a \in F$.
3. T is linear if and only if ~~for~~ $x_1, \dots x_n \in V$ and $a_1 \dots a_n \in F$ we have
$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

Example:

Define $T: R^2 \to R^2$ by $T(a_1, a_2) = (2a_1 + a_2, a_1)$

To show that T is linear, let $c \in F$ and $x, y \in R^2$, where $x = \{b_1, b_2\}$ and $y = \{d_1, d_2\}$

Since $cx + y = (cb_1 + d_1, cb_2 + d_2)$

We have

$T(cx+y) = \{2(cb_1+d_1) + cb_2+d_2,\ cb_1+d_1\}$

Also $cT(x)+T(y) = c(2b_1+b_2, b_1) + (2d_1+d_2, d_1)$

$= (2cb_1 + cb_2 + 2d_1 + d_2,\ cb_1+d_1)$

$= (2(cb_1+d_1) + cb_2+d_2,\ cb_1+d_1)$

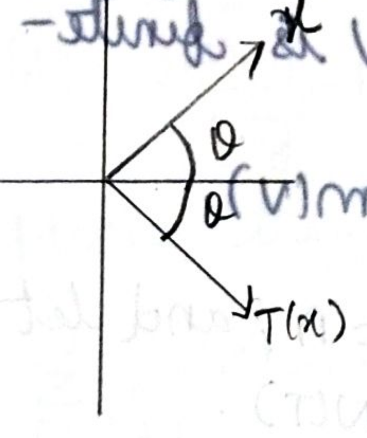
So T is linear.

Example:

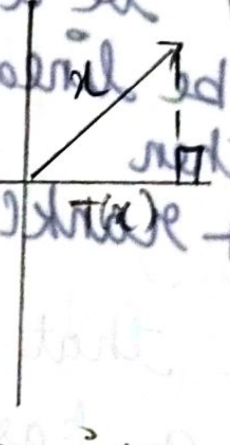
* Define $T: R^2 \to R^2$ by $T(a_1, a_2) = (a_1, -a_2)$. T is called the reflection about the x-axis

* Define $T: R^2 \to R^2$ by $T(a_1, a_2) = (a_1, 0)$. T is called the projection on the x-axis.

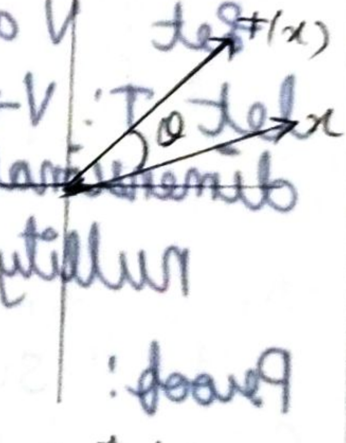
* For $0 \leq \theta < 2\pi$ we define $T_\theta: R^2 \to R^2$ by $T_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)$. T_θ is called the rotation by θ.



Reflection



Projection



Rotation

Definition: Null space

Let V and W be vector spaces, and let $T: V \to W$ be linear. We define the null space (or Kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$;
i.e., $N(T) = \{x \in V : T(x) = 0\}$

Range:

The range (or image) $R(T)$ of T to be the subset of W consisting of all image (under T) of elements of V: i.e., $R(T) = \{T(x), x \in V\}$

Example:

Define $T: R^3 \to R^2$ by

$T(a_1, a_2, a_3) = (a_1, -a_2, 2a_3)$

$N(T) = \{(a, a, 0) : a \in R\}$ and $R(T) = R^2$.

Theorem: (Dimension Theorem).

Let V and W be a vector spaces, and let $T: V \to W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Proof: suppose that $\dim(V) = n$, and let $\{x_1 \ldots x_n\}$ be a basis for $N(T)$.

We extend $\{x_1 \ldots x_n\}$ to a basis $\beta = \{x_1 \ldots x_n\}$ for V. We will S.T the set $S = \{T(x_{k+1} \ldots T(x_n)\}$ is a basis for $R(T)$.

First we prove that S generates R(T).

(Let V and W be vector spaces, and let $T: V \to W$ be linear. If V has a basis $\beta = \{x_1 \dots x_n\}$ then $R(T) = \mathrm{Span}\{T(x_1) \dots T(x_n)\}$

By using above theorem and the fact that $T(x_i) = 0$ for $1 \le i \le k$, we have that

$R(T) = \mathrm{Span}\{T(x_1) \dots T(x_n)\} = \mathrm{Span}(S)$,

Now we prove that S is linearly independent.

Suppose that

$$\sum_{i=k+1}^{n} b_i T(x_i) = 0 \quad \text{for } b_{k+1}, \dots b_n \in F.$$

Using the fact that T is linear,

We have

$$T\left(\sum_{i=k+1}^{n} b_i x_i\right) = 0$$

So $\sum_{i=k+1}^{n} b_i x_i \in N(T)$

Hence there exist $c_1 \dots c_k \in F$ such that

$$\sum_{i=k+1}^{n} b_i x_i = \sum_{i=1}^{k} c_i x_i \quad (\text{or}) \quad \sum_{i=1}^{k} (-c_i) x_i + \sum_{i=k+1}^{n} b_i x_i = 0$$

Since β is a basis for V, we have $b_i = 0$ for all i. Hence S is linearly independent.

Theorem: Let V and W be vector space of equal (finite) dimension, and let $T: V \to W$ be linear. Then, T is one-to-one if and only if T is onto.

Proof:

From the dimension theorem
$\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

(Let V and W be vector spaces, and let $T: V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$)

By using above theorem we have T is one-to-one if and only if $N(T) = \{0\}$, if and only if $\text{nullity}(T) = 0$, if and only if $\text{rank}(T) = \dim(V)$, if and only if $\text{rank}(T) = \dim(W)$ and if and only if $\dim(R(T)) = \dim(W)$.

(Let W be a subspace of a finite-dimensional vector space V, then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $W = V$.

By using above theorem this equality is equivalent to $R(T) = W$. the defenition of T being onto.

Problems:

1) Label the following statements as being true or false. For the following V and W are finite-dimensional vector space (over F) and T is a function of V into W.

(a) If $T(x+y) = T(x)+T(y)$, then T is linear

(b) T is one-to-one if and only if $N(T) = \{0\}$

(c) If T is linear, then $T(0_V) = 0_W$

(d) If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$

solution:

(a) NO, consider a map 'f' from C over C to C over C by letting $f(x+iy) = x$

Then we have

$$f(x_1+iy_1+x_2+iy_2) = x_1+x_2$$

But $f(iy) = 0 = if(y) = iy$.

(b) NO, This is right when 'T' is a linear transformation but not right in general.

For example

$$T: R \to R$$

$$x \to x+1$$

It's one-to-one but not $T(x) = 0$ means $x = -1$

(c) Yes. We have $T(0_V) = T(0x) = 0T(x) = 0_W$ for arbitrary $x \in V$.

(d) No. It is $(\dim(V))$. For example, the transformation mapping the real line to $\{0\}$.

(2) For the following $T: R^2 \to R^2$, state why T is not linear.

(a) $T(a_1, a_2) = (1, a_2)$

(b) $T(a_1, a_2) = (a_1, a_1^2)$

(c) $T(a_1, a_2) = (\sin a_1, 0)$

(d) $T(a_1, a_2) = (|a_1|, a_2)$

solution.

(a) In this $T(a_1, a_2) = (1, a_2)$

$$T(0,0) \neq (0,0)$$

so, T is not a linear.

(b) In this $T(a_1, a_2) = (a_1, a_1^2)$

$$T(2(0,1)) = (0,4) \neq 2T(0,1) = (0,2)$$

$\therefore$ T is not a linear.

(c) In this $T(a_1, a_2) = (\sin a_1, 0)$

$$T(2\tfrac{\pi}{2}, 0) = (0,0) \neq 2T(\pi/2, 0) = (2,0)$$

$\therefore$ T is not a linear.

(d) $T(1,0) + (-1,0) = (0,0) \neq T(1,0) + T(0,1) = (2,0)$

$\therefore$ T is not a linear.

③ The mapping $T: V_3(R) \to V_2(R)$ is defined by $T(x, y, z) = (x-y, x+x)$. Show that T is a linear transformation.

Soln..

Let $\alpha = (x_1, y_1, z_1)$ and

$\beta = (x_2, y_2, z_2)$ be two vectors of

$V_3(R)$. For $a, b \in R$.

$$T[a\alpha + b\beta] = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$$

$$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$= ((ax_1 + bx_2) - (ay_1 + by_2), ax_1 + bx_2 - (az_1 + bz_2))$$

$$= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 - z_1) + b(x_2 - z_2))$$

$$= (a(x_1 - y_1), a(x_1 - z_1)) + (b(x_2 - y_2), b(x_2 - z_2))$$

$$= a(x_1 - y_1, x_1 - z_1) + b(x_2 - y_2, x_2 - z_2)$$

$$= aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$$

$$= aT(\alpha) + bT(\beta)$$

$\Rightarrow T$ is a linear transformation from $V_3(R)$ to $V_2(R)$

④ Is the mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (|x|, 0)$ a linear Transformation.

Soln., We have $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x,y,z) = (|x|, 0)$$

Let $\alpha, \beta \in \mathbb{R}^3$ where

$\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$

For $a, b \in \mathbb{R}$,

$$a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$\therefore T(a\alpha + b\beta) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$
$$= (|ax_1 + bx_2|, 0)$$

and
$$aT(\alpha) + bT(\beta) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$$
$$= a(|x_1|, 0) + b(|x_2|, 0)$$
$$= (a|x_1| + b|x_2|, 0).$$

clearly $T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$

Hence T is not a linear transformation.

5) Describe explicitly the linear transformation $T: R^2 \to R^2$ such that $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$

Soln.,

First we have to show that the vectors $(2,3)$ and $(1,0)$ are linearly Independent.

Let $a(2,3) + b(1,0) = 0$

$\Rightarrow [2a+b,\ 3a+0] = (0,0)$

$\Rightarrow 2a+b=0,\ 3a=0$

$\Rightarrow a=0,\ b=0$

$\therefore S = \{(2,3),(1,0)\}$ is linearly Independent.

Let us prove that $L(S) = R^2$

Let $(x,y) \in R^2$ and

$(x,y) = a(2,3) + b(1,0)$

$= (2a+b,\ 3a)$

$\Rightarrow 2a+b = x\ ,\ 3a = y$

$\Rightarrow a = \frac{y}{3}\ ;\ b = x - 2a$

$= x - 2\left(\frac{y}{3}\right)$

$= \frac{3x-2y}{3}$

$\Rightarrow (x,y) = \frac{y}{3}(2,3) + \frac{3x-2y}{3}(1,0)$

Hence S space R^2.

Now $T(x,y) = T\left[\frac{y}{3}(2,3) + \frac{3x-2y}{3}(1,0)\right]$

$$T(x,y) = \frac{y}{3} T(2,3) + \frac{3x-2y}{3} T(1,0)$$

$$= \frac{y}{3}(4,5) + \frac{3x-2y}{3}(0,0)$$

$$= \frac{4y}{3}, \frac{5y}{3}$$

which is the required transformation.

The Matrix Representation of a Linear Transformation.

Definition:-

Let A be a $m \times n$ matrix defined by $A_{ij} = a_{ij}$ the matrix that represents T is the ordered bases β and γ and will write $A = [T]_\beta^\gamma$

If $V = W$ and $\beta = \gamma$, we write $A = [T]_\beta$

Example:

Define $T: P_3(R) \to P_2(R)$ by $T(f) = f'$

Let β and γ be standard ordered bases for $P_3(R)$ and $P_2(R)$ respectively.

Then
$$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$
$$T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2$$

So
$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Note that the coefficients of $T(x^j)$ when vectors as a linear combination of elements of γ given the entries of the jth column.

Soln, we have

$$T(1,0) = (2,3,1)$$
$$= 2(1,0,0) + 3(0,1,0) + 1(0,0,1)$$

and $T(0,1) = (-1,4,0)$.

$$= -1(1,0,0) + 4(0,1,0) + 0(0,0,1)$$

Hence we get

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$

(b) Similarly $[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

③ Let $T: R^2 \to R^3$ be defined as $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$ compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$ compute $[T]_\alpha^\beta$.

Soln., since $T(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)$

$T(0,1) = (-1,0,1) = -1(1,1,0) + 1(0,1,1) + 0(2,2,3)$

$T(1,2) = (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$

$T(2,3) = (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$

Hence we have $[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$

and $[T]_\alpha^\beta = \begin{pmatrix} -\frac{7}{3} & -11/3 \\ 2 & 3 \\ \frac{2}{3} & 4/3 \end{pmatrix}$

Example:-

Define $T: R^2 \to R^3$ by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let β and γ be the standard ordered bases for R^2 and R^3, respectively. Now

$$T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3$$

and $T(0,1) = (3,0,-4) = 3e_1 + 0e_2 - 4e_3$

Hence $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$

If we let $\gamma' = \{e_3, e_2, e_1\}$ then

$$[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$$

Problems:-

① Let β and γ be the standard ordered bases for R^n and R^m respectively. For the following transformations $T: R^n \to R^m$, compute $[T]_\beta^\gamma$.

(a) $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

(b) $T: R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

Applications of Vector Spaces

Introduction :

Vector spaces have many applications as they occur frequently in common circumstances, namely wherever functions with values in some field are involved. They provide a framework to deal with analytical and geometrical problems, or are used in the Fourier transform.

For example, in optimization. The minimax theorem of game theory stating the existence of a unique payoff when all players play optimally can be formulated and proven using vector spaces methods.

Representation theory transfers the good understanding of linear Algebra and vector spaces to other mathematical domains such as group theory.

Some Aplications of the vector Spaces

1. physics :-

It is easy to highlight the need for linear algebra for physicists - Quantum Mechanics is entirely based on it. Also important for time domain (State Space) control theory and stresses in materials using tensors.

2. Circuit Theory :-

Matrices are used to solve for current or voltage. In electromagnetic field theory which is fundamental course for communication engineering, conception of divergence, curl are important.

3. Computer Graphics :-

Linear operator plays a key role in computer graphics. For many CAD software generates drawing using linear operators. Matrices can be used in Criptography. and MATLAB'S default data type is matrix.

4. Least Square

Least Square estimation is used in digital filter design, tracking (Kalman filters), Control systems, etc.

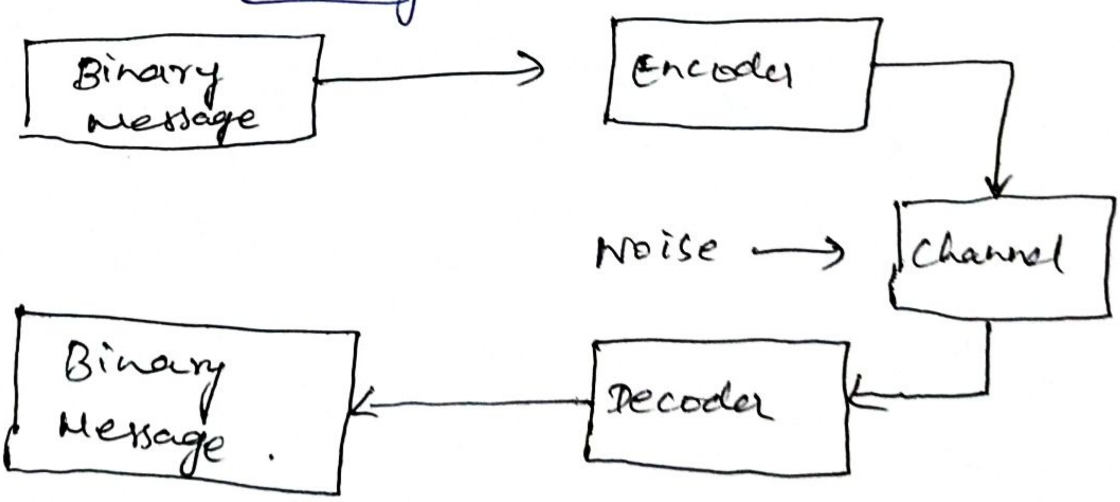
The method of least squares is a standard approach to the approximate solution of overdetermined systems, (i.e) sets of equations in which there are more equation are unknowns;

5. Fourier Analysis :-

The discrete Fourier transform is a finite dimensional example, and the FFT algorithm is just fun to learn about. Continuous time is an infinite dimensional space.

Data Communication Model.

Sending Coded Message.



Encoding :-

To encode a message means to convert the message ~~from~~ to a vector code space.

Decoding :-

To decode a message means to check the message to determine if there has been an error, and if so, correct the error and entract the original message

PART - A

1. Suppose u and v belong to a vector space V, simplify the following expression. $E = 3(2u-4v)+5u+7v$.

2. Simplify the following expression $E = 5u - \frac{3}{v} + 5u$ where $u, v \in V$ a vector space.

3. Let $V = R^3$. Is $W = \{(a,b,c); a \geq 0\}$ a subspace of V?

4. Let $V = R^3$. Is $W = \{(a,b,c) : a^2+b^2+c^2 \leq 1\}$ a subspace of V?

5. Let $V = P(t)$ the vector space of real polynomials. Let W consists of all polynomials with integral coefficients. Is W a subspace of V.

6. Let $P(t)$ be the space of real polynomials. Let W consists of all polynomial with only even powers of t. Is W a subspace of V.

7. Let V be the vector space of function $f: R \to R$. Let $W = \{f(x) : f(1) = 0\}$. Is W a subspace of V.

8. Check whether $u = (1,2)$ and $v = (3,-5)$ are linearly dependent

9. Check whether $u = (1,2,-3)$ and $v = (4,5,-6)$ are linearly dependent.

10. Determine if $u = (1,-3)$ and $v = (-2,6)$ are linearly independent.

11. Determine if $u = (2,4,-8)$ and $v = (3,6,-12)$ are linearly independent.

12. Determine if $u = 2t^2+4t-3$ and $v = 4t^2+8t-6$ are linearly dependent.

13. Determine whether or not u and v are linearly dependent where $u = 2t^2 - 3t + 4$, $v = 4t^2 - 3t + 2$.

14. Show that $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$.

15. Find a basis and dimension of the subspace W of $\mathbb{R}^3$ where $W = \{(a, b, c) : a + b + c = 0\}$

16. Find a basis and dimension of the subspace W of $\mathbb{R}^3$ where $W = \{(a, b, c) : (a = b = c)\}$.

17. Let A be a n-square matrix. Show that A is invertible if and only if $\operatorname{rank}(A) = n$.

20. Relative to the basis $S = \{u_1, u_2\} = \{(1,1), (2,3)\}$ of $\mathbb{R}^2$, find the coordinate vector of v, where $v = (4, -3)$.

21. Consider the vector space $P_3(t)$ of polynomials of degree ≤ 3. Check if $S = \{(t-1)^3, (t-1)^2, t-1, 1\}$ is a basis of $P_3(t)$.

22. Consider the vectors $u = (1, 2, 3)$ and $v = (2, 3, 1)$ in $\mathbb{R}^3$, write $w = (1, 3, 8)$ as a linear combination of u and v.

23. Show that the following functions $f = e^t$, $g(t) = \sin t$, $h(t) = t^2$ are linear independent.

24. Consider the mapping $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (yz, x^2)$. Find $F(2, 3, 4)$

25. If $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (xy, x)$ linear?

26. Find the matrix of the linear transformation $F(x, y, z) = (x + y + z, 2x - 3y + 4z)$

27. Consider the linear operator T on $\mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Show that T is invertible.

28. Show that $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x^2, y)$ is not linear.

29. Find a 2×2 matrix A that maps $(1, 3)$ and $(1, 4)$ into $(-2, 5)$ and $(3, -1)$ respectively.

PART B

1. Show that the vectors $u = (1,2,3)$, $v = (2,5,7)$, $w = (1,3,5)$ are linearly independent.

2. Show that the vectors $u = (1,2,3)$, $v = (2,5,7)$ and $w = (1,3,5)$ are linearly independent.

3. Let $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 5 & 6 & 4 & 5 \\ 3 & 7 & 6 & 11 & 6 & 9 \\ 1 & 5 & 10 & 8 & 9 & 9 \\ 2 & 6 & 8 & 11 & 9 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis of row space of A
(b) find a basis of column space of A
(c) Find the rank of A.

4. Consider the real space $\mathbb{R}^3$. Let $u_1 = (1,-1,0)$, $u_2 = (1,1,0)$, $u_3 = (0,1,1)$. Find the coordinates of $v = (5,3,4)$ relative to this basis.

5. Express $v = (1,-2,5)$ in $\mathbb{R}^3$ as a linear combination of vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ and $u_3 = (2,-1,1)$.

6. Express $v = (2,-5,3)$ in $\mathbb{R}^3$ as a linear combination of $u_1 = (1,-3,2)$, $u_2 = (2,-4,-1)$, $u_3 = (1,-5,7)$.

7. Show that the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ and $u_3 = (1,5,8)$ span $\mathbb{R}^3$.

8. Find the condition on a, b, c so that $v = (a,b,c)$ in $\mathbb{R}^3$ belongs to the span (u_1, u_2, u_3) where $u_1 = (1,2,0)$, $u_2 = (-1,1,2)$ and $u_3 = (3,0,-4)$.

9. Determine whether or not the vectors $u = (1,1,2)$, $v = (2,3,1)$ and $w = (4,5,5)$ in $\mathbb{R}^3$ are linearly dependent.

10. Check whether the following sets in $\mathbb{R}^3$ are linearly dependent.
(a) $u_1 = (1,2,5)$, $u_2 = (1,3,1)$, $u_3 = (2,5,7)$, $u_4 = (3,1,4)$.
(b) $u = (1,2,5)$, $v = (2,5,1)$, $w = (1,5,2)$.
(c) $u = (1,2,3)$, $v = (0,0,0)$, $w = (1,5,6)$.

11. Suppose that u, v, w are linearly independent. Show that vectors $u+v$, $u-v$, $u-2v+w$ are also linearly independent.

12. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $u_1 = (1,-2,5,-3)$, $u_2 = (2,3,1,-4)$, $u_3 = (3,8,-3,-5)$. Find a basis and dimension of W.

13. Find the rank and basis of the row space of each of the following matrices.

(i) $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

14. Find the dimension and a basis of the solution space W of each homogeneous system.

(a) $x+2y+2z-s+3t=0$, $x+2y+3z+s+t=0$, $3x+6y+8z+s+5t=0$.

(b) $x+2y+z-2t=0$, $2x+4y+4z-3t=0$, $3x+6y+7z-4t=0$

(c) $x+y+2z=0$, $2x+3y+3z=0$, $x+3y+5z=0$.

15. Relative to the basis $S = \{u_1, u_2\} = \{(1,1),(2,3)\}$ of $\mathbb{R}^2$, find the coordinate vector of v, where (a) $v = (4,-3)$. (b) $v = (5,2)$.

16. Find the coordinate vector of $v = (a,b,c)$ in $\mathbb{R}^3$, relative to the basis $u_1 = (1,1,1)$, $u_2 = (1,1,0)$, $u_3 = (1,0,0)$.

17. Show that $S = \{(t-1)^3, (t-1)^2, t-1, 1\}$ is a basis of $P_3(t)$, vector space of polynomials of degree ≤ 3.

18. Determine whether the following vectors in $\mathbb{R}^4$ are linearly dependent or independent.

(a) $(1,2,-3,1)$, $(3,7,1,-2)$, $(1,3,7,-4)$. (b) $(1,3,1,-2)$, $(2,5,-1,3)$, $(1,3,7,-2)$

19. Check whether $u = t^3-4t^2+3t+3$, $v = t^3+2t^2+4t-1$, $w = 2t^3-t^2-3t+5$ are linearly dependent.

20. Suppose u, v, w are linearly independent. Show that $u+v-2w$, $u-v-w$, $u+w$ is linearly independent.