Havel–Hakimi

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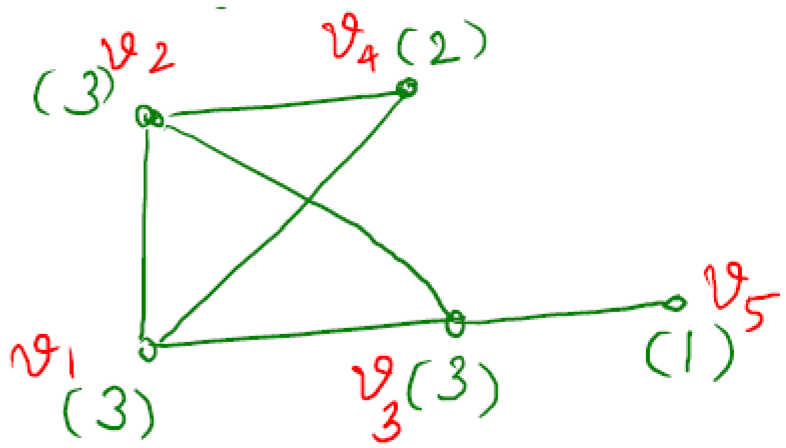
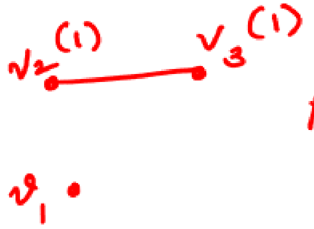
- is an algorithm in graph theory solving the graph realization problem
- Given a finite list of nonnegative integers in non-increasing order, is there a simple graph such that its degree sequence is exactly this list?
- The degree sequence is a list of numbers in non-increasing order indicating the number of edges incident to each vertex in the graph
- If a simple graph exists for exactly the given degree sequence, the list of integers is called graphic



$3, \; 3, \; 3, \; 2, \; 1$ deg seq:

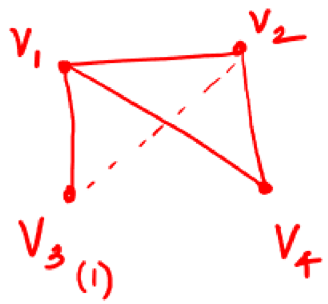
$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5$

$S_1 : \langle 1, 1, 1 \rangle$



Not graphical

$S_2 : \langle \overset{v_1}{3}, \overset{v_2}{3}, \overset{v_4}{3}, \overset{v_3}{1} \rangle$



Not graphical.

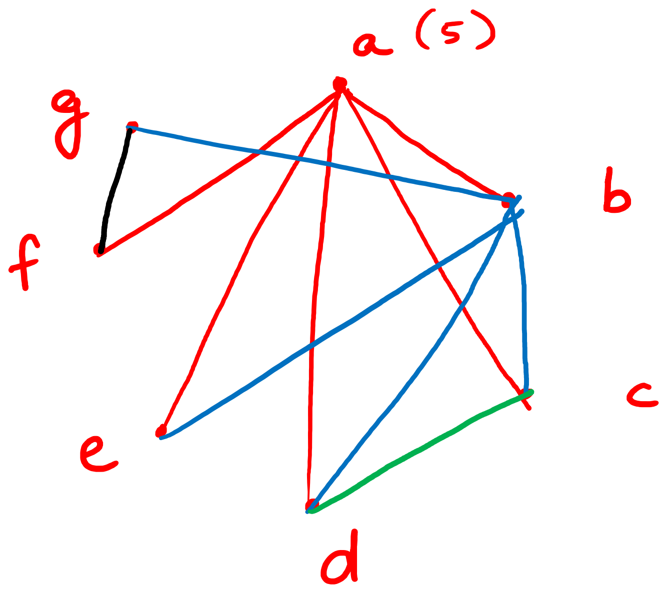
- A graphical sequence may be degree sequence of more than one graph
- Ex: 3,3,2,2,1,1

Degree Sequence

A sequence $d_1, d_2 \ldots d_n$ of non-negative integers is called a degree sequence of a graph G if the vertices of G can be labelled $v_1, v_2, \ldots v_n$ $\therefore$ S $deg(v_i) = d_i$ for all $i = 1, 2, \ldots n$.

$$S_0 : \langle \overset{a}{5}, \overset{b}{5}, \overset{c}{3}, \overset{d}{3}, \overset{e}{2}, \overset{f}{2}, \overset{g}{2} \rangle$$

G



$$S_1 : \langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$$

$$S_2 : \langle *_a, 4_b, 2_c, 2_d, 1_e, 1_f, 2_g \rangle$$

$$S_2' : \langle *a, 4_b, \underbrace{2_c, 2d, 2_g, 1e,}\, 1_f \rangle$$

$$S_3 : \langle *a, *b, 1c, 1d, 1g, 0e, 1f \rangle$$

$$S_3' : \langle *a, *b, 1c, 1d, 1g, 1f, 0e \rangle$$

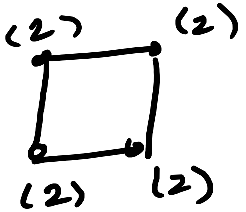
$$S_4 : \langle *a, *b, *c, *d, 1g, 1f, 0e \rangle$$

$$S_5 : \langle *a, *b, *c, *d, *g, 0f, 0e \rangle$$

Graphical

$S: \langle 5, 5, 5, 5, 2, 2, 2 \rangle$

$S_1: \langle \underline{5}, \underbrace{5, 5, 5, 2, 2}, 2 \rangle$



$S_2: \langle *, 4, 4, 4, 1, 1, 2 \rangle$

$S_2': \langle *, \underline{4}, \underbrace{4, 4, 2, 1}, 1 \rangle$

$S_3: \langle *, *, 3, 3, 1, 0, 1 \rangle$

$S_3': \langle *, *, \underline{3}, \underbrace{3, 1, 1}, 0 \rangle$

$S_4: \langle *, *, *, 2, \underbrace{0, 0}, 0 \rangle$

$S_5: \langle *, *, *, *, -1, -1, 0 \rangle$ Not Graphical

Havel Hakimi Theorem

A degree sequence $S = d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n$, where $n \geq 2$ and $d_1 \leq n-1$ and $d_1 \geq 1$, is graphic if and only if the reduced sequence $S' = \{*, d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n\}$ is graphic.

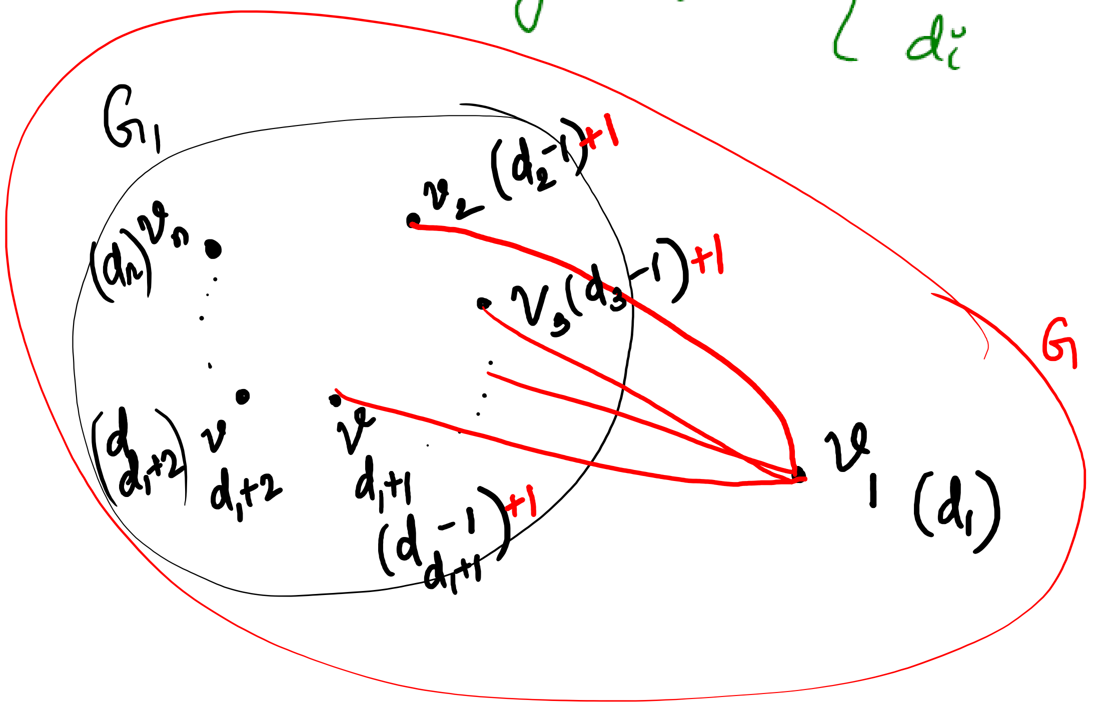
$S : d_1, d_2 \ldots\ldots d_n \qquad n \geq 2 \qquad d_1 \geq 1 \qquad d_1 \leq n-1$

$S_1 : d_2 - 1, d_3 - 1, \ldots\ldots d_{d_1+1} - 1, d_{d_1+2}, \ldots\ldots d_n$

① Suppose S_1 is graphical, $\exists$ a graph G_1 of order $n-1$ with deg. seq S

Hence, label $V(G_1)$ as $v_2, v_3, \ldots v_n$ S.

$$\deg(v_i) = \begin{cases} d_i - 1 & \text{for } i = 2, 3 \ldots d_1 + 1 \\ d_i & \text{for } i = d_1 + 2, \ldots d_n \end{cases}$$



Adding v_1 and its edges to G_1, we can obtain G which is Graphical.

edges $\Rightarrow v_1 v_i$, $\forall i$ from 2 to $d_1 + 1$

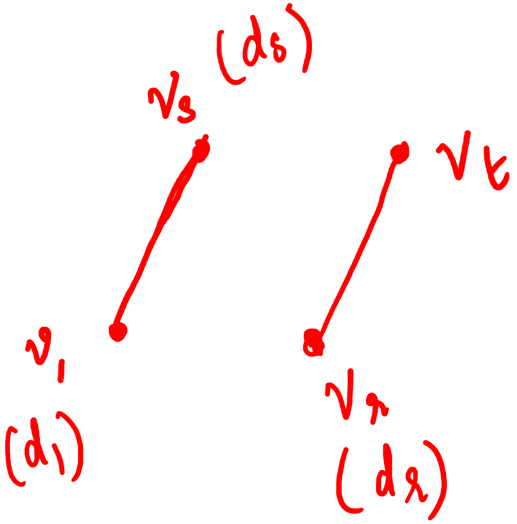
$\deg(v_i) = d_i$

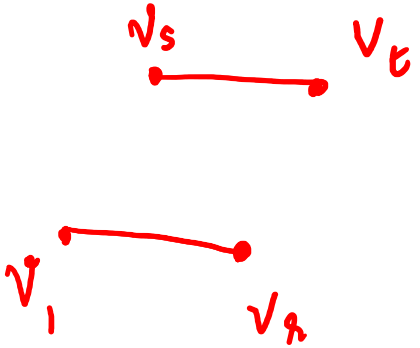
② S is graphical, $\exists$ graph order n with degree seq S.

$V(G) = \{v_1, v_2, \ldots v_n\}$ $\deg(v_i) = d_i$ for $i = 1, 2, \ldots n$

Claim : v_1 is adjacent to vertices having degrees $d_2, d_3, \ldots d_{d_1+1}$

The sum of the degrees of the vertices adjacent to v_1 is maximum





$d_r > d_s$

$d_1, d_r \ldots d_s$

Thus, the graph G-v_1 has the degree sequence

$S_1 : d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, d_{d_1+2}, \ldots d_n$