📘 Linear Algebra Study Guide

(Vector Spaces, Subspaces, Linear Combinations, Basis, Transformations, Null & Range, Dimension Theorem, and Applications)

🧩 1️⃣ Vector Spaces

⚙️ Definition

A **vector space V** over a **field F** is a non-empty set of elements (vectors) that satisfy eight axioms under two operations — vector addition and scalar multiplication.

🧮 Eight Conditions of a Vector Space

No.	Property	Description
1	Closure under addition	If $(u,v \in V)$, then $(u+v \in V)$.
2	Commutative addition	$(u+v=v+u)$
3	Associative addition	$((u+v)+w=u+(v+w))$
4	Existence of zero vector	There exists $(0 \in V)$ such that $(u+0=u)$.
5	Existence of additive inverse	For each $(u \in V)$, $\exists$ $((-u))$ such that $(u+(-u)=0)$.
6	Closure under scalar multiplication	$(c \cdot u \in V)$ for all $(c \in F)$.
7	Distributive property	$(c(u+v)=cu+cv)$.
8	Compatibility of scalar multiplication	((ab)u=a(bu);;1u=u.)

💡 Examples

Example Type	Vector Space	Explanation
Coordinate	(\mathbb{R}^2,\mathbb{R}^3,\mathbb{R}^n)	Tuples ((x_1,…,x_n)) under normal addition/scalar mult.
Polynomials	(P_n)	All polynomials of degree ≤ n
Matrices	(M_{m×n})	All (m×n) real matrices
Functions	(C[a,b])	All continuous functions
Zero Space	({0})	Contains only the zero vector

⚙️ Zero Vector & Inverses

- Zero vector → unique element that leaves every vector unchanged when added.
- Additive inverse → vector which cancels another to yield zero.
- Multiplicative inverse (scalar side) → reciprocal of scalar.

🧠 MCQ Practice

Theory
1️⃣ Vector space must be closed under addition & scalar multiplication ✅
2️⃣ Set of positive integers under addition → not vector space ✅
3️⃣ Zero vector is unique ✅

Numerical → Add vectors, find inverses, check closure.
Practical → Temperature readings, pixel intensities, voltage signals behave as vector spaces.

🧩 2️⃣ **Subspaces**

⚙️ Definition

A **subspace W** of a vector space V is a subset that itself is a vector space under the same operations.

✅ Conditions:
1️⃣ $(0 \in W)$ (contains zero)
2️⃣ Closed under addition
3️⃣ Closed under scalar multiplication

💡 Examples

Subspace	**Description**
Line $(y=2x)$ in $(\mathbb{R}^2)$	Passes through origin → subspace
Plane $(x+y+z=0)$ in $(\mathbb{R}^3)$	Subspace of $(\mathbb{R}^3)$
$(\{0\})$	Zero subspace
Even functions	Subspace of continuous functions
Solutions of $(A x=0)$	Null space → subspace

🧠 Key Properties

Concept	**Meaning**
Intersection of subspaces	Always a subspace
Union of subspaces	Not always a subspace
Sum of subspaces	Always a subspace
Smallest subspace	$(\{0\})$
Largest subspace	V itself

🧮 Examples

1️⃣ $(W=\{(x,y):y=2x\})$ → subspace ✅
2️⃣ $(W=\{(x,y):y=x+1\})$ → not subspace ❌ (fails zero vector)
3️⃣ Plane through origin → subspace ✅

🧠 MCQs

Theory → Subspace must include zero ✅ ; Intersection → always subspace ✅
Numerical → ($y=2x$) → yes ✅ ; ($x+y+z=0$) → 2D subspace ✅
Practical → Null space and planes through origin ✅

🧩 3️⃣ Linear Combinations and Systems of Linear Equations

⚙️ Definition

A **linear combination** of vectors ($v_1,v_2,\dots,v_n$):

$$c_1v_1+c_2v_2+\dots+c_nv_n$$

where (c_i) are scalars.
The set of all such combinations = ($\text{span}\{v_1,\dots,v_n\}$).

💡 Matrix Relation

For matrix A with columns ($a_1,a_2,\dots,a_n$):

$$A x=b \Rightarrow b=x_1a_1+x_2a_2+\dots+x_na_n$$

So (b) is a linear combination of columns of A.
System is consistent iff ($b\in\text{Col}(A)$).

⚙️ Homogeneous Systems

$$A x=0$$

Always consistent (($x=0$) works); solution set = null space → subspace.

🧮 Examples

1️⃣ $\text{Span}\{(1,0),(0,1)\}=\mathbb{R}^2$
2️⃣ $\text{Span}\{(1,2),(2,4)\}$= line ($y=2x$) (dependent)
3️⃣ $A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$, $b=(3,6)$ → consistent ✅

🧠 MCQs

Theory → (b) combination of A's columns ↔ solution exists ✅
Homogeneous always consistent ✅
Practical → regression, circuits, data fitting ✅

🧩 4️⃣ **Linear Independence, Dependence, and Basis**

⚙️ Definition

- Independent → only trivial combination gives 0: $c_1v_1+\ldots+c_nv_n=0 \Rightarrow c_i=0$
- Dependent → some non-zero combination gives 0
- Basis → independent set that spans V
- Dimension → number of basis vectors

💡 Examples

Example	**Independence**	**Dimension**
$\{(1,0),(0,1)\}$ in $\mathbb{R}^2$	Independent	2
$\{(1,2),(2,4)\}$	Dependent	1 (line)
$\{1,x,x^2\}$ in P_2	Independent	3

⚙️ Check for Independence

Method	Criterion
Definition	Solve (c_1v_1+…+c_nv_n=0)
Determinant	(\det A ≠ 0) → independent
Rank test	rank = #vectors → independent
RREF	Pivot in each column → independent
Geometry	Not parallel / coplanar → independent

💡 Relation Summary

Concept	Condition
Independent	Only trivial solution
Dependent	At least one non-trivial solution
Basis	Independent + spans
Dimension	#basis vectors
Span	Subspace generated by vectors

🧠 MCQs

Zero vector → dependent ✅
All bases same #vectors ✅
(\det=0) → dependent ✅
(\text{rank}+\text{nullity}=n) ✅

🧩 5️⃣ Linear Transformations, Null & Range Spaces

⚙️ Definition

A linear transformation (T:V→W) satisfies
(T(u+v)=T(u)+T(v),;T(cu)=cT(u)).

Every linear map ($\mathbb{R}^n \to \mathbb{R}^m$) can be represented by an ($m \times n$) matrix A: ($T(x)=A x$).

💡 Example

($T(x,y)=(x+y,2x-y)$)
($T(e_1)=(1,2)$, $T(e_2)=(1,-1)$)

$$[T]=\begin{bmatrix}1&1\\2&-1\end{bmatrix}$$

🧮 Null and Range

Concept	**Definition**	**Subspace of**	**Dimension**
Null space / Kernel	($\{x : A x=0\}$)	Domain	Nullity
Range / Image	($\{A x : x \in V\}$)	Codomain	Rank

Rank–Nullity Theorem: ($\text{rank}(A)+\text{nullity}(A)=n$)

Example: Projection ($T(x,y,z)=(x,y,0)$)

$$[T]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

→ rank = 2, nullity = 1.

🧠 MCQs

Linear map → additive + homogeneous ✅
Columns of [T] → images of basis vectors ✅
rank = 1 ⇒ nullity = n−1 ✅

🧩 6️⃣ Dimension Theorem (Rank–Nullity)

⚙️ Statement

For linear map $T:V\to W$:

$$\dim(V)=\dim(\ker T)+\dim(\operatorname{im} T)$$

or matrix form $\text{rank}(A)+\text{nullity}(A)=n$.

🧮 Examples

1️⃣ $A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$: rank 1, nullity 1 → 1+1=2 ✅
2️⃣ $A=\text{diag}(1,1,0)$: rank 2, nullity 1 ✅
3️⃣ $T(p)=p'$ on P_3: rank 3, nullity 1 ✅

🧠 Consequences

Property	Meaning
Nullity = 0	Injective (one-to-one)
Rank = dim(W)	Surjective (onto)
Square full rank	Invertible (bijective)
row rank = col rank	Always true

🧩 7️⃣ Matrix Representation of Linear Transformation

⚙️ Procedure

1️⃣ Choose bases for domain & codomain.
2️⃣ Apply T to each basis vector.
3️⃣ Express $T(v_i)$ *in codomain basis.*
4️⃣ *Form matrix columns* → $[T]_{B_W}^{B_V}$.

Example: Differentiation ($T(p)=p'$) in basis ($\{1,x,x^2\}$):

$$
T(1)=0,\;T(x)=1,\;T(x^2)=2x
\Rightarrow
[T]=\begin{bmatrix}
0&1&0\\
0&0&2\\
0&0&0
\end{bmatrix}
$$

🧠 Key Properties

Operation	Matrix Rule
Composition	($[T_2\circ T_1]=[T_2][T_1]$)
Sum	($[T_1+T_2]=[T_1]+[T_2]$)
Inverse	Exists iff A invertible
Change of basis	($[T]'=P^{-1}[T]P$)

🧩 8️⃣ Applications of Vector Spaces

💻 Computer Science

Area	Use
Machine Learning	Feature vectors, embeddings
Graphics	Transformations, projections
Data Compression	Basis transform (PCA, DCT)
Computer Vision	Images as matrices, filters as linear ops
Cryptography	Linear block codes

⚡ Electrical & Electronics

Use	Vector-space aspect
Circuit analysis	Linear system (A x=b)
Signal processing	Function spaces, Fourier basis
Control systems	State-space models
Communications	Coding subspaces

⚙️ Physics

Domain	Vector role
Mechanics	Forces, velocities ((\mathbb{R}^3))
Quantum mechanics	States in Hilbert spaces
Relativity	4-D vector space with metric tensor

📈 Math & Data

Use	Vector idea
PCA	Orthogonal basis transformation
Regression	Span of columns in design matrix
Differential equations	Solutions form subspace

🎨 Visualization

Application	Meaning
Projection	Linear map to subspace
Rotation	Orthogonal linear map
Scaling	Diagonal transformation

🌍 Summary

Vector spaces provide the language of linearity —
everything from sound, image, motion, and data lives in them.

🧾 Final Summary Table

Concept	Definition	Key Idea
Vector Space	Set with addition & scalar mult.	8 axioms
Subspace	Subset closed under ops.	Contains 0
Span	All linear combinations	Generates subspace
Linear Independence	Only trivial combo gives 0	Unique representation
Basis	Independent spanning set	Defines coordinates
Dimension	#basis vectors	Size of space
Linear Transformation	(T(u+v)=T(u)+T(v))	Represented by matrix
Null Space	({x:A x=0})	Lost directions
Range Space	({A x})	Outputs of map
Rank	dim(range)	Info preserved
Nullity	dim(kernel)	Info lost
Dimension Theorem	rank + nullity = dim(domain)	Conservation of dimensions
Matrix Representation	Columns = images of basis	Implements T
Applications	ML, circuits, physics, graphics	Universal framework

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