INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

AE 593 - DUAL DEGREE PROJECT I

A Comparative Study of Turbulence Modelling for Smoothed Particle Hydrodynamics

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Indian Institute of Technology Bombay

Abstract

A Comparative Study of Turbulence Modelling for Smoothed Particle Hydrodynamics

by K T Prajwal Prathiksh

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Keywords: Fluid Mechanics, Turbulence Modelling, Smoothed Particle Hydrodynamics, Reynolds Averaging, Large Eddy Simulation, Lagrangian Averaging

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List of Symbols

Symbol	Description
$\mathbf{a}$	Vector Field
$\underline{\underline{\boldsymbol{A}}}$	Second-rank Tensor Field
$\hat{\mathbf{e}}_i$	i^{th} Basis
$\frac{\mathrm{d}}{\mathrm{d}\,t}$	Lagrangian Derivative
$\langle \underline{\underline{\boldsymbol{A}}}, \underline{\underline{\boldsymbol{B}}} \rangle_F$	Frobenius Inner Product of $\underline{\underline{\boldsymbol{A}}}$ and $\underline{\underline{\boldsymbol{B}}}$
$\lvert\lvert A \rvert\rvert_F$	Frobenius Norm of $\underline{\underline{\boldsymbol{A}}}$ $(= \sqrt{\langle \underline{\underline{\boldsymbol{A}}}, \underline{\underline{\boldsymbol{A}}} \rangle_F})$
Δ	Component-wise Laplacian Operator
$(...)_i$	Property of i^{th} SPH particle
$(...)_j$	Property of j^{th} SPH particle
$(...)_{ij}$	$(...)_i - (...)_j$
t	Time
$\mathbf{r}$	Position
$\mathbf{v}$	Velocity
m	Mass
P	Pressure
ρ	Density
W_h	SPH Interpolating Kernel
h	Kernel Smoothing Length
$\nabla_i W_{h,ij}$	$\nabla_i W_h(\mathbf{r}_{ij})$
$\mathcal{V}_i$	Volume of i^{th} SPH particle
$\mathbf{F}$	External Body Force
ν	Kinematic Viscosity
η	Dynamic Viscosity
ε	Machine Epsilon

Table 1 continued from previous page

Symbol	Description
P_0	Reference Pressure
ρ_0	Reference Density
c_s	Speed of Sound
γ	Exponent - Equation of State
ν_t	Turbulent Eddy Viscosity
$\underline{\underline{\boldsymbol{S}}}$	Strain-Rate Tensor $(= [1/2][\nabla \mathbf{v} + \nabla \mathbf{v}^T])$
$\underline{\underline{\boldsymbol{\tau}}}$	Sub-Particle Stress Tensor

Chapter 1

Introduction

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

1.1 Project Motivation

1.2 Research Aims & Objectives

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Chapter 2

Turbulence Modelling

2.1 Viscosity-Based Models

Violeau et al. (VIOLEAU, PICCON, and CHABARD 2002) were amongst the early pioneers who tried to incorporate a turbulence model in SPH. They came up with two techniques to tackle the problem of turbulence in a Lagrangian framework, which so far had been neglected till then in research, namely, the eddy viscosity model and a generalised Langevin model. For each of their techniques, they considered the following equation of state 2.1, continuity equation 2.2 and momentum equation 2.3, based on the work of (Monaghan 1992):

$$P_i = \frac{\rho_0 c_s^2}{\gamma}\left[\left(\frac{\rho_i}{\rho_0}\right)^{\gamma} - 1\right] \tag{2.1}$$

$$\frac{\mathrm{d}\,\rho_i}{\mathrm{d}\,t} = \sum_j m_j \mathbf{v}_{ij} \cdot \nabla_i W_{h,ij} \tag{2.2}$$

$$\frac{\mathrm{d}\,\mathbf{v}_i}{\mathrm{d}\,t} = \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij}\right) \nabla_i W_{h,ij} + \mathbf{F}_i \tag{2.3}$$

Where the viscous term is defined as:

$$\Pi_{ij} = -\frac{16\nu}{\rho_i + \rho_j} \frac{\mathbf{v}_{ij} \cdot \mathbf{r}_{ij}}{|\mathbf{r}_{ij}|^2 + \varepsilon^2} \tag{2.4}$$

2.1.1 Eddy Viscosity Model

The eddy viscosity model was devised as a first-order closure model, which consisted of a relationship between the Reynolds stress tensor and the mean velocity gradients. Therefore, the momentum equation is similar to the momentum equation, except that the kinematic viscosity is replaced by the eddy viscosity (ν_t), and the velocities are Reynolds-averaged. In the SPH formalism, the diffusion term occurring is therefore defined as given in 2.5, with the eddy viscosity defined according to 2.6.

$$\tilde{\Pi}_{ij} = -8\frac{\nu_{t,i} + \nu_{t,j}}{\rho_i + \rho_j} \frac{\langle \mathbf{v} \rangle_{ij} \cdot \mathbf{r}_{ij}}{|\mathbf{r}_{ij}|^2 + \varepsilon^2} \tag{2.5}$$

$$\nu_t = L_m^2 ||\underline{\underline{\boldsymbol{S}}}||_F \tag{2.6}$$

Where $\langle \mathbf{v} \rangle$ is Reynolds-averaged velocity, and L_m refers to the mixing length scales. The SPH formulation for the mean velocity gradients are given in 2.7.

$$\nabla\langle\mathbf{v}\rangle_i = -\frac{1}{\rho_i}\sum_j m_j \langle\mathbf{v}\rangle_{ij} \otimes \nabla_i W_{h,ij} \tag{2.7}$$

On simulating Poiseuille flow for a high Reynolds number case, the authors could show that the velocity profile showed only a slight discrepancy with theory, with the expected log-law profile near the walls 2.1. This indicated that the model is appropriate for turbulent mixing problems or for cases involving spatially-varying viscosity while restricted to shear flows.

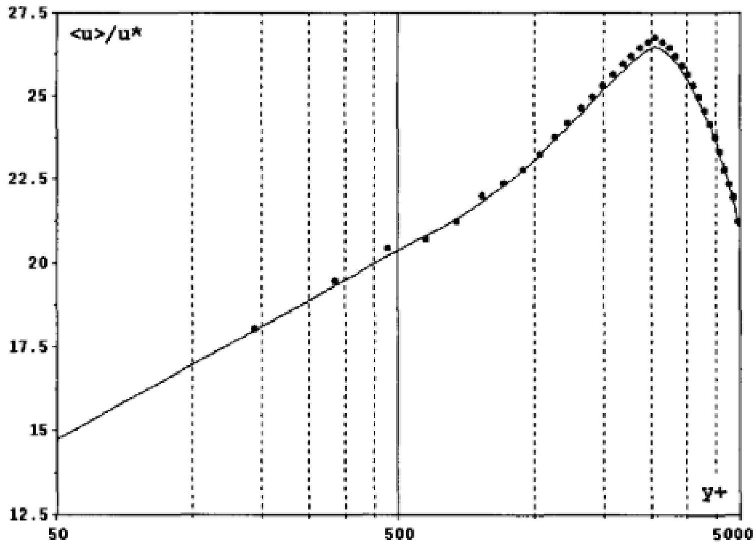

FIGURE 2.1: Turbulent Poiseuille flow in a pipe ($Re = 64e3$) modelled using the eddy viscosity model. Computed mean velocity profiles after ($t = 1s$) (solid circles), against theory (solid line). Ref: (VIOLEAU, PICCON, and CHABARD 2002)

2.1.2 Generalized Langevin Model

Violeau et al. also considered a stochastic approach, where the main idea is built on the concept of prescribing particle velocities as a random process, with properties fulfilling the theoretical turbulence hypotheses (Pope 1994). Hence, came about the Generalised Langevin model (GLM), where the particle acceleration is defined as:

$$d\mathbf{v} = -\frac{1}{\rho}\nabla\langle P\rangle + \underline{\underline{G}}(\mathbf{v} - \langle\mathbf{v}\rangle)dt + \sqrt{C_0\epsilon dt}\,\vec{\xi} \tag{2.8}$$

Where $\vec{\xi}$ is a random vector statistically non-correlated with velocities. The closure for this model was defined by specifying $\underline{\underline{G}}$ as:

$$\underline{\underline{G}} = \frac{1}{2}C_1\frac{\epsilon}{k}\underline{\underline{I}} + C_2\nabla\langle\mathbf{v}\rangle \tag{2.9}$$

Where (k) is the turbulent kinetic energy, (E) the dissipation rate, and (C_i) being constants - ($C_1 = 1.8, C_2 = 0.6$). By modelling turbulence as GLM in SPH, the momentum equation derived was given by:

$$\frac{\mathrm{d}\,\mathbf{v}_i}{\mathrm{d}\,t} = -\sum_j m_j \left(\frac{\langle P \rangle_i}{\rho_i^2} + \frac{\langle P \rangle_j}{\rho_j^2} \right) \nabla_i W_{h,ij} - \frac{1}{2} C_1 \frac{\epsilon_i}{k_i} \mathbf{v}_i' + C_2 \nabla \langle \mathbf{v} \rangle_i \cdot \mathbf{v}_i' + \sqrt{\frac{C_0 \epsilon_i}{\delta t}} \vec{\xi}_i \tag{2.10}$$

$$\langle \mathbf{v} \rangle = \sum_j \frac{m_j}{\rho_j} \mathbf{u}_j W_h(\mathbf{r}_j) \tag{2.11}$$

Where the fluctuations are defined as $\mathbf{v}' = \mathbf{v} - \langle \mathbf{v} \rangle$, and the local values of turbulent kinetic energy and dissipation are:

$$\epsilon_i = 2\nu_{t,i} + ||\underline{\underline{S}}_i||_F^2 \tag{2.12}$$

$$k_i = \frac{\epsilon_i \nu_{t,i}}{C_\mu}, C_\mu = 0.009 \tag{2.13}$$

It is to be noted that the authors did not estimate the dissipation rate through the proper velocity gradients since the fluctuations of random velocities do not reproduce the small eddies. The same test case as mentioned in 2.1.1 was considered for the performance of GLM. The authors observed large fluctuations. They attributed the discrepancy to the mean operator being redefined as given by 2.11 instead of being a Reynolds average. In fact, by redefining the mean operator in such a fashion, they appeared to have constructed a rudimentary LES filter. As observed in 2.2, the fluctuations have an order of magnitude of $k^{1/2}$. However, as claimed by the authors, unlike the eddy viscosity model, the GLM method can be used for different flows instead of being restricted to only shear flows.

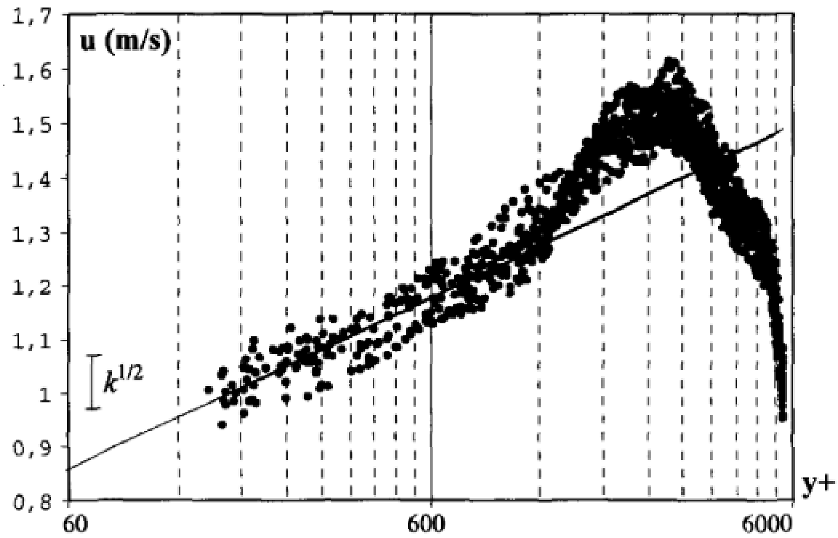

FIGURE 2.2: Turbulent Poiseuille flow in a pipe ($Re = 64e3$) modelled using the generalised Langevin model. Computed mean velocity profiles after ($t = 1s$) (solid circles), against theory (solid line). Ref: (VIOLEAU, PICCON, and CHABARD 2002)

2.2 mSPH

Adami et al. (Adami, X. Y. Hu, and N. A. Adams 2012) devised a model built on their observation of SPH simulations, wherein the absence of viscosity in typical SPH formulations produced purely noisy particle motion. At finite viscosities, the method would over-predict dissipation. Hence to counter this, they essentially "modified" (hence the name: Modified SPH [mSPH]) the momentum equation and the equation of state to advect the particles in order to homogenise the particle distribution, in turn stabilising the numerical scheme. They were also able to reduce the artificial dissipation in transitional flows.

The authors considered summation density (2.15), which is a function of the volume of the respective SPH particle as given by 2.14, as opposed to evolving density through the continuity equation (Xiang Yu Hu and Nikolaus A Adams 2006). The modified equation of state as given by 2.16, is equivalent to the classical SPH equation-of-state with $\gamma = 1$.

$$\mathcal{V}_i = \frac{1}{\sum_j W_{ij}} \tag{2.14}$$

$$\rho_i = \frac{m_i}{\mathcal{V}_i} = m_i \sum_j W_{ij} \tag{2.15}$$

$$P_i = c_s^2(\rho_i - \rho_0) \tag{2.16}$$

The momentum equation, which provides the acceleration of the particle, is a function of just the gradient and viscous shear forces as given by 2.17. The corresponding SPH formulation was derived as given by 2.18, which built on the earlier work of Hu and Adams (X. Hu and Nikolaus A Adams 2007).

$$\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{v} \tag{2.17}$$

$$\frac{\mathrm{d}\,\mathbf{v}_i}{\mathrm{d}\,t} = -\frac{1}{m_i}\sum_j(\mathcal{V}_i^2 + \mathcal{V}_j^2)\frac{P_i\rho_j + P_j\rho_i}{\rho_i + \rho_i}\nabla_i W_{h,ij} - \frac{\eta}{m_i}\sum_j(\mathcal{V}_i^2 + \mathcal{V}_j^2)\frac{\mathbf{v}_{ij}}{|\mathbf{r}_{ij}|}\nabla_i W_{h,ij} + \mathbf{F}_i \tag{2.18}$$

This scheme takes advantage of the regularisation of the particle motion stemming from the additional background pressure ($P_0 = \rho_0 c_s^2$). The additional force exerted by the background pressure counteracts non-homogeneous particle distributions, therein reducing numerical dissipation.

The authors estimated the energy spectra of the flow simulations in order to analyse the results of their test cases, using first and second-order moving-least-squares (MLS) method (GOSSLER 2001) and its subsequent Fourier transform (Frigo and Johnson 2005). Their first test case, the $2D$ variant of the Taylor-Green Vortex (TGV) problem, involved 8×8 counter-rotating vortices, requiring 64^2 particles. They considered the viscosity to be zero. As seen in the time evolution of the

The time evolution of the velocity field is given in 2.3, where it can be observed that the $2D$ turbulence is characterised by merging and pairing of small vortices. The energy spectra given in 2.4 show that at low wave numbers, both interpolation schemes give the same results, but at high wave numbers, the results differ. The energy spectrum of the standard SPH has a linear slope of magnitude $m = 1$ in a log-log scale equivalent to a purely noisy velocity field. Theoretically, however, $2D$ turbulence has an energy cascade with a slope of $m = -3$ in the inertial range, which is reasonably predicted using mSPH.

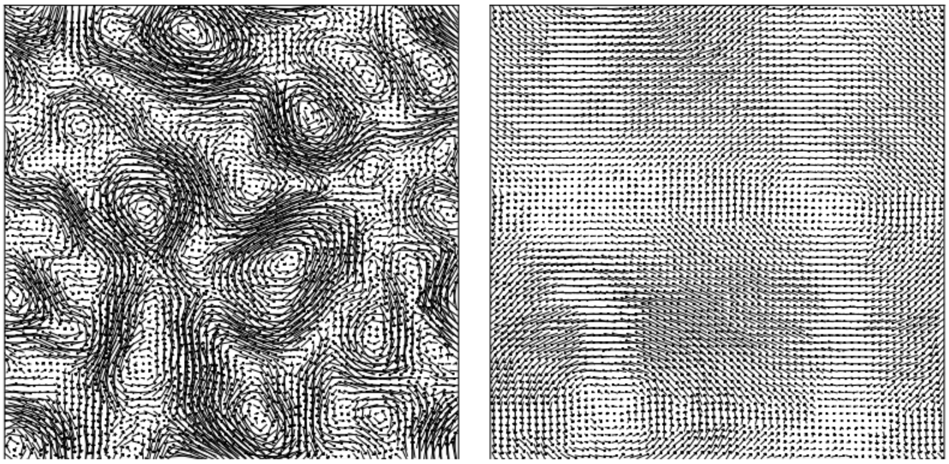

FIGURE 2.3: Velocity vector plot at $t = 2$ (left) and $t = 30$ (right). $Re = \infty$. Ref: (Adami, X. Y. Hu, and N. A. Adams 2012)

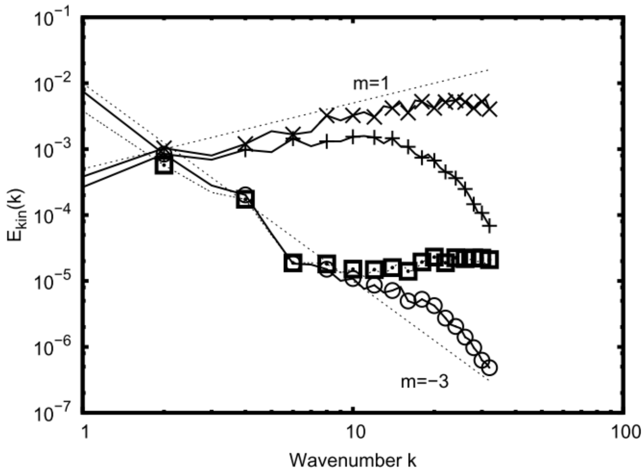

FIGURE 2.4: Comparison of energy spectra $t = 10$. $+$ and $\times$ denote standard SPH results with quintic spline and MLS interpolation; $\circ$ and $\Box$ denote mSPH results with quintic spline and MLS interpolation. Ref: (Adami, X. Y. Hu, and N. A. Adams 2012)

The second test case employed by the authors was that of the 3*D* TGV problem requiring 64^3 particles for a wide range of Reynolds numbers. The dissipation rate of the flow simulations are shown in 2.5 and 2.6. It can be observed that the standard SPH is unable to simulate transitional flows due to excessive dissipation. In contrast, mSPH can reproduce the dissipation rate reasonably well. This implies that the corrected particle transport velocity is an analogous eddy-viscosity model on scales below the numerical resolution.

2.3 Large Eddy Simulation-based Models

2.3.1 Implicit Pressure Poisson-based Solvers

2.3.2 Explicit Pressure Equation of State-based Solvers

Weakly compressible solvers

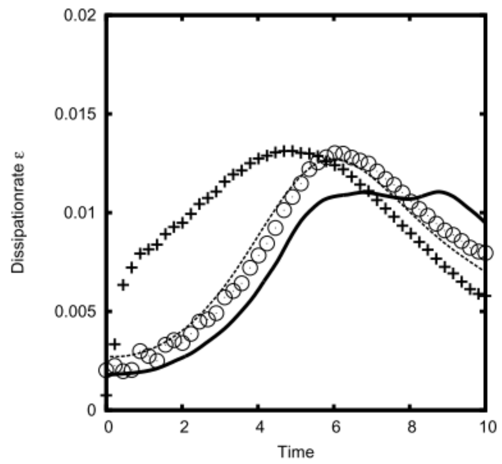

FIGURE 2.5: Dissipation rate at $Re = 400$ using DNS (solid line), Smagorinsky model (dashed line), standard SPH (+) and mSPH (∘). Ref: (Adami, X. Y. Hu, and N. A. Adams 2012)

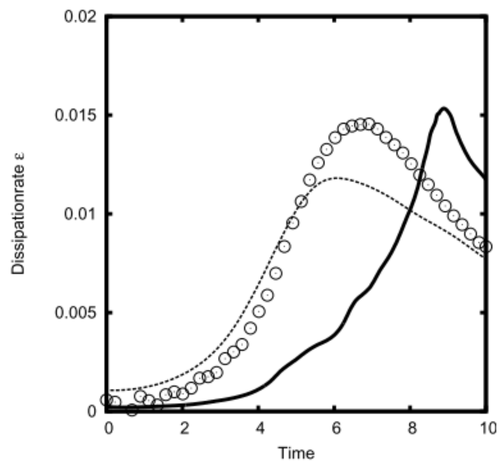

FIGURE 2.6: Dissipation rate at $Re = 3000$ using DNS (solid line), Smagorinsky model (dashed line) and mSPH (∘). Ref: (Adami, X. Y. Hu, and N. A. Adams 2012)

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