30-08-22

Ajay

Module - II.

Fourier Transforms.

The F.T or infinite F.T of a real valued function $f(x)$ is defined by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$, provided the integral exist. On integration we obtain a fun of u which is usually denoted by $F(u)$ or $\hat{f}(u)$.

The inverse F.T of $F(u)$ is defined by, $$F^{-1}[F(u)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux} du$$

Properties of F.T

(1) If $c_1, c_2, --- c_n$ are constants then
$F[c_1 f_1(x) + c_2 f_2(x) + --- + c_n f_n(x)] =$
$c_1 F[f_1(x)] + c_2 F(f_2(x)] + --- + c_n F[f_n(x)]$

(2) Change the scale property
If $F[f(x)] = \hat{f}(u)$ then $F[f(ax)] = 1/a \hat{f}(u/a)$

3. Shifting property
$F[f(x-a)] = e^{iua} \hat{f}(u)$

4. Modulation property
$$F[f(x)\cos ax] = \frac{1}{2}[\hat{f}(u+a) + \hat{f}(u-a)]$$

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1. Find the F.T of $f(x) = e^{-|x|}$

<u>Soln:-</u> we have by F.T

$$F[f(x)] = \int_{-\infty}^{\infty} e^{iux}\, dx.$$

But,

$$f(x) = e^{|x|} = \begin{cases} e^{-(+x)}, & x \geq 0 \\ e^{-(-x)}, & x < 0 \end{cases}$$

$$F(u) = \int_{-\infty}^{0} e^{x} e^{iux}\, dx + \int_{0}^{\infty} e^{-x} . e^{iux}\, dx$$

$$= \int_{-\infty}^{0} e^{(1+iu)x}\, dx + \int_{0}^{\infty} e^{-(1-iu)x}\, dx$$

$$F(u) = \left[\frac{e^{(1+iu)x}}{(1+iu)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_{0}^{\infty}$$

$$= \frac{1}{1+iu}[1-0] - \frac{1}{1-iu}[0-1]$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu}$$

$$= \frac{1-iu+1+iu}{(1+iu)(1-iu)}$$

$$\begin{cases} e^{0} = 1 \\ e^{-\infty} = 0 \\ e^{\infty} = \infty \end{cases}$$

$$= \frac{2}{1^2 - u^2 i^2}$$

$$\boxed{F(u) = \frac{2}{1+u^2}} \qquad (i^2 = -1)$$

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2. Find the F.T of the function 9

$f(x) = \begin{cases} 1 & \text{for } |x| \leq a \\ 0 & \text{for } |x| \geq a. \end{cases}$

S. Hence, evaluate $\int_0^\infty \frac{\sin x}{x} dx.$

Soln :- we have by F.T

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux} dx.$$

$$= \int_{-\infty}^{-a} \cancel{0 e^{iux}} dx + \int_{-a}^{a} 1 e^{iux} dx + \int_{a}^{\infty} \cancel{0 e^{iux}} dx.$$

$$= \left[\frac{e^{iux}}{iu} \right]_{-a}^{a}$$

$$F(u) = \frac{1}{iu} \left[e^{iua} - e^{-iua} \right]$$

$$F(u) = \frac{1}{iu} \left[\cos au + i\sin au - (\cos au - i\sin au) \right] \qquad \begin{cases} \cos\theta + i\sin\theta = e^{i\theta} \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

$$= \frac{1}{iu} \left[\cancel{\cos au} + i\sin au - \cancel{\cos au} + i\sin au \right]$$

$$= \frac{1}{iu} \left[2i\sin au \right]$$

$$\boxed{F(u) = \frac{2\sin au}{u}}$$

Now, let us evaluate, $\int_0^\infty \frac{\sin x}{x} dx.$

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consider the inverse F.T

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{iux}\, dx.$$

$$f(x) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin au}{u}\,(e^{-iux})\, du$$

Put $x=0$ ($\because x=0$ is a point of continuity as $-a \le x \le a$)

$$f(0) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\,(e^{-iu(0)})\, du.$$

$\langle e^{0} = 1$

$$1 = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\,(1)\, du$$

$$\pi = 2\int_{0}^{\infty} \frac{\sin au}{u}\, du \quad \left(\because \frac{\sin au}{u} \text{ is an even fun}\right)$$

Put $a=1$

$$\int_{0}^{\infty} \frac{\sin u}{u}\, du = \frac{\pi}{2}.$$

$\therefore$ change the variable u to x, we get;

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

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3. Find the F.T of $f(x) = \begin{cases} 1-|x| & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

& hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2} = \frac{\pi}{2}$

Soln:- we have by F.T,

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx.$$

$$= \int_{-\infty}^{-1} 0 + \int_{-1}^{1} [1-|x|] e^{iux}\, dx + \int_{1}^{\infty} 0$$

$$= \int_{-1}^{0} [1-|-x|]\, e^{iux}\, dx + \int_{0}^{1} [1+(-x)] e^{iux}\, dx$$

$$F(u) = \int_{-1}^{0} (1+x) e^{iux}\, dx + \int_{0}^{1} (1-x)\, e^{iux}\, dx$$

$$= \left[(1+x) \frac{e^{iux}}{iu} - (0+1) \frac{e^{iux}}{(iu)^2} \right] + \left[(1-x) \frac{e^{iux}}{iu} - (0-1) \frac{e^{iux}}{(iu)^2} \right]$$

$$= \left[\frac{(1+x)e^{iux}}{iu} - \frac{e^{iux}}{(iu)^2} \right]_{-1}^{0} + \left[\frac{(1-x)e^{iux}}{iu} + \frac{e^{iux}}{(iu)^2} \right]_{0}^{+1}$$

$$= \left[\frac{1}{iu} - \frac{1}{(iu)^2} \right] - \left[(1-1) - \right.$$

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$$= \frac{1}{iu}[1-0] - \frac{1}{i^2u^2}[1-e^{-iu}] + \frac{1}{iu}[0-1] + \frac{1}{i^2u^2}[e^{iu}-1]$$

$$= \frac{1}{iu} + \frac{1}{u^2}(1-e^{-iu}) - \frac{1}{iu} - \frac{1(e^{iu}-1)}{u^2}$$

$$= \frac{1}{u^2}[1-e^{-iu}-e^{iu}+1]$$

$$= \frac{1}{u^2}[2-(e^{-iu}-e^{iu})]$$

$\left\{ \begin{array}{l} 2\cos\theta = e^{\theta} + e^{-i\theta} \\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} \end{array} \right.$

$$= \frac{1}{u^2}[2-2\cos u]$$

$$= \frac{2[1-\cos u]}{u^2}$$

$$= \frac{2\times 2\sin^2(u/2)}{u^2} \quad \left(\because 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right) \right.$$

$$\boxed{F(u) = \frac{4\sin^2(u/2)}{u^2}}$$

Now, let us evaluate $\int_{0}^{\infty} \frac{\sin^2 t}{t^2}$

we have by inverse F.T,

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$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2(u/2)}{u^2} e^{-iux}\, du.$$

let us put $x=0$ ($\because x$ is a point of continuity only $(-1 \le x \le 1)$ & $f(0)=1$)

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2(u/2)\, du}{u^2/4}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2(u/2)\, du}{(u/2)^2}$$

put $\frac{u}{2} = t \Rightarrow u = 2t.$

$du = 2dt.$

$$1 = \frac{\cancel{2}\times 1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt.$$

$\left(\frac{\sin^2 t}{t^2} \right.$ is an even fun.

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt.$$

$$= \frac{1}{\pi} \times 2\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dx.$$

$$\therefore \boxed{\int_0^{\infty} \frac{\sin^2 t}{t^2} = \frac{\pi}{2}.}$$

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Fourier sine & cosine transform:

Fourier cosine transform:

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx .$$

Inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(u) \cos ux \, dx .$$

Fourier sine transform

$$F_S(u) = \int_0^\infty f(x) \sin ux \cdot dx .$$

~~F~~ Inverse F.S sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(u) \sin ux \, dx$$

Assign 1. Find the Fourier sine & cosine transform of $f(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$

Soln:- we have,

$$F_S(u) = \int_0^\infty f(x) \sin ux \, dx .$$

$$F_c(u) = \int_0^\infty f(x) \cdot \cos ux \, dx .$$

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i.e, $F_S(u) = \int_0^2 x \sin ux\, dx + \int_2^\infty 0$

$= \int_0^2 x \sin ux\, dx$

Apply Bernoli's rule of integration.

$F(S)u = \left[x \frac{(-\cos ux)}{u} - (1) \frac{(-\sin ux)}{u^2} \right]_0^2$

$= \frac{1}{u} \{ \cancel{2} [-2\cos 2u - 0] + \frac{1}{u^2} [\sin 2u - 0]$

$= \frac{-2\cos 2u}{u} + \frac{\sin 2u}{u^2}$ or

$$\boxed{F_S(u) = \frac{-2u\cos 2u + \sin 2u}{u^2}.}$$

$|||^{ly}$,

$F_C(u) = \int_0^\infty f(x) \cos ux\, dx$

$= \int_0^2 x \cos ux\, dx$

$F_C(u) = \left[x \frac{\sin ux}{u} - (1) \frac{(-\cos ux)}{u^2} \right]_0^2$

$= \frac{1}{u} [2\sin 2u - 0] + \frac{1}{u^2} [\cos 2u - \cos 0]$

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$$= \frac{2\sin 2u}{u} + \frac{\cos 2u}{u^2} - \frac{1}{u^2}$$

$$\boxed{F_c(u) = \frac{2u\sin 2u + \cos 2u - 1}{u^2}.}$$

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2. Find the Fourier sine transform of $f(x) = e^{-|x|}$ & hence evaluate

$$\int_0^\infty \frac{x \sin mx}{1+x^2}, \quad m>0.$$

Soln: We have by Fourier sine transform,

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$= \int_0^\infty e^{-|x|} \sin ux \, dx$$

$$= \int_0^\infty e^{-(+x)} \sin ux \, dx \quad (\because |x| = +x \text{ for } x>0)$$

$$F_s(u) = \int_0^\infty e^{-x} \sin ux \, dx.$$

FORM:-

use $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a\sin bx - b\sin ax]$

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$a = -1 \quad b = u$

$$F_s(u) = \left\{ \frac{e^{-x}}{(-1)^2 + u^2} \left[-\sin ux - u\cos ux \right] \right\}_0^{\infty}$$

$$= \frac{1}{1+u^2} \left\{ 0 - e^0(-\sin 0 - u\cos 0) \right\}$$

$\{ \because e^{-\infty} = 0, \; \cos 0 = 1, \; \sin 0 = 0.$

$$= \frac{1}{1+u^2} \left[-(0 - u) \right]$$

$$F_s(u) = \frac{u}{1+u^2}$$

Now let us evaluate $\int_0^{\infty} \frac{x \sin mx}{1+x^2}$

we have by inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(u) \cdot \sin ux \, dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{u}{1+u^2} \cdot \sin ux \, du$$

Put $x = m$, we get,

$$\int_0^{\infty} \frac{u \sin mu \, du}{1+u^2} = \frac{\pi}{2} f(m)$$

$$\int_0^{\infty} \frac{u \sin mu \, du}{1+u^2} = \frac{\pi}{2} \times e^{-(m)}$$

change the variable u to x we get,

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2} e^{-m}.$$

3. Find the inverse Fourier sine transform of $\hat{f}(\alpha) = \frac{1}{\alpha} e^{-a\alpha}$, $a > 0$

soln:- we have by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty f_s(\alpha) \sin \alpha x \, d\alpha$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} e^{-a\alpha} \sin \alpha x \, d\alpha.$$

let us use Leibnitz rule of differentiation under integral sign (i.e, diff w.r.t x)

$$\frac{d}{dx}[f(x)] = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} e^{-a\alpha} \frac{\partial}{\partial x}[\sin \alpha x]\, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\cancel{\alpha}} e^{-a\alpha} \cos \alpha x \cdot \cancel{\alpha}\, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty e^{-a\alpha} \cos \alpha x \, d\alpha$$

< FORM: $\int e^{ax} \cos \alpha \, d\alpha = \frac{e^{a\alpha}}{a^2+b^2}[a \cos b\alpha + b \sin b\alpha]$

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$a = -a \quad b = x$

$$= \frac{2}{\pi}\left\{\frac{e^{-a\alpha}}{(-a)^2+x^2}\left[-a\cos\alpha x + x\sin\alpha x\right]\right\}_0^\infty$$

$$= \frac{2}{\pi} \times \frac{1}{a^2+x^2}\left[0 - e^0\left[-a(1)+0\right]\right]$$

$$\frac{d[f(x)]}{dx} = \frac{2}{\pi} \times \frac{a}{a^2+x^2}$$

integrate w.r.t x

$$\int \frac{d}{dx}[f(x)] = \frac{2}{\pi}\int \frac{a}{a^2+b^2}\, dx$$

$$f(x) = \frac{2a}{\pi} \times \frac{1}{a}\tan^{-1}(x/a)$$

$$\boxed{\therefore f(x) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right)} \qquad \left\langle \frac{1}{a}\tan^{-1}(x/a) = \int \frac{dx}{x^2+a^2}\right.$$

4. Solve the integral eqn $\int_0^\infty f(x)\cos\alpha\, dx = e^{-ax}$

Soln: we have by inverse cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha$$

By given data $F_c(\alpha) = e^{-a\alpha}$

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$\therefore f(x) = \frac{2}{\pi} \int_0^\infty e^{-a\alpha} \cos \alpha x \, d\alpha$

$\int e^{a\alpha} \cos b\alpha \, d\alpha = \frac{e^{ax}}{a^2+b^2} [a \cos b\alpha + b \sin b\alpha]$

$a = -a \quad b = x.$

$= \frac{2}{\pi} \left\{ \frac{e^{-a\alpha}}{(-a)^2+x^2} [-a \cos \alpha x + x \sin \alpha x] \right\}_0^\infty$

$= \frac{2}{\pi} \times \frac{1}{a^2+x^2} (0 - e^0 (-a(1) + 0))$

$= \frac{2}{\pi} \times \frac{a}{a^2+x^2}.$

$$\boxed{f(x) = \frac{2a}{\pi(a^2+x^2)}}$$

Complex Line Integral.

Let $f(z) = u(x,y) + iv(x,y)$ be a complex valued function defined over a region R and C any curve in region then,

$$\int_C f(z) dz = \int_C (u dx - v dy) + i \int_C v dx + u dy$$

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$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx + iudy + ivdx - vdy$$

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C vdx + udy$$

Properties of complex integral:-

1. If c denotes the curve traversed from q to p then,

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

2. If c is split into a number of parts C_1, C_2, C_3 ---- then,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \text{-----}$$

Cauchy's theorem:

State:- If $f(z)$ is analytic at all points inside and on a simple closed curve C then,

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$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

we have by Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Applying this theorem to the above eqn we get,

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

since $f(z)$ is analytic we have by cauchy's Rieman (C-R) eqn's

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\therefore \int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

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$\therefore \int_C f(z)\,dz = 0.$

This proves the Cauchy's theorem.

Remark:

Consequence of cauchy's theorem:-

1. If $f(z)$ is analytic in a region R and if P & Q are any two points in it then,

$$\int_P^Q f(z)\,dz$$ is independent of the path joining P & Q.

2. If C_1 & C_2 are two simple closed curves such that C_2 lies entirely within C_1 then,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

IMP Cauchy's Integral Formula:-

Statement: If $f(z)$ is analytic inside in a simple closed curve C and if 'a' is any point within C then,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz.$$

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since, a is a point within c, we shall enclose it by a circle c1 with $z=a$ as a center & r as a radius. such that, c1 lies entirely within c.

The function $\frac{f(z)}{z-a}$ is analytic.

it is inside and on the boundary of annual region (b/w c1 to c) b/w c1 & c.

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C1} \frac{f(z)}{(z-a)}\,dz \quad \text{--- (1)}$$

The eqn of c ~~at~~ (of circle with centre a & radius r) can be written in the form,

$$|z-a| = r$$

$$z-a = re^{i\theta}$$

$$z = a+re^{i\theta}$$

$$\boxed{dz = ire^{i\theta}\,d\theta}$$

Now, the RHS of eqn (1) becomes,

$$\int_C \frac{f(z)}{(z-a)}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a_0 + re^{i\theta})}{re^{i\theta}}\, i re^{i\theta}\, d\theta$$

$$\int_C \frac{f(z)}{(z-a)}\,dz = i\int_{\theta=0}^{2\pi} f(a + re^{i\theta})\, d\theta$$

$$\int_C \frac{f(z)}{(z-a)}\,dz = i\int_{0}^{2\pi} f(a + re^{i\theta})\, d\theta.$$

This is true for any r>0.
However rt is small,
Hence, as r→0,
we get,

$$= i\int_0^{2\pi} f(a)\,d\theta$$

$$= i\, f(a)\,[2\pi - 0]$$

$$\int \frac{f(z)}{(z-a)}\,dz = \cancel{d}\,2\pi i\, f(a)$$

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$$\boxed{\therefore f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)}\, dz}$$

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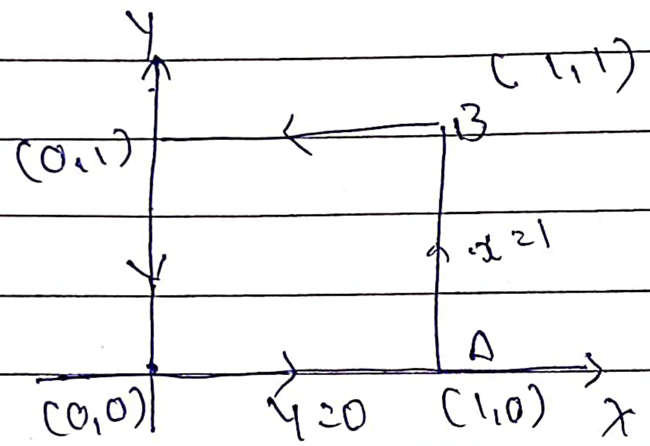
(1) verify cauchy's theorem for :- $f(x) = z^2$ where C is the square having the vertices (0,0), (1,0), (1,1), (0,1)



Here, C is the square OABC. & we have by cauchy's theorem

$$\int_C f(z)\,dz = 0$$

$$\int_C z^2\,dz = 0$$

we have to show that,

$$\int_{OA} z^2\,dz + \int_{AB} z^2\,dz + \int_{BC} z^2\,dz + \int_{CO} z^2\,dz = 0$$

Along OA $= y = 0 \Rightarrow dy = 0$; $0 \le x \le 1$

$$z^2\,dz = (x+iy)^2\,(dx + i\,dy)$$

$$= (x+0)^2\,(dx + i(0))$$

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$z^2 dz = x^2 dx$

$\therefore \int_{OA} z^2 dz = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1$

$= \left[\frac{1}{3} - 0\right]$

$= \int_{OA} z^2 dz = \frac{1}{3}$ —— (1)

Along AB: $x = 1 \Rightarrow dx = 0$; $0 \le y \le 1$

$z^2 dz = (x+iy)^2 (dx + idy)$

$= (1+iy)^2 (0 + idy)$

$\therefore \int_{AB} z^2 dz = \int_{y=0}^{1} (1+iy)^2 dz = \int_{y=0}^{1} ((1+(iy)^2 + 2iy) \, idy$

$= i\left[y - \frac{y^3}{3} + 2i \times \frac{y^2}{2}\right]_0^1$

$= i\left[\left(1 - \frac{1}{3} + i^2\right) - 0\right]_0^1$

$= i\left[\frac{2}{3} + i\right]_0^1$

$= \frac{2}{3} i + i^2$

$\int_{AB} z^2 dz = \frac{-1 + i \cdot 2}{3}$ —— (2)

Along BC : $y=1 \Rightarrow dy=0$; $1 \geq x \geq 0$

$$z^2 dz = (x+iy)^2 (dx+idy)$$
$$= [x+i(1)]^2 [dx+0]$$
$$= (x+i)^2 dx$$

$$\int_{BC} z^2 dz = \int_{x=1}^{0} (x^2+i^2+2ix)\, dx.$$

$x=1$

$$= \left[\frac{x^3}{3} - x + 2i\frac{x^2}{2}\right]_1^0$$

$$= \left[0-0+0-\left(\frac{1}{3}-1+i(1)^2\right)\right]$$

$$= -\left(\frac{-2}{3}+i\right)$$

$$\int_{BC} z^2 dz = \frac{2}{3} - i$$

Along CO : $x=0 \Rightarrow dx=0$; $1 \geq y \geq 0$

$$z^2 dz = (x+iy)^2 (dx+idy)$$
$$= (0+iy)^2 (0+idy)$$
$$= i^2 y^2 (0+idy)$$
$$= i^2 y^2 i dy$$
$$z^2 dz = -iy^2 dy$$

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$$\therefore \int_{CO} z^2 dz = \int_{y=1}^{0} -i y^2 dz$$

$$= -i\left[\frac{y^3}{3}\right]_1^0$$

$$= -i\left[\frac{0-1}{3}\right]$$

$$\int_{CO} z^2 dz = \frac{i}{3} \quad \text{--- (4)}$$

Add eqn (1) (2) (3) & (4) we get,

$$\int_{OA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dc + \int_{CO} z^2 dx$$

$$= \frac{-1+i\cdot 2}{3} + \frac{1}{3} + \frac{2}{3} - i + \frac{i}{3}$$

$$= \frac{-3+2i+1+2-3i+i}{3}$$

$$= \frac{-3+3+3i-3i}{3}$$

$$= 0$$

3. verify cauchy's theorem, $f(z) = ze^{-z}$ over the unit circle with origin as a centre as origin & radius = 1.

$|z| = 1$

$z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$, $0 \le \theta \le 2\pi$

$$\therefore \int_c ze^{-z} dz = \int_{\theta=0}^{2\pi} e^{i\theta} e^{-e^{i\theta}} \cdot i \cdot e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} e^{2i\theta} \cdot e^{-e^{i\theta}} d\theta$$

Put $e^{i\theta} = t \Rightarrow i \cdot e^{i\theta} d\theta = dt$

$\Rightarrow d\theta = \frac{dt}{ie^{i\theta}} = \frac{dt}{it}$.

where,

$\theta = 0 \Rightarrow t = e^0 = 1$

when, $\theta = 2\pi \Rightarrow t = e^{i2\pi} = \cos 2\pi + \sin 2\pi (i)$

$= 1 + i(0)$

$\boxed{t = 1}$

eqn (1) $\Rightarrow$

$$\int_c ze^{-z} dz = i \int_{t=1}^{1} t^2 e^{-t} \times \frac{dt}{it}$$

$$\int_C^{t=1} z e^{-z} dz = \int_{t=1}^{1} t \cdot e^{-t} dt .$$

Since, both the limits are same. The value of the integral is zero.

$$\therefore \int_C z e^{-z} dz = 0 .$$

Hence, the theorem is verified.

14-09-22

Problems on Cauchy's integral formula.

Working procedure:-

i) We need to evaluate the integral of the form, $\int_C \frac{f(z)}{(z-a)} dz$ over a given closed curve C.

ii) Firstly, we have to find out, whether the point $z=a$ lies inside or outside the given curve.

iii) If $z=a$ is inside C then, we use the Cauchy's integral formula in the form, $\int \frac{f(z)}{(z-a)} dz = 2\pi i . f(a)$

(v) If $z=a$ is outside c, then the value of the integral is 0.

1. Evaluate $\int_c \frac{e^z}{z-i\pi}$ where, c is the circle (a) $|z| = 2\pi$ & (b) $|z| = \pi/2$

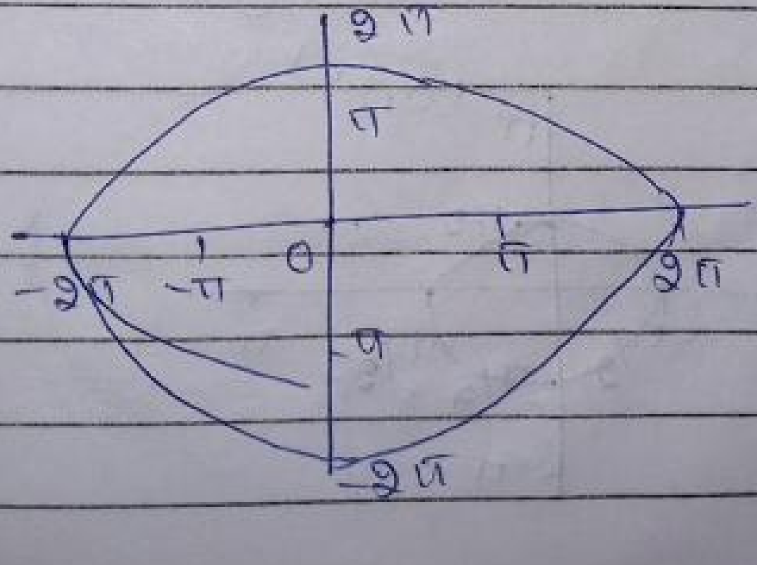
soln:-

we need to evaluate the integral

$$\int_c \frac{e^z}{(z-i\pi)} dz$$

Here, $z=a=i\pi = (0,\pi)$

(a) $|z| = 2\pi$ is the eqn of circle



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∴ The point $(0,\pi)$ is lies inside C. we have by cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)}\, dz = 2\pi i \cdot f(a)$$

$$\int_C \frac{e^z}{(z-a)}\, dz = 2\pi i \cdot f(i\pi)$$

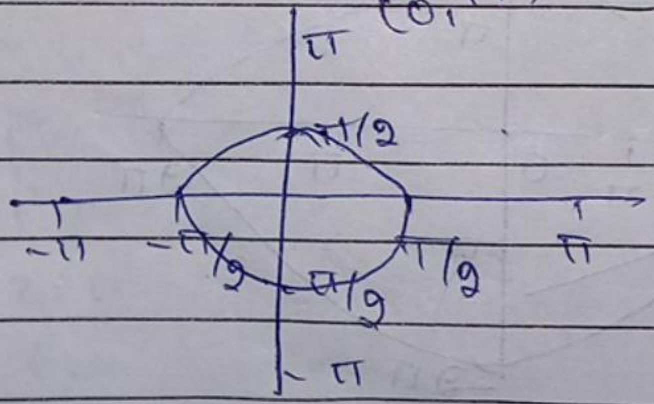
$$= 2\pi i \cdot e^{i\pi}$$

$$= 2\pi i\,[\cos\pi + i\sin\pi]$$

$$= 2\pi i\,[-1 + i(0)] \quad [\because \cos\pi = -1,\ \sin\pi = 0]$$

$$\boxed{\int_C \frac{e^z}{(z-a)}\, dz = -2\pi i}$$

(b) $|z| = \pi/2$ is a eqn of a circle with centre as origin & radius $= \pi/2$.



The point $(0, \pi)$ lies outside the given curve.

Hence, the integral value is zero

i.e, $\int_C \frac{e^z}{(z-i\pi)} dz = 0.$

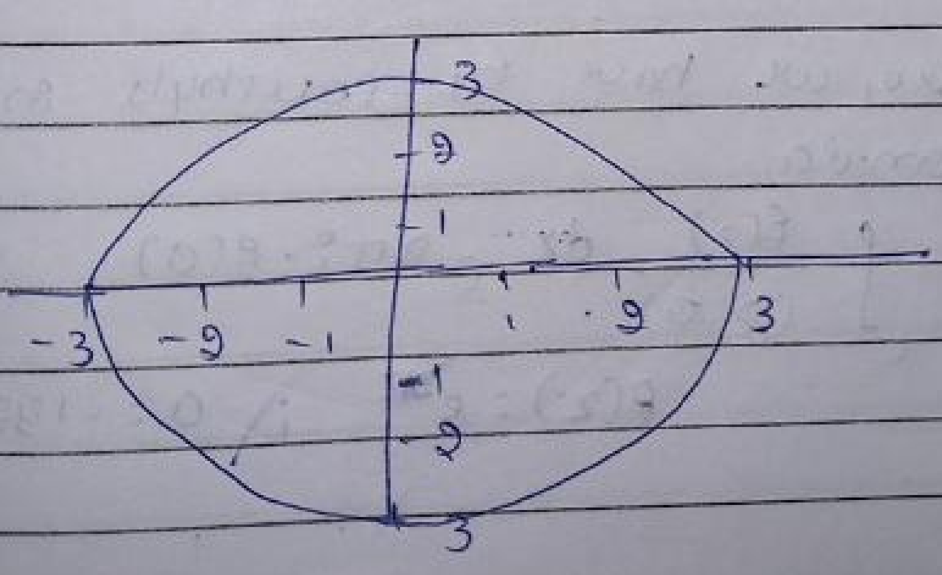
2. Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)} dz$ where C is the circle $|z| = 3$

we need to evaluate,

$$\int_C \frac{e^{2z}}{[(z-(-1)](z-2)}$$

Here, $z = a = -1$ & 2 i.e. $(-1, 0)$ & $(2, 0)$

Given, $|z| = 3$ is a circle with centre as origin & radius $= 3$.



Let us Resolve into partial fraction,

$$\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$$

$$\frac{1}{\cancel{(z+1)(z-2)}} = \frac{A(z-2)+B(z+1)}{\cancel{(z+1)(z-2)}}$$

$$1 = A(z-2) + B(z+1)$$

Put $z=2$; $1 = A(0) + B(2+1)$

$$1 = 3B$$

$$\boxed{B = 1/3}$$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{1}{3(z+1)} + \frac{1}{3(z-2)}$$

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{1}{3}\left[-\int \frac{e^{2z}}{(z+1)} dz + \right. \quad a=-1$$

$$\int_C \frac{e^{2z}}{(z-2)} dz. \quad a=2$$

Now, we have by cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)} dz = 2\pi i \cdot f(a)$$

$$f(z) = e^{2z} \quad ; \quad a = -1 \& 2.$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = \frac{1}{3}\left[-2\pi i \cdot f(-1) + 2\pi i \cdot f(2)\right]$$

$$= \frac{1}{3}\left[-2\pi i \cdot e^{2(-1)} + 2\pi i \cdot e^{2(2)}\right]$$

$$= \frac{2\pi i}{3}\left[-e^{-2} + e^{4}\right]$$

$$\boxed{\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = \frac{2\pi i}{3}\left[e^{4} - \frac{1}{e^{2}}\right]}$$