Combinatorics and Graph Theory

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Objective

- To introduce basic concepts of combinatorics and graph theory
- To study graphs, trees and networks
- To discuss Euler formula, Hamilton paths, planar graphs and coloring problem
- To practice useful algorithms on networks such as shortest path algorithm, minimal spanning tree algorithm and min-flow max-cut algorithm

Course content

- Unit – I
  - Introduction to combinatorics, permutation of multisets. Combinations of Multisets, distribution of distinct objects into distinct cells, distribution of non-distinct objects into distinct cells, Shamire secret sharing. Catalan number. Principle of inclusion and exclusion, Derangement.
- Unit – II
  - Generating functions, Partitions of integer, Ferrer graph. Solving recurrence relations using generating functions, Generating permutations and combinations. Pigeonhole principle: simple and strong Form, A THEOREM OF RAMSEY
- Unit – III
  - Graph, simple graph, graph isomorphism, incidence and adjacency matrices, Haveli-Hakimi criterion. Subgraphs Tree, minimum spanning tree, Kruskal, Prims algorithm, Caleys’ formula, Kirchoff-Matrix- tree Theorem, Fundamental circuits, Algorithms for fundamental circuits, Cut-sets and Cut-vertices, fundamental cut-sets.

Course content

- Unit – IV
  - Eular graph, Fleury’s algorithm Hamiltonian graph, Planar and Dual Graphs, Kuratowski's graphs. Coloring, Greedy coloring algorithm, chromatic polynomial.
- Unit – V
  - Mycielski’s theorem, Matching, halls marriage problem. Independent set, Dominating set, Vertex cover, clique, approximation algorithms

Course Outcomes

- Comprehend the fundamentals of combinatorics and apply combinatorial ideas in mathematical arguments in analysis of algorithms, queuing theory, etc.
- Comprehend graph theory fundamentals and tackle problems in dynamic programming, network flows,etc.
- Design and develop real time application using graph theory
- Construct and communicate proofs of theorems

Text Books

- Ralph P. Grimaldi, “Discrete and Combinatorial Mathematics”, 5th Edition, PHI/Pearson Education,2004
- G. Chartrand and P. Zhang, “Introduction to Graph Theory”, McGraw-Hill,2006
- Narsingh Deo, “Graph Theory with Applications to Engineering and Computer Science”, PHI

Reference Books

- Kenneth H. Rosen, “Discrete Mathematics and its Applications”, 7th edition, McGraw- Hill, 2012
- John Harris, Jeffry L. Hirst, Michael Mossinghoff, “Combinatorics and Graph Theory”, 2nd edition, Springer Science & Business Media, 2008
- J. H. Van Lint and R. M. Wilson, “A course in Combinatorics”, 2nd edition, Cambridge Univ. Press,2001
- Dr. D. S. Chandrasekharaiah, "Graph Theory and Combinatorics", Prism,2005.

Combinatorics has emerged as a new subject standing at the crossroads between pure and applied mathematics, the center of bustling activity, a simmering pot of new problems and exciting speculations.

-- Gian-Carlo Rota

Combinatorics

5, 6, 10, 12, 15, 18, 20

- is the mathematics of counting
- the study of arrangements: pairings and groupings, rankings and orderings, selections and allocations
- Three principal branches
  - Enumerative combinatorics is the science of counting.
  - Existential combinatorics studies problems concerning the existence of arrangements that possess some specified property.
  - Constructive combinatorics is the design and study of algorithms for creating arrangements with special properties.

Basic principles of counting

- Rules of Sum
- Rules of Product

- A college library has 40 textbooks on Data structures and 50 text books on algorithms. If a student can take only one book at a time, how many choices he/she has?

$40 + 50 = 90$

- If CS faculty has 2 colleagues, one of them has 3 textbooks on algorithms and other has 5 such textbooks. If *n* denotes the number of different books on this topic that the faculty can borrow from them, what is the range of n?

$1 \leq n \leq 8$

$5 \leq n \leq 8$

Basic principles of counting …

- Rules of Sum:
  - If a first task can be performed in *m* ways, while a second task can be performed in *n* ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of *m* + *n* ways.

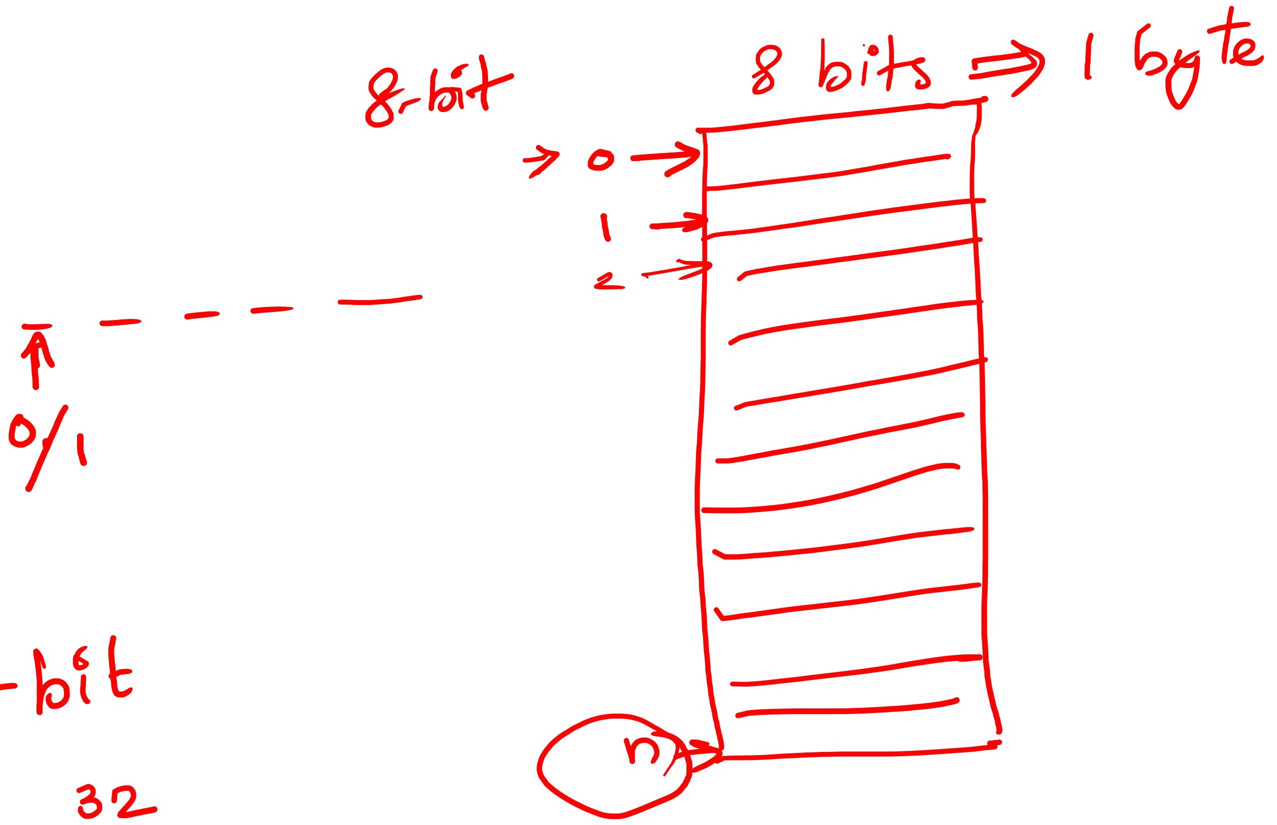
$$\begin{array}{ccccc} T_1 & T_2 & T_3 & \cdots & T_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ n_1 + & n_2 + & n_3 + & \cdots & + n_m \end{array}$$

- How many eight-character passwords are possible if each character is either an uppercase letter A–Z, a lowercase letter a–z, or a digit 0–9?

$(26 + 26 + 10) = 62$ $\qquad$ 62^8

$\underset{1}{\overset{a}{\underline{62}}}\ \underset{\uparrow}{\overset{a}{\underline{62}}}\ \underline{62}\ \_\ \_\ \_$ $\qquad$ $\underline{\underline{(62)^8}}$

Memory addressing



32-bit

2^{32}

2^{64}

Basic principles of counting…

- Rules of Product
  - If a procedure can be broken down into first and second stages, and if there are *m* possible outcomes for the first stage and if for each of these outcomes, there are *n* possible outcomes for the second stage, then the total procedure can be carried out in the designated order, in *mn* ways.

$$\begin{array}{ccccccc} S_1 & & S_2 & & \cdots & & S_n \\ \downarrow & & \downarrow & & \cdots & & \downarrow \\ n_1 & \times & n_2 & \times & \cdots & \times & n_m \end{array}$$

Basic principles of counting…

- At times it is necessary to combine different counting principles to solve a problem
  - For example: A coffee shop menu is limited to: 6 kinds of muffins, 8 kinds of sandwiches and 5 beverages (hot coffee, hot tea, iced tea, cola, and orange juice). A manager sends his assistant to the shop to get his breakfast – either a muffin and a hot beverage or a sandwich and a cold beverage.

$$(6 \times 2) + (8 \times 3) = 36$$

- Given nine players, in how many different ways can a manager write out a batting lineup?

$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times \ldots 1 \Rightarrow 9!$

$=$

- License plates consisting of 2 letter words followed by 4 digits

Permutation

- Given a collection of *n* distinct objects, any (linear) arrangement of these objects – permutation
- If there are *n* distinct objects and *r* is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size *r* for *n* objects is

$$P(n,r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r-1)$$

nPr

$$= n \times (n-1) \times (n-2) \times \cdots \times (n-r-1) \times \frac{(n-r) \times (n-r-1) \times \cdots 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \cdots 3 \times 2 \times 1}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

- If repetitions are allowed then, there are n^r possible arrangements.

COMPUTER

${}^{8}P_{5}$

$n = 8$

$r = 5$

$8 \times 7 \times 6 \times 5 \times 4 = {}^{8}P_{5}$

BALL

$n = 4$ $\quad r = 4$

$BAL_1L_2 \Rightarrow {}^4P_4 \Rightarrow 4!$

$\boxed{4!/2!}$

L

$BL_1AL_2 \Rightarrow BLAL$

$BL_2AL_1 \Rightarrow BLAL$

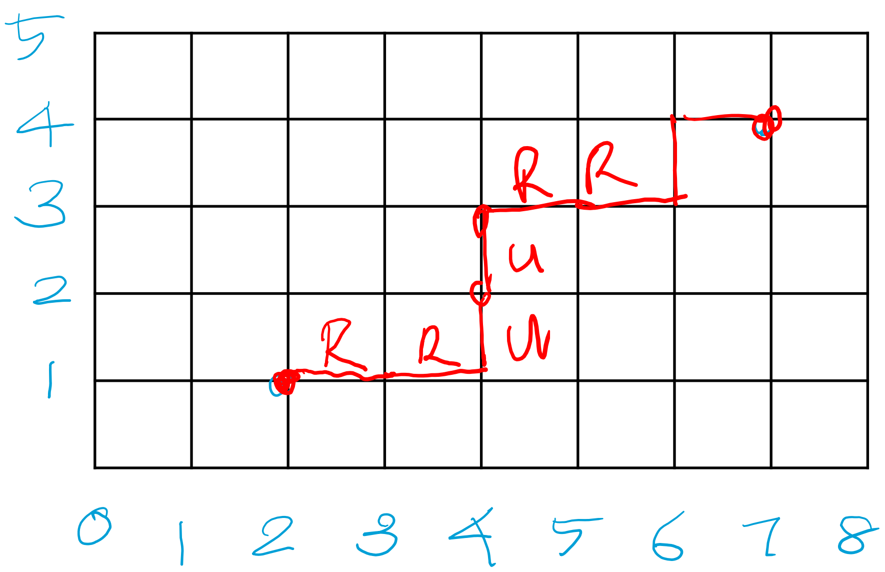
DATABASES

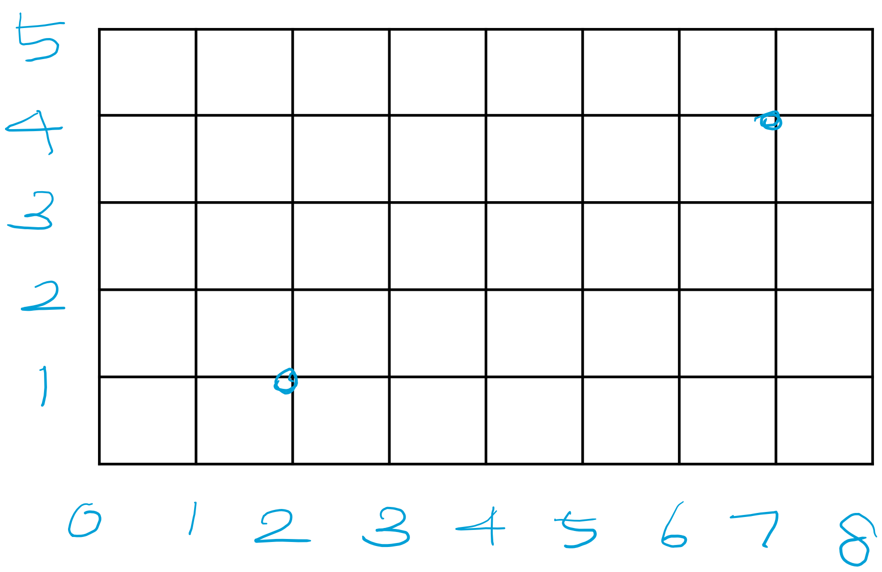
$9!/(3! \times 2!)$

- If there are n distinct objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable of second type, … and n_r indistinguishable of an r^{th} type, where $n_1 + n_2 + \ldots + n_r = n$, then there are $\frac{n!}{n_1!n_2!\ldots n_r!}$ (linear) arrangements of the given n objects

- Determine the number of paths in the xy-plane from (2,1) to (7,4) where each path is made up of individual steps going one unit to the right (R) or one unit upward (U).

RRUURRUR





$$\frac{8!}{5!\,3!}$$

- If n and k are positive integers, and n=2k, the number of ways in which we can arrange all of these n symbols is

a a

b b

k - distinct

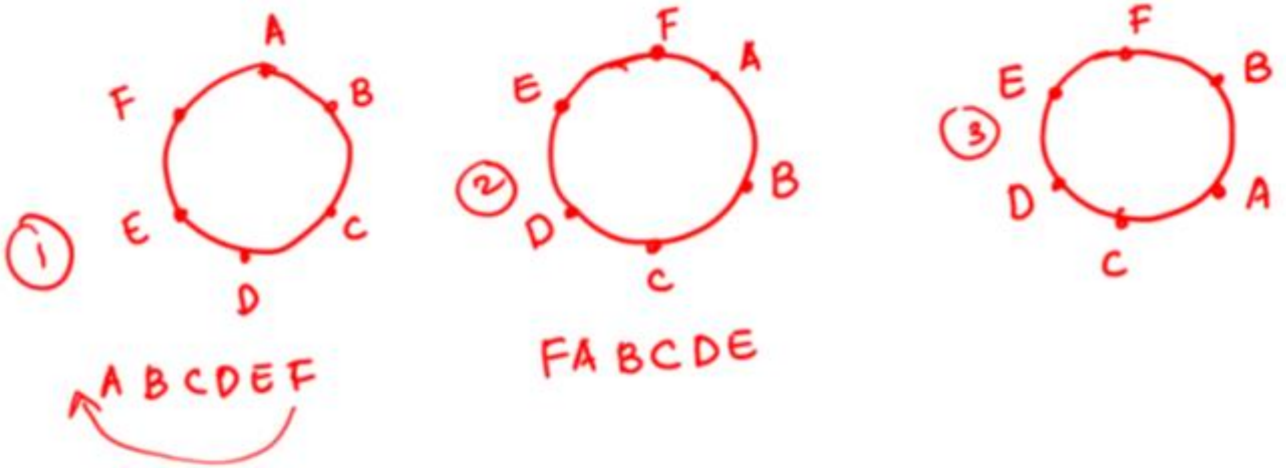
$n = 4$

$k = 2$

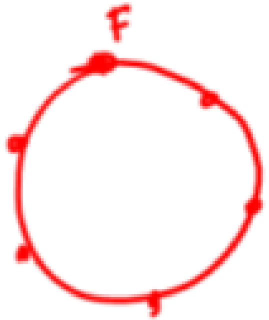
$$\frac{n!}{2! \cdot 2! \cdot 2! \cdots 2!}$$

$$n!/(2!)^k$$

If 6 people, designated as A, B, C, D, E, and F are seated in a round table, how many circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

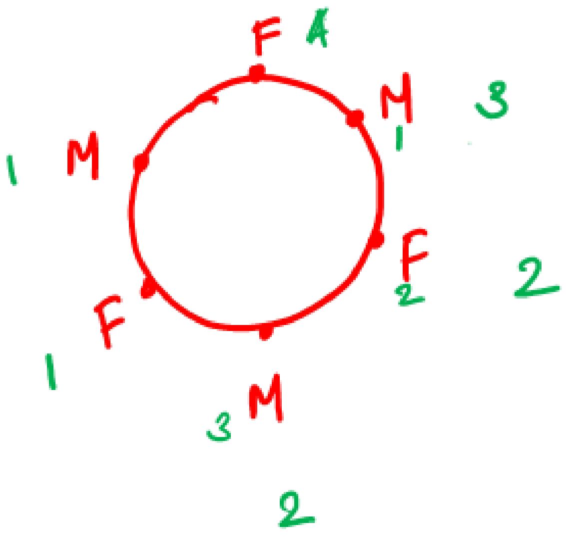


$$6!/6 = 5!$$



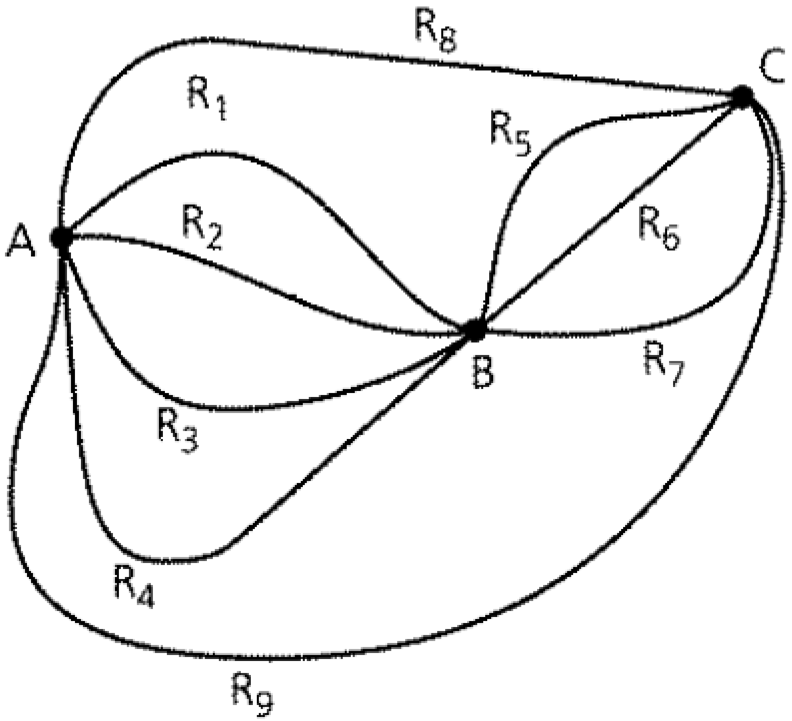
$\Rightarrow 5!$

Suppose in that 6 people, A, B, and C are female and D, E, and F are male. If we want to arrange people such that sexes alternate, how many ways can we arrange them?



$$\boxed{3! \times 2!}$$

Three small towns, designated by A, B and C are interconnected by a system of two-way roads, as shown in the figure



a) In how manv wavs can Linda travel from town A to town C? $2 + (4 \times 3) = 14$ ways

b) How many different round trips can Linda travel from town A to town C and back to town A? 14×14

c) How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.) 14×13

a) Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, and k are integer variables.)

```
counter := 0
for i := 1 to 12 do
   counter := counter + 1
for j := 5 to 10 do
   counter := counter + 2
for k := 15 downto 8 do
   counter := counter + 3
```

12×1
$+$
6×2
$+$
8×3

b) Which counting principle is at play in part (a)?

Combinations

- If we start with n distinct objects, each selection or combination of r of these objects with no reference to order, corresponds to r! permutations of size r from the n objects. Thus, the number of combinations of size r from a collection of size n is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}\ , \qquad 0 \le r \le n$$

$_nC_r$ $\quad \binom{n}{r}$

- How many n-digit binary numbers have exactly k 1s?

^nC_k

5- digit 2

11000

$\boxed{01001}$

10001

$^5C_2$

- Lynn and Patt decide to buy a powerball ticket. To win the grand prize, one must match 5 numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the 5 numbers and Patt selects the powerball. How many ways can they select the numbers for powerball ticket?

$$^{49}C_5 \times {}^{42}C_1$$

- A student taking an examination is directed to answer 7 of the 10 questions where at least 3 are selected from the first 5. How many ways can the students answer?

$$5C_3 \cdot 5C_4 + 5C_4 \cdot 5C_3 + 5C_5 \cdot 5C_2$$

7 10

	5	5
$\Rightarrow$ (1)	3	4
(2)	4	3
(3)	5	2

- A gym teacher must make up for 4 volleyball teams of nine girls each from the 36 girls in her P.E class. In how many ways can she select these four teams?

36 4

A A A A . . . . . . A 9

B B B B 9

C 9

D 9

$$\frac{36!}{9!\,9!\,9!\,9!}$$

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}$$

- How many number of arrangements of the letters in TALLAHASSEE is possible without adjacent A's?

3 A's

T L L H S S E E

$$\frac{8!}{(2!)^3} \times {}^{9}C_{3}$$

Concise way of writing the sum of a list of n+1 terms

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i$$

Results related to Combinations

- For integers n and r with $n \geq r \geq 0, C(n,r) = C(n,n-r)$
- Binomial Theorem: If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

In view of this theorem, $\binom{n}{k}$ is often referred to as a *binomial coefficient*.

For each integer $n > 0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, and

b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof: Part (a) follows from the binomial theorem when we set $x = y = 1$. When $x = -1$ and $y = 1$, part (b) results.

Proof: In the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\ \underset{\text{2nd factor}}{(x+y)}\ \underset{\text{3rd factor}}{(x+y)}\cdots\underset{n\text{th factor}}{(x+y)}$$

the coefficient of $x^k y^{n-k}$, where $0 \leq k \leq n$, is the number of different ways in which we can select k x's [and consequently $(n-k)$ y's] from the n available factors. (One way, for example, is to choose x from the first k factors and y from the last $n-k$ factors.) The total number of such selections of size k from a collection of size n is $C(n, k) = \binom{n}{k}$, and from this the binomial theorem follows.

- Multinomial Theorem: generalization of binomial theorem

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_t!},$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, . . . , and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.22, in

$$\binom{n}{n_1}\binom{n - n_1}{n_2}\binom{n - n_1 - n_2}{n_3} \cdots \binom{n - n_1 - n_2 - n_3 - \cdots - n_{t-1}}{n_t}$$

ways.

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient.* (When $t = 2$ this reduces to a binomial coefficient.)

Seven high school students stop at a restaurant, where each of them has one of the following: a cheeseburger, hot dog, taco and fish burger. How many different purchases are possible?

c h t f 7

1.	c, c, h, h, t, t, f	1.	x x | x x | x x | x
2.	c, c, c, c, h, t, f	2.	x x x x | x | x | x
3.	c, c, c, c, c, c, f	3.	x x x x x x | | | x
4.	h, t, t, f, f, f, f	4.	| x | x x | x x x x
5.	t, t, t, t, t, f, f	5.	| | x x x x x | x x
6.	t, t, t, t, t, t, t	6.	| | x x x x x x x |
7.	f, f, f, f, f, f, f	7.	| | | x x x x x x x

$n=4$, $r=\underline{7}$

$(n+r-1)C_r$

$10C_7$

Combinations with repetitions

- The number of combinations of n objects taken r at a time, with repetition is

$$C(n+r-1,r)=\frac{(n+r-1)!}{r!\,(n-1)!}$$

$${}^{n}C_{r}=\frac{n!}{r!\,(n-r)!}$$

$n+r-1-r$

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them $1000 in Christmas bonus checks, where each check will be written for a multiple of $100.

- Allowing the situation in which one or more of the vice presidents get nothing, then in how many ways can President Helen distribute the bonus checks?

$n=4$ \$1000 \$100

$C(n+r-1, r) = C(4+10-1, 10)$ $\hookrightarrow 10 \leftarrow r$

$= C(13, 10)$

- If each vice president must get at least $100, then in how many ways can President Helen distribute the bonus checks?

$n=4$

$C(4+6-1, 6) = {}^9C_6$

$\$1000 - \$400$

$= \$600$

$r=6$

- If each vice president must get at least \$100 and Mona gets at least \$500, then in how many ways can President Helen distribute the bonus checks?

\$100

$\Downarrow$

\$400

Mona $\Leftarrow$ \$500

\$400

\$1000 − \$800 = \$200

$n = 4$

$r = 2$

$\boxed{{}^{5}C_{2}}$

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

$n = 4 \quad r_b = 3$

$C(4+3-1, 3) = {}^6C_3$

$r_o = 6 \quad n = 4$

$C(4+6-1, 6)$

$= C(9,6)$

$\Rightarrow \boxed{{}^6C_3 \times {}^9C_6}$

Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \qquad \text{where } x_i \geq 0 \quad \text{for all } 1 \leq i \leq 4.$$

$n = 4$ $\qquad r = 7$

$C(10, 7)$

The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

The number of selections, with repetition, of size r from a collection of size n.

The number of ways r identical objects can be distributed among n distinct containers.

How many solutions are there for

$$x_1 + x_2 + \cdots + x_6 \underset{=}{<} 10$$

$x_i \geq 0$

$1 \leq i \leq 6$

$x_1 + x_2 + \cdots + x_6 + x_7 = 10$

$x_7 > 0$

$n = 7 \qquad r = 9$

$C(7+9-1, 9) = {}^{15}C_9$

Total number of terms in a binomial expansion

$(x+y)^{\underline{n}}$

$\binom{n}{k} x^{k} y^{n-k}$

$k+n-k \Rightarrow n$

$x^{n_1} y^{n_2} \Rightarrow \boxed{n_1+n_2 = n}$

$n_\alpha = 2 \qquad r = n$

$C(n_\alpha + r - 1, r) = C(2+n-1, n) = C(n+1, n) = \dfrac{(n+1)!}{n!\,(\cancel{n}+1-\cancel{n})!}$

$= \dfrac{n!\,(n+1)}{n!}$

$= \underline{\underline{n+1}}$

Total number of terms in a multinomial expansion

$(x+y+z+w)^{10}$

$\boxed{p+q+r+s = 10}$

$(x+y+z+\cdots)^n$

$x^p y^q z^r w^s$

$n=4 \qquad r=10$

$C(n+r-1, r)$

$C(4+10-1, 10) = \underline{C(13, 10)}$

Determine the number of compositions for the number 4

$\boxed{C(n+r-1, r)}$

T_0 1. 4

T_1 2. 1+3

3. 3+1

4. 2+2

T_2 5. 1+1+2

6. 1+2+1

7. 2+1+1

T_3 8. 1+1+1+1

$x_1 + x_2 = 4$

$\boxed{x_1' + x_2' = 2}$ $n=2, r=2 \Rightarrow 3C_2$

$y_1 + y_2 + y_3 = 4$

$\boxed{y_1' + y_2' + y_3' = 1}$ $n=3, r=1 \Rightarrow 3C_1$

$z_1 + z_2 + z_3 + z_4 = 4$

$\boxed{z_1' + z_2' + z_3' + z_4' = 0}$ $n=4, r=0 \Rightarrow 3C_0$

$3C_3 + 3C_2 + 3C_1 + 3C_0 = \sum_{k=0}^{3} 3C_k = 2^3$

Determine the number of compositions for the number 7

The counter at Patti and Terri's Bar has 15 bar stools. Upon entering the bar Darrell finds the stools occupied as follows:

$$\underbrace{O\ O}_{1}\ \underbrace{E}_{2}\ \underbrace{O\ O\ O\ O}_{3}\ \underbrace{E\ E\ E}_{4}\ \underbrace{O\ O\ O}_{5}\ \underbrace{E}_{6}\ \underbrace{O}_{7},$$

where O indicates an occupied stool and E an empty one. (Here we are not concerned with the occupants of the stools, just whether or not a stool is occupied.) In this case we say that the occupancy of the 15 stools determines seven runs.

find the total number of ways five E's and 10 O's can determine seven runs.

Case 1: Starting with O

4 runs Os

3 runs Es

$$x_1 + x_3 + x_5 + x_7 = 10$$

$$x_2 + x_4 + x_6 = 5$$

$$\boxed{\begin{gathered} y_1 + y_3 + y_5 + y_7 = 6 \\ y_2 + y_4 + y_6 = 2 \end{gathered}}$$

Case 2: Starting with E

4 runs E

3 runs O

$$\underbrace{E}_{1}\ \underbrace{OOO}_{2}\ \underbrace{EE}_{3}\ \underbrace{OO}_{4}\ \underbrace{E}_{5}\ \underbrace{OOOOO}_{6}\ \underbrace{E}_{7}$$

$$x_1 + x_3 + x_5 + x_7 = 5$$

$$x_2 + x_4 + x_6 = 10$$

$$\boxed{\begin{gathered} y_1 + y_3 + y_5 + y_7 = 1 \\ y_2 + y_4 + y_6 = 7 \end{gathered}}$$

Principle of inclusion and exclusion

Let S represent the set of 100 students enrolled in the freshman engineering program at Central College. Then $|S| = 100$. Now let c_1, c_2 denote the following conditions (or properties) satisfied by some of the elements of S:

c_1: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Freshman Composition.

c_2: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Economics.

Suppose that 35 of these 100 students are enrolled in Freshman Composition and that 30 of them are enrolled in Introduction to Economics.
If nine of these 100 students are enrolled in both Freshman Composition and Introduction to Economics then we write $N(c_1c_2) = 9$.

$N = 100 \leftarrow |S|$ $\quad N(C_1) = 35$ $\quad N(C_2) = 30$ $\quad N(C_1C_2) = 9$

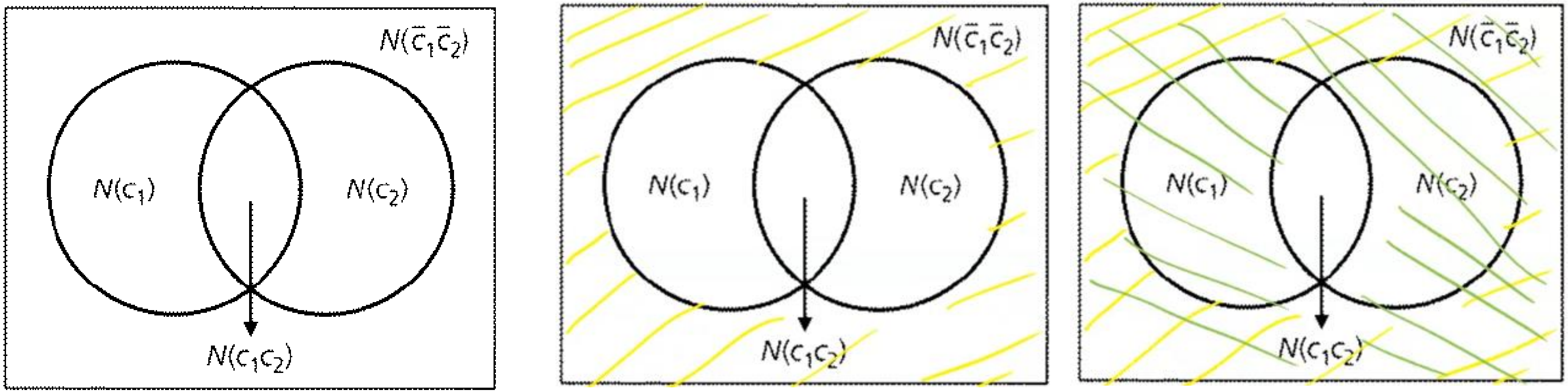
$N(\bar{C}_1) = \underset{N}{100} - \underset{N(C_1)}{35} = 65$

$N(\bar{C}_2) = N - N(C_2) = 70$

$N(C_1\bar{C}_2) = N(C_1) - N(C_1C_2) = 26$

$N(\bar{C}_1C_2) = N(C_2) - N(C_1C_2)$

$= 21$



$$N(\bar{c}_1\bar{c}_2) = N - [N(c_1) + N(c_2)] + N(c_1 c_2)$$

$$N(\bar{c}_1\bar{c}_2) = N(\overline{c_1 c_2})$$

c_3: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Fundamentals of Computer Programming.

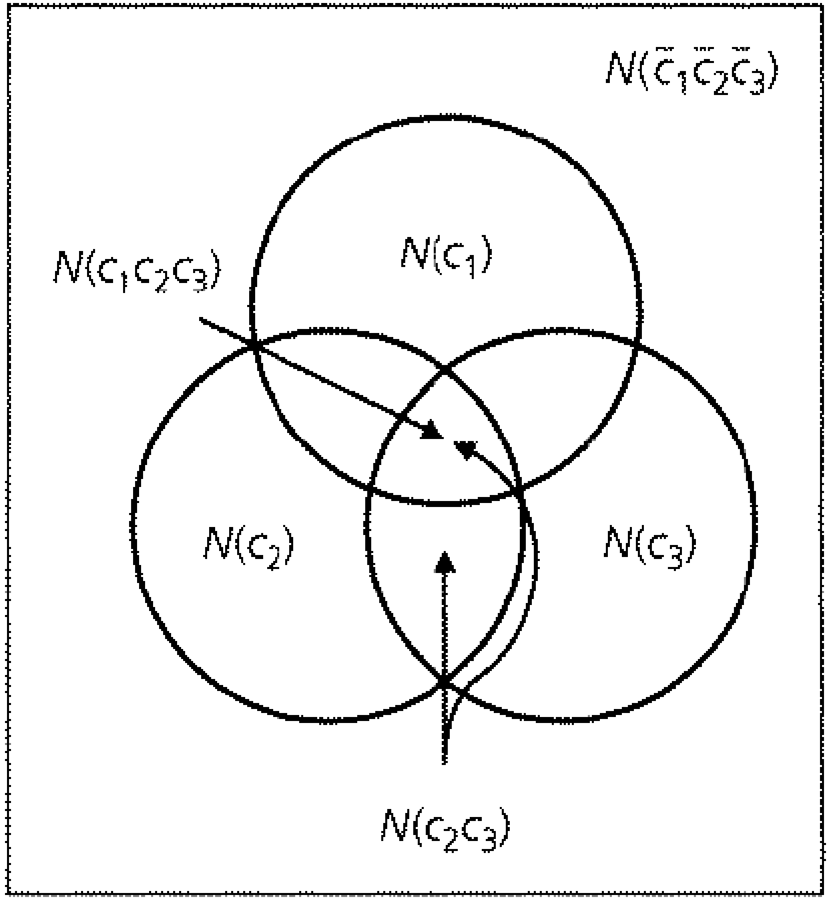
also given that $N(c_3) = 30$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$

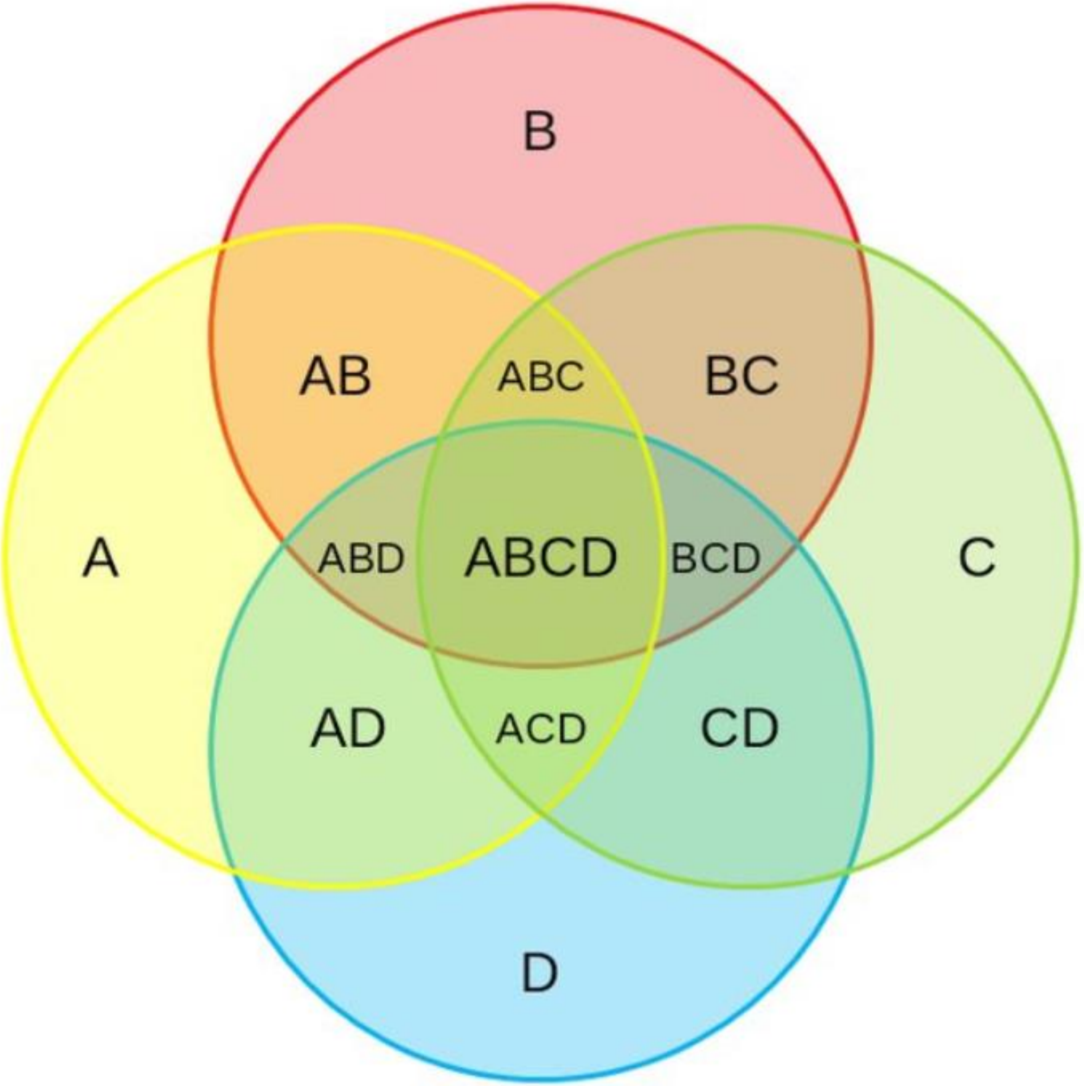
$N(\overline{c}_1\overline{c}_2\overline{c}_3) =$ $N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_1C_3) + N(C_2C_3)] - N(C_1C_2C_3)$

$N(\overline{c}_3) =$ $N - N(C_3) = 70$

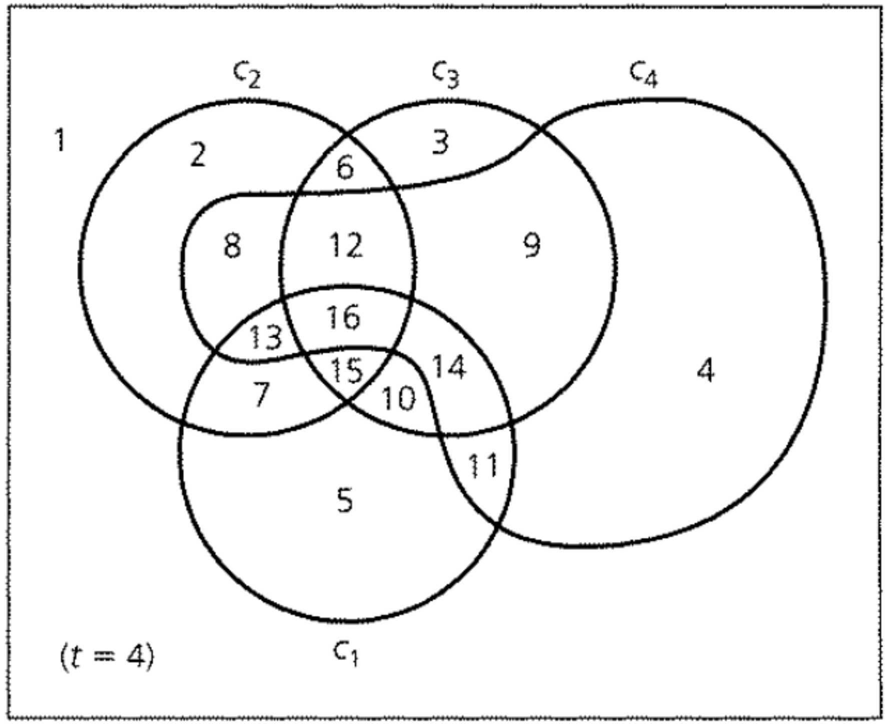
$N(\overline{c}_1\overline{c}_3) =$ $N - [N(C_1) + N(C_3)] + N(C_1C_3) = 46$

$N(\overline{c}_2\overline{c}_3)$




B
AB
ABC
BC
A
ABD
ABCD
BCD
C
AD
ACD
CD
D

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$
$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)]$$
$$+ N(c_1c_2c_3c_4).$$



$x \in S$

0) once in LHS , RHS N

1) $x \in C_1$; LHS 0 times ; RHS $N - N(C_1)$

$1 - 1 = 0$

$1 - \binom{1}{1} = 0$

2) $x \in C_2, x \in C_4$; LHS $= 0$; RHS

$N - [N(C_2) + N(C_4)] + N(C_2C_4)$

$1 - [1 + 1] + 1 = 0$

$1 - \binom{2}{1} + \binom{2}{2}$

3) $x \in C_1, C_2, C_4$; LHS = 0 ; RHS $N - [N(C_1) + N(C_2) + N(C_4)] + [N(C_1C_2) + N(C_1C_4) + N(C_2C_4)] - N(C_1C_2C_4) = 1 - [1+1+1] + [1+1+1] - 1$

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3}$$

4) $x \in C_1C_2C_3C_4$

RHS = $1 - [1+1+1+1] + [1+1+1+1+1+1] - [1+1+1+1] + 1$

$$1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$$

c_4: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Design.

We already know that $N(c_1) = 35$, $N(c_2) = 30$, $N(c_3) = 30$, $N(c_1c_2) = 9$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$. If $N(c_4) = 41$, $N(c_1c_4) = 13$, $N(c_2c_4) = 14$, $N(c_3c_4) = 10$, $N(c_1c_2c_4) = 6$, $N(c_1c_3c_4) = 6$, $N(c_2c_3c_4) = 6$, and $N(c_1c_2c_3c_4) = 4$, then, using

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) =$$

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) =$$

$$N(\overline{c}_2\overline{c}_3\overline{c}_4) = N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) + N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$$

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$$

$$
\begin{aligned}
N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) &= N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) \\
&= \Big\{N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\
&\quad - N(c_2c_3c_4)\Big\} - \Big\{N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\
&\quad + [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\
&\quad - [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\Big\}, \text{ or}
\end{aligned}
$$

$$
\begin{aligned}
N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) &= N(c_1) - [N(c_1c_2) + N(c_1c_3) + N(c_1c_4)] \\
&\quad + [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4)] - N(c_1c_2c_3c_4).
\end{aligned}
$$

The Principle of Inclusion and Exclusion. Consider a set S, with $|S| = N$, and conditions c_i, $1 \leq i \leq t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy *none* of the conditions c_i, $1 \leq i \leq t$, is denoted by $\overline{N} = N(\overline{c}_1\overline{c}_2\overline{c}_3 \cdots \overline{c}_t)$ where

$$\begin{aligned}\overline{N} = N &- [N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_t)] \\ &+ [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots \\ &+ N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)] + \cdots + (-1)^t N(c_1c_2c_3 \cdots c_t),\end{aligned}$$

or

$$\begin{aligned}\overline{N} = N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i < j \leq t} N(c_ic_j) - \sum_{1 \leq i < j < k \leq t} N(c_ic_jc_k) + \cdots \\ + (-1)^t N(c_1c_2c_3 \cdots c_t).\end{aligned}$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms in Eq. Consequently, x contributes a count of 1 to each side of the equation.

The other possibility is that x satisfies *exactly* r of the conditions where $1 \leq r \leq t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. x is counted

(1) One time in N.

(2) r times in $\sum_{1 \leq i \leq t} N(c_i)$. (Once for each of the r conditions.)

(3) $\binom{r}{2}$ times in $\sum_{1 \leq i < j \leq t} N(c_i c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

(4) $\binom{r}{3}$ times in $\sum_{1 \leq i < j < k \leq t} N(c_i c_j c_k)$.

. . . .

$\binom{r}{r} = 1$ time in $\sum N(c_{i_1} c_{i_2} \cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq., x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times}$$

the number of elements in S that satisfy at least one of the conditions c_i, where $1 \leq i \leq t$, is given by $N(c_1 \text{ or } c_2 \text{ or } \ldots \text{ or } c_t) = N - \overline{N}$.

Determine the number of positive integers n where $1 \leq n \leq 100$ and n is *not* divisible by 2, 3, or 5.

Here $S = \{1, 2, 3, \ldots, 100\}$ and $N = 100$. For $n \in S$, n satisfies

a) condition c_1 if n is divisible by 2,

b) condition c_2 if n is divisible by 3, and

c) condition c_3 if n is divisible by 5.

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

$N(c_1) = 50$ $\quad N(c_2) = 33$ $\quad N(c_3) = 20$

$N(c_1c_2) = 16$ $\quad N(c_1c_3) = 10$ $\quad N(c_2c_3) = 6$

$N(c_1c_2c_3) = 3$

$$\bar{N} = 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 =$$

the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

with the extra restriction that $x_i \le 7$, for all $1 \le i \le 4$.

$S_0 = N = 21C_{18}$

4 conditions

$\boxed{C_1 : x_1 > 7 \quad x_1 \geq 8}$ $\quad C_2 : x_2 > 7 \quad x_2 \geq 8$

$C_3 : x_3 \geq 8 \quad C_4 : x_4 \geq 8$

$\overset{8}{x_1} + x_2 + x_3 + x_4 = 18$

$y_1 + y_2 + y_3 + y_4 = 10$

$y_1 + y_2 + y_3 + y_4 = 2$

$\bar{N} = N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = S_0 - S_1 + S_2 - \underbrace{S_3 + S_4}_{0}$

$N(C_1) = 13C_{10} \qquad N(C_2) = 13C_{10}$

$S_1 = \binom{4}{1} 13C_{10}$

$N(C_1C_2) = 5C_2$

$S_2 = \binom{4}{2} 5C_2$

$N(C_1C_2C_3) = 0$

$\bar{N} = 21C_{18} - 4C_1 \, 13C_{10} + 4C_2 \cdot 5C_2 - 0 + 0 = 246$

In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns *car*, *dog*, *pun*, or *byte* occurs?

$N = 26!$

$N(c_1) = 24!$ car, b, d, e, ... - 24

$N(c_2) = N(c_3) = 24!$ $N(c_4) = 23!$ $26-4 = \underline{22}+1$

$N(c_1 c_2) = 22!$ $N(c_1 c_3) = N(c_2 c_3) = 22!$

$N(c_i c_4) = 21!$ $i \neq 4$

$N(c_1 c_2 c_3) = 20!$ $\underline{N(c_i c_j c_4)} = \underline{19!}$ $1 \le i < j \le 3$

$N(c_1 c_2 c_3 c_4) = 17!$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$

$=$

Six married couples are to be seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband?

$S_0 = N = 11!$

c_1: couple 1

$10 + 1 = 11$

c_2 : couple 2

$N(c_1) = 2 \cdot 10!$

$S_1 = \binom{6}{1} 2 \cdot 10!$

$N(c_1 c_2) = 2^2 \cdot 9!$

$S_2 = \binom{6}{2} 2^2 \cdot 9!$

$N(c_1 c_2 c_3) = 2^3 \cdot 8!$

$S_3 = \binom{6}{3} \cdot 2^3 \cdot 8!$

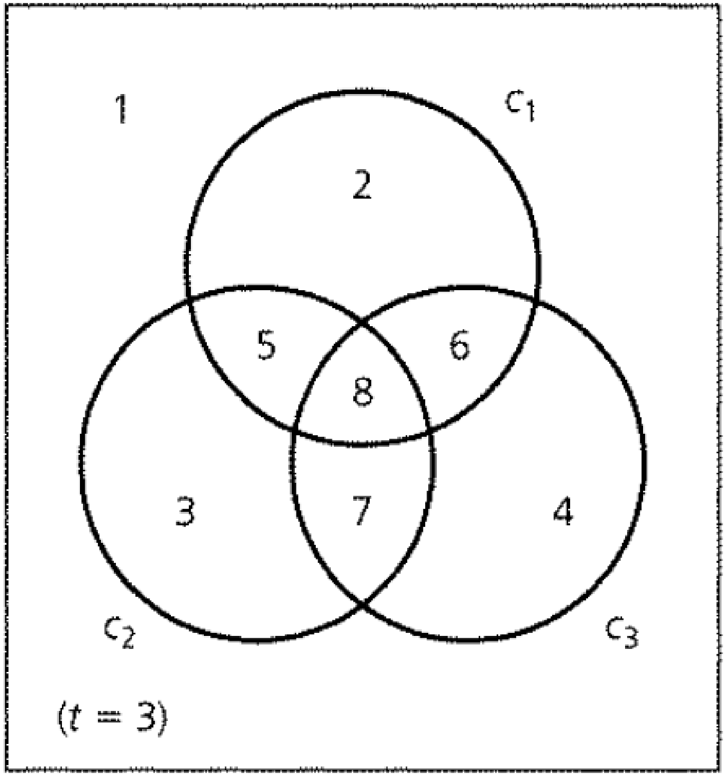
$S_4 = \binom{6}{4} 2^4 \cdot 7!$

$S_5 = \binom{6}{5} 2^5 \cdot 6!$

$S_6 = \binom{6}{6} 2^6 \; 5!$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$

Generalization principle



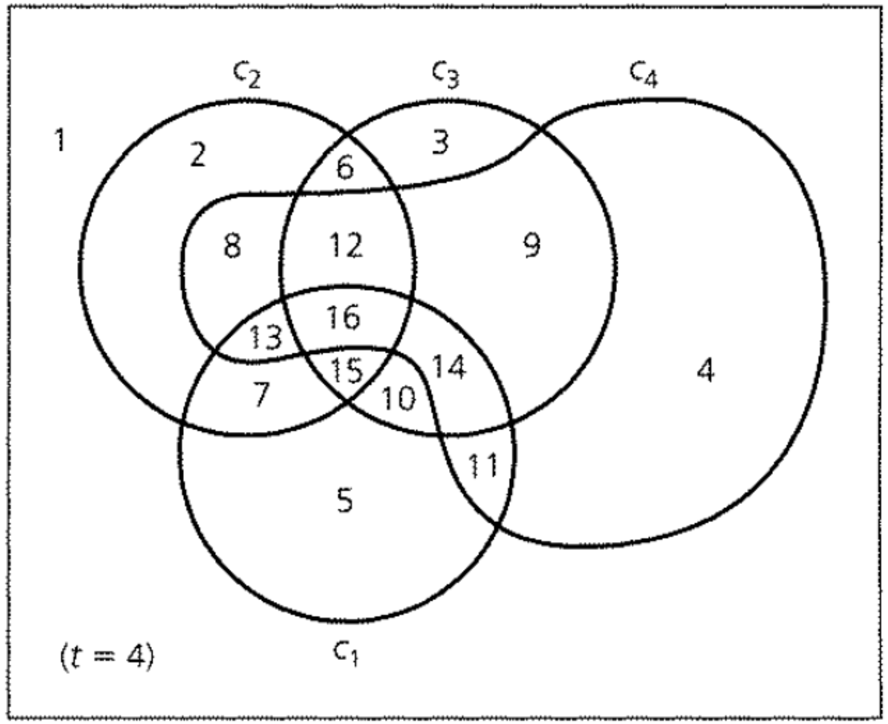
$$E_1 = N(c_1) + N(c_2) + N(c_3) - 2\,[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3).$$

$$E_1 = S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$$

$$E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3) = S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$$

$$E_3 = N(c_1c_2c_3) = S_3$$

E_1 [regions 2, 3, 4, 5]:



$$\begin{aligned} E_1 &= [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\ &\quad - 2\,[N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &\quad + 3\,[N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] \\ &\quad - 4N(c_1c_2c_3c_4) \\ &= S_1 - 2S_2 + 3S_3 - 4S_4 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4. \end{aligned}$$

E_2 [regions 6–11]:

$$E_2 = S_2 - 3S_3 + 6S_4 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4.$$

S_2	S_3	S_4
$N(c_1c_2)$: 7, 13, 15, 16 $N(c_1c_3)$: 10, 14, 15, 16 $N(c_1c_4)$: 11, 13, 14, 16 $N(c_2c_3)$: 6, 12, 15, 16 $N(c_2c_4)$: 8, 12, 13, 16 $N(c_3c_4)$: 9, 12, 14, 16	$N(c_1c_2c_3)$: 15, 16 $N(c_1c_2c_4)$: 13, 16 $N(c_1c_3c_4)$: 14, 16 $N(c_2c_3c_4)$: 12, 16	$N(c_1c_2c_3c_4)$: 16

$$E_3 = S_3 - 4S_4 = S_3 - \binom{4}{1}S_4 \qquad E_4 = S_4$$

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t .$$

COROLLARY

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t .$$

a) When x satisfies fewer than m conditions, it contributes a count of 0 to each of the terms E_m, S_m, S_{m+1}, $\ldots$, S_t, so it is not counted on either side of the equation.

b) When x satisfies exactly m of the conditions, it is counted once in E_m and once in S_m, but not in S_{m+1}, $\ldots$, S_t. Consequently, it is included once in the count for either side of the equation.

c) Suppose x satisfies r of the conditions, where $m < r \le t$. Then x contributes nothing to E_m. Yet it is counted $\binom{r}{m}$ times in S_m, $\binom{r}{m+1}$ times in S_{m+1}, $\ldots$, and $\binom{r}{r}$ times in S_r, but 0 times for any term beyond S_r. So on the right-hand side of the equation, x is counted $\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$ times.

For $0 \le k \le r-m$,

$$\begin{aligned}\binom{m+k}{k}\binom{r}{m+k} &= \frac{(m+k)!}{k!\,m!}\cdot\frac{r!}{(m+k)!(r-m-k)!}\\ &= \frac{r!}{m!}\cdot\frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!}\cdot\frac{(r-m)!}{k!(r-m-k)!}\\ &= \binom{r}{m}\binom{r-m}{k}.\end{aligned}$$

Consequently, on the right-hand side of Eq. (1), x is counted

$$\begin{aligned}
&\binom{r}{m}\binom{r-m}{0} - \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m} \\
&\qquad = \binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r-m}{r-m}\right] \\
&\qquad = \binom{r}{m}[1-1]^{r-m} = \binom{r}{m}\cdot 0 = 0 \text{ times,}
\end{aligned}$$

Derangement

- A derangement is a permutation with no fixed points
- Example: the derangements of {1,2,3} are {2, 3, 1} and {3, 1, 2} {3, 2, 1}
- number of derangements of an n-element set is called the n^{th} derangement number or rencontres number, or the subfactorial of n and is sometimes denoted !n or D_n.

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$n! \cdot e^{-1}$$

$$n! - \underline{\binom{n}{1}(n-1)!} + \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)!$$

$$\binom{n}{k}(n-k)! = \frac{n!}{k!\,\cancel{(n-k)!}}\cancel{(n-k)!}$$

$$n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots (-1)^n \frac{n!}{n!} = \frac{n!}{k!}$$

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \cdot \boxed{\frac{1}{n!}}\right.$$

$$= n! \boxed{\sum_{k=0}^{n} (-1)^k \cdot 1/k!} = \boxed{n!\,e^{-1}}$$

While at the racetrack, Ralph bets on each of the ten horses in a race to come in according to how they are favored. In how many ways can they reach the finish line so that he loses all of his bets?

d_{10}

Peggy has seven books to review for the C–H Company, so she hires seven people to review them. She wants two reviews per book, so the first week she gives each person one book to read and then redistributes the books at the start of the second week. In how many ways can she make these two distributions so that she gets two reviews (by different people) of each book?

7 b 7 p

1st week $= 7!$

2nd week $= d_7 = 7!\,e^{-1}$

$7! \cdot d_7$

$$7! \cdot 7!\,e^{-1}$$

Generating Functions

$x_1 + x_2 + x_3 = a$

While shopping one Saturday, Mildred buys 12 oranges for her children, Grace, Mary, and Frank. In how many ways can she distribute the oranges so that Grace gets at least four, and Mary and Frank get at least two, but Frank gets no more than five?

$g \geq 4$

$m \geq 2$

$\boxed{5 \leq} f \geq 2$

$$c_1 + c_2 + c_3 = 12 \text{ where } 4 \leq c_1,\ 2 \leq c_2, \text{ and } 2 \leq c_3 \leq 5$$

$$(x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5).$$

x^{12}

$x^4 \cdot x^3 \cdot x^5 = x^{12}$

$x^5 \cdot x^4 \cdot x^3 = x^{12}$

G	M	F	G	M	F
4	3	5	6	2	4
4	4	4	6	3	3
4	5	3	6	4	2
4	6	2	7	2	3
5	2	5	7	3	2
5	3	4	8	2	2
5	4	3			
5	5	2			

If there is an unlimited number (or at least 24 of each color) of red, green, white, and black jelly beans, in how many ways can Douglas select 24 of these candies so that he has an even number of white beans and at least six black ones?

$$f(x) = (1 + x + x^2 + \cdots + x^{24})^2(1 + x^2 + x^4 + \cdots + x^{24})(x^6 + x^7 + \cdots + x^{24})$$

How many integer solutions are there for the equation $c_1 + c_2 + c_3 + c_4 = 25$ if $0 \le c_i$ for all $1 \le i \le 4$?

$$f(x) = (1 + x + x^2 + \cdots + x^{25})^4$$

co-efficient x^{25}.

$$g(x) = (1 + x + x^2 + \cdots + x^{25} + x^{26} + \cdots)^4$$

Definition

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function

$a_0 \, H_0(x) + a_1 \, H_1(x) + a_2 \, H_2(x)$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

For any $n \in \mathbf{Z}^+$,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

so $(1+x)^n$ is the generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots.$$

a, b & c

$a \quad b \quad c \qquad a+b+c$

$ab+ac+bc$

abc

$$(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+bc+ca)x^2+(abc)x^3$$

$\Rightarrow a_0, a_1, a_2, a_3 \ldots$ $H_0(x)$ $H_1(x)$ $H_2(x) \ldots$ E_1

E_2

$b_0, b_1, b_2, b_3 \ldots$

For $n \in \mathbf{Z}^+$,

$$(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + x^3 + \overset{x^{n-1}}{\cdots} + x^n).$$

$$1 + \underbrace{\cancel{x} + x^2 + x^3 + \cdots + x^n}_{} - \underbrace{\cancel{x} - x^2 - x^3 - \cdots - x^n}_{} - x^{n+1}$$

$$= 1 - x^{n+1}$$

So

$$\frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n,$$

and $(1 - x^{n+1})/(1 - x)$ is the generating function for the sequence 1, 1, 1, . . . , 1, 0, 0, 0, . . . , where the first $n + 1$ terms are 1.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$$

valid for all real x where $|x| < 1$

$\frac{1}{1-x}$ is the generating function for the sequence 1, 1, 1, 1, . . .

taking the derivative yields

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$\frac{1}{(1-x)^2}$ is the generating function for the sequence 1, 2, 3, 4, . . .

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

is the generating function for the sequence 0, 1, 2, 3, ...

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + \cdots)$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots$$

$\dfrac{x+1}{(1-x)^3}$ generates $1^2, 2^2, 3^2, \ldots$

$\dfrac{x(x+1)}{(1-x)^3}$ generates $0^2, 1^2, 2^2, 3^2, \ldots$.

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$f_1(x) = x\frac{d}{dx}f_0(x) = \frac{x}{(1-x)^2}$$

$$= 0 + x + 2x^2 + 3x^3 + \cdots$$

$$f_2(x) = x\frac{d}{dx}f_1(x) = \frac{x^2+x}{(1-x)^3}$$

$$= 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots$$

$$f_3(x) = x\frac{d}{dx}f_2(x) = \frac{x^3+4x^2+x}{(1-x)^4}$$

$$= 0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \cdots$$

$$f_4(x) = x\frac{d}{dx}f_3(x) = \frac{x^4+11x^3+11x^2+x}{(1-x)^5}$$

$$= 0^4 + 1^4x + 2^4x^2 + 3^4x^3 + \cdots$$

$1/(1-2x)$ is the generating function for

$$\frac{1}{1-x} \qquad \frac{1}{1-y} \qquad \boxed{x} = 2x$$

$$\frac{1}{1-2x} \qquad y = 2x$$

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots$$

$$= 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots \Rightarrow 1, 2, 2^2, 2^3, 2^4, \ldots$$

for each $a \in \mathbf{R}$, it follows that $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots = 1 + ax + a^2x^2 + a^3x^3 + \cdots$, so $1/(1-ax)$ is the generating function for the sequence $1\ (= a^0), a\ (= a^1), a^2, a^3, \ldots$.

$g(x) = f(x) - x^2$ is the generating function for

$f(x) = 1/1-x$

$= 1+x+x^2+x^3+ \cdots \quad - x^2$

$= 1+x+0x^2+x^3+ \cdots$

$1, 1, 0, 1, 1, \ldots$

$h(x) = f(x) + 2x^3$

$1, 1, 1, 3, 1, 1, \ldots$

find a generating function for the sequence 0, 2, 6, 12, 20, 30, 42, . . .

$a_0 = 0 \quad = 0^2 + 0$

$a_1 = 2 \quad = 1^2 + 1$

$a_2 = 6 \quad = 2^2 + 2$

$a_3 = 12 \quad = 3^2 + 3$

$a_n = n^2 + n$

$$\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(x+1) + x(1-x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

With $n, r \in \mathbf{Z}^+$ and $n \geq r > 0$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

If $n \in \mathbf{R}$, we use

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

as the definition of $\binom{n}{r}$.

for each *real* number n, we define $\binom{n}{0} = 1$.

$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} \\ &= \frac{(-1)^r(n)(n+1)(n+2)\cdots(n+r-1)}{r!} \\ &= \frac{(-1)^r(n+r-1)!}{(n-1)!\,r!} = (-1)^r\binom{n+r-1}{r}.\end{aligned}$$

For $n \in \mathbf{Z}^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is given by

$$\begin{aligned}(1+x)^{-n} &= 1 + (-n)x + (-n)(-n-1)x^2/2! \\ &\quad + (-n)(-n-1)(-n-2)x^3/3! + \cdots \\ &= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} x^r \\ &= \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r.\end{aligned}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = \sum_{r=0}^{\infty} \binom{-n}{r} x^r$$

$(1+x)^{-n}$ is the generating function for the sequence $\binom{-n}{0}, \binom{-n}{1}, \binom{-n}{2}, \binom{-n}{3}, \ldots$.

Find the coefficient of x^5 in $(1-2x)^{-7}$.

$(1-\underset{\sim}{2x})^{-7}$ $\qquad x^5$ $\qquad y = -2x$

$(1+y)^{-7}$ $\qquad n = -7$ $\qquad r = 5$

$\binom{-7}{5} y^5 = (-1)^5 \binom{7+5-1}{5} (-2)^5$

$= \binom{11}{5} 32$

For each real number n, the Maclaurin series expansion for $(1+x)^n$ is

$$1 + nx + n(n-1)x^2/2! + n(n-1)(n-2)x^3/3! + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r.$$

$(1+8x)^{1/3}$

$$(1+3x)^{-1/3} = 1 + \sum_{r=1}^{\infty} \frac{(-1/3)\overset{-4/3}{(-1/3-1)}\overset{-7/3}{(-1/3-2)}\cdots\cdots}{r!}(3x)^r.$$

Determine the coefficient of x^{15} in $f(x) = (x^2 + x^3 + x^4 + \cdots)^4$.

$$(x^2+x^3+x^4+\cdots) = x^2(1+x+x^2+x^3+\cdots)$$

$$f(x) = [x^2(1+x+x^2+x^3+\cdots)]^4 = x^8 \cdot (1+x+x^2+x^3+\cdots)^4$$

$$15-8 = 7$$

$$x^7 \text{ in } (1+x+x^2+\cdots)^4 \Rightarrow (1-x)^{-4}$$

$$\Rightarrow 1/1-x$$

$$\binom{-4}{7}(-1)^7 = (-1)^7\binom{4+7-1}{7}(-1)^7 = {}^{10}C_7$$

Identities

For all $m, n \in \mathbf{Z}^+$, $a \in \mathbf{R}$,

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$
2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$
3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$
4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + \cdots + x^n$
5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$
6) $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$

$$= \sum_{i=0}^{\infty}(ax)^i = \sum_{i=0}^{\infty} a^i x^i$$

$$= 1 + ax + a^2x^2 + a^3x^3 + \cdots$$

$$\textbf{7)}\ 1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$$
$$= \textstyle\sum_{i=0}^{\infty} \binom{-n}{i}x^i$$
$$= 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$$
$$= \textstyle\sum_{i=0}^{\infty}(-1)^i\binom{n+i-1}{i}x^i$$
$$\textbf{8)}\ 1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots$$
$$= \textstyle\sum_{i=0}^{\infty} \binom{-n}{i}(-x)^i$$
$$= 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$$
$$= \textstyle\sum_{i=0}^{\infty} \binom{n+i-1}{i}x^i$$

If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, and $h(x) = f(x)g(x)$, then $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where for all $k \geq 0$,

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 = \sum_{j=0}^{k} a_j b_{k-j}.$$

In how many ways can we select, with repetitions allowed, r objects from n distinct objects?

$$(1 + x + x^2 + x^3 + \cdots)^n$$

co-efficient of x^r.

$$\left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$

$$\boxed{\binom{n+r-1}{r}}$$

In how many ways can a police captain distribute 24 rifle shells to four police officers so that each officer gets at least three shells, but not more than eight?

$$f(x) = (x^3 + x^4 + x^5 + \cdots + x^8)^4$$

Co-efficient of x^{24} in $f(x)$

$$f(x) = \left[x^3 (1 + x + x^2 + \cdots + x^5)\right]^4 = x^{12} \cdot (1 + x + x^2 + \cdots x^5)^4$$

$$= x^{12} \cdot \left[\frac{1 - x^6}{(1-x)}\right]^4$$

co-efficient of x^{12} in $(1-x^6)^4 \cdot (1-x)^{-4}$ $\binom{-4}{12}(-x)^{12}$

$$= \left[1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + x^{24}\right] \cdot \left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \cdots\right]$$

$$\binom{-4}{12}(-1)^{12} + \binom{4}{1}(-1)\binom{-4}{6}(-1)^6 + \binom{4}{2}\binom{-4}{0} = \binom{4+12-1}{12} - \binom{4}{1}\binom{4+6-1}{6} + \binom{4}{2}$$

$= 125$

Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$.

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$= \frac{A(x-2)^2 + B(x-3)(x-2) + C(x-3)}{(x-3)(x-2)^2}$$

$$1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$$

$$= A(x^2-4x+4) + B(x^2-5x+6) + C(x-3)$$

$$0.x^2 + 0.x + 1 = (A+B)x^2 + (-4A-5B+C)x + (4A+6B-3C)$$

$A+B=0 \qquad -4A-5B+C=0 \qquad 4A+6B-3C=1$

$A=1,\ B=-1,\ C=-1$

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= \left(\frac{-1}{3}\right)\frac{1}{1-x/3} + \frac{1}{2}\frac{1}{1-x/2} + \left(\frac{-1}{4}\right)\frac{1}{(1-x/2)^2}$$

$$\frac{1-x}{x-3}$$

$$\frac{1}{x-a} = \frac{-1}{a}\frac{1}{(1-x/a)}$$

$\to 1/a$

$$\frac{1}{x-3} = \frac{-1}{3}\cdot\frac{1}{(1-x/3)}$$

x^2

$y = 2x$

$y = 1/3^x$

$$= \left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^{i} + \left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^{i} + \left(\frac{-1}{4}\right)\left[\binom{-2}{0} + \binom{-2}{1}\frac{-x}{2} + \binom{2}{2}\left(\frac{-x}{2}\right)^{2} + \cdots\right]$$

co-efficient of x^{8} is

$$\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)^{8} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{8} + \left(\frac{-1}{4}\right)\binom{-2}{8}\left(\frac{-1}{2}\right)^{8} =$$

$$= -\left(\frac{1}{3}\right)^{9} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{8} + \frac{-1}{4}\binom{2+8-1}{8}(-1)^{8}\left(\frac{-1}{2}\right)^{8}$$

$$= -\left(1/3\right)^{9} + \left(1/2\right)^{9} + \frac{-1}{4}\cdot 9\cdot\left(\frac{1}{2}\right)^{8} = -\left(\frac{1}{3}\right)^{9} + \left(\frac{1}{2}\right)^{9}\left[1 - 9/2\right]$$

$$= -\left[\left(1/3\right)^{9} + 7\left(1/2\right)^{10}\right]$$

Partitions of Integers

Partitions of Integers

- Partitioning a positive integer n into positive summands and seeking the number of such partitions without regard to the order

$$
\begin{aligned}
p(1) &= 1: \quad 1 \\
p(2) &= 2: \quad 2 = 1 + 1 \\
p(3) &= 3: \quad 3 = 2 + 1 = 1 + 1 + 1 \\
p(4) &= 5: \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \\
p(5) &= 7: \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\
&\qquad\quad = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

determine $p(10)$

$$f(x) = \overset{1}{(1 + x + x^2 + x^3 + x^4 + \ldots)}\overset{2}{(1 + x^2 + x^4 + x^6 + \ldots)}\overset{3}{(1 + x^3 + x^6 + x^9 + \ldots)}\ldots$$

$$\ldots (1 + x^{10} + x^{20} + \ldots)$$

$$f(x) = \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^{10}} = \prod_{i=1}^{10} \frac{1}{(1-x^i)}$$

$$P(x) = \prod_{i=1}^{\infty} \left[1/(1-x^i) \right]$$

$$\prod_{i=1}^{\ell} \frac{1}{1-x^i}$$

x^n

Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into *distinct* summands.

let us consider the 11 partitions of 6:

1) $1+1+1+1+1+1$	**2)** $1+1+1+1+2$
3) $1+1+1+3$	**4)** $1+1+4$
5) $1+1+2+2$	**6)** $1+5$
7) $1+2+3$	**8)** $2+2+2$
9) $2+4$	**10)** $3+3$
11) 6	

Partitions (6), (7), (9), and (11) have distinct summands, so $p_d(6) = 4$.

In calculating $p_d(n)$, for each $k \in \mathbf{Z}^+$ there are two choices: Either k is not used as one of the summands of n, or it is. This can be accounted for by the polynomial $1 + x^k$, and consequently, the generating function for these partitions is

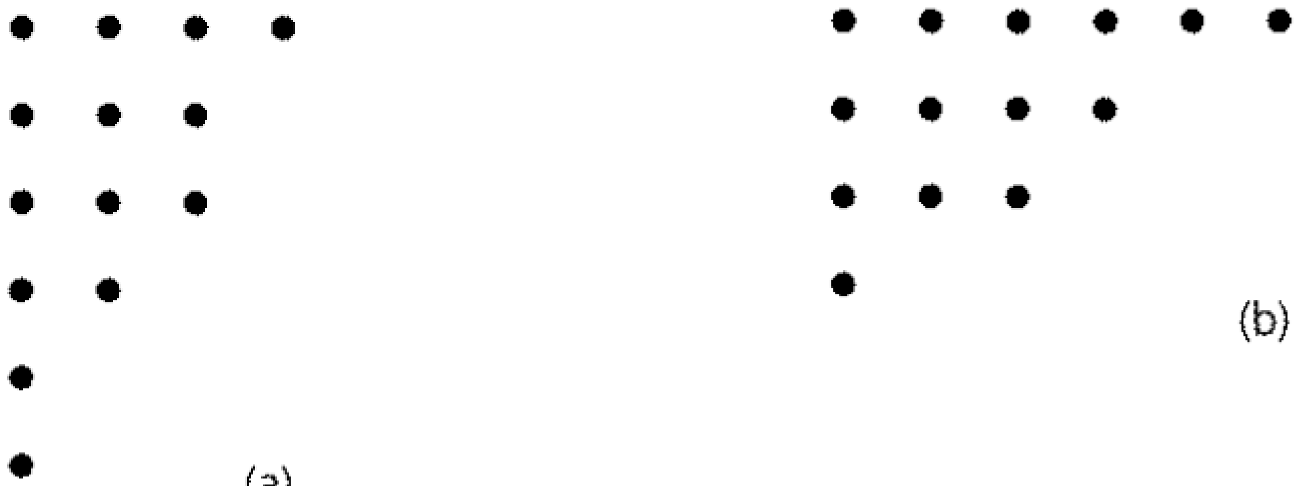
$$P_d(x) = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{i=1}^{\infty}(1+x^i).$$

For each $n \in \mathbf{Z}^+$, $p_d(n)$ is the coefficient of x^n in $(1+x)(1+x^2)\cdots(1+x^n)$.

Ferrers graph

- This graph uses rows of dots to represent a partition of an integer where the number of dots per row does not increase as we go from any row to the one below it

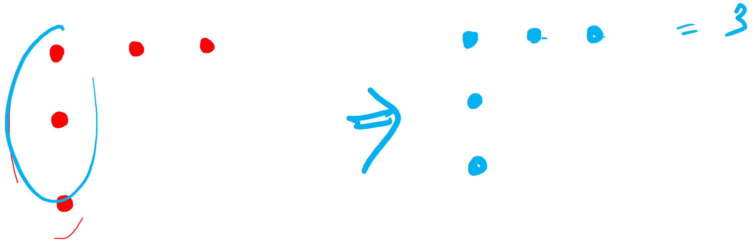
Ferrers graphs for two partitions of 14: (a) $4 + 3 + 3 + 2 + 1 + 1$ and (b) $6 + 4 + 3 + 1$. The graph in part (b) is said to be the *transposition* of the graph in part (a), and vice versa, because one graph can be obtained from the other by interchanging rows and columns.



(a) (b)

There is a one-to-one correspondence between a Ferrers graph and its transposition, so this example demonstrates a particular instance of the general result: The number of partitions of an integer n into m summands is equal to the number of partitions of n into summands where m is the largest summand.

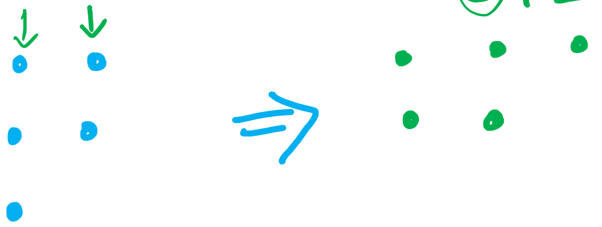
$P(5) = 3 + 1 + 1$



$\Rightarrow$ self conjugate

$P(5) = 2 + 2 + 1$ $\quad m = 3$

$3 + 2$



$P(4) = 2+2$

$P(8) = 3+3+2$

Exponential Generating Functions

- Ordinary generating functions are used in selection problems where order was not relevant

Now for all $0 \le r \le n$,

$$C(n, r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n, r),$$

where $P(n, r)$ denotes the number of permutations of n objects taken r at a time. So

$$\begin{aligned}(1+x)^n &= C(n,0) + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \cdots + C(n,n)x^n \\ &= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + \cdots + P(n,n)\frac{x^n}{n!}.\end{aligned}$$

For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the *exponential generating function* for the given sequence.

Examining the Maclaurin series expansion for e^x, we find

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

so e^x is the exponential generating function for the sequence 1, 1, 1, ... (The function e^x is the ordinary generating function for the sequence 1, 1, 1/2!, 1/3!, 1/4!, ...)

In how many ways can four of the letters in ENGINE be arranged?

E & N $\left(1+x+x^2/2!\right)$

G & I $(1+x)$

E	E	N	N	4!/(2! 2!)	E	G	N	N	4!/2!
E	E	G	N	4!/2!	E	I	N	N	4!/2!
E	E	I	N	4!/2!	G	I	N	N	4!/2!
E	E	G	I	4!/2!	E	I	G	N	4!

$$f(x) = [1 + x + (x^2/2!)][1 + x + (x^2/2!)](1 + x)(1 + x)$$

$$\left(x^2/2!\right)\left(x^2/2!\right)(1)(1) \Rightarrow x^4/2!.2! \Rightarrow \frac{4!}{2!.2!}\cdot\frac{x^4}{4!}$$

$$\left(x^2/2!\right)(1)(x)(x) \Rightarrow x^4/2! \Rightarrow \frac{4!}{2!}\cdot\frac{x^4}{4!}$$

In the complete expansion of $f(x)$, the term involving x^4 is

$$\left(\frac{x^4}{2!\,2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+x^4\right)$$

$$=\left[\left(\frac{4!}{2!\,2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+4!\right]\left(\frac{x^4}{4!}\right)$$

Consider the Maclaurin series expansions of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Adding these series together, we find that

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right),$$

or

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

Subtracting e^{-x} from e^x yields

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

A ship carries 48 flags, 12 each of the colors red, white, blue, and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.

a) How many of these signals use an even number of blue flags and an odd number of black flags?

The exponential generating function

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$f(x) = (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right) (e^{2x})(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{4x} - 1)$$

$$= \frac{1}{4} \left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \left(\frac{1}{4}\right) \sum_{i=1}^{\infty} \frac{(4x)^i}{i!},$$

coefficient of $x^{12}/12!$ in $f(x)$ yields $(1/4)(4^{12}) = 4^{11}$

b) How many of the signals have at least three white flags or no white flags at all? In this situation we use the exponential generating function

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$= e^x\left(e^x - x - \frac{x^2}{2!}\right)(e^x)^2 = e^{3x}\left(e^x - x - \frac{x^2}{2!}\right) = e^{4x} - xe^{3x} - \left(\frac{1}{2}\right)x^2e^{3x}$$

$$= \sum_{i=0}^{\infty}\frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty}\frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right).$$

$\sum_{i=0}^{\infty}\frac{(4x)^i}{i!}$ — Here we have the term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$, so the coefficient of $x^{12}/12!$ is 4^{12}

$x\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$ — to get $x^{12}/12!$ consider the term

$x[(3x)^{11}/11!] = 3^{11}(x^{12}/11!) = (12)(3^{11})(x^{12}/12!)$, and here the coefficient of $x^{12}/12!$ is $(12)(3^{11})$

$$(x^2/2)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$$

$(x^2/2)[(3x)^{10}/10!] = (1/2)(3^{10})(x^{12}/10!) = (1/2)(12)(11)(3^{10})(x^{12}/12!)$,
where this time the coefficient of $x^{12}/12!$ is $(1/2)(12)(11)(3^{10})$.

Consequently, the number of 12 flag signals with at least three white flags, or none at all, is

$$4^{12} - 12(3^{11}) - (1/2)(12)(11)(3^{10}) = 10{,}754{,}218.$$

Recurrence Relation

Recurrence Relations

- Definition:

An equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely a_0, a_1, a_2,…, a_{n-1} for all integers n with $n \geq n_0$, where n_0 is the nonnegative integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

Recurrence Relations

- In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

- Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

- If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

What is the recurrence equation for the series

- 5,15,45,135, …..

$$a_{n+1} = 3a_n, \quad n \geq 0, \; a_0 = 5$$

- 3,9,27,…..

$$a_{n+1} = 3a_n, \quad n \geq 0, \; a_0 = 3$$

Since, a_{n+1} depends only on its immediate predecessor, the relation is said to be first order

The general form of such an equation can be written $a_{n+1} = da_n$, $n \geq 0$, where d is a constant.

Values such as a_0 or a_1, given in addition to the recurrence relations, are called *boundary conditions*. The expression $a_0 = A$, where A is a constant, is also referred to as an *initial condition*.

- The Unique solution of the recurrence relation $a_{n+1} = d a_n$, where $n \geq 0$, d is a constant and $a_0 = A$ is given by

$$\mathbf{a_n = Ad^n, \ n \geq 0}$$

$$a_{n+1} = 3a_n, \qquad n \geq 0, \qquad a_0 = 5.$$

The first four terms of this sequence are

$$\begin{aligned} a_0 &= 5, \\ a_1 &= 3a_0 = 3(5), \\ a_2 &= 3a_1 = 3(3a_0) = 3^2(5), \quad \text{and} \\ a_3 &= 3a_2 = 3(3^2(5)) = 3^3(5). \end{aligned}$$

These results suggest that for each $n \geq 0$, $a_n = 5(3^n)$. This is the *unique solution* of the given recurrence relation.

Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$.

$a_n = a_0 (7^n)$

$a_2 = a_0 (7^2)$

$98 = a_0 \, 49$

$a_0 = 2$

$a_n = 2(7^n)$, $n \geq 0$ is the unique solution.

A bank pays 6% (annual) interest on savings, compounding the interest monthly. If Bonnie deposits \$1000 on the first day of May, how much will this deposit be worth a year later?

The annual interest rate is 6%, so the monthly rate is $6\%/12 = 0.5\% = 0.005$.
let p_n denote the value of Bonnie's deposit at the end of n months.
$p_{n+1} = p_n + 0.005p_n$, where $0.005p_n$ is the interest earned on p_n during month $n+1$, for $0 \leq n \leq 11$, and $p_0 = \$1000$.

The relation $p_{n+1} = (1.005)p_n$, $p_0 = \$1000$, has the solution $p_n = p_0(1.005)^n = \$1000(1.005)^n$. Consequently, at the end of one year, Bonnie's deposit is worth $\$1000(1.005)^{12} = \$1061.68$.

Find the recurrence relation of composition of numbers.

let a_n count the number of compositions of n, for $n \in \mathbf{Z}^+$,

$$a_{n+1} = 2a_n, \qquad n \geq 1, \qquad a_1 = 1.$$

		(1′)	4
		(2′)	1 + 3
(1)	3	(3′)	2 + 2
(2)	1 + 2	(4′)	1 + 1 + 2
(3)	2 + 1		
(4)	1 + 1 + 1	(1″)	3 + 1
		(2″)	1 + 2 + 1
		(3″)	2 + 1 + 1
		(4″)	1 + 1 + 1 + 1

to apply the formula for the unique solution (where $n \geq 0$) to this recurrence relation, we let $b_n = a_{n+1}$.

$$b_{n+1} = 2b_n, \qquad n \geq 0, \qquad b_0 = 1,$$

so $b_n = b_0(2^n) = 2^n$, and $a_n = b_{n-1} = 2^{n-1}$, $n \geq 1$.

Find a_{12} if $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ for $n \geq 0$, and $a_0 = 2$.

let $b_n = a_n^2$,

$b_{n+1} = 5b_n$ for $n \geq 0$, and $b_0 = 4$, is a linear relation

solution is $b_n = 4 \cdot 5^n$

Therefore, $a_n = 2(\sqrt{5})^n$ for $n \geq 0$

$a_{12} = 2(\sqrt{5})^{12} = 31{,}250$

The general first-order linear recurrence relation with constant coefficients has the form $a_{n+1} + ca_n = f(n)$, $n \geq 0$, where c is a constant and $f(n)$ is a function on the set $\mathbf{N}$ of nonnegative integers.

When $f(n) = 0$ for all $n \in \mathbf{N}$, the relation is called *homogeneous*; otherwise it is called *nonhomogeneous*.

Bubble sort

```
procedure BubbleSort(n: positive integer; x1, x2, x3, ..., xn: real numbers)
begin
  for i := 1 to n - 1 do
    for j := n downto i + 1 do
      if xj < xj-1 then
        begin        {interchange}
          temp := xj-1
          xj-1 := xj
          xj := temp
        end
end
```

i = 1									
x_1	7		7		7		7	j = 2	2
x_2	9		9		9	j = 3	2		7
x_3	2		2	j = 4	2		9		9
x_4	5	j = 5	5		5		5		5
x_5	8		8		8		8		8

Four comparisons and two interchanges.

i = 2							
x_1	2		2		2		2
x_2	7		7		7	j = 3	5
x_3	9		9	j = 4	5		7
x_4	5	j = 5	5		9		9
x_5	8		8		8		8

Three comparisons and two interchanges.

i = 3					
x_1	2		2		2
x_2	5		5		5
x_3	7		7	j = 4	7
x_4	9	j = 5	8		8
x_5	8		9		9

Two comparisons and one interchange.

i = 4		
x_1	2	
x_2	5	
x_3	7	
x_4	8	j = 5
x_5	9	

One comparison but no interchanges.

To determine the time-complexity function $h(n)$ when this algorithm is used on an input (array) of size $n \geq 1$, we count the total number of *comparisons* made in order to sort the n given numbers into ascending order.

If a_n denotes the number of comparisons needed to sort n numbers in this way, then we get the following recurrence relation:

$$a_n = a_{n-1} + (n - 1), \qquad n \geq 2, \qquad a_1 = 0.$$

Given a list of n numbers, we make $n - 1$ comparisons to bubble the smallest number up to the start of the list. The remaining sublist of $n - 1$ numbers then requires a_{n-1} comparisons in order to be completely sorted.

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_1 + (2 - 1) = 1 \\
a_3 &= a_2 + (3 - 1) = 1 + 2 \\
a_4 &= a_3 + (4 - 1) = 1 + 2 + 3 \\
\cdots \quad & \cdots \quad \cdots \quad \cdots
\end{aligned}
$$

In general, $a_n = 1 + 2 + \cdots + (n - 1) = [(n - 1)n]/2 = (n^2 - n)/2$.

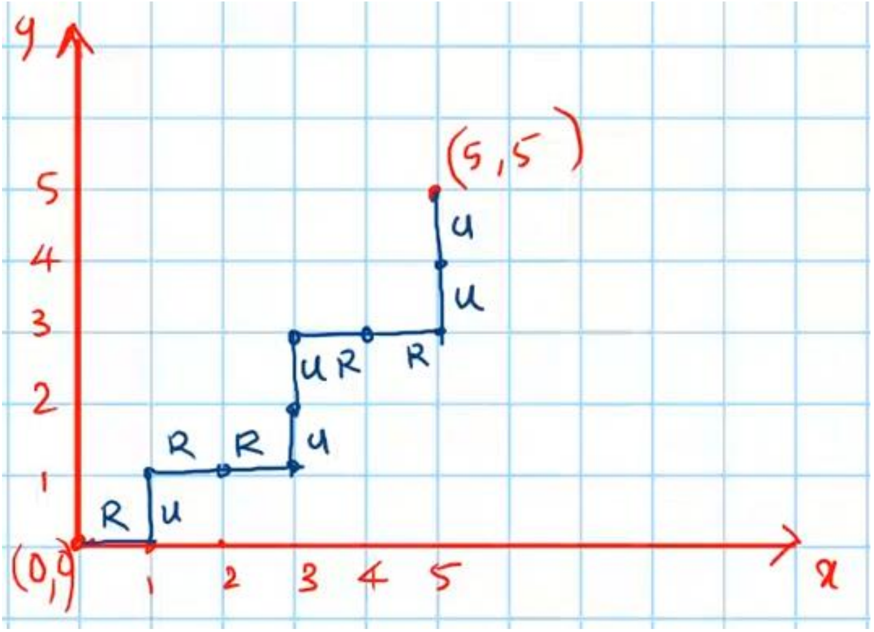
Catalan Numbers

CATALAN NUMBERS

(0,0) to (5,5)

R ⇒ (x,y) to (x+1,y)

U ⇒ (x,y) to (x,y+1)



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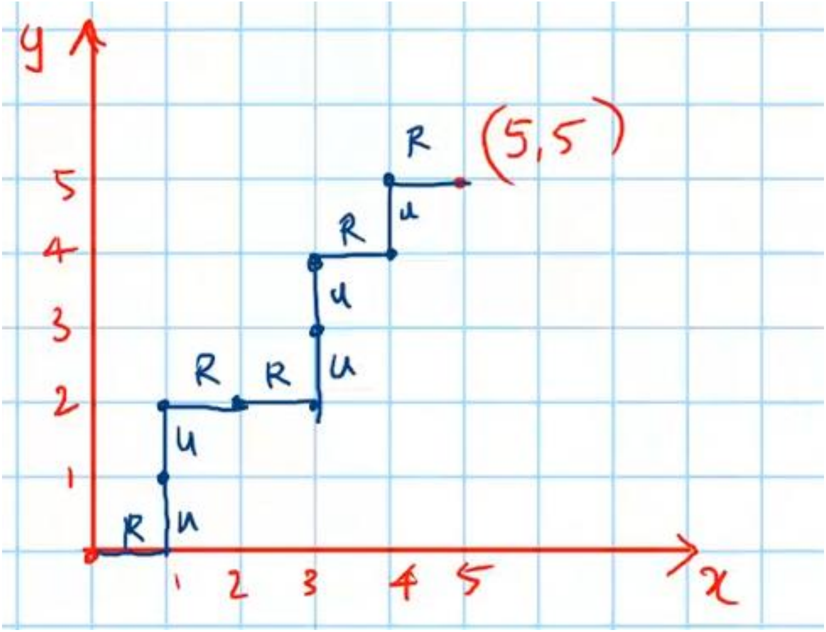

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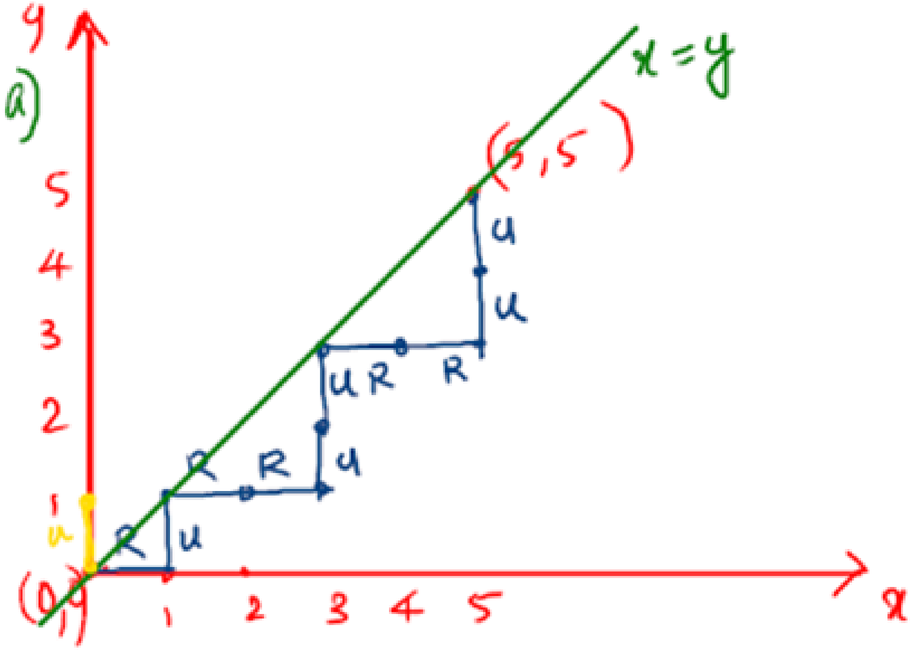
CATALAN NUMBERS

$(0,0)$ to $(5,5)$

${}^{10}C_5$

$R \Rightarrow (x,y)$ to $(x+1,y)$

$U \Rightarrow (x,y)$ to $(x,y+1)$



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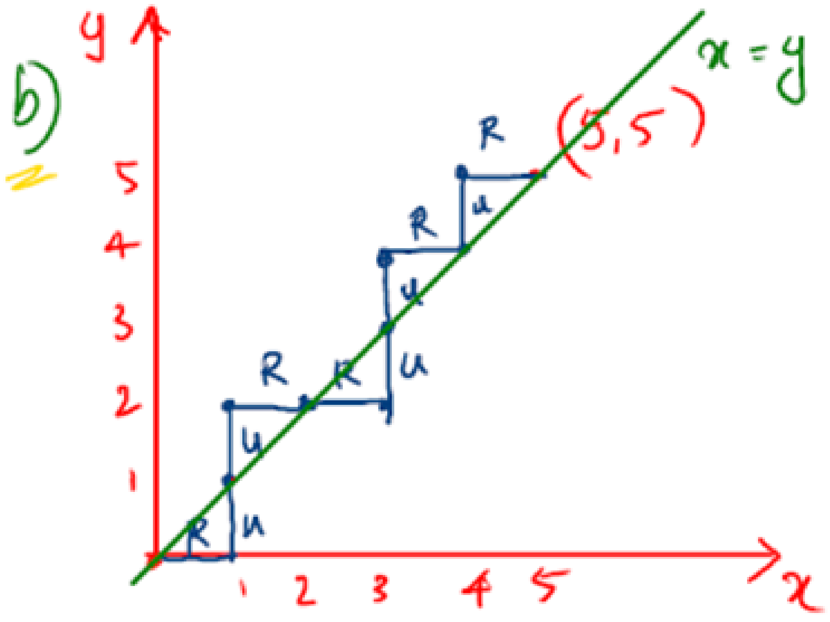

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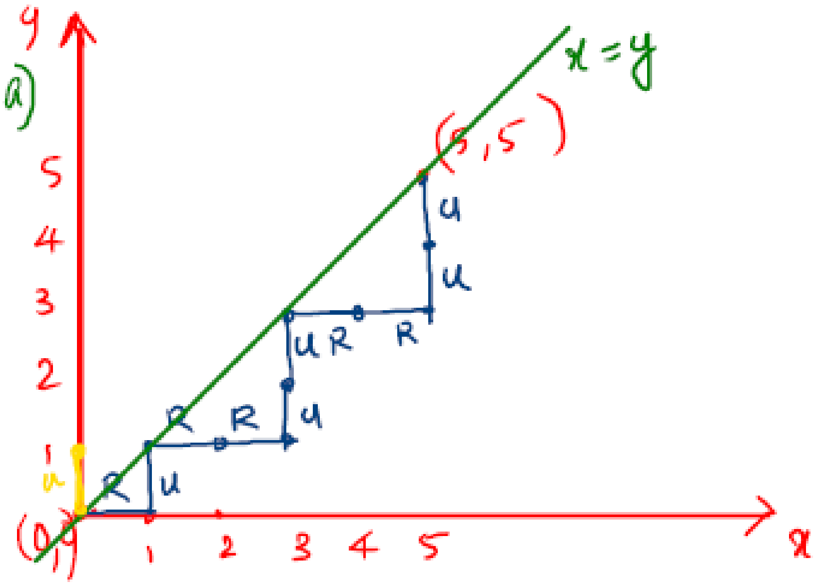
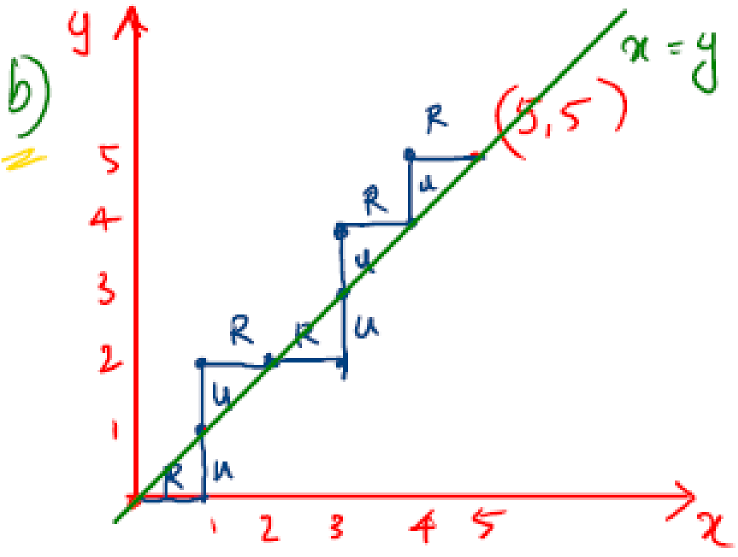
CATALAN NUMBERS

$(0,0)$ to $(5,5)$ to ${}^{10}C_5$

$R \Rightarrow (x,y)$ to $(x+1,y)$

$U \Rightarrow (x,y)$ to $(x,y+1)$





R=5 U=5 ${}^{10}C_5$ – N

a) R U R R U U R R U U

X b) R U U R R U U R U R   R=5, U=5

R U U U U R R U R U   R=4, U=6 $\boxed{{}^{10}C_4}$

${}^{10}C_5 - {}^{10}C_4$

R U R U U U U U R R

R U R U U R R R U U ⇒ R's 5 U's 5

$${}^{10}C_5 - {}^{10}C_4 = \frac{10!}{5!\,5!} - \frac{10!}{4!\,6!}$$

$${}^{2n}C_n \quad - {}^{2n}C_{n-1}$$

$$= \frac{6 \cdot 10!}{5!\,6!} - \frac{5 \cdot 10!}{5!\,6!} = \frac{10!}{5!\,6!}$$

$$= \frac{1}{6} \cdot \frac{10!}{5!\,5!} = \frac{1}{(5+1)} \; {}^{10}C_5$$

$$b_5 = \frac{1}{(5+1)} \; {}^{(2\times5)}C_5$$

$$\boxed{b_n = \frac{1}{n+1}\binom{2n}{n}}$$

$(0,0)$ to (n,n)

Catalan Numbers

$$b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 1, \qquad b_0 = 1.$$

- $b_0, b_1, b_2, \ldots$ are called the Catalan numbers

How many ways one can parenthesize the product abcd ?

S_1	S_2	S_3
1. (((ab)c)d)	(((abc	111000
2. ((a(bc))d)	((a(bc	110100
3. ((ab)(cd))	((ab(c	110010
4. (a((bc)d))	(a((bc	101100
5. (a(b(cd)))	(a(b(c	101010

$a_1 a_2 a_3 a_4 \cdots a_n$

b_{n-1}

((ab(c $\Rightarrow$ ((ab)(cd))

(((abc $\rightarrow$ (((ab)c)d)

$b_3 = 5$

Let us arrange the integers 1, 2, 3, 4, 5, 6 in two rows of three so that (1) the integers increase in value as each row is read, from left to right, and (2) in any column the smaller integer is on top. For example, one way to do this is

1 $\quad$ 1 2 4
$\quad$ 3 5 6

How many ways can you arrange this?

2 $\quad$ 1 3 4
$\quad$ 2 5 6

$\cancel{1 \ 5 \ 6}$
$\cancel{2 \ 3 \ 4}$

3 $\quad$ 1 2 5
$\quad$ 3 4 6

	1	2	3	4	5	6
1 $\Rightarrow$	1	1	0	1	0	0
2 $\Rightarrow$	1	0	1	1	0	0
3 $\Rightarrow$	1	1	0	0	1	0

$b_3 = 5$ ways

Twelve patrons, six each with a \$5 bill and the other six each with a \$10 bill, are the first to arrive at a movie theater, where the price of admission is five dollars. In how many ways can these 12 individuals (all loners) line up so that the number with a \$5 bill is never exceeded by the number with a \$10 bill (and, as a result, the ticket seller is always able to make any necessary change from the bills taken in from the first 11 of these 12 patrons)?

\$5 | \$10
1 | 0
~ |
6 | 6

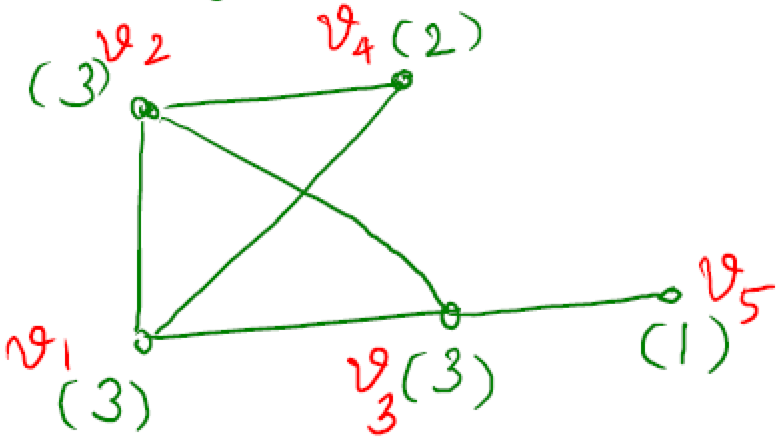
$\Rightarrow$ 12 length

b_6

Havel–Hakimi

Havel–Hakimi

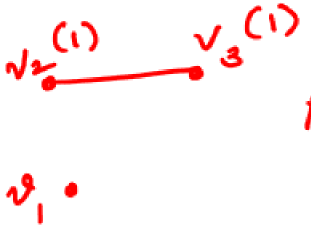
- is an algorithm in graph theory solving the graph realization problem
- Given a finite list of nonnegative integers in non-increasing order, is there a simple graph such that its degree sequence is exactly this list?
- The degree sequence is a list of numbers in non-increasing order indicating the number of edges incident to each vertex in the graph
- If a simple graph exists for exactly the given degree sequence, the list of integers is called graphic



deg seq: $3, 3, 3, 2, 1$

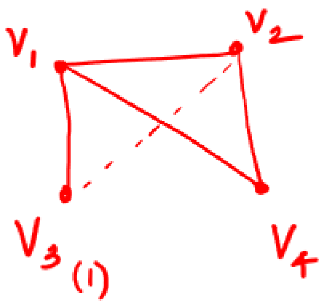
$\begin{array}{ccccc} 3, & 3, & 3, & 2, & 1 \\ | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{array}$

$S_1 : \langle 1, 1, 1 \rangle$



Not graphical

$S_2 : \langle \overset{v_1}{3}, \overset{v_2}{3}, \overset{v_4}{3}, \overset{v_3}{1} \rangle$



Not graphical.

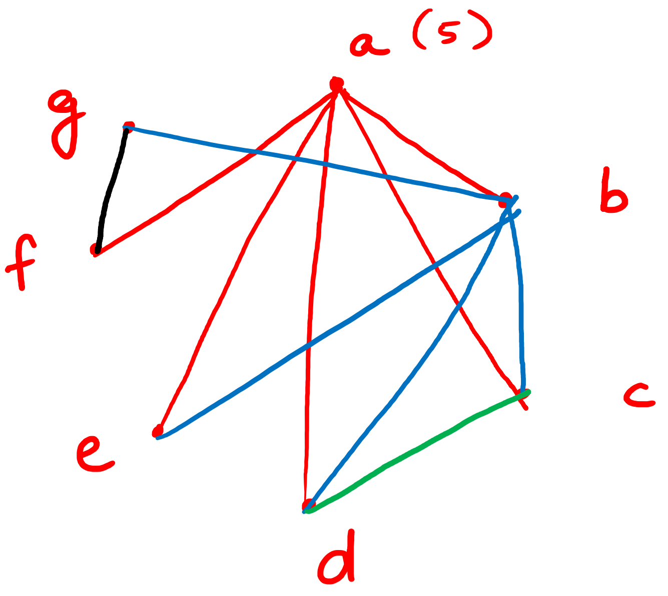
- A graphical sequence may be degree sequence of more than one graph
- Ex: 3,3,2,2,1,1

Degree Sequence

A sequence $d_1, d_2 \ldots d_n$ of non-negative integers is called a degree sequence of a graph G if the vertices of G can be labelled $v_1, v_2, \ldots v_n$ $\therefore$ $\deg(v_i) = d_i$ for all $i = 1, 2, \ldots n$.

$S_0 : \langle \overset{a}{5}, \overset{b}{5}, \overset{c}{3}, \overset{d}{3}, \overset{e}{2}, \overset{f}{2}, \overset{g}{2} \rangle$

G



$S_1 : \langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$

$S_2 : \langle *_a, 4_b, 2_c, 2_d, 1_e, 1_f, 2_g \rangle$

$S_2' : \langle *a, 4_b, \underbrace{2c, 2d, 2g, 1e,} 1f \rangle$

$S_3 : \langle *a, *b, 1c, 1d, 1g, 0e, 1f \rangle$

$S_3' : \langle *a, *b, 1c, 1d, 1g, 1f, 0e \rangle$

$S_4 : \langle *a, *b, *c, *d, 1g, 1f, 0e \rangle$

$S_5 : \langle *a, *b, *c, *d, *g, 0f, 0e \rangle$

Graphical

$S: \langle 5, 5, 5, 5, 2, 2, 2 \rangle$

$S_1: \langle \underline{5}, \underbrace{5, 5, 5, 2, 2}, 2 \rangle$

$S_2: \langle *, 4, 4, 4, 1, 1, 2 \rangle$

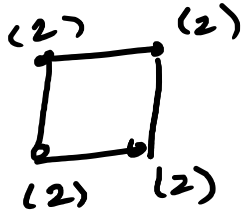
$S_2': \langle *, \underline{4}, \underbrace{4, 4, 2, 1}, 1 \rangle$

$S_3: \langle *, *, 3, 3, 1, 0, 1 \rangle$

$S_3': \langle *, *, \underline{3}, \underbrace{3, 1, 1}, 0 \rangle$

$S_4: \langle *, *, *, 2, \underbrace{0, 0}, 0 \rangle$

$S_5: \langle *, *, *, *, -1, -1, 0 \rangle$ Not Graphical



Havel Hakimi Theorem

A degree sequence $S = d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n$, where $n \geq 2$ and $d_1 \leq n-1$ and $d_1 \geq 1$, is graphic if and only if the reduced sequence $S' = \{*, d_2-1, d_3-1, \dots d_{d_1+1}-1, d_{d_1+2}, \dots, d_n\}$ is graphic.

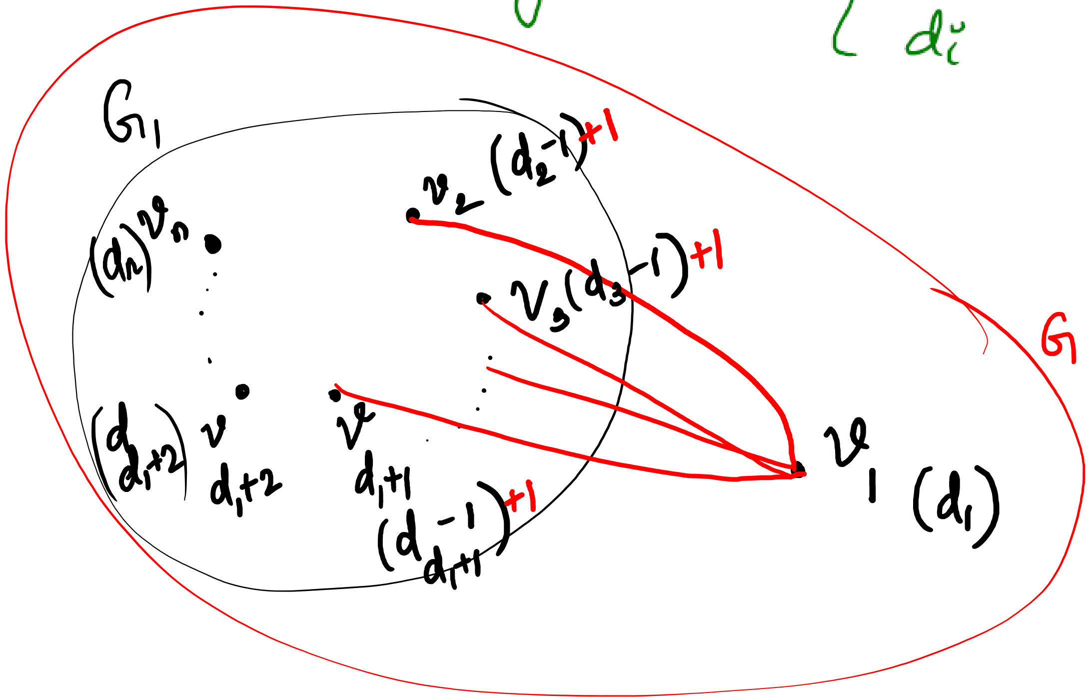
✓ $S: d_1, d_2 \ldots d_n$ $\quad n \geq 2 \quad d_1 \geq 1 \quad d_1 \leq n-1$

$S_1: d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, d_{d_1+2}, \ldots d_n$

① Suppose S_1 is graphical, $\exists$ a graph G_1 of order $n-1$ with deg. seq S_1

Hence, label $V(G_1)$ as $v_2, v_3, \ldots v_n$ $\therefore$

$$\deg(v_i) = \begin{cases} d_i - 1 & \text{for } i = 2, 3 \ldots d_1 + 1 \\ d_i & \text{for } i = d_1 + 2, \ldots d_n \end{cases}$$



Adding v_1 and its edges to G_1, we can obtain G which is Graphical.

edges $\Rightarrow v_1 v_i$, $\forall i$ from 2 to $d_1 + 1$

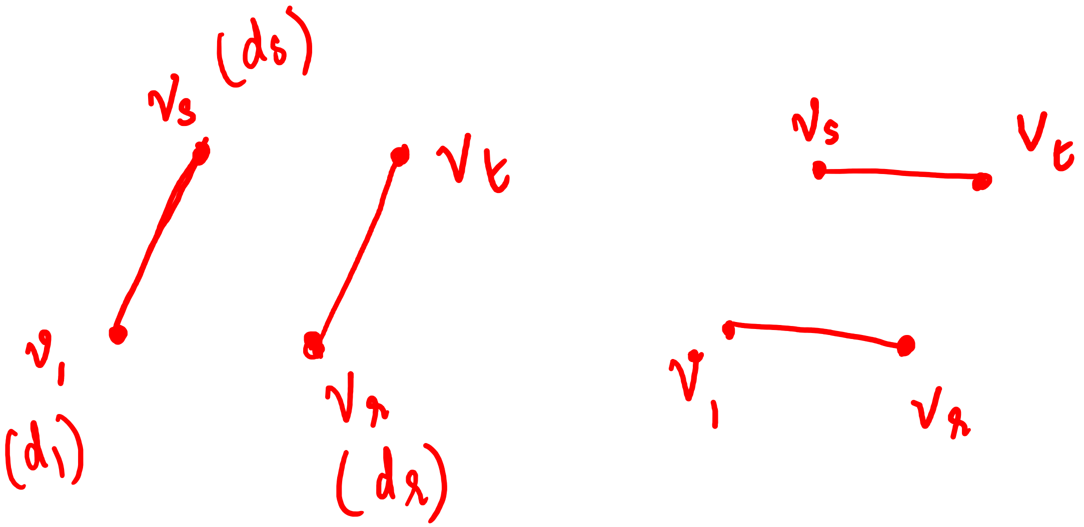
$\deg(v_i) = d_i$

(2) S is graphical, $\exists$ graph ~~order~~ n with degree seq S.

$V(G) = \{v_1, v_2, \ldots v_n\}$ $\deg(v_i) = d_i$ for $i = 1, 2, \ldots n$

Claim : v_1 is adjacent to vertices having degrees $d_2, d_3, \ldots d_{d_1+1}$

The sum of the degrees of the vertices adjacent to v_1 is maximum



$d_r > d_s$

$d_1, d_r \ldots d_s$

Thus, the graph G-v_1 has the degree sequence

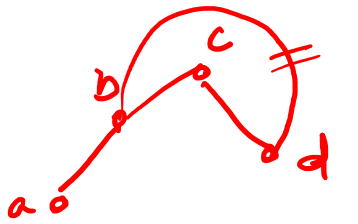
$S_1 : d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, d_{d_1+2}, \ldots d_n$

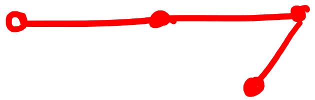
Tree

Tree

- A tree is a connected graph without any circuits.
- It has to be a simple graph.

Some Properties of Tree





THEOREM 1

There is one and only one path between every pair of vertices in a tree, *T*.

Proof: Since T is a connected graph, there must exist at least one path between every pair of vertices in *T*. Now suppose that between two vertices *a* and *b* of *T* there are two distinct paths. The union of these two paths will contain a circuit and *T* cannot be a tree. ■

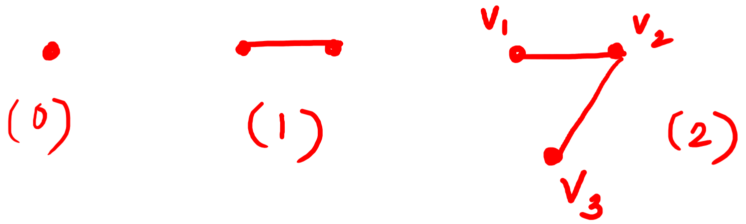
Some Properties of Tree

THEOREM 2

If in a graph G there is one and only one path between every pair of vertices, G is a tree.

Proof: Existence of a path between every pair of vertices assures that G is connected. A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G can have no circuit. Therefore, G is a tree. ■

Some Properties of Tree

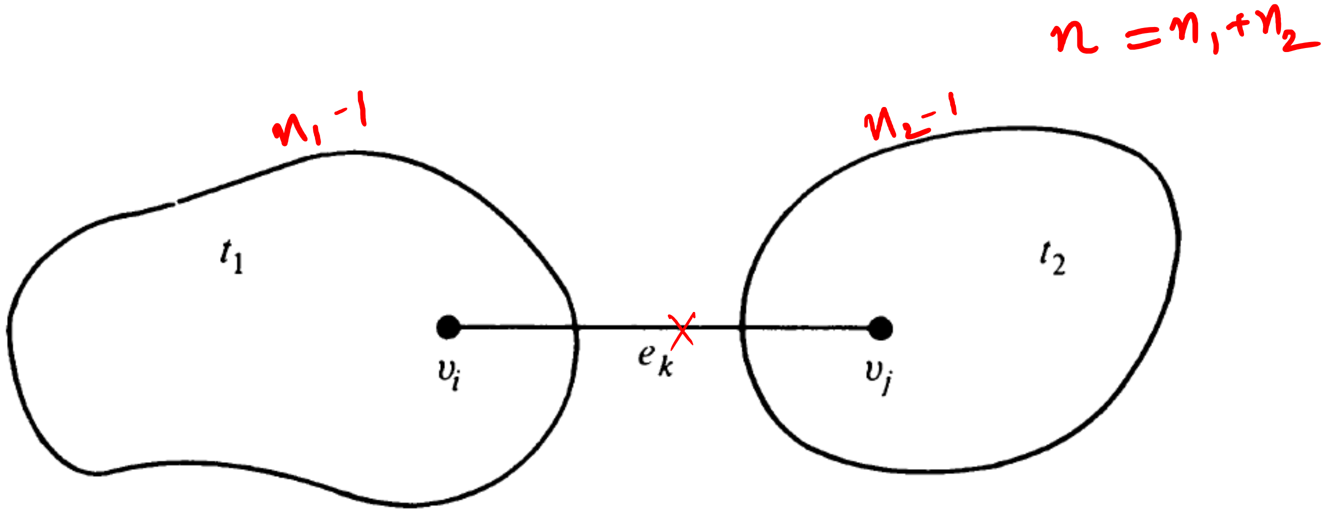


THEOREM 3

A tree with n vertices has $n - 1$ edges.

THEOREM 4

Any connected graph with n vertices and $n - 1$ edges is a tree.



Some Properties of Tree

THEOREM 5

A graph is a tree if and only if it is minimally connected.

- A connected graph is said to be *minimally connected* if removal of any one edge from it disconnects the graph. A minimally connected graph cannot have a circuit; otherwise, we could remove one of the edges in the circuit and still leave the graph connected. Thus a minimally connected graph is a tree.
- Conversely, if a connected graph G is not minimally connected, there must exist an edge e_i in G such that $G - e_i$ is connected. Therefore, e_i is in some circuit, which implies that G is not a tree.

Some Properties of Tree

THEOREM 6

A graph G with n vertices, $n - 1$ edges, and no circuits is connected.

- ***Proof:***
  - Suppose there exists a circuitless graph G with n vertices and n − 1 edges which is disconnected. In that case G will consist of two or more circuitless components.
  - Without loss of generality, let G consist of two components, g1 and g2. Add an edge e between a vertex v1 in g1 and v2 in g2. Since there was no path between v1 and v2 in G, adding e did not create a circuit.
  - Thus, G ∪ e is a circuitless, connected graph (i.e., a tree) of n vertices and n edges, which is not possible, because of Theorem 3.

a graph G with n vertices is called a tree if

1. G is *connected* and is *circuitless*, or
2. G is *connected* and has $n - 1$ *edges*, or
3. G is *circuitless* and has $n - 1$ *edges*, or
4. There is *exactly one path* between every pair of vertices in G, or
5. G is a *minimally connected* graph.

Pendant vertices in a Tree

In any tree (with two or more vertices), there are at least two pendant vertices.

vertices - n $\qquad$ $(n-1)$ edges

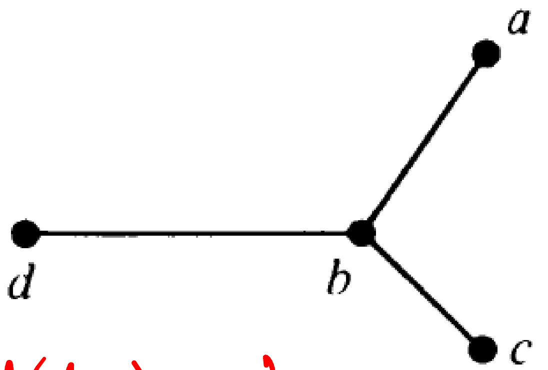
$n \geq 2.$

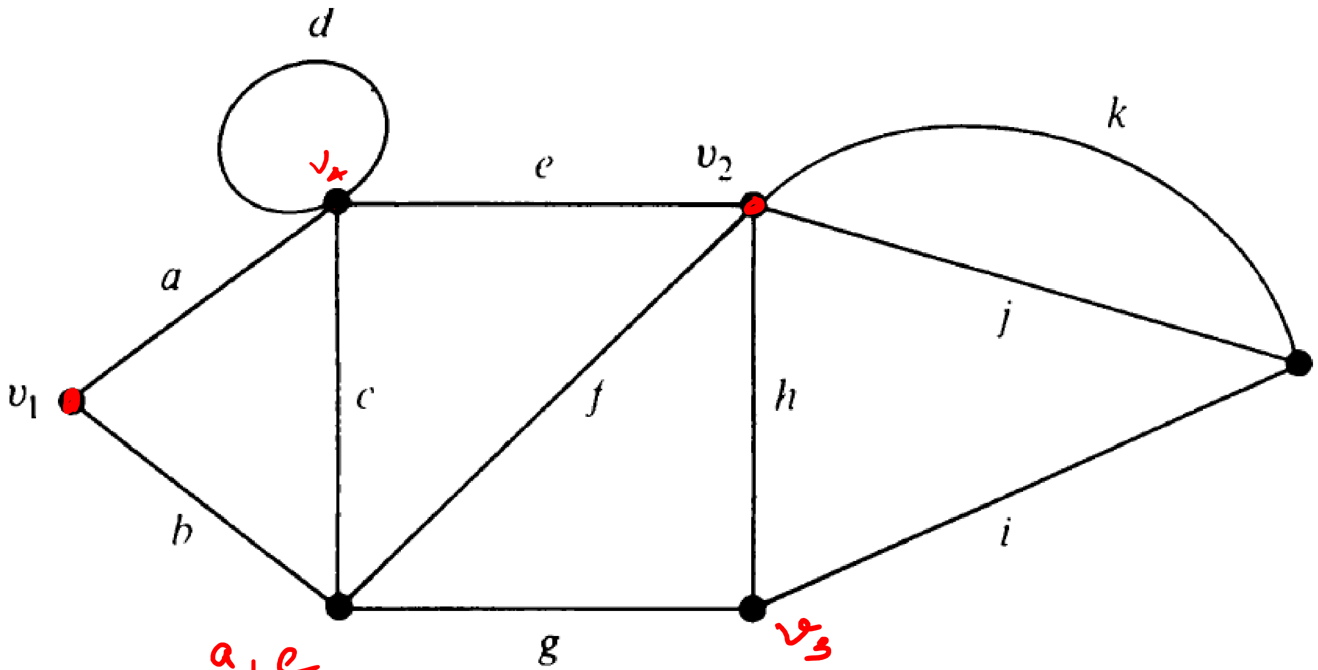
$$\sum 2e = 2(n-1)$$

$$= \boxed{2n-2.}$$

Distance in a Tree

In a connected graph G, the *distance* $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path (i.e., the number of edges in the shortest path) between them.





$d(d,a) = 2$

$d(d,c) = 2$

$d(d,b) = 1$

$d(a,d) = 2$

$d(c,d) = 2$

$d(b,d) = 1$

a, e

$f(v_1,v_2) \leq f(v_1,v_3) + f(v_3,v_2)$

$2 \qquad\qquad 2 + 1 = 3$

Distance in a Tree

A Metric: Before we can legitimately call a function $f(x, y)$ of two variables a "distance" between them, this function must satisfy certain requirements. These are

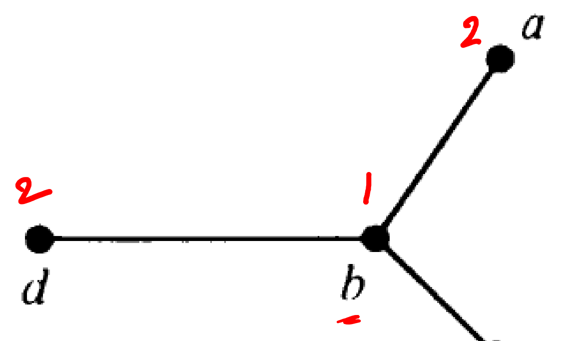
1. Nonnegativity: $f(x, y) \geq 0$, and $f(x, y) = 0$ if and only if $x = y$.
2. Symmetry: $f(x, y) = f(y, x)$.
3. Triangle inequality: $f(x, y) \leq f(x, z) + f(z, y)$ for any z.

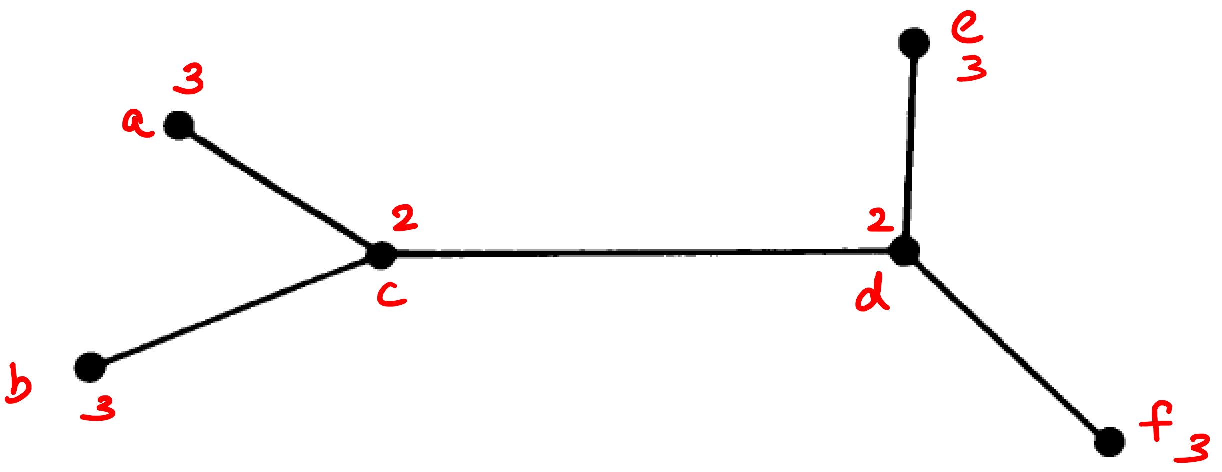
Centers in a Tree

The eccentricity $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G; that is,

$$E(v) = \max_{v_i \in G} d(v, v_i).$$

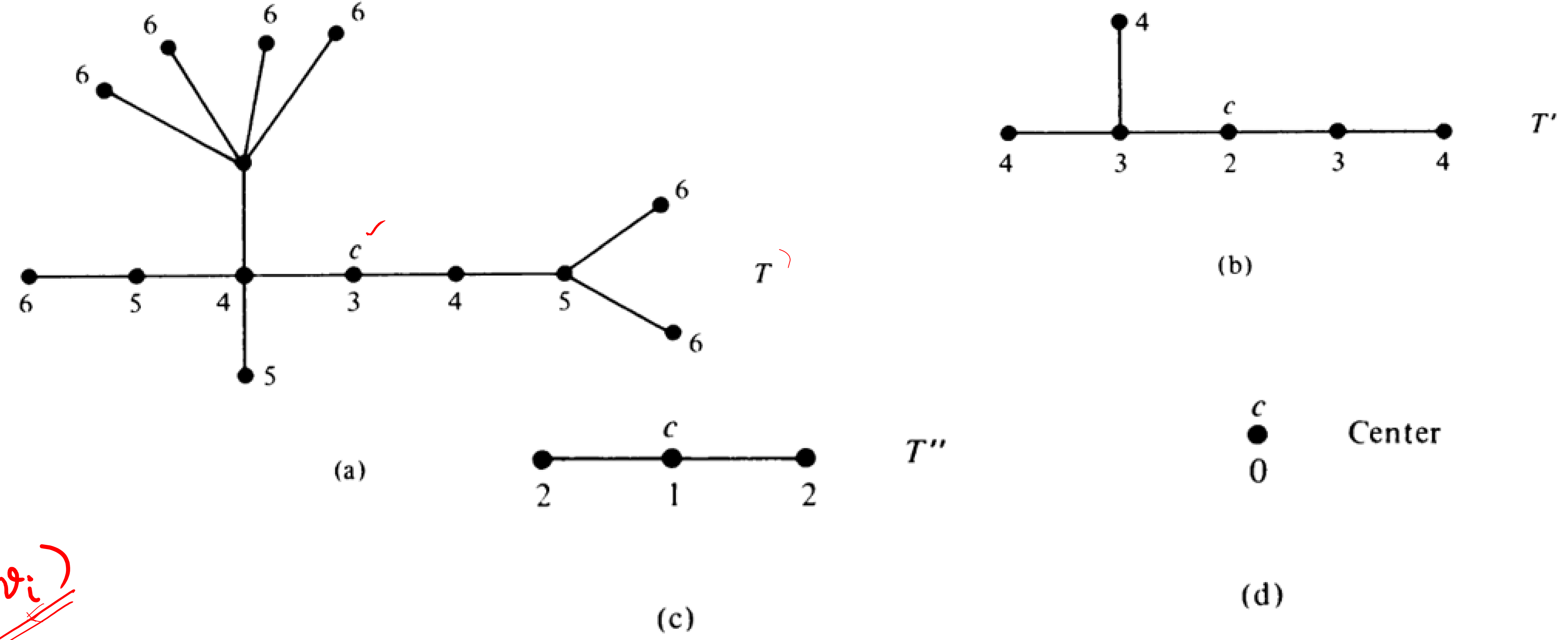
A vertex with minimum eccentricity in graph G is called a *center* of G.





THEOREM

Every tree has either one or two centers.



max(v, v_i)

COROLLARY

From the argument used in proving Theorem, we see that if a tree T has two centers, the two centers must be adjacent.

- Proof:
  - The maximum distance, max $d(v, v_i)$, from a given vertex v to any other vertex v_i occurs only when v_i is a pendant vertex.
  - With this observation, let us start with a tree T having more than two vertices. Tree T must have two or more pendant vertices (Theorem-7).
  - Delete all the pendant vertices from T. The resulting graph T' is still a tree. The removal of all pendant vertices from T uniformly reduced the eccentricities of the remaining vertices (i.e., vertices in T') by one. Therefore, all vertices that T had as centers will still remain centers in T'.
  - From T' we can again remove all pendant vertices and get another tree T''. We continue this process until there is left either a vertex (which is the center of T) or an edge (whose end vertices are the two centers of T). Thus the theorem.

Radius and Diameter

The eccentricity of a center (which is the distance from the center of the tree to the farthest vertex) in a tree is defined as the *radius* of the tree.

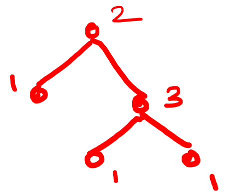
Diameter of a tree T, is defined as the length of the longest path in T.

Rooted Trees

A tree in which one vertex (called the *root*) is distinguished from all the others is called a *rooted tree*.

Generally, the term *tree* means trees without any root. However, for emphasis they are sometimes called *free trees* (or *nonrooted trees*) to differentiate them from the rooted kind.

Binary tree



A *binary tree* is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three

Two properties:

1. The number of vertices n in a binary tree is always odd. This is because there is exactly one vertex of even degree, and the remaining $n - 1$ vertices are of odd degrees. Since from Theorem 1-1 the number of vertices of odd degrees is even, $n - 1$ is even. Hence n is odd.

2. Let p be the number of pendant vertices in a binary tree T. Then $n - p - 1$ is the number of vertices of degree three. Therefore, the number of edges in T equals

$$\frac{1}{2}[p + 3(n - p - 1) + 2] = n - 1;$$

hence $$p = \frac{n+1}{2}.$$

Spanning Trees

G (5)

T' $(3) - 2$

T'' $(4) - 3$

T_5 $(5) - 4$

A tree T is said to be a *spanning tree* of a connected graph G if T is a subgraph of G and T contains all vertices of G.

Since the vertices of G are barely hanging together in a spanning tree, it is a sort of skeleton of the original graph G. This is why a spanning tree is sometimes referred to as a *skeleton* or *scaffolding* of G.

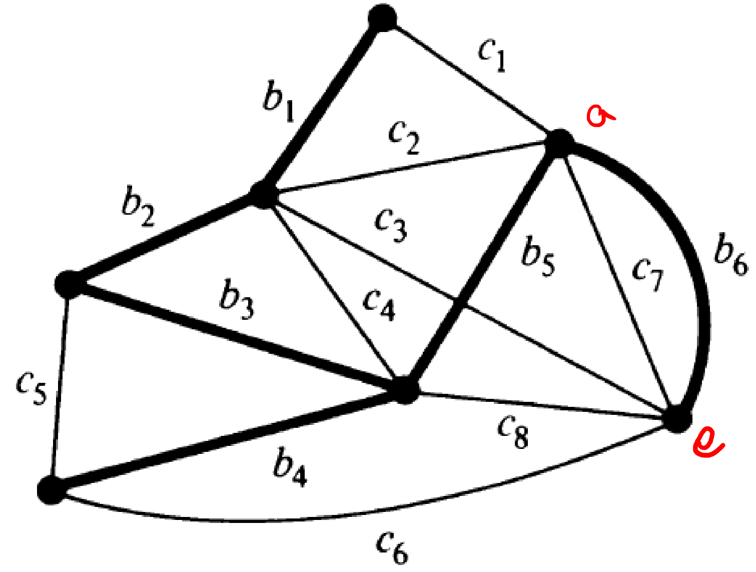
Since spanning trees are the largest trees among all trees in G, it is also quite appropriate to call a spanning tree a *maximal tree subgraph* or *maximal tree* of G.

a disconnected graph with k components has a *spanning forest* consisting of k spanning trees.

THEOREM

Every connected graph has at least one spanning tree.

$b_1 \dots b_5\, c_7 \qquad b_6$



An edge in a spanning tree T is called a *branch* of T. An edge of G that is not in a given spanning tree T is called a *chord*.

$$T \cup \bar{T} = G,$$

where T is a spanning tree, and $\bar{T}$ is the complement of T in G. Since the subgraph $\bar{T}$ is the collection of chords, it is quite appropriately referred to as the *chord set* (or *tie set* or *cotree*) of T.

THEOREM

With respect to any of its spanning trees, a connected graph of n vertices and e edges has $n - 1$ tree branches and $e - n + 1$ chords.

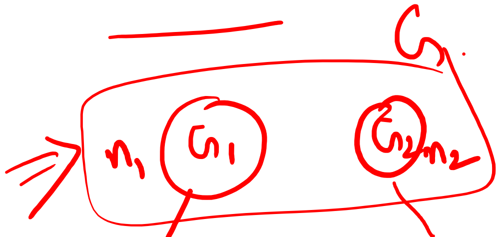
$G \Rightarrow |V| = n \qquad |E| = e$

$T \Rightarrow |V| = n \qquad |E_T| = n-1$

branch

$e - (n-1)$

$\boxed{e - n + 1}$



rank $\quad r = n - k,$

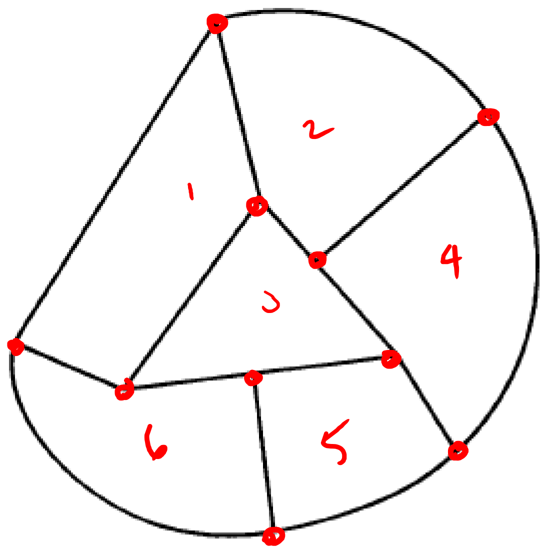
nullity $\quad \mu = e - n + k$

$T_1 \; |V| = n_1 \qquad T_2 \; |V| = n_2$

$|E_1| = n_1 - 1 \qquad |E_2| = n_2 - 1$

$e - (n_1 + n_2 - 2) = \boxed{e - n + 2} \qquad n_1 - 1 + n_2 - 1 = n_1 + n_2 - 2$

- The nullity of a graph is also referred to as its cyclomatic number, or first Betti number farm consisting of six walled plots of land, as shown in Fig. and these plots are full of water, how many walls will have to be broken so that all the water can be drained out?



Here $n = 10$ and $e = 15$.

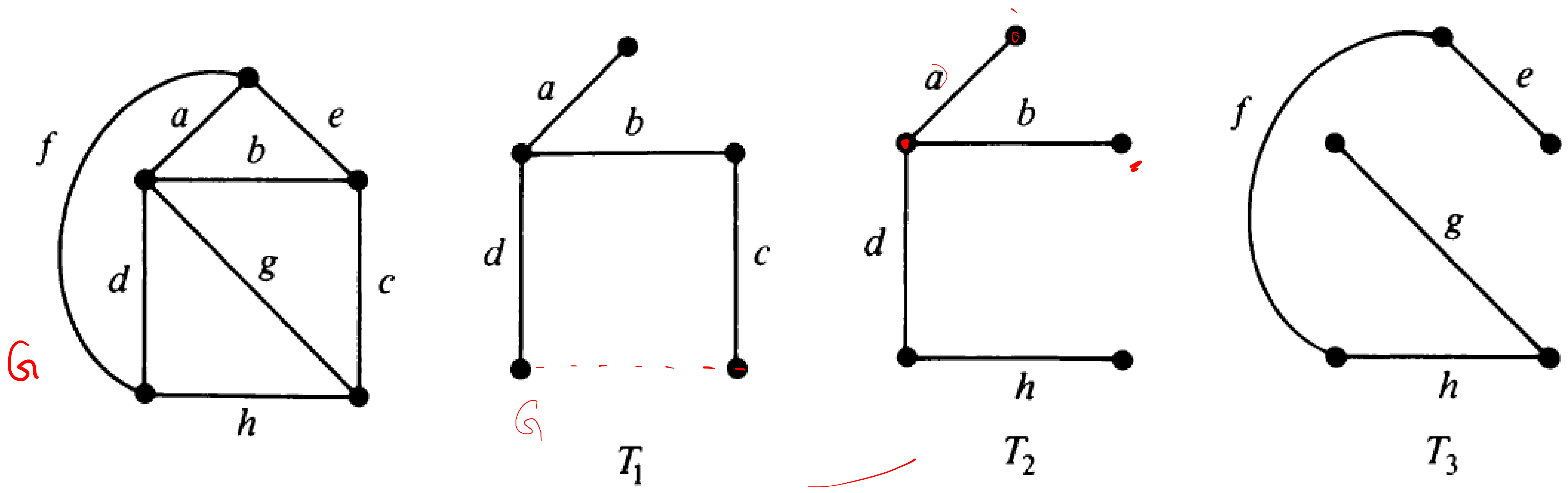
$n - 1 = 9$

$e - n + 1$

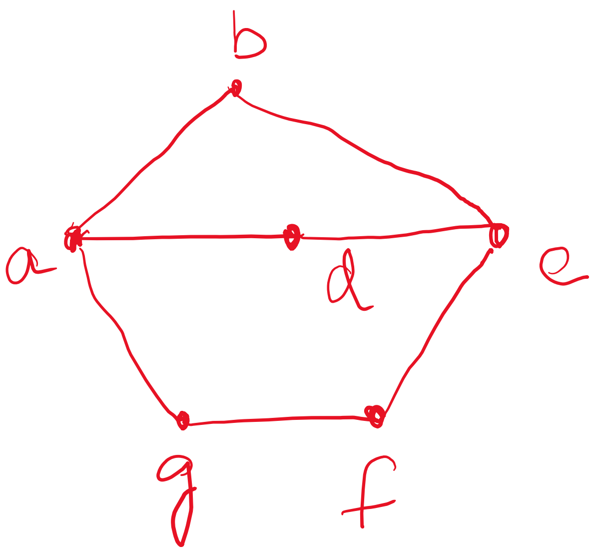
$15 - 10 + 1 = \underline{6}$

Finding all spanning trees of a graph

$T_2 \oplus T_3 = \{a, b, d, e, f, g\}$



This generation of one spanning tree from another, through addition of a chord and deletion of an appropriate branch, is called a *cyclic interchange* or *elementary tree transformation*.


b
a
d
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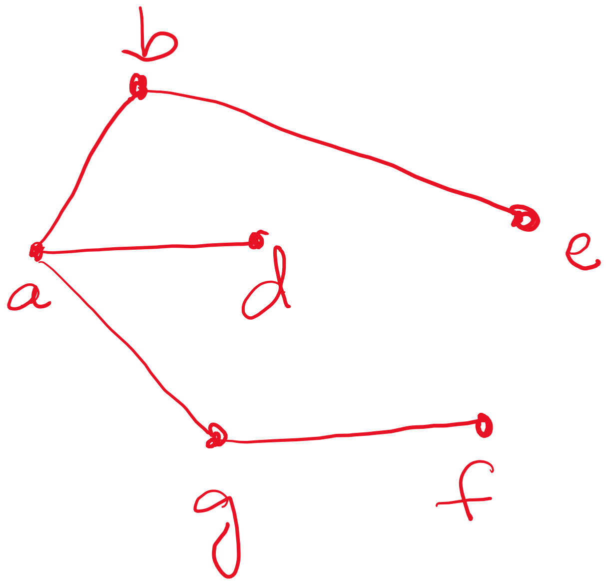
g
f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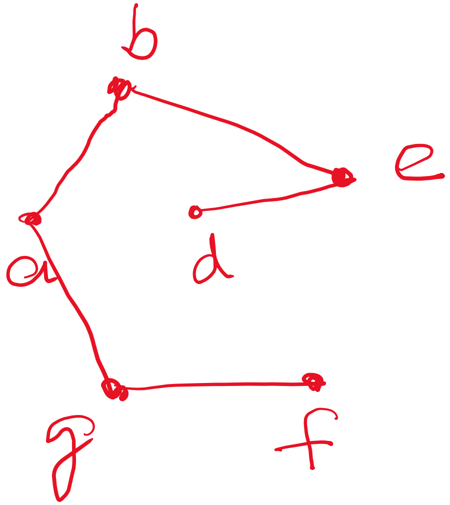
b
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g
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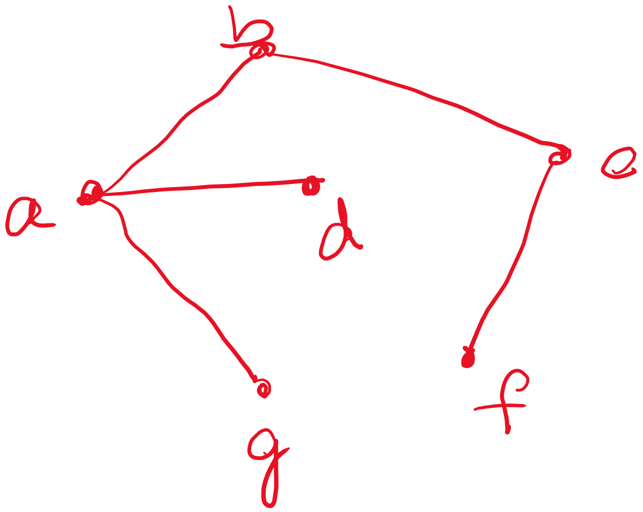

b
a
d
e
g
f

The distance between two spanning trees T_i and T_j of a graph G is defined as the number of edges of G present in one tree but not in the other. This distance may be written as $d(T_i, T_j)$.

Let $T_i \oplus T_j$ be the ring sum of two spanning trees T_i and T_j of G, Let $N(g)$ denote the number of edges in a graph g. Then, from definition,

$$d(T_i, T_j) = \frac{1}{2} N(T_i \oplus T_j)$$

The distance between the spanning trees of a graph is a *metric*. That is, it satisfies

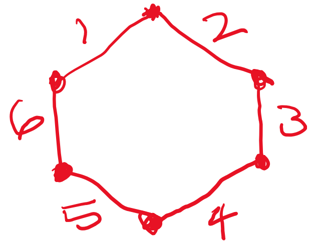
$$d(T_i, T_j) \geq 0 \quad \text{and} \quad d(T_i, T_j) = 0 \text{ if and only if } T_i = T_j,$$
$$d(T_i, T_j) = d(T_j, T_i),$$
$$d(T_i, T_j) \leq d(T_i, T_k) + d(T_k, T_j).$$

Starting from any spanning tree of a graph G, we can obtain every spanning tree of G by successive cyclic exchanges.

$n = r+1$ $\quad e = n-1 = r$

Since in a connected graph G of rank r (i.e., of $r + 1$ vertices) a spanning tree has r edges, we have the following results:

The maximum distance between any two spanning trees in G is

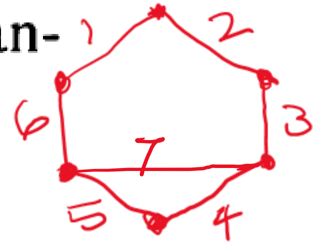


$$\max d(T_i, T_j) = \frac{1}{2} \max N(T_i \oplus T_j)$$

$$\leq r, \text{ the rank of } G.$$

$^6C_5$

Also, if μ is the nullity of G, we know that no more than μ edges of a spanning tree T_i can be replaced to get another tree T_j.



Hence $$\max d(T_i, T_j) \leq \mu;$$

$^7C_5 - 2$

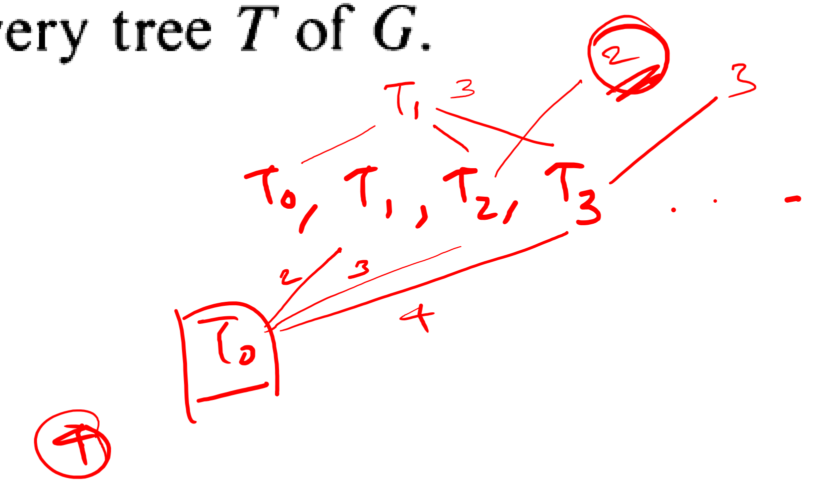
combining the two,

$$\max d(T_i, T_j) \leq \min(\mu, r),$$

Central Tree

Central Tree: For a spanning tree T_0 of a graph G, let $\max_i d(T_0, T_i)$ denote the maximal distance between T_0 and any other spanning tree of G. Then T_0 is called a *central tree* of G if

$$\max_i d(T_0, T_i) \leq \max_j d(T, T_j) \qquad \text{for every tree } T \text{ of } G.$$



Spanning Trees in a Weighted Graph

$1+1+1+1+1=5$

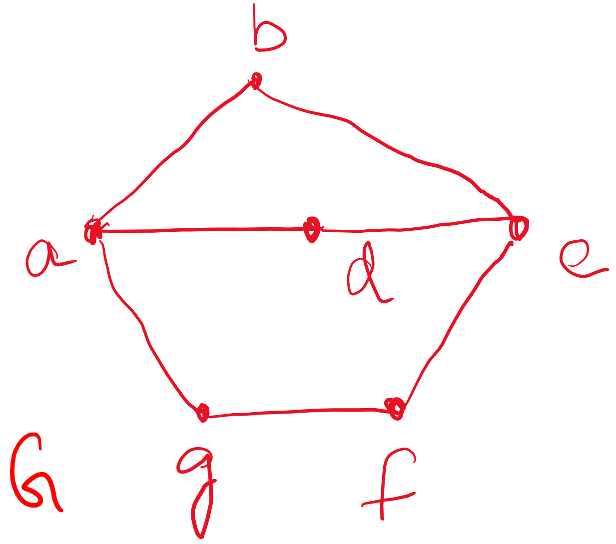
the *weight of a spanning tree T* of G is defined as the sum of the weights of all the branches in T. In general, different spanning trees of G will have different weights.

A spanning tree with the smallest weight in a weighted graph is called a *shortest spanning tree* or *shortest-distance spanning tree* or *minimal spanning tree*.

THEOREM

A spanning tree T (of a given weighted connected graph G) is a shortest spanning tree (of G) if and only if there exists no other spanning tree (of G) at a distance of one from T whose weight is smaller than that of T.

(5)


b
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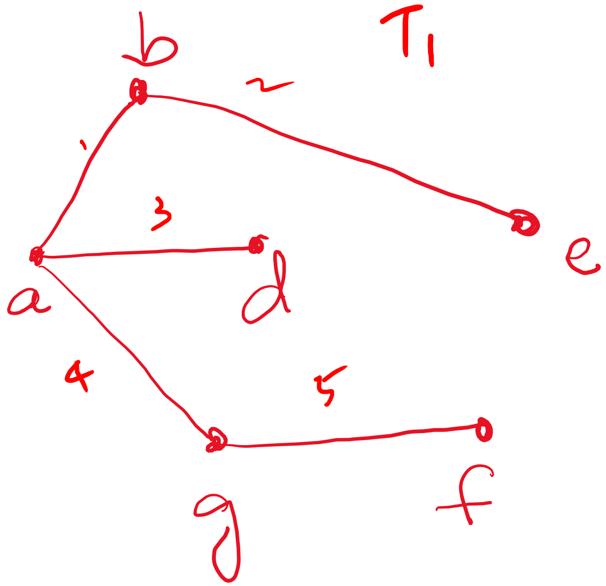
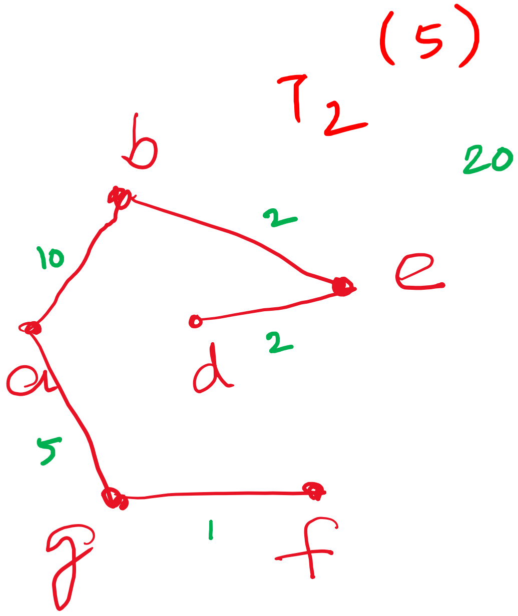
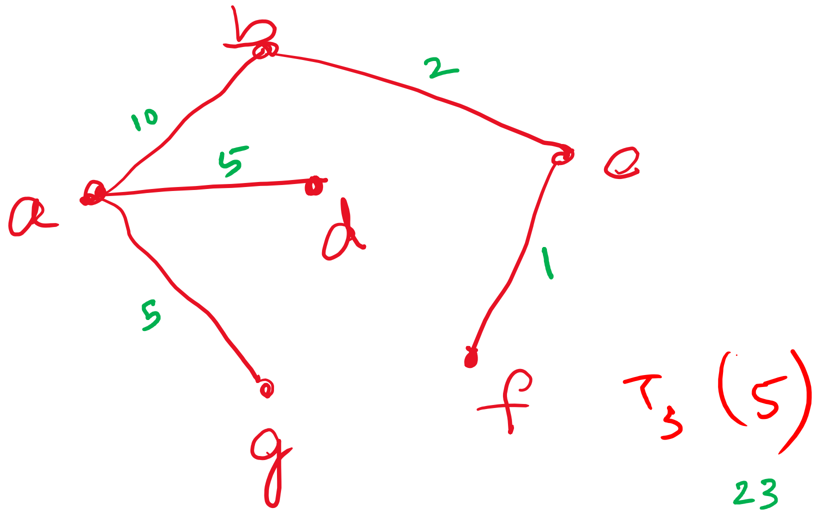
G
g
f


T1
b
1
2
3
a
d
e
4
5
g
f


(5)
T2
20
b
10
2
e
a
d
2
5
g
1
f


b
10
2
a
5
d
e
5
1
g
f
T3 (5)
23

MST Methods

- Kruskal’s
- Prim’s

Select n-1 edges from a weighted graph of n vertices with minimum cost.

Greedy Strategy

- An optimal solution is constructed in stages
- At each stage, the best decision is made at this time
- Since this decision cannot be changed later,
  we make sure that the decision will result in a feasible solution
- Typically, the selection of an item at each stage is based on a least cost or a highest profit criterion

Kruskal’s Idea

- Build a minimum cost spanning tree T by adding edges to T one at a time
- Select the edges for inclusion in T in nondecreasing order of the cost
- An edge is added to T if it does not form a cycle
- Since G is connected and has $n > 0$ vertices, exactly $n-1$ edges will be selected

Examples for Kruskal's Algorithm

0 —10— 5

2 —12— 3

1 —14— 6

1 —16— 2

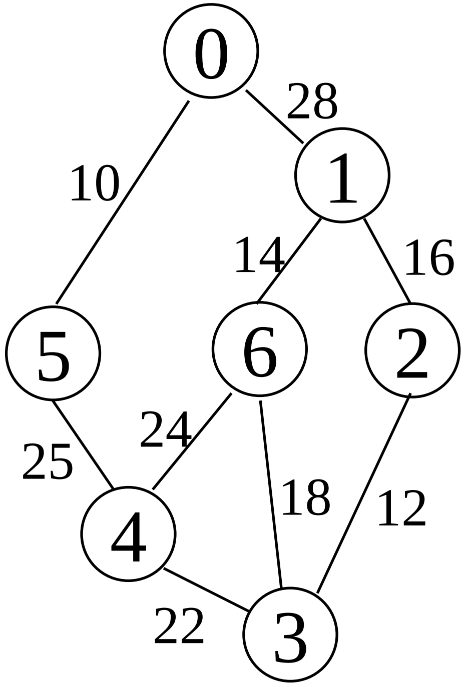
3 —18— 6

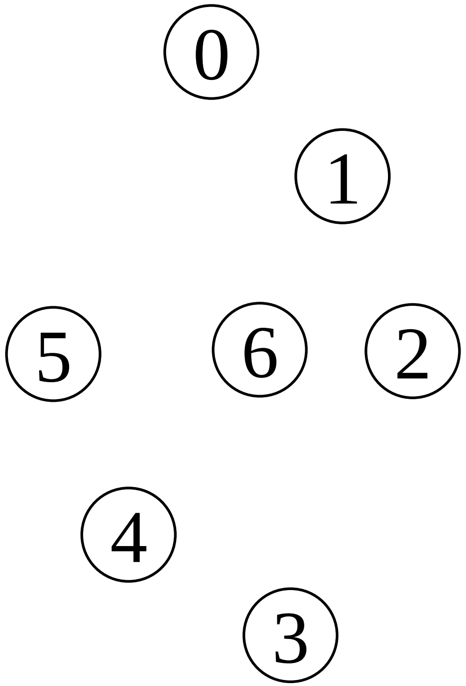
3 —22— 4

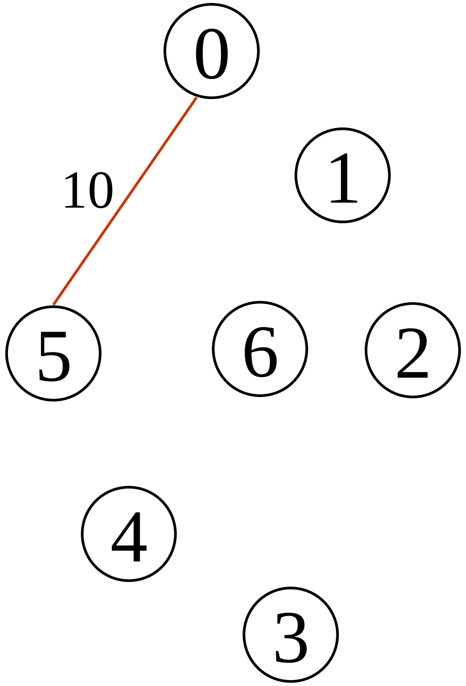
4 —24— 6

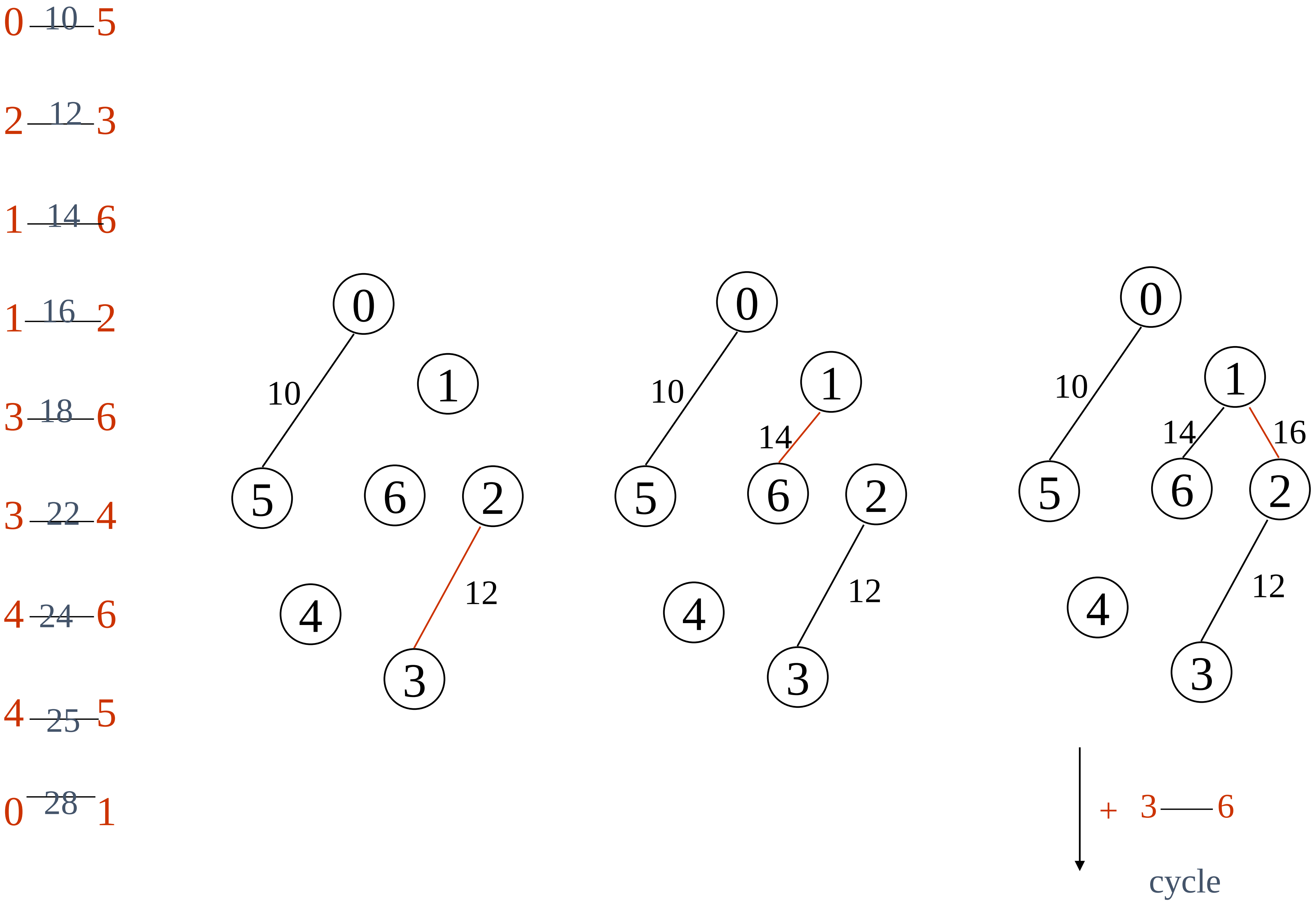
4 —25— 5

0 —28— 1









0 10 5
2 12 3
1 14 6
1 16 2
3 18 6
3 22 4
4 24 6
4 25 5
0 28 1
0
1
5
6
2
4
3
10
12
14
16
+ 3 6
cycle

0 —10— 5

2 —12— 3

1 —14— 6

1 —16— 2

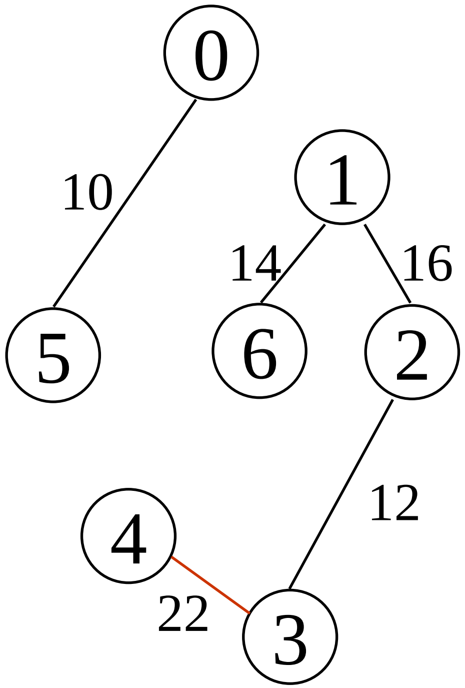
3 —18— 6

3 —22— 4

4 —24— 6

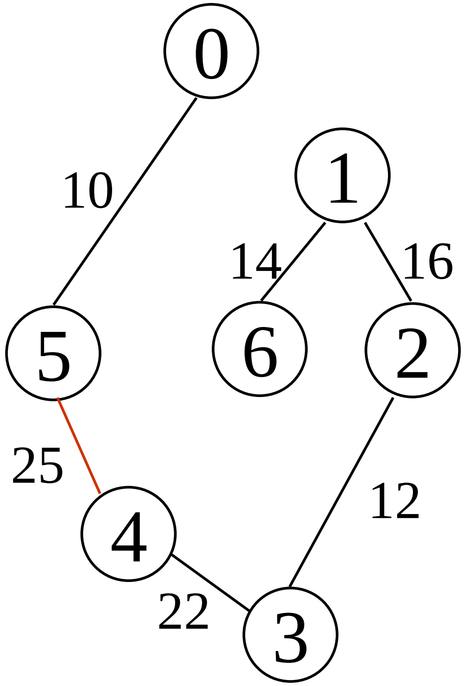
4 —25— 5

0 —28— 1



+ 4 — 6

cycle

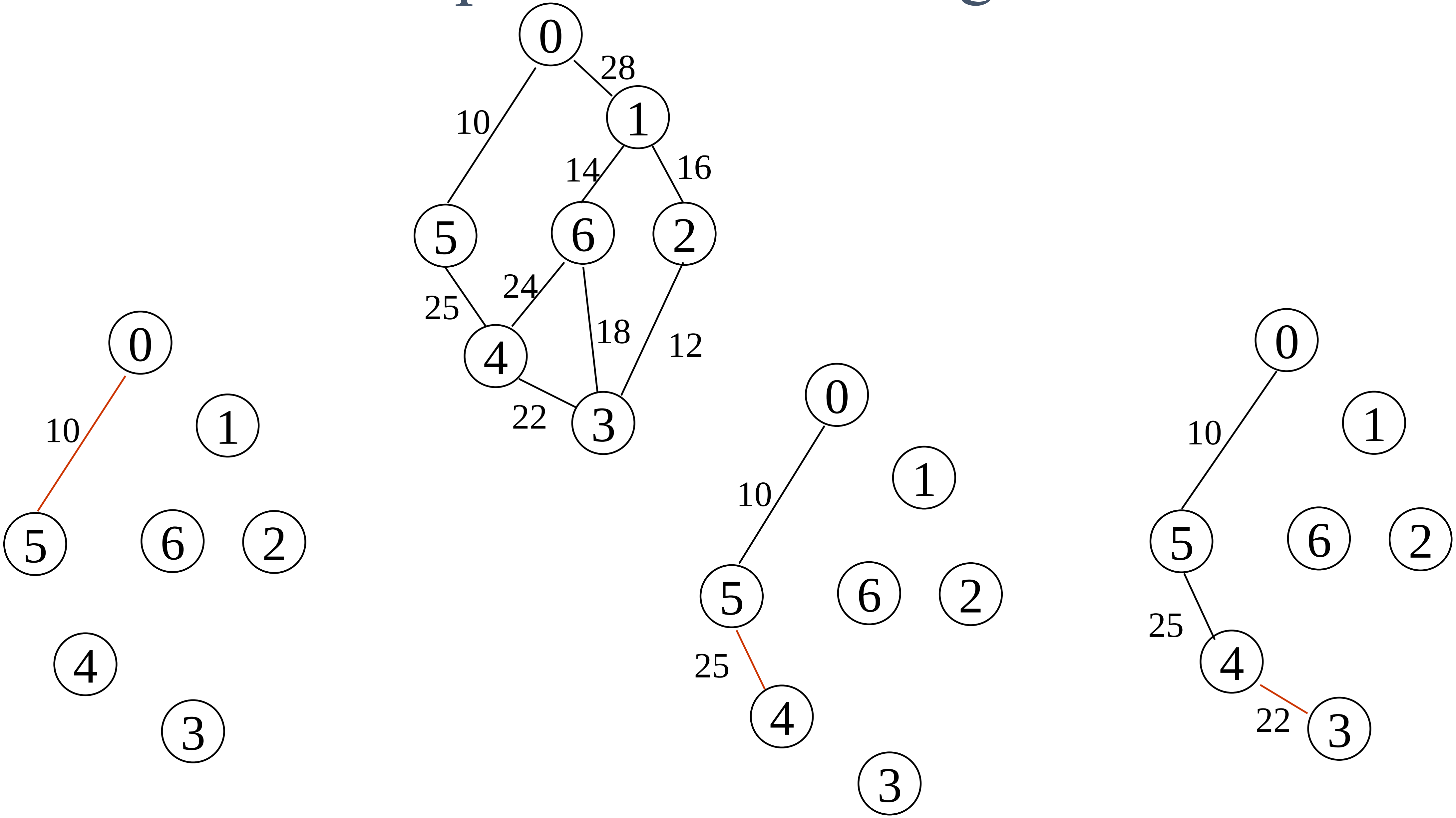


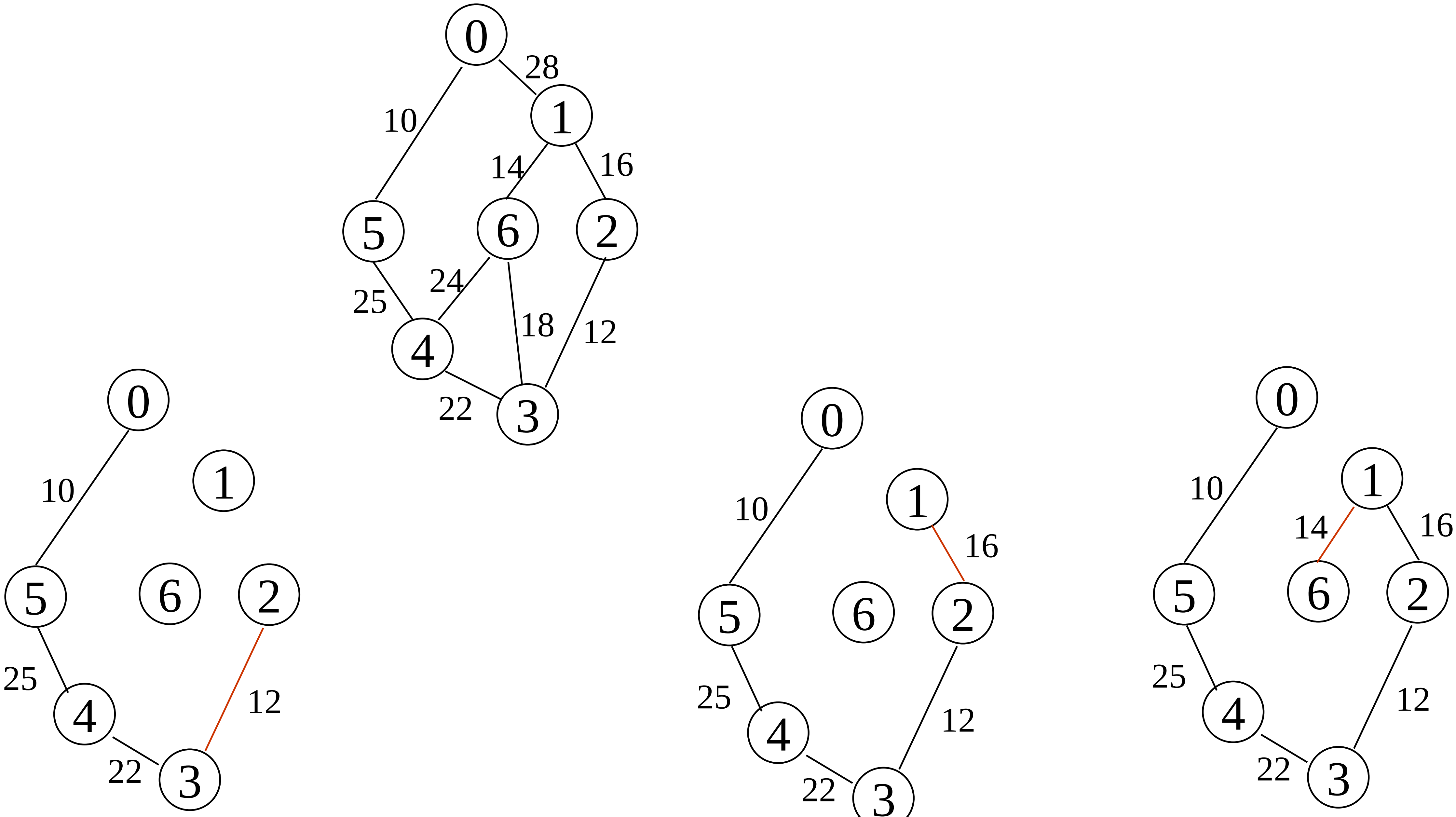
cost = 10 +25+22+12+16+14

Kruskal's Algorithm

```
T= {};
while (T contains less than n-1 edges && E is not empty)
{
  choose a least cost edge (v,w) from E;
  delete (v,w) from E;
  if ((v,w) does not create a cycle in T)
     add (v,w) to T
  else discard (v,w);
}
if (T contains fewer than n-1 edges)
  printf("No spanning tree\n");
```

Examples for Prim's Algorithm

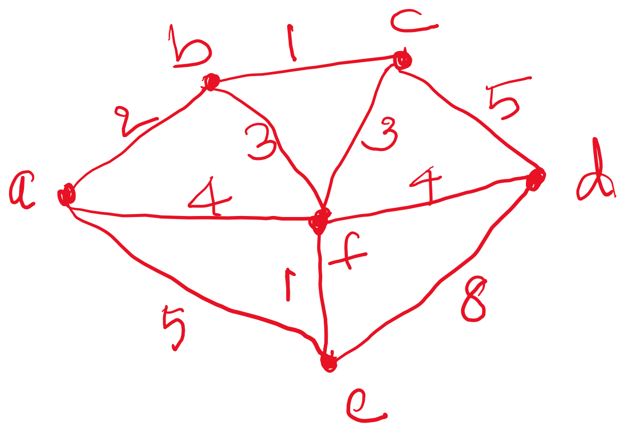



0 1 2 3 4 5 6 28 10 14 16 24 25 18 12 22
0 1 5 6 2 4 3 10 25 12 22
0 1 5 6 2 4 3 10 16 25 12 22
0 1 5 6 2 4 3 10 14 16 25 12 22

Prim's Algorithm

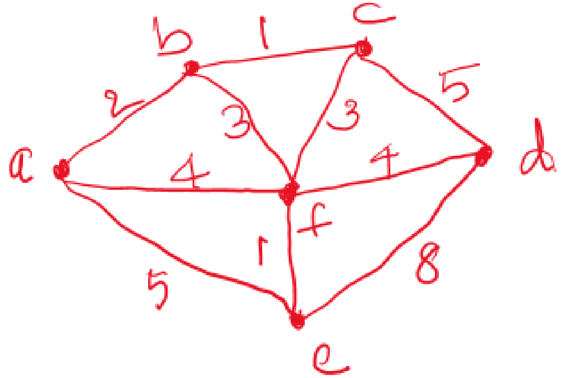
```
T={};
TV={0};
while (T contains fewer than n-1 edges)
{
  let (u,v) be a least cost edge such
    that u ∈ TV and v ∉ TV
  if (there is no such edge ) break;
  add v to TV;
  add (u,v) to T;
}
if (T contains fewer than n-1 edges)
  printf("No spanning tree\n");
```

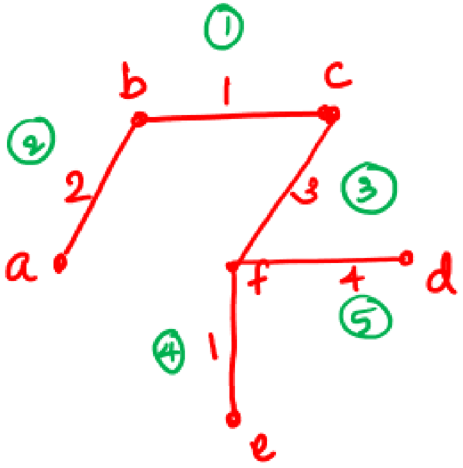
Prim’s Methods

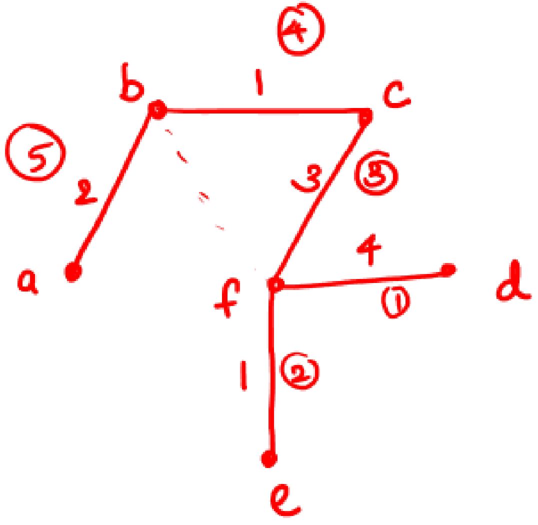


	a	b	c	d	e	f
a	–	2	∞	∞	5	4
b	2	–	1	∞	∞	3
c	∞	1	–	5	∞	3
d	∞	∞	5	–	8	4
e	5	∞	∞	8	–	1
f	4	3	3	4	1	–

Prim's Methods





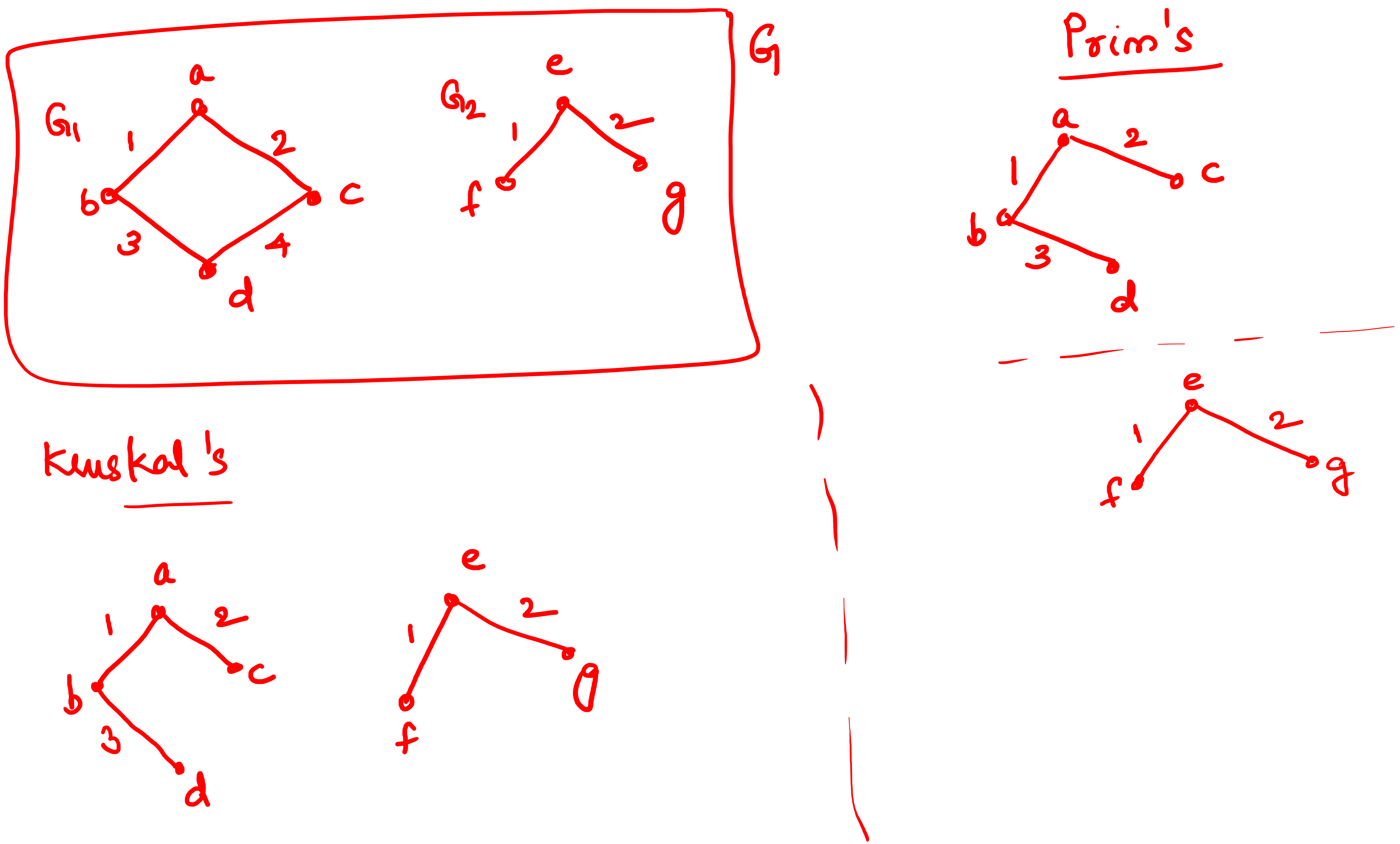


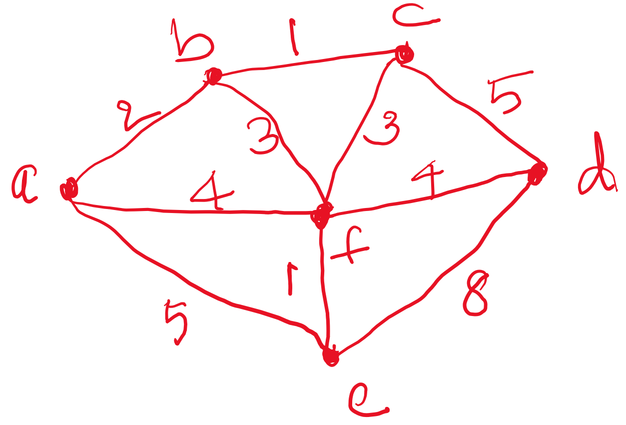
	a	b	c	d	e	f
→ a	–	2	∞	∞	5	4
→ b	2	–	1	∞	∞	3
→ c	∞	1	–	5	∞	3
→ d	∞	∞	5	–	8	4
→ e	5	∞	∞	8	–	1
→ f	4	3	3	4	1	–

cost = 2 + 1 + 3 + 1 + 4

= 11

MST in Disconnected Graph





a
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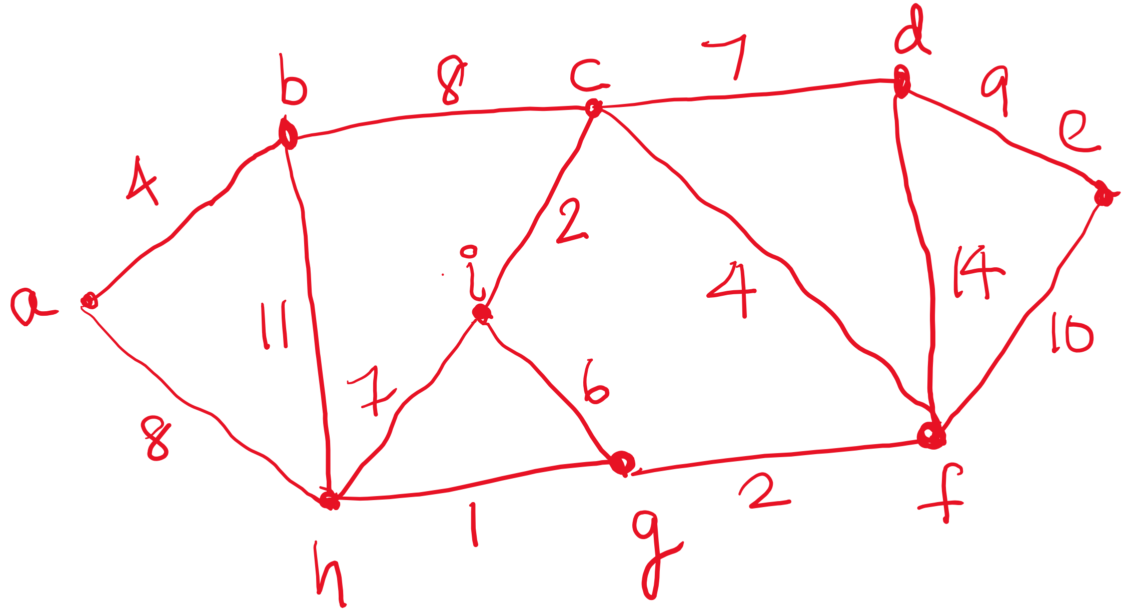
4
4
5
1
8



a
b
c
d
e
f
g
h
i
4
8
8
7
9
11
7
2
4
14
10
6
1
2

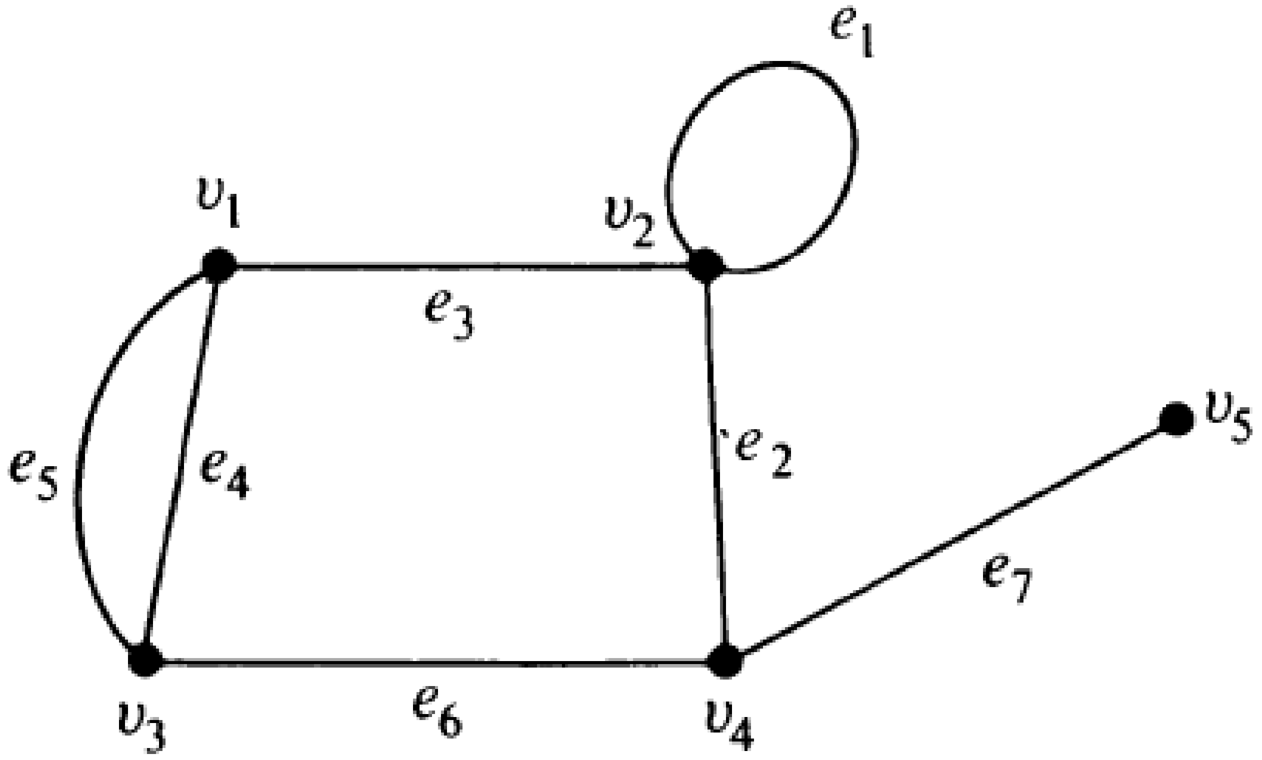
Graphs

Graph

A *linear graph* (or simply a *graph*) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k.

The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.



$V = \{v_1, v_2, \ldots v_5\}$

$E = \{e_1, e_2, \ldots e_7\}$

$e_2 = (v_2, v_4)$ or (v_4, v_2)

unordered

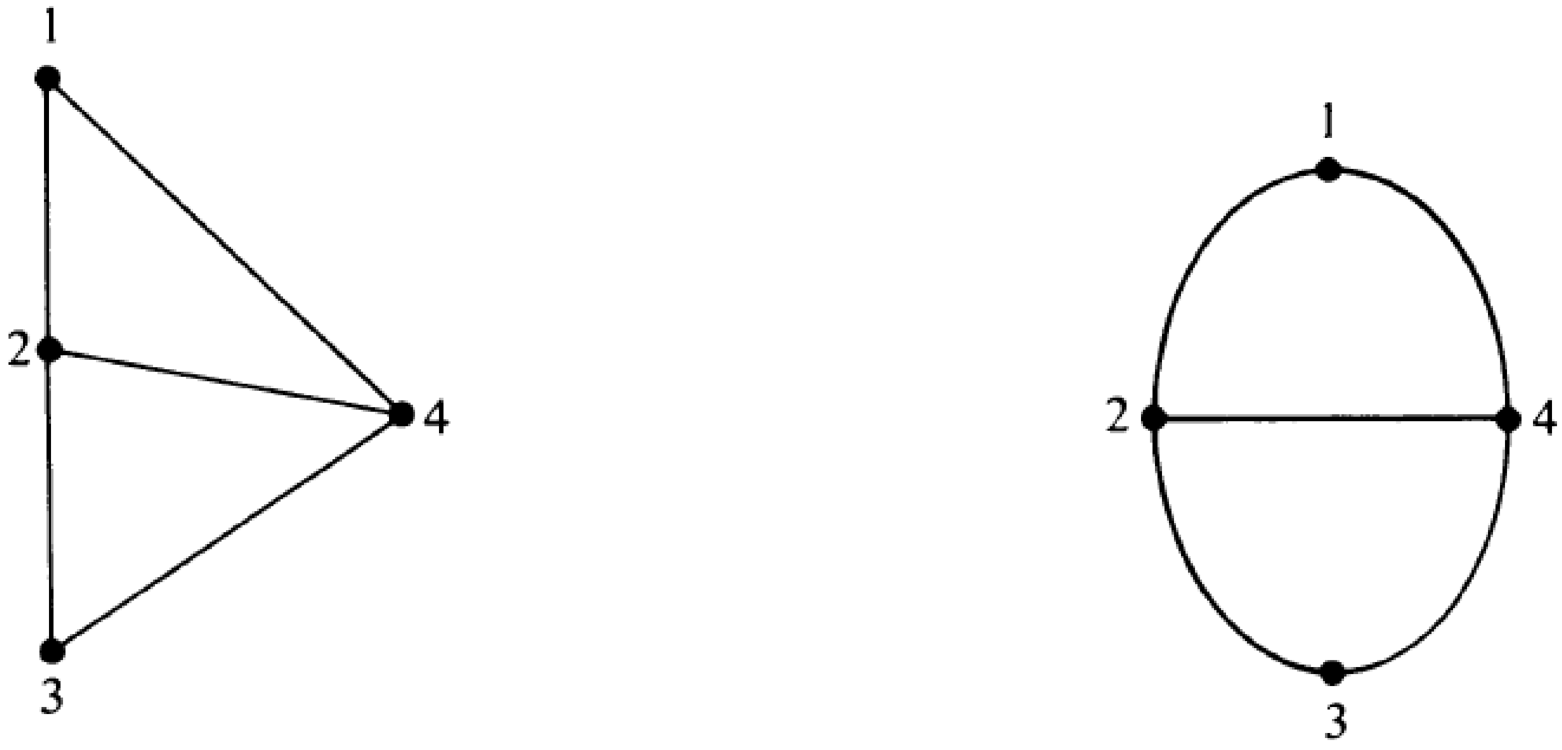
$e_1 = (v_2, v_2)$

an edge having the same vertex as both its end vertices is called a *self-loop*

vertex pair (v_i, v_i)

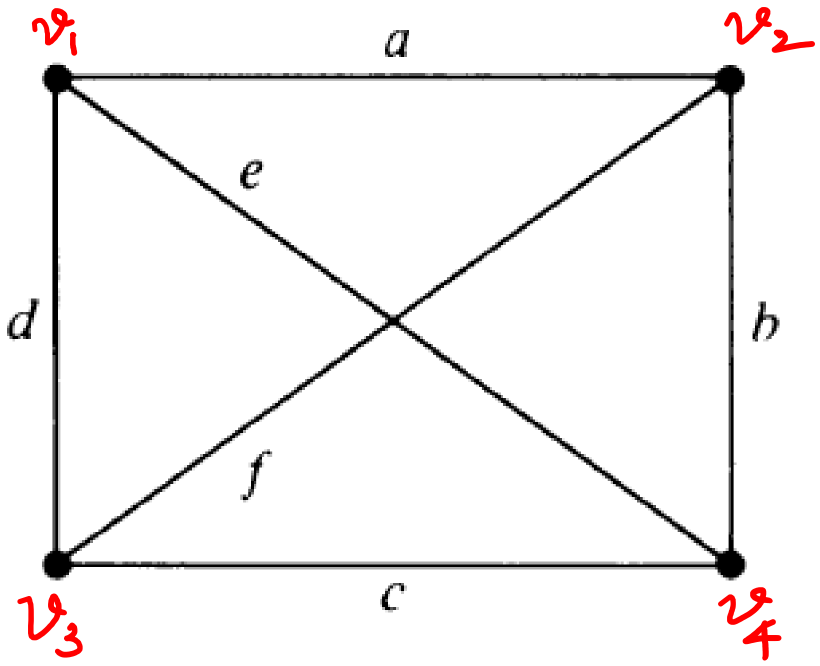
the definition allows more than one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in Fig.
Such edges are referred to as *parallel edges*.

A graph that has neither self-loops nor parallel edges is called a *simple graph*. In some graph-theory literature, a graph is defined to be only a simple graph, but in most engineering applications it is necessary that parallel edges and self-loops be allowed; this is why our definition includes graphs with self-loops and/or parallel edges. Some authors use the term *general graph* to emphasize that parallel edges and self-loops are allowed.



Same graph drawn differently.

How many vertices are there?



A graph is also called a *linear complex*, a *1-complex*, or a *one-dimensional complex*.

A vertex is also referred to as a *node*, a *junction*, a *point*, *0-cell*, or an *0-simplex*.

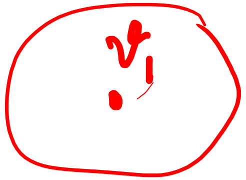
Other terms used for an edge are a *branch*, a *line*, an *element*, a *1-cell*, an *arc*, and a *1-simplex*.

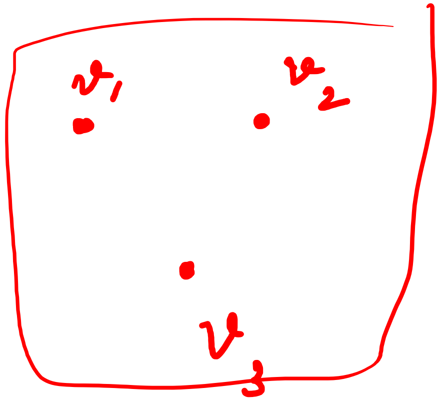
The number of vertices in G is often called the **order** of G, while the number of edges is its **size**.

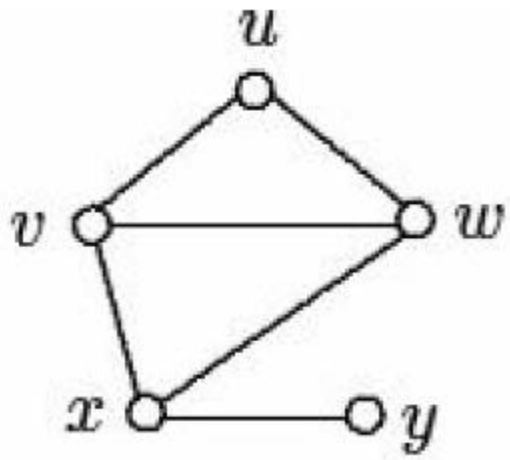
Since the vertex set of every graph is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a **trivial graph**, implying that the order of a **nontrivial graph** is at least 2.

$V \neq \emptyset$

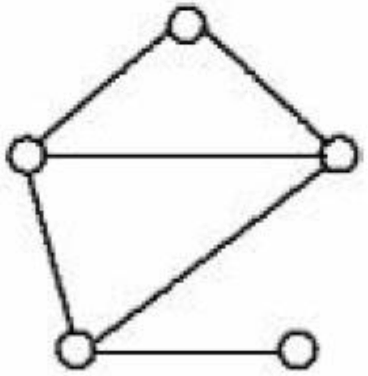
$E = \emptyset$







labeled graph



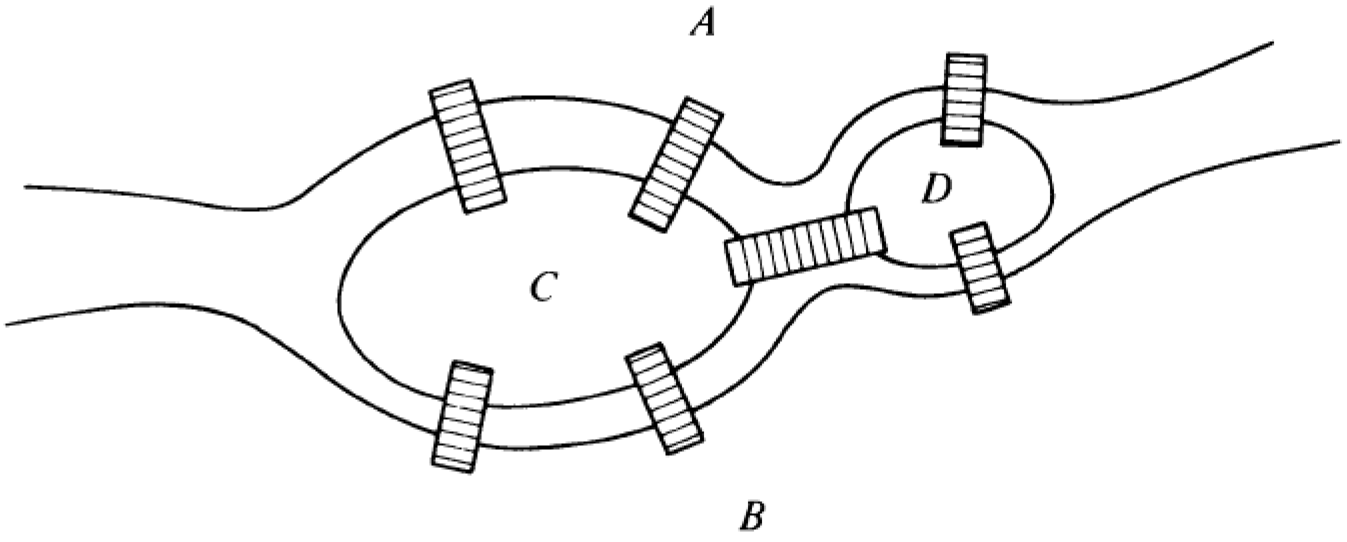
unlabeled graph

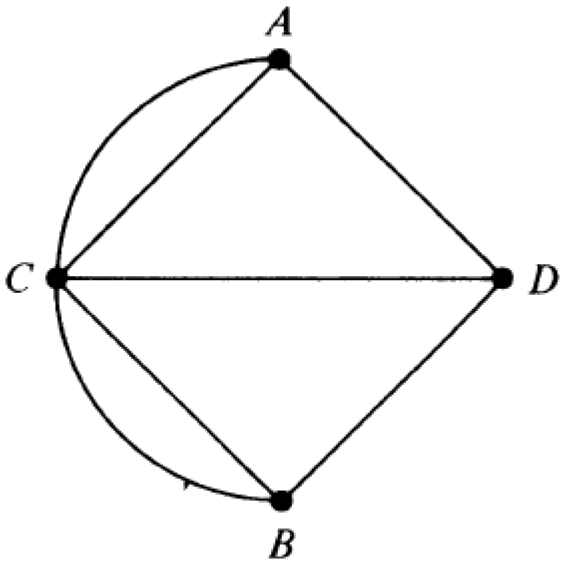
Applications of Graph theory

A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them.

Königsberg Bridge Problem.

Leonard Euler





Finite and infinite graphs

A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph;* otherwise, it is an *infinite graph*.

Incidence and Degree

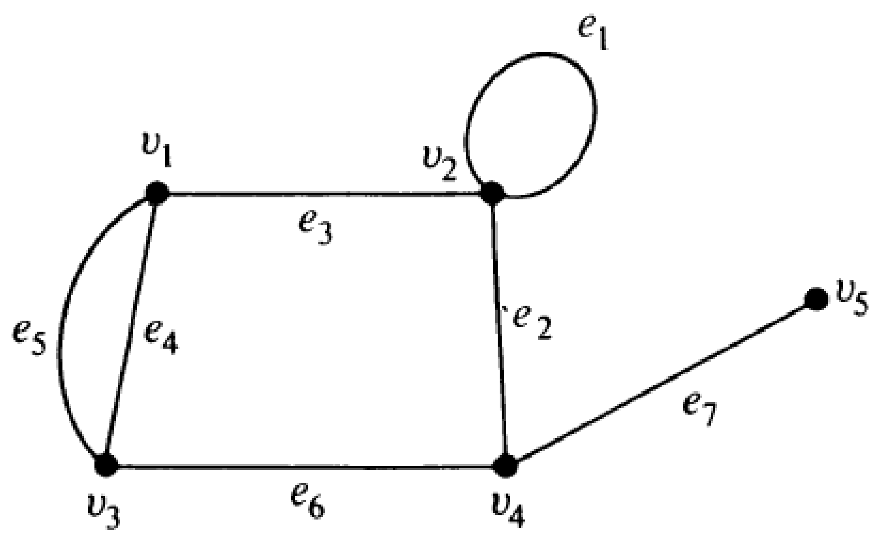
When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident with* (*on* or *to*) each other.

Two nonparallel edges are said to be *adjacent* if they are incident on a common vertex.

Two vertices are said to be adjacent if they are the end vertices of the same edge.

The number of edges incident on a vertex v_i, with self-loops counted twice, is called the *degree*, $d(v_i)$, of vertex v_i.

The degree of a vertex is also referred to as its *valency*.



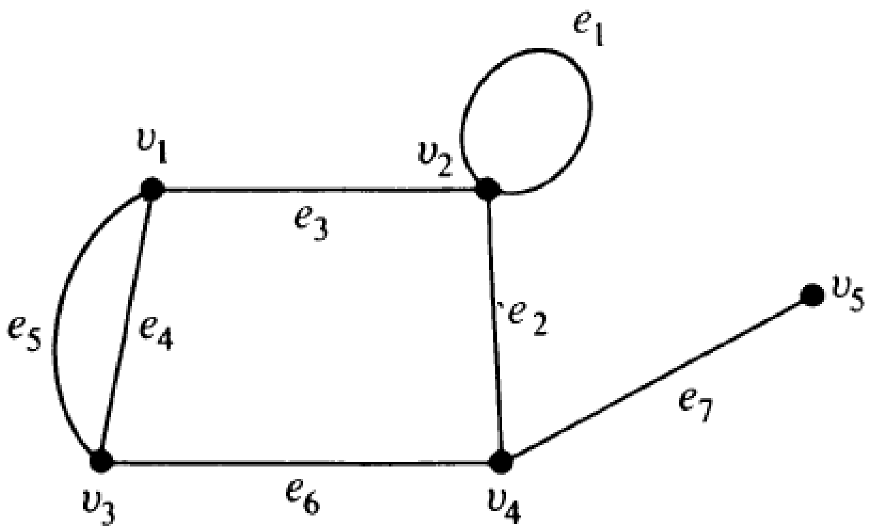
$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 3$

$d(v_4) = 3$

$d(v_5) = 1$



Let us now consider a graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

THEOREM 1-1

The number of vertices of odd degree in a graph is always even.

Proof

$$\sum_{i=1}^{n} d(v_i) = 2e \quad = \text{even}$$

$2+4+6+\ldots$ $\qquad$ $1+3+5+\ldots$

$$\underbrace{\sum_{i=1}^{n} d(v_i)}_{\text{even}} = \underbrace{\sum_{\text{even}} d(v_j)}_{\text{even}} + \underbrace{\sum_{\text{odd}} d(v_k)}_{\text{even}}$$

$$\sum_{i=1}^{n} d(v_i) - \sum_{\text{even}} d(v_j) = \sum_{\text{odd}} d(v_k)$$

even − even is even. To form LHS = RHS,

$\sum_{\text{odd}} d(v_k)$ should be even. So, count of odd degree vertices should be even.

$\overset{12}{\text{even}} + \overset{3}{\text{odd}} = \text{odd}$ ×

$\text{even} + \underbrace{(\overset{3}{\text{odd}} + \overset{5}{\text{odd}})}_{\text{even}} = \text{even}$

× ev $\quad (\overset{1}{\text{odd}} + \overset{7}{\text{odd}} + \overset{3}{\text{odd}}) =$

$\underbrace{0+0+0+0}$

A graph in which all vertices are of equal degree is called a *regular graph* (or simply a *regular*).

2-regular

3

K-regular

A graph is said to be k-regular if the degree, $\mathrm{d(v) = k}, \forall v \in V$

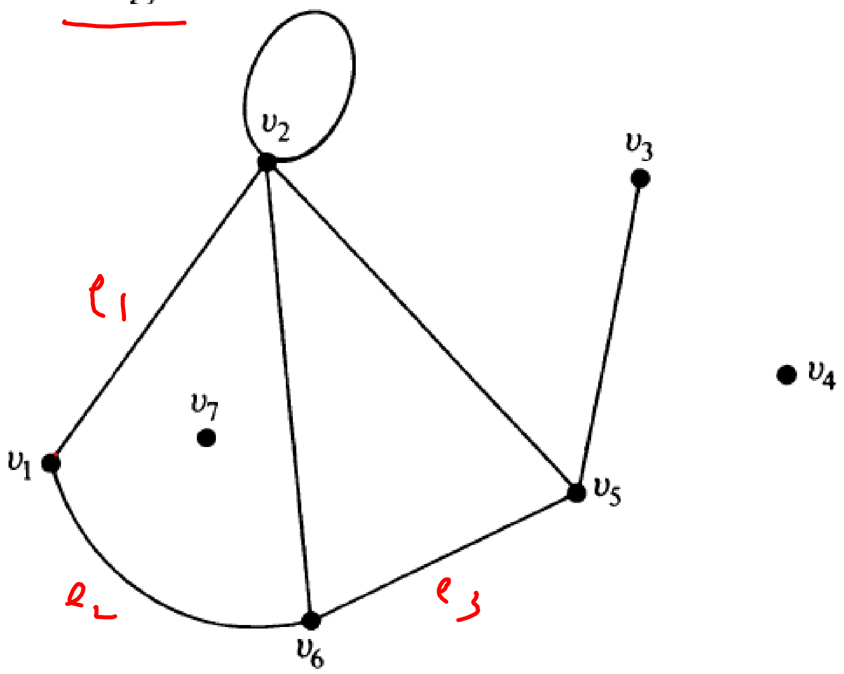
A complete graph with n vertices is (n-1) regular

Isolated vertex & pendent vertex

A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are vertices with zero degree.

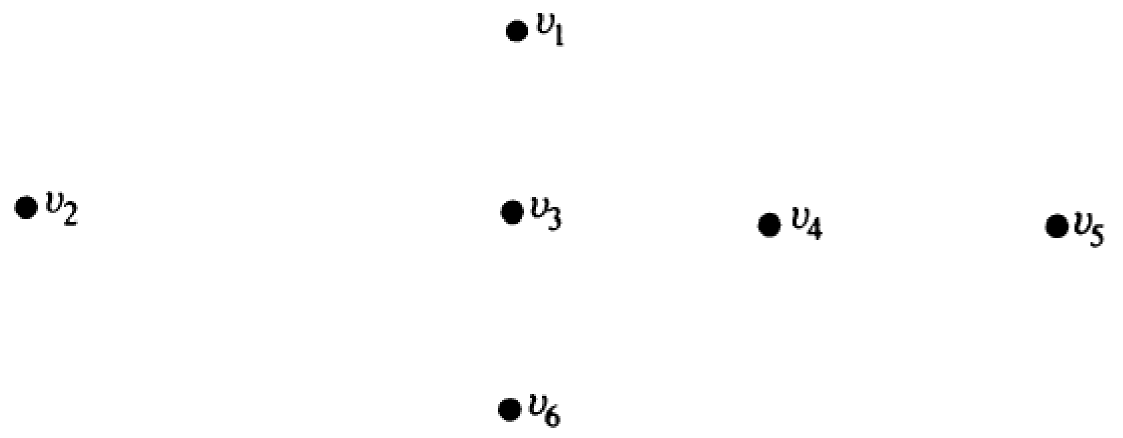
A vertex of degree one is called a *pendant vertex* or an *end vertex*.

Two adjacent edges are said to be in *series* if their common vertex is of degree two.



Null graph

In the definition of a graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph, without any edges, is called a *null graph*. In other words, every vertex in a null graph is an isolated vertex.



Directed graph

Let V be a finite nonempty set, and let $E \subseteq V \times V$. The pair (V, E) is then called a *directed graph* (on V), or *digraph* (on V), where V is the set of *vertices*, or *nodes*, and E is its set of (*directed*) *edges* or *arcs*.

- Degree of a vertex
  - In-degree
  - Out-degree

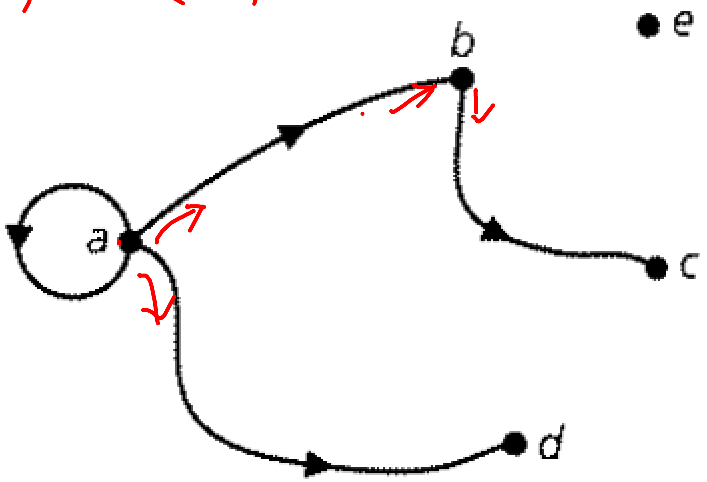
$I(e) = O(e) = 0$

$g(b) = 1$

$O(b) = 1$

$g(a) = 1$

$O(a) = 3$



Subgraph

A graph H is called a **subgraph** of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G.

If a subgraph of a graph G has the same vertex set as G, then it is a **spanning subgraph** of G.

A subgraph F of a graph G is called an **induced subgraph** of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well.

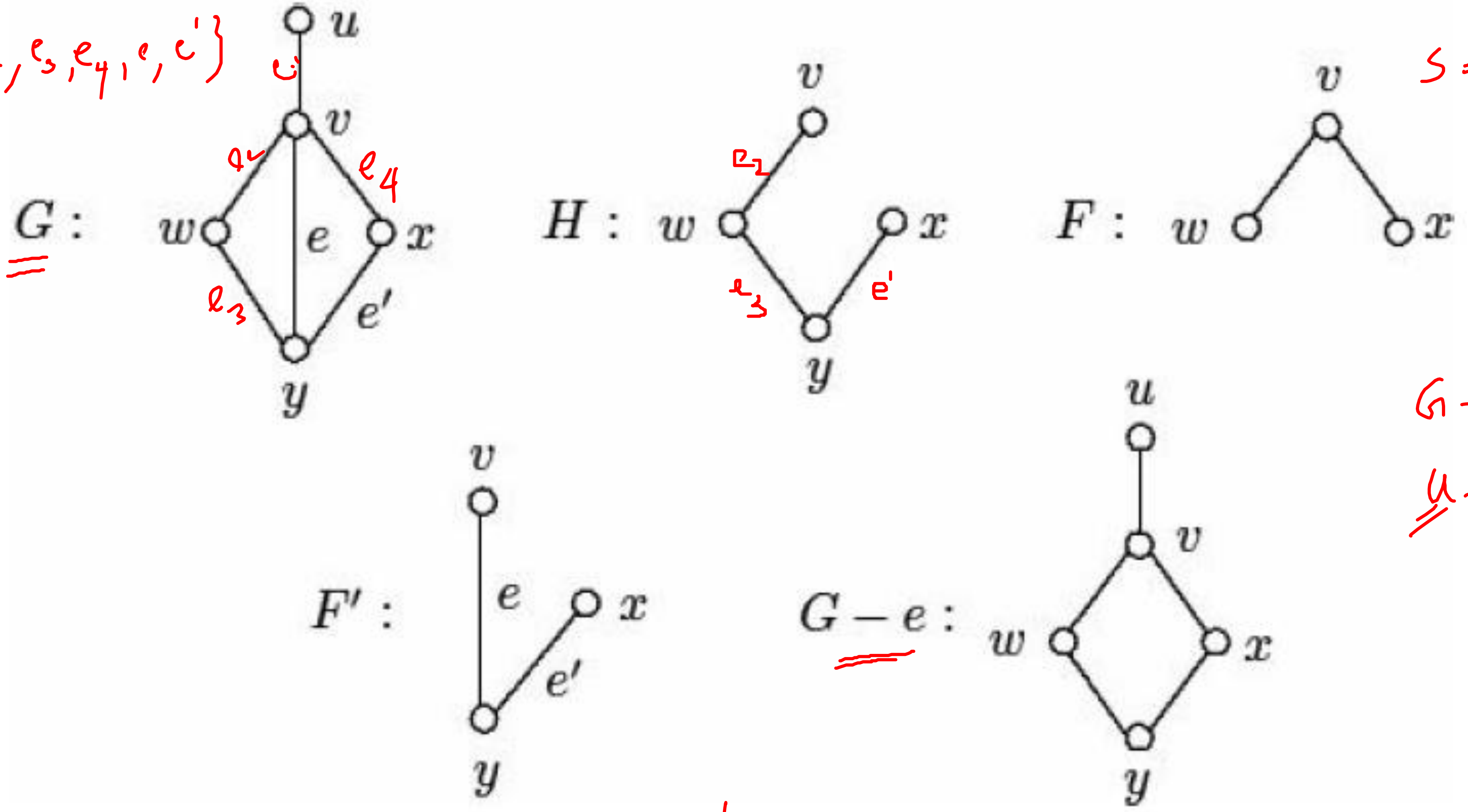
If S is a nonempty set of vertices of a graph G, then the **subgraph of** G **induced by** S is the induced subgraph with vertex set S. This induced subgraph is denoted by $G[S]$.

For a nonempty set X of edges,the **subgraph** $G[X]$ **induced by** X has edge set X and consists of all vertices that are incident with at least one edge in X is called an **edge-induced subgraph** of G.

$V(G) = \{u, v, w, x, y\}$

$E(G) = \{e_1, e_2, e_3, e_4, e, e'\}$

$S = \{v, w, x\}$



$G - e_1 - e_2$

$U = \{e_1, e_2\}$

$G - U$

$X = \{e, e'\}$

spanning

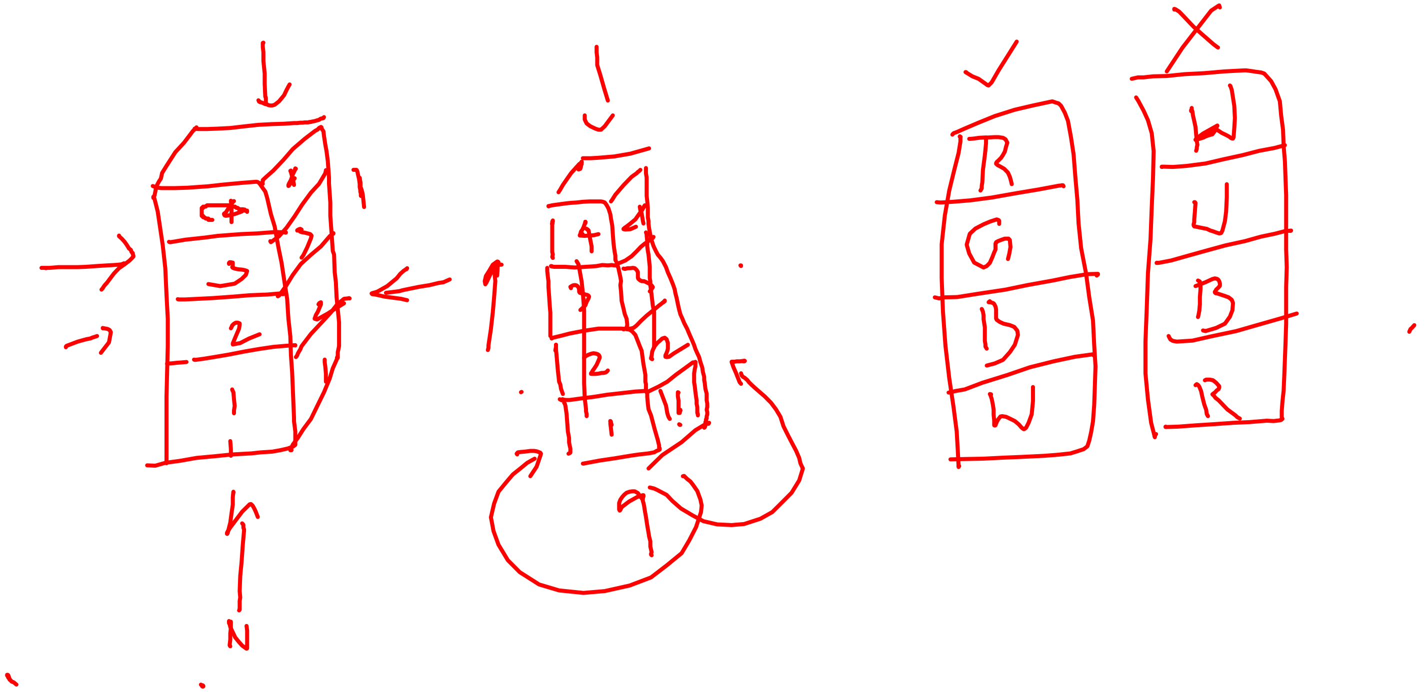
G

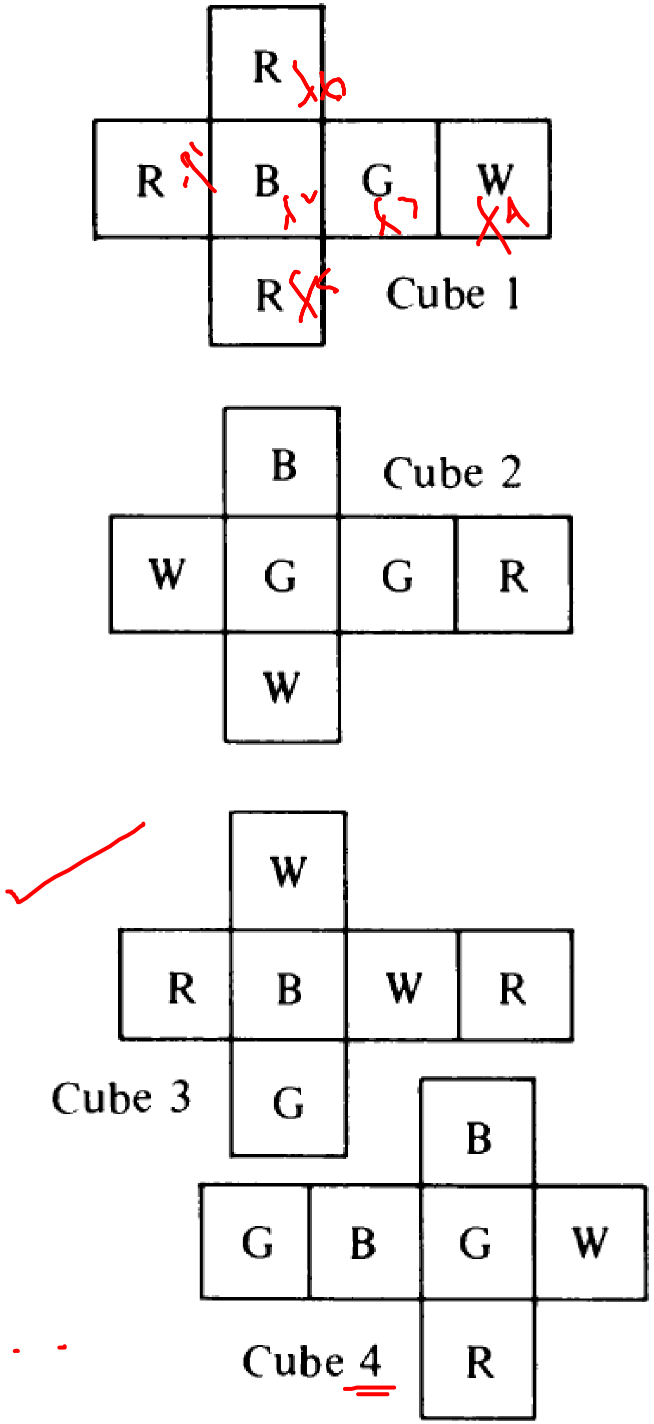
$G + \underline{e}$

$G + v$

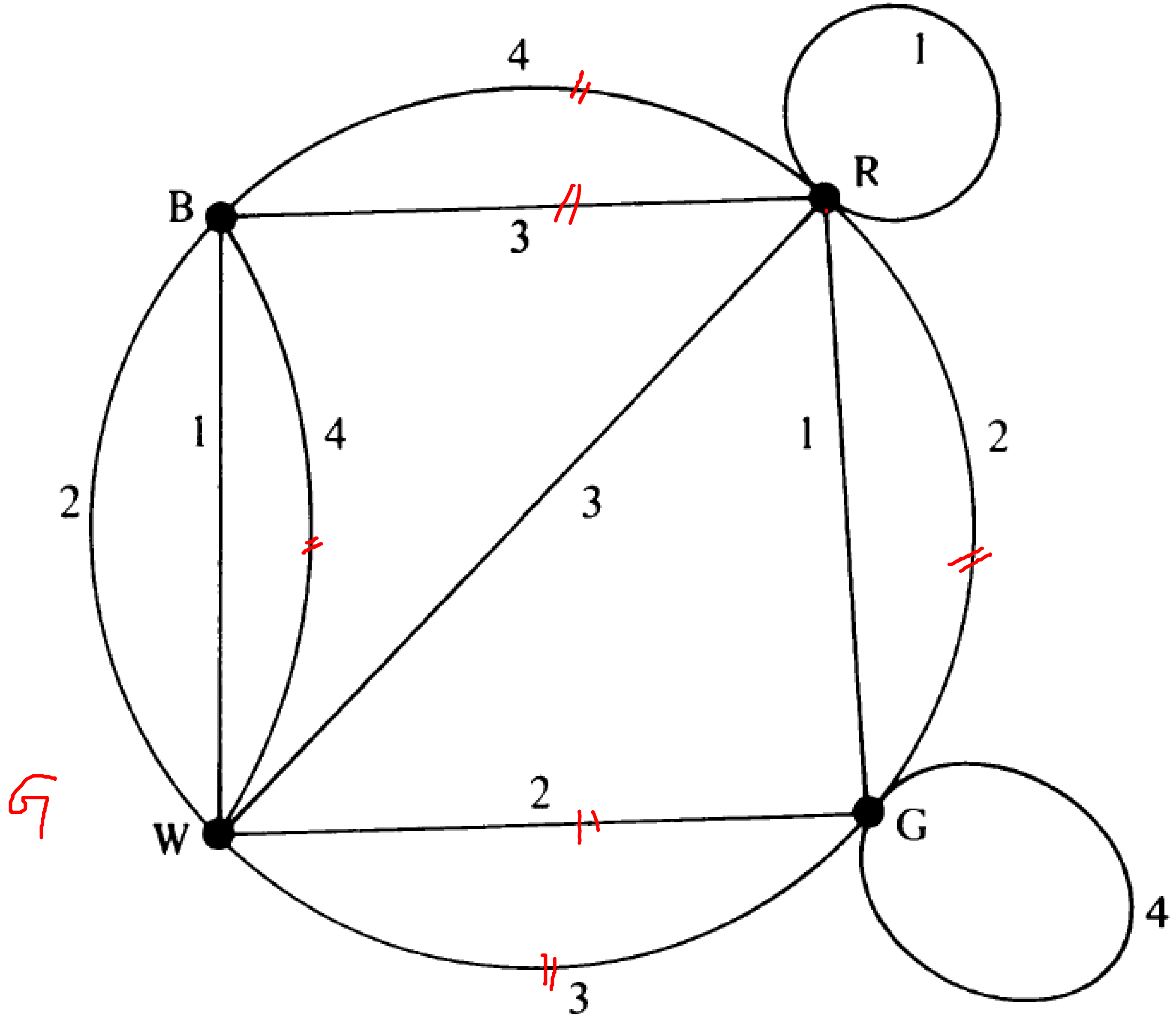
Puzzle with Multicolored Cubes

Problem: We are given four cubes. The six faces of every cube are variously colored blue, green, red, or white. Is it possible to stack the cubes one on top of another to form a column such that no color appears twice on any of the four sides of this column?

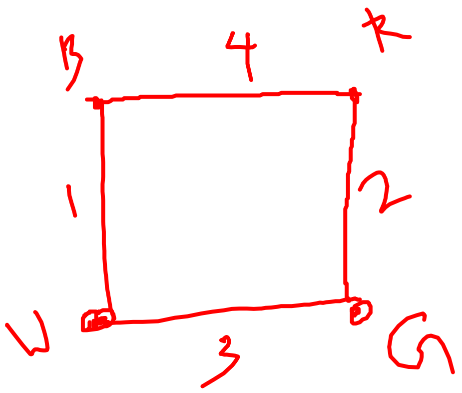




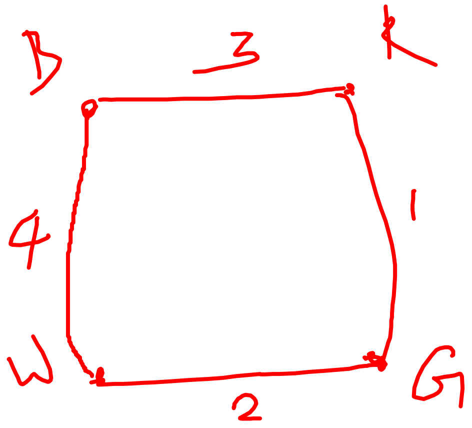
R
R B G W
R
Cube 1
B
Cube 2
W G G R
W
W
R B W R
G
Cube 3
B
G B G W
R
Cube 4



B
R
W
G
4
3
1
1
4
2
3
1
2
2
3
4



N.S

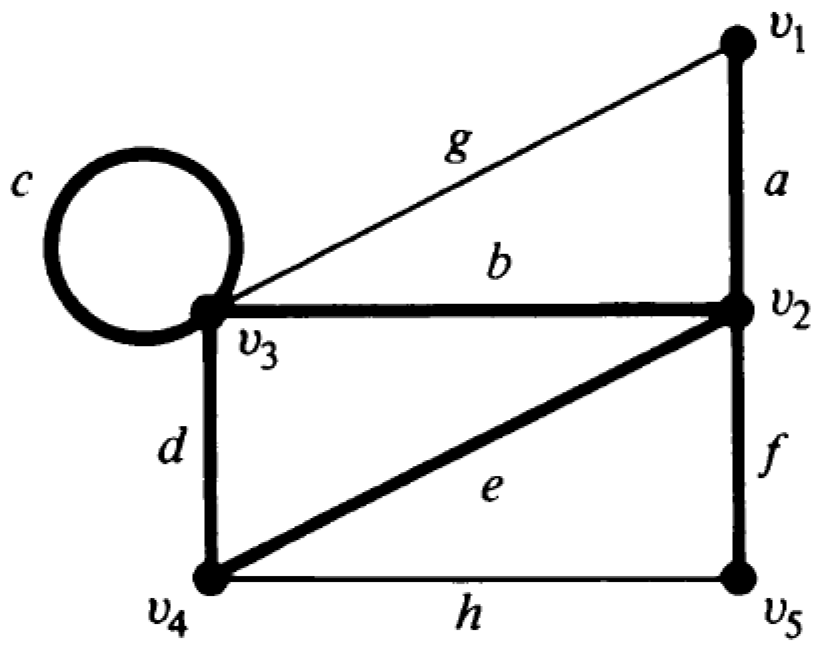


E-W

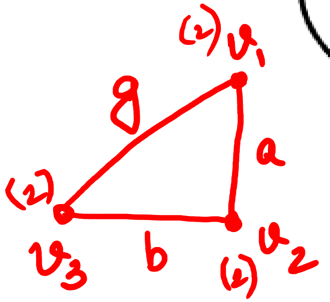
Walk

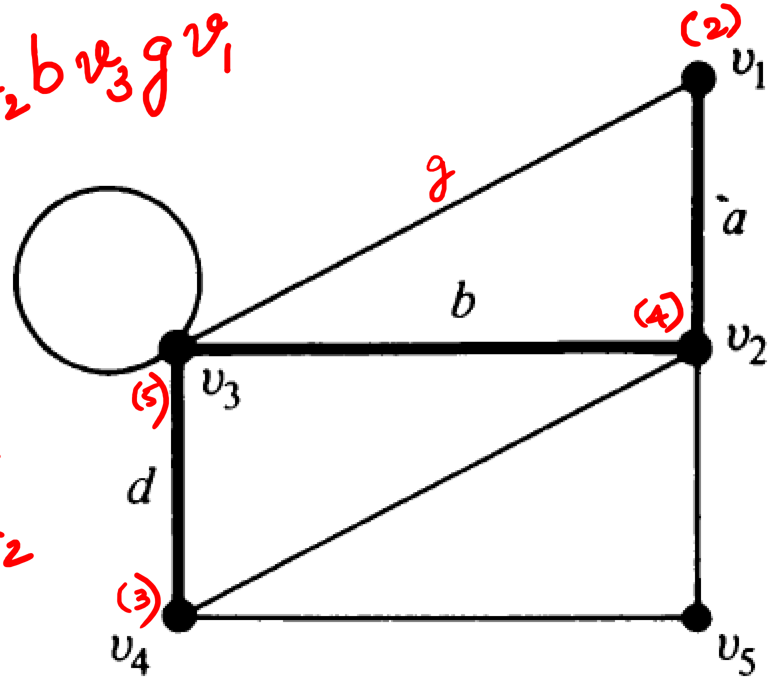
A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears (is covered or traversed) more than once in a walk. A vertex, however, may appear more than once.

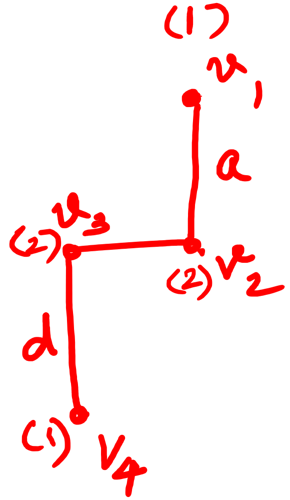
$v_3 c v_3$



w3: $v_1 a v_2 b v_3 g v_1$







w1: $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$

w2: $v_1\ a\ v_2\ b\ v_3\ d\ v_4$ $\Rightarrow$ path

A walk is also referred to as an *edge train* or a *chain*.

Vertices with which a walk begins and ends are called its *terminal vertices*.

It is possible for a walk to begin and end at the same vertex. Such a walk is called a *closed walk*. A walk that is not closed (i.e., the terminal vertices are distinct) is called an *open walk*

Path

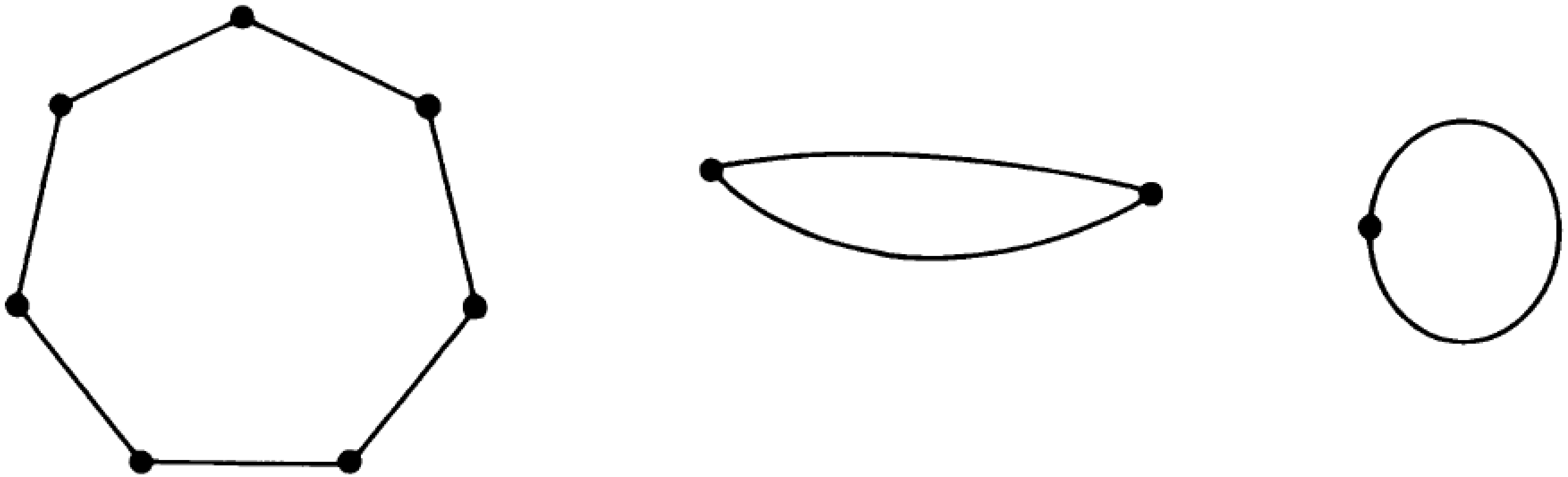
An open walk in which no vertex appears more than once is called a *path* (or a *simple path* or an *elementary path*).

A path does not intersect itself. The number of edges in a path is called the *length of a path*. It immediately follows, then, that an edge which is not a self-loop is a path of length one.

The terminal vertices of a path are of degree one, and the rest of the vertices (called *intermediate vertices*) are of degree two. This degree, of course, is counted only with respect to the edges included in the path and not the entire graph in which the path may be contained.

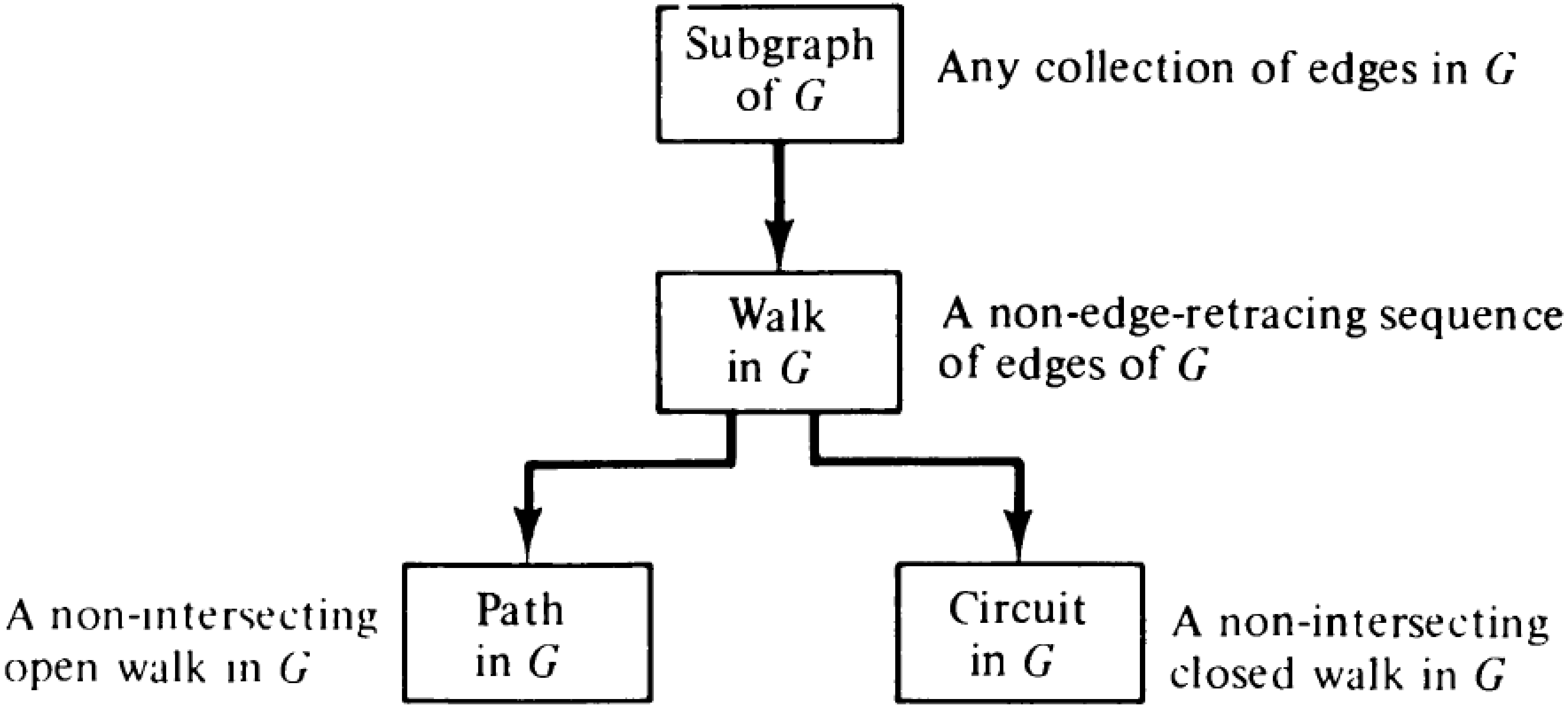
Circuit

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit*. That is, a circuit is a closed, non-intersecting walk.



every vertex in a circuit is of degree two; again, if the circuit is a subgraph of another graph, one must count degrees contributed by the edges in the circuit only.

A circuit is also called a *cycle*, *elementary cycle*, *circular path*, and *polygon*.



Walks, paths, and circuits as subgraphs.

Connected Graphs

- A graph is connected if we can reach any vertex from any other vertex by travelling along the edges.

A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. Otherwise, G is *disconnected*.

A null graph of more than one vertex is disconnected.

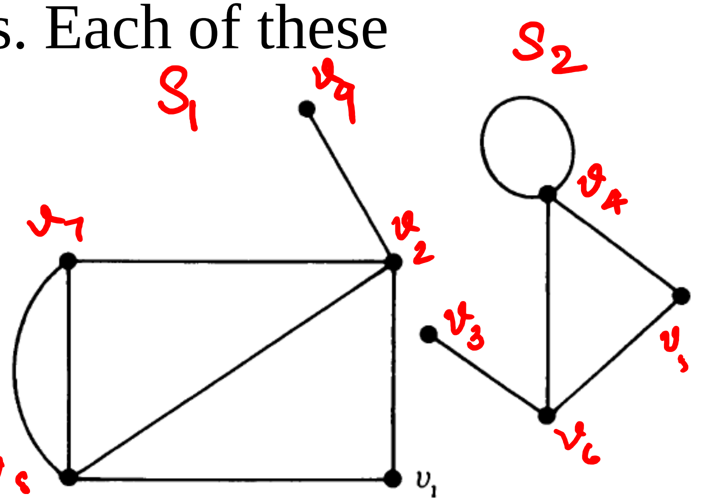
A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

$V(G) = \{v_1, \ldots v_9\}$

- A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

$v(S_1) = \{v_1, v_2, v_7, v_8, v_9\}$

$V(S_2) = \{v_3, v_4, v_5, v_6\}$



Theorem 1-2

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

- Proof:
  - Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.
  - Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition.

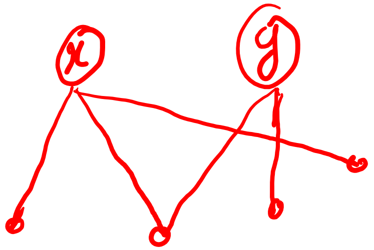
THEOREM 1-3

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

- *Proof:* Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd. From Theorem 1-1, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, $v1$ and $v2$ must belong to the same component, and hence must have a path between them.

Theorem 1-4

- Let G be a graph of order n. If d(u)+d(v) >= n-1 for every two non-adjacent vertices u and v of G, then G is connected.
- Proof – We need to prove that every two vertices of G are connected by a path
    - Let x,y $\in V$ of G. If (x,y) $\in E$, x and y are adjacent. Assume that (x,y)$\notin E$.
    - d(x) + d(y) >= n-1 implies that there must be a vertex, v_i adjacent to x and y. When x and y are not adjacent, there is a path between x and y always.
    - Hence, G is connected



$n=6$

$d(x)+d(y) \geq 5$

Theorem 1-5

- Let G be a graph of order n with d(G) >= (n-1)/2, then G is connected.
- Proof –
  - For every two non-adjacent vertices u and v of G
  - d(x) + d(y) >= (n-1)/2 + (n-1)/2 = n-1
  - Hence, according to Theorem 1-4, G is connected

THEOREM 1-6

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

$$n_1 + n_2 + n_3 + \dots n_k = n$$

$$\frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2}$$

$${}^nC_2$$

$$\frac{1}{2} n_1 (n_1 - 1)$$

$$\frac{1}{2}\sum_{i=1}^{k} n_i (n_i - 1)$$

Complement of graph

- Has the same set of vertices as G, but 2 vertices are adjacent in G' if they are adjacent in G.

Theorem 1-7

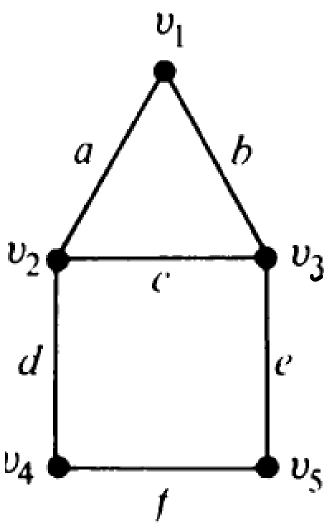
- If G is disconnected then G' is connected.

Operations on Graphs

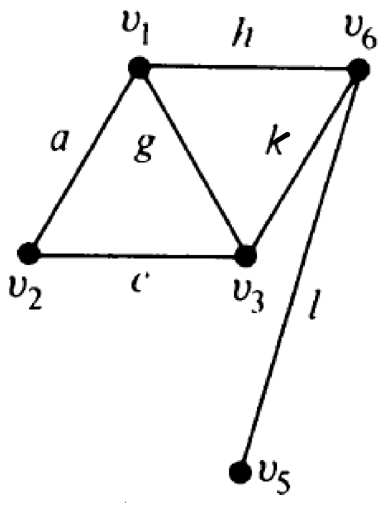
The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3 (written as $G_3 = G_1 \cup G_2$) whose vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$.

The *intersection* $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2.

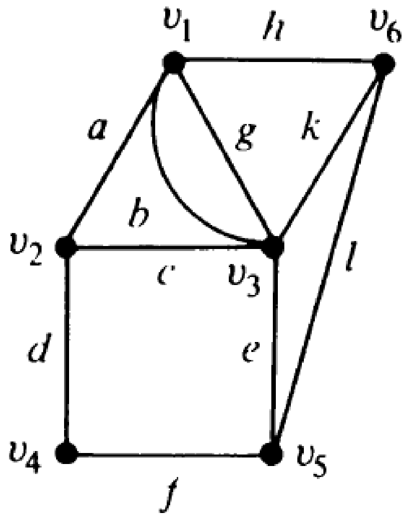
The *ring sum* of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but *not* in both.



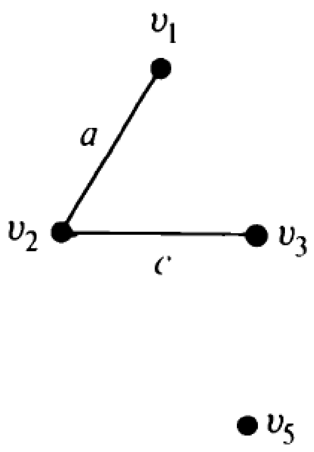
G_1



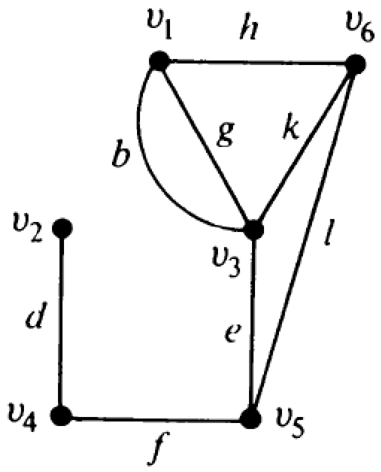
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \oplus G_2$

It is obvious from their definitions that the three operations just mentioned are commutative. That is,

$$G_1 \cup G_2 = G_2 \cup G_1, \qquad G_1 \cap G_2 = G_2 \cap G_1,$$
$$G_1 \oplus G_2 = G_2 \oplus G_1.$$

If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$. If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.

For any graph G,

$$G \cup G = G \cap G = G,$$

and

$$G \oplus G = \text{a null graph.}$$

If g is a subgraph of G, then $G \oplus g$ is, by definition, that subgraph of G which remains after all the edges in g have been removed from G. Therefore, $G \oplus g$ is written as $G - g$, whenever $g \subseteq G$. Because of this complementary nature, $G \oplus g = G - g$ is often called the complement of g in G.

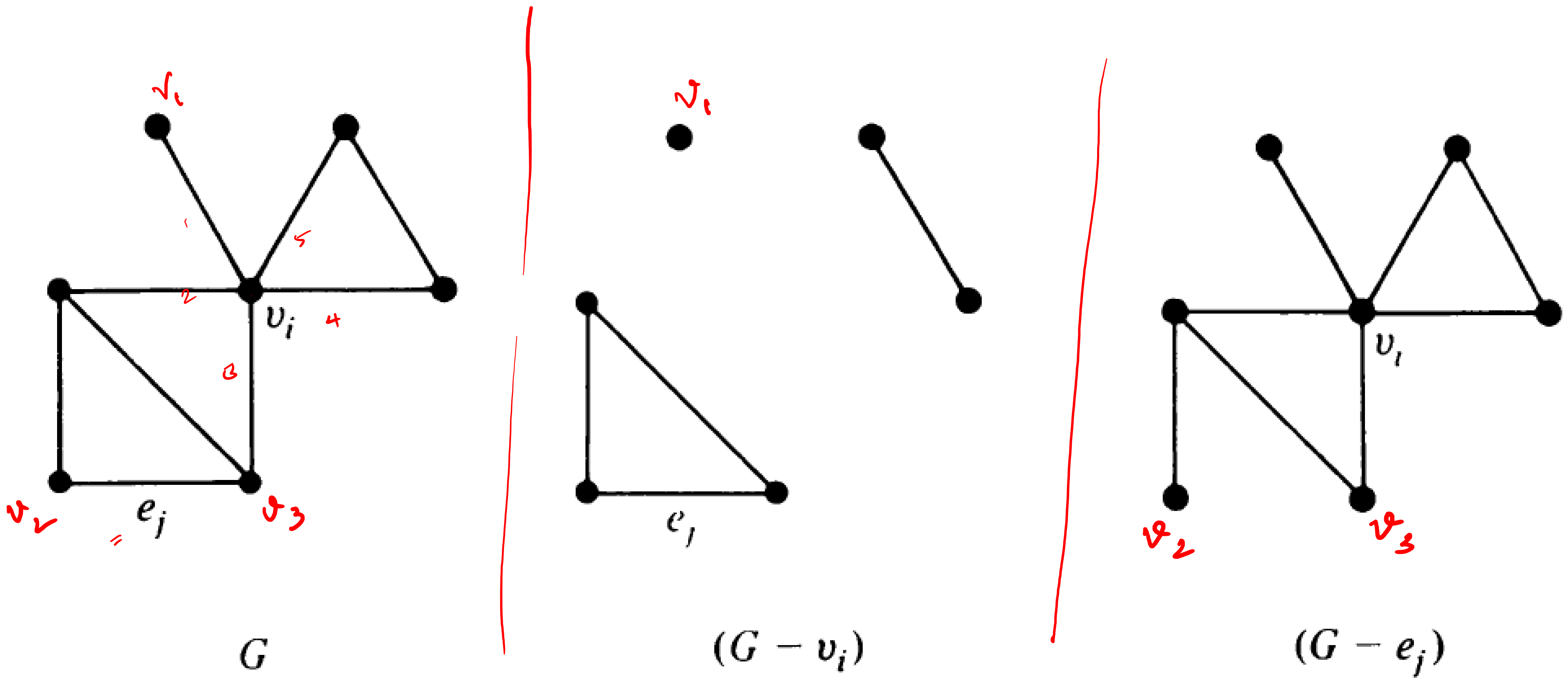
Decomposition: A graph G is said to have been *decomposed* into two subgraphs g_1 and g_2 if

$$g_1 \cup g_2 = G,$$

and

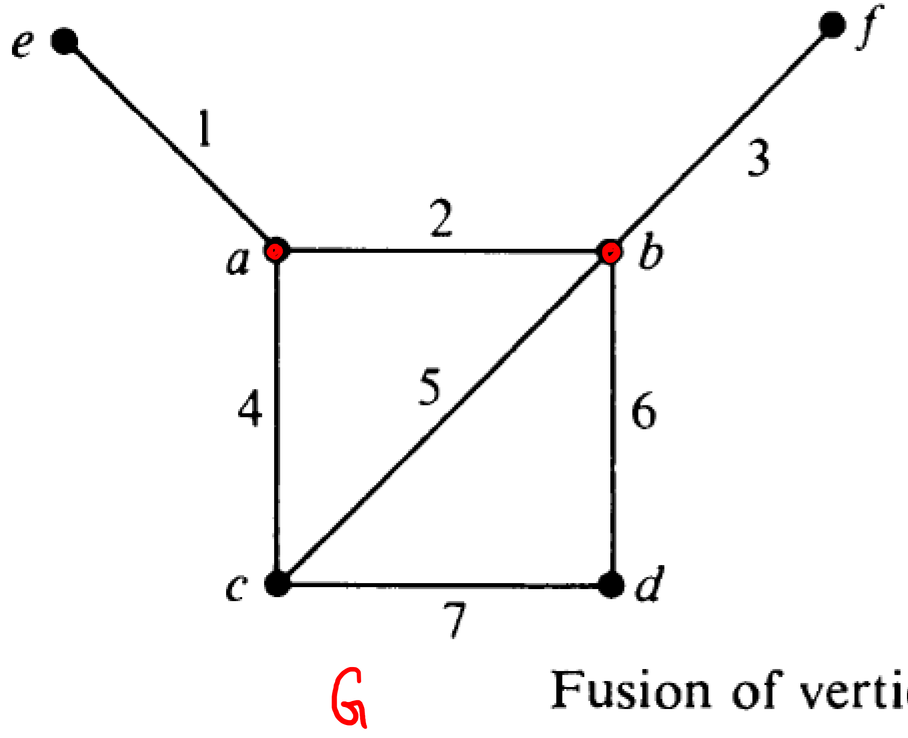
$$g_1 \cap g_2 = \text{a null graph.}$$

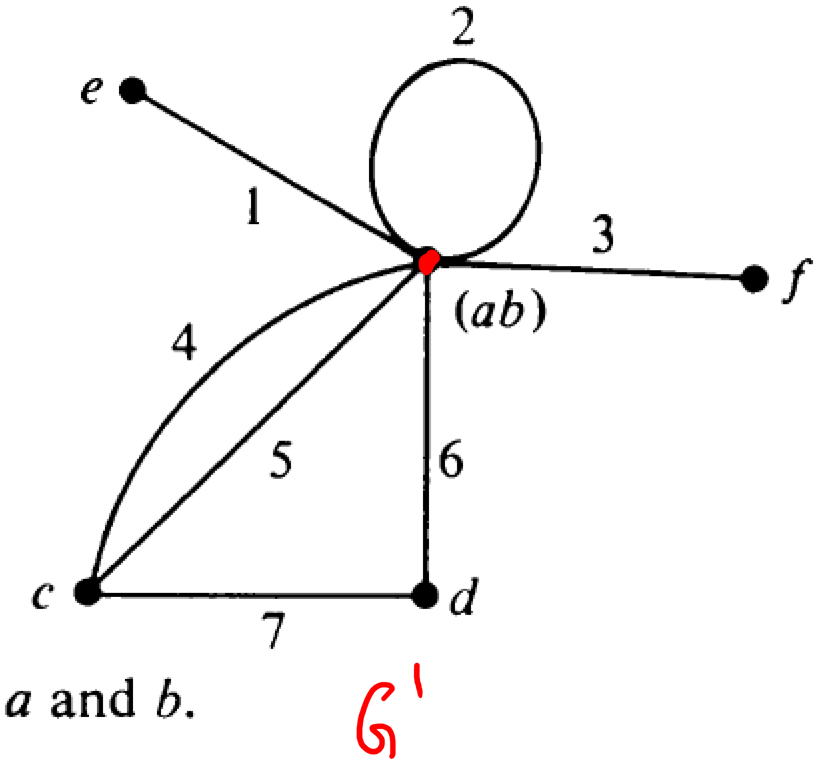
Deletion: If v_i is a vertex in graph G, then $G - v_i$ denotes a subgraph of G obtained by deleting (i.e., removing) v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex. If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices. Therefore $G - e_j = G \oplus e_j$.



Vertex deletion and edge deletion.

Fusion: A pair of vertices a, b in a graph are said to be *fused* (*merged* or *identified*) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. Thus fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one.





Fusion of vertices a and b.

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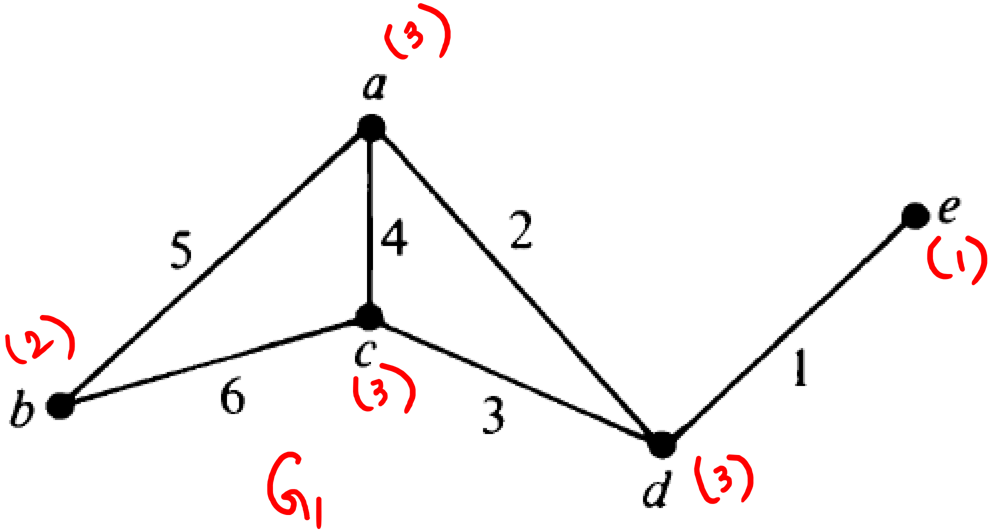
Matrix Representation of Graphs

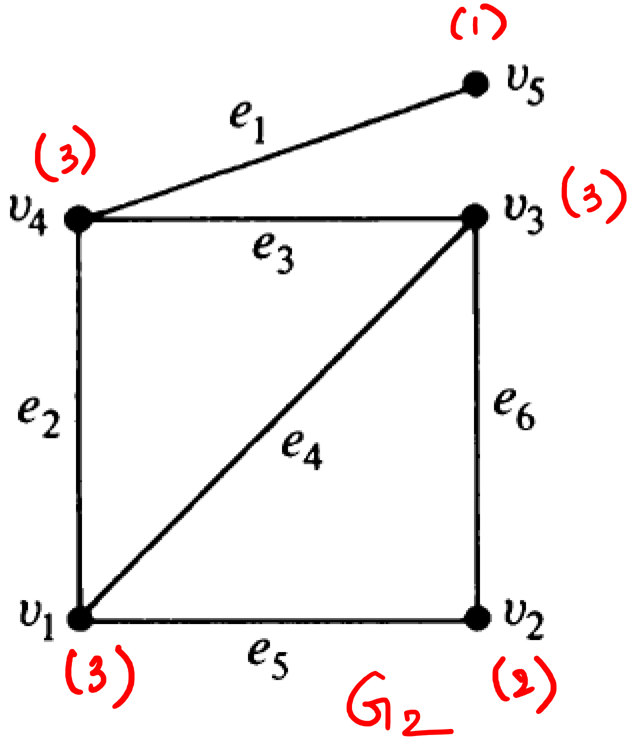
Isomorphism

Two graphs are thought of as equivalent (and called *isomorphic*) if they have identical behavior in terms of graph-theoretic properties.

Two graphs G and G' are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

suppose that edge e is incident on vertices v_1 and v_2 in G; then the corresponding edge e' in G' must be incident on the vertices v'_1 and v'_2 that correspond to v_1 and v_2, respectively.



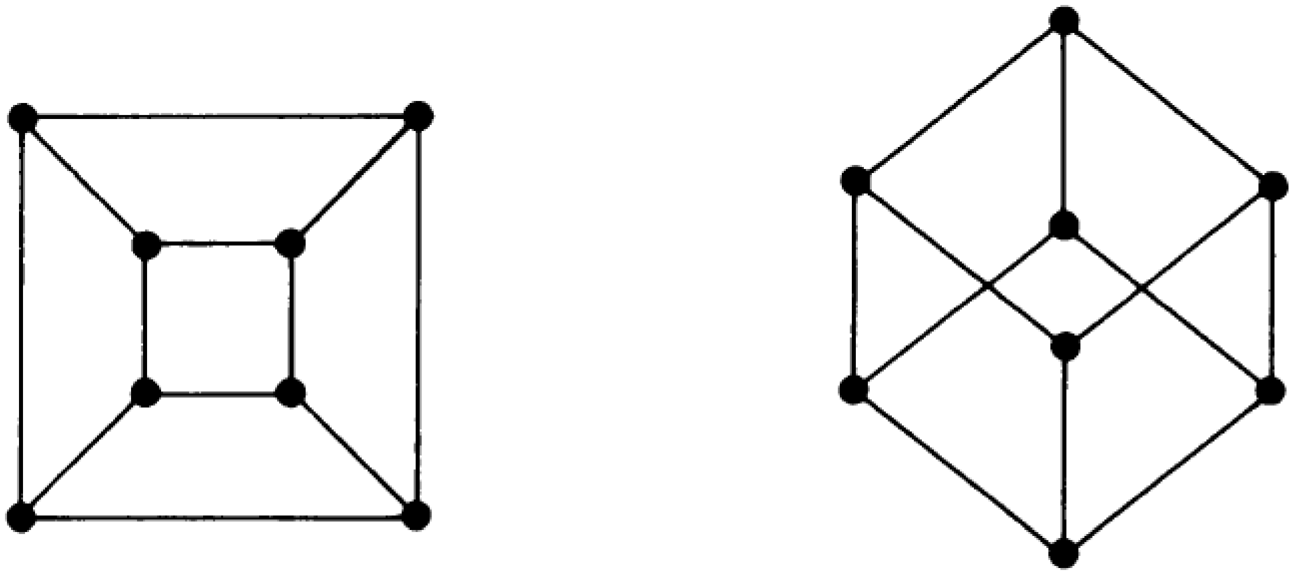


Isomorphic graphs.

The vertices a, b, c, d, and e correspond to v_1, v_2, v_3, v_4, and v_5, respectively.

The edges 1, 2, 3, 4, 5, and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

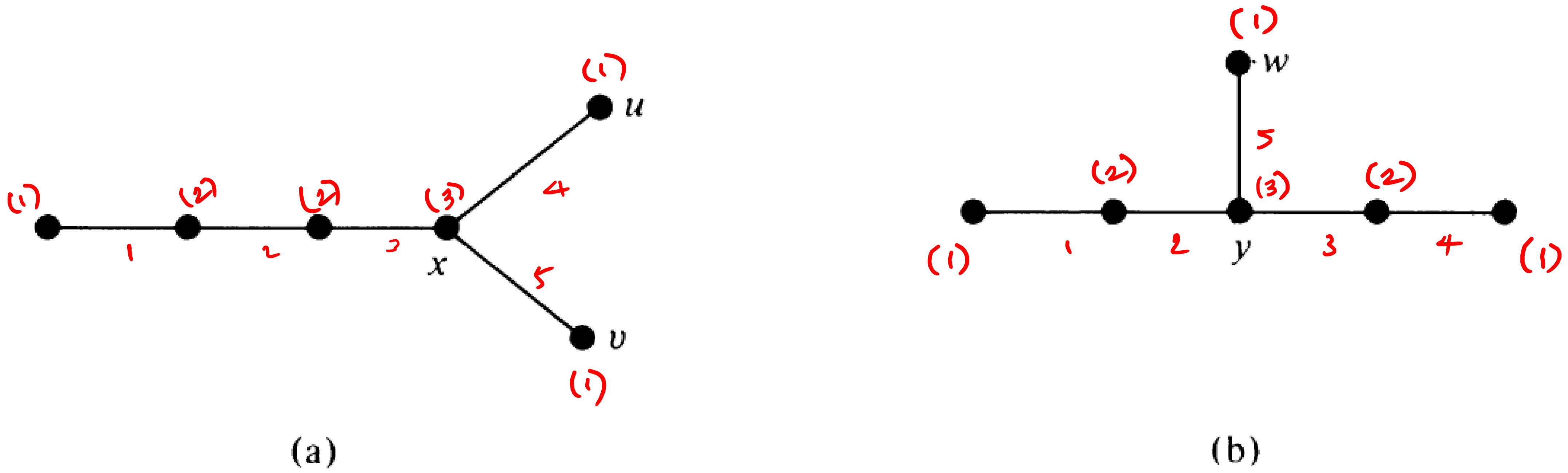
Except for the labels (i.e., names) of their vertices and edges, isomorphic graphs are the same graph, perhaps drawn differently.



Isomorphic graphs.

It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.
4. Same diameter
5. Same number of components
6. Same length of longest path
7. If one of them contains a cycle of particular length then the same must be true for the other graph



Two graphs that are not isomorphic.

- Two graphs are isomorphic if and only if there complements are isomorphic.

Matrix Representation of Graphs

A matrix is a convenient and useful way of representing a graph to a computer.

Matrices lend themselves easily to mechanical manipulations.

Many known results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. In many applications of graph theory, such as in electrical network analysis and operations research, matrices also turn out to be the natural way of expressing the problem.

Incidence matrix

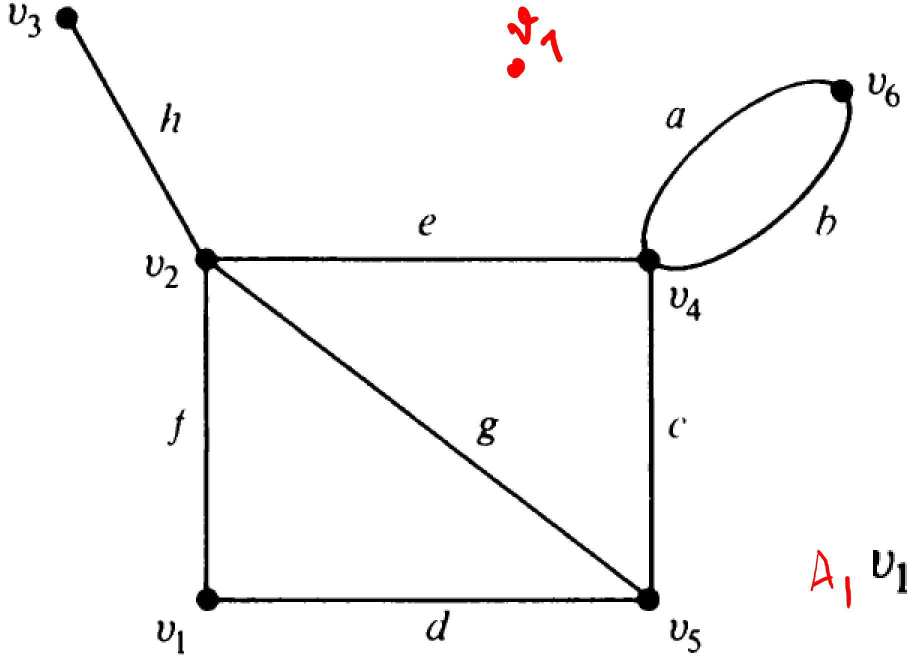
$G \quad A_{(n \times e)}$

Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix $\mathsf{A} = [a_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows:

The matrix element

$$a_{ij} = 1, \qquad \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \text{, and}$$
$$= 0, \qquad \text{otherwise.}$$

Such a matrix A is called the *vertex-edge incidence matrix*, or simply *incidence matrix*. Matrix A for a graph G is sometimes also written as $\mathsf{A}(G)$.



$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_6 \end{bmatrix}$$

5×8
$n \times e$

$$A(G) = \begin{array}{ll} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{array}{r} A_1 \; v_1 \\ A_2 \; v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \; v_7 & \end{array}$$

The incidence matrix contains only two elements, 0 and 1. Such a matrix is called a *binary matrix* or a *(0, 1)-matrix*.

Given any geometric representation of a graph without self-loops, we can readily write its incidence matrix.

On the other hand, if we are given an incidence matrix $\mathsf{A}(G)$, we can construct its geometric graph G without ambiguity.

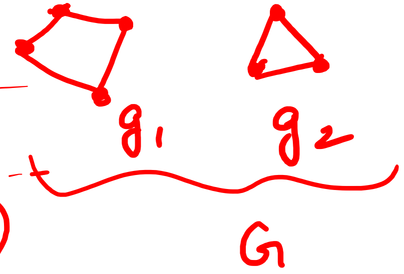
The following observations about the incidence matrix A can readily be made:

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example, columns 1 and 2 in Fig.

5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $A(G)$ of graph G can be written in a block-diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right],$$

$$\begin{bmatrix} A(g_1) & & \\ & A(g_2) & \\ & & A(g_3) \end{bmatrix}$$



where $A(g_1)$ and $A(g_2)$ are the incidence matrices of components g_1 and g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2, and vice versa. Obviously, this remark is also true for a disconnected graph with any number of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

THEOREM

Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $\mathsf{A}(G_1)$ and $\mathsf{A}(G_2)$ differ only by permutations of rows and columns.

THEOREM

If $\mathsf{A}(G)$ is an incidence matrix of a connected graph G with n vertices, the rank of $\mathsf{A}(G)$ is $n - 1$.

rank of $\mathsf{A}(G)$ is $n - k$, if G is a disconnected graph with n vertices and k components

If we remove any one row from the incidence matrix of a connected graph, the remaining $(n - 1)$ by e submatrix is of rank $n - 1$ (Theorem).
In other words, the remaining $n - 1$ row vectors are linearly independent. Thus we need only $n - 1$ rows of an incidence matrix to specify the corresponding graph completely, for $n - 1$ rows contain the same amount of information as the entire matrix.

$n \times e$

Such an $(n - 1)$ by e submatrix A_f of A is called a *reduced incidence matrix*. The vertex corresponding to the deleted row in A_f is called the *reference vertex*. Clearly, any vertex of a connected graph can be made the reference vertex.

Since a tree is a connected graph with n vertices and $n - 1$ edges, its reduced incidence matrix is a square matrix of order and rank $n - 1$.

$n \times \underline{e}$ $\qquad$ n $\qquad$ $\underline{n-1}$

$n \times (n-1)$

$(n-1) \times (n-1)$

COROLLARY

The reduced incidence matrix of a tree is nonsingular.

Submatrices of A(G)

Let g be a subgraph of a graph G, and let $\mathsf{A}(g)$ and $\mathsf{A}(G)$ be the incidence matrices of g and G, respectively. $\mathsf{A}(g)$ is a submatrix of $\mathsf{A}(G)$.

there is a one-to-one correspondence between each n by k submatrix of $\mathsf{A}(G)$ and a subgraph of G with k edges, k being any positive integer less than e and n being the number of vertices in G.

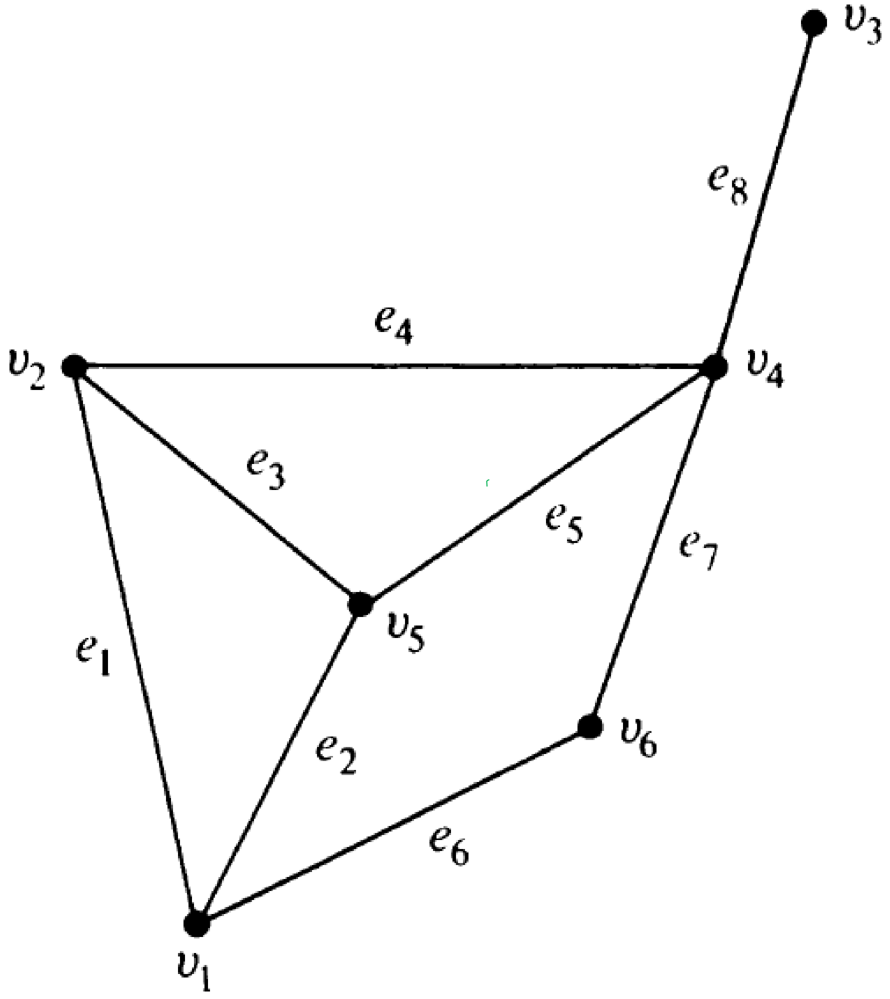
THEOREM

Let $\mathsf{A}(G)$ be an incidence matrix of a connected graph G with n vertices. An $(n-1)$ by $(n-1)$ submatrix of $\mathsf{A}(G)$ is nonsingular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree in G.

Adjacency matrix/ Connection matrix

Adjacency matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $\mathsf{X} = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = 1, \quad \text{if there is an edge between } i\text{th and } j\text{th vertices, and}$$
$$= 0, \quad \text{if there is no edge between them.}$$



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Observations that can be made immediately about the adjacency matrix X of a graph G are

1. The entries along the principal diagonal of X are all 0's if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs without parallel edges.
3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of 1's in the corresponding row or column of X.

4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $\mathsf{X}(G_1)$ and $\mathsf{X}(G_2)$ are related:

$$\mathsf{X}(G_2) = \mathsf{R}^{-1} \cdot \mathsf{X}(G_1) \cdot \mathsf{R},$$

where R is a permutation matrix.

5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $\mathsf{X}(G)$ can be partitioned as

$$\mathsf{X}(G) = \left[\begin{array}{c|c} \mathsf{X}(g_1) & 0 \\ \hline 0 & \mathsf{X}(g_2) \end{array}\right],$$

where $\mathsf{X}(g_1)$ is the adjacency matrix of the component g_1 and $\mathsf{X}(g_2)$ is that of the component g_2.

6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

Powers of X

The value of an off-diagonal entry in X^2, that is, ijth entry ($i \neq j$) in X^2,

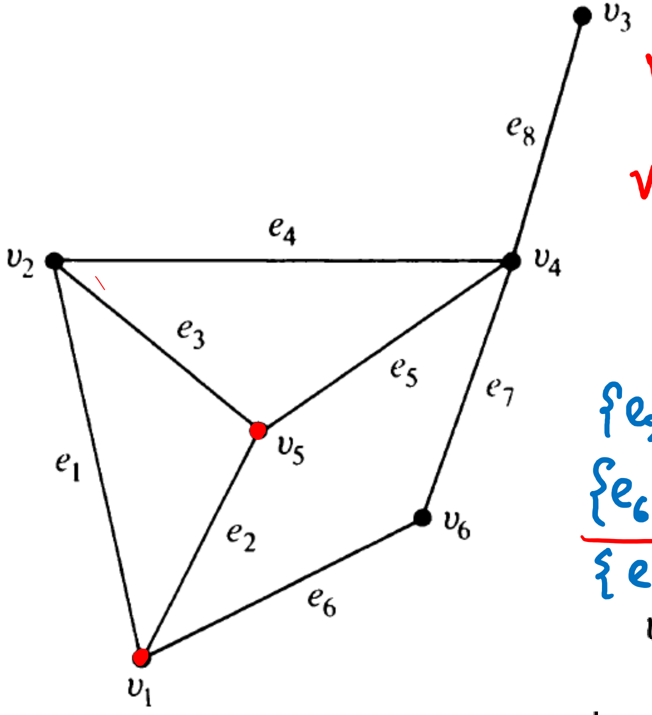
= number of 1's in the dot product of ith row and jth column (or jth row) of X.

= number of positions in which both ith and jth rows of X have 1's.

= number of vertices that are adjacent to both ith and jth vertices.

= number of different paths of length two between ith and jth vertices.

Similarly, the ith diagonal entry in X^2 is the number of 1's in the ith row (or column) of matrix X. Thus the value of each diagonal entry in X^2 equals the degree of the corresponding vertex, if the graph has no self-loops.



$v_1 \; (0,1,0,0,1,1)$ } 1

$v_2 \; (1,0,0,1,1,0)$ } 7

$(3,1,0,3,1,0)$

$\{e_2,e_2,e_2\}, \{e_2,e_3,e_3\}, \{e_2,e_5,e_5\}$,
$\{e_6,e_7,e_5\}, \{e_1,e_4,e_5\}, \{e_6,e_6,e_2\}$
$\{e_1,e_1,e_2\}$

$$X = \begin{array}{c|cccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 1 & 1 & 0 & 1 & 1 \\ v_5 & 1 & 1 & 0 & 1 & 0 & 0 \\ v_6 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

X^2_{12}

$$X^2 = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 3 & 1 & 0 & 3 & 1 & 0 \\ 2 & 1 & 3 & 1 & 1 & 2 & 2 \\ 3 & 0 & 1 & 1 & 0 & 1 & 1 \\ 4 & 3 & 1 & 0 & 4 & 1 & 0 \\ 5 & 1 & 2 & 1 & 1 & 3 & 2 \\ 6 & 0 & 2 & 1 & 0 & 2 & 2 \end{array}$$

$$X^3 = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 7 & 3 & 2 & 7 & 6 \\ 2 & 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 3 & 1 & 0 & 4 & 1 & 0 \\ 4 & 2 & 8 & 4 & 2 & 8 & 7 \\ 5 & 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 6 & 2 & 0 & 7 & 2 & 0 \end{array}$$

$X^2 \cdot X$

Let us now consider the ijth entry of X^3. $(i \neq j)$

ijth entry of X^3 = dot product of ith row X^2 and jth column (or row) of X.

$= \sum_{k=1}^{n} ik\text{th entry of } X^2 \cdot kj\text{th entry of } X.$

$= \sum_{k=1}^{n}$ number of all different edge sequences of three edges from ith to jth vertex via kth vertex.

= number of different edge sequences of three edges between ith and jth vertices.

the iith entry in X^3 equals twice the number of different circuits of length three (i.e., triangles) in the graph passing through the corresponding vertex v_i.

For example, consider how the 1,5th entry on $\mathbf{X}^3$ for the graph of Fig. is formed. It is given by the dot product

$$\begin{aligned}\text{row 1 of } \mathbf{X}^2 \cdot \text{row 5 of } \mathbf{X} &= (3, 1, 0, 3, 1, 0) \cdot (1, 1, 0, 1, 0, 0) \\ &= 3 + 1 + 0 + 3 + 0 + 0 = 7.\end{aligned}$$

These seven different edge sequences of three edges between v_1 and v_5 are

$$\{e_1, e_1, e_2\}, \quad \{e_2, e_2, e_2\}, \quad \{e_6, e_6, e_2\}, \quad \{e_2, e_3, e_3\},$$
$$\{e_6, e_7, e_5\}, \quad \{e_2, e_5, e_5\}, \quad \{e_1, e_4, e_5\}.$$

Theorem

Let X be the adjacency matrix of a simple graph G. Then the ijth entry in X^r is the number of different edge sequences of r edges between vertices v_i and v_j.

Corollary A

In a connected graph, the distance between two vertices v_i and v_j (for $i \neq j$) is k, if and only if k is the smallest integer for which the i, jth entry in x^k is nonzero.

This is a useful result in determining the distances between different pairs of vertices.

Corollary B

If X is the adjacency matrix of a graph G with n vertices, and

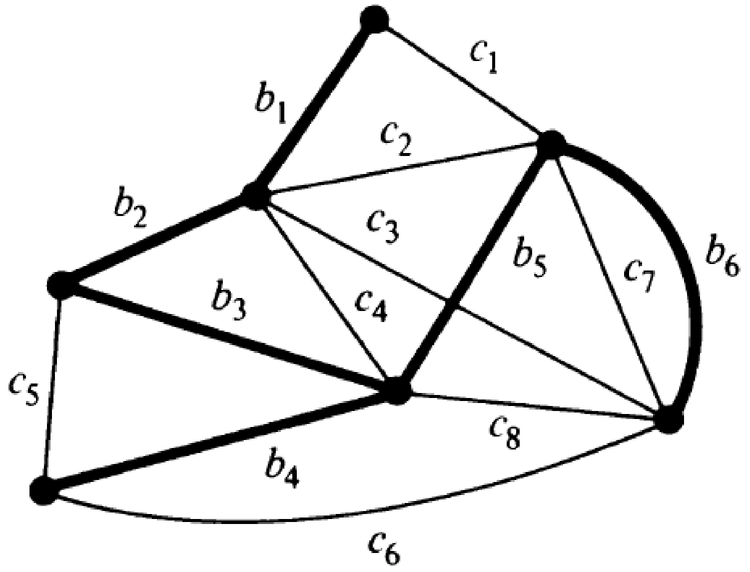
$$Y = X + X^2 + X^3 + \cdots + X^{n-1}, \qquad \text{(in the ring of integers)},$$

then G is disconnected if and only if there exists at least one entry in matrix Y that is zero.

Fundamental Circuits

Fundamental Circuits

Let us now consider a spanning tree T in a connected graph G. Adding any one chord to T will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called a *fundamental circuit*.



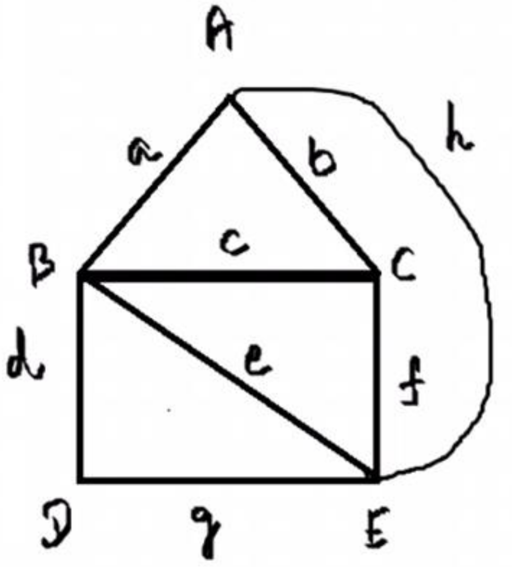
Let us look at the tree $\{b_1, b_2, b_3, b_4, b_5, b_6\}$

$\{b_1, b_2, b_3, b_4, b_5, b_6, c_1\} \Rightarrow \{b_1, b_2, b_3, b_5, c_1\}$

$\{b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2\} \Rightarrow \{b_1, b_2, b_3, b_5, c_1\}$

$\Rightarrow \{b_2, b_3, b_5, c_2\}$

$\Rightarrow \{b_1, c_1, c_2\}$ ✗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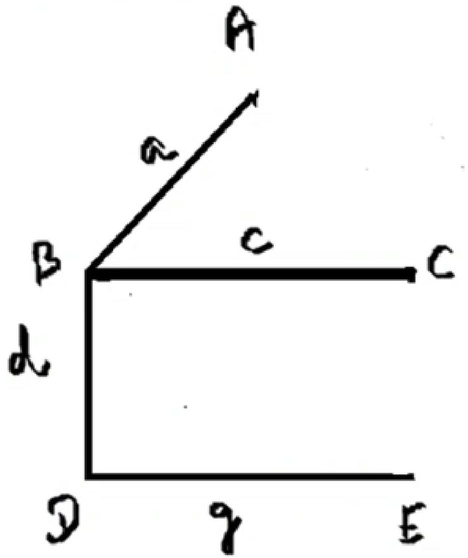
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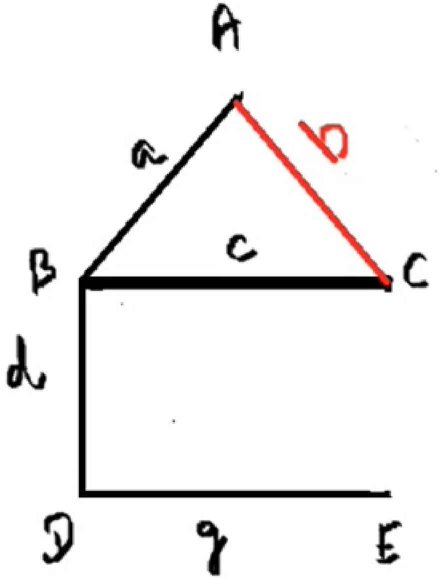
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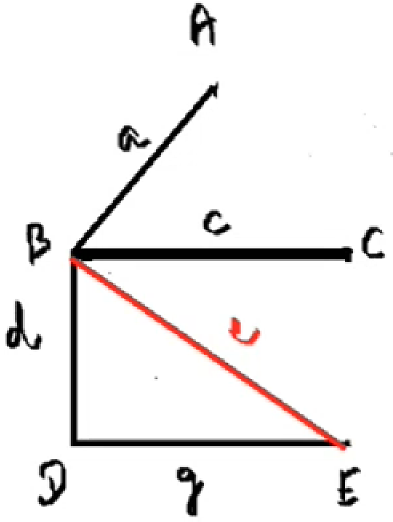
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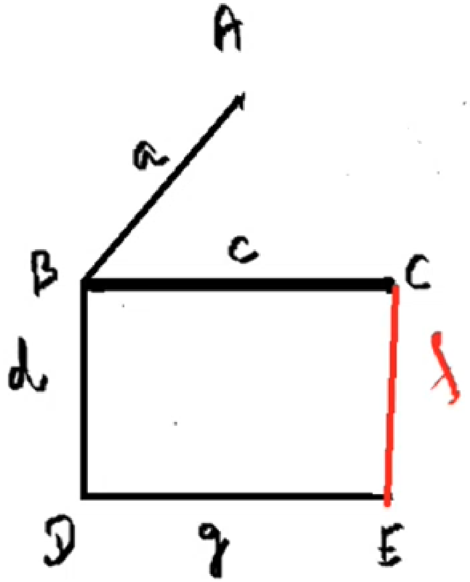
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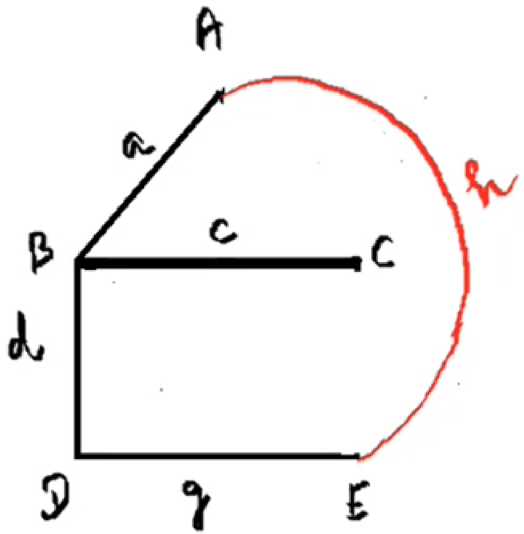
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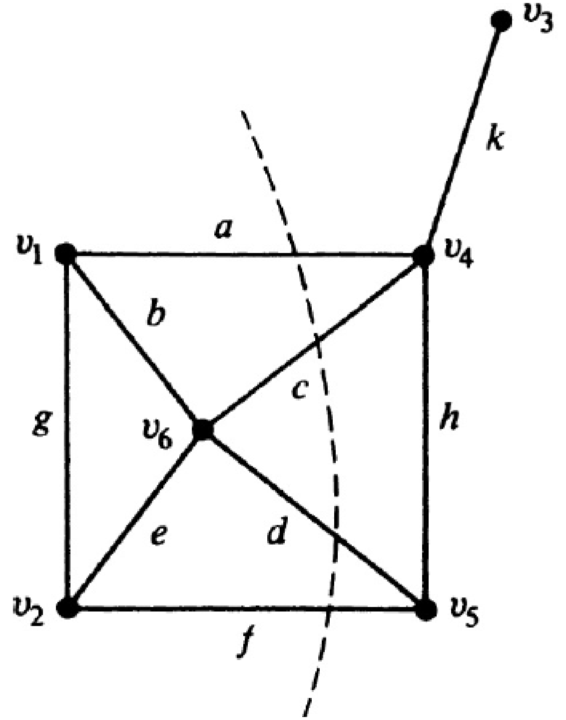
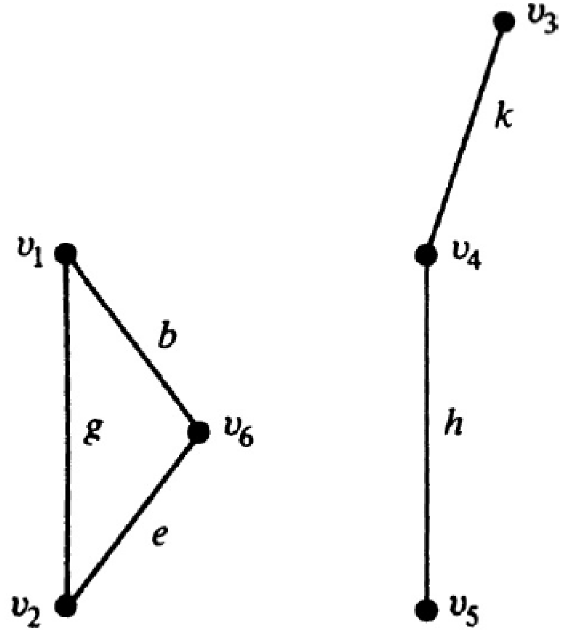
Cut-sets or edge cut

- In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.
- For instance, in Fig. the set of edges {a, c, d, f} is a cut-set.
- There are many other cut-sets, such as {a, b, g} , {a, b, e, f} , and {d, h, f}. Edge {k} alone is also a cut-set.
- The set of edges {a, c, h, d} , on the other hand, is not a cut-set, because one of its proper subsets, {a, c, h}, is a cut-set.

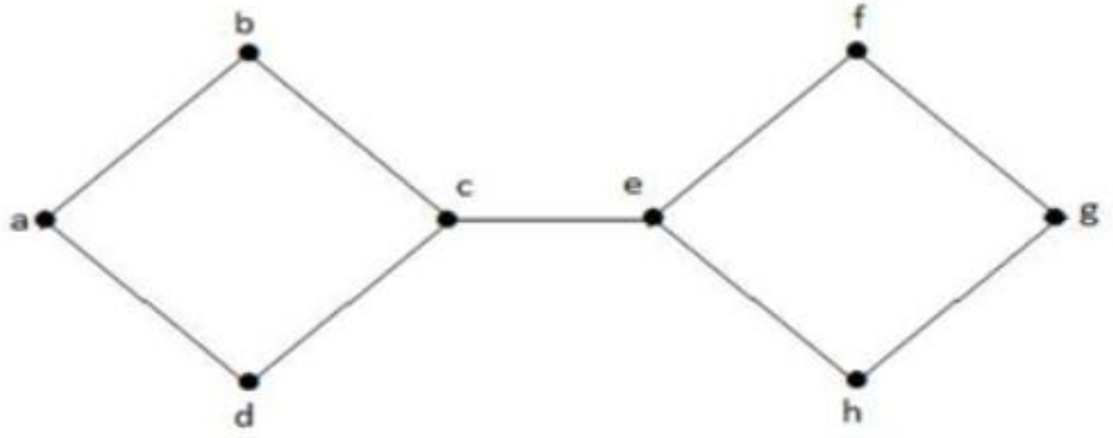


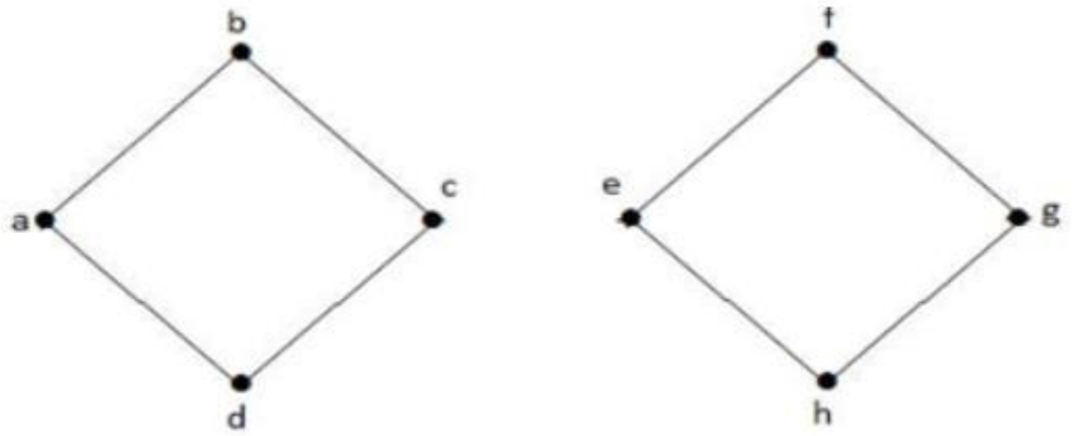


- To emphasize the fact that no proper subset of a cut-set can be a cut-set, some authors refer to a cut-set as a minimal cut-set, a proper cut-set, or a simple cut-set.
- A cut-set always “cuts” a graph into two.
- Therefore, a cut-set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph by one.
- If we partition all the vertices of a connected graph G into two mutually exclusive subsets, a cut-set is a minimal number of edges whose removal from G destroys all paths between these two sets of vertices.
    - For example, in Fig. cut-set {a, c, d, f} connects vertex set {v1, v2 , v6} with {v3, v4, v5}.
- Since removal of any edge from a tree breaks the tree into two parts, every edge of a tree is a cut-set.
- Edge connectivity, K’(G), is the size of the smallest edge cut.

Bridge or cut edge

- A **cut- Edge or bridge** is a single edge whose removal disconnects a graph.
- Let G be a connected graph. An edge e of G is called a cut edge of G, if G-e results a disconnected graph.

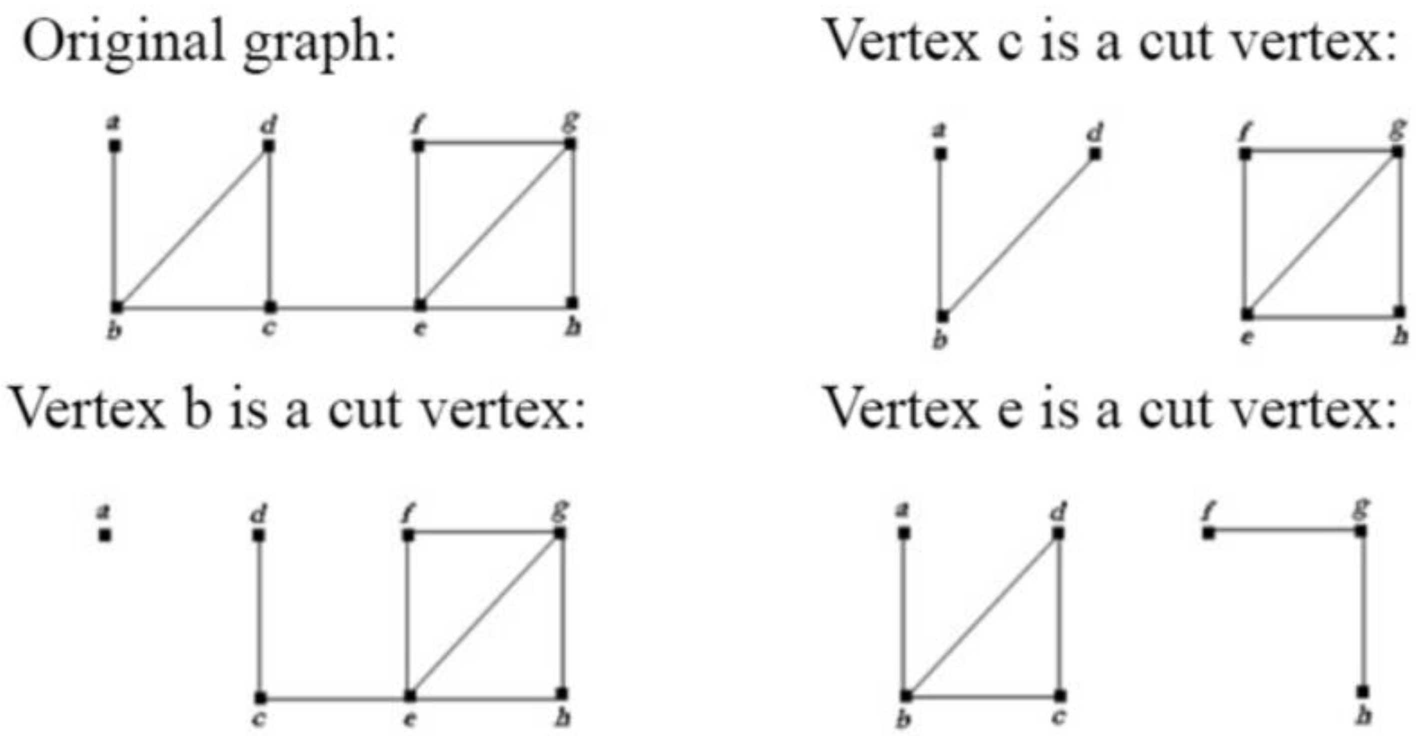




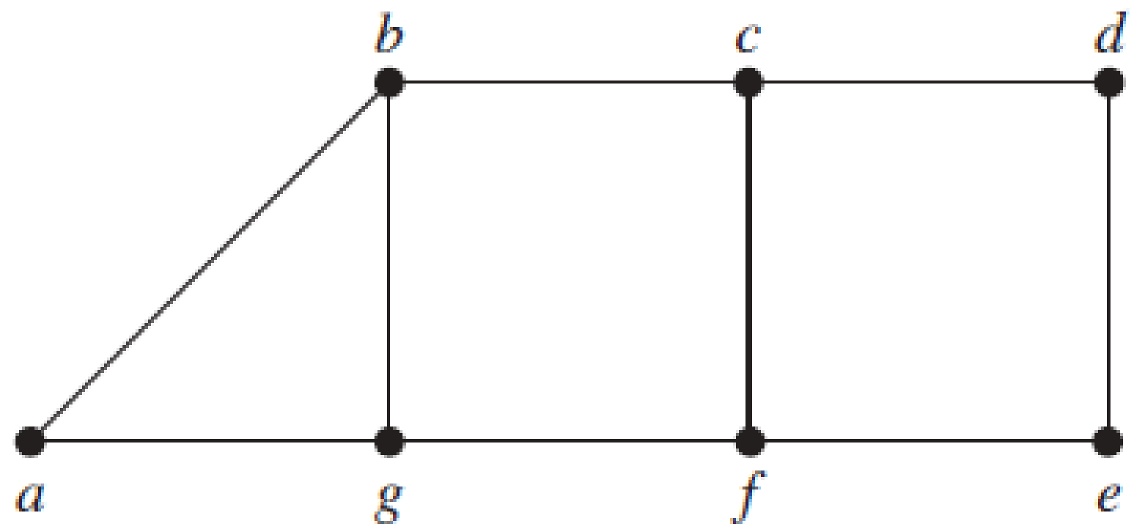
- Theorem: An edge $e \in E(G)$ is a bridge if and only if $\exists u, w \in V(G)$, $u \neq w$ such that e is on every u-w path of G

Vertex Cut

- A subset V’ of the vertex set V of G = (V, E) is a vertex cut, or separating set, if G − V’ is disconnected.
- A vertex v Є V(G) is called a cut vertex, a cut-node, or an articulation point if G-v has more connected components than G.



- A graph is k-connected if there does not exist a set of k-1 vertices whose removal disconnects the graph.
- The **vertex connectivity K(G)** of G is defined as the largest k such that G is k-connected. It is the minimum size of a vertex set $S \subseteq V\ such\ that\ G - S$ is disconnected or has only one vertex (in case of complete graph).
    - **vertex connectivity** of a noncomplete graph *G* is the minimum number of vertices in a vertex cut
    - When G is a complete graph, it has no vertex cuts, because removing any subset of its vertices and all incident edges still leaves a complete graph. Consequently, we cannot define κ(G) as the minimum number of vertices in a vertex cut when G is complete. Instead, we set $\kappa(K_n) = n - 1$, the number of vertices needed to be removed to produce a graph with a single vertex.



G_3

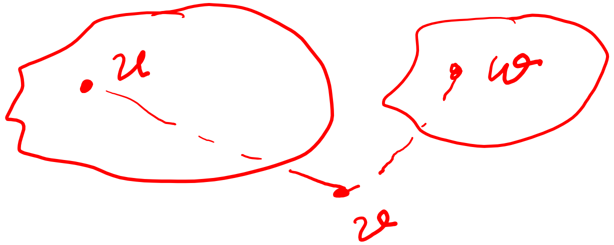
- $G3$ has no cut vertices, but that $\{b, g\}$ is a vertex cut. Hence, $\kappa(G3) = 2$.
- There is no cut-vertex in this graph.

- Disconnected graphs and K_1 have $\kappa(G) = 0$
- Connected graphs with cut vertices and K_2 have $\kappa(G) = 1$
- Graphs without cut vertices that can be disconnected by removing two vertices and K_3 have $\kappa(G) = 2$, and so on
- If G is a k-connected graph, then G is a j-connected graph for all j with $0 \leq j \leq k$.

- Theorem: A vertex $v \in V(G)$ is a cut vertex of G if and only if $\exists u, w \in V(G),\ u, w \neq v$ such that v is on every u-w path of G

Proof: Assume G is connected. (otherwise repeat the argument on connected components).

$\Rightarrow$ Let $v \in V(G)$ be a cut vertex. Then $G - v$ is disconnected. Let u, w be vertices in different components of $G - v$. So $\nexists$ any u-w path in $G - v$.



But G is connected so $\exists$ u-w path in G.

$\therefore$ all such paths went through v.

$\Leftarrow$ Suppose $\exists\, u, w \in V(G),\ u, w \neq v$ such that v lies on every u-w path.

Then removing v means $\nexists$ any u-w path in $G - v$.

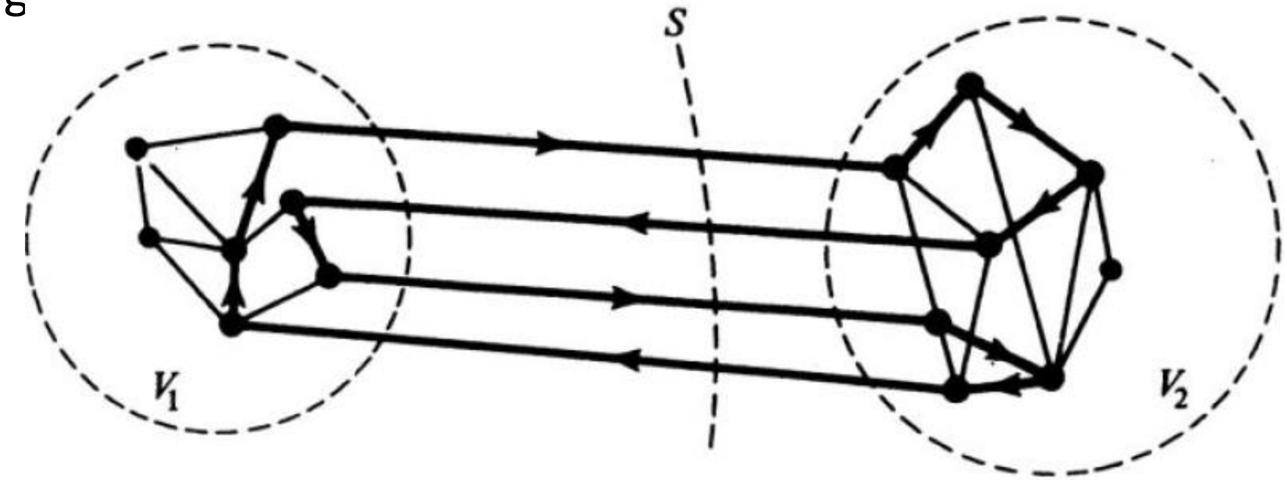
$\therefore$ $G - v$ is disconnected & v is a cut-vertex.

- Theorem 1: Every non-trivial connected graph contains at least two vertices that are not cut-vertices.
- Proof (by contradiction):
  - There exists a non-trivial connected graph with at most 1 non cut-vertex.
  - Let $u, v \in V(G)$ with distance, $d(u, v) = Diam(G)$.
  - At least one of u, v is a cut vertex, say v. So $\mathrm{G} - v$ is disconnected. Let $w \in V(G)$ be in a different component of $G - v$ than u. Then, every uw path v.
  - Therefore, $d(u, w) \; > d(u, v) = Diam(G)$ which is a contradiction.
  - Hence, the theorem is true.

- A non-trivial connected graph is called non-separable or 2-connected if it has no cut-vertices

- THEOREM 2: Every cut-set in a connected graph *G* must contain at least one branch of every spanning tree of G.
- THEOREM 3: In a connected graph *G*, any minimal set of edges containing at least one branch of every spanning tree of *G* is a cut-set.
- Proof:
    - In a given connected graph *G*, let *Q* be a minimal set of edges containing at least one branch of every spanning tree of *G*. Consider $G - Q$, the subgraph that remains after removing the edges in *Q* from *G*. Since the subgraph $G - Q$ contains no spanning tree of G, $G - Q$ is disconnected. Also, since *Q* is a minimal set of edges with this property, any edge *e* from *Q* returned to $G - Q$ will create at least one spanning tree. Thus the subgraph $G - Q + e$ will be a connected graph. Therefore, *Q* is a minimal set of edges whose removal from *G* disconnects G. This, by definition, is a cut-set.

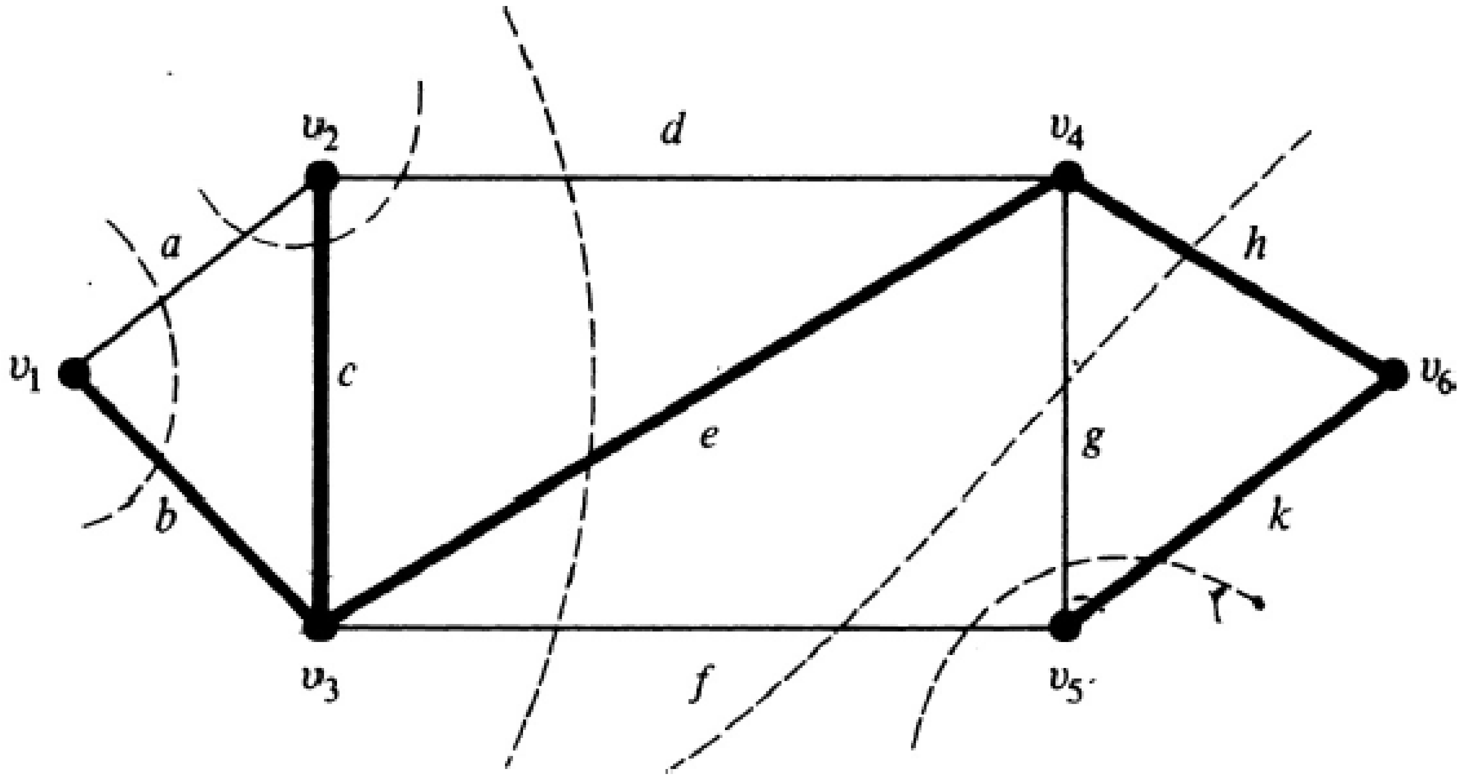
- THEOREM 4: Every circuit has an even number of edges in common with any cut-set.
- *Proof:*
  - Consider a cut-set *S* in graph *G*. Let the removal of *S* partition the vertices of *G* into two (mutually exclusive or disjoint) subsets *V*1 and *V*2. Consider a circuit Γ in *G*.
  - If all the vertices in Γ are entirely within vertex set *V1* (or *V*2), the number of edges common to *S* and Γ is zero; that is, $N(S \cap \Gamma) = 0$, an even number.
  - If, on the other hand, some vertices in Γ are in *V*1and some in *V*2, we traverse back and forth between the sets *V*1 and *V*2 as we traverse the circuit. Because of the closed nature of a circuit, the number of edges we traverse between *V*1 and *V*2 must be even. And since very edge in *S* has one end in *V*1 and the other in *V*2, and no other edge in *G* has this property (of separating sets *V*1and *V*2), the number of edges common to S and Γ is even.



Circuit Γ shown in heavy lines, and is traversed along the direction of the arrows

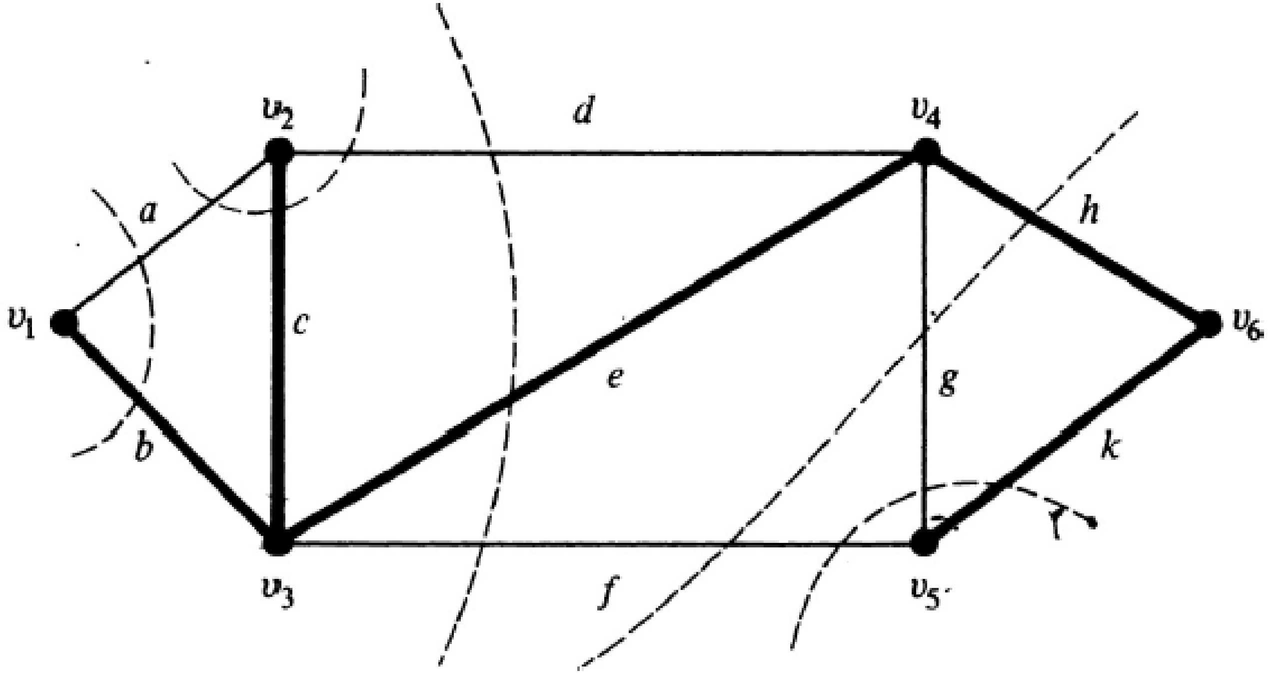
Fundamental Cut-Sets

- Consider a spanning tree T of a connected graph G. Take any branch b in T. Since $\{b\}$ is a cut-set in T, $\{b\}$ partitions all vertices of T into two disjoint sets—one at each end of b.
- Consider the same partition of vertices in G, and the cut set S in G that corresponds to this partition.
- Cut-set S will contain only one branch b of T, and the rest (if any) of the edges in S are chords with respect to T. Such a cut-set S containing exactly one branch of a tree T is called a *fundamental cut-set* with respect to T.
- Sometimes a fundamental cut-set is also called a *basic cut-set*.



- A spanning tree T (in heavy lines) and all five of the fundamental cut-sets with respect to T are shown (broken lines “cutting” through each cut-set).

- THEOREM 5: The ring sum of any two cut-sets in a graph is either a third cut-set or an edge-disjoint union of cut-sets.

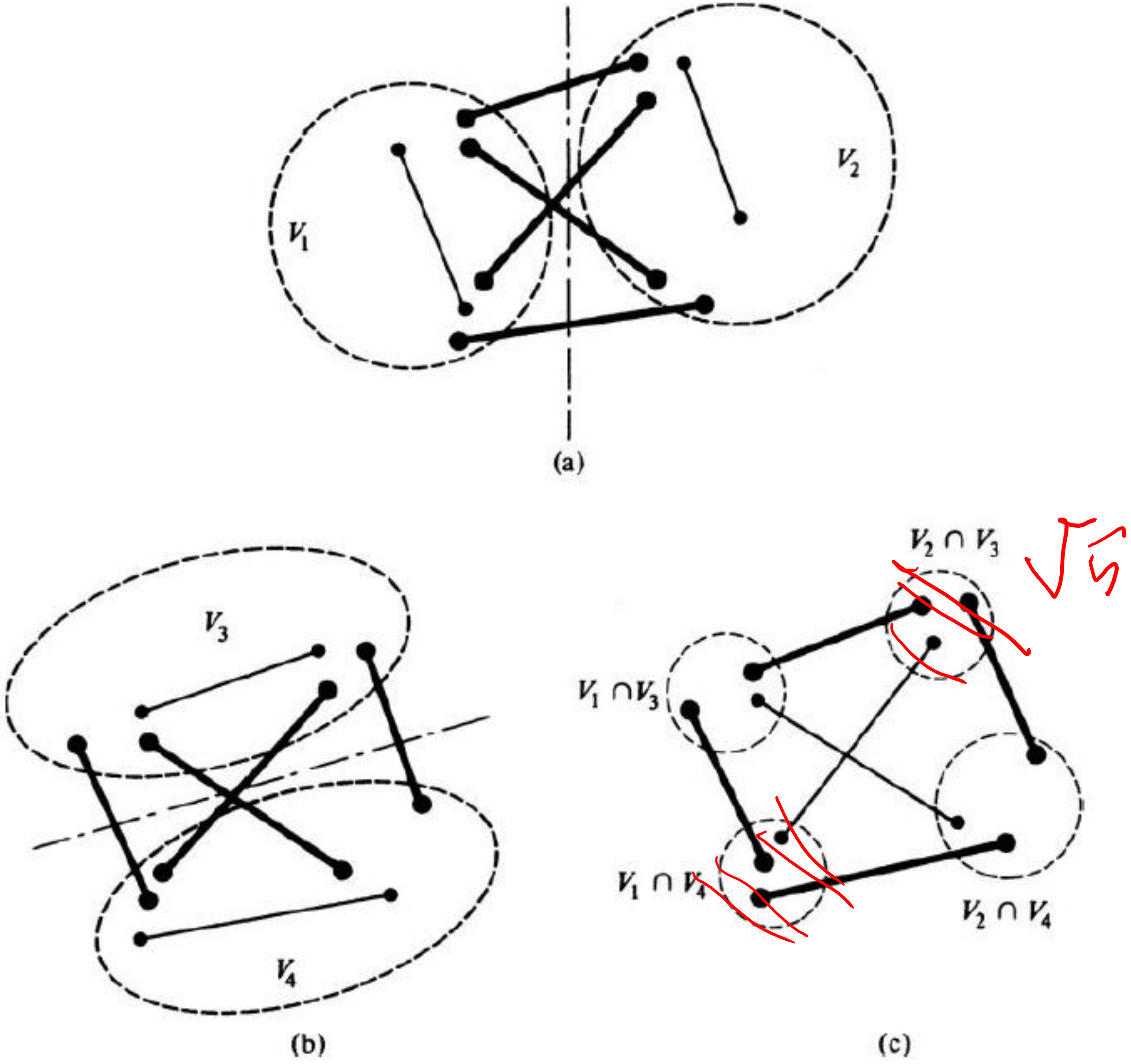


$$\{d, e, f\} \oplus \{f, g, h\} = \{d, e, g, h\}, \qquad \text{another cut-set,}$$

$$\{a, b\} \oplus \{b, c, e, f\} = \{a, c, e, f\}, \qquad \text{another cut-set,}$$

$$\begin{aligned} \{d, e, g, h\} \oplus \{f, g, k\} &= \{d, e, f, h, k\} \\ &= \{d, e, f\} \cup \{h, k\}, \text{ an edge-disjoint} \\ &\quad \text{union of cut-sets.} \quad \blacksquare \end{aligned}$$

- ***Proof:*** Let $S1$ and $S2$ be two cut-sets in a given connected graph G.
- Let $V1$ and $V2$ be the (unique and disjoint) partitioning of the vertex set V of G corresponding to $S1$. Let $V3$ and $V4$ be the partitioning corresponding to $S2$.
- $V1 \cup V2 = V$ and $V1 \cap V2 = \emptyset$,
- $V3 \cup V4 = V$ and $V3 \cap V4 = \emptyset$.
- Now let the subset $(V1 \cap V4) \cup (V2 \cap V3)$ be called $V5$, and this by definition is the same as the ring sum $V1 \oplus V3$. Similarly, let the subset $(V1 \cap V3) \cup (V2 \cap V4)$ be called $V6$, which is the same as $V2 \oplus V3$.
- The ring sum of the two cut-sets $S1 \oplus S2$ can be seen to consist only of edges that join vertices in $V5$ to those in $V6$. Also, there are no edges outside $S1 \oplus S2$ that join vertices in $V5$ to those in $V6$.
- Thus, the set of edges $S1 \oplus S2$ produces a partitioning of V into $V5$ and $V6$ such that
- $V5 \cup V6 = V$ and $V5 \cap V6 = \emptyset$.
- Hence, $S1 \oplus S2$ is a cut-set if the subgraphs containing $V5$ and $V6$ each remain connected after $S1 \oplus S2$ is removed from G. Otherwise, $S1 \oplus S2$ is an edge-disjoint union of cut-sets.

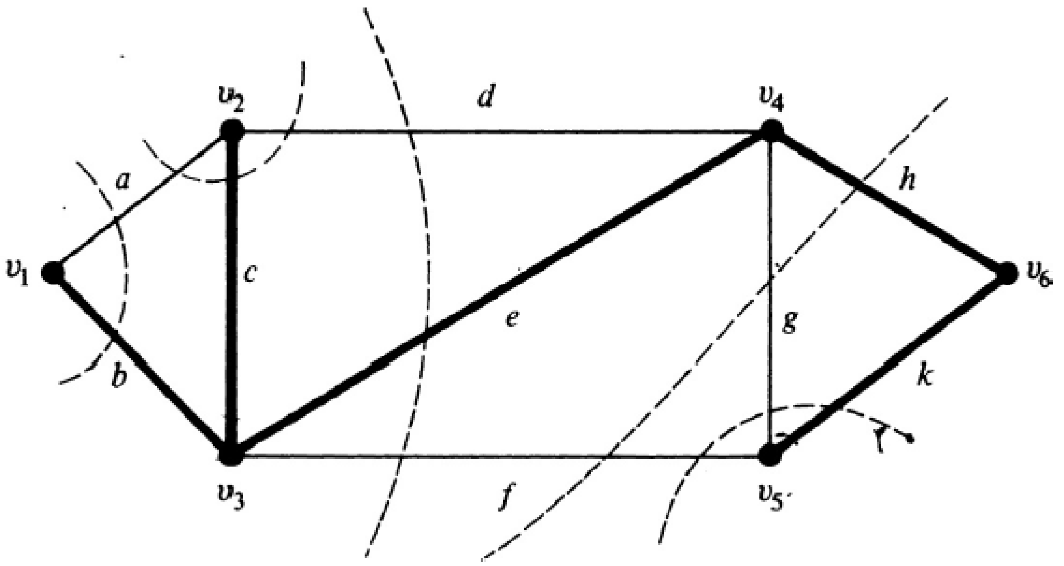


Two cut-sets and their partitioning's.

Fundamental Circuits and Cut-sets

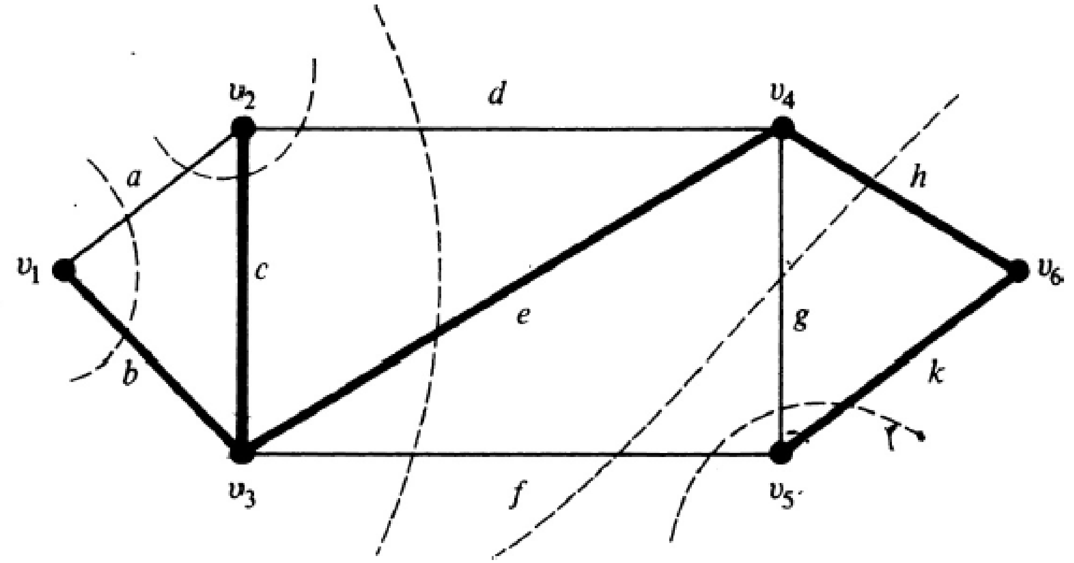
- THEOREM 6: With respect to a given spanning tree T, a chord c_i that determines a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other.
- THEOREM 7: With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.

- **Proof of Theorem 6:** Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T, and let the fundamental circuit made by c_i be called Γ, consisting of k branches $b_1, b_2, \ldots, b_k$ in addition to the chord c_i; that is,
    - Γ = {*ci*, *b*1, *b*2, . . . , *bk*) is a fundamental circuit with respect to T
- Every branch of any spanning tree has a fundamental cut-set associated with it. Let $S1$ be the fundamental cut-set associated with b_1, consisting of q chords in addition to the branch b_1; that is,
    - $S1 = \{b_l, c_l, c_2, \ldots, c_q\}$ is a fundamental cut-set with respect to T
- Because of Theorem 4, there must be an even number of edges common to Γ and $S1$. Edge $b1$ is in both Γ and $S1$, and there is only one other edge in Γ (which is c_i) that can possibly also be in $S1$. Therefore, we must have two edges b_1 and c_i common to $S1$ and Γ. Thus, the chord c_i is one of the chords $c_1, c_2, \ldots c_q$.
- Exactly the same argument holds for fundamental cut-sets associated with $b_2, b_3, \ldots, b_k$. Therefore, the chord c_i is contained in every fundamental cut-set associated with branches in Γ.
- It is not possible for the chord c_i to be in any other fundamental cut-set S' (with respect to T) besides those associated with $b_1, b_2, \ldots$ and b_k. Otherwise (since none of the branches in Γ are in S'), there would be only one edge c_i common to S' and Γ, a contradiction to Theorem 4.



- Consider the spanning tree {*b*, *c*, *e*, *h*, *k*}.
- The fundamental circuit made by chord *f* is {*f*, *e*, *h*, *k*}.
- The three fundamental cut-sets determined by the three branches *e*, *h*, and *k* are
  - determined by branch *e*: {*d*, *e*, *f*},
  - determined by branch *h*: {*f*, *g*, *h*},
  - determined by branch *k*: {*f*, *g*, *k*}.
- Chord *f* occurs in each of these three fundamental cut-sets, and there is no other fundamental cut-set that contains *f*.

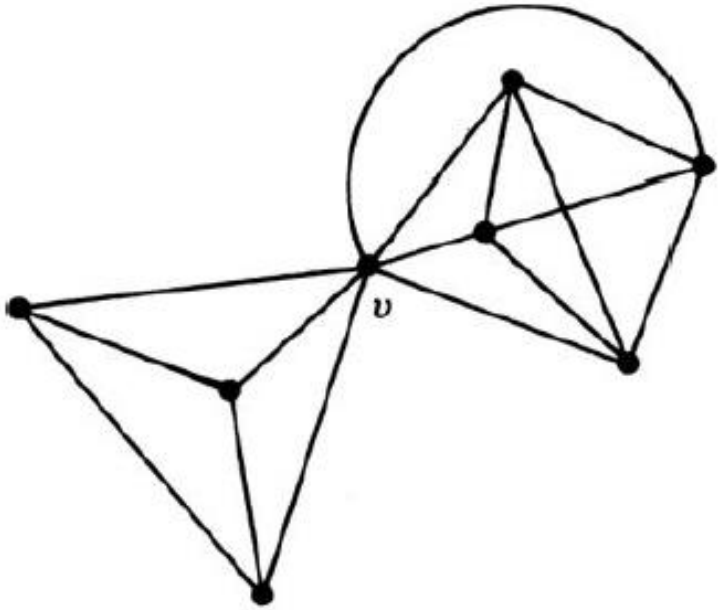
- **Proof of Theorem 7:** Let the fundamental cut-set S determined by a branch b_i be $S = \{b_i, c_1, c_2, \ldots, c_p\}$, and let Γ_1 be the fundamental circuit determined by chord c_1: $\Gamma_1 = \{c_1, b_1, b_2, \ldots, b_q\}$.
- Since the number of edges common to S and Γ_1 must be even, b_i must be in Γ_1.
- The same is true for the fundamental circuits made by chords $c_2, c_3, \ldots, c_p$.
- On the other hand, suppose that b_i occurs in a fundamental circuit Γ_{p+1} made by a chord other than $c_1, c_2, \ldots, c_p$. Since none of the chords $c_1, c_2, \ldots, c_p$ is in Γ_{p+1}, there is only one edge b_i common to a circuit Γ_{p+1} and the cut-set S, which is not possible. Hence, the theorem.

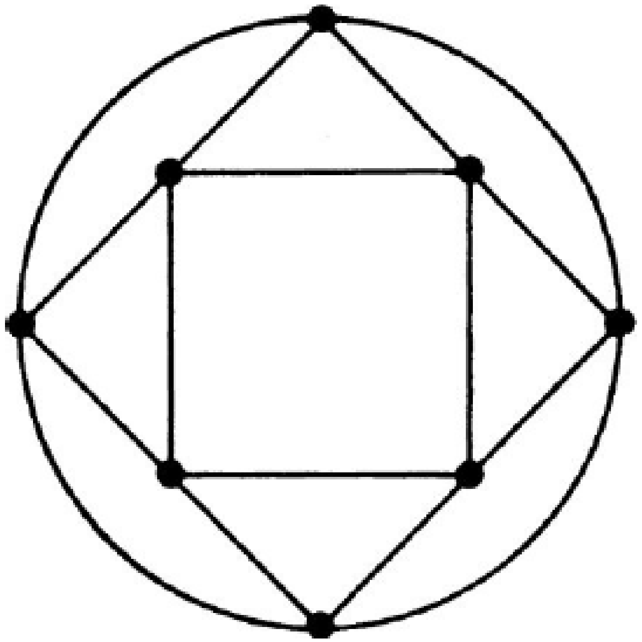


- Consider branch *e* of the spanning tree {*b*, *c*, *e*, *h*, *k*}.
- The fundamental cut-set determined by *e* is {*e*, *d*, *f*}.
- The two fundamental circuits determined by chords *d* and *f* are
  - determined by chord *d:* {*d*, *c*, *e*},
  - determined by chord *f*: {*f*, *e*, *h*, *k*}.
- Branch *e* is contained in both these fundamental circuits, and none of the remaining fundamental circuits contains branch *e*.

Separable Graph

- A connected graph is said to be separable if its vertex connectivity is one.
- All other connected graphs are called non-separable.
- An equivalent definition is that a connected graph G is said to be separable if there exists a subgraph g in G such that g'(the complement of g in G) and g have only one vertex in common.





- Two graphs with 8 vertices and 16 edges.
- First graph has Vertex connectivity, $K(G) = 1$ and edge connectivity, $K'(G) = 3$
- Second one has edge connectivity as well as the vertex connectivity of four
- Thus, the network of Second is better connected than that of First.

- **THEOREM 8:**
- The edge connectivity of a graph *G* cannot exceed the degree of the vertex with the smallest degree in *G*.
- ***Proof:***
- Let vertex v_i be the vertex with the smallest degree in G.
- Let $d(v_i)$ be the degree of v_i.
- Vertex v_i can be separated from *G* by removing the $d(v_i)$ edges incident on vertex v_i. Hence the theorem.

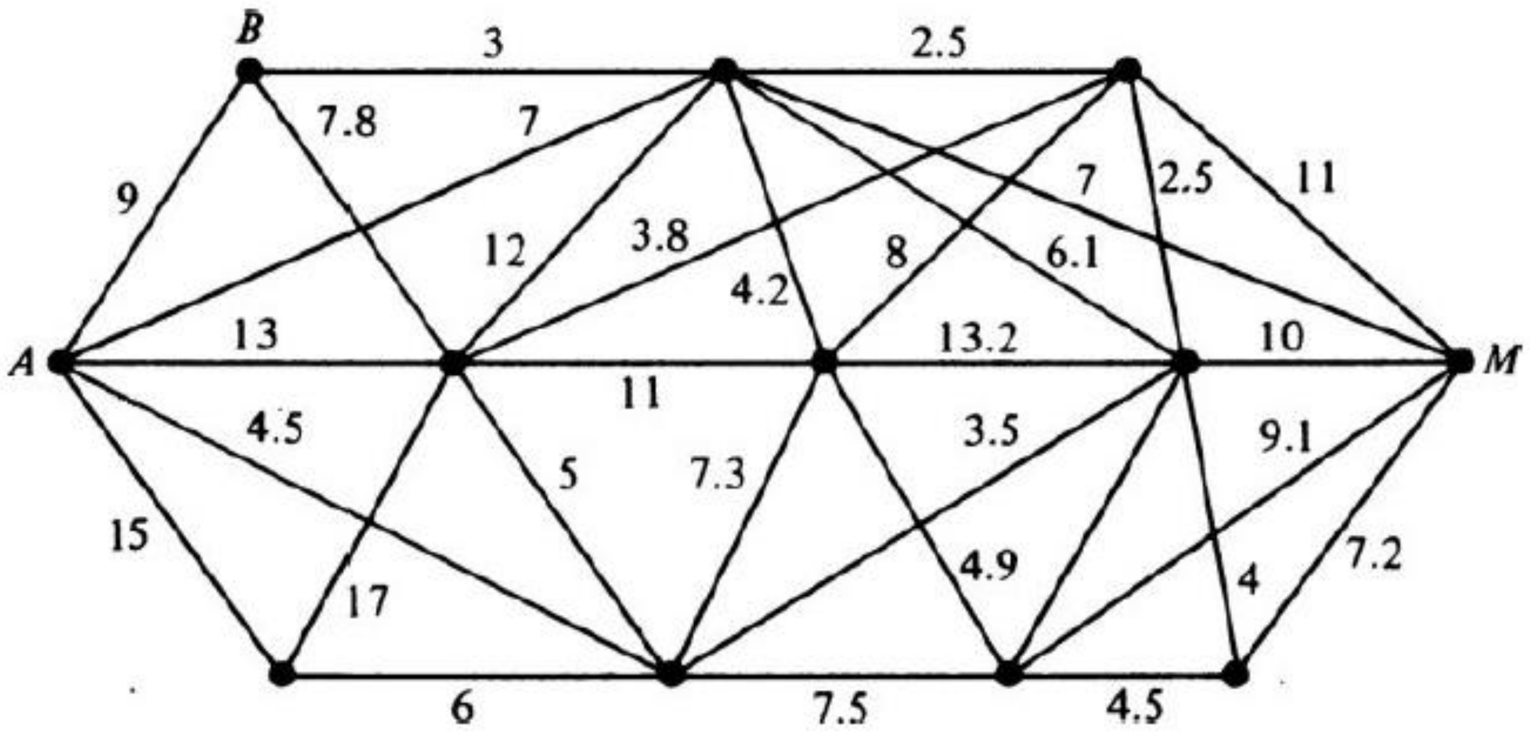
- **THEOREM 9:**
- The vertex connectivity of any graph G can never exceed the edge connectivity of G.
- ***Proof:***
- Let α denote the edge connectivity of G. Therefore, there exists a cut-set S in G with α edges.
- Let S partition the vertices of G into subsets $V1$ and $V2$.
- By removing at most α vertices from $V1$ (or $V2$) on which the edges in S are incident, we can effect the removal of S (together with all other edges incident on these vertices) from G. Hence the theorem.

- If G is a simple graph, then $K(G) \leq K'(G) \leq \delta(G)$
- Every cut-set in a non-separable graph with more than two vertices contains at least two edges.
- **THEOREM 10:** The maximum vertex connectivity one can achieve with a graph *G* of *n* vertices and *e* edges ($e \geq n - 1$) is the integral part of the number $2e/n$; that is, $\lfloor 2e/n \rfloor$.
- ***Proof:***
  - Every edge in *G* contributes two degrees. The total ($2e$ degrees) is divided among *n* vertices. Therefore, there must be at least one vertex in *G* whose degree is equal to or less than the number $2e/n$. The vertex connectivity of *G* cannot exceed this number, in light of Theorems 8 and 9.

- **THEOREM 11:** A connected graph *G* is *k*-connected if and only if every pair of vertices in *G* is joined by *k* or more paths that do not intersect, and at least one pair of vertices is joined by exactly *k* non-intersecting paths.
- **THEOREM 12:** The edge connectivity of a graph *G* is *k*: if and only if every pair of vertices in *G* is joined by *k* or more edge-disjoint paths and at least one pair of vertices is joined by exactly *k* edge-disjoint paths.
- A graph G is nonseparable if and only if any pair of vertices in G can be placed in a circuit.

Network Flows

- In a network of telephone lines, highways, railroads, pipelines of oil (or gas or water), and so on, it is important to know the maximum rate of flow that is possible from one station to another in the network.
- This type of network is represented by a weighted connected graph in which the vertices are the stations and the edges are lines through which the given commodity (oil, gas, water, number of messages, number of cars, etc.) flows.
- The weight, a real positive number, associated with each edge represents the capacity of the line, that is, the maximum amount of flow possible per unit of time.



- The graph represents a flow network consisting of 12 stations and 31 lines. The capacity of each of these lines is also indicated in the figure.

- It is assumed that at each intermediate vertex the total rate of commodity entering is equal to the rate leaving. In other words, there is no accumulation or generation of the commodity at any vertex along the way.
- The flow through a vertex is limited only by the capacities of the edges incident on it. In other words, the vertex itself can handle as much flow as allowed through the edges.
- The lines are lossless.
- **THEOREM 13:** The maximum flow possible between two vertices a and b in a network is equal to the minimum of the capacities of all cut-sets with respect to a and b.

- **Proof:** Consider any cut-set S with respect to vertices a and b in G.
- In the subgraph G − S (the subgraph left after removing S from G) there is no path between a and b.
- Therefore, every path in G between a and b must contain at least one edge of S. Thus every flow from a to b (or from b to a) must pass through one or more edges of S.
- Hence the total flow rate between these two vertices cannot exceed the capacity of S. Since this holds for all cut-sets with respect to a and b, the flow rate cannot exceed the minimum of their capacities.

Fundamental Circuits – Algorithm

- Each edge is tested to see if it forms a circuit with the tree constructed so far; but instead of taking the edges themselves in an arbitrary order, we select a vertex z and examine this vertex by looking at every edge incident on z.
- Vertex z, is the vertex added most recently to the partially formed tree.
- Let the vertices of the given connected graph G = (V, E) be labeled 1, 2, . . . , n, and the graph be given by its adjacency matrix X.
- Let T be the current set of vertices in the partially formed tree, and let W be the set of vertices that are yet to be examined (i.e., those vertices, in T as well as not in T, which have one or more unexamined edges incident on them).
- Initially, T = Ø and W = V, the entire set of vertices.
- We start the algorithm by setting T = 1, the first vertex, and W = V. Vertex 1 will be regarded as the root of the tree to be formed

1. If $T \cap W = \emptyset$, then the algorithm is terminated.
2. If $T \cap W \neq \emptyset$, choose a vertex z in $T \cap W$.
3. Examine z by considering every edge incident on z. If there is no such edge left, remove z from W, and go to step 1.
4. If there is such an edge (z, p), test if vertex p is in T.
5. If $p \in T$, find the fundamental circuit consisting of edge (z, p) together with the unique path from z to p in the tree (formed so far). Delete edge (z, p) from the graph, and go to step 3.
6. If $p \notin T$, add edge (z, p) to the tree and vertex p to set T. Delete edge (z, p) from the graph, and go to step 3.

Cayley’s Theorem

Cayley's Formula

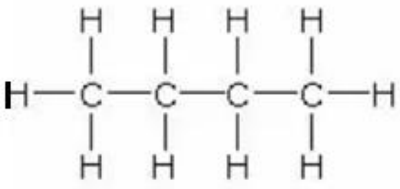
- In 1857, Arthur Cayley discovered trees while he was trying to count the number of structural isomers of the saturated hydrocarbons (or paraffin series) C_kH_{2k+2} .
- He used a connected graph to represent the C_kH_{2k+2} molecule. Corresponding to their chemical valencies, a carbon atom was represented by a vertex of degree four and a hydrogen atom by a vertex of degree one (pendant vertices). The total number of vertices in such a graph is

$$n = 3k + 2,$$

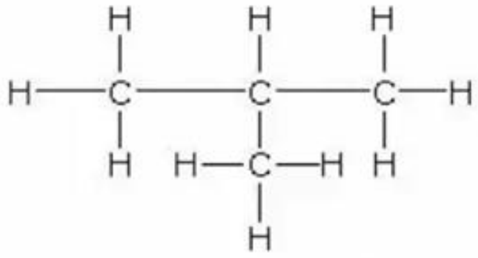
and the total number of edges is

$$e = \frac{1}{2}(\text{sum of degrees}) = \frac{1}{2}(4k + 2k + 2)$$
$$= 3k + 1.$$

Since the graph is connected and the number of edges is one less than the number of vertices, it is a tree. Thus the problem of counting structural isomers of a given hydrocarbon becomes the problem of counting trees.



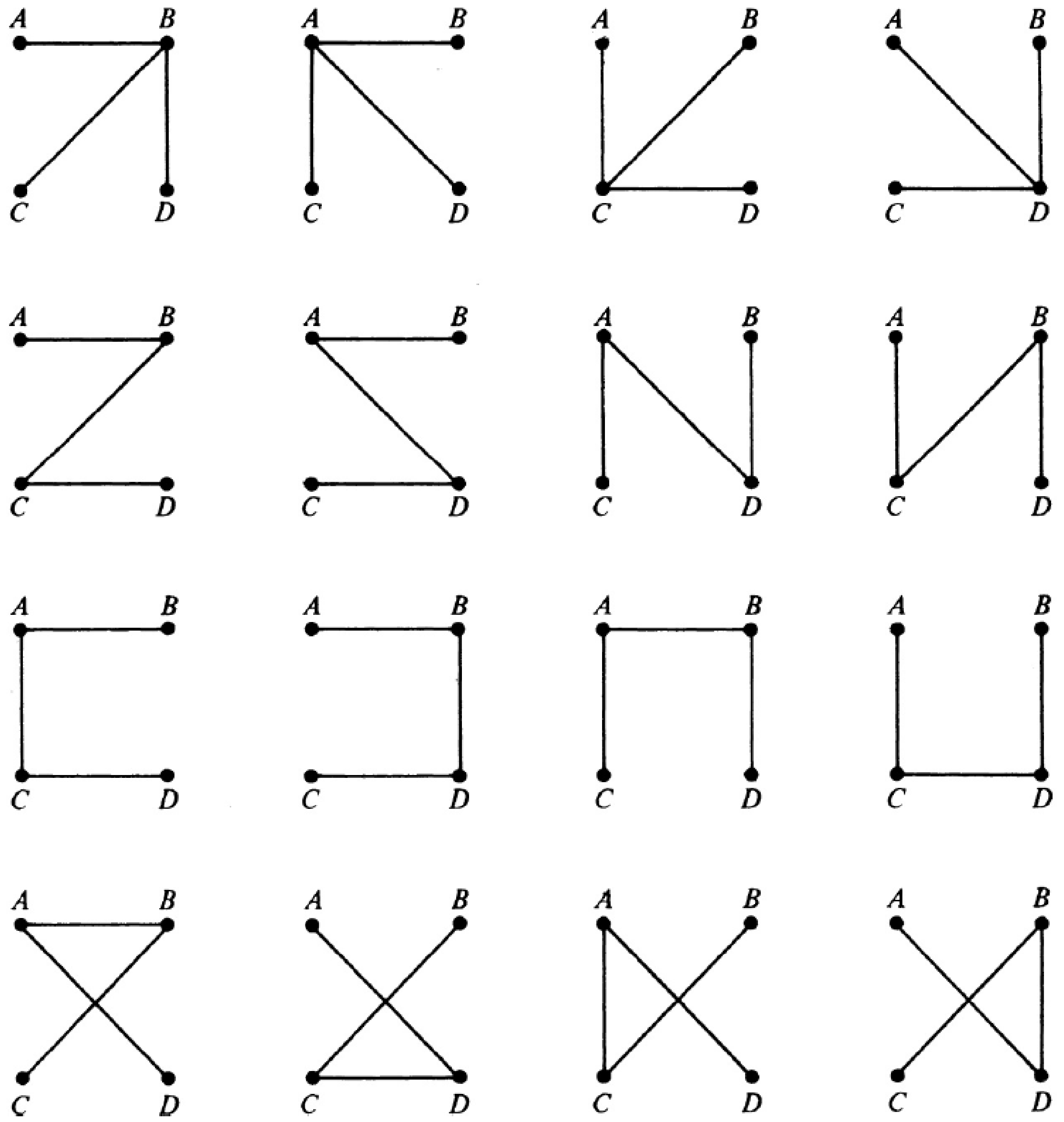
Butane (C_4H_{10})



2-methyl propane (C_4H_{10})

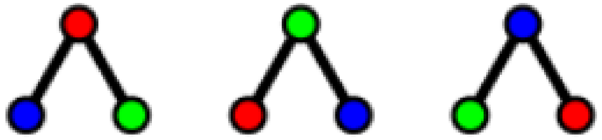
what is the number of different trees that one can construct with n distinct (or labeled) vertices?

n=4



n=1

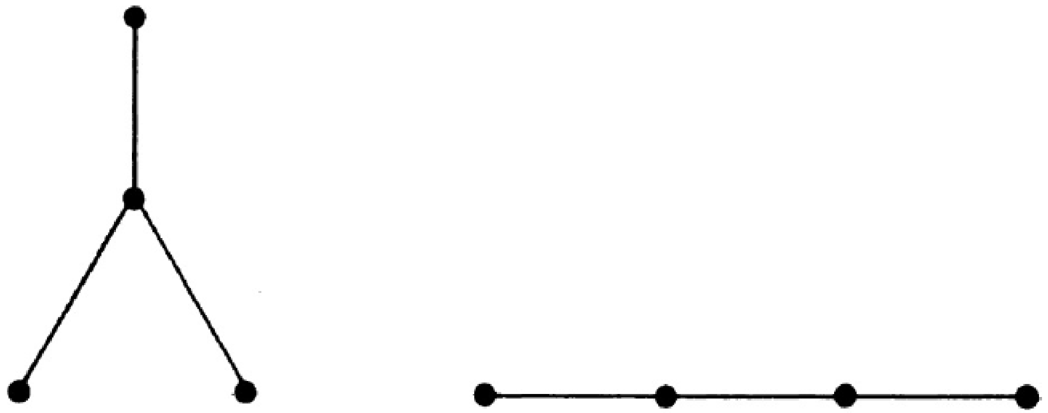
n=2



n=3

Cayley's Theorem

- The number of labeled trees with n vertices ($n \geq 2$) is n^{n-2}.



All trees of Unlabeled vertices, n=4

Proof of Cayley's

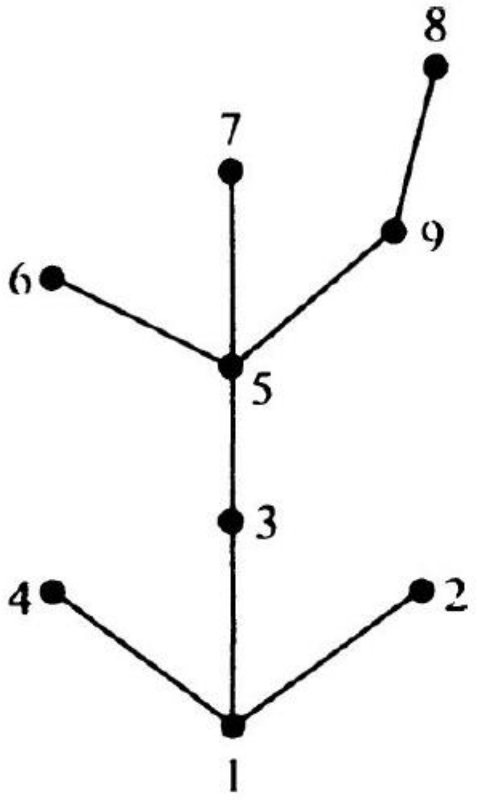
- Let the n vertices of a tree T be labeled 1, 2, 3, . . . , n.
- Remove the pendant vertex (and the edge incident on it) having the smallest label, which is, say, a1. Suppose that b1 was the vertex adjacent to a1. Among the remaining n − 1 vertices let a2 be the pendant vertex with the smallest label, and b2 be the vertex adjacent to a2. Remove the edge (a2, b2).
- This operation is repeated on the remaining n − 2 vertices, and then on n − 3 vertices, and so on.
- The process is terminated after n − 2 steps, when only two vertices are left. The tree T defines the sequence

$$B = (b_1, b_2, \ldots, b_{n-2})$$

- Conversely, given a sequence of n − 2 labels, an n-vertex tree can be constructed uniquely, as follows:
- Determine the first number in the vertex sequence that does not appear in the sequence.

$$(b_1, b_2, \ldots, b_{n-2})$$

- This number clearly is a1. And thus the edge (a1, b1) is defined. Remove b1 from B sequence and a1 from vertex sequence. In the remaining vertex sequence find the first number that does not appear in the remainder of B. This would be a2, and thus the edge (a2, b2) is defined.
- The construction is continued till the sequence B has no element left.
- Finally, the last two vertices remaining in the vertex sequence are joined.



- Nine-vertex labeled tree, which yields sequence

  (1, 1, 3, 5, 5, 5, 9).

Euler Graph,
Fleury’s Algorithm,
Hamiltonian Graph

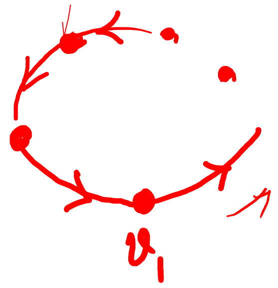
Euler Graph

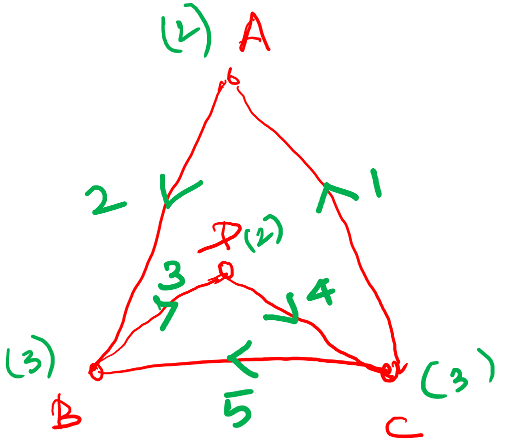
If some closed walk in a graph contains all the edges of the graph, then the walk is called an *Euler line* and the graph an *Euler graph*.

THEOREM

A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

- Proof: Suppose that G is an Euler graph.
- It therefore contains an Euler line (which is a closed walk).
- In tracing this walk we observe that every time the walk meets a vertex v it goes through two “new” edges incident on v—with one we “entered” v and with the other “exited.” This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we “exited” and “entered” the same vertex at the beginning and end of the walk, respectively.
- Thus if G is an Euler graph, the degree of every vertex is even.





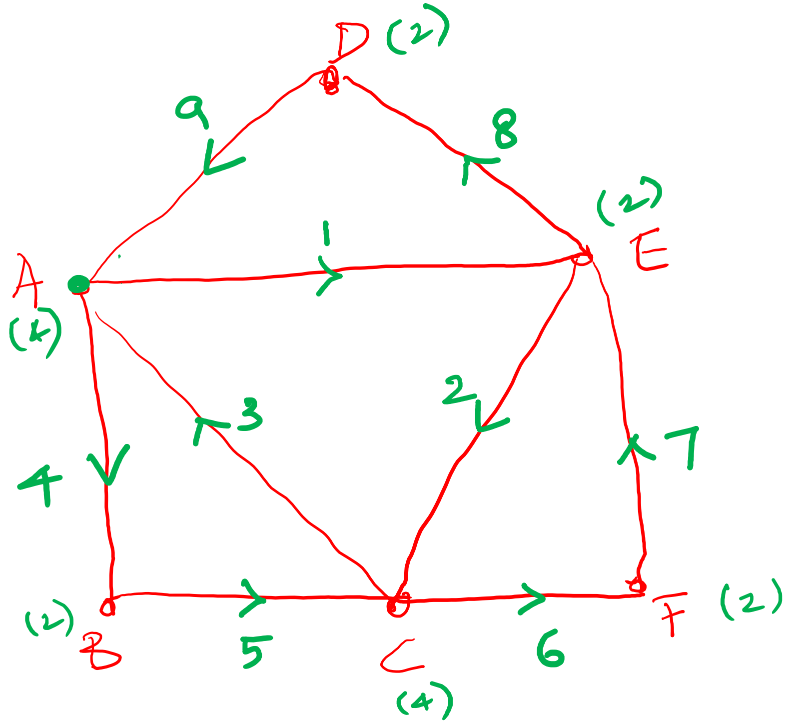
C 1 A 2 B 3 D 4 C 5 B

Open walk

Euler path

Open Euler line

Unicursal line



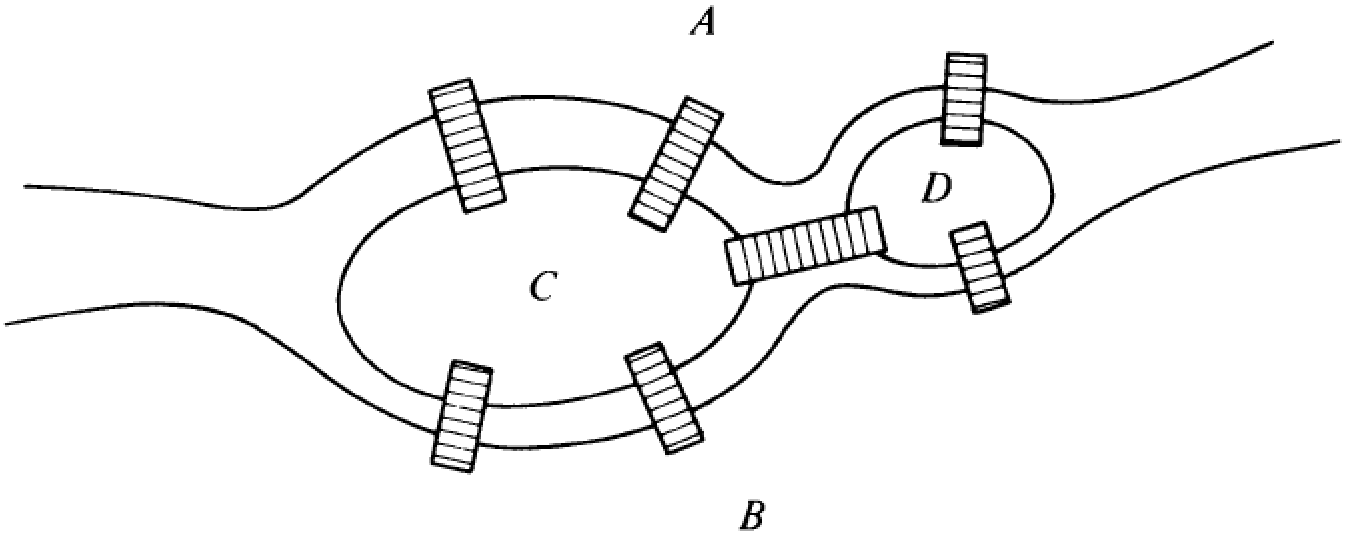
✓ A 1 E 2 C 3 A 4 B 5 C 6 F 7 E 8 D 9 A

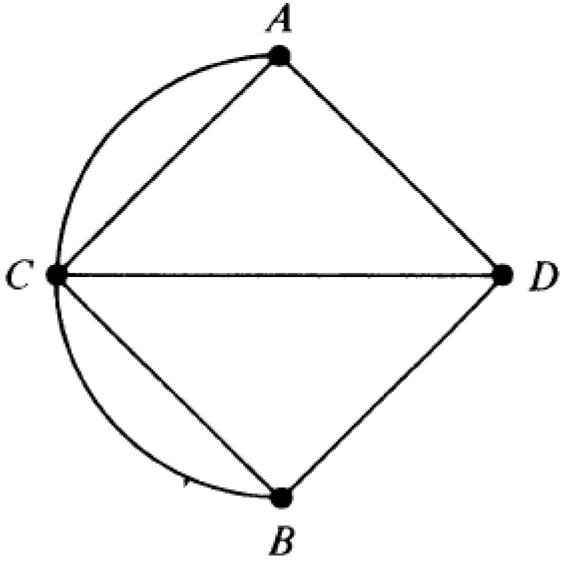
Closed walk ✓

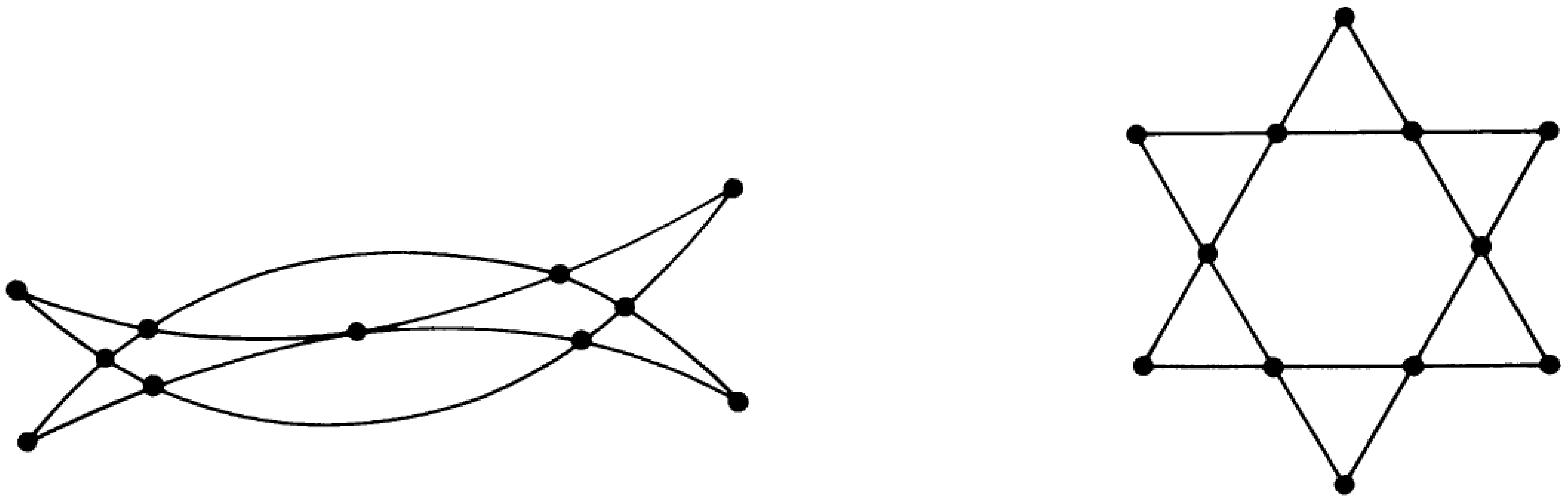
Euler circuit / Euler line

Euler Graph

Königsberg Bridge Problem



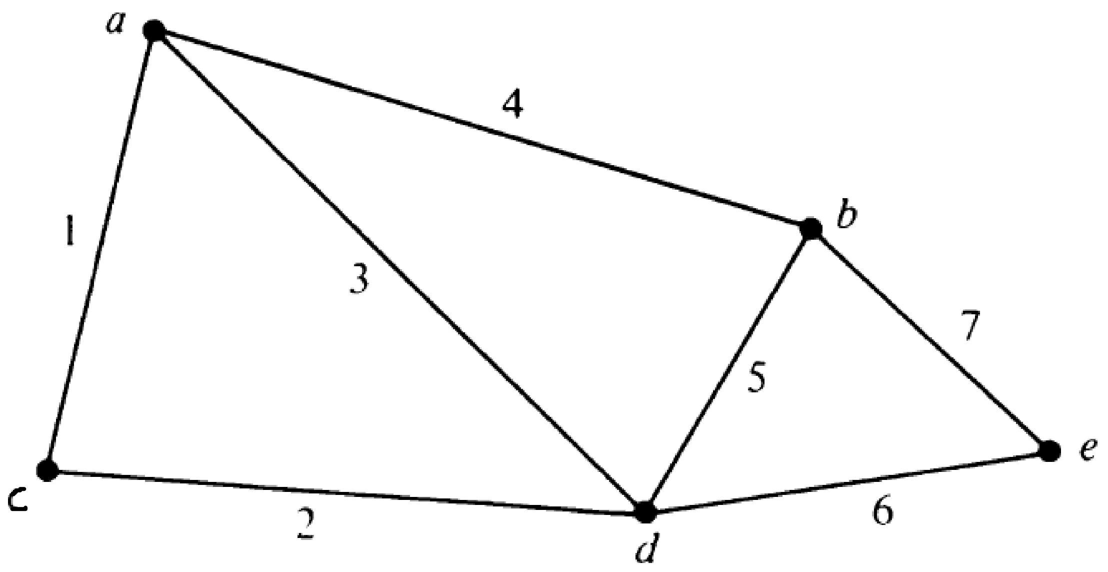




(a) (b)

Two Euler graphs.

an open walk that includes (or traces or covers) all edges of a graph without retracing any edge a *unicursal line* or an *open Euler line*. A (connected) graph that has a unicursal line will be called a *unicursal graph*.



a 1 *c* 2 *d* 3 *a* 4 *b* 5 *d* 6 *e* 7 *b*

It is clear that by adding an edge between the initial and final vertices of a unicursal line we shall get an Euler line. Thus a connected graph is unicursal if and only if it has exactly two vertices of odd degree.

THEOREM

In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

$v_1, v_2, \ldots \ldots v_k$ $w_1, w_2, w_3 \ldots \ldots w_k$

THEOREM

In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

- Proof: Let the odd vertices of the given graph G be named v1, v2, . . . , vk; w1, w2, . . . , wk in any arbitrary order.
- Add k edges to G between the vertex pairs (v1, w1), (v2, w2), . . . , (vk, wk) to form a new graph G′.
- Since every vertex of G′ is of even degree, G′ consists of an Euler line p.
- Now if we remove from p the k edges we just added (no two of these edges are incident on the same vertex), p will be split into k walks, each of which is a unicursal line: The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them.
- Thus the theorem.

Euler Graphs continued...

THEOREM

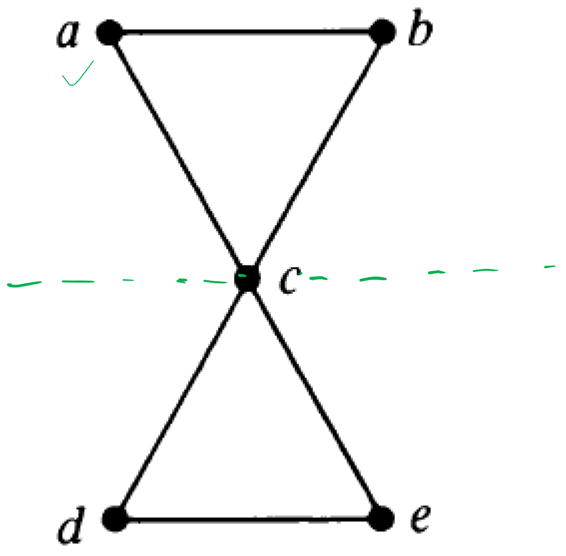
A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

- Proof:
- Suppose graph G can be decomposed into circuits; that is, G is a union of edge-disjoint circuits. Since the degree of every vertex in a circuit is two, the degree of every vertex in G is even. Hence G is an Euler graph.
- Conversely, let G be an Euler graph. Consider a vertex v1. There are at least two edges incident at v1. Let one of these edges be between v1 and v2. Since vertex v2 is also of even degree, it must have at least another edge, say between v2 and v3. Proceeding in this fashion, we eventually arrive at a vertex that has previously been traversed, thus forming a circuit Γ. Let us remove Γ from G. All vertices in the remaining graph (not necessarily connected) must also be of even degree. From the remaining graph remove another circuit in exactly the same way as we removed Γ from G. Continue this process until no edges are left.
- Hence the theorem.

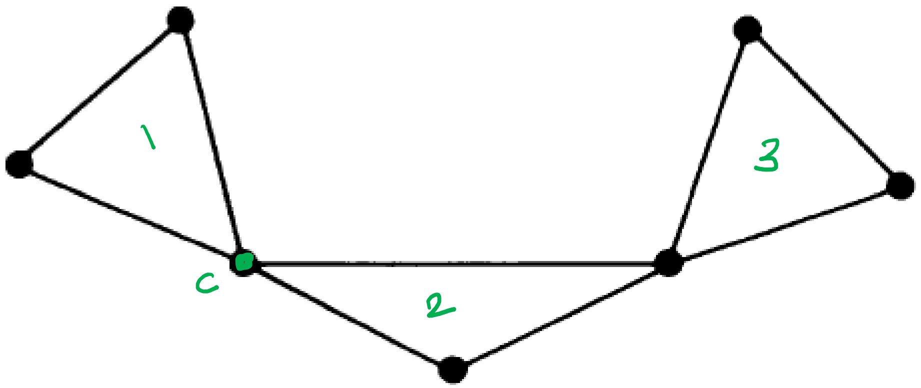
Euler Graphs continued…

THEOREM

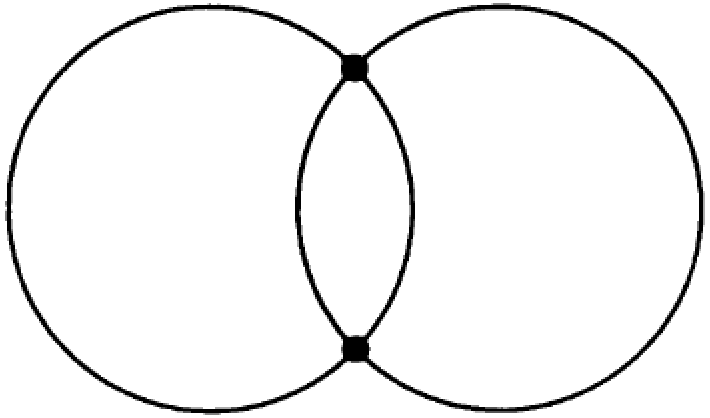
An Euler graph G is arbitrarily traceable from vertex v in G if and only if every circuit in G contains v.



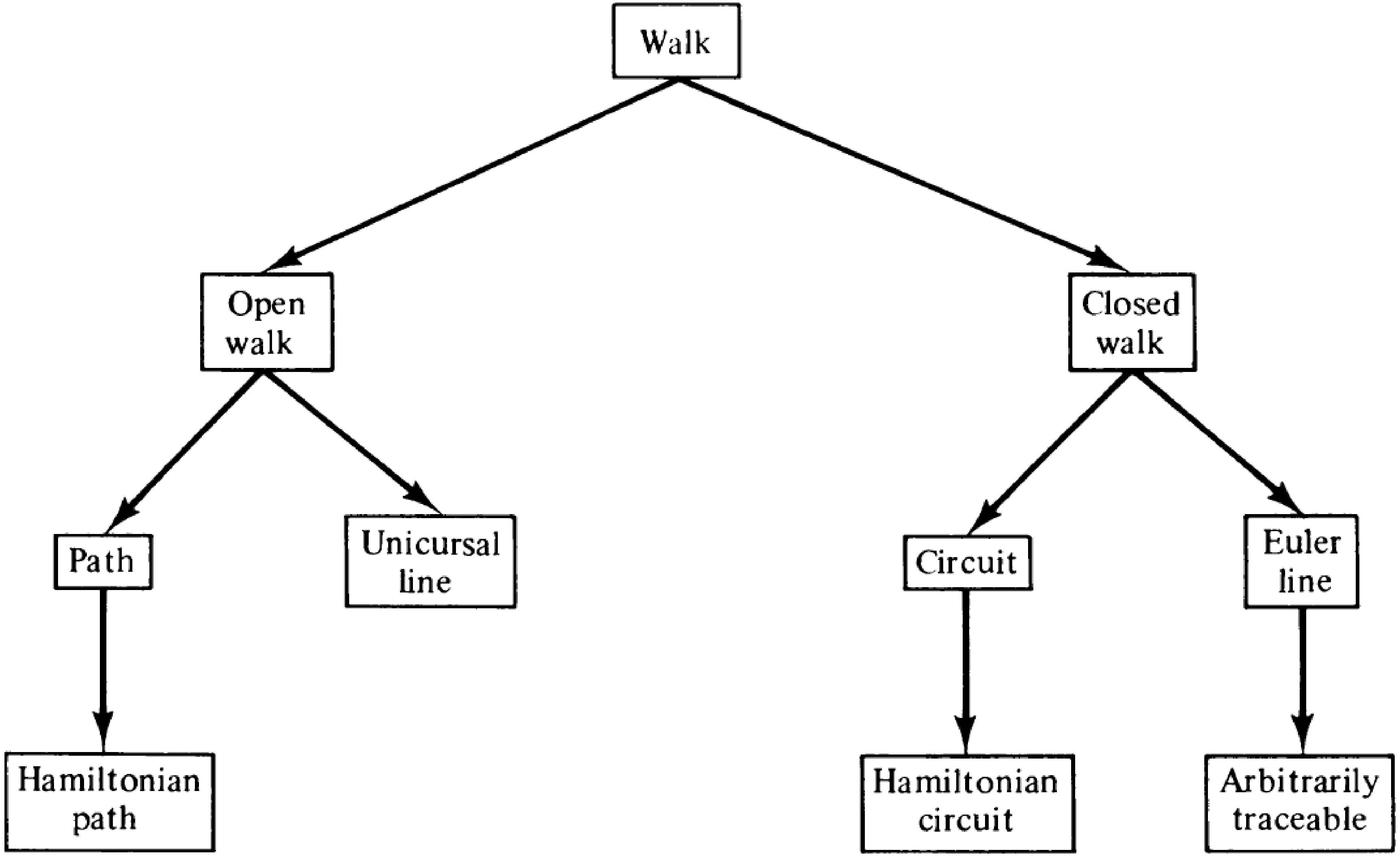
Arbitrarily traceable graph from c.



Euler graph; not arbitrarily traceable.



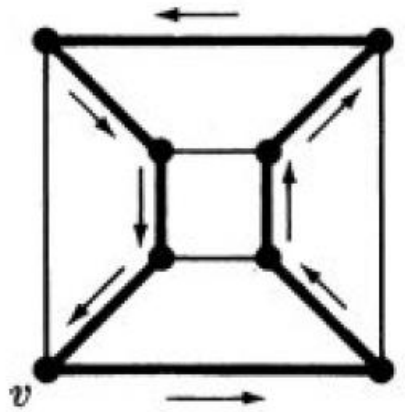
Arbitrarily traceable graph from all vertices.

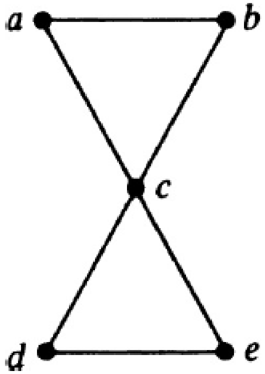


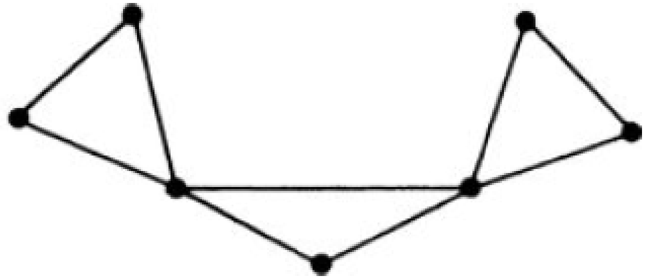
Different types of walks.

Hamiltonian Paths and Circuits

- A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except of course the starting vertex, at which the walk also terminates.
- A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.
- If we remove any one edge from a Hamiltonian circuit, we are left with a path, called a Hamiltonian path.
- A Hamiltonian path in a graph G traverses every vertex of G.
- Since a Hamiltonian path is a subgraph of a Hamiltonian circuit (which in turn is a subgraph of another graph), every graph that has a Hamiltonian circuit also has a Hamiltonian path
- The length of a Hamiltonian path in a connected graph of n vertices is n — 1.







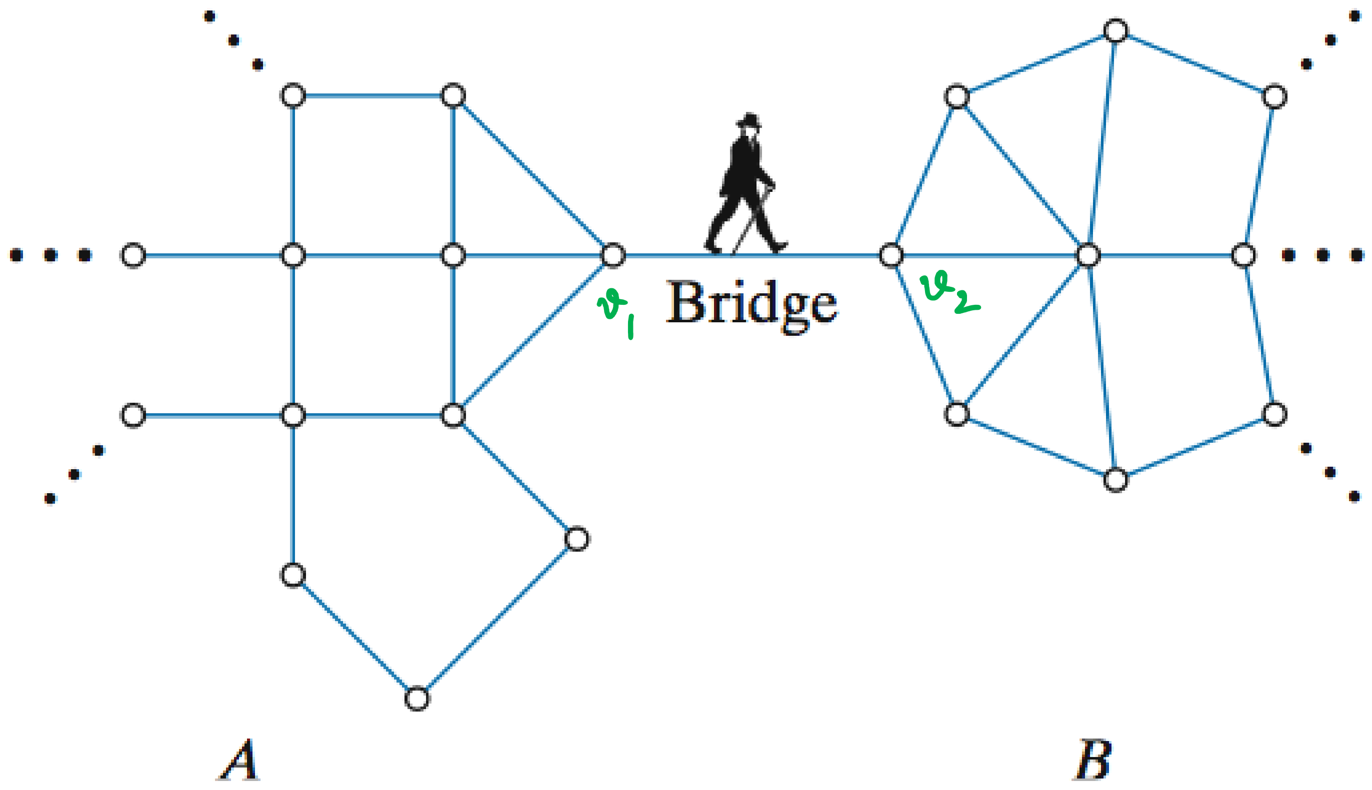
Second and third graphs don't have Hamiltonian circuit

Fleury’s Algorithm

- We will now turn our attention to an algorithm that finds an *Euler circuit* or an *Euler path* in a connected graph. Technically speaking, these are two separate algorithms, but in essence they are identical, so they can be described as one.
- The idea behind Fleury’s algorithm can be paraphrased by that old piece of folk wisdom: *Don’t burn your bridges behind you*.

Fleury's Algorithm

In graph theory the word bridge has a very specific meaning–it is the only edge connecting two separate sections (subgraph *A* and *B*) of a graph, as illustrated in Fig.



Fleury’s Algorithm

Fleury’s algorithm is based on a simple principle: To find an Euler circuit or an Euler path, *bridges are the last edges you want to cross*.

Our concerns lie only on how we are going to get around the *yet-to-be-traveled* part of the graph.

Thus, when we talk about bridges that we want to leave as a last resort, we are really referring to *bridges of the to-be-traveled part of the graph*.

FLEURY’S ALGORITHM FOR FINDING AN EULER CIRCUIT (PATH)

- **Preliminaries.** Make sure that the graph is connected and either (1) has no odd vertices (circuit) or (2) has just two odd vertices (path).
- **Start.** Choose a starting vertex. [In case (1) this can be any vertex; in case (2) it must be one of the two odd vertices.]

FLEURY'S ALGORITHM FOR FINDING AN EULER CIRCUIT (PATH)

- **Intermediate steps.** At each step, if you have a choice, don't choose a bridge of the yet-to-be-traveled part of the graph. However, if you have only one choice, take it.
- **End.** When you can't travel any more, the circuit (path) is complete. [In case (1) you will be back at the starting vertex; in case (2) you will end at the other odd vertex.]

Fleury’s Algorithm Bookkeeping

In implementing Fleury’s algorithm it is critical to separate the past (the part of the graph that has already been traveled) from the future (the part of the graph that still needs to be traveled).

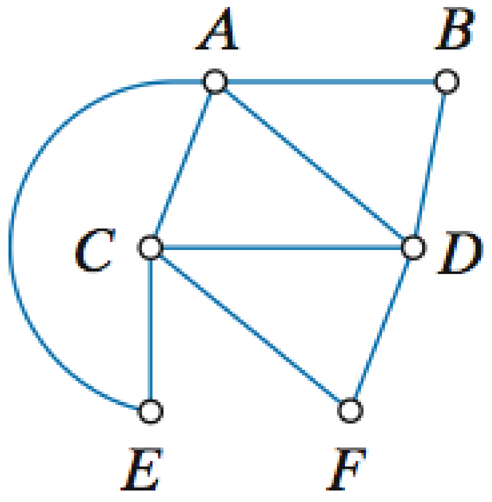
While there are many different ways to accomplish this (you are certainly encouraged to come up with one of your own), a fairly reliable way goes like this: Start with two copies of the graph. Copy 1 is to keep track of the “future”; copy 2 is to keep track of the “past.”

Fleury’s Algorithm Bookkeeping

Every time you travel along an edge, erase the edge from copy 1, but mark it (say in red) and label it with the appropriate number on copy 2. As you move forward, copy 1 gets smaller and copy 2 gets redder. At the end, copy 1 has disappeared; copy 2 shows the actual Euler circuit or path.

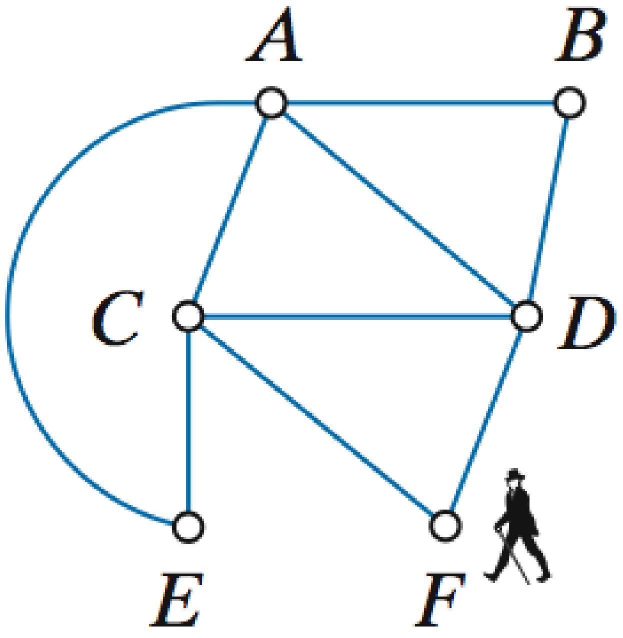
Example: Implementing Fleury's Algorithm

The graph in Fig is a very simple graph – it would be easier to find an Euler circuit just by trial-and-error than by using Fleury's algorithm. Nonetheless, we will do it using Fleury's algorithm. The real purpose of the example is to see the algorithm at work. Each step of the algorithm is explained in Figs.



Example Implementing Fleury's Algorithm…

Start: We can pick any starting point we want. Let's say we start at *F*.



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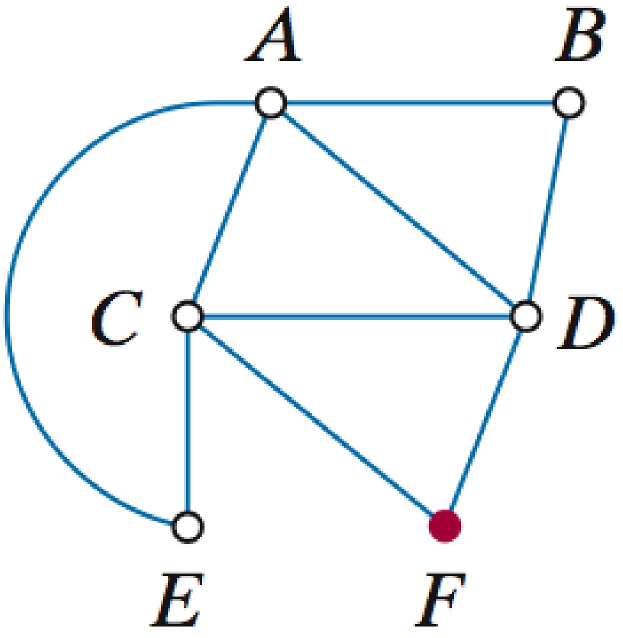

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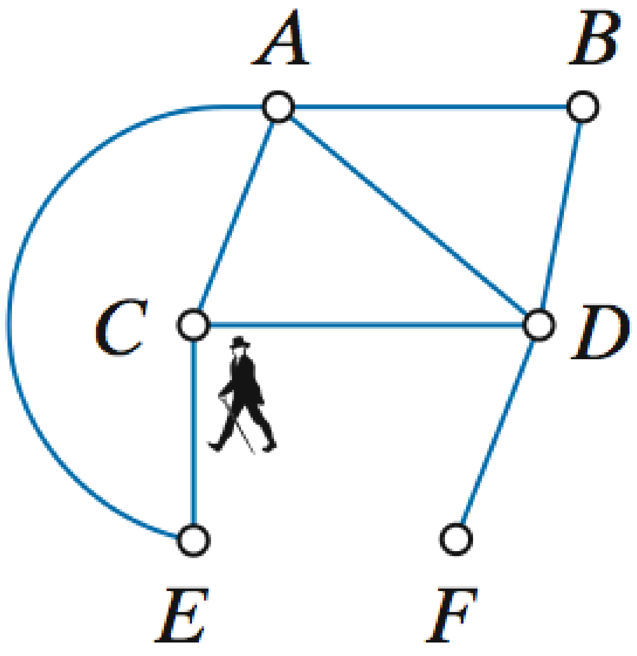
Example Implementing Fleury's Algorithm…

Step 1: Travel from *F* to *C*. (Could have also gone from *F* to *D*.)



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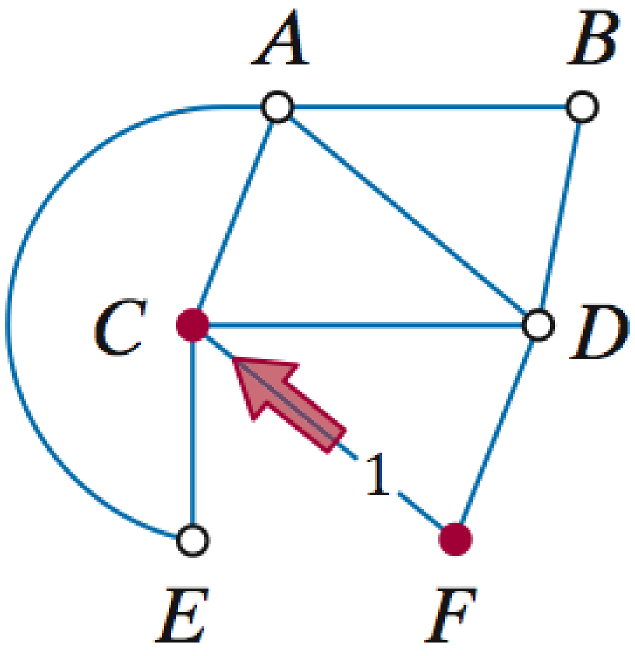

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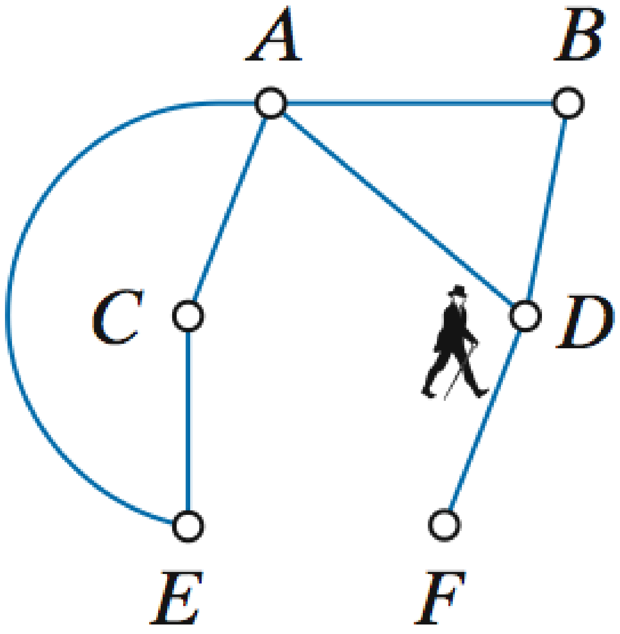
Example Implementing Fleury's Algorithm…

Step 2: Travel from *C* to *D*. (Could have also gone to *A* or to *E*.)



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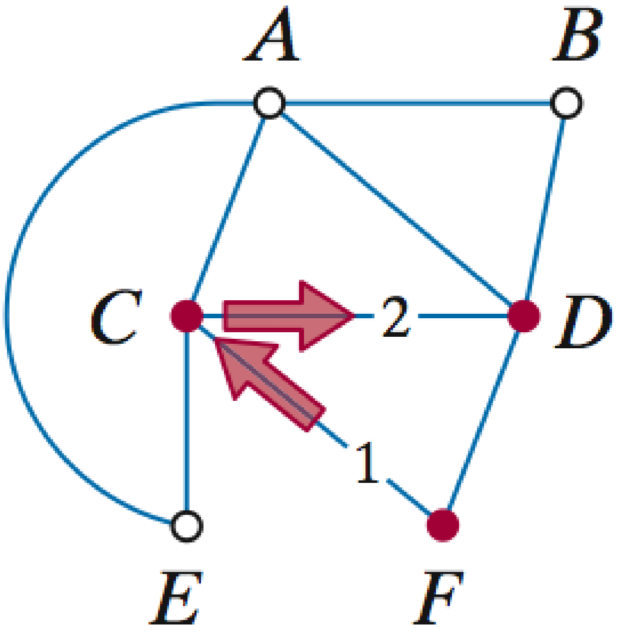

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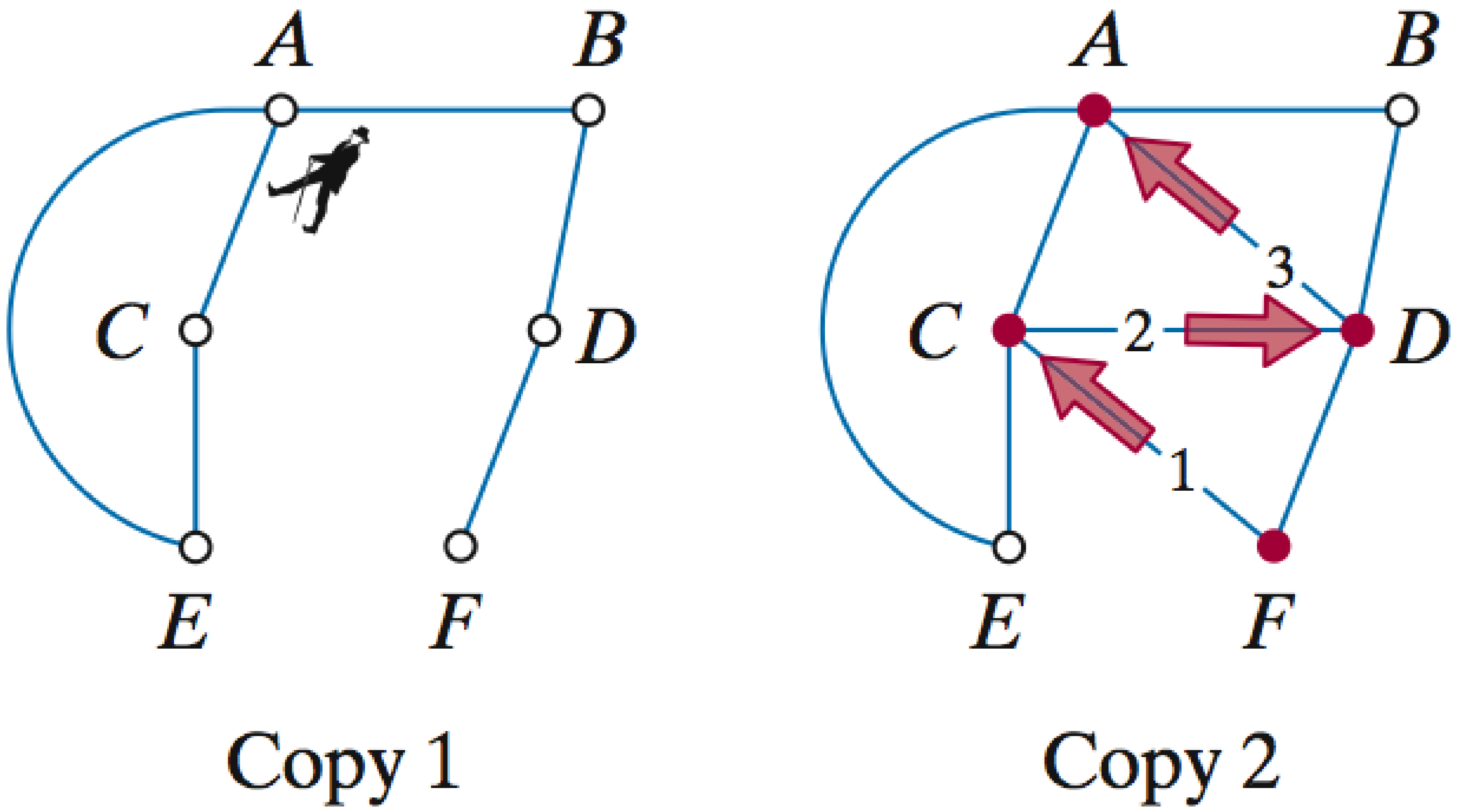
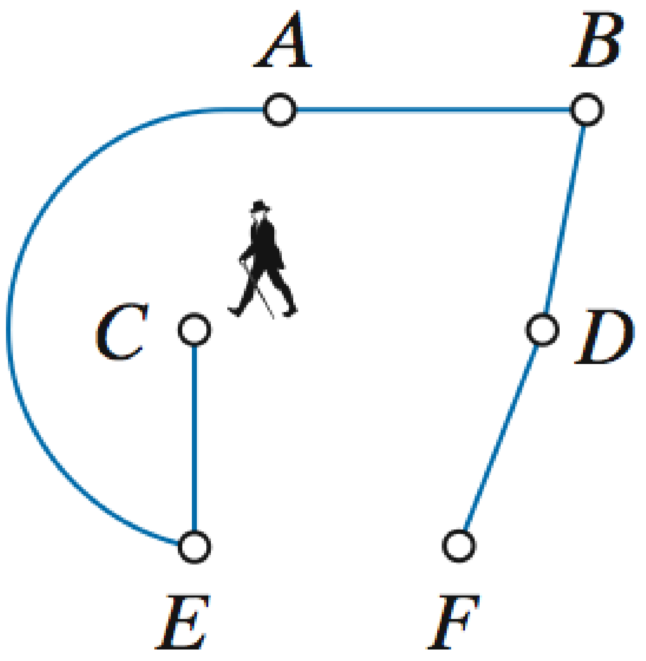
Example Implementing Fleury's Algorithm…

Step 3: Travel from *D* to *A*. (Could have also gone to *B* but not to *F* – *DF* is a bridge!)



Example Implementing Fleury's Algorithm…

Step 4: Travel from *A* to *C*. (Could have also gone to *E* but not to *B* – *AB* is a bridge!)



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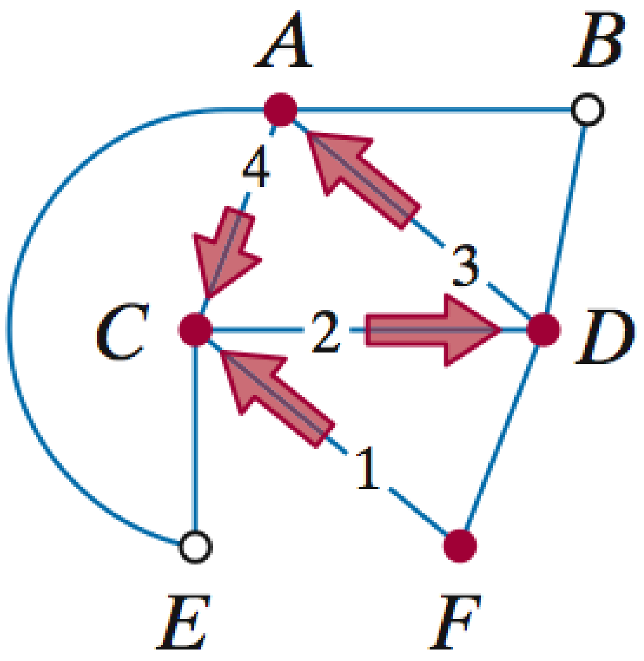

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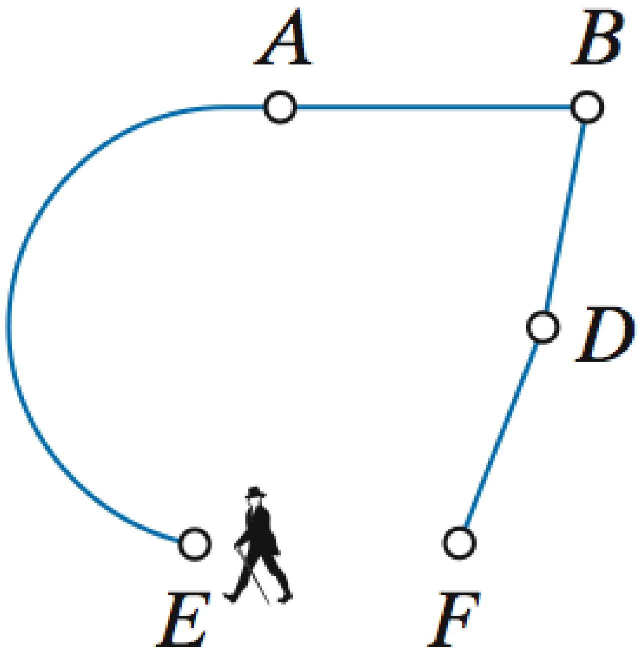
Example Implementing Fleury's Algorithm…

Step 5: Travel from *C* to *E*. (There is no choice!)



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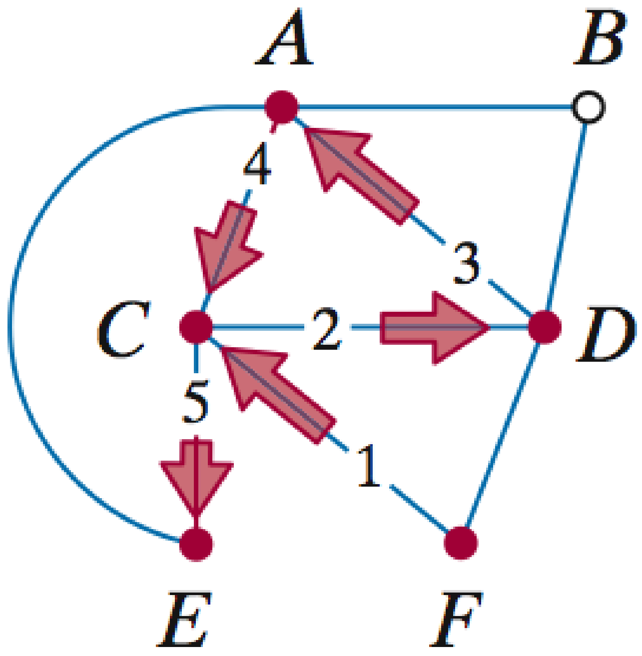

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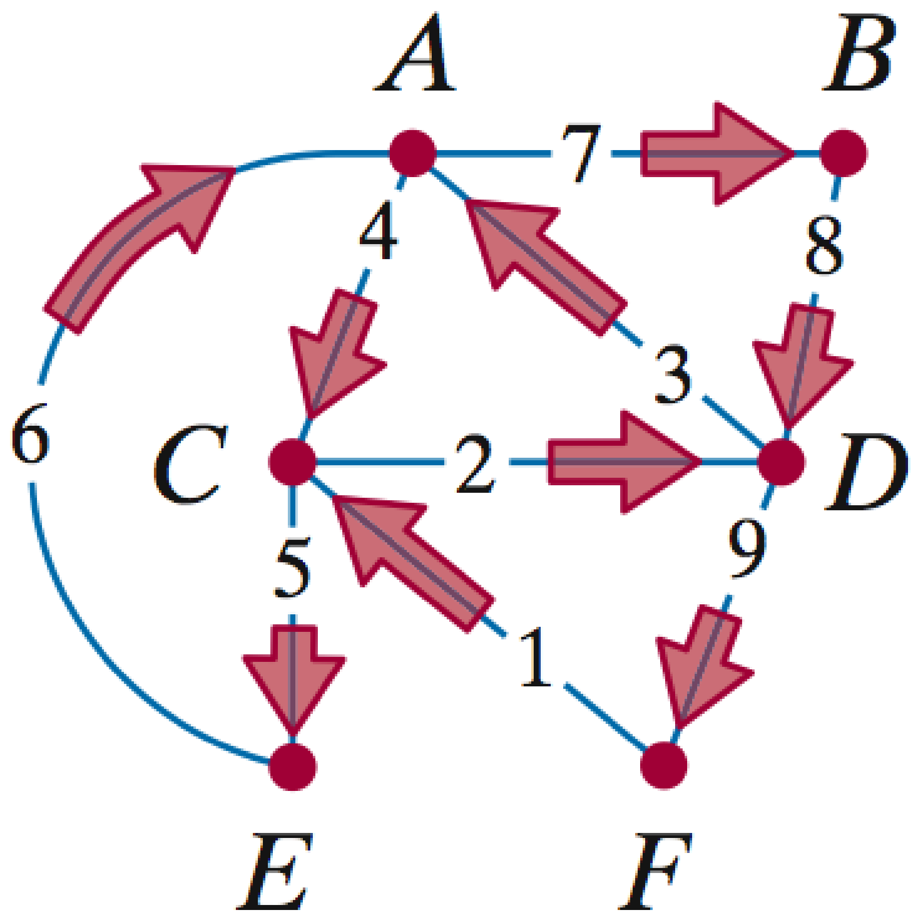
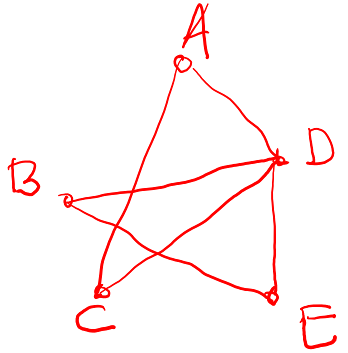
Example Implementing Fleury's Algorithm…

Steps 6, 7, 8, and 9: Only one way to go at each step.



F 1 C 2 D 3 A 4 C 5 E 6 A 7 B 8 D 9 F

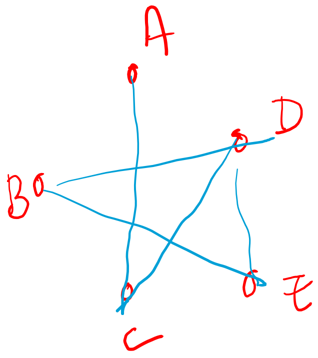
Example: Fleury's Algorithm for Euler Circuit



Starting from A

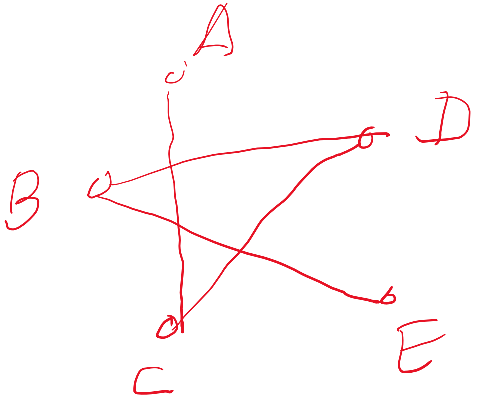
Choose A - D

circuit : AD



- D is current
- can't choose DC
- Can choose B or E
- Choosing DE

circuit: ADE

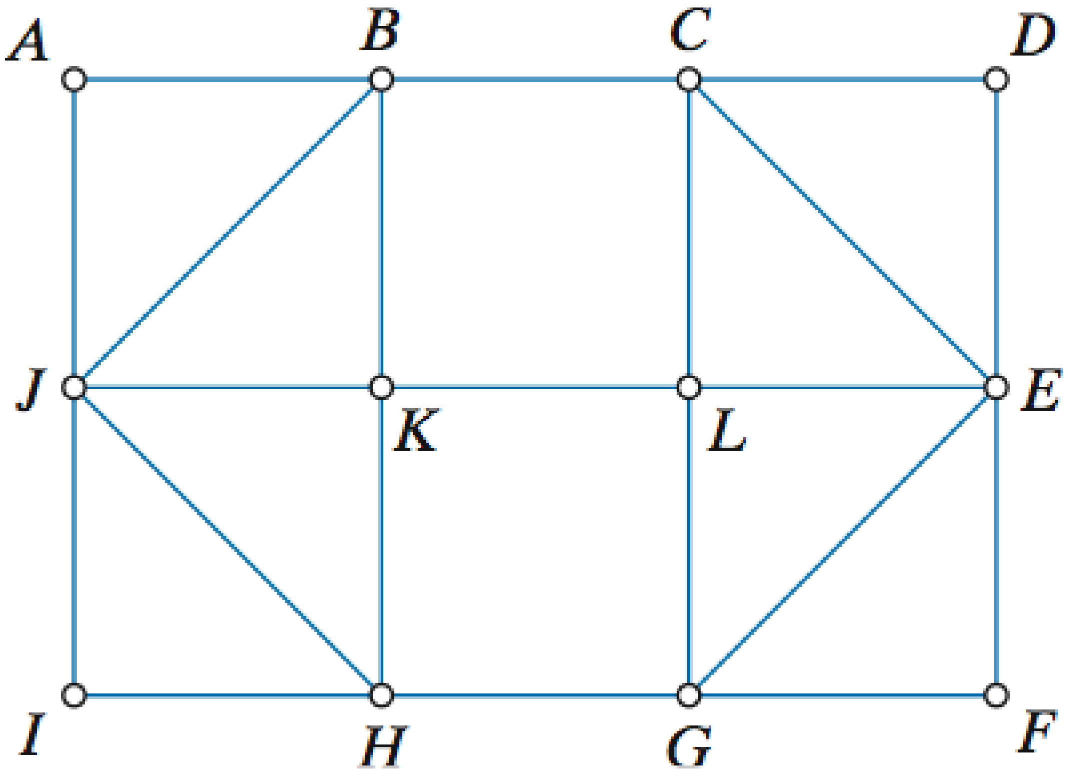


- E is current
- from here only one option

circuit : ADEBDCA

Example: Fleury’s Algorithm for Euler Paths

We will apply Fleury’s algorithm to the graph in Fig.



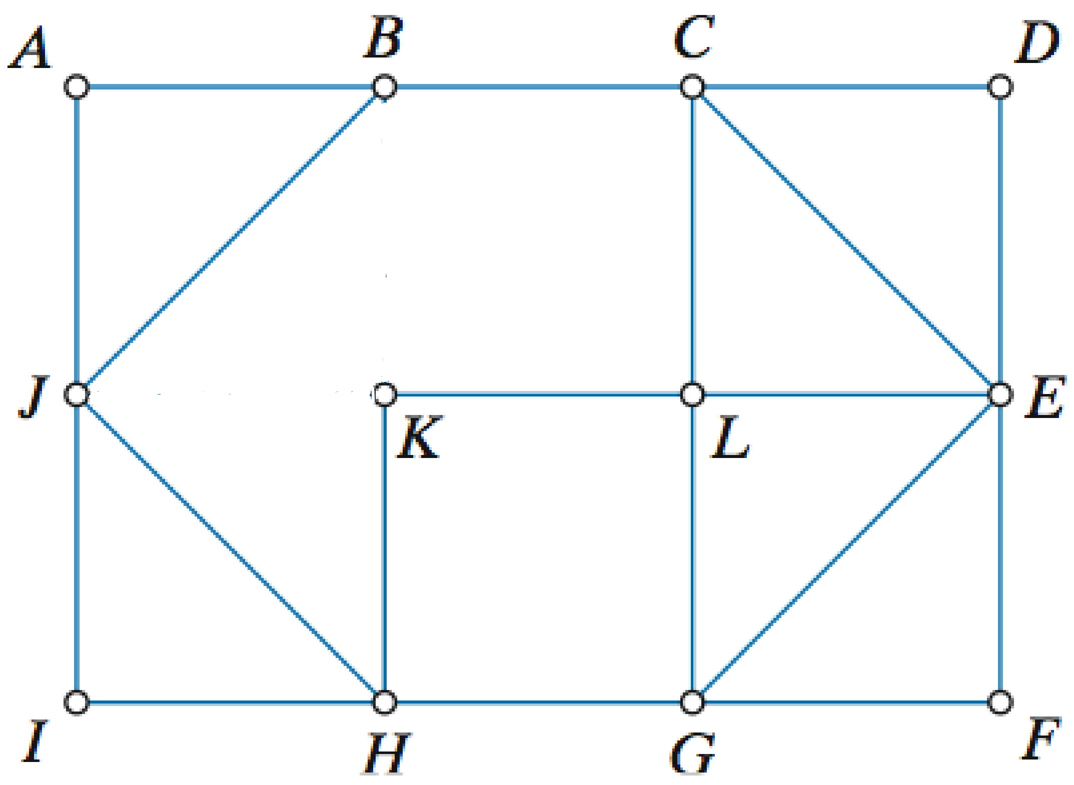
Example: Fleury's Algorithm for Euler Paths…

- **Start.** This graph has two odd vertices, *E* and *J*. We can pick either one as the starting vertex. Let's start at *J*.

Example: Fleury's Algorithm for Euler Paths…

- **Step 1.** From *J* we have five choices, all of which are OK. We'll randomly pick *K*. (Erase *JK* on copy 1, and mark and label *JK* with a 1 on copy 2.)

- **Step 2.** From *K* we have three choices (*B*, *L*, or *H*). Any of these choices is OK. Say we choose *B*. (Now erase *KB* from copy 1 and mark and label *KB* with a 2 on copy 2.)



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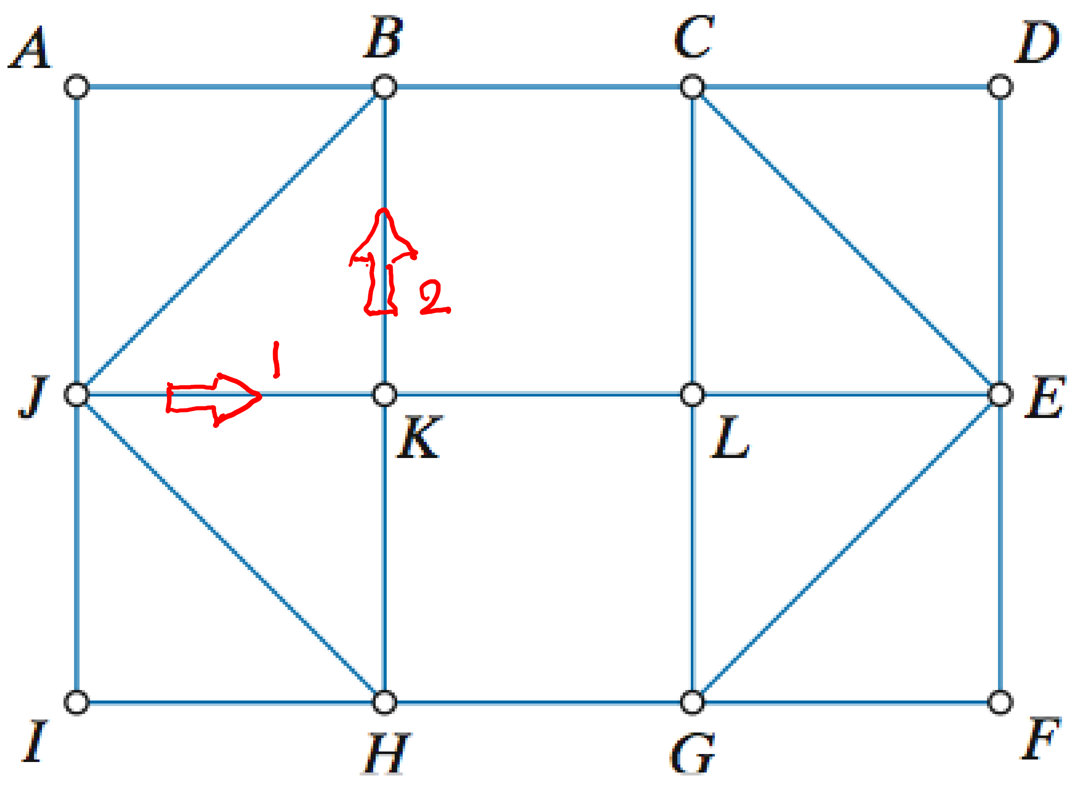

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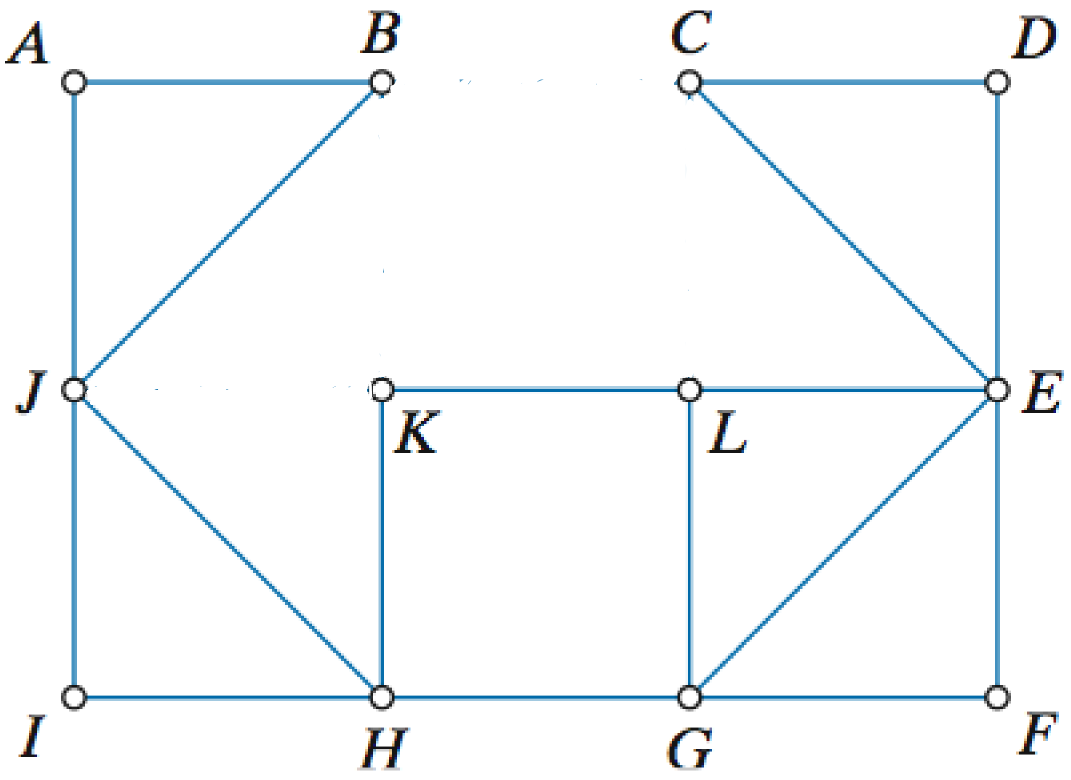
Example: Fleury's Algorithm for Euler Paths…

- **Step 3.** From *B* we have three choices (*A*, *C*, or *J*). Any of these choices is OK. Say we choose *C*. (Now erase *BC* from copy 1 and mark and label *BC* with a 3 on copy 2.)

- **Step 4.** From *C* we have three choices (*D*, *E*, or *L*). Any of these choices is OK. Say we choose *L*. (EML–that's shorthand for erase, mark, and label.)



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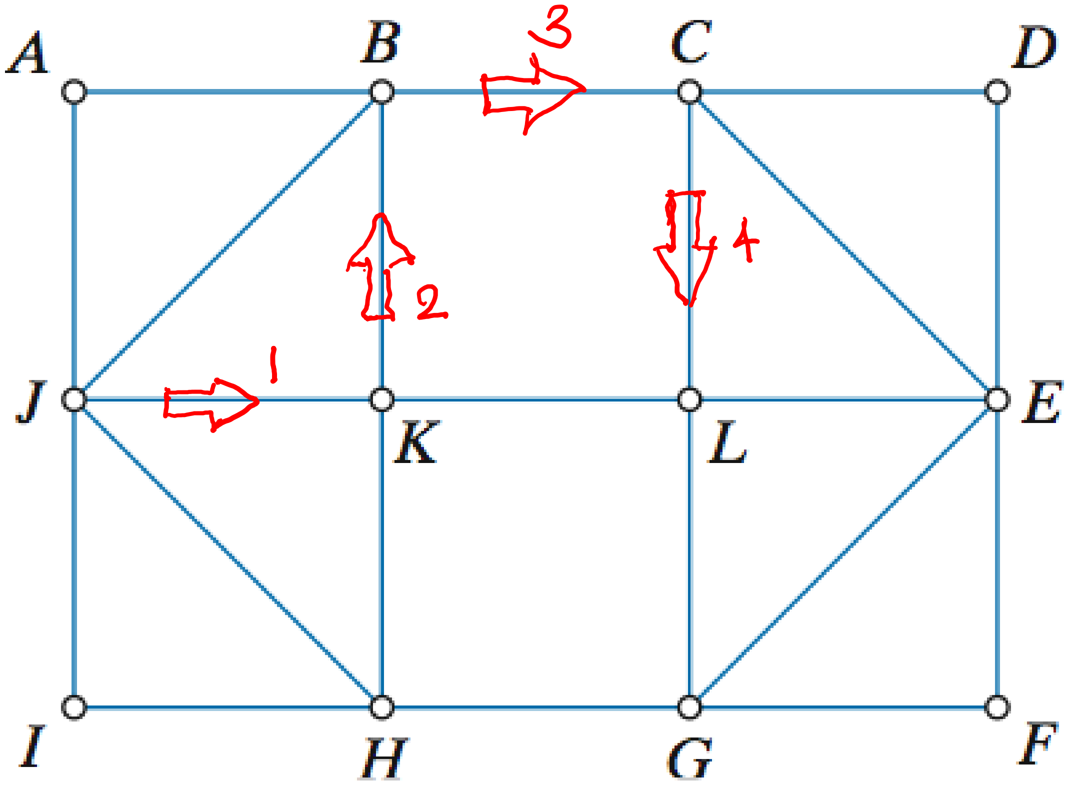

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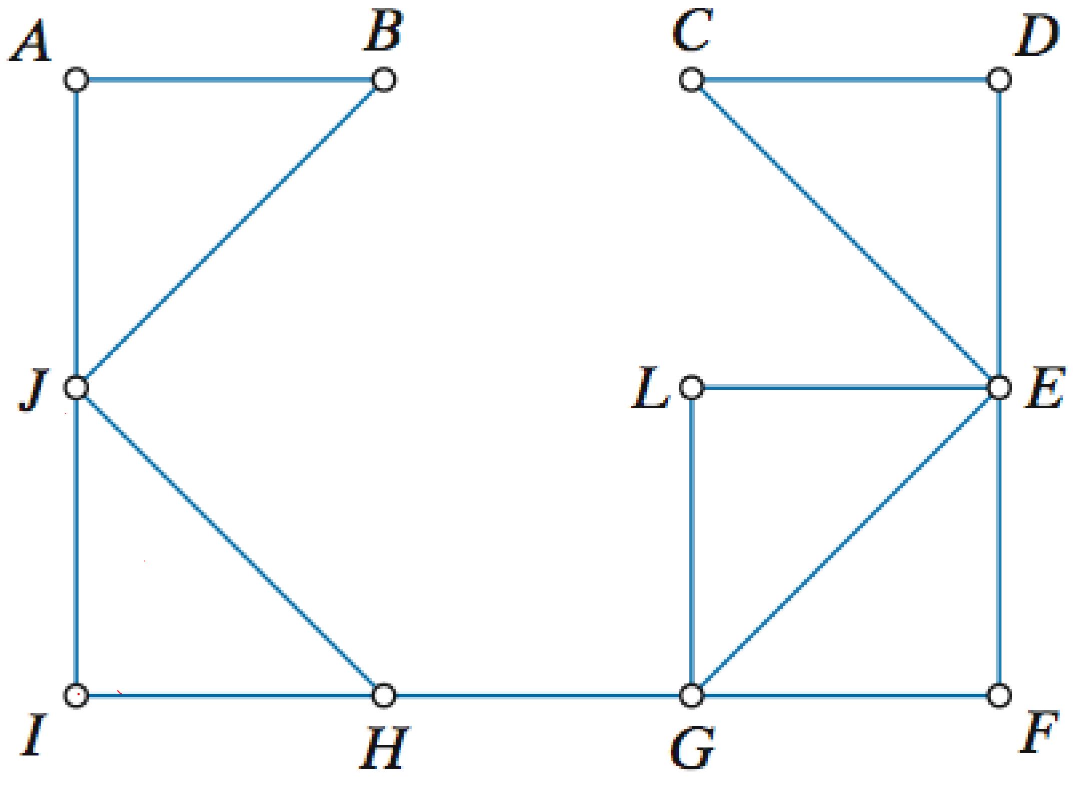
Example: Fleury's Algorithm for Euler Paths…

- **Step 5.** From L we have three choices (*E*, *G*, or *K*). Any of these choices is OK. Say we choose *K*. (EML.)

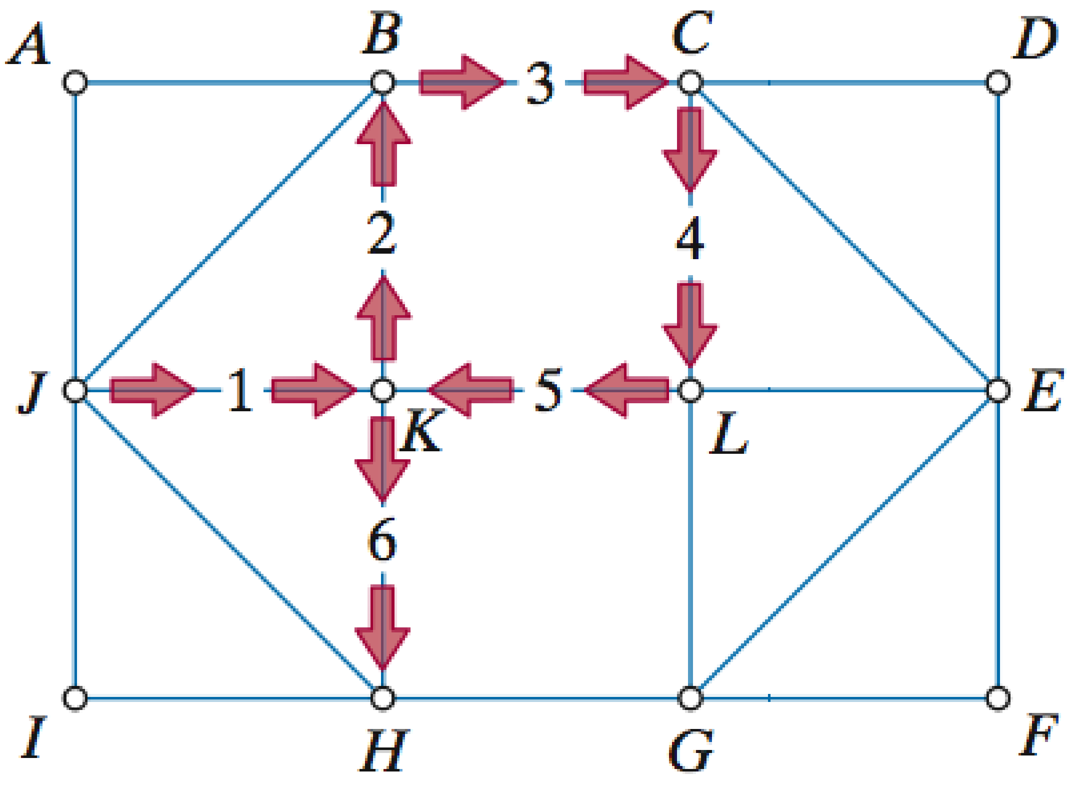
- **Step 6.** From *K* we have only one choice– to H. We choose *H*. (EML.)

Example: Fleury's Algorithm for Euler Paths…

- **Step 7.** From H we have three choices (*G*, *I*, or *J*). We should not choose *G*, as *HG* is a bridge of the yet-to-be-traveled part of the graph



Copy 1 at Step 7



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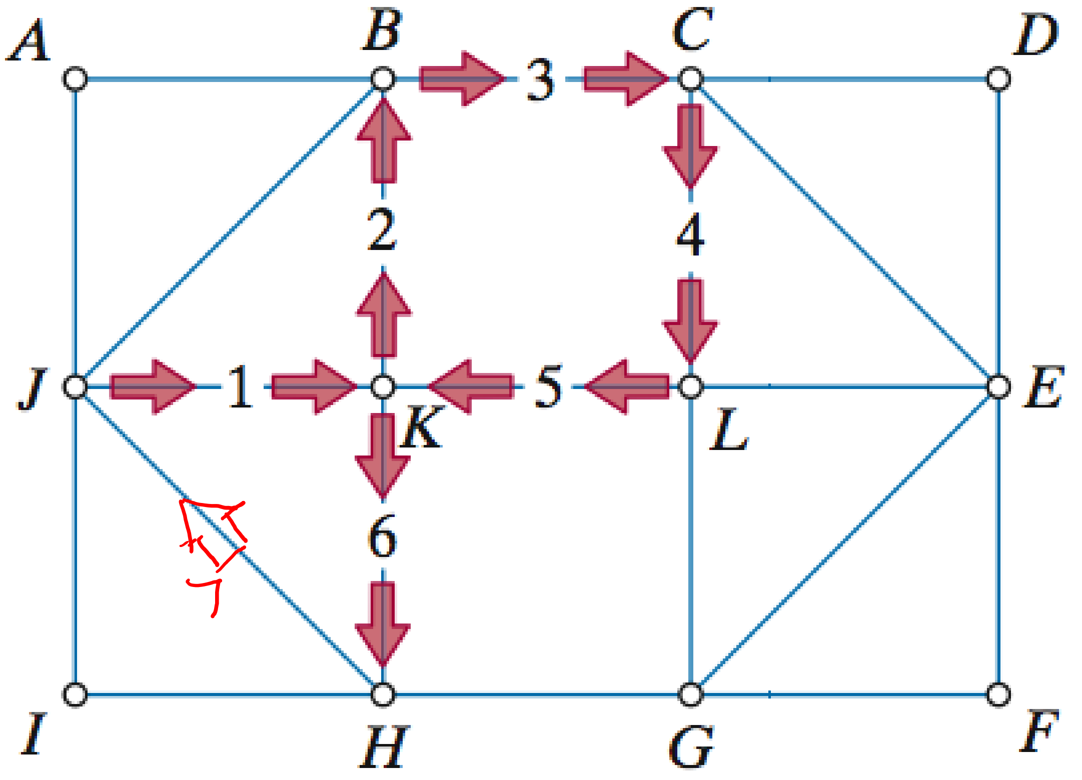
Example: Fleury's Algorithm for Euler Paths…

- **Step 7.** Either of the other two choices is OK. Say we choose J. (EML.)



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Example: Fleury's Algorithm for Euler Paths…

- **Step 8.** From *J* we have three choices (*A*, *B*, or *I*), but we should not choose *I*, as *JI* has just become a bridge. Either of the other two choices is OK. Say we choose *B*. (EML)

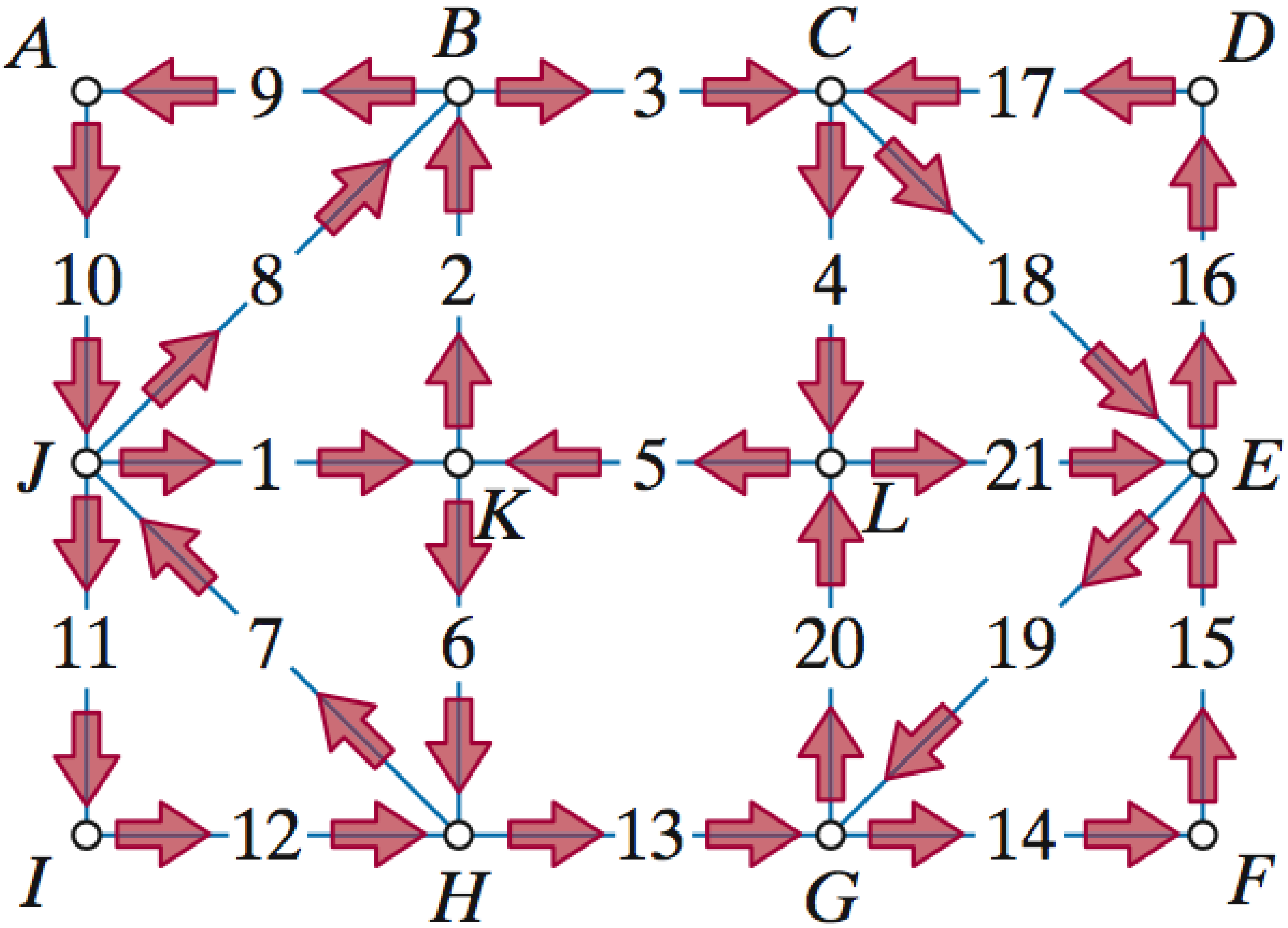
- **Step 9 through 13.** Each time we have only one choice. From *B* we have to go to *A*, then to *J*, *I*, *H*, and *G*.

Example: Fleury's Algorithm for Euler Paths…

- **Step 14 through 21.** Not to belabor the point, let's just cut to the chase. The rest of the path is given by *G*, *F*, *E*, *D*, *C*, *E*, *G*, *L*, *E*. There are many possible endings, and you should find a different one by yourself.

The completed Euler path (one of hundreds of possible ones) is shown in Fig.

Example: Fleury's Algorithm for Euler Paths…



Hamiltonian Paths and Circuits Continued…

- What general class of graphs is guaranteed to have a Hamiltonian circuit?
- Complete graphs of three or more vertices constitute one such class.
- It is easy to construct a Hamiltonian circuit in a complete graph of n vertices.
- Let the vertices be numbered v1, v2 ,. . . , vn. Since an edge exists between any two vertices, we can start from v1 and traverse to v2, and v3, and so on to vn, and finally from vn to v1. This is a Hamiltonian circuit.

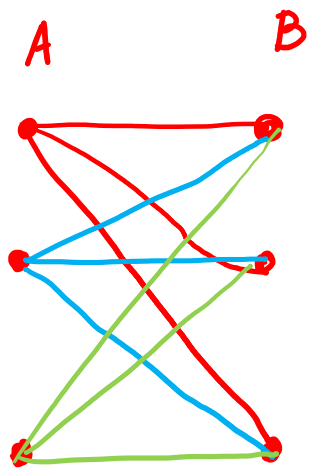
Number of Hamiltonian Circuits in a Graph

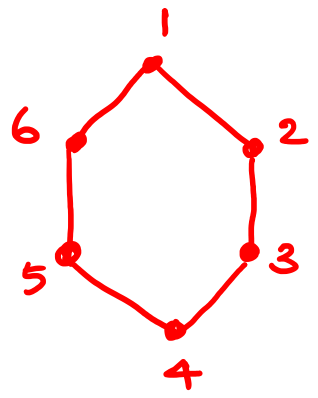
- A given graph may contain more than one Hamiltonian circuit.
- **THEOREMS**
  - A sufficient (but by no means necessary) condition for a simple graph G to have a Hamiltonian circuit is that the degree of every vertex in G be at least n/2, where n is the number of vertices in G.
  - In a complete graph with n vertices there are $(n - 1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.
  - The total number of different (not edge disjoint) Hamiltonian circuits in a complete graph of n vertices is $(n - 1)!/2$.
  - A bipartite graph is Hamiltonian only if $n = m \geq 2$, where n and m are the cardinality of vertex partitions

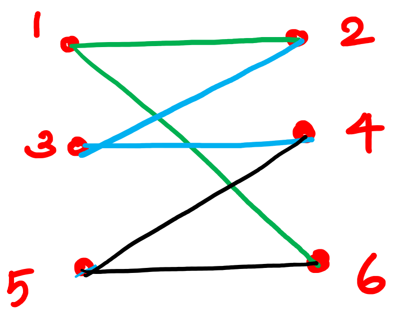
- A bipartite graph is Hamiltonian only if n = m ≥ 2, where n and m are the cardinality of vertex partitions
- **Proof:**
- Let $K_{m,n}$ have bipartition X, Y where $|x| = n$ and $|y| = m$.
- Let C be a Hamiltonian cycle of $K_{m,n}$
- Each cycle in $K_{m,n}$ has equal length and thus the cycle visits X and Y equally many times.
- Then C visits X and Y equally many times and is incident with all the vertices. Therefore, $|x| = |y|$

Bipartite graph

- Graph whose vertices can be divided into two parts A and B such that every edge connects a vertex in A to a vertex in B.





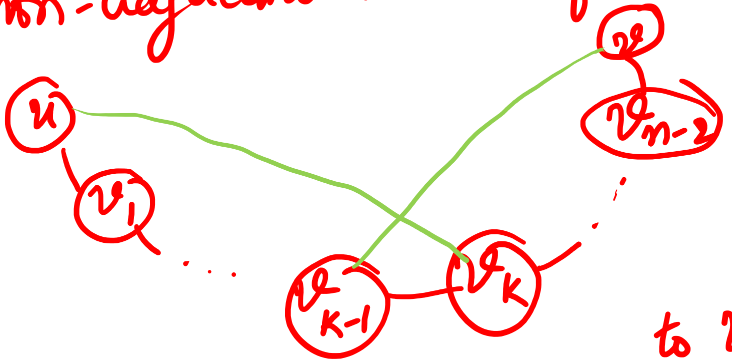


- The total number of different (not edge disjoint) Hamiltonian circuits in a complete graph of n vertices is $(n - 1)!/2$.
- Proof
- This follows from the fact that starting from any vertex we have $n - 1$ edges to choose from the first vertex, $n - 2$ from the second, $n - 3$ from the third, and so on.
- These being independent choices, we get $(n - 1)!$ possible number of choices.
- This number is divided by 2, because each Hamiltonian circuit has been counted twice

If G is a simple graph with $n\ (\geq 3)$ vertices and if $d(v) \geq n/2$ for each vertex v, then G is Hamiltonian.

Proof by Contradiction

Let G be such a counter example to the theorem so that no graph on n vertices with more edges than G is also a counter example. Let u, v be 2 non-adjacent vertices of G. Then, there is a Hamiltonian path joining u & v in G.



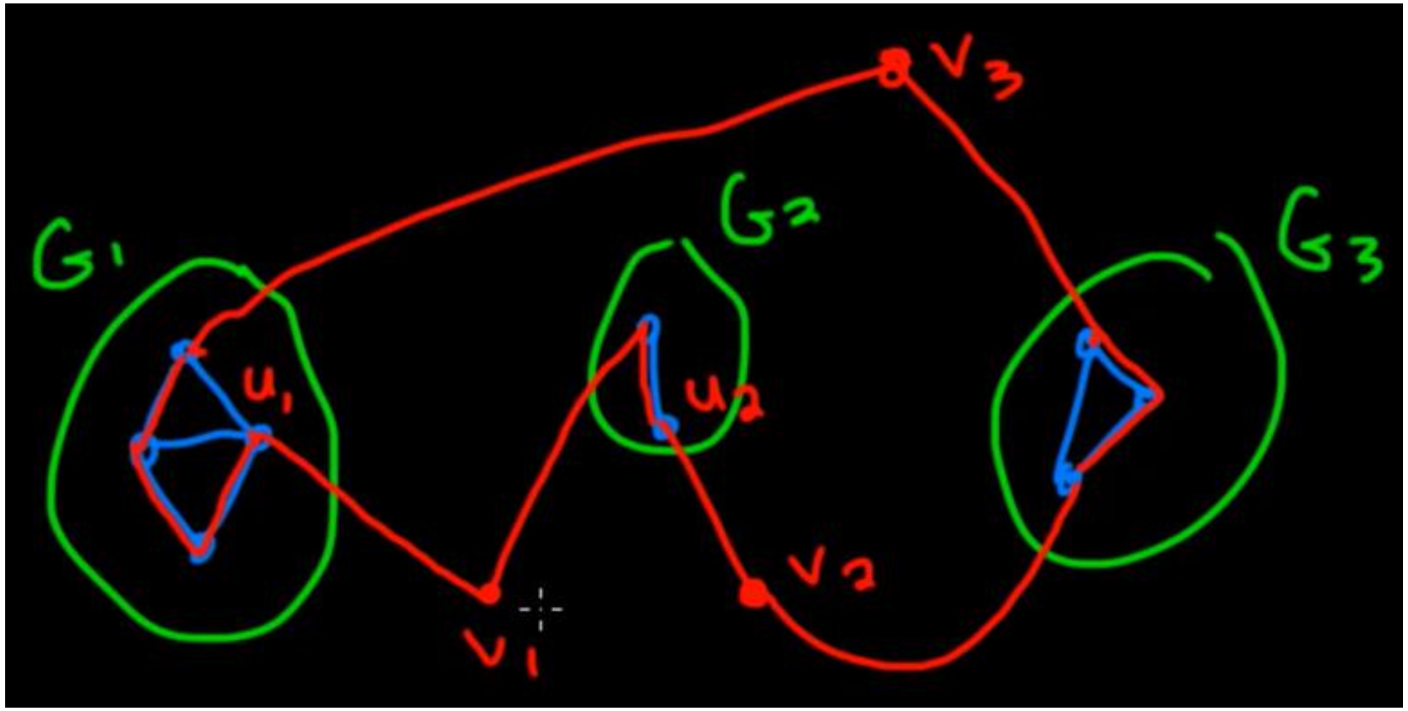
Let $d(u) = k \geq n/2$

Now, we prove that if u is adjacent to v_k, then v_{k-1} can't be adjacent to v. If possible, let v be adjacent to v_{k-1}. Then, we have the Hamiltonian cycle,

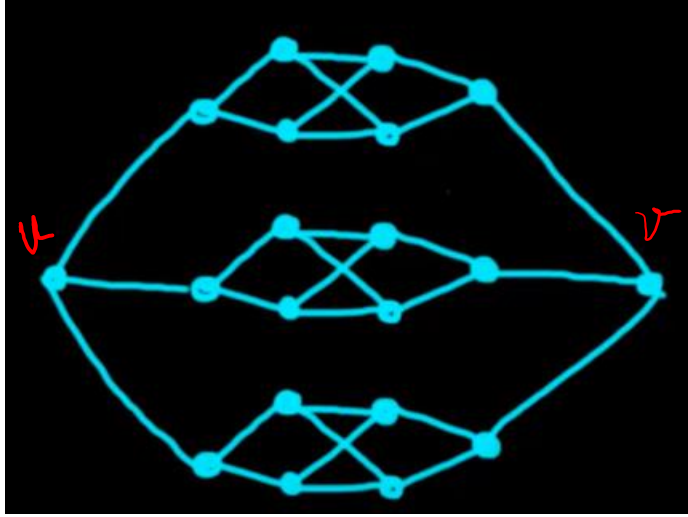
$v v_{n-2} \ldots v_k u v_1 \ldots v_{k-1} v$. Since, we assumed that G is not Hamiltonian, this is not possible. So, v is not adjacent to v_{k-1}. Since u is adjacent to k vertices, v can't be adjacent to at least k of the $n-1$ vertices. Thus $d(v) \leq n-1-k \leq n-1-n/2 = n/2 - 1$; Contradiction. Hence, G must be Hamiltonian

- Let G be a simple graph G on n (≥ 3) vertices. Suppose u, v ∈ V(G) and (u, v) ∉ E(G) and d(u) + d(v) $\geq$ n then G is Hamiltonian if G + (u, v) is Hamiltonian.
- **Proof:** G is Hamiltonian then obviously G+(u, v) is Hamiltonian.
- Given G+(u,v) is ian, we need to prove that G is Hamiltonian.
- Suppose G is not Hamiltonian. Then, there is a Hamiltonian path or spanning path in G joining u and v. If u is adjacent to v_k, then v can't be adjacent to v_{k-1}. Let d(u) = k. So, v is not adjacent to at least k of the n-1 vertices.
- Thus d(v) $\leq$ n – 1 – k
- d(u) + d(v) $\leq$ n – 1 – k + k $\leq$ n – 1 which is a contradiction.
- So, G is Hamiltonian.

- If G is Hamiltonian, then for every non empty subset S ⊆ V(G), number of connected components c(G-S) $\leq |S|$
- **Proof:**
- G is Hamiltonian, let S ⊆ V(G), $S \neq \oint$ and let w∈ S and $C: w \longrightarrow w$ be a Hamiltonian cycle of G.
- Assume c(G-S) = k and let G_1, G_2, … G_k be the connected components of G-S.
- If k=1, $|S| \geq 1$
- So for k>1, consider an orientation of C.
- Let u_i be the last vertex of C that belongs to G_i. Let v_i be the next vertex of C after u_i. $v_i \notin V(Gi)$ by the definition of u_i and $v_i \notin V(Gj)$ since the components are mutually disjoint. So, $v_i \in S$.
- Also, $v_i \neq v_j$ for $i \neq j$, because C is a Hamiltonian cycle and $u_i v_i \in E(G)$ for all i.
- For each i = 1, 2, … , k there is a $v_i \in S$ and $|S| \geq k$
- K = c(G-S) $\leq |S|$

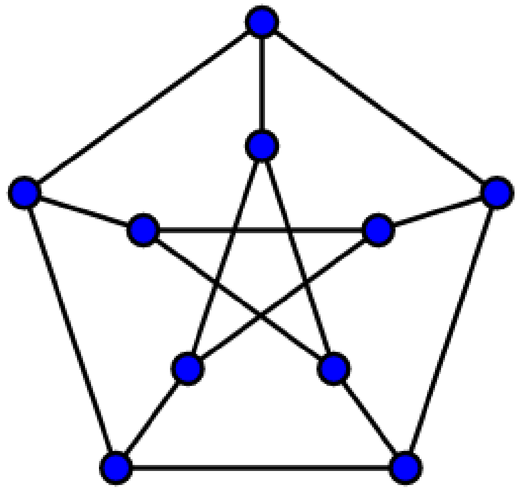


G1
G2
G3
u1
u2
v1
v2
v3



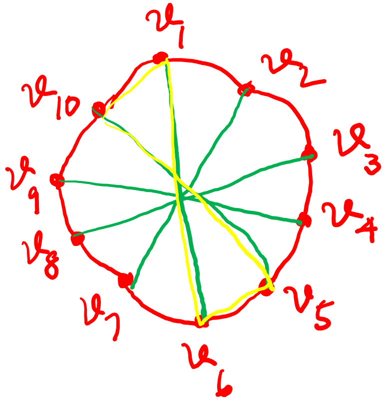
- S = {u, v}
- c(G-S) = 3 but |S| = 2.
- Hence, the graph is not Hamiltonian

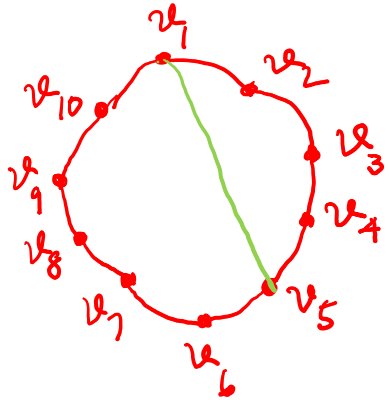
Petersen graph



- Not bipartite as cycles of odd length is present.
- Consists of Hamiltonian path
- It has no cycles on ≤ 4 vertices.
- It is not Hamiltonian.

- Proof by contradiction: Suppose C is a Hamiltonian cycle of the Petersen graph. |V| = 10, |E| = 15 and $d(v_i) = 3$ for all *i*.
- C is a 10-cycle together with 5 chords (edges that connect non-adjacent vertices in C).
- If each chord joins vertices opposite in C, then there exists a 4-cycle. It is contradiction to Petersen graph.
- There is a chord that connects vertices at distance <5 in C. These vertices must be at distance 4 in C. (Without loss of generality $v_1v_5 \in E(G)$).
- Now v_6 cannot be incident with any chord edge without making a cycle of length ≤ 4. It is a contradiction to Petersen graph.
- Hence it is not Hamiltonian.





Traveling-salesman Problem

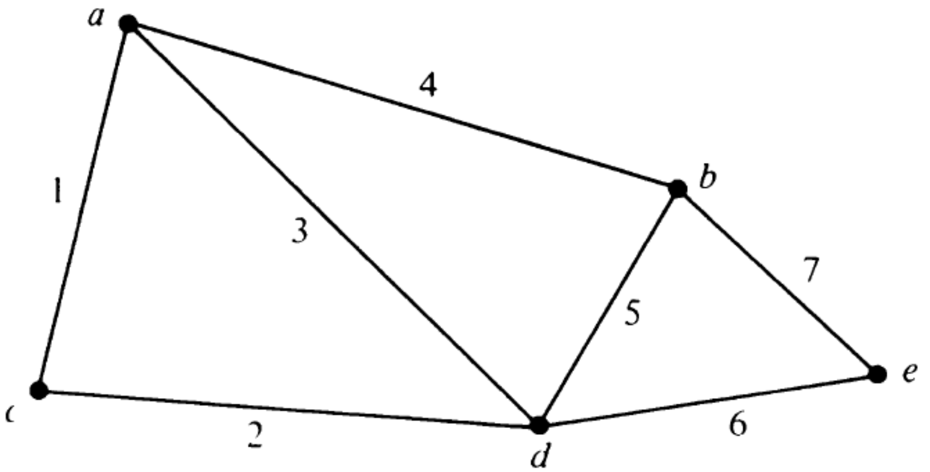
- A salesman is required to visit a number of cities during a trip. Given the distances between the cities, in what order should he travel so as to visit every city precisely once and return home, with the minimum mileage traveled?
- Representing the cities by vertices and the roads between them by edges, we get a graph. In this graph, with every edge e_i there is associated a real number (the distance in miles, say), $w(e_i)$.
- In our problem, if each of the cities has a road to every other city, we have a complete weighted graph. This graph has numerous Hamiltonian circuits, and we are to pick the one that has the smallest sum of distances (or weights).
- Theoretically, the problem of the traveling salesman can always be solved by enumerating all $(n - l)!/2$ Hamiltonian circuits, calculating the distance traveled in each, and then picking the shortest one. However, for a large value of n, the labor involved is too great even for a digital computer.
- You can be greedy in selecting edges.

PLANAR AND DUAL GRAPHS

Combinatorial versus Geometric Graphs

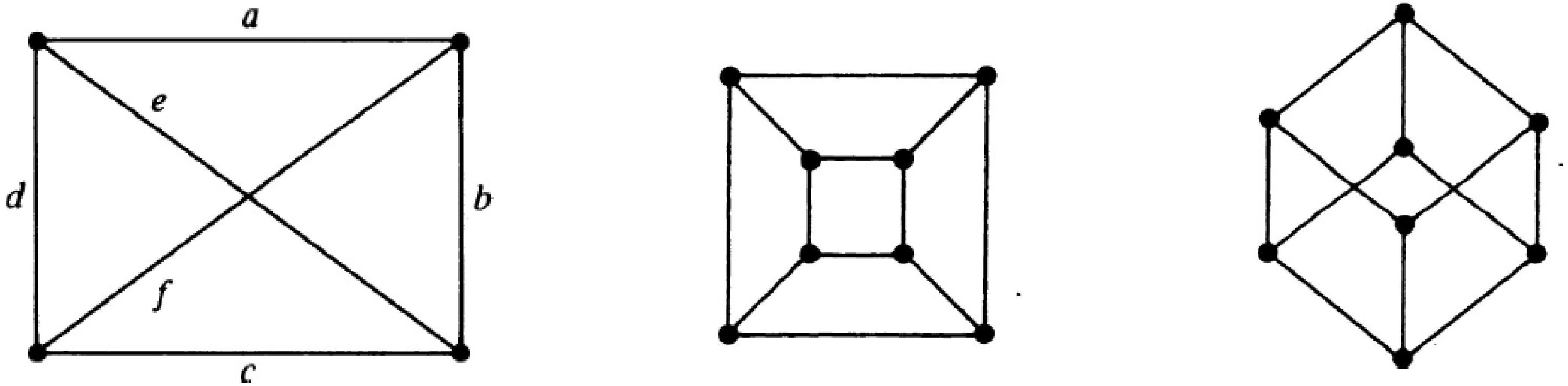
- An abstract graph (or combinatorial) G_1 can be defined as, $G_1 = (V, E, \Psi)$
- where the set V consists of the five objects named a, b, c, d, and e, that is, V = {a, b, c, d, e},
- the set E consists of seven objects (none of which is in set V) named 1, 2, 3, 4, 5, 6, and 7, that is, E = {1, 2, 3, 4, 5, 6, 7},
- and the relationship between the two sets is defined by the mapping Ψ
- Here, the symbol 1 → (a, c) says that object 1 from set E is mapped onto the (unordered) pair (a, c) of objects from set V.

$$\Psi = \begin{bmatrix} 1 \longrightarrow (a, c) \\ 2 \longrightarrow (c, d) \\ 3 \longrightarrow (a, d) \\ 4 \longrightarrow (a, b) \\ 5 \longrightarrow (b, d) \\ 6 \longrightarrow (d, e) \\ 7 \longrightarrow (b, e) \end{bmatrix}.$$

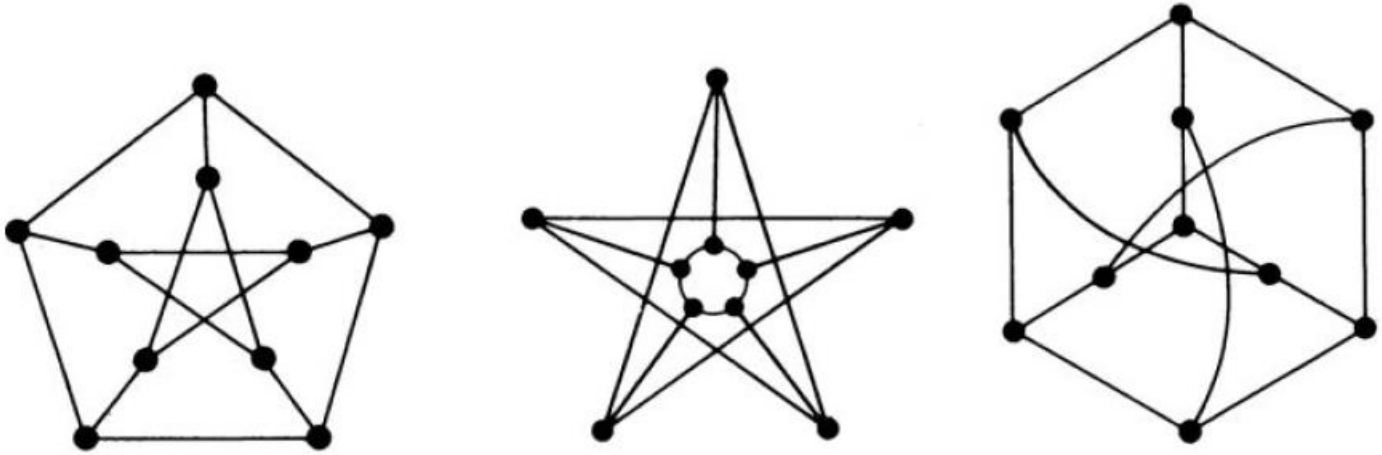


Planar Graphs

- A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.
- A graph that cannot be drawn on a plane without a crossover between its edges is called nonplanar.
- A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.
- Thus, to declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane.
- Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is nonplanar.
- An embedding of a planar graph G on a plane is called a plane representation of G.



Planar graphs

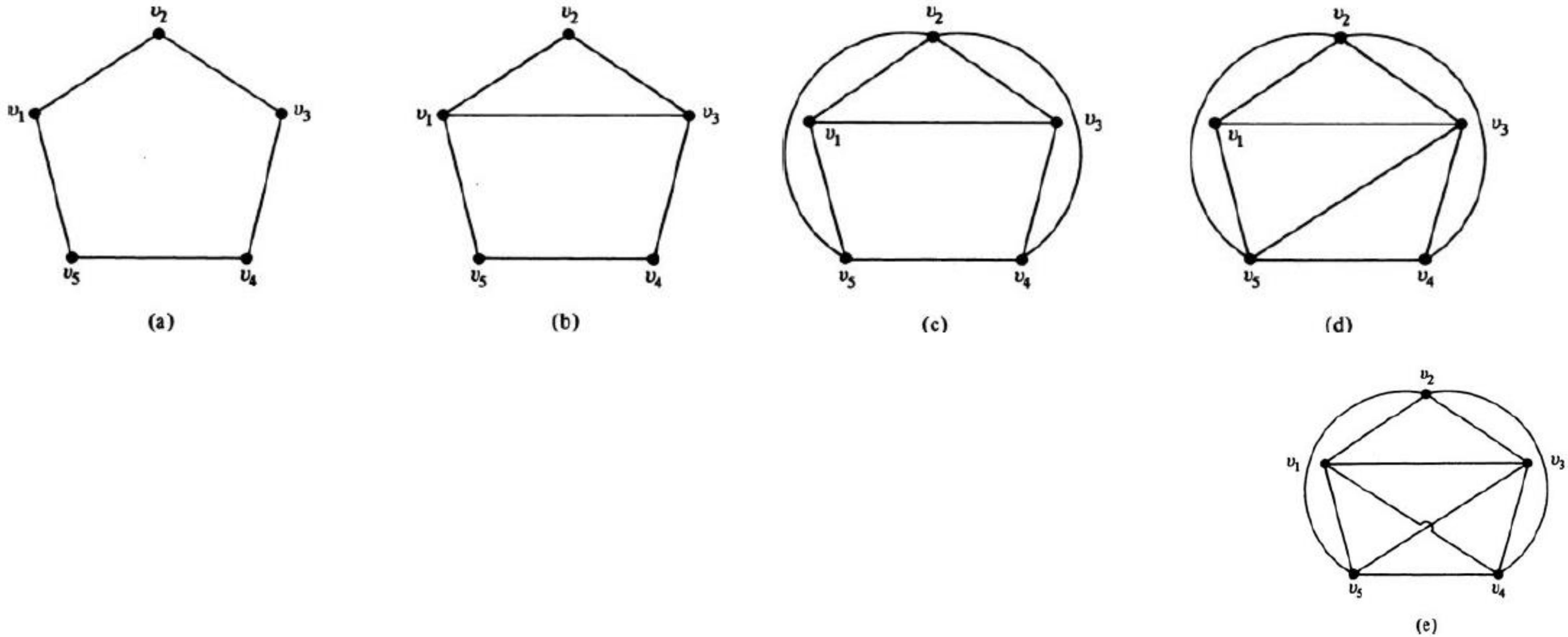


Nonplanar graphs

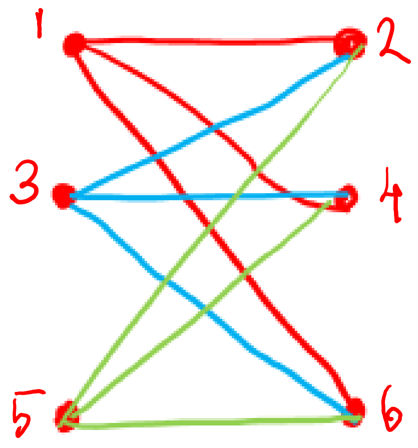
Kuratowski’s Graphs

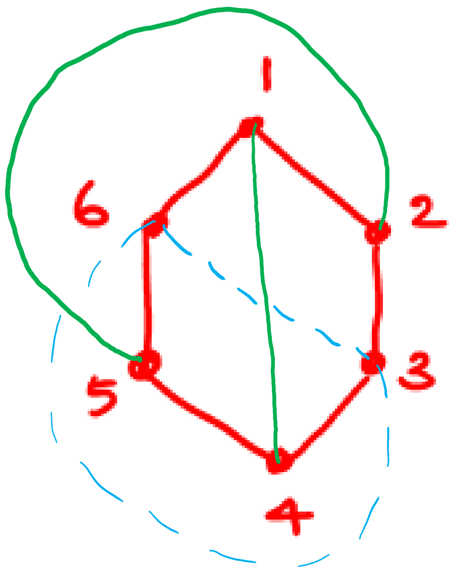
- Theorem 1: The complete graph of five vertices is nonplanar.
- Theorem 2: $K_{3,3}$ is nonplanar.

- Theorem 1: The complete graph of five vertices is nonplanar.
- Let the five vertices in the complete graph be named v1 v2, v3, v4, and v5.



- Theorem 2: $K_{3,3}$ is nonplanar.

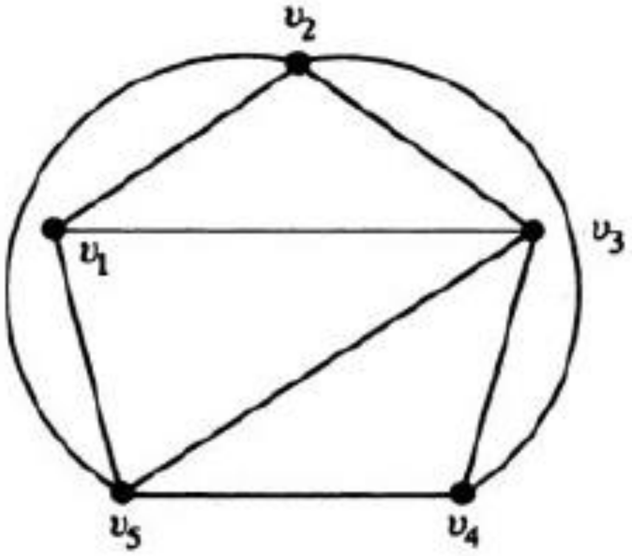


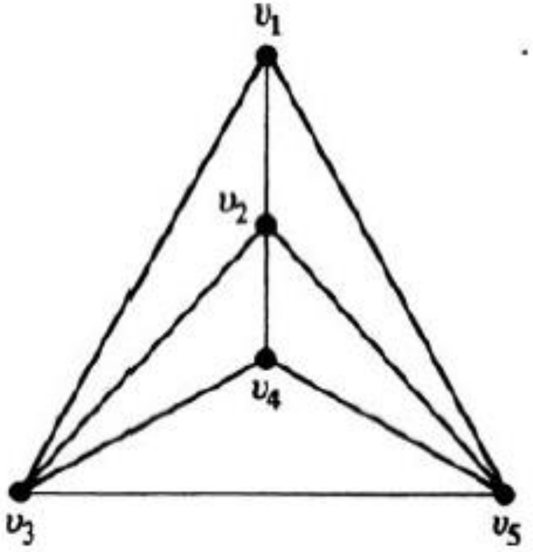


Properties common to the two graphs of Kuratowski

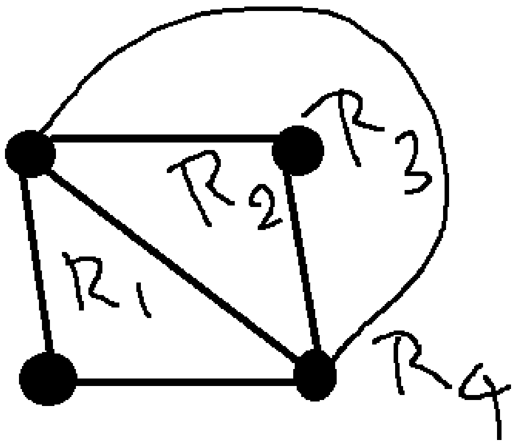
- Both are regular graphs.
- Both are nonplanar.
- Removal of one edge or a vertex makes each a planar graph.
- Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges. Thus both are the simplest nonplanar graphs.

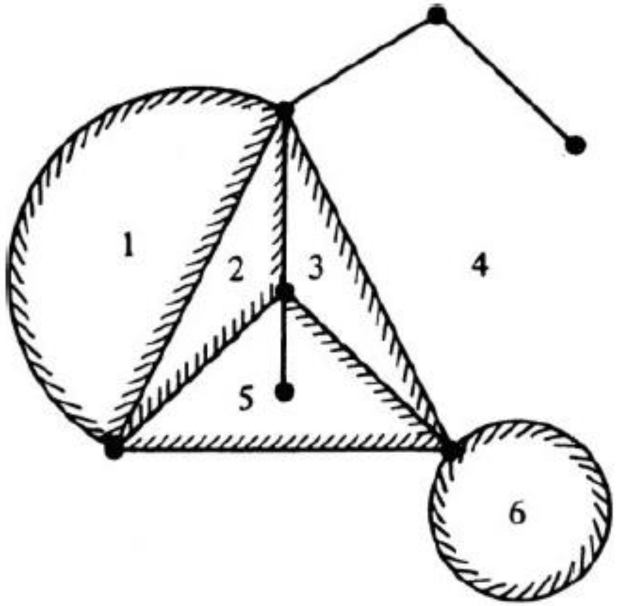
- THEOREM 3: Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.



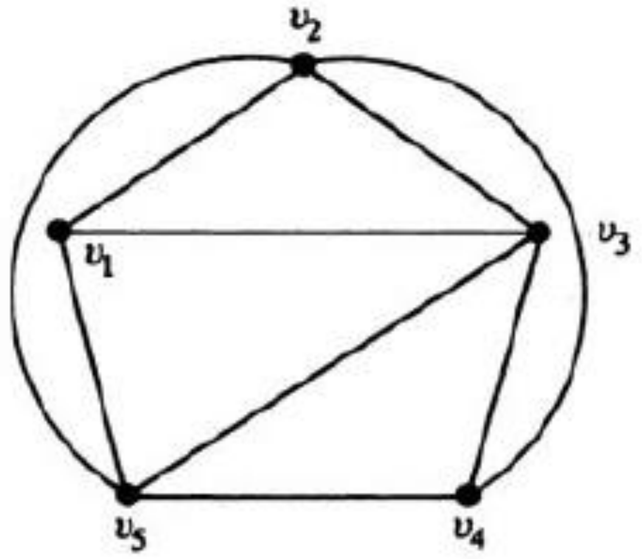


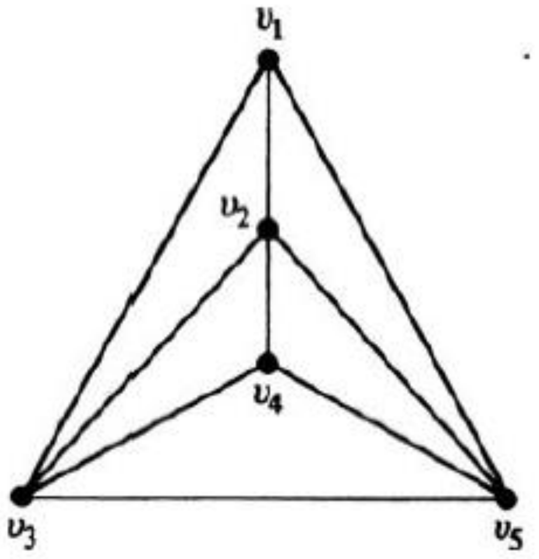
- A plane graph divides the plane into regions (also called windows, faces, or meshes).
- The boundary of a region in a plane graph is characterized by the set of vertices and edges that outine it.
- The portion of the plane lying outside a graph embedded in a plane, such as region 4, is infinite in its extent. Such a region is called the infinite, unbounded, outer, or exterior region for that particular plane representation. Infinite region is also characterized by a set of edges (or vertices).
- Every tree has only region which is the unbounded region.
- Isomorphic graphs can be drawn in different ways in a plane.
- A region is a property of the specific plane representation of a graph and not of an abstract graph per se.





- By changing the embedding of a given planar graph, we can change the infinite region.
- Figures show two different embeddings of the same graph. The finite region v1 v3 v5 in first figure becomes the infinite region in second.
- Any region can be made the infinite region by proper embedding.

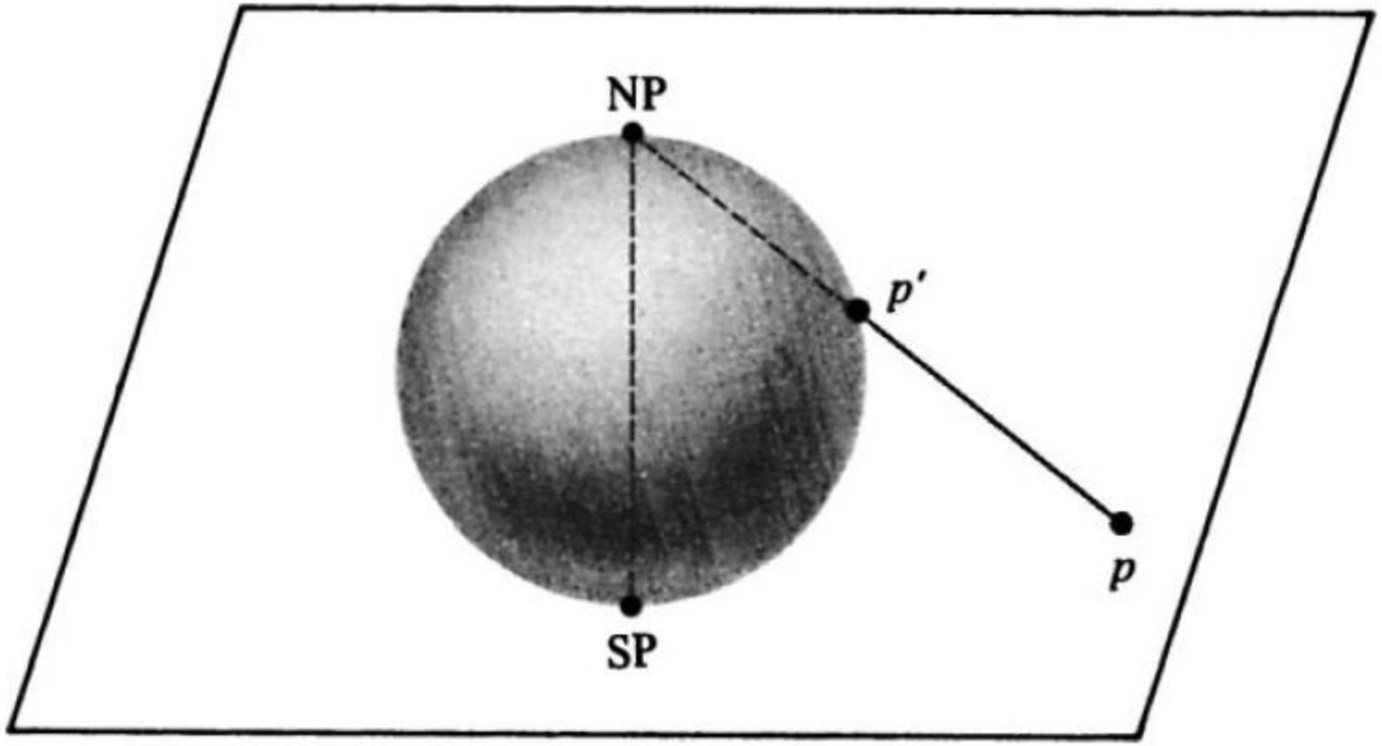




Embedding on a Sphere

- To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere.
- It is accomplished by stereographic projection of a sphere on a plane.
- Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole).
- Corresponding to any point p on the plane, there exists a unique point p′ on the sphere and vice versa, where p′ is the point at which the straight line from point p to point NP intersects the surface of the sphere.
- Thus there is a one-to-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane correspond to the point NP on the sphere.

Stereographic projection



- **THEOREM 4:** A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.
- A planar graph embedded in the surface of a sphere divides the surface into different regions.
- Each region on the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP.
- Now it is clear that by suitably rotating the sphere we can make any specified region map onto the infinite region on the plane.
- **THEOREM 5:** A planar graph may be embedded in a plane such that any specified region (i.e., specified by the edges forming it) can be made the infinite region.

Euler’s Formula

- **THEOREM 6:** A connected planar graph with n vertices and e edges has e − n + 2 regions.
- Proof:
  - It will suffice to prove the theorem for a simple graph, because adding a self-loop or a parallel edge simply adds one region to the graph and simultaneously increases the value of e by one. We can also remove all edges that do not form boundaries of any region. Addition (or removal) of any such edge increases (or decreases) e by one and increases (or decreases) n by one, keeping the quantity e − n unaltered.

Proof…

- Since any simple planar graph can have a plane representation such that each edge is a straight line (Theorem 3), any planar graph can be drawn such that each region is a polygon (a polygonal net).
- Let the polygonal net representing the given graph consist of f regions or faces, and let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3 \cdot k_3 + 4 \cdot k_4 + 5 \cdot k_5 + \cdots + r \cdot k_r = 2 \cdot e,$$

- where k_r is the number of polygons, with maximum edges. Also,

$$k_3 + k_4 + k_5 + \cdots + k_r = f.$$

- The sum of all angles subtended at each vertex in the polygonal net is $2\pi n$.

Proof…

- Recalling that the sum of all interior angles of a p-sided polygon is π(p − 2), and the sum of the exterior angles is π(p + 2), let us compute the sum of angles as the grand sum of all interior angles of f − 1 finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\begin{aligned} &\pi(3-2)\cdot k_3 + \pi(4-2)\cdot k_4 + \cdots + \pi(r-2)\cdot k_r + 4\pi \\ &= \pi(2e - 2f) + 4\pi. \end{aligned}$$

- Equating, we get

$$\begin{aligned} 2\pi(e-f) + 4\pi &= 2\pi n, \\ e - f + 2 &= n. \end{aligned}$$

- Therefore, the number of regions is $f = e - n + 2.$

COROLLARY

In any simple, connected planar graph with f regions, n vertices, and e edges ($e > 2$), the following inequalities must hold :

$$e \geq \frac{3}{2}f,$$

$$e \leq 3n - 6.$$

Proof: Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f$$

or

$$e \geq \frac{3}{2}f.$$

Substituting for f from Euler's formula in inequality

$$e \geq \frac{3}{2}(e - n + 2)$$

or

$$e \leq 3n - 6. \quad \blacksquare$$

in the case of K_5, the complete graph of five vertices

$$n = 5, \qquad e = 10, \qquad 3n - 6 = 9 < e.$$

the graph violates inequality and hence it is not planar.

- Kuratowski's second graph, $K_{3,3}$, satisfies the inequality, because
  - e = 9
  - 3n − 6 = 3·6 − 6 = 12.
- To prove the nonplanarity of Kuratowski's second graph, we make use of the additional fact that no region in this graph can be bounded with fewer than four edges. Hence, if this graph were planar, we would have

$$2e \geq 4f,$$

and, substituting for f from Euler's formula,

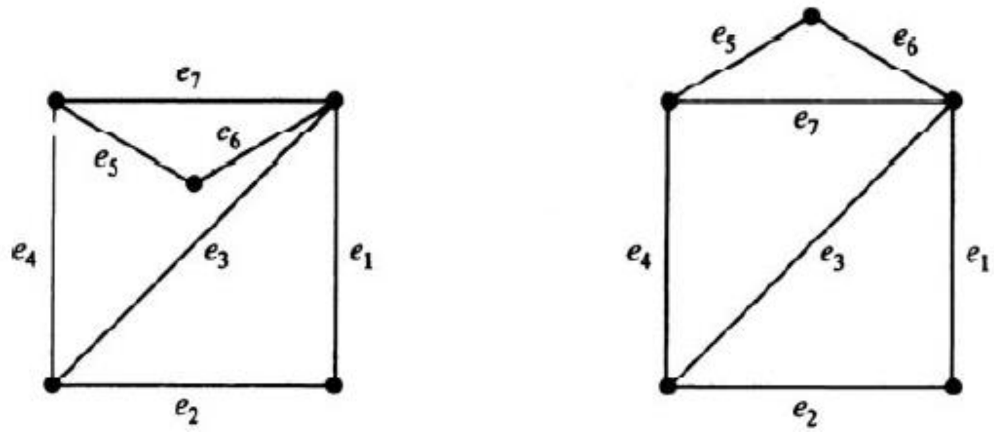
$$2e \geq 4(e - n + 2),$$

or $$2 \cdot 9 \geq 4(9 - 6 + 2),$$

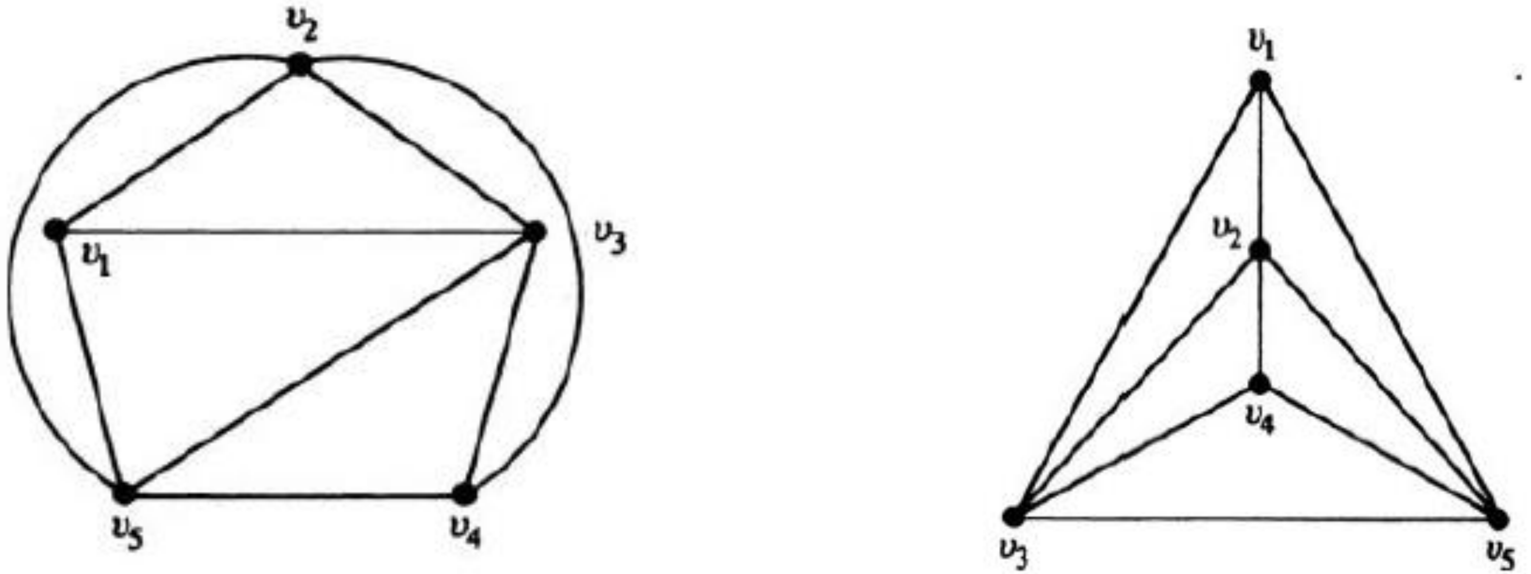
or $$18 \geq 20, \text{ a contradiction.}$$

Plane Representation and Connectivity

- In a disconnected graph the embedding of each component can be considered independently. Therefore, a disconnected graph is planar if and only if each of its components is planar.
- In a separable (or 1-connected) graph the embedding of each block (i.e., maximal non-separable subgraph) can be considered independently.
- Hence a separable graph is planar if and only if each of its blocks is planar.
- Therefore, in questions of embedding or planarity, one need consider only nonseparable graphs.
- Two embeddings of a planar graph on spheres are not distinct if the embeddings can be made to coincide by suitably rotating one sphere with respect to the other and possibly distorting regions.
- If of all possible embeddings on a sphere no two are distinct, the graph is said to have a unique embedding on a sphere (or a unique plane representation).



Two distinct plane representations of the same graph.



Unique plane representations of the same graph.

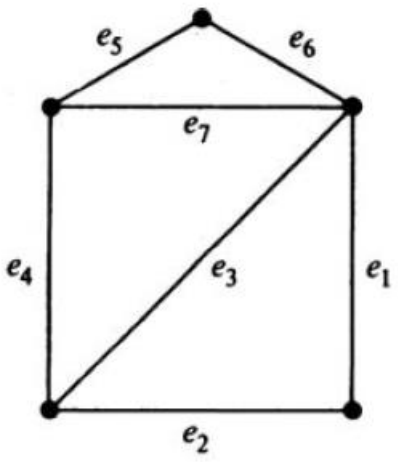
- **THEOREM 7:**
- The spherical embedding of every planar 3-connected graph is unique.

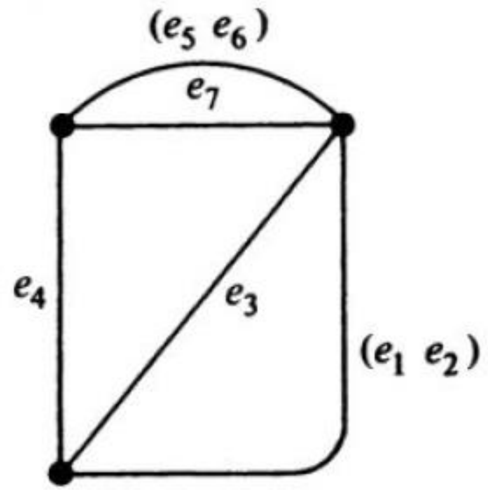
DETECTION OF PLANARITY

- **Elementary Reduction**
- Step 1: Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set $G = \{G_1, G_2, \ldots, G_k\}$, where each G_i is a non-separable block of G. Then we have to test each G_i for planarity.
- Step 2: Since addition or removal of self-loops does not affect planarity, remove all self-loops.
- Step 3: Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.
- Step 4: Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series.
- Repeated application of steps 3 and 4 will usually reduce a graph drastically.

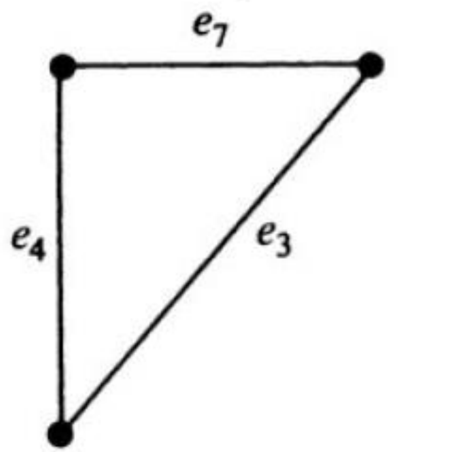
- **THEOREM 8:**
- Graph H_i (graph obtained after elementary reduction) is
    1. A single edge, or
    2. A complete graph of four vertices, or
    3. A non-separable, simple graph with $n \geq 5$ and $e \geq 7$.
- All H_i falling in categories 1 or 2 are planar and need not be checked further.
- Therefore, we need to investigate only simple, connected, non-separable graphs of at least five vertices and with every vertex of degree three or more.
- Next, we can check to see if $e \leq 3n - 6$. If this inequality is not satisfied, the graph Hi is nonplanar. If the inequality is satisfied, we have to test the graph further using Kuratowski's theorem (Theorem 9).

Series-parallel reduction of a graph

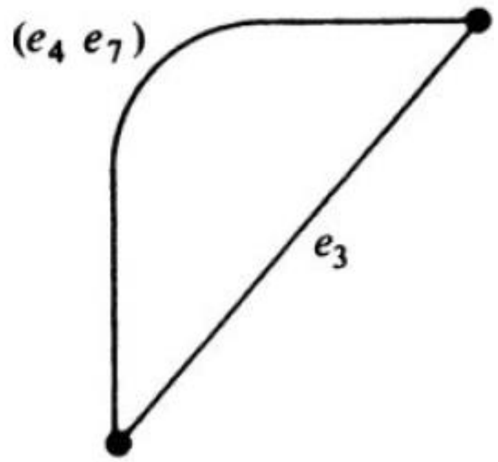




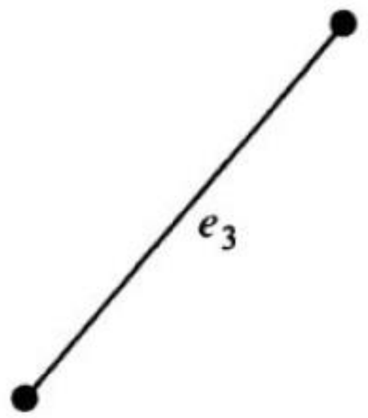
(a) Series Reduced



(b) Parallel Reduced

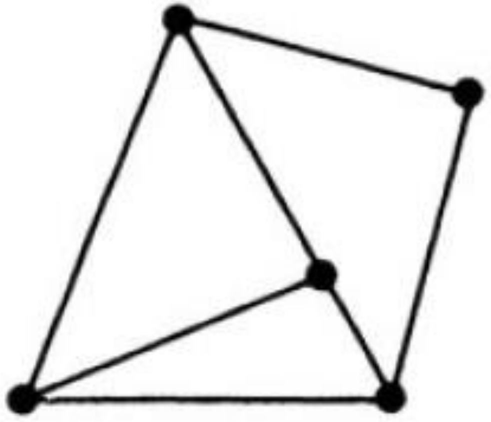
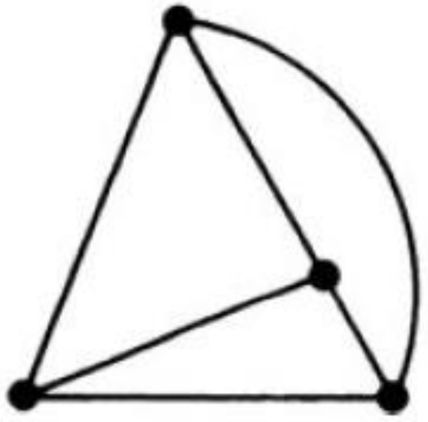
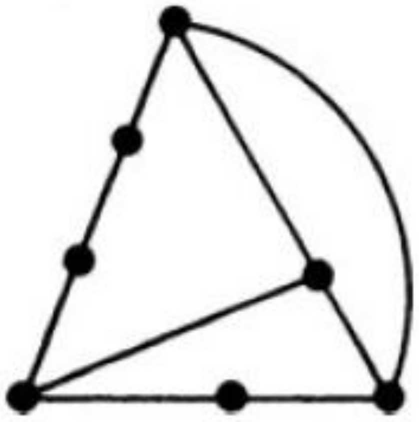


(c) Series Reduced



(d) Parallel Reduced

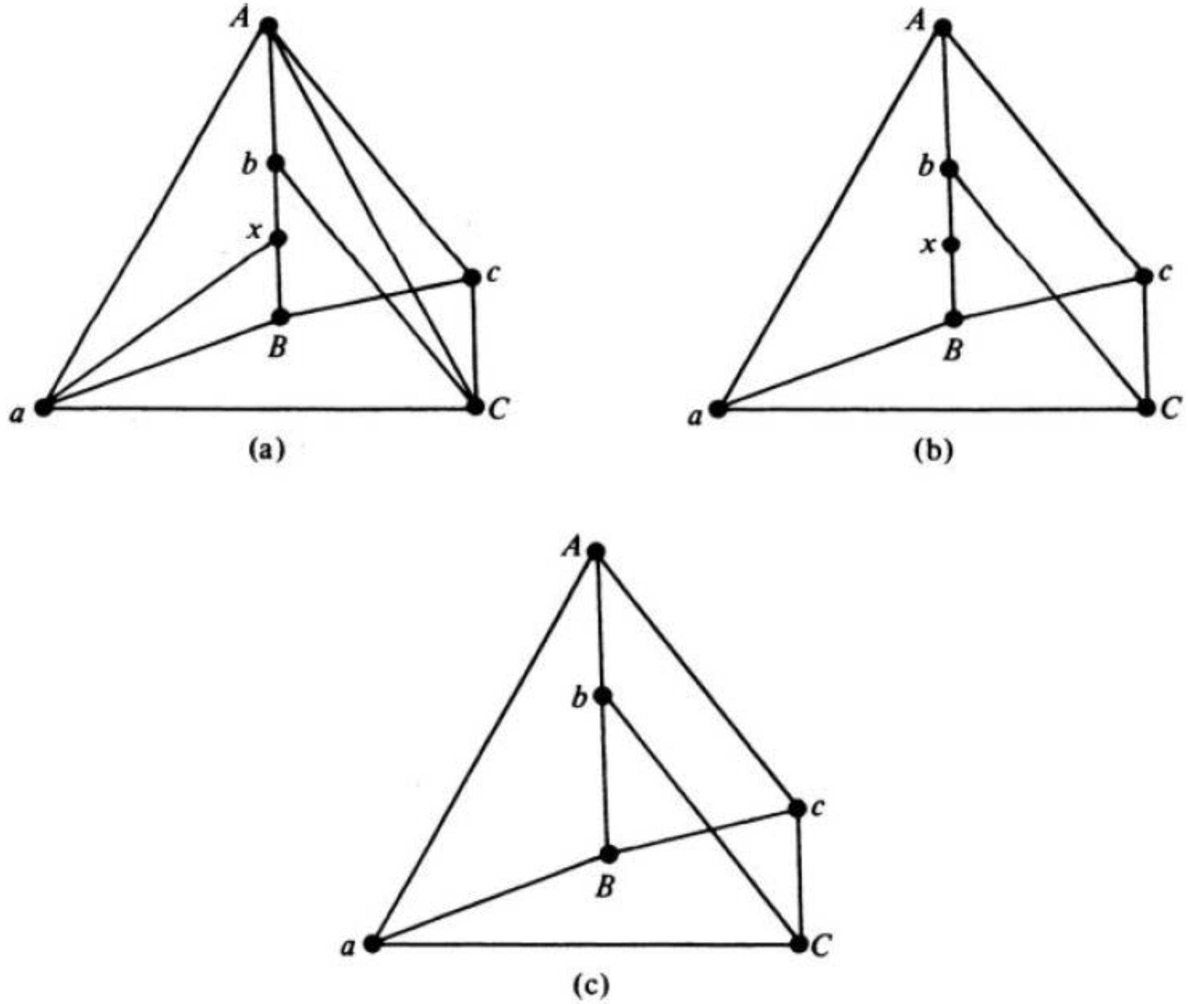
- **Homeomorphic Graphs:**
  - Two graphs are said to be homeomorphic if one graph can be obtained from the other by the creation of edges in series (i.e., by insertion of vertices of degree two) or by the merger of edges in series.
- A graph G is planar if and only if every graph that is homeomorphic to G is planar.

Three graphs homeomorphic to each other.

- **THEOREM 9:**
- A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski’s two graphs or any graph homeomorphic to either of them.

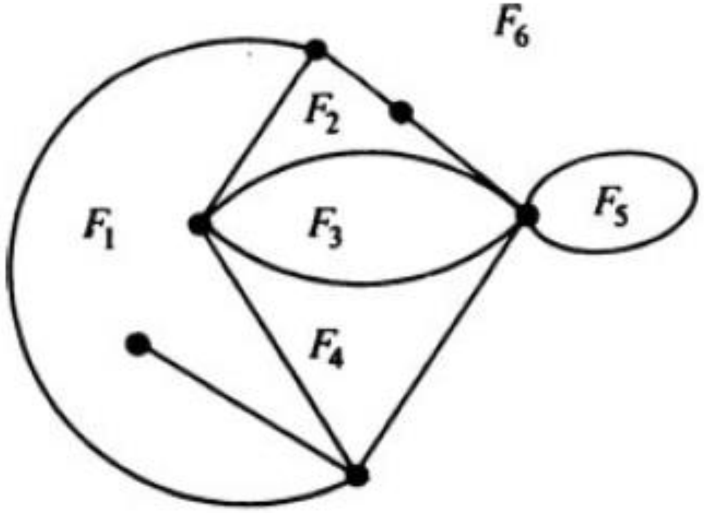
- Note that it is not necessary for a nonplanar graph to have either of the Kuratowski graphs as a subgraph. The nonplanar graph may have a subgraph homeomorphic to a Kuratowski graph.



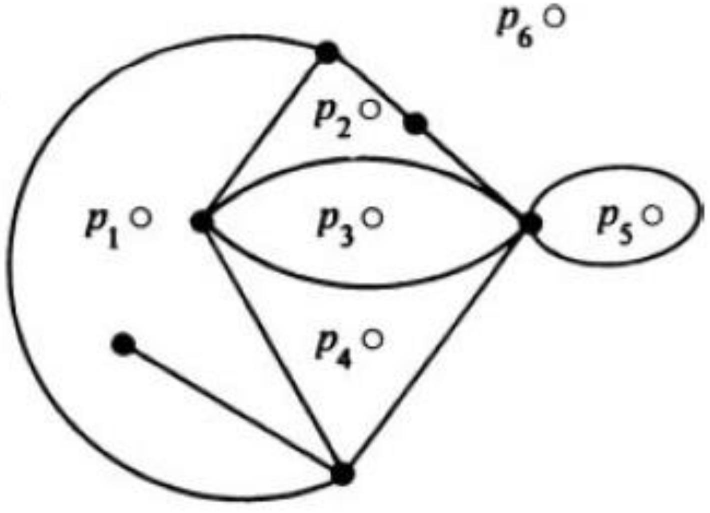
Nonplanar graph with a subgraph homeomorphic to $K_{3,3}$.

GEOMETRIC DUAL

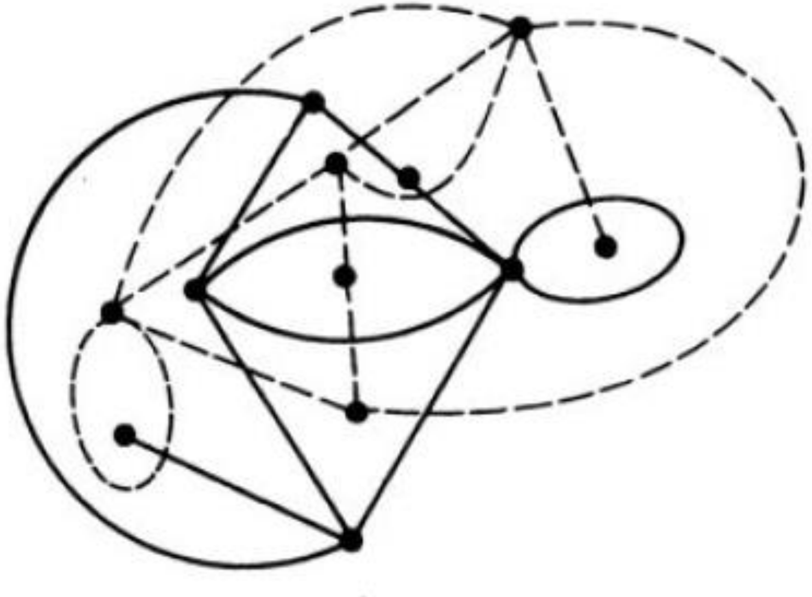
- Consider the plane representation of a graph G with n regions or faces F_1, F_2, F_3, … F_n. Let us place points $p_1, p_2, \ldots, p_n$, one in each of the regions. Next let us join these points according to the following procedure:
- If two regions F_i and F_j are adjacent (i.e., have a common edge), draw a line joining points p_i and p_j that intersects the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j, draw one line between points p_i and p_j for each of the common edges.
- For an edge e lying entirely in one region, say F_k, draw a self-loop at point p_k intersecting e exactly once.
- Such a graph G* is called a dual (a geometric dual) of G.



(a)



(b)



(c)

- There is a one-to-one correspondence between the edges of graph G and its dual G*−one edge of G* intersecting one edge of G. Some simple observations that can be made about the relationship between a planar graph G and its dual G* are
  - An edge forming a self-loop in G yields a pendant edge in G*.
  - A pendant edge in G yields a self-loop in G*.
  - Edges that are in series in G produce parallel edges in G*.
  - Parallel edges in G produce edges in series in G*.
  - The number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex p_i in G*, and vice versa.
  - Graph G* is also embedded in the plane and is therefore planar.
  - If n, e, f, r, and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n*, e*, f*, r*, and μ* are the corresponding numbers in dual graph G*, then

$$n^* = f, \qquad r^* = \mu,$$

$$e^* = e, \qquad \mu^* = r.$$

$$f^* = n.$$

- **THEOREM 10:**
- All duals of a planar graph G are 2-isomorphic; and every graph 2-isomorphic to a dual of G is also a dual of G.
- **THEOREM 11:**
- A necessary and sufficient condition for two planar graphs G1 and G2 to be duals of each other is as follows: There is a one-to-one correspondence between the edges in G1 and the edges in G2 such that a set of edges in G1 forms a circuit if and only if the corresponding set in G2 forms a cut-set.
- **THEOREM 12:**
- A graph has a dual if and only if it is planar.

Dual of a Subgraph

- Let G be a planar graph and G* be its dual.
- Let a be an edge in G, and the corresponding edge in G* be a*.
- Suppose that we delete edge a from G and then try to find the dual of G − a.
- If edge a was on the boundary of two regions, removal of a would merge these two regions into one. Thus the dual (G − a)* can be obtained from G* by deleting the corresponding edge a* and then fusing the two end vertices of a* in G* − a*.
- On the other hand, if edge a is not on the boundary, a* forms a self-loop. In that case G* − a* is the same as (G − a)*.
- Thus if a graph G has a dual G*, the dual of any subgraph of G can be obtained by successive application of this procedure.

Dual of a Homeomorphic Graph

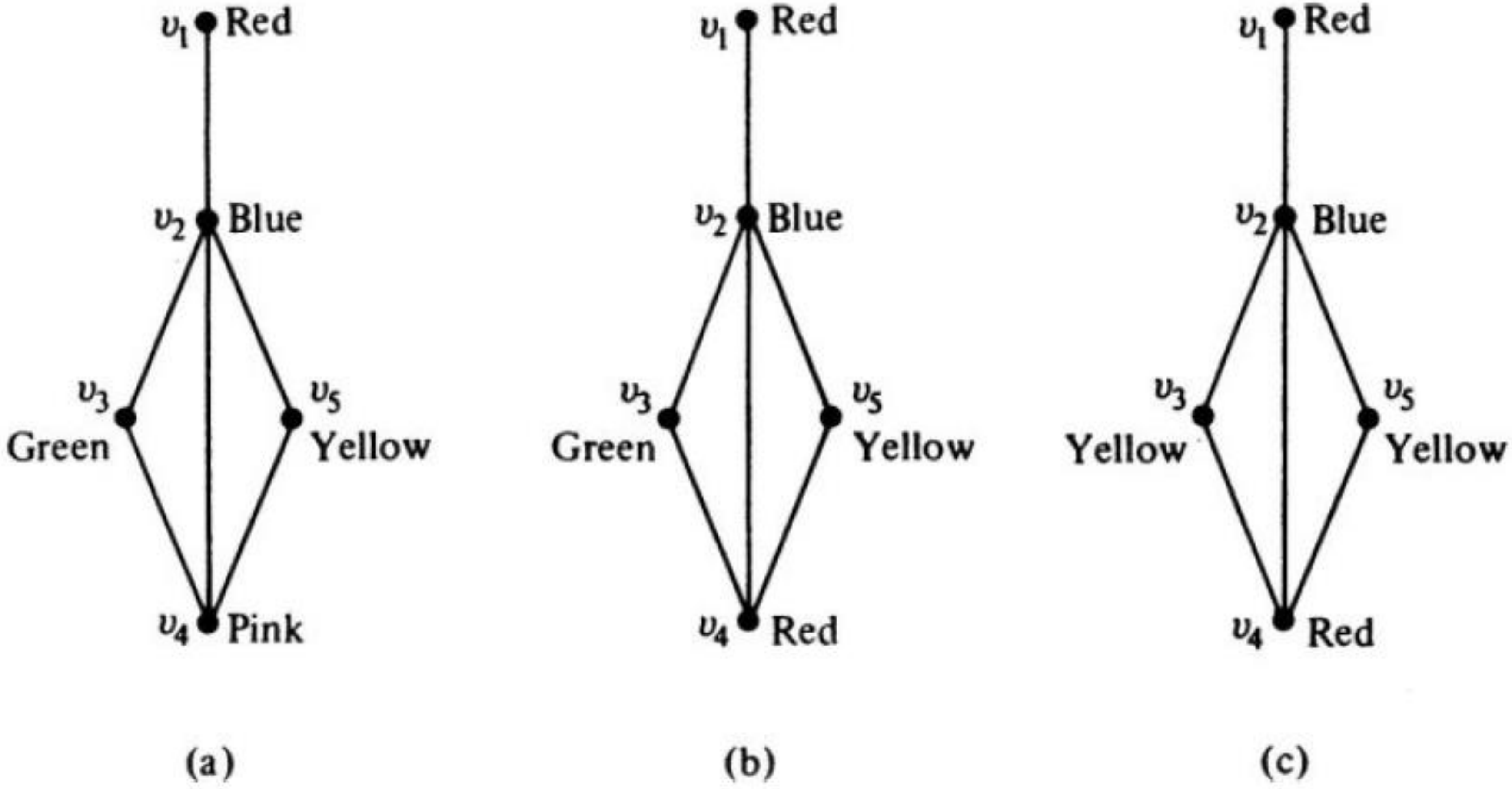
- Let G be a planar graph and G* be its dual.
- Let a be an edge in G, and the corresponding edge in G* be a*.
- Suppose that we create an additional vertex in G by introducing a vertex of degree two in edge a (i.e., a now becomes two edges in series). It will simply add an edge parallel to a* in G*.
- Likewise, the reverse process of merging two edges in series will simply eliminate one of the corresponding parallel edges in G*.
- Thus if a graph G has a dual G*, the dual of any graph homeomorphic to G can be obtained from G* by the above procedure.

Graph Coloring

CHROMATIC NUMBER

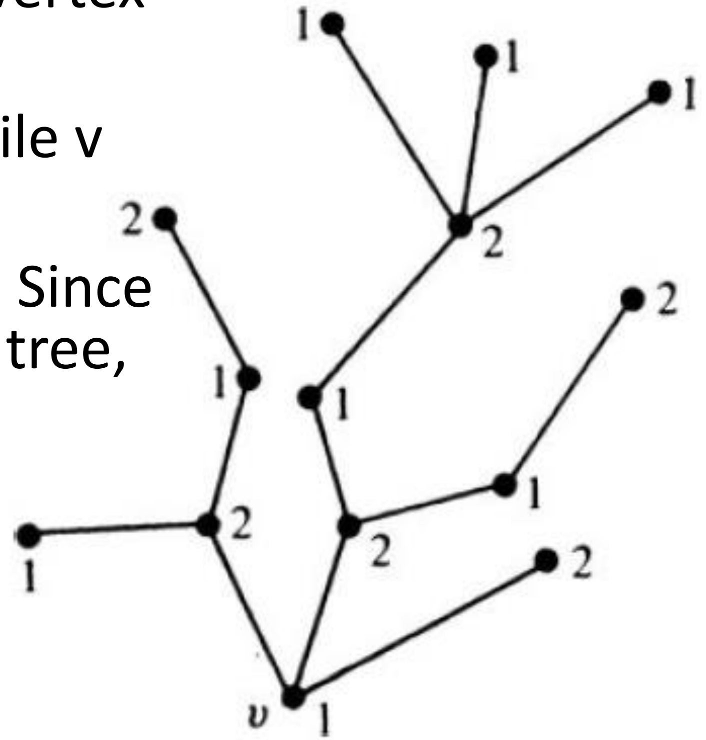
- Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the proper coloring (or coloring) of a graph.
- A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph.
- Usually a given graph can be properly colored in many different ways.
- The proper coloring which is of interest to us is one that requires the minimum number of colors.
- A graph G that requires K different colors for its proper coloring, and no less, is called a K-chromatic graph, and the number K is called the chromatic number of G.

Proper colorings of a graph



- Some observations on simple graph:
  - A graph consisting of only isolated vertices is 1-chromatic.
  - A graph with one or more edges is at least 2-chromatic (also called bichromatic).
  - A complete graph of n vertices is n-chromatic, as all its vertices are adjacent. Hence a graph containing a complete graph of r vertices is at least r-chromatic.
  - Every graph having a triangle is at least 3-chromatic.
  - A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.
- Proper coloring of a given graph is simple enough, but a proper coloring with the minimum number of colors is a difficult task.

- **THEOREM 1:**
- Every tree with two or more vertices is 2-chromatic.
- **Proof:** Select any vertex v in the given tree T.
- Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all vertices adjacent to v with color 2.
- Next, paint the vertices adjacent to these (those that just have been colored with 2) using color 1. Continue this process till every vertex in T has been painted.
- Now in T, all vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1.
- Now along any path in T the vertices are of alternating colors. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same color.
- Thus T has been properly colored with two colors.



- **THEOREM 2:**
- A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.
- **Proof:** Let G be a connected graph with circuits of only even lengths.
- Consider a spanning tree T in G. Using the coloring procedure and the result of Theorem 1, let us properly color T with two colors.
- Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every chord being replaced are differently colored in T. Thus G is colored with two colors, with no adjacent vertices having the same color.
- That is, G is 2-chromatic.
- Conversely, if G has a circuit of odd length, we would need at least three colors just for that circuit. Thus the theorem.

- **THEOREM 3:**
  - If d_{max} is the maximum degree of the vertices in a graph G, chromatic number of G ≤ 1 + d_{max}.
- If G has no complete graph of d_{max} + 1 vertices. Then, chromatic number of G ≤ d_{max}.

- Every 2-chromatic graph is bipartite because the coloring partitions the vertex set into two subsets V_1 and V_2 such that no two vertices in V_1 (or V_2) are adjacent.
- Every bipartite graph is 2-chromatic, with one trivial exception; a graph of two or more isolated vertices and with no edges is bipartite but is 1-chromatic.
- A graph G is called p-partite if its vertex set can be decomposed into p disjoint subsets $V_1, V_2, \ldots, V_p$, such that no edge in G joins the vertices in the same subset. Clearly, a κ-chromatic graph is p-partite if and only if $\kappa \leq p$.

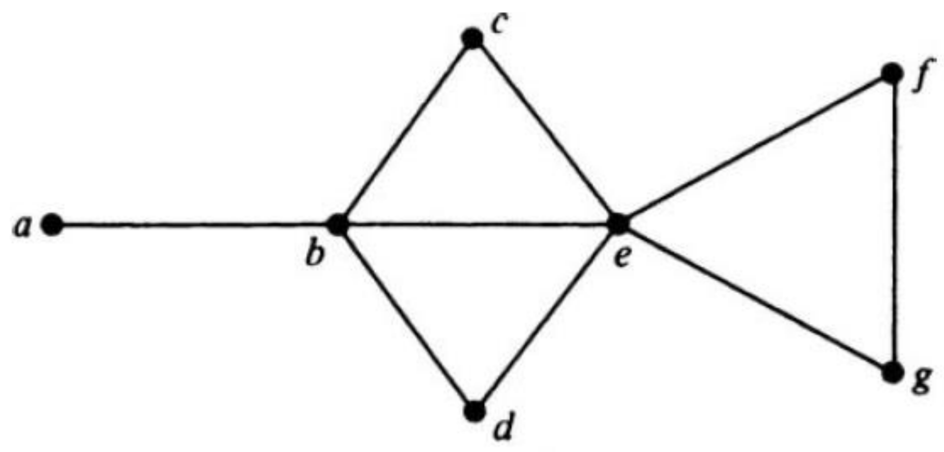
CHROMATIC PARTITIONING

- A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets.
- No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set.
- A set of vertices in a graph is said to be an independent set of vertices or simply an independent set (or an internally stable set) if no two vertices in the set are adjacent.
- The number of vertices in the largest independent set of a graph G is called the independence number (or coefficient of internal stability), β{G).
- Consider a κ-chromatic graph G of n vertices properly colored with κ different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number β(G), we have the inequality

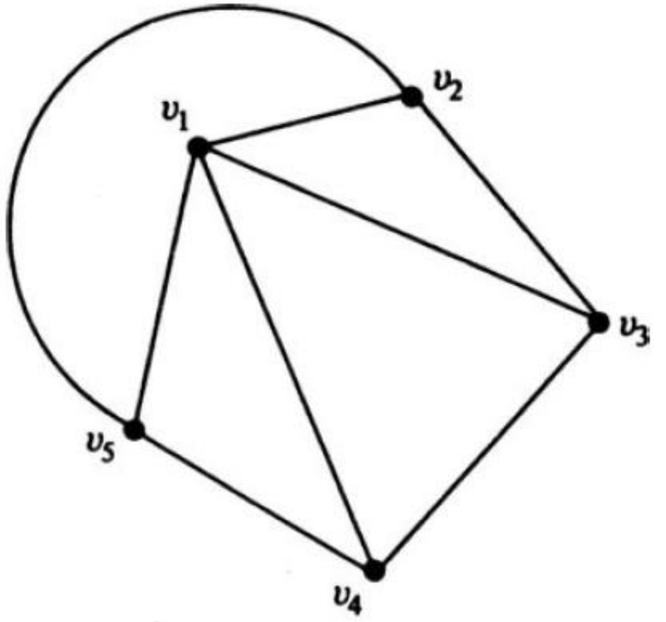
$$\beta(G) \geq \frac{n}{\kappa}.$$

- To find the **chromatic number of G**, we must find the minimum number of maximal independent sets, which collectively include all the vertices of G.
- For the graph in Fig., there are five maximal independent sets: {a, c, d, f}, {a, c, d, g}, {b, g}, {b, f}, and {a, e}.
- Sets {a, c, d, f}, {b, g}, and {a, e}, for example, collectively includes all vertices. Thus the graph is 3-chromatic.
- **Chromatic Partitioning:** Given a simple, connected graph G, partition all vertices of G into the smallest possible number of disjoint, independent sets.
- The following four are some chromatic partitions of the graph

$$\{(a, c, d, f), (b, g), (e)\},$$
$$\{(a, c, d, g), (b, f), (e)\},$$
$$\{(c, d, f), (b, g), (a, e)\},$$
$$\{(c, d, g), (b, f), (a, e)\}.$$



- Uniquely Colorable Graphs: A graph that has only one chromatic partition is called a uniquely colorable graph



CHROMATIC POLYNOMIAL

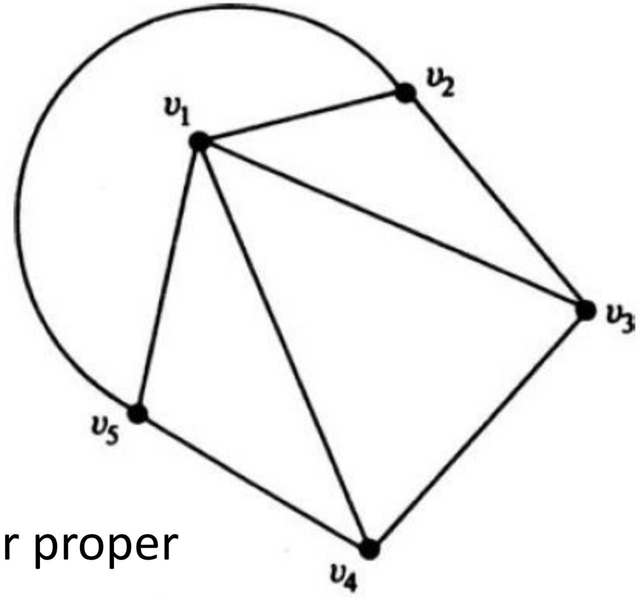
- A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial, called the chromatic polynomial of G.
- The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.
- Let c_i be the different ways of properly coloring G using exactly i different colors.
- Since i colors can be chosen out of λ colors in $C(\lambda, i)$ different ways, there are c_i $C(\lambda, i)$ different ways of properly coloring G using exactly i colors out of λ colors.

- Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms;

$$\begin{aligned}P_n(\lambda) &= \sum_{i=1}^{n} c_i \binom{\lambda}{i} \\ &= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots \\ &\quad + c_n \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}.\end{aligned}$$

- Each c_i has to be evaluated individually for any given graph.
- For example, any graph with even one edge requires at least two colors for proper coloring, and therefore $c_1 = 0$.
- A graph with n vertices and using n different colors can be properly colored in n! ways; that is, $c_n = n!$.

Find the chromatic polynomial of the graph given



$$P_5(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$
$$+ c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}.$$

- Since the graph has a triangle, it will require at least three different colors for proper coloring. Therefore, $c_1 = c_2 = 0$ and $c_5 = 5!$.
- To evaluate c_3, suppose that we have three colors x, y, and z. These three colors can be assigned properly to vertices v_1, v_2, and v_3 in $3! = 6$ different ways. Having done that, we have no more choices left, because vertex v_5 must have the same color as v_3, and v_4 must have the same color as v_2. Therefore, $c_3 = 6$.
- With four colors, v_1, v_2, and v_3 can be properly colored in $4\cdot 6 = 24$ different ways. The fourth color can be assigned to v_4 or v_5, thus providing two choices. The fifth vertex provides no additional choice. Therefore, $c_4 = 24\cdot 2 = 48$.
- Substituting these coefficients in $P_5(\lambda)$,

The presence of factors $\lambda - 1$ and $\lambda - 2$ indicates that G is at least 3-chromatic.

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7).$$

- **THEOREM 4:**
- A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1).$$

- **Proof:** With λ colors, there are λ different ways of coloring any selected vertex of a graph. A second vertex can be colored properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, the fourth in $\lambda - 3$ ways, ... , and the nth in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other. That is, if and only if the graph is complete.
- **THEOREM 5:**
- An n-vertex graph is a tree if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)^{n - 1}$$

- **THEOREM 6:**
- Let a and b be two nonadjacent vertices in a graph G. Let G′ be a graph obtained by adding an edge between a and b. Let G″ be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

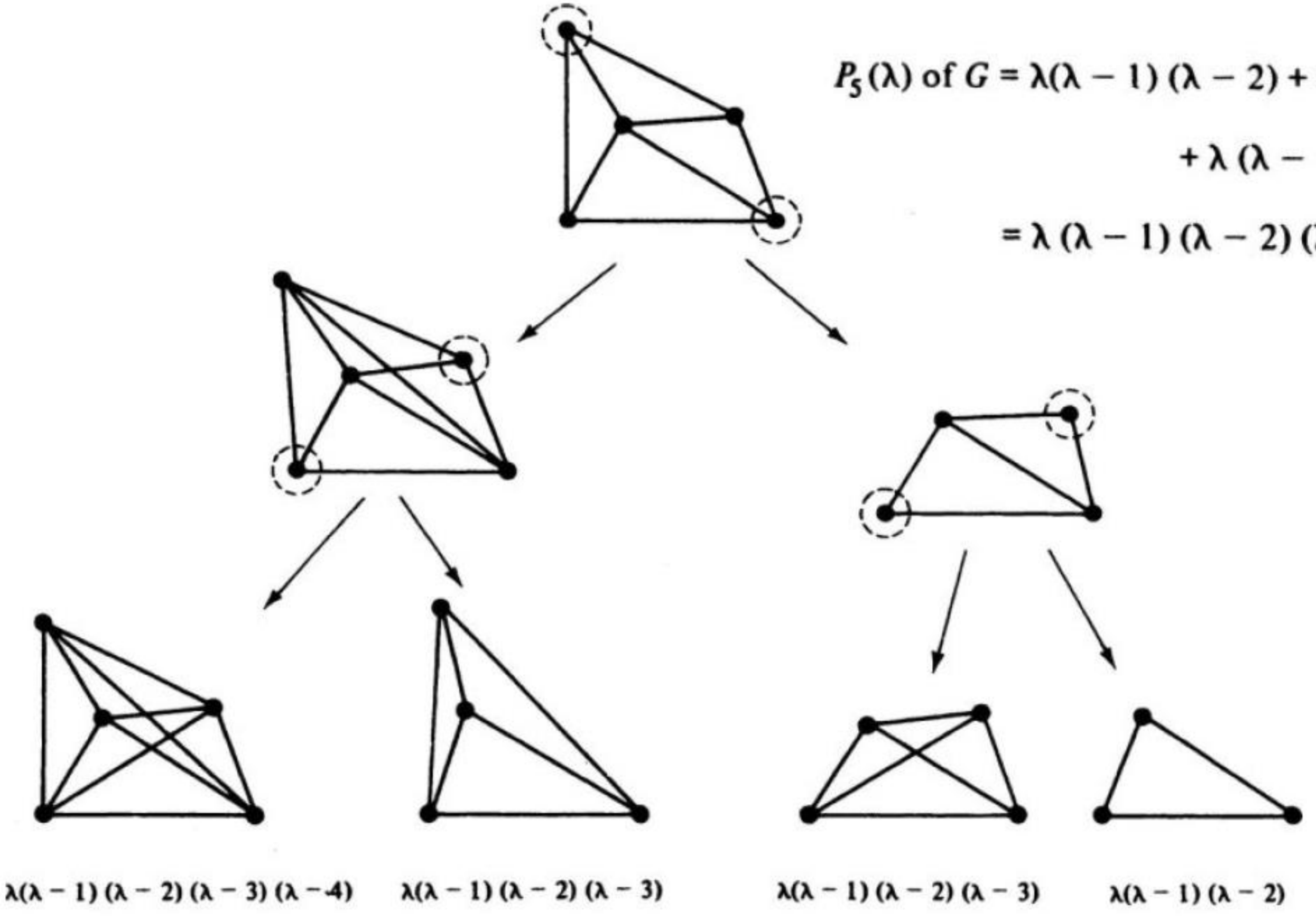
- **Proof:** The number of ways of properly coloring G can be grouped into two cases, one such that vertices a and b are of the same color and the other such that a and b are of different colors.
- Since the number of ways of properly coloring G such that a and b have different colors = number of ways of properly coloring G′, and number of ways of properly coloring G such that a and b have the same color = number of ways of properly coloring G″,

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

$$P_5(\lambda) \text{ of } G = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$+ \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)$$



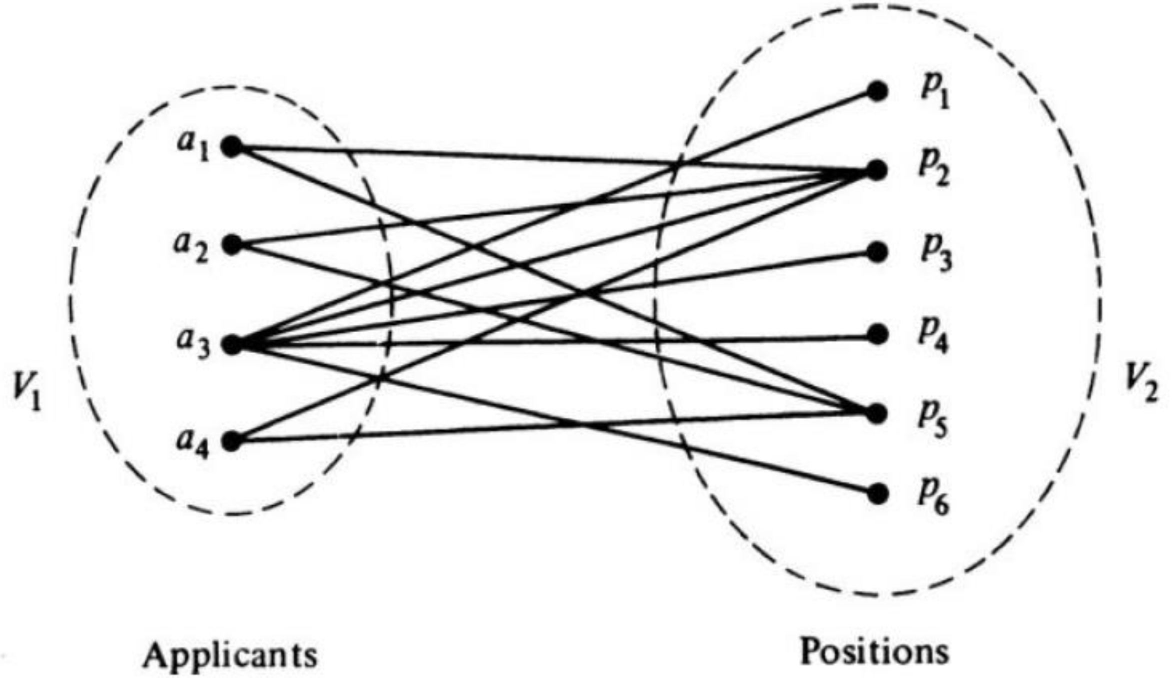
MATCHINGS

- A set of edges in which no two are adjacent is called a *matching*.
- A matching in a graph is a subset of edges in which no two edges are adjacent.
- A single edge in a graph is obviously a matching.
- A maximal matching is a matching to which no edge in the graph can be added.
  - For example, in a complete graph of three vertices (i.e., a triangle) any single edge is a maximal matching.
- A graph may have many different maximal matchings, and of different sizes. Among these, the maximal matchings with the largest number of edges are called the largest maximal matchings.
- The number of edges in a largest maximal matching is called the matching number of the graph.

(a) (b)

- The edges shown by heavy lines in Fig. are two maximal matchings.
- A largest maximal matching is shown in heavy lines in (b).

- Suppose that four applicants a_1, a_2, a_3, and a_4 are available to fill six vacant positions p_1, p_2, p_3, p_4, p_5, and p_6. Applicant a_1 is qualified to fill position p_2 or p_5. Applicant a_2 can fill p_2 or p_5. Applicant a_3 is qualified for p_1, p_2, p_3, p_4, or p_6. Applicant a_4 can fill jobs p_2 or p_5.
- Is it possible to hire all the applicants and assign each a position for which he is suitable? If the answer is no, what is the maximum number of positions that can be filled from the given set of applicants?
- This situation is represented by the graph in Fig. The vacant positions and applicants are represented by vertices. The edges represent the qualifications of each applicant for filling different positions. The graph clearly is bipartite, the vertices falling into two sets $V_1 = \{a_1, a_2, a_3, a_4\}$ and $V_2 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$.



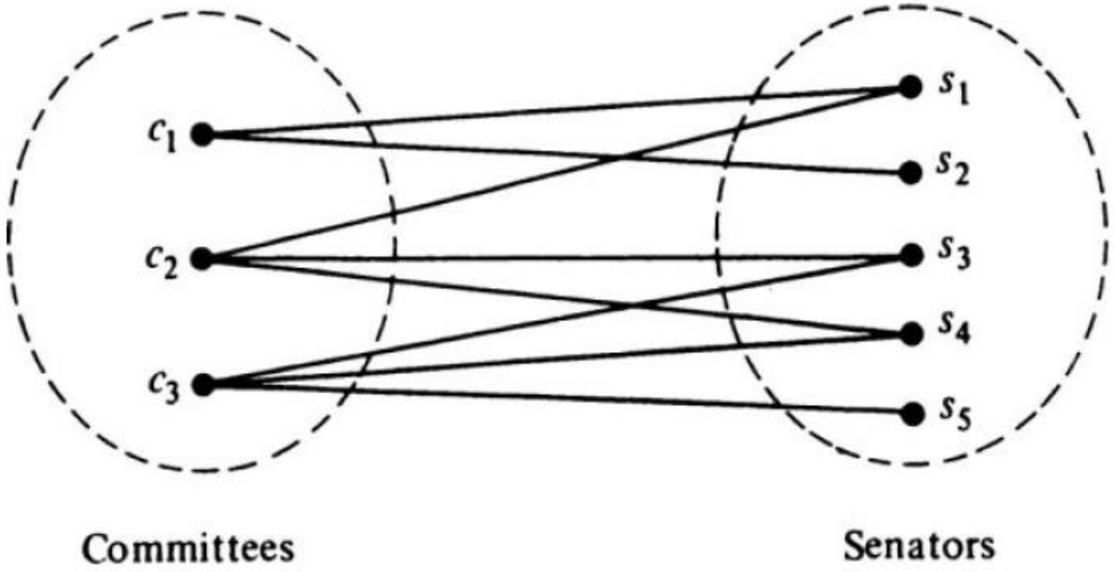
- Although there are 6 positions and 4 applicants, a complete matching does not exist.
- Of the three applicants a_1, a_2; and a_4, each qualifies for the same two positions p_2 and p_5, and therefore one of the three applicants cannot be matched.

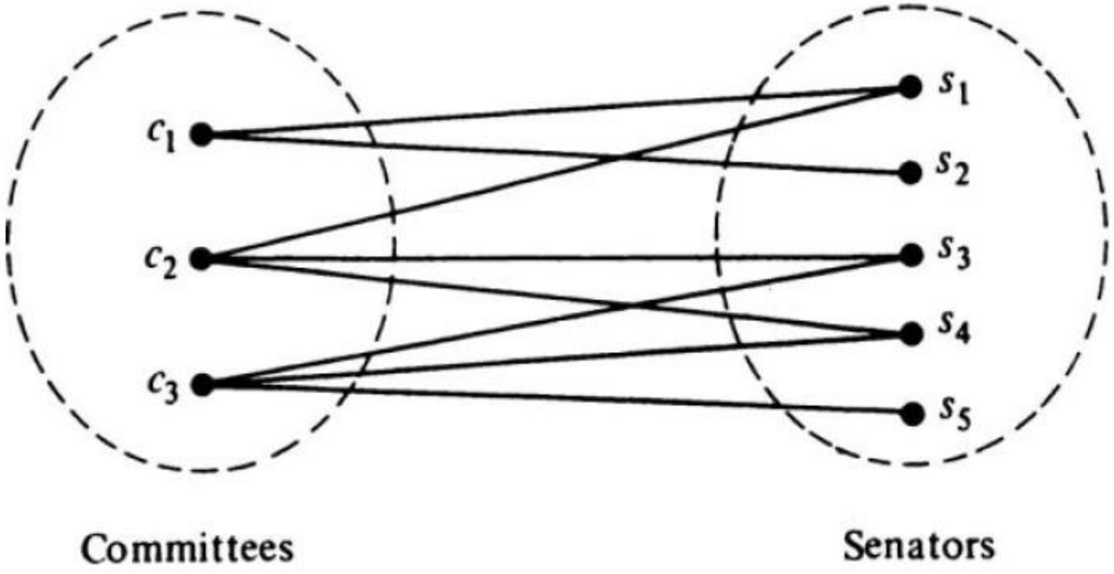
- In a bipartite graph having a vertex partition V_1 and V_2, a complete matching of vertices in set V_1 into those in V_2 is a matching in which there is one edge incident with every vertex in V_1.
- In other words, every vertex in V_1 is matched against some vertex in V_2.
- Clearly, a complete matching (if it exists) is a largest maximal matching, whereas the converse is not necessarily true.
- For the existence of a complete matching of set V_1 into set V_2, first we must have at least as many vertices in V_2 as there are in V_1.
- This condition, however, is not sufficient.

- **THEOREM 7:**
- A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.

Problem of Distinct Representatives

- Five senators s_1, s_2, s_3, s_4, and s_5 are members of three committees, c_1, c_2, and c_3. The membership is shown in Fig. One member from each committee is to be represented in a super-committee. Is it possible to send one distinct representative from each of the committee?
- This problem is one of finding a complete matching of a set V_1 into set V_2 in a bipartite graph.
- Let us use Theorem 7 and check if r vertices from V_1 are collectively adjacent to at least r vertices from V_2, for all values of r.





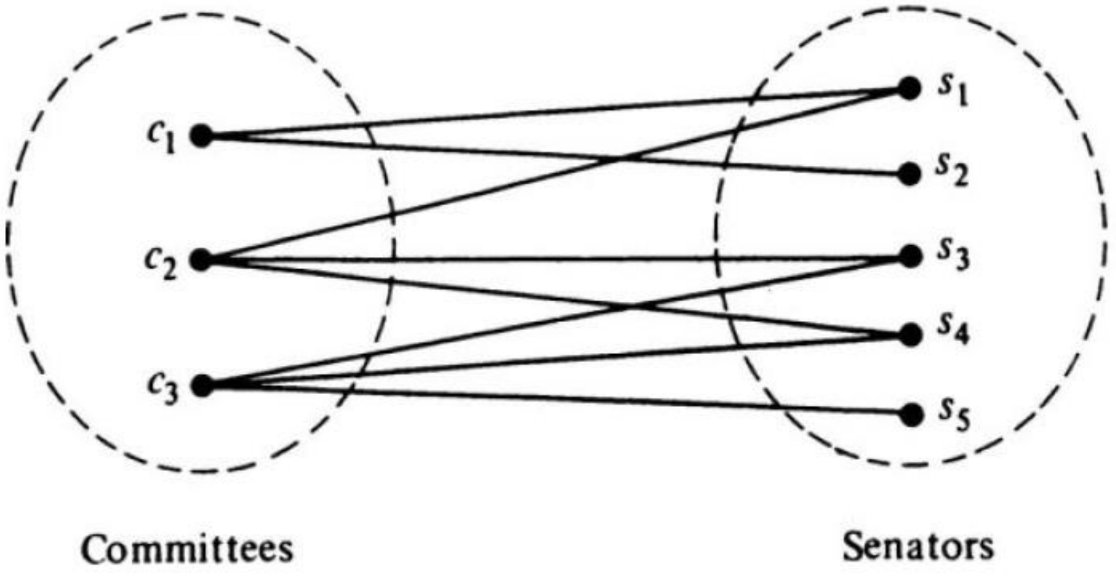
The condition for the existence of a complete matching is satisfied as stated in Theorem 7. Hence it is possible to form the supercommittee with one distinct representative from each committee.

	V_1	V_2
$r = 1$	$\{c_1\}$	$\{s_1, s_2\}$
	$\{c_2\}$	$\{s_1, s_3, s_4\}$
	$\{c_3\}$	$\{s_3, s_4, s_5\}$
$r = 2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$
	$\{c_3, c_1\}$	$\{s_1, s_2, s_3, s_4, s_5\}$
$r = 3$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$

- **THEOREM 8:**
- In a bipartite graph a complete matching of V_1 into V_2 exists if (but not only if) there is a positive integer m for which the following condition is satisfied : degree of every vertex in

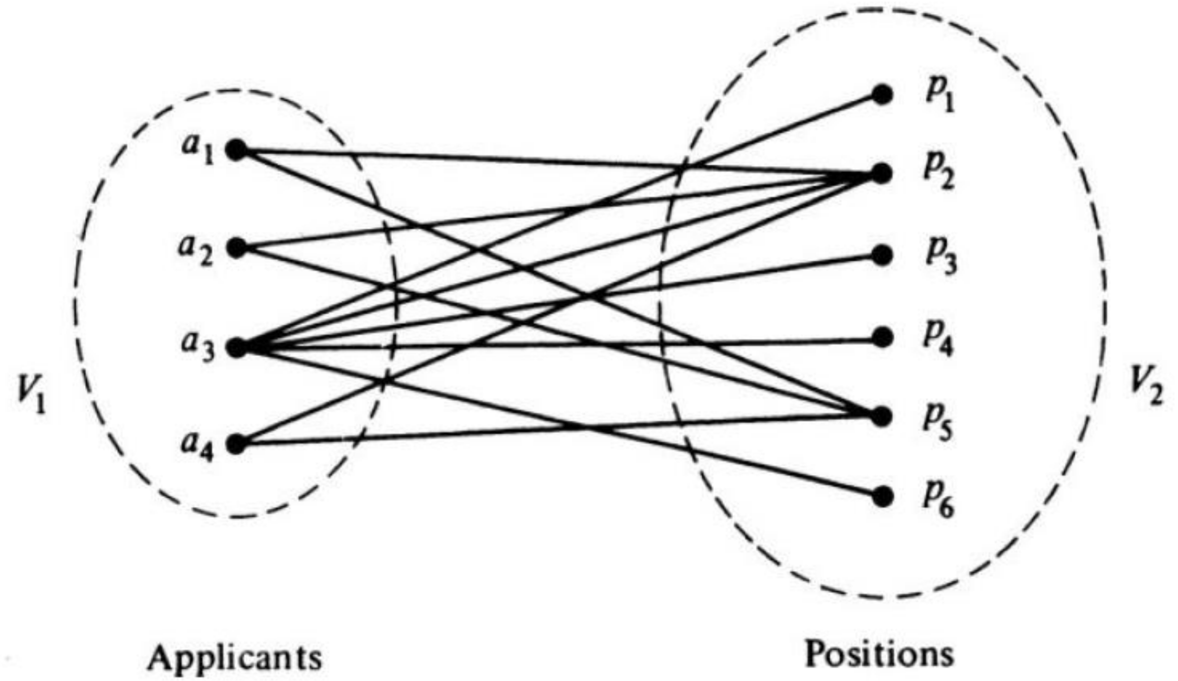
$$V_1 \geq m \geq \text{degree of every vertex in } V_2$$

- **Proof:** Consider a subset of r vertices in V_1.
- These r vertices have at least $m \cdot r$ edges incident on them. Each $m \cdot r$ edge is incident to some vertex in V_2. Since the degree of every vertex in set V_2 is no greater than m, these $m \cdot r$ edges are incident on at least $(m \cdot r)/m = r$ vertices in V_2.
- Thus any subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2.
- Therefore, according to Theorem 7, there exists a complete matching of V_1 into V_2.



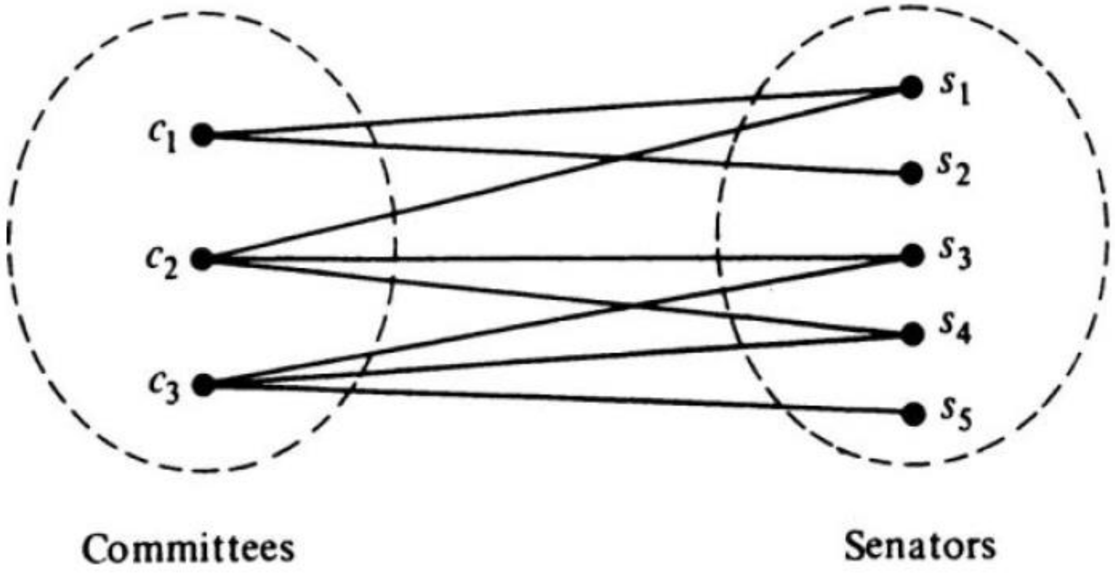
degree of every vertex in $V_1 \geq 2 \geq$ degree of every vertex in V_2.

Therefore, there exists a complete matching.



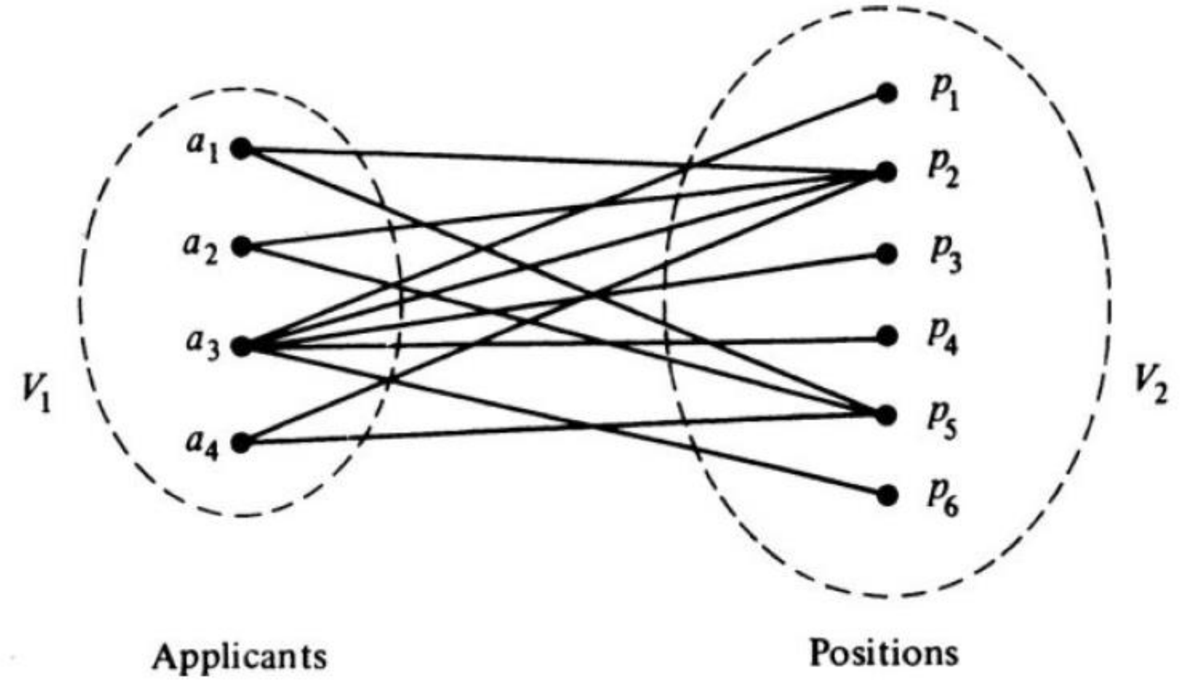
- no such number is found, because the degree of p_2 = 4 > degree of a_1.
- Complete matching does not exist.

- If one fails to find a complete matching, he is most likely to be interested in finding a maximal matching, that is, to pair off as many vertices of V_1 with those in V_2 as possible.
- For this purpose, let us define a new term called deficiency, $\delta(G)$, of a bipartite graph G.
- A set of r vertices in V_1 is collectively incident on, say, q vertices of V_2. Then the maximum value of the number $r - q$ taken over all values of $r = 1, 2, . . .$ and all subsets of V_1 is called the deficiency $\delta(G)$ of the bipartite graph G.
- Theorem 7, expressed in terms of the deficiency, states that a complete matching in a bipartite graph G exists if and only if $\delta(G) \leq 0$.



	V_1	V_2	$r-q$
$r=1$	$\{c_1\}$	$\{s_1, s_2\}$	-1
	$\{c_2\}$	$\{s_1, s_3, s_4\}$	-2
	$\{c_3\}$	$\{s_3, s_4, s_5\}$	-2
$r=2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$	-2
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$	-2
	$\{c_3, c_1\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-3
$r=3$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-2

- **THEOREM 9:**
- The maximal number of vertices in set V_1 that can be matched into V_2 is equal to number of vertices in $V_1 - \delta(G)$.



- The size of a maximal matching = number of vertices in $V_1 - \delta(G)$ $= 4 - 1 = 3$.

Matching and Adjacency Matrix

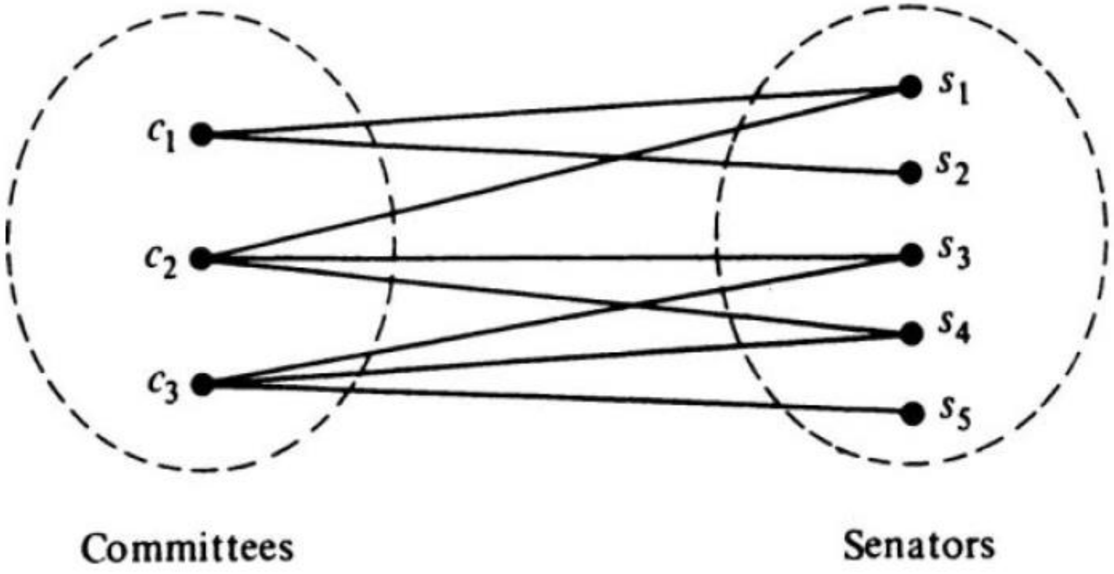
- Consider a bipartite graph G with non-adjacent sets of vertices V_1 and V_2, having number of vertices n_1 and n_2, respectively, and let $n_1 \leq n_2$, $n_1 + n_2 = n$, the number of vertices in G.
- The adjacency matrix X(G) of G can be written in the form

$$\mathrm{X}(G) = \left[\begin{array}{c|c} 0 & \mathrm{X}_{12} \\ \hline \mathrm{X}_{12}^{T} & 0 \end{array}\right]$$

- where the submatrix X_{12} is the n_1 by n_2, (0, 1)-matrix containing the information as to which of the n_1 vertices of V_1 are connected to which of the n_2 vertices of V_2.
- X_{12}^{T}Matrix is the transpose of X_{12}.
- All the information about the bipartite graph G is contained in its X_{12} matrix.
- A matching V_1 into V_2 corresponds to a selection of the 1's in the matrix X_{l2} such that no line (i.e., a row or a column) has more than one 1.

- The matching is complete if the n_1 by n_2 matrix made of selected 1's has exactly one 1 in every row.
- A maximal matching corresponds to the selection of a largest possible number of 1's from X_{12} such that no row in it has more than one 1.
- Therefore, according to Theorem 9, in matrix X_{12} the largest number of 1's, no two of which are in one row, is equal to number of vertices in $V_1 - \delta(G)$.

$$\mathbf{X}_{12} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array},$$



$n_1 = 3, \quad n_2 = 5, \quad n = 8, \quad \text{and} \quad n_1 \leq n_2,$

$V_1 = \{c_1, c_2, c_3\}$

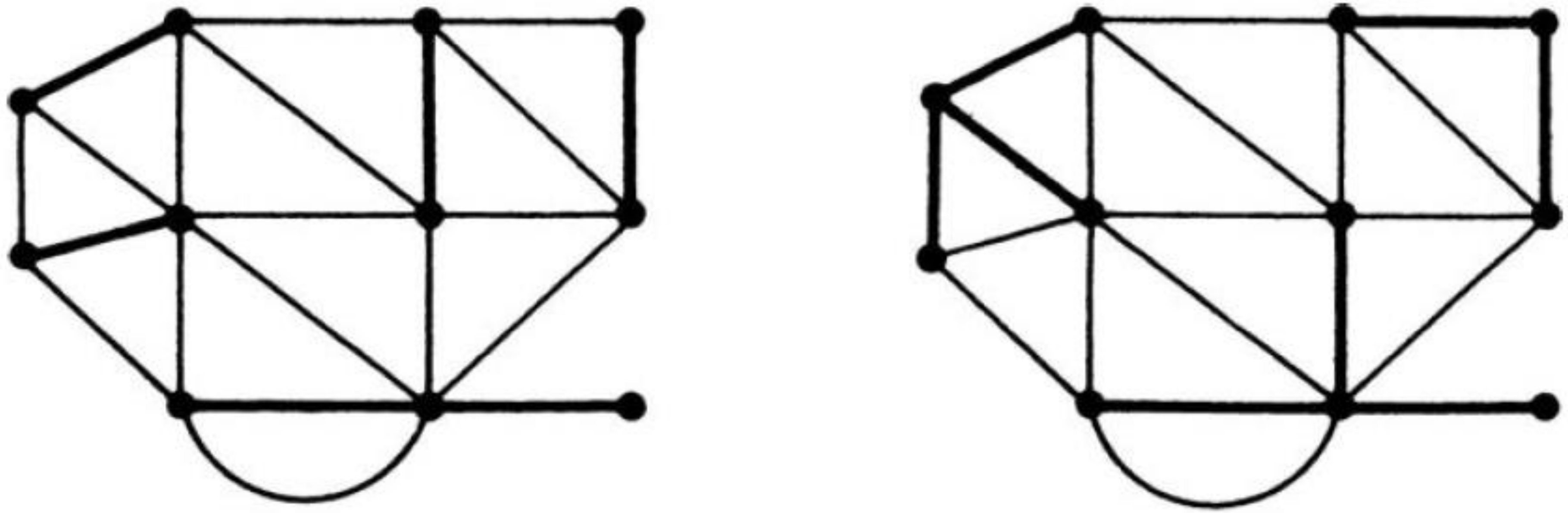
$V_2 = \{s_1, s_2, s_3, s_4, s_5\}.$

A complete matching of V_1 into V_2 is given by

$$\mathbf{M} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Covering

- In a graph G, a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g.
- A set of edges that covers a graph G is said to be an edge covering, a covering subgraph, or simply a covering of G.
- For example, a graph G is trivially its own covering. A spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering. A Hamiltonian circuit (if it exists) in a graph is also a covering.
- Minimal covering–a covering from which no edge can be removed without destroying its ability to cover the graph.

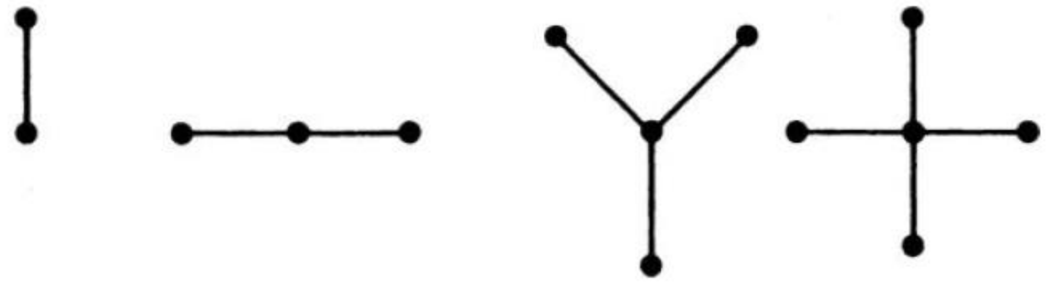


(a) (b)

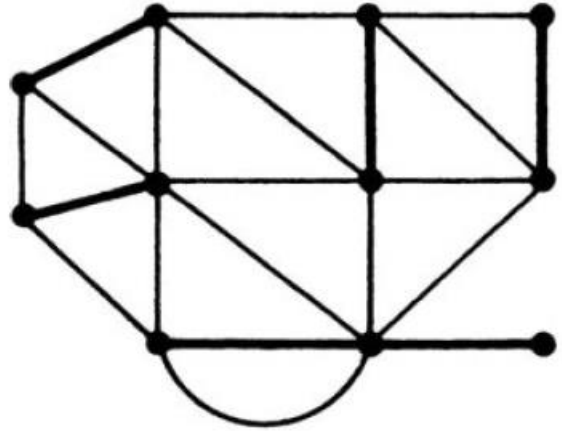
Graph and two of its minimal coverings

- Observations:
  - A covering exists for a graph if and only if the graph has no isolated vertex.
  - A covering of an n-vertex graph will have at least $\lceil n/2 \rceil$ edges.
  - Every pendant edge in a graph is included in every covering of the graph.
  - Every covering contains a minimal covering.
  - If we denote the remaining edges of a graph by (G – g), the set of edges g is a covering if and only if, for every vertex V, the degree of vertex in (G –g) ≤ (degree of vertex v in G) – 1.
  - No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than n – 1 edges.
  - A graph, in general, has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges).
  - The number of edges in a minimal covering of the smallest size is called the covering number of the graph.

- **THEOREM 10:**
  - A covering g of a graph is minimal if and only if g contains no paths of length three or more.
- **Proof:**
  - Suppose that a covering g contains a path of length three, and it is $v_1 e_1 v_2 e_2 v_3 e_3 v_4$.
  - Edge e_2 can be removed without leaving its end vertices v_2 and v_3 uncovered.
  - Therefore, g is not a minimal covering.
  - Conversely, if a covering g contains no path of length three or more, all its components must be star graphs (i.e., graphs in the shape of stars; see Fig.). From a star graph no edge can be removed without leaving a vertex uncovered. That is, g must be a minimal covering.

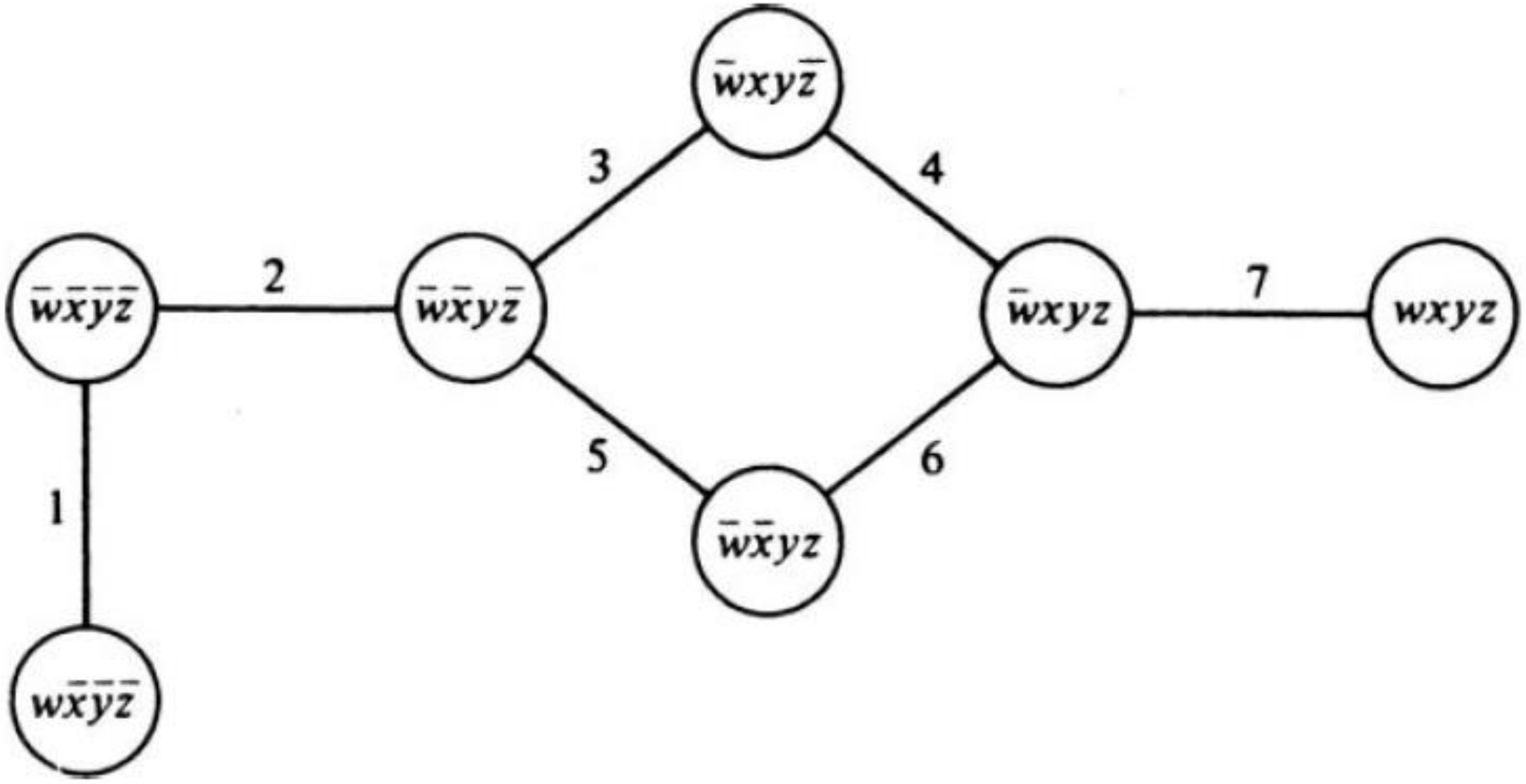


- Suppose that the graph in Fig. represents the street map of a part of a city. Each of the vertices is a potential trouble spot and must be kept under the surveillance of a patrol car. How will you assign a minimum number of patrol cars to keep every vertex covered?
- The answer is a smallest minimal covering.
- The covering shown in heavy lines is an answer, and it requires six patrol cars.
- Since there are 11 vertices and no edge can cover more than two, less than six edges cannot cover the graph.



Minimization of Switching Functions

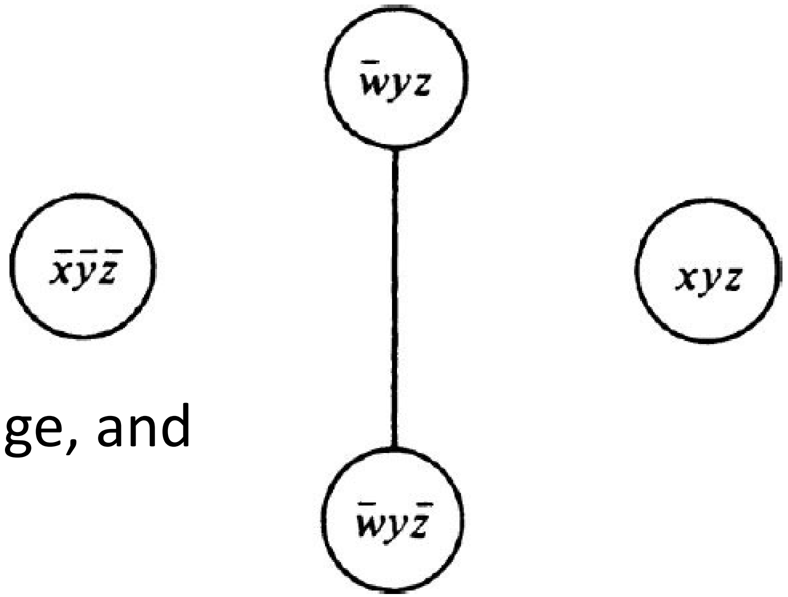
$$F = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}yz + \bar{w}xy\bar{z} + \bar{w}xyz + wxyz$$



- Let us represent each of the seven terms in F by a vertex, and join every pair of vertices that differ only in one variable.
- An edge between two vertices represents a term with three variables.
- A minimal cover of this graph will represent a simplified form of F, performing the same function as F, but with less logic hardware.
- The pendant edges 1 and 7 must be included in every covering of the graph.
- Two additional edges 3 and 6 (or 4 and 5 or 3 and 5) will cover the remainder. Thus a simplified version of F is

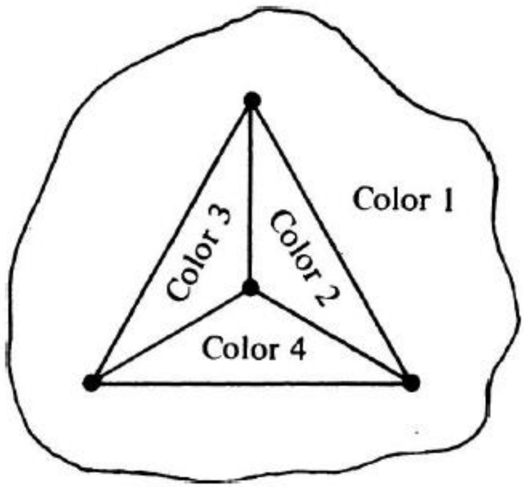
$$F = \bar{x}\bar{y}\bar{z} + xyz + \bar{w}y\bar{z} + \bar{w}yz$$

- This expression can again be represented by a graph of four vertices



- The essential terms and xyz cannot be covered by any edge, and hence cannot be minimized further.
- One edge will cover the remaining two vertices.
- Thus the minimized Boolean expression is

$$F = \bar{x}\bar{y}\bar{z} + xyz + \bar{w}y.$$



- Every planar graph has a chromatic number of four or less.
- **THEOREM 11**
- The vertices of every planar graph can be properly colored with five colors.