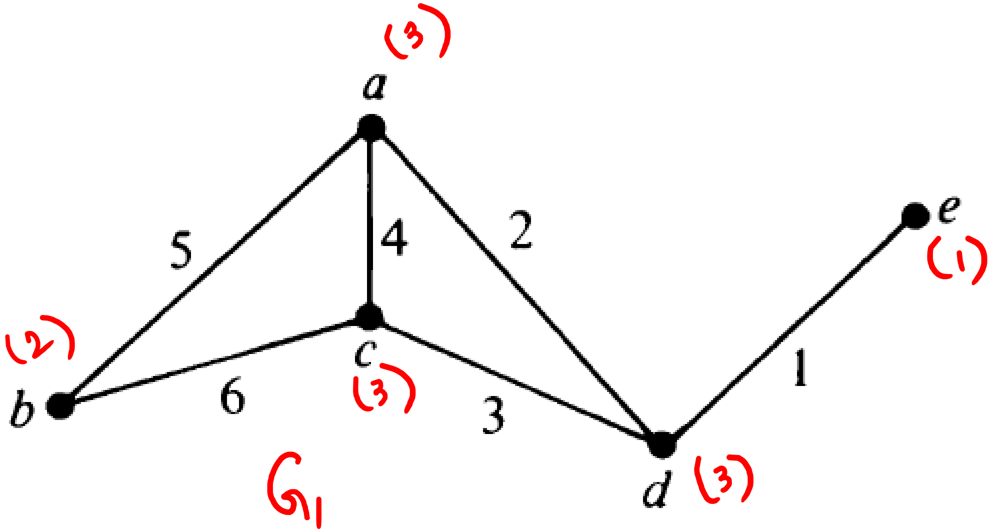
Matrix Representation of Graphs

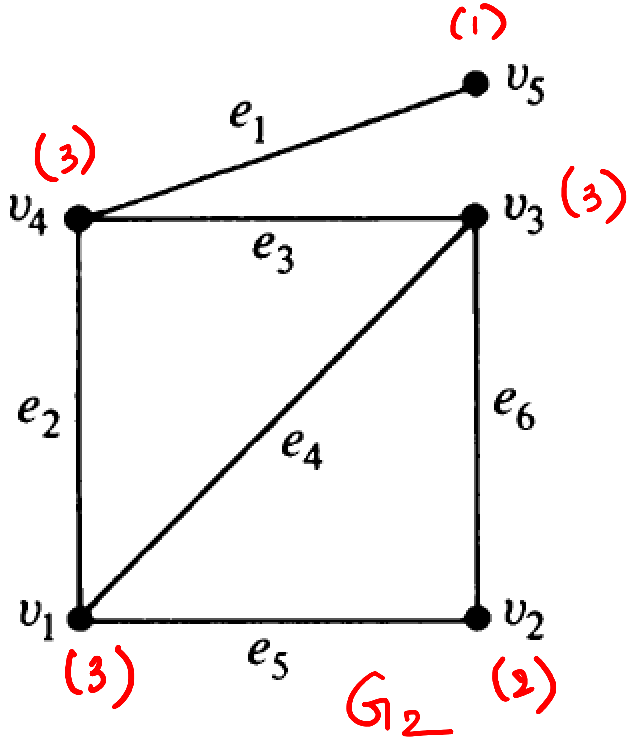
Isomorphism

Two graphs are thought of as equivalent (and called *isomorphic*) if they have identical behavior in terms of graph-theoretic properties.

Two graphs G and G' are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

suppose that edge e is incident on vertices v_1 and v_2 in G; then the corresponding edge e' in G' must be incident on the vertices v'_1 and v'_2 that correspond to v_1 and v_2, respectively.



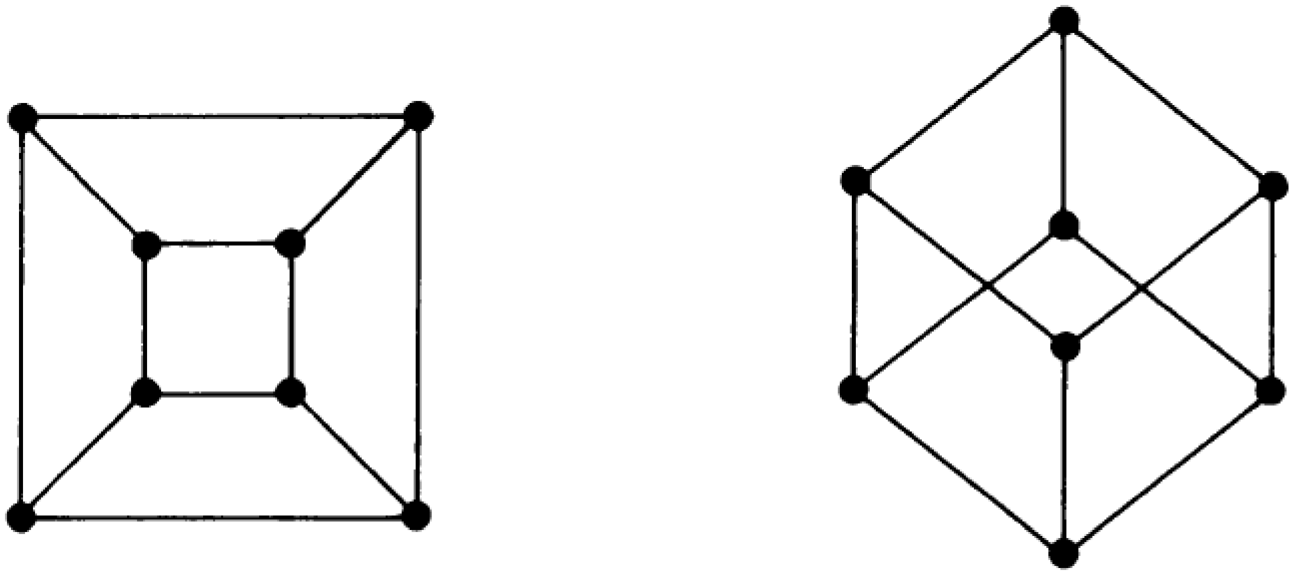


Isomorphic graphs.

The vertices a, b, c, d, and e correspond to v_1, v_2, v_3, v_4, and v_5, respectively.

The edges 1, 2, 3, 4, 5, and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

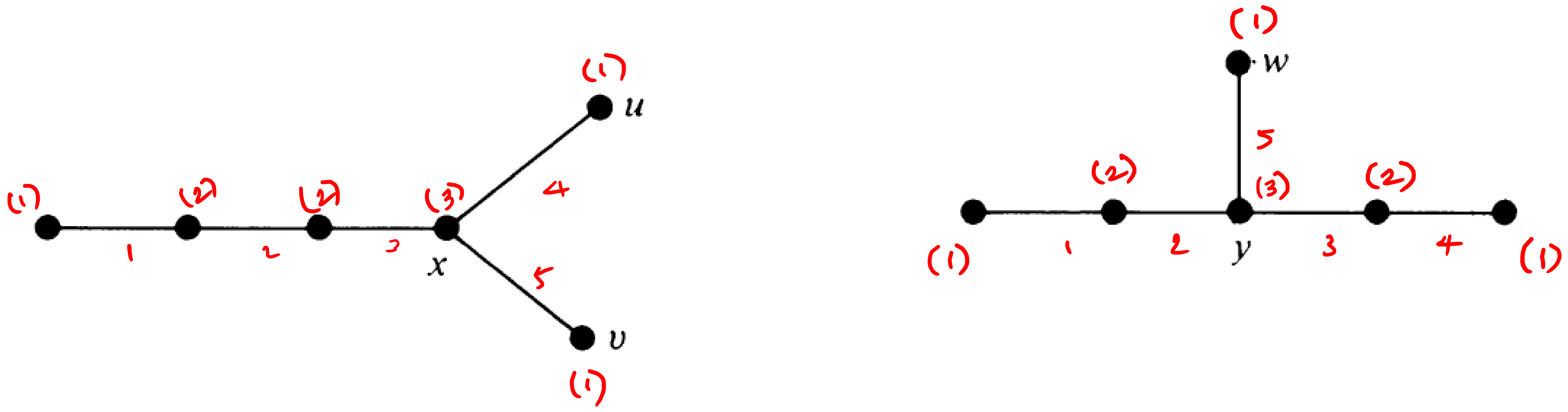
Except for the labels (i.e., names) of their vertices and edges, isomorphic graphs are the same graph, perhaps drawn differently.



Isomorphic graphs.

It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.
4. Same diameter
5. Same number of components
6. Same length of longest path
7. If one of them contains a cycle of particular length then the same must be true for the other graph



(a) (b)

Two graphs that are not isomorphic.

- Two graphs are isomorphic if and only if there complements are isomorphic.

Matrix Representation of Graphs

A matrix is a convenient and useful way of representing a graph to a computer.

Matrices lend themselves easily to mechanical manipulations.

Many known results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. In many applications of graph theory, such as in electrical network analysis and operations research, matrices also turn out to be the natural way of expressing the problem.

Incidence matrix

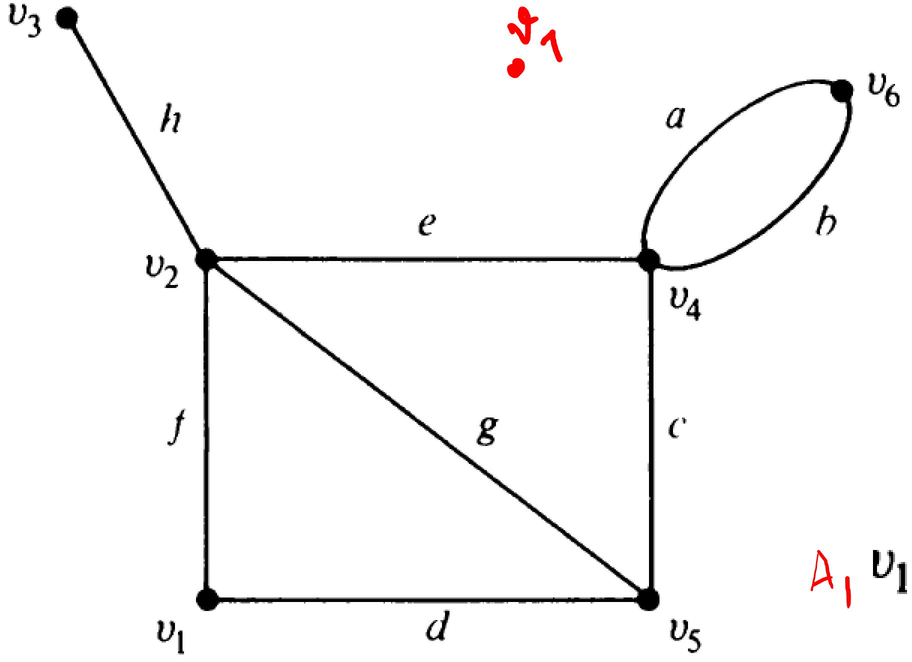
G $\quad A_{(n \times e)}$

Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix A $= [a_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows:

The matrix element

$$a_{ij} = 1, \quad \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i, \text{ and}$$
$$= 0, \quad \text{otherwise.}$$

Such a matrix A is called the *vertex-edge incidence matrix*, or simply *incidence matrix*. Matrix A for a graph G is sometimes also written as A(G).



$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_6 \end{bmatrix}$$

5x8
nxe

$$A(G) = \begin{array}{r c} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{array}{r} A_1\ v_1 \\ A_2\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

v_7

The incidence matrix contains only two elements, 0 and 1. Such a matrix is called a *binary matrix* or a *(0, 1)-matrix*.

Given any geometric representation of a graph without self-loops, we can readily write its incidence matrix.

On the other hand, if we are given an incidence matrix $\mathsf{A}(G)$, we can construct its geometric graph G without ambiguity.

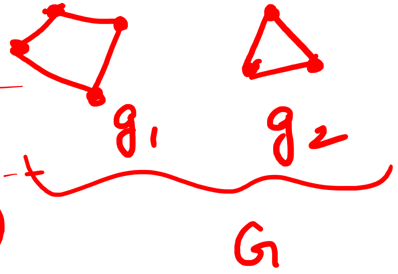
The following observations about the incidence matrix A can readily be made:

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example, columns 1 and 2 in Fig.

5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $\mathsf{A}(G)$ of graph G can be written in a block-diagonal form as

$$\mathsf{A}(G) = \left[\begin{array}{c|c} \mathsf{A}(g_1) & 0 \\ \hline 0 & \mathsf{A}(g_2) \end{array}\right],$$

$$\begin{bmatrix} A(g_1) & & \\ & A(g_2) & \\ & & A(g_3) \end{bmatrix}$$



where $\mathsf{A}(g_1)$ and $\mathsf{A}(g_2)$ are the incidence matrices of components g_1 and g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2, and vice versa. Obviously, this remark is also true for a disconnected graph with any number of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

THEOREM

Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $\mathsf{A}(G_1)$ and $\mathsf{A}(G_2)$ differ only by permutations of rows and columns.

THEOREM

If $\mathsf{A}(G)$ is an incidence matrix of a connected graph G with n vertices, the rank of $\mathsf{A}(G)$ is $n - 1$.

rank of $\mathsf{A}(G)$ is $n - k$, if G is a disconnected graph with n vertices and k components

If we remove any one row from the incidence matrix of a connected graph, the remaining $(n - 1)$ by e submatrix is of rank $n - 1$ (Theorem). In other words, the remaining $n - 1$ row vectors are linearly independent. Thus we need only $n - 1$ rows of an incidence matrix to specify the corresponding graph completely, for $n - 1$ rows contain the same amount of information as the entire matrix.

$n \times e$

Such an $(n - 1)$ by e submatrix A_f of A is called a *reduced incidence matrix*. The vertex corresponding to the deleted row in A_f is called the *reference vertex*. Clearly, any vertex of a connected graph can be made the reference vertex.

Since a tree is a connected graph with n vertices and $n - 1$ edges, its reduced incidence matrix is a square matrix of order and rank $n - 1$.

$n \times e$ $\quad n \quad n-1$

$n \times (n-1)$

$(n-1) \times (n-1)$

COROLLARY

The reduced incidence matrix of a tree is nonsingular.

Submatrices of A(G)

Let g be a subgraph of a graph G, and let $\mathsf{A}(g)$ and $\mathsf{A}(G)$ be the incidence matrices of g and G, respectively. $\mathsf{A}(g)$ is a submatrix of $\mathsf{A}(G)$.

there is a one-to-one correspondence between each n by k submatrix of $\mathsf{A}(G)$ and a subgraph of G with k edges, k being any positive integer less than e and n being the number of vertices in G.

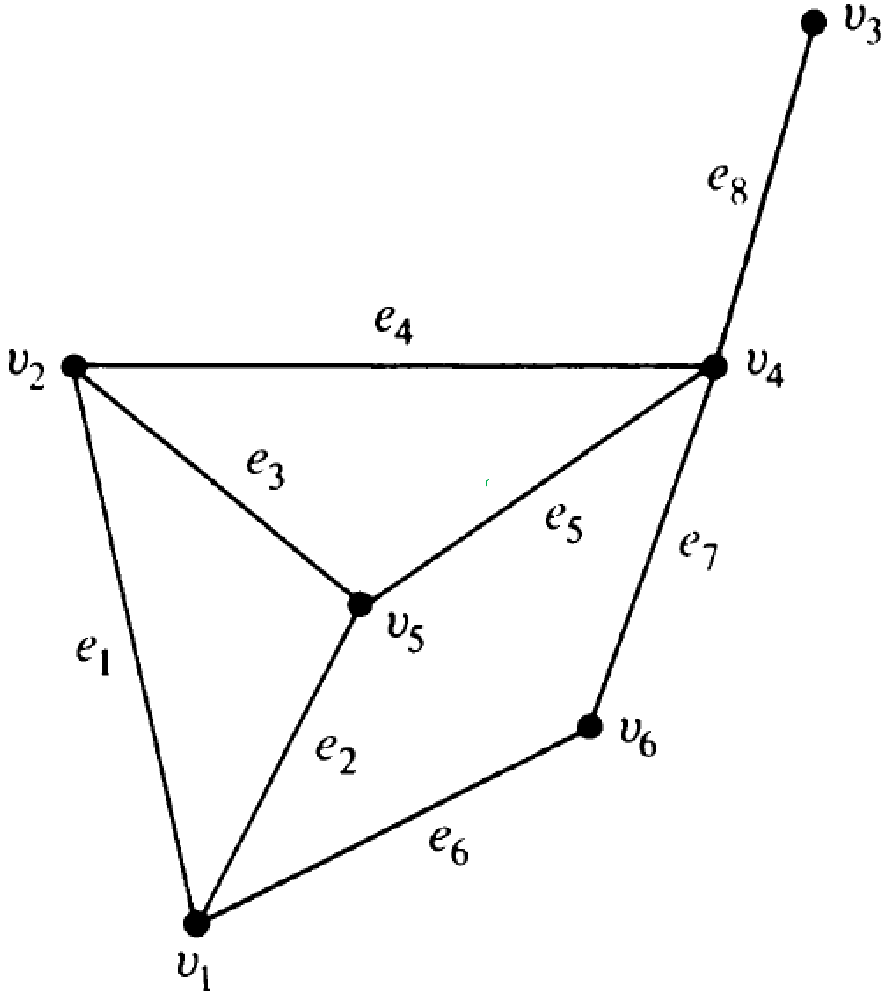
THEOREM

Let $\mathsf{A}(G)$ be an incidence matrix of a connected graph G with n vertices. An $(n-1)$ by $(n-1)$ submatrix of $\mathsf{A}(G)$ is nonsingular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree in G.

Adjacency matrix/ Connection matrix

Adjacency matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $\mathsf{X} = [x_{ij}]$ defined over the ring of integers such that

$$
\begin{aligned}
x_{ij} &= 1, && \text{if there is an edge between } i\text{th and } j\text{th vertices, and} \\
&= 0, && \text{if there is no edge between them.}
\end{aligned}
$$



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Observations that can be made immediately about the adjacency matrix X of a graph G are

1. The entries along the principal diagonal of X are all 0's if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs without parallel edges.
3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of 1's in the corresponding row or column of X.

4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $\mathsf{X}(G_1)$ and $\mathsf{X}(G_2)$ are related:

$$\mathsf{X}(G_2) = \mathsf{R}^{-1} \cdot \mathsf{X}(G_1) \cdot \mathsf{R},$$

where R is a permutation matrix.

5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $\mathsf{X}(G)$ can be partitioned as

$$\mathsf{X}(G) = \left[\begin{array}{c|c} \mathsf{X}(g_1) & 0 \\ \hline 0 & \mathsf{X}(g_2) \end{array}\right],$$

where $\mathsf{X}(g_1)$ is the adjacency matrix of the component g_1 and $\mathsf{X}(g_2)$ is that of the component g_2.

6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

Powers of X

The value of an off-diagonal entry in X^2, that is, ijth entry ($i \neq j$) in X^2,

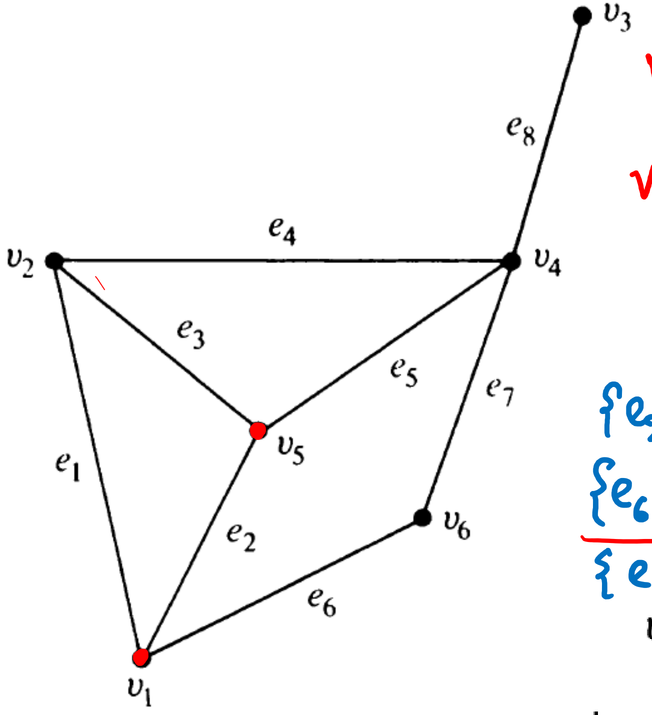
= number of 1's in the dot product of ith row and jth column (or jth row) of X.

= number of positions in which both ith and jth rows of X have 1's.

= number of vertices that are adjacent to both ith and jth vertices.

= number of different paths of length two between ith and jth vertices.

Similarly, the ith diagonal entry in X^2 is the number of 1's in the ith row (or column) of matrix X. Thus the value of each diagonal entry in X^2 equals the degree of the corresponding vertex, if the graph has no self-loops.



$v_1 \; (0,1,0,0,1,1)$

$v_2 \; (1,0,0,1,1,0) \; > 1$

$(3,1,0,3,1,0) \; > 7$

$\{e_2,e_2,e_2\}, \{e_2,e_3,e_3\}, \{e_2,e_5,e_5\}$
$\{e_6,e_7,e_5\}, \{e_1,e_4,e_5\}, \{e_6,e_6,e_2\}$
$\{e_1,e_1,e_2\}$

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

X^2_{12}

$$X^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix} \end{array}$$

$$X^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix} \end{array}$$

$X^2 \cdot X$

Let us now consider the ijth entry of X^3. $(i \neq j)$

ijth entry of X^3 = dot product of ith row X^2 and jth column (or row) of X.

$$= \sum_{k=1}^{n} ik\text{th entry of } X^2 \cdot kj\text{th entry of } X.$$

$= \sum_{k=1}^{n}$ number of all different edge sequences of three edges from ith to jth vertex via kth vertex.

= number of different edge sequences of three edges between ith and jth vertices.

the iith entry in X^3 equals twice the number of different circuits of length three (i.e., triangles) in the graph passing through the corresponding vertex v_i.

For example, consider how the 1,5th entry on $\mathbf{X}^3$ for the graph of Fig. is formed. It is given by the dot product

$$\begin{aligned}\text{row 1 of } \mathbf{X}^2 \cdot \text{row 5 of } \mathbf{X} &= (3, 1, 0, 3, 1, 0) \cdot (1, 1, 0, 1, 0, 0) \\ &= 3 + 1 + 0 + 3 + 0 + 0 = 7.\end{aligned}$$

These seven different edge sequences of three edges between v_1 and v_5 are

$$\{e_1, e_1, e_2\}, \quad \{e_2, e_2, e_2\}, \quad \{e_6, e_6, e_2\}, \quad \{e_2, e_3, e_3\},$$
$$\{e_6, e_7, e_5\}, \quad \{e_2, e_5, e_5\}, \quad \{e_1, e_4, e_5\}.$$

THEOREM

Let X be the adjacency matrix of a simple graph G. Then the ijth entry in X^r is the number of different edge sequences of r edges between vertices v_i and v_j.

COROLLARY A

In a connected graph, the distance between two vertices v_i and v_j (for $i \neq j$) is k, if and only if k is the smallest integer for which the i, jth entry in x^k is nonzero.

This is a useful result in determining the distances between different pairs of vertices.

COROLLARY B

If X is the adjacency matrix of a graph G with n vertices, and

$$Y = X + X^2 + X^3 + \cdots + X^{n-1}, \qquad \text{(in the ring of integers)},$$

then G is disconnected if and only if there exists at least one entry in matrix Y that is zero.