Recurrence Relation

Recurrence Relations

- Definition:

An equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots, a_{n-1}$ for all integers n with $n \geq n_0$, where n_0 is the nonnegative integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

Recurrence Relations

- In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

- Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

- If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

What is the recurrence equation for the series

- 5,15,45,135, ...

$$a_{n+1} = 3a_n, \qquad n \geq 0, \quad a_0 = 5$$

- 3,9,27,.....

$$a_{n+1} = 3a_n, \qquad n \geq 0, \quad a_0 = 3$$

Since, a_{n+1} depends only on its immediate predecessor, the relation is said to be first order

The general form of such an equation can be written $a_{n+1} = da_n$, $n \geq 0$, where d is a constant.

Values such as a_0 or a_1, given in addition to the recurrence relations, are called *boundary conditions*. The expression $a_0 = A$, where A is a constant, is also referred to as an *initial condition*.

- The Unique solution of the recurrence relation $a_{n+1} = da_n$, where $n \geq 0$, d is a constant and $a_0 = A$ is given by

$$\mathbf{a_n = Ad^n,\ n \geq 0}$$

$$a_{n+1} = 3a_n, \qquad n \geq 0, \qquad a_0 = 5.$$

The first four terms of this sequence are

$$\begin{aligned}
a_0 &= 5, \\
a_1 &= 3a_0 = 3(5), \\
a_2 &= 3a_1 = 3(3a_0) = 3^2(5), \qquad \text{and} \\
a_3 &= 3a_2 = 3(3^2(5)) = 3^3(5).
\end{aligned}$$

These results suggest that for each $n \geq 0$, $a_n = 5(3^n)$. This is the *unique solution* of the given recurrence relation.

Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$.

$$a_n = a_0 (7^n)$$

$$a_2 = a_0 (7^2)$$

$$98 = a_0 \, 49$$

$$a_0 = 2$$

$a_n = 2(7^n)$, $n \geq 0$ is the unique solution.

A bank pays 6% (annual) interest on savings, compounding the interest monthly. If Bonnie deposits \$1000 on the first day of May, how much will this deposit be worth a year later?

The annual interest rate is 6%, so the monthly rate is $6\%/12 = 0.5\% = 0.005$.
let p_n denote the value of Bonnie's deposit at the end of n months.
$p_{n+1} = p_n + 0.005p_n$, where $0.005p_n$ is the interest earned on p_n during month $n+1$, for $0 \leq n \leq 11$, and $p_0 = \$1000$.

The relation $p_{n+1} = (1.005)p_n$, $p_0 = \$1000$, has the solution $p_n = p_0(1.005)^n = \$1000(1.005)^n$. Consequently, at the end of one year, Bonnie's deposit is worth $\$1000(1.005)^{12} = \$1061.68$.

Find the recurrence relation of composition of numbers.

		(1′)	4
		(2′)	1 + 3
(1)	3	(3′)	2 + 2
(2)	1 + 2	(4′)	1 + 1 + 2
(3)	2 + 1		
(4)	1 + 1 + 1	(1″)	3 + 1
		(2″)	1 + 2 + 1
		(3″)	2 + 1 + 1
		(4″)	1 + 1 + 1 + 1

let a_n count the number of compositions of n, for $n \in \mathbf{Z}^+$,

$$a_{n+1} = 2a_n, \qquad n \geq 1, \qquad a_1 = 1.$$

to apply the formula for the unique solution (where $n \geq 0$) to this recurrence relation, we let $b_n = a_{n+1}$.

$$b_{n+1} = 2b_n, \qquad n \geq 0, \qquad b_0 = 1,$$

so $b_n = b_0(2^n) = 2^n$, and $a_n = b_{n-1} = 2^{n-1}$, $n \geq 1$.

Find a_{12} if $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ for $n \geq 0$, and $a_0 = 2$.

let $b_n = a_n^2$,

$b_{n+1} = 5b_n$ for $n \geq 0$, and $b_0 = 4$, is a linear relation

solution is $b_n = 4 \cdot 5^n$

Therefore, $a_n = 2(\sqrt{5})^n$ for $n \geq 0$

$a_{12} = 2(\sqrt{5})^{12} = 31{,}250$

The general first-order linear recurrence relation with constant coefficients has the form $a_{n+1} + ca_n = f(n)$, $n \geq 0$, where c is a constant and $f(n)$ is a function on the set $\mathbf{N}$ of nonnegative integers.

When $f(n) = 0$ for all $n \in \mathbf{N}$, the relation is called *homogeneous*; otherwise it is called *nonhomogeneous*.

Bubble sort

```
procedure BubbleSort(n: positive integer; x_1, x_2, x_3, ..., x_n: real numbers)
begin
   for i := 1 to n − 1 do
      for j := n downto i + 1 do
         if x_j < x_{j−1} then
            begin       {interchange}
               temp := x_{j−1}
               x_{j−1} := x_j
               x_j := temp
            end
end
```

i = 1									
x_1	7		7		7		7	j = 2	2
x_2	9		9		9	j = 3	2		7
x_3	2		2	j = 4	2		9		9
x_4	5	j = 5	5		5		5		5
x_5	8		8		8		8		8

Four comparisons and two interchanges.

i = 2							
x_1	2		2		2		2
x_2	7		7		7	j = 3	5
x_3	9		9	j = 4	5		7
x_4	5	j = 5	5		9		9
x_5	8		8		8		8

Three comparisons and two interchanges.

i = 3					
x_1	2		2		2
x_2	5		5		5
x_3	7		7	j = 4	7
x_4	9	j = 5	8		8
x_5	8		9		9

Two comparisons and one interchange.

i = 4		
x_1	2	
x_2	5	
x_3	7	
x_4	8	j = 5
x_5	9	

One comparison but no interchanges.

To determine the time-complexity function $h(n)$ when this algorithm is used on an input (array) of size $n \geq 1$, we count the total number of *comparisons* made in order to sort the n given numbers into ascending order.

If a_n denotes the number of comparisons needed to sort n numbers in this way, then we get the following recurrence relation:

$$a_n = a_{n-1} + (n - 1), \qquad n \geq 2, \qquad a_1 = 0.$$

Given a list of n numbers, we make $n - 1$ comparisons to bubble the smallest number up to the start of the list. The remaining sublist of $n - 1$ numbers then requires a_{n-1} comparisons in order to be completely sorted.

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_1 + (2 - 1) = 1 \\
a_3 &= a_2 + (3 - 1) = 1 + 2 \\
a_4 &= a_3 + (4 - 1) = 1 + 2 + 3 \\
\cdots &\quad \cdots \quad \cdots \quad \cdots
\end{aligned}
$$

In general, $a_n = 1 + 2 + \cdots + (n - 1) = [(n - 1)n]/2 = (n^2 - n)/2$.