Module - V

If X and Y are two district random variables, we define the joint probability function of X and Y by
$P(X=x, Y=y) = f(x,y)$.
where, $f(x,y)$ satisfies the condition
$f(x,y) \geq 0$
$\sum_x \sum_y f(x,y) = 1$

Suppose $X = \{x_1, x_2, ---- x_n\}$
$Y = \{y_1, y_2, ---- y_n\}$

Then $P(X = x_i, Y = y_j) = f(x_i, y_j)$ is denoted by J_{ij} and set of values of this function $f(x_i, y_j) = J_{ij}$ is called joint probability of distribution of X + Y, and this values are presented in the form of two way table as follows:

X \ Y	y_1	y_2	- - -	y_n	sum
x_1	J_{11}	J_{12}	- - -	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	- - -	J_{2n}	$f(x_2)$
⋮	⋮	⋮	⋮	⋮	⋮
x_m	J_{m1}	J_{m2}	- - -	J_{mn}	$f(x_m)$
sum	$g(y_1)$	$g(y_2)$	- - -	$g(y_n)$	1

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Marginal Probability distribution:-

In the joint probability table $f(x_1), f(x_2), \ldots f(x_n)$ represents the sum of all the entries in the first row, second row upto n^{th} row and $g(y_1), g(y_2) \ldots g(y_n)$ represents the sum of all the entries in the first column, second column upto n^{th} column and these are called marginal probability distribution of x and y.

* Independent Random Variables:-

The district random variables x and y are said to be independent if these entries J_{ij} in the table is equal to the product of its marginal entries i.e., $f(x_i) \cdot g(y_j) = J_{ij}$.

otherwise x and y are said to be dependent.

Expectation, Variance, Covariance + correlation:

If x and y are two district random variables having the joint probability function $f(x, y)$ then the expectation (mean) of x and y are defined as follows.

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$$\mu_x = E(X) = \sum x_i \cdot f(x_i)$$
$$\mu_y = E(Y) = \sum y_j \cdot g(y_j)$$
Further, $E(X,Y) = \sum x_i \cdot \sum y_j \cdot J_{ij}$

} Mean

variance(σ^2) = $\sigma_x^2 = E(X^2) - \mu_x^2$

$$\sigma_y^2 = E(Y^2) - \mu_y^2$$

where, $E(X^2) = \sum x_i^2 \cdot f(x_i)$

$$E(Y^2) = \sum y_j^2 \cdot g(y_j)$$

covariance: $\mathrm{cov}(X,Y) = E(XY) - \mu_x \cdot \mu_y$

correlation: $\rho(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\sigma_x \sigma_y}$

Note: If X and Y are independent random variables then

i) $E(XY) = E(X) \cdot E(Y)$

ii) $\mathrm{cov}(X,Y) = 0$

$\Rightarrow \rho(X,Y) = 0$

1) The joint distribution of two random variables X and Y is also follows

	y_1	y_2	y_3
X \ Y	-4	2	7
x_1 1	1/8	1/4	1/8
x_2 5	1/4	1/8	1/8

Compute the following

a) $E(X)$ and $E(Y)$, b) $E(XY)$

c) σ_X and σ_Y, d) $COV(X, Y)$

e) $\rho(X, Y)$.

Solⁿ: (The marginal distribution are obtained by adding the each row and columns entries). The Marginal distribution of X and Y are as follows.

Distribution of X

x_i	1	5
$f(x_i)$	1/2	1/2

Distribution of Y

y_j	-4	2	7
$g(y_j)$	3/8	3/8	1/4

a) $E(X) = \Sigma x_i \cdot f(x_i)$

$= (1)(1/2) + 5(1/2)$

$= \frac{6}{2} = 3$

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$E(Y) = \mu_y = \sum Y_j . g(Y_j)$

$= (-4)\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 7\left(\frac{1}{4}\right)$

$E(Y) = 1$

b) $E(XY) = \sum x_i . y_j . J_{ij}$

$= 1(4)\left(\frac{1}{8}\right) + 1(2)\left(\frac{1}{4}\right) + (1)(7)\left(\frac{1}{8}\right)$

$+ 5(-4)\left(\frac{-1}{4}\right) + 5(2)\left(\frac{1}{8}\right) + 5(7)\left(\frac{1}{8}\right)$

$\therefore E(XY) = 10.5$

c) $\sigma_x^2 = E(X^2) - \mu_x^2$

Here, $E(X^2) = \sum x_i^2 . f(x_i)$

$= (1)^2\left(\frac{1}{2}\right) + 5^2\left(\frac{1}{2}\right)$

$E(X^2) = \frac{26}{2} = 13$

$\therefore \sigma_x^2 = 13 - (3)^2$

$= 13 - 9$

$\sigma_x^2 = 4$

$\sigma_x = \sqrt{4}$

$\underline{\underline{\sigma_x = 2}}$

|||ly $\sigma_Y^2 = E(Y^2) - \mu_y^2$

Here, $E(Y^2) = \sum Y_j^2 . g(Y_j)$

$E(Y^2) = (-4)^2\left(\frac{3}{8}\right) + (2)^2\left(\frac{3}{8}\right) + (7)^2\left(\frac{1}{4}\right)$

$$E(Y^2) = 6 + \frac{3}{2} + \frac{49}{4}$$

$$= \frac{79}{4}$$

$$E(Y^2) = 19.75$$

$$\therefore \sigma_Y^2 = 19.75 - (1)^2$$

$$\sigma_Y^2 = 18.75$$

$$\sigma_Y = \sqrt{18.75}$$

$$\underline{\underline{\sigma_Y = 4.33}}$$

d) $\text{cov}(X, Y) = E(XY) - \mu_x . \mu_y$

$$= 1.5 - (3)(1)$$

$$= 1.5 - 3$$

$$= -1.5$$

e) $\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x . \sigma_y}$

$$= \frac{-1.5}{2(4.33)}$$

$$\rho(X, Y) = \frac{-1.5}{8.66}$$

$$\rho(X, Y) = -0.1732$$ //

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Q) The joint probability distribution table for two random variables X and Y is as follows.

$X \backslash Y$	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal probability distribution of X and Y, also construct
a) Expectations of X, Y and XY
b) S.Ds of X, Y
c) Covariance of X and Y
d) Correlation of X and Y.

Soln:

x_i	1	2
$f(x_i)$	0.6	0.4

y_j	-2	-1	4	5
$g(y_j)$	0.3	0.3	0.1	0.3

a) $E(X) = \Sigma x_i \cdot f(x_i)$
$= 1(0.6) + 2(0.4)$
$= 1.4$

$E(Y) = \Sigma y_j \cdot g(y_j)$
$= -2(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3)$
$= -0.6 - 0.3 + 0.4 + 1.5$
$= 1$

$$E(XY) = (1)(-2)(0.1) + (1)(-1)(0.2) + 1(4)(0)$$
$$+ 1(5)(0.3) + 2(-2)(0.2) + 2(-1)(0.1)$$
$$+ 2(4)(0.1) + 2(5)(0)$$

$$E(XY) = -0.2 - 0.2 + 1.5 - 0.8 - 0.2 + 0.8$$
$$= -0.6 + 1.5$$
$$\therefore E(XY) = 0.9$$

b) $\sigma_x^2 = E(X^2) - \mu_x^2$

$$E(X^2) = \Sigma x_i^2 . f(x_i)$$
$$= (1)^2(0.6) + 2^2(0.4)$$
$$= 0.6 + 1.6$$
$$= 2.2$$

$$\sigma_x^2 = 2.2 - (1.4)^2$$
$$= 2.2 - 1.96$$
$$\sigma_x^2 = 0.24$$
$$\sigma_x = 0.489$$

$$\therefore \sigma_Y^2 = E(Y^2) - \mu_j^2$$

$$E(Y^2) = \Sigma y_j^2 . g(y_j)$$
$$= (-2)^2(0.3) + (-1)^2(0.3) + (4)^2(0.1)$$
$$+ 5^2(0.3)$$
$$= 1.2 + 0.3 + 1.6 + 7.5$$
$$= 10.6$$

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$\sigma_Y^2 = 10.6 - 1$

$\sigma_Y^2 = 9.6$

$\therefore \sigma_Y = 3.098$

c) $cov(X,Y) = E(XY) - \mu_x \mu_y$

$= 0.9 - (1.4)(1)$

$= 0.9 - 1.4$

$= -0.5$

d) $\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$

$= \frac{-0.5}{(0.489)(3.098)}$

$= \frac{-0.5}{1.5149}$

$\rho(X,Y) = -0.33$

If X & Y are independent, we must have $f(x_i).g(y_j) = J_{ij}$

Here, $f(x_1).g(y_1) = (0.6)(0.3) = 0.18$

$J_{11} = 0.1$

$\Rightarrow f(x_1).g(y_1) \neq J_{11}$

similarly, otherwise will not satisfied

$\therefore$ X and Y are dependent.

1) Suppose X and Y are independent random variables with the following respective distribution, find the joint distribution of X and Y, Also verify that COV(X, Y) = 0.

x_i	1	2	Y_j	-2	5	8
$f(x_i)$	0.7	0.3	$g(Y_j)$	0.3	0.5	0.2

Soln: Since X and Y are independent random variables, the joint distribution is obtained by using the definition

$f(x_i).g(Y_j) = J_{ij}$

i.e, J_{ij} are obtained on multiplication of X marginal entries

Consider the following table

X \ Y	-2	5	8	$f(x_i)$	
1	J_{11}	J_{12}	J_{13}	0.7	$f(x_1)$
2	J_{21}	J_{22}	J_{23}	0.3	$f(x_2)$
$g(Y_j)$	0.3	0.5	0.2	1	

$J_{11} = f(x_1).g(Y_1) = (0.7)(0.3) = 0.21$

$J_{12} = f(x_1).g(Y_2) = (0.7)(0.5) = 0.35$

$J_{13} = (0.7)(0.2) = 0.14$

$J_{21} = f(x_2).g(Y_1) = (0.3)(0.3) = 0.09$

$J_{22} = (0.3)(0.5) = 0.15$

$J_{23} = (0.3)(0.2) = 0.06$

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Now we have the joint distribution of X and Y as

$X \backslash Y$	-2	5	8
1	0.21	0.35	0.14
2	0.09	0.15	0.06

We have, $COV(X,Y) = E(XY) - \mu_x \mu_y$ — ①

$$\mu_x = E(X) = \Sigma x_i . f(x_i)$$
$$= (1)(0.7) + 2(0.3)$$
$$= 0.7 + 0.6$$
$$= 1.3$$

$$\mu_y = E(Y) = \Sigma y_j . g(y_j)$$
$$= (-2)(0.3) + 5(0.5) + 8(0.2)$$
$$= -0.6 + 2.5 + 1.6$$
$$= 3.5$$

$$E(XY) = \Sigma x_i . y_j . J_{ij}$$
$$= (1)(-2)(0.21) + (1)(5)(0.35) + 1(8)(0.14)$$
$$+ 2(-2)(0.09) + 2(5)(0.15) + 2(8)(0.06)$$

$$E(XY) = 4.55$$

① ⇒ $COV(X,Y) = 4.55 - (1.3)(3.5)$
$$= 4.55 - 4.55$$
$$= 0$$

2) X and Y are independent random variables. X takes the values 2, 5, 7 with the probability 1/2, 1/4, 1/4 respectively. Y takes the values 3, 4, 5 with the probabilities 1/3, 1/3, 1/3 respectively.
a) Find the joint probability distribution of X and Y
b) Show that the covariance of X and Y is equal to zero.

Soln:- Given data

x_i	2	5	7
$f(x_i)$	1/2	1/4	1/4

y_j	3	4	5
$g(y_j)$	1/3	1/3	1/3

X \ Y	3	4	5	$f(x_i)$
2	J_{11}	J_{12}	J_{13}	1/2
5	J_{21}	J_{22}	J_{23}	1/4
7	J_{31}	J_{32}	J_{33}	1/4
$g(y_j)$	1/3	1/3	1/3	1

$J_{11} = (1/2)(1/3) = 1/6 = 0.166$
$J_{12} = (1/2)(1/3) = 0.166$
$J_{13} = (1/2)(1/3) = 0.166$

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$J_{21} = (1/4)(1/3) = 1/12 = 0.0833$
$J_{22} = (1/4)(1/3) = 1/12 = 0.0833$
$J_{23} = (1/4)(1/3) = 1/12 = 0.0833$

$J_{31} = (1/4)(1/3) = 1/12 = 0.0833$
$J_{32} = (1/4)(1/3) = 1/12 = 0.0833$
$J_{33} = (1/3)(1/4) = 1/12 = 0.0853$

Now we have joint distribution as follows:

X \ Y	3	4	5
2	1/6	1/6	1/6
5	1/12	1/12	1/12
7	1/12	1/12	1/12

$$\mathrm{cov}(X, Y) = E(XY) - \mu_x \cdot \mu_y \quad \text{—— (1)}$$

$$\mu_x = E(X) = \Sigma x_i \cdot f(x_i)$$
$$= 2(1/2) + 5(1/4) + 7(1/4)$$
$$= 1 + 1.25 + 1.75$$
$$= 4$$

$$\mu_y = E(Y) = \Sigma y_j \cdot g(y_j)$$
$$= 3(1/3) + 4(1/3) + 5(1/3)$$
$$= 1 + 1.33 + 1.66$$
$$= 4$$

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$$E(XY) = 2(3)(1/6) + 2(4)(1/6) + 2(5)(1/6) + 5(3)(1/12) + 5(4)(1/12) + 5(5)(1/12) + 7(3)(1/12) + 7(4)(1/12) + 7(5)(1/12)$$

$$= 1 + 1.33 + 1.66 + 1.25 + 1.66 + 2.083 + 1.75 + 2.33 + 2.916$$

$$= 16$$

① → $cov(X, Y) = 16 - (4)(4)$

$= 16 - 16$

$= 0$ //

3) If X & Y are independent random variables prove the following results.

a) $E(XY) = E(X) . E(Y)$

b) $cov(X, Y) = 0$

c) $\sqrt{x+y}^2 = \sqrt{x}^2 + \sqrt{y}^2$

Soln: a) $E(XY) = \Sigma x_i . y_j . J_{ij}$

$= \Sigma x_i y_j . f(x_i) . g(y_j)$

[∵ X & Y independent $J_{ij} = f(x_i) g(y_j)$]

$E(XY) = \Sigma x_i . f(x_i) . \Sigma y_j . g(y_j)$

$E(XY) = E(X) . E(Y)$

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b) $\operatorname{cov}(X,Y) = E(XY) - \mu_x . \mu_y$

$= E(X).E(Y) - E(X).E(Y)$

$\therefore \operatorname{cov}(X,Y) = 0$

c) $\sigma^2_{x+y} = \sum (x_i + y_j)^2 J_{ij} - \mu^2_{x+y}$

$= \sum (x_i^2 + y_j^2 + 2x_i y_j) J_{ij} - [E(x+y)]^2$

$= \sum x_i^2 J_{ij} + \sum y_j^2 J_{ij} + 2\sum x_i . y_j . J_{ij} - [\sum (x_i + y_j) J_{ij}]^2$

$= \sum x_i^2 f(x_i) g(y_j) + \sum y_j^2 f(x_i) g(y_j) + 2\sum x_i y_j f(x_i) . g(y_j)$

$= \sum x_i^2 f(x_i) \sum g(y_j) + \sum y_j^2 g(y_j) \sum f(x_i) + 2[\sum x_i f(x_i)][\sum y_j g(y_j)] - [\sum x_i f(x_i) g(y_j) + \sum y_j . f(x_i) . g(y_j)]^2$

$= E(X^2).1 + E(Y^2).1 + 2E(X)E(Y) - [E(X)+E(Y)]^2$

$= E(X^2) + E(Y^2) + 2\cancel{E(X)E(Y)} - [E(X)]^2 - [E(Y)]^2 - 2\cancel{E(X)E(Y)}$

$= [E(X^2) - \mu_x^2] + [E(Y^2) - \mu_y^2]$

$\sigma^2_{x+y} = \sigma^2_x + \sigma^2_y$ //

* Stochastic Process:-

A stochastic process is such that a generation of the probability distribution depends only on the present state. (The collection of random variables).

a) Probability Vector:-

A vector $v = [v_1, v_2, v_3 ... v_n]$ is called a probability vector if each one of its components are non-negative and their sum is equal to unity.

Ex: $u = (1, 0)$; $v = (1/2, 1/2)$

$w = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = 1$

Note:- If v is not a probability vector but each one of that v are non-negative then λv is a probability vector. where λ is equal to $\frac{1}{\sum_{i=1}^{n} v_i}$

Ex: $v = \{1, 2, 3\}$

$\lambda = \frac{1}{6}$

$\therefore \lambda v = \frac{1}{6}[1, 2, 3]$

$= \left[\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right] = 1$

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b) stochastic Matrix:-
A square matrix $P = [P_{ij}]$ having every row in the form of a probability vector is called a stochastic matrix.

Ex: ① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ③ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

② $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

④ $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

c) Regular stochastic Matrix:-
A stochastic matrix P is said to be a regular stochastic matrix if all the entries of sum power P^n are positive

* Properties of Regular stochastic Matrix:-

→ If P is a regular stochastic matrix of order n then
1) P has a unique fixed point $x = [x_1, x_2, x_3 \ldots x_n]$ such that $x.P = x$.

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3) P has a unique fixed probability vector $v = (v_1, v_2, \ldots v_n)$ such that $vP = v$.

i) If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_1 \end{bmatrix}$ is a stochastic matrix and $v = [v_1, v_2]$ is a probability vector show that $v.A$ is also a probability vector.

Soln. Given that $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a stochastic matrix and v is a probability vector.

$$\left.\begin{aligned} a_1 + a_2 &= 1 \\ b_1 + b_2 &= 1 \end{aligned}\right\} \text{——} (1) \quad \& \quad v_1 + v_2 = 1 \text{——} (2)$$

consider, $vA = [v_1 \ v_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

$$vA = [v_1a_1 + v_2b_1 \quad v_1a_2 + v_2b_2]$$

we have to prove that

$$v_1a_1 + v_2b_1 + v_1a_2 + v_2b_2 = 1$$

consider

$$\begin{aligned} LHS &= v_1a_1 + v_2b_1 + v_1a_2 + v_2b_2 \\ &= v_1(a_1 + a_2) + v_2(b_1 + b_2) \\ &= v_1(1) + v_2(1) \qquad [\because (1)] \end{aligned}$$

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$= V_1 + V_2$

$= 1 \quad [\because \text{②}]$

$\therefore$ LHS = RHS

$\therefore$ Hence VA is also a Probability vector.

2) Prove with reference to two second order stochastic matrices that their product is also a stochastic matrices.

Soln: Let, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

are stochastic matrices

$$\Rightarrow \left.\begin{matrix} a_{11} + a_{12} = 1 & \& & b_{11} + b_{12} = 1 \\ a_{21} + a_{22} = 1 & & b_{21} + b_{22} = 1 \end{matrix}\right\} \text{— ①}$$

Consider,

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

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We have to prove that,

$a_{11}b_{11} + a_{12}b_{21} + a_{11}b_{12} + a_{12}b_{22} = 1$ ——— (1)

$a_{21}b_{11} + a_{22}b_{21} + a_{21}b_{12} + a_{22}b_{22} = 1$ ———→ (3)

Consider, LHS of eqⁿ (2).

$= a_{11}b_{11} + a_{12}b_{21} + a_{11}b_{12} + a_{12}b_{22}$

$= a_{11}(b_{11}+b_{12}) + a_{12}(b_{21}+b_{22})$

$= a_{11}(1) + a_{12}(1)$

$= a_{11} + a_{12}$

$= 1 = RHS$

Consider, LHS of eqⁿ (3)

$= a_{21}b_{11} + a_{22}b_{21} + a_{21}b_{12} + a_{22}b_{22}$

$= a_{21}(b_{11}+b_{12}) + a_{22}(b_{21}+b_{22})$

$= a_{21}(1) + a_{22}(1)$

$= a_{21} + a_{22}$

$= 1 = RHS$

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1) If A is a square matrix of order n whose rows are each of the same vector $a = (a_1, a_2, a_3 \dots a_n)$ and if $v = (v_1, v_2, v_3 \dots v_n)$ is a probability vector prove that $vA = a$.

Sol: Given that,

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \\ \vdots & & & \\ a_1 & a_2 & \dots & a_n \end{bmatrix}$$ and

$$v = v_1 + v_2 + v_3 \dots + v_n = 1$$

Consider,

$$vA = \underset{(1 \times n)}{[v_1, v_2, v_3 \dots v_n]} \underset{(n \times n)}{\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \\ \vdots & & & \\ a_1 & a_2 & \dots & a_n \end{bmatrix}}$$

$$vA = [v_1 a_1 + v_2 a_1 + \dots + v_n a_1 \quad v_1 a_2 + v_2 a_2 + \dots + v_n a_2 \quad \dots v_1 a_n + v_2 a_n \dots v_n a_n]$$

$$vA = [a_1(v_1 + v_2 + \dots v_n) \quad a_2(v_1 + v_2 + \dots v_n) \quad \dots a_n(v_1 + v_2 \dots v_n)]$$

$$= [a_1(1) \quad a_2(1) \dots a_n(1)]$$

$$= [a_1 \quad a_2 \quad \dots \quad a_n]$$

$vA = a$ //

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Q. Find the unique fixed probability vector of the regular stocastic matrix

$$A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Soln: We have to find $v = (x, y)$ such that $vA = v$. where $x + y = 1$ —— (a)

consider, $vA = v$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} \frac{3x}{4} + \frac{y}{2}, & \frac{x}{4} + \frac{y}{2} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\Rightarrow \frac{3x}{4} + \frac{y}{2} = x \quad \text{—— (1)}$$

$$\frac{x}{4} + \frac{y}{2} = y \quad \text{—— (2)}$$

We can solve the either of the equation by using (a)

$$x + y = 1$$
$$y = 1 - x.$$

$$(1) \Rightarrow \frac{3x}{4} + \frac{(1-x)}{2} = x$$

$$\frac{3x + 2 - 2x}{4} = x$$

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$x + 2 = 4x$

$4x - x = 2$

$3x = 2$

$x = \frac{2}{3}$ //

we have, $y = 1 - x$

$y = 1 - \frac{2}{3}$

$y = \frac{3-2}{3}$

$y = \frac{1}{3}$ //

$\therefore v = (x, y) = (2/3, 1/3)$ is the required unique fixed probability vector.

3) Find the unique fixed probability vector of the regular stochastic matrix given

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

Soln: We have to find $v = (a, b, c, d)$ such that $vP = v$

where, $a + b + c + d = 1$ —— (i)

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consider,

$$VP = V$$

$$\begin{array}{c} \begin{bmatrix} a & b & c & d \end{bmatrix}_{1\times 4} \end{array} \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$= \begin{bmatrix} a & b & c & d \end{bmatrix}$$

$$\begin{bmatrix} b/2 + c/2 + d/2 & a/2 + c/2 + d/2 & a/4 + b/4 & a/4 + b/4 \end{bmatrix}$$
$$= \begin{bmatrix} a & b & c & d \end{bmatrix}$$

$b/2 + c/2 + d/2 = a$ — (1) $\qquad b + c + d = 2a$ — (1)

$a/2 + c/2 + d/2 = b$ — (2) $\qquad a + c + d = 2b$ — (2)

$a/4 + b/4 = c$ — (3) $\qquad a + b = 4c$ — (3)

$a/4 + b/4 = d$ — (4) $\qquad a + b = 4d$ — (4)

using the (1) we can solve

$$a = 1 - b - c - d$$

(1) ⇒ $b/2 + c/2 + d/2 = 1 - b - c - d$

$b/2 + b + c/2 + c + d/2 + d = 1$

$3b/2 + 3c/2 + 3d/2 = 1$

$3b + 3c + 3d = 2$

$b + c + d = 2/3$

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On solving these equation using $a+b+c+d=1$

$\Rightarrow b+c+d=1-a$

① $\Rightarrow$ $1-a=2a$

$1=2a+a$

$3a=1$

$a=\frac{1}{3}$ //

IIIly $a+c+d=1-b$

② $\Rightarrow$ $1-b=2b$

$2b+b=1$

$3b=1$

$b=\frac{1}{3}$ //

$\therefore$ ③ $\Rightarrow$ $a+b=4c$

$4c=\frac{1}{3}+\frac{1}{3}$

$4c=\frac{2}{3}$

$c=\frac{2}{12} \Rightarrow c=\frac{1}{6}$ //

④ $\Rightarrow$ $a+b=4d$

$4d=\frac{2}{3}$

$d=\frac{2}{12} \Rightarrow d=\frac{1}{6}$ //

$\therefore V=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$ //

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* Markov chain:-

The term markov chain refers to any system in which there are certain number of states and gives the probabilities that the system changes from one state to another state and these probabilities (P_{ij}) which are non-zero real numbers are called transition probabilities and they form a square matrix of order m called the transition probability matrix (t.p.m) denoted by "P".

* Higher Transition

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

* Higher Transition Probabilities:-

The probability that the system changes from state a_i to the state a_j in exactly n steps is denoted by $P_{ij}^{(n)}$

Remark: ① If $P^{0} = [P_1^{(0)}, P_2^{(0)}, P_3^{(0)} \cdots P_n^{(0)}]$ denotes the initial probability distribution then the n^{th} step probability distribution is given by.

$$p^{(1)} = p^{(0)}.P \quad , \quad p^{(2)} = p^{(0)}.P^2 \quad , \quad p^{(3)} = p^{(0)}.P^3$$

$$\text{-------} \quad p^{(n)} = p^{(0)}.P^n$$

(2) The Markov chain is said to be regular if the associated with TPM is regular.

1) 3 boys A, B, C are throughing ball to each other. A always throughs the ball to B and B always throughs the ball to C and C is just has lightly to through the ball to B or as to A.

If C was the first person to through the ball find the probabilities that after 3 throughs i) A has the ball
ii) B has the ball
iii) C has the ball

Soln: We have the state space {A, B, C} and the associated t.P.M is

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

If C is the first person to throw the ball gives the initial probability distribution.

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$p^{(0)} = [0, 0, 1]$

we need to find

$p^{(3)} = p^{(0)}.p^3$

Consider

$$p^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$p^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$p^3 = p^2.p = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$p^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\therefore p^{(3)} = p^{(0)}.p^3 = [0 \ 0 \ 1] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\therefore p^{(3)} = [1/4 \ 1/4 \ 1/2]$$

$$p^{(3)} = [p_A, p_B, p_C]$$

i) Probability that A has the ball is $1/4$

ii) " " B " " " " $1/4$

iii) " " C " " " " $1/2$

Q) The transition matrix 'P' of a Markov chain is given by $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with the initial probability distribution $P^{(0)} = [1/4, 3/4]$ define and find the following.

i) $P_{21}^{(2)}$ ii) $P_{12}^{(2)}$ iii) $P^{(2)}$ iv) $P_1^{(2)}$

v) The vector $P^{(0)}.P^{n}$ approaches.

Solⁿ: Given,

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} ; \quad P^{(0)} = [1/4, 3/4]$$

i) $P_{21}^{(2)} = a_2 \rightarrow a_1$ in 2 steps

$$P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 + 3/8 & 1/4 + 1/8 \\ 3/8 + 3/16 & 3/8 + 1/16 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{bmatrix}$$

i) $\therefore P_{21}^{(2)} = \frac{9}{16}$ A

ii) $P_{12}^{(2)} = a_1 \longrightarrow a_2$ in two steps.

$\therefore P_{12}^{(2)} = \frac{3}{8}$

iii) $p^{(2)} = p^{(0)} . P^2$

$= \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$

$p^{(2)} = \begin{bmatrix} 5/32 + 27/64 & 3/32 + 21/64 \end{bmatrix}$

$p^{(2)} = \begin{bmatrix} 37/64 , & 27/64 \end{bmatrix}$

$p^{(2)} = \begin{bmatrix} p_1^{(2)} , & p_2^{(2)} \end{bmatrix}$

iv) $p_1^{(2)} = \frac{37}{64}$

v) $p^{(0)} . P^n$ approaches the unique fixed probability vector such that $v = (x, y)$, $vP = v$.
where $x + y = 1$ — (a)

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} x/2 + 3y/4 & x/2 + y/4 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

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$$\frac{x}{2} + \frac{3y}{4} = x \text{ ——— (1)}$$

$$\frac{x}{2} + \frac{y}{4} = y \text{ ——— (2)}$$

using equ^n (1)

$$y = 1 - x.$$

$$(1) \Rightarrow \frac{x}{2} + \frac{3(1-x)}{4} = x$$

$$\frac{x}{2} + \frac{3-3x}{4} = x$$

$$\frac{2x + 3 - 3x}{4} = x.$$

$$3 - x = 4x$$

$$3 = 4x + x$$

$$5x = 3$$

$$x = \frac{3}{5}$$

$$(2) \Rightarrow x + y = 1$$

$$\frac{3}{5} + y = 1$$

$$y = 1 - \frac{3}{5}$$

$$y = \frac{5-3}{5}$$

$$\therefore y = \frac{2}{5}$$

$$\therefore v = [3/5, \; 2/5]$$

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⇒ Prove that the Markov chain whose t.p.m is $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducable.

Find the corresponding stationary probability vector.

Soln: Here we need to prove that P is a regular stochastic matrix.

Consider,

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 2/6 + 1/6 & 1/6 & 2/6 \\ 1/4 & 2/6 + 1/4 & 1/6 \\ 1/4 & 2/6 & 1/6 + 1/4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 4+3/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Since all the entries in P^2 are positive, hence we conclude that the t.p.m P is regular. Hence the Markov chain having t.p.m P is irreducable.

Next, we shall find unique fixed probability vector.

Consider $v = [x \quad y \quad z]$ where $x + y + z = 1$ — ⓐ

Such that $vP = v$.

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\begin{bmatrix} y/2 + z/2 & 2x/3 + z/2 & x/3 + y/2 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\Rightarrow \frac{y}{2} + \frac{z}{2} = x$$

$$y + z = 2x \longleftarrow (1)$$

$$\Rightarrow \frac{2x}{3} + \frac{z}{2} = y$$

$$4x + 3z = 6y \longleftarrow (2)$$

$$\Rightarrow \frac{x}{3} + \frac{y}{2} = z$$

$$2x + 3y = 6z \longleftarrow (3)$$

Using equation ⓐ

$$y + z = 1 - x$$

$$(1) \Rightarrow 1 - x = 2x$$

$$3x = 1 \Rightarrow x = \frac{1}{3}$$

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Put $x = 1/3$ in eqn (2) + (3)

(2) => $4(1/3) + 3z = 6y$

$\frac{4}{3} + 3z = 6y$

$6y - 3z = \frac{4}{3}$ —— (4)

(3) => $2(1/3) + 3y = 6z$

$3y - 6z = -\frac{2}{3}$

$-3y + 6z = \frac{2}{3}$ —— (5)

Compairing (4) + (5)

$6y - 3z = 4/3$

$-3y + 6z = 2/3$

$+ \quad - \quad -$

$9y - 9z = 2/3$

(4) => $\cancel{6y} - 3z = 4/3$

(5) => $-\cancel{6y} + 12z = 4/3$ [$\because$ Multiply by 2]

$9z = 8/3$

$z = \frac{8}{27}$

(4) => $6y - 3(8/27) = 4/3$

$6y = \frac{4}{3} + \frac{8}{9}$

$6y = \frac{12+8}{9}$

$$6Y = \frac{20}{9}$$

$$3Y = \frac{10}{9}$$

$$Y = \frac{10}{27}$$

4) A students study habits are as follows.

i) If he studies one night, he is 70% sure not to study the next day night.

ii) On the other hand if he does study one night, he is 60% sure not to study the next night.

In the long run how often does he study.

soln: state space is {A, B}

The associated t.P.M is

$$P = \begin{array}{cc} & \\ \left[\begin{array}{cc} 0.3 & 0.7 \\ 0.4 & 0.6 \end{array}\right] & \begin{array}{c} A \\ B \end{array} \\ \begin{array}{cc} A & \quad B \end{array} & \end{array}$$

In order to find the study in the long run, we need to find its fixed prob vector.

Let $V = [x \quad y]$ be a vector such that $VP = V$ where

$$x + y = 1 \longleftarrow (a)$$

$$[x \quad y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \quad y]$$

$$[0.3x + 0.4y \quad 0.7x + 0.6y] = [x \quad y]$$

$$0.3x + 0.4y = x \quad \text{——— (1)}$$

$$0.7x + 0.6y = y \quad \text{——— (2)}$$

$(1) \Rightarrow$

$$0.7x = 0.4y$$
$$0.7x = (0.4)(1-x)$$
$$0.7x = 0.4 - 0.4x$$
$$1.1x = 0.4$$
$$x = \frac{0.4}{1.1}$$

$$\therefore \quad x = 0.3636$$

$(1) \Rightarrow$

$$x + y = 1$$
$$y = 1 - x$$
$$y = 1 - 0.3636$$
$$y = 0.6364$$

$$\therefore \quad v = [x \quad y] = [0.3636, \quad 0.6364]$$
$$= [P_A, \quad P_B]$$

Hence we conclude that in a long run a student may study 36.36% of the time.