1 A Quantum Oscillator

1.1 Theory

The Hamiltonian is given by –

$$\hat{H} = \frac{\hbar^2 \hat{L}^2}{2I} + U(\phi) \tag{1}$$

Where, $\hbar\hat{L}$ is the quantum-mechanical angular momentum operator for a rotation around a fixed axis by the angle ϕ. The operator is given by –

$$\hat{L} = -i\frac{\partial}{\partial\phi} \tag{2}$$

And the potential is given by –

$$U(\phi) = Mga(1 - \cos\phi) \tag{3}$$

Where M is the mass, a is the distance between the center of mass and the pivot point, I is the moment of inertia. A plot of $U(\phi)$ is given bellow (used *gnuplot* to plot) –

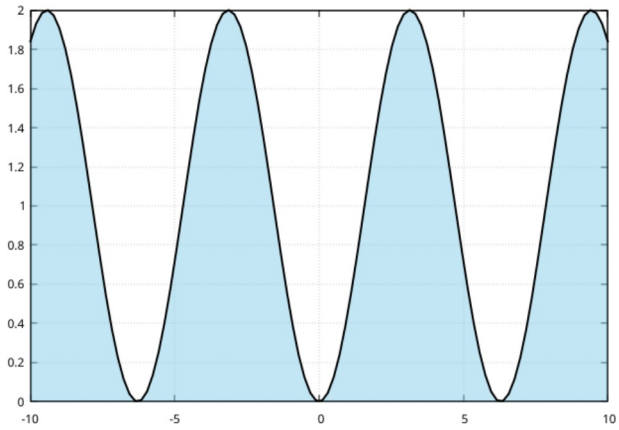

Abbildung 1: Potential $U(\phi)$ as a function of angle ϕ

As it can be seen in fig 1, the potential has a periodicity of 2π. Hence, the wave-function should have a 2π periodicity –

$$\Psi(\phi, t) = \Psi(\phi + 2n\pi, t) \tag{4}$$

Hence we can safely choose $-\pi \leq \phi\pi$, and required $\Psi(-\pi, t) = \Psi(\pi, t)$. A possible basis is:

$$\phi_m = \frac{1}{\sqrt{2\pi}} e^{im\phi} \qquad m = 0, \pm1, \pm2, \pm3... \tag{5}$$

Hence the Hamiltonian matrix elements –

$$\begin{aligned} H_{mn} &= \int_{-\pi}^{\pi} \chi_n^*(\phi)\hat{H}\chi_m(\phi)\, d\phi \\ &= \int_{-\pi}^{\pi} e^{-im\phi}\left[-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2} + Mga(1-\cos\phi)\right] e^{in\phi}\, d\phi \\ &= \left(\frac{\hbar^2 m^2}{2I} + Mga\right)\delta_{mn} - \frac{Mga}{2\pi}\int_{-\pi}^{\pi} e^{-im\phi}\,\cos\phi\, e^{in\phi}\, d\phi \end{aligned} \tag{6}$$

Now, considering $\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}$,

$$H_{mn} = \left(\frac{\hbar^2 m^2}{2I} + Mga\right)\delta_{mn} - \frac{Mga}{2}(\delta_{m(n+1)} + \delta_{m(n-1)}) \tag{7}$$

Using this *Hamiltonian* we solved the problem using *python3*.

1.2 Results

Assuming $M = 1$, $a = 1$, $g = 1$ and $I = 300$, as asked in the question. Also in natural units $\hbar = 1$. The first 300 eigenvalues are calculated and plotted:

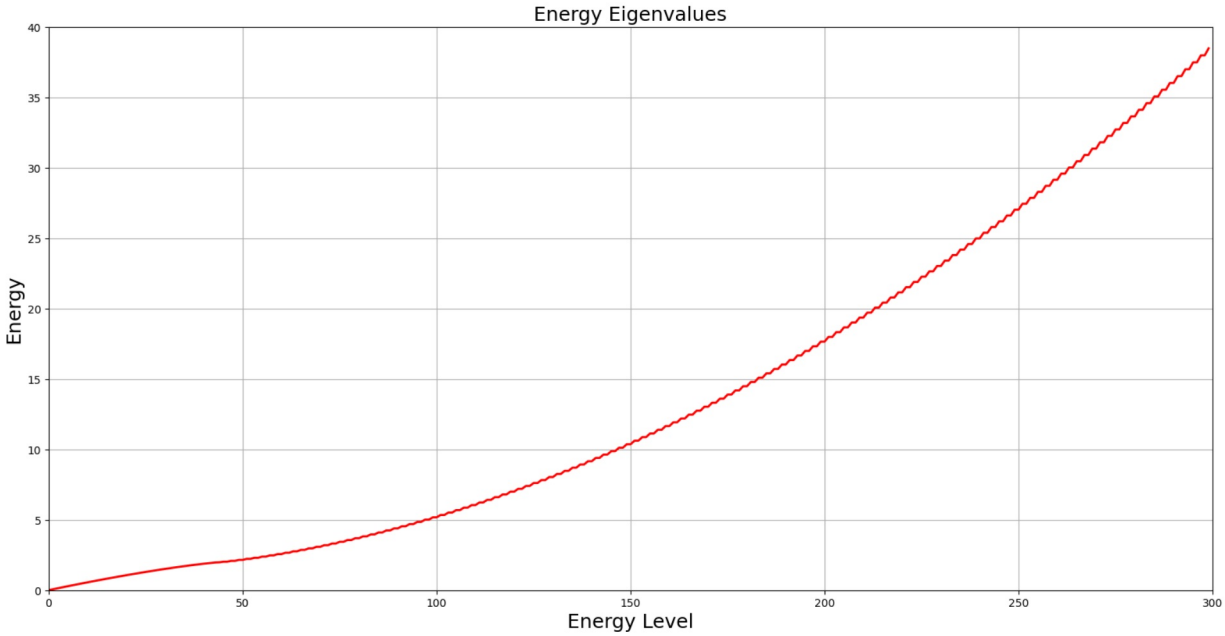

Abbildung 2: Energy Eigenvalues vs Eigenstates

The 1st, 50th and 200th eigenvalues are printed:

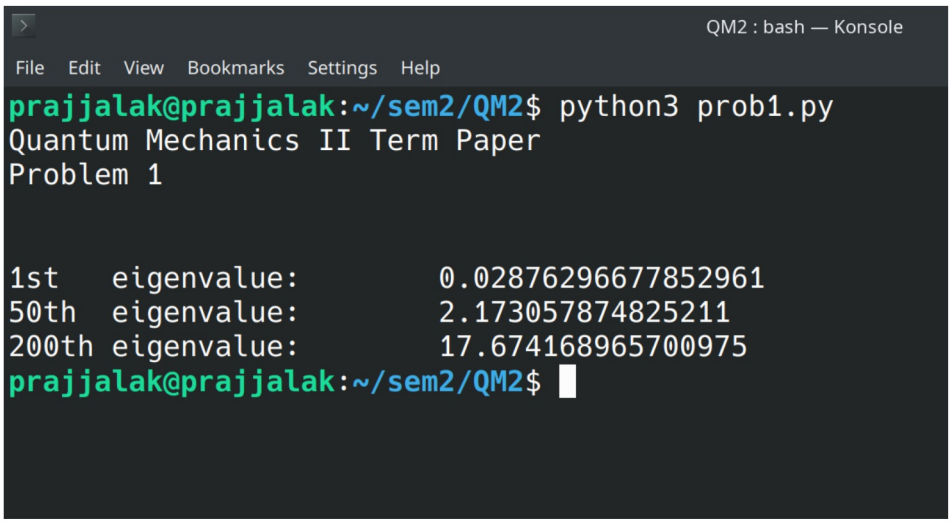

Abbildung 3: Output of the *python* code

The first 80 eigenvalues are plotted to observe the nature closely (zoomed version of fig. 2:

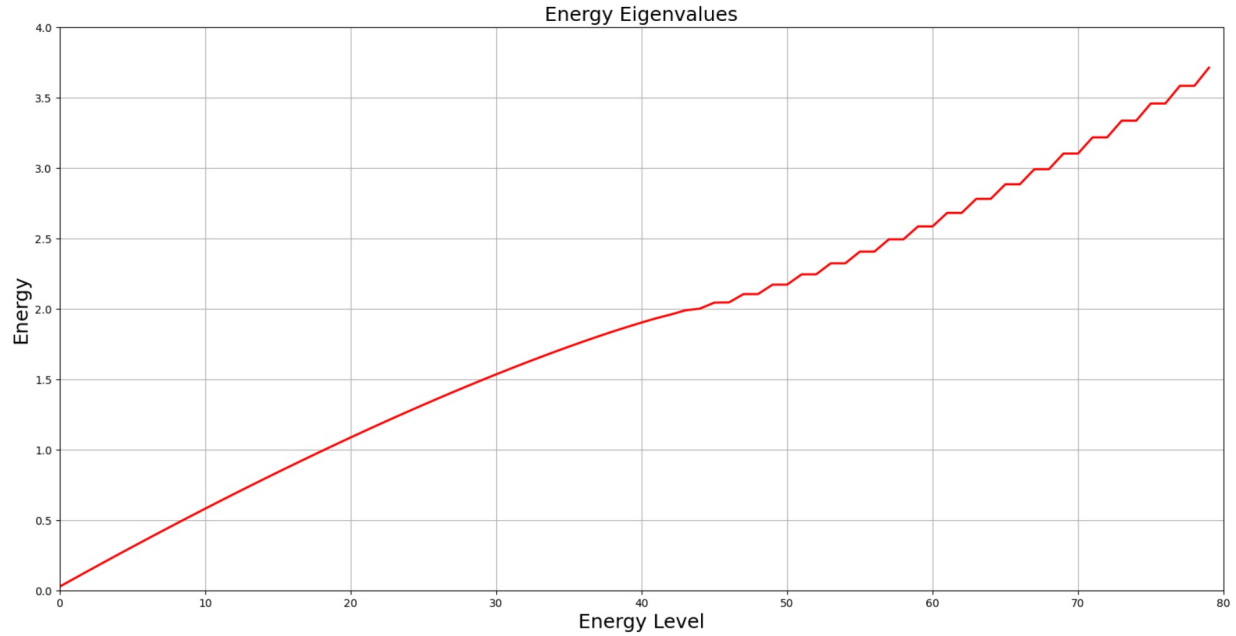

Abbildung 4: Energy Eigenvalues vs Eigenstates (first 80)

From fig. 4, it is clear that the degeneracy starts from roughly 42th state.
And the differences between two consecutive levels are plotted:

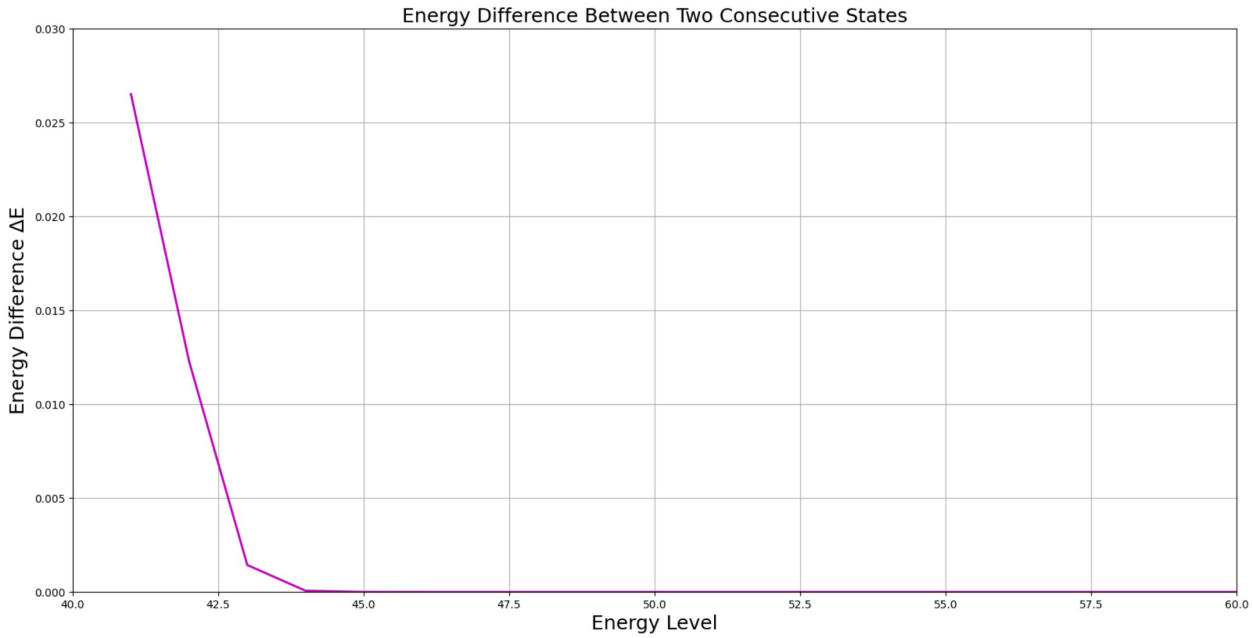

Abbildung 5: ΔE for consecuitive levels

From fig. 5 above, it is clear that, from about 44th level, consecutive eigenvalues are very closed. I think they are degenerate states.