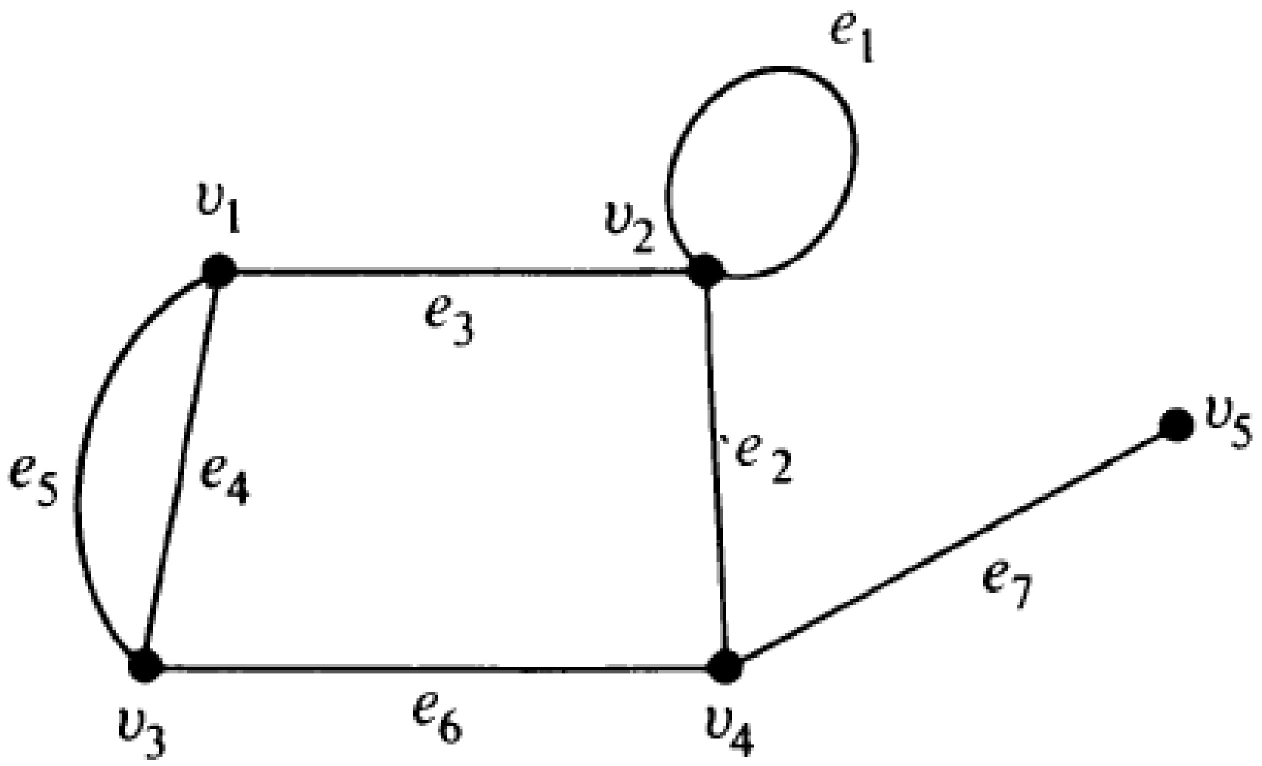
Graphs

Graph

A *linear graph* (or simply a *graph*) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k.

The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.



$V = \{v_1, v_2, \dots v_5\}$

$E = \{e_1, e_2, \dots e_7\}$

$e_2 = (v_2, v_4)$ or (v_4, v_2)

unordered

$e_1 = (v_2, v_2)$

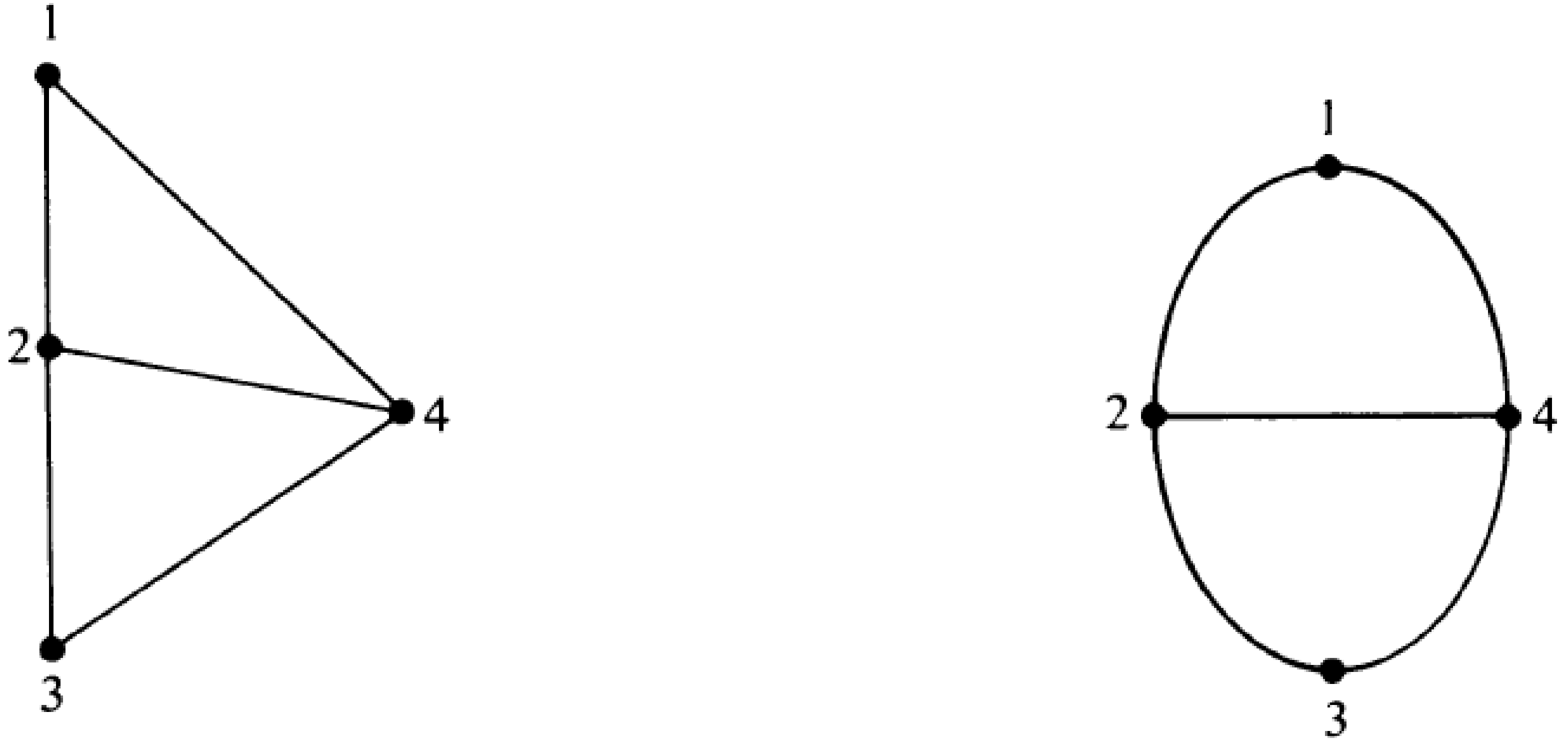
an edge having the same vertex as both its end vertices is called a *self-loop*

vertex pair (v_i, v_i)

the definition allows more than one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in Fig.

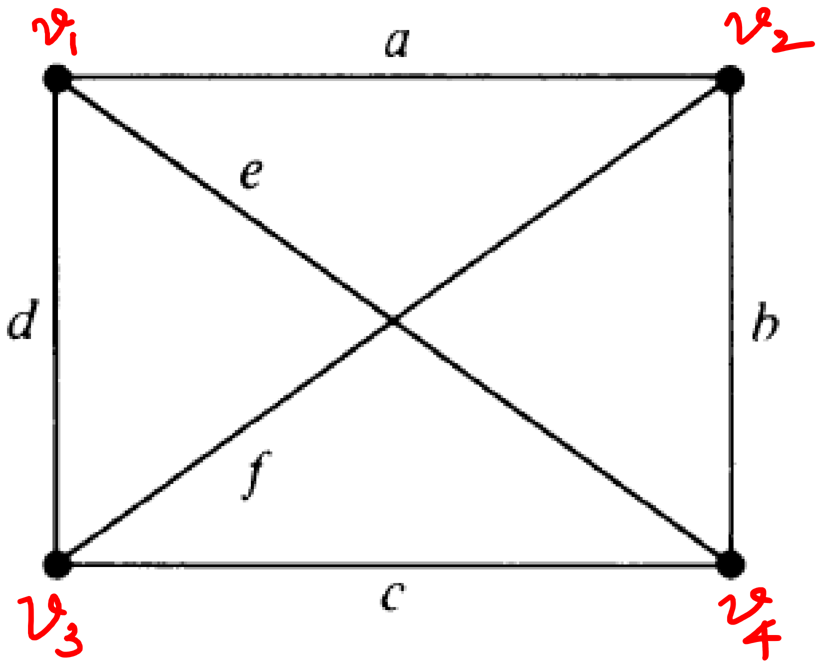
Such edges are referred to as *parallel edges.*

A graph that has neither self-loops nor parallel edges is called a *simple graph*. In some graph-theory literature, a graph is defined to be only a simple graph, but in most engineering applications it is necessary that parallel edges and self-loops be allowed; this is why our definition includes graphs with self-loops and/or parallel edges. Some authors use the term *general graph* to emphasize that parallel edges and self-loops are allowed.



Same graph drawn differently.

How many vertices are there?



A graph is also called a *linear complex*, a *1-complex*, or a *one-dimensional complex*.

A vertex is also referred to as a *node*, a *junction*, a *point*, *0-cell*, or an *0-simplex*.

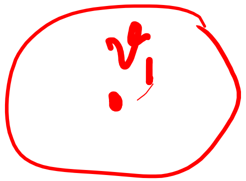
Other terms used for an edge are a *branch*, a *line*, an *element*, a *1-cell*, an *arc*, and a *1-simplex*.

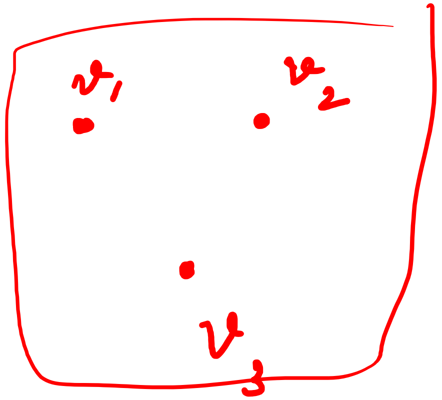
The number of vertices in G is often called the **order** of G, while the number of edges is its **size**.

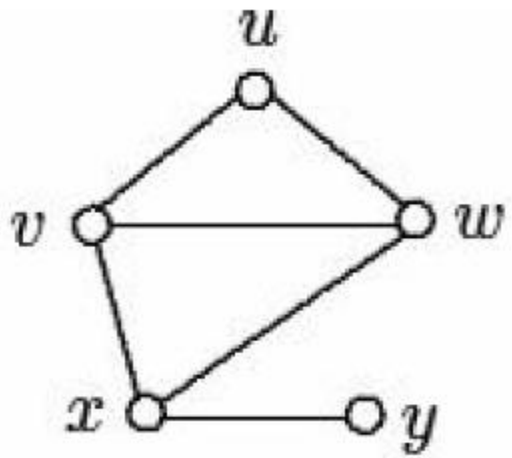
$V \neq \emptyset$

$E = \emptyset$

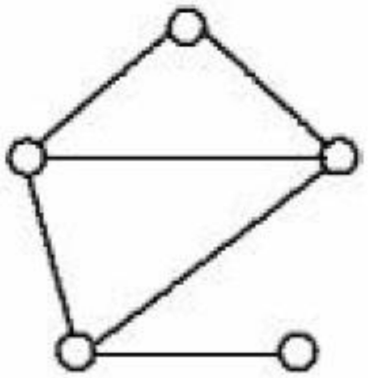
Since the vertex set of every graph is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a **trivial graph**, implying that the order of a **nontrivial graph** is at least 2.







labeled graph



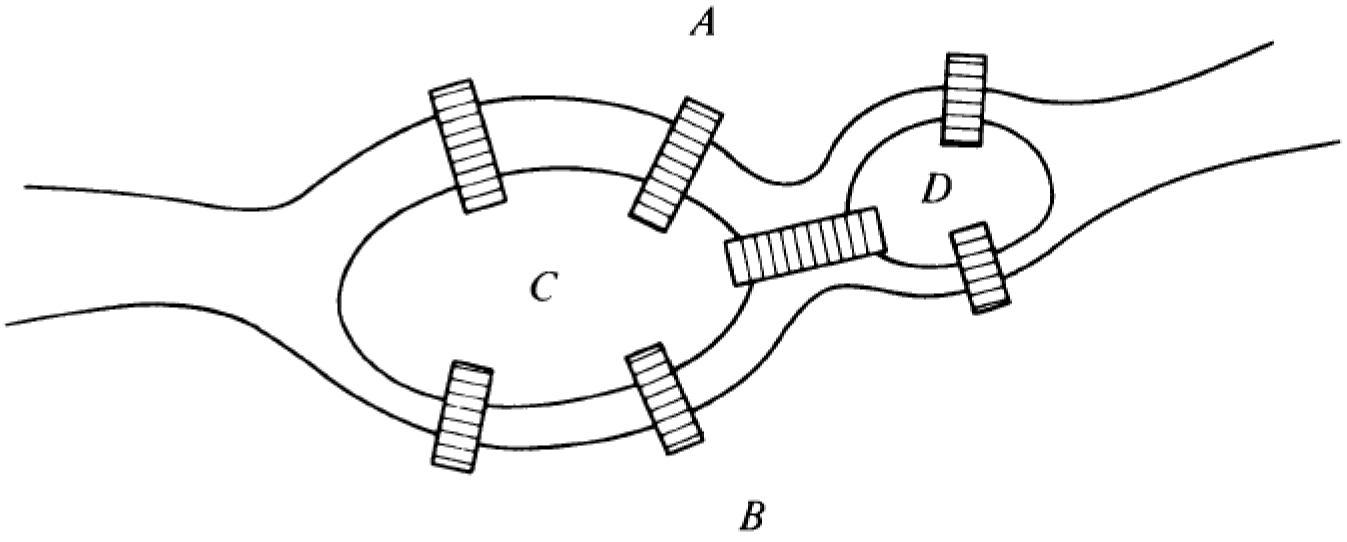
unlabeled graph

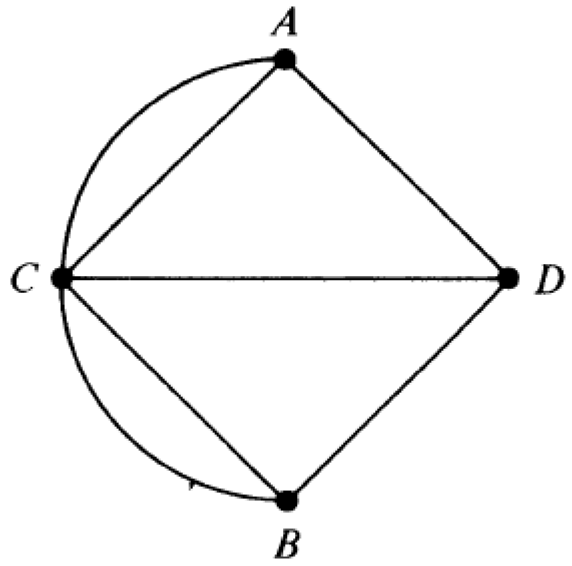
Applications of Graph theory

A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them.

Königsberg Bridge Problem.

Leonard Euler





Finite and infinite graphs

A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph;* otherwise, it is an *infinite graph*.

Incidence and Degree

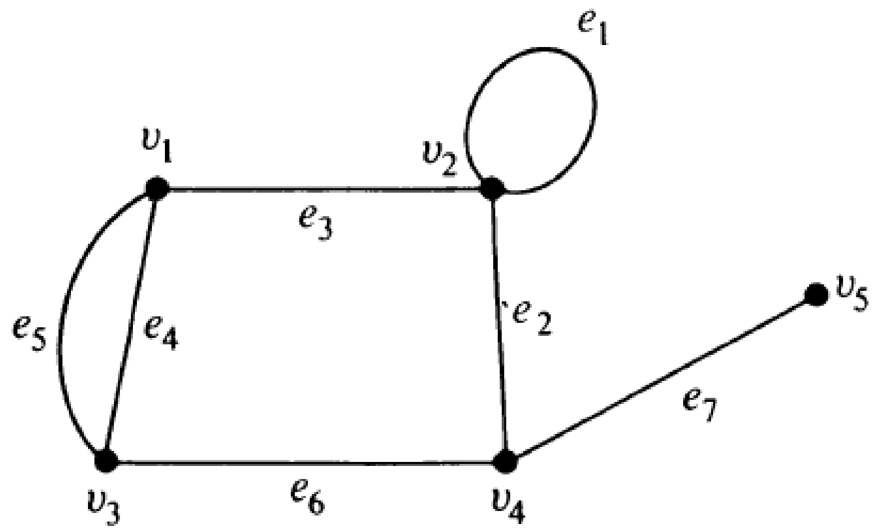
When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident with* (*on* or *to*) each other.

Two nonparallel edges are said to be *adjacent* if they are incident on a common vertex.

Two vertices are said to be adjacent if they are the end vertices of the same edge.

The number of edges incident on a vertex v_i, with self-loops counted twice, is called the *degree*, $d(v_i)$, of vertex v_i.

The degree of a vertex is also referred to as its *valency*.



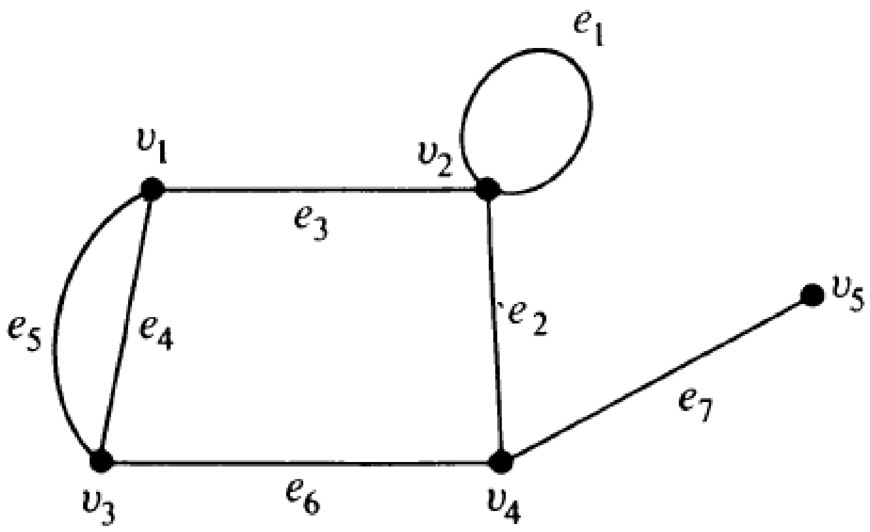
$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 3$

$d(v_4) = 3$

$d(v_5) = 1$



Let us now consider a graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

THEOREM 1-1

The number of vertices of odd degree in a graph is always even.

Proof

$$\sum_{i=1}^{n} d(v_i) = 2e \quad = \text{even}$$

$2+4+6+\ldots$ $\qquad$ $1+3+5+\ldots$

$$\underbrace{\sum_{i=1}^{n} d(v_i)}_{\text{even}} = \underbrace{\sum_{\text{even}} d(v_j)}_{\text{even}} + \underbrace{\sum_{\text{odd}} d(v_k)}_{\text{even}}$$

$$\sum_{i=1}^{n} d(v_i) - \sum_{\text{even}} d(v_j) = \sum_{\text{odd}} d(v_k)$$

even − even is even. To form LHS = RHS, $\sum_{\text{odd}} d(v_k)$ should be even. So, count of odd degree vertices should be even.

$\overset{12}{\text{even}} + \overset{3}{\text{odd}} = \text{odd}$ ×

$\text{even} + \underbrace{(\overset{3}{\text{odd}} + \overset{5}{\text{odd}})}_{\text{even}} = \text{even}$

× ev $\quad (\overset{1}{\text{odd}} + \overset{7}{\text{odd}} + \overset{3}{\text{odd}}) =$

$\underbrace{0+0+0+0}$

A graph in which all vertices are of equal degree is called a *regular graph* (or simply a *regular*).

2-regular

3

K-regular

A graph is said to be k-regular if the degree, $\mathrm{d(v)} = \mathrm{k}, \forall v \in V$

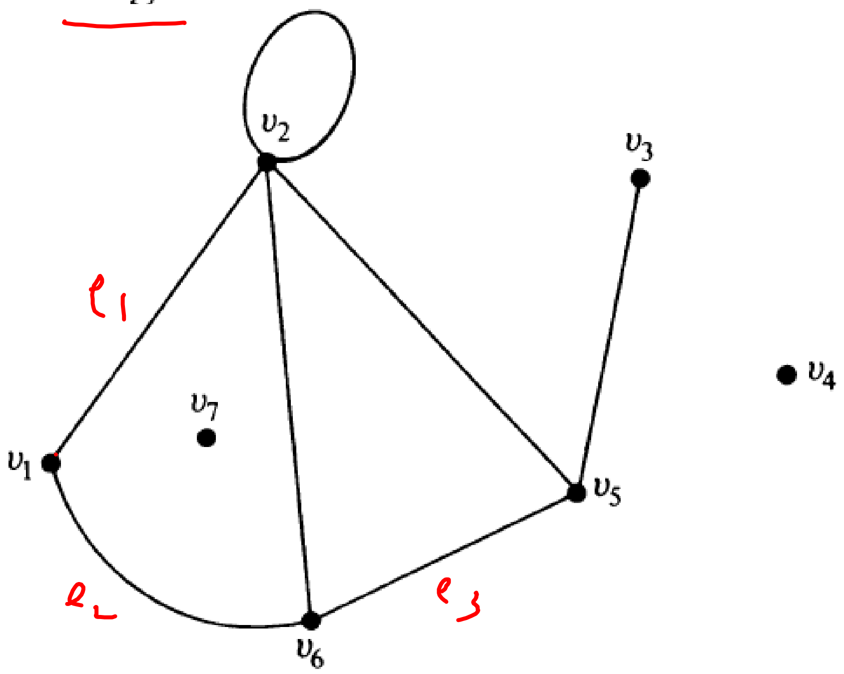
A complete graph with n vertices is (n-1) regular

Isolated vertex & pendent vertex

A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are vertices with zero degree.

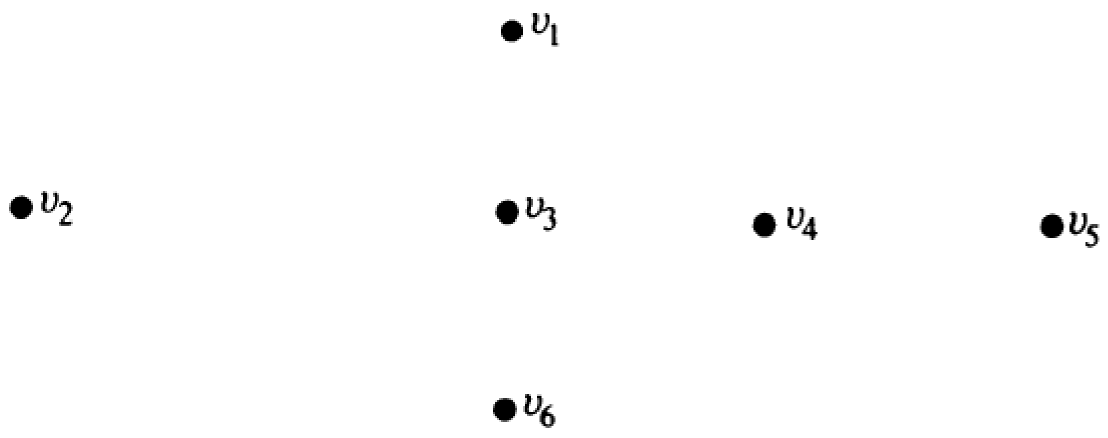
A vertex of degree one is called a *pendant vertex* or an *end vertex*.

Two adjacent edges are said to be in *series* if their common vertex is of degree two.



Null graph

In the definition of a graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph, without any edges, is called a *null graph*. In other words, every vertex in a null graph is an isolated vertex.



Directed graph

Let V be a finite nonempty set, and let $E \subseteq V \times V$. The pair $(V,\ E)$ is then called a *directed graph* (on V), or *digraph* (on V), where V is the set of *vertices*, or *nodes*, and E is its set of (*directed*) *edges* or *arcs*.

- Degree of a vertex
    - In-degree
    - Out-degree

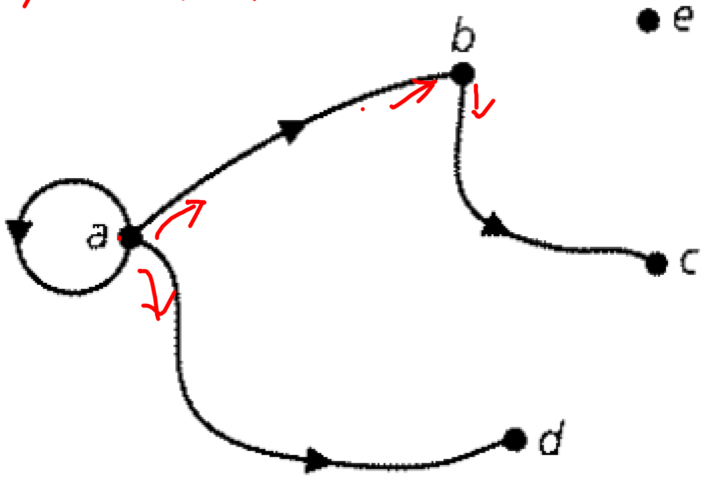
$I(e) = O(e) = 0$

$I(b) = 1$

$O(b) = 1$

$I(a) = 1$

$O(a) = 3$



Subgraph

A graph H is called a **subgraph** of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G.

If a subgraph of a graph G has the same vertex set as G, then it is a **spanning subgraph** of G.

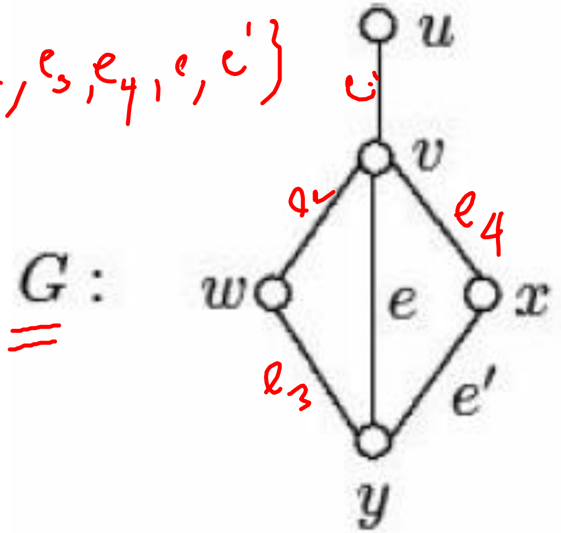
A subgraph F of a graph G is called an **induced subgraph** of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well.

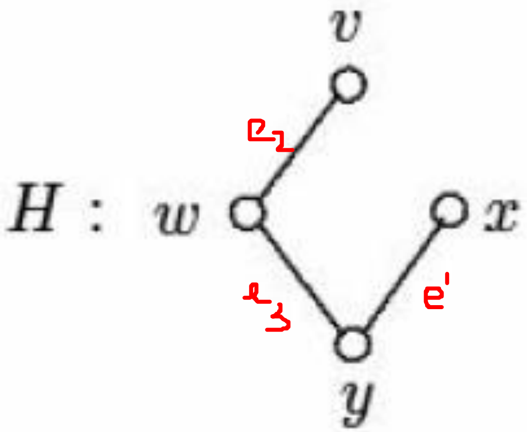
If S is a nonempty set of vertices of a graph G, then the **subgraph of** G **induced by** S is the induced subgraph with vertex set S. This induced subgraph is denoted by $G[S]$.

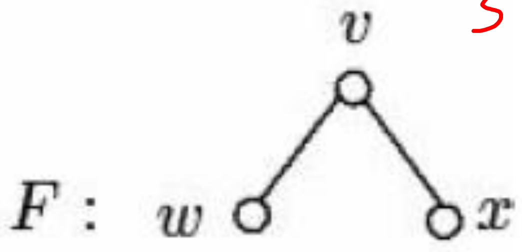
For a nonempty set X of edges,the **subgraph** $G[X]$ **induced by** X has edge set X and consists of all vertices that are incident with at least one edge in X is called an **edge-induced subgraph** of G.

$V(G) = \{u, v, w, x, y\}$

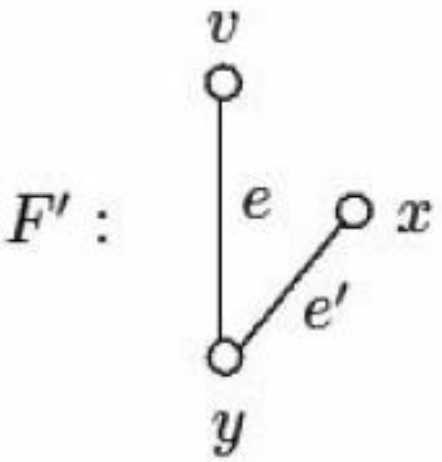
$E(G) = \{e_1, e_2, e_3, e_4, e, e'\}$



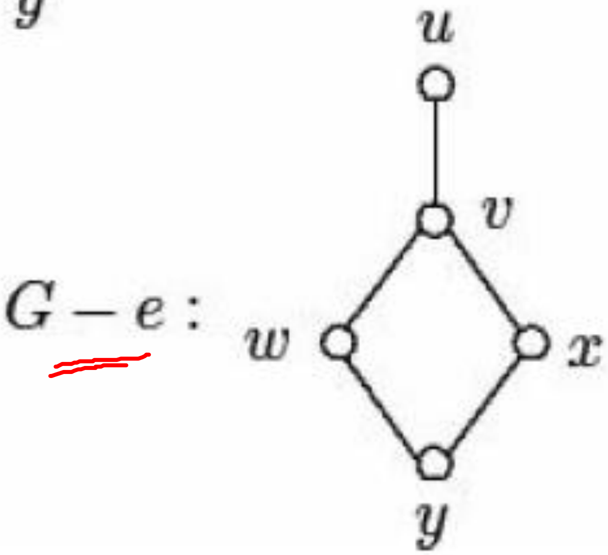




$S = \{v, w, x\}$



$X = \{e, e'\}$



spanning

$G - e_1 - e_2$

$U = \{e_1, e_2\}$

$G - U$

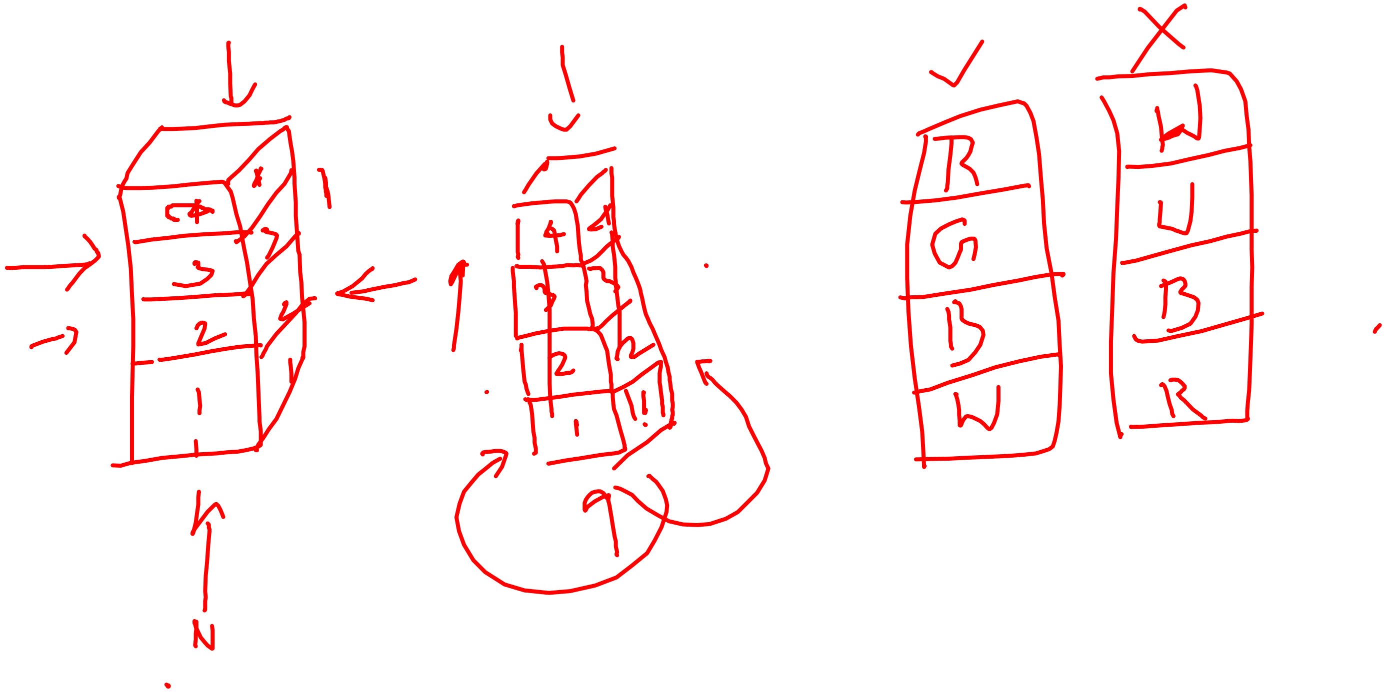
G

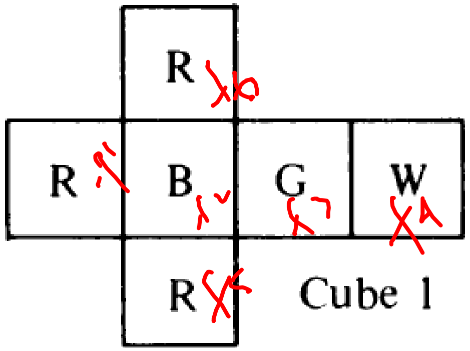
$G + \underline{e}$

$G + v$

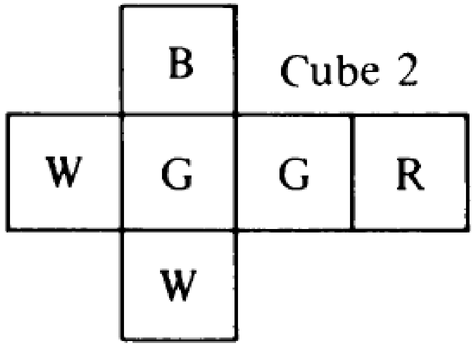
Puzzle with Multicolored Cubes

Problem: We are given four cubes. The six faces of every cube are variously colored blue, green, red, or white. Is it possible to stack the cubes one on top of another to form a column such that no color appears twice on any of the four sides of this column?

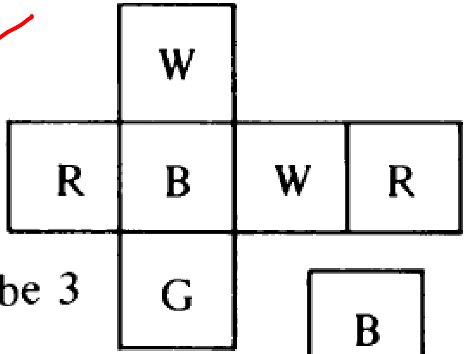




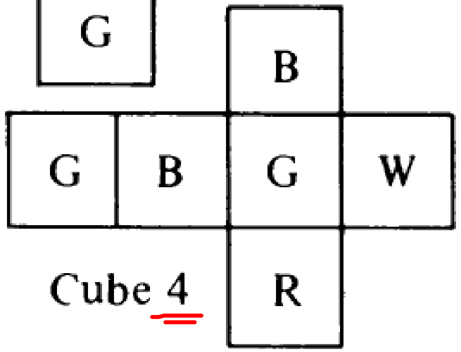
Cube 1



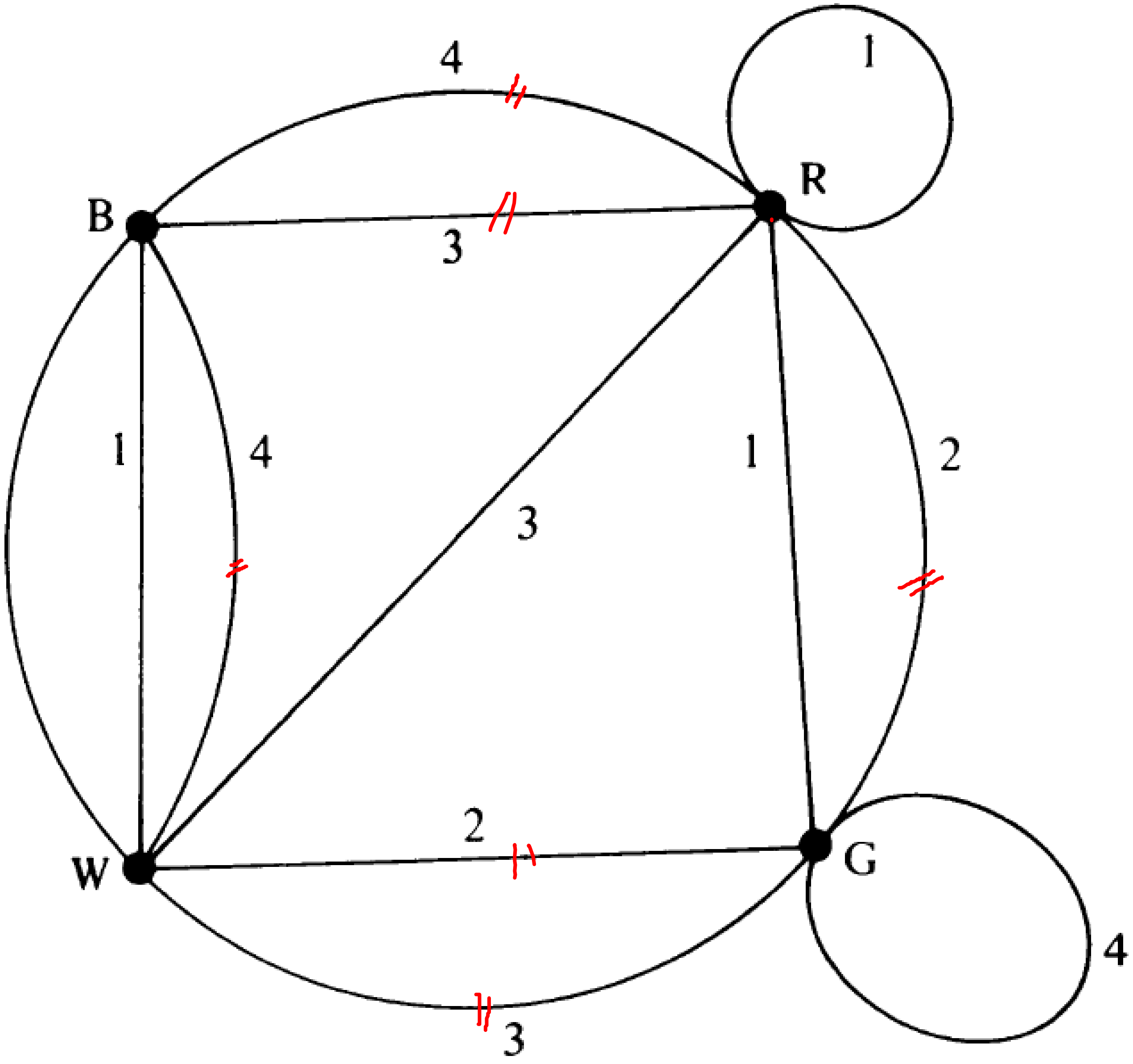
Cube 2

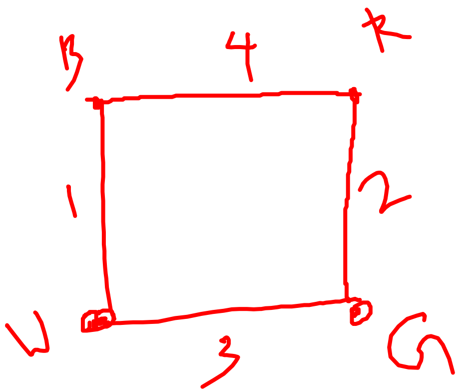


Cube 3

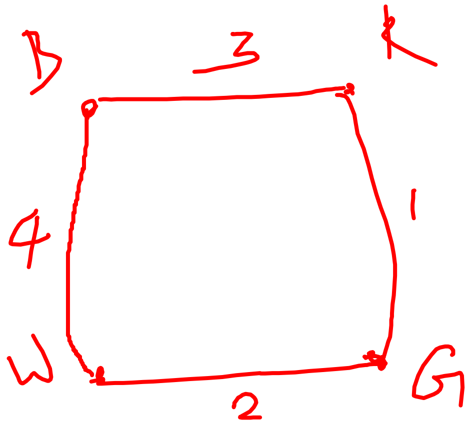


Cube 4





N. S



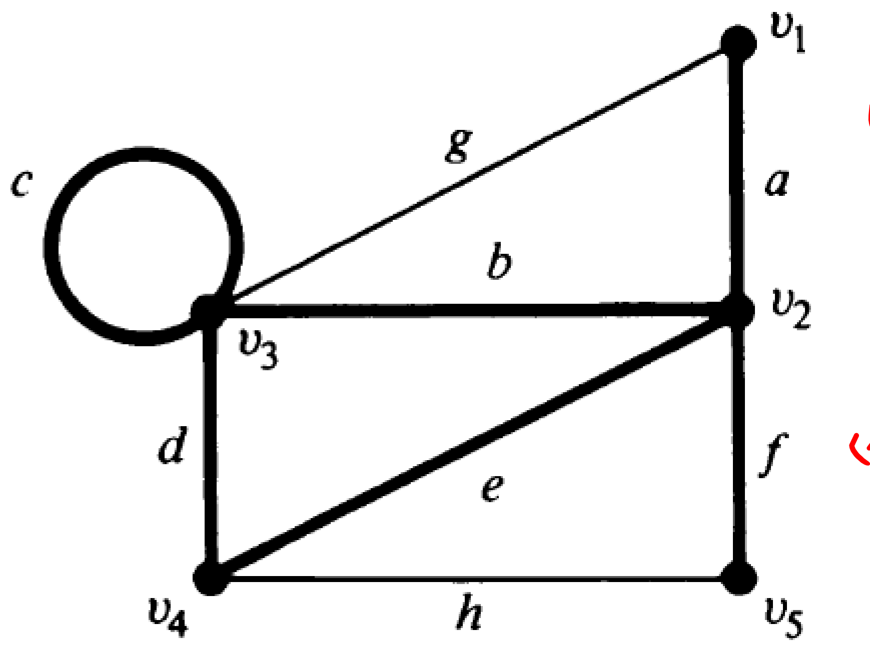
E-W

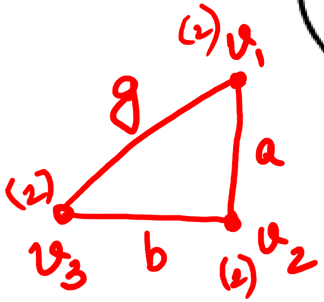
Walk

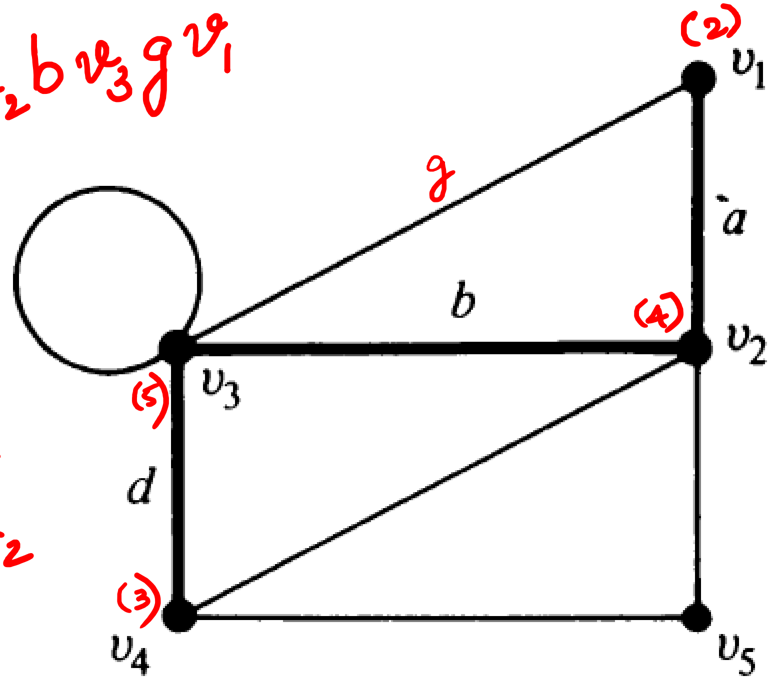
A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears (is covered or traversed) more than once in a walk. A vertex, however, may appear more than once.

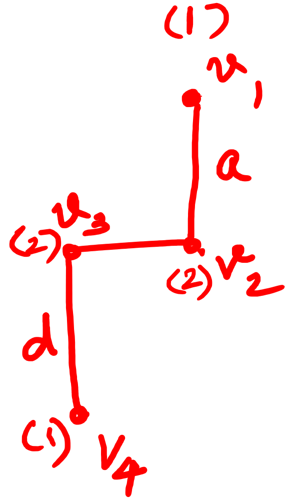
$w3: v_1 a v_2 b v_3 g v_1$

$v_3 c v_3$









$w1: v_1 \, a \, v_2 \, b \, v_3 \, c \, v_3 \, d \, v_4 \, e \, v_2 \, f \, v_5$

$w2: v_1 \, a \, v_2 \, b \, v_3 \, d \, v_4$ ⇒ path

A walk is also referred to as an *edge train* or a *chain*.

Vertices with which a walk begins and ends are called its *terminal vertices*.

It is possible for a walk to begin and end at the same vertex. Such a walk is called a *closed walk*. A walk that is not closed (i.e., the terminal vertices are distinct) is called an *open walk*

Path

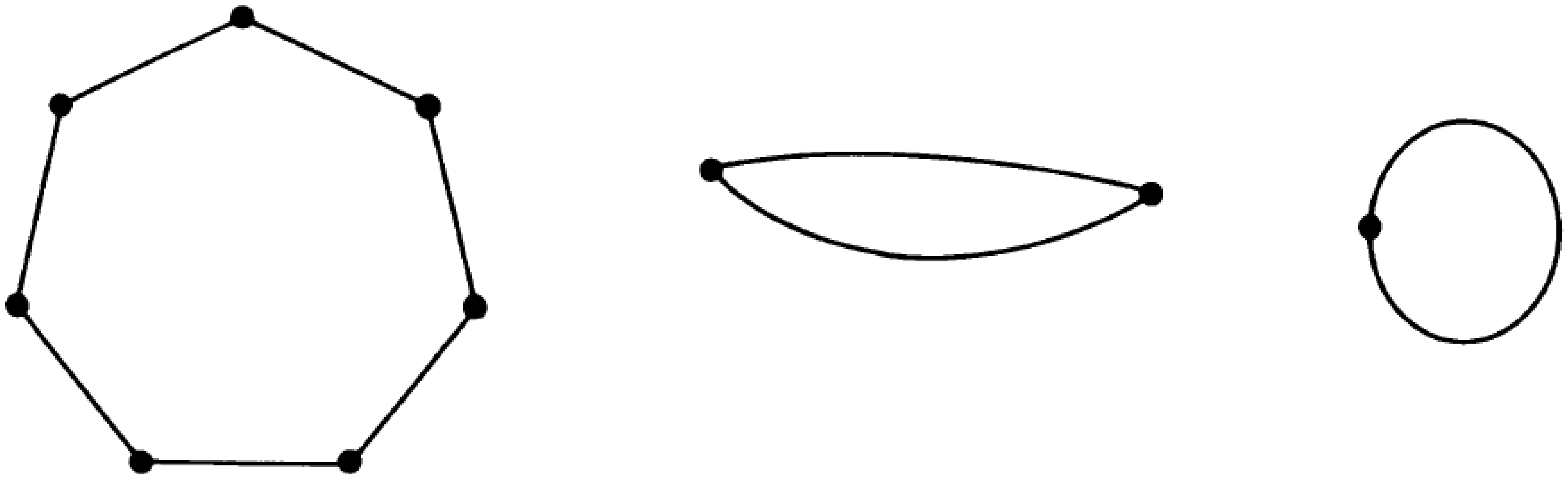
An open walk in which no vertex appears more than once is called a *path* (or a *simple path* or an *elementary path*).

A path does not intersect itself. The number of edges in a path is called the *length of a path*. It immediately follows, then, that an edge which is not a self-loop is a path of length one.

The terminal vertices of a path are of degree one, and the rest of the vertices (called *intermediate vertices*) are of degree two. This degree, of course, is counted only with respect to the edges included in the path and not the entire graph in which the path may be contained.

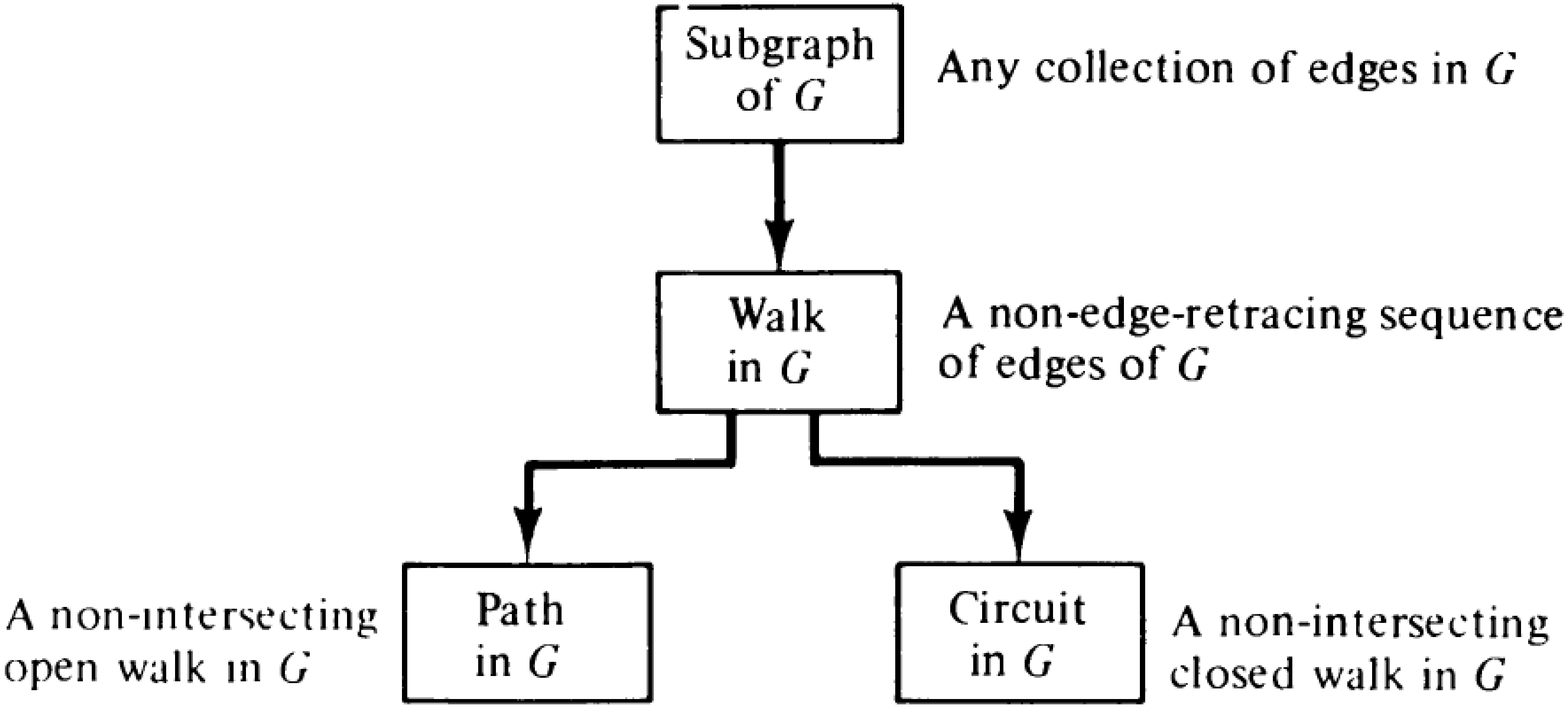
Circuit

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit*. That is, a circuit is a closed, non-intersecting walk.



every vertex in a circuit is of degree two; again, if the circuit is a subgraph of another graph, one must count degrees contributed by the edges in the circuit only.

A circuit is also called a *cycle*, *elementary cycle*, *circular path*, and *polygon*.



Walks, paths, and circuits as subgraphs.

Connected Graphs

- A graph is connected if we can reach any vertex from any other vertex by travelling along the edges.

A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. Otherwise, G is *disconnected*.

A null graph of more than one vertex is disconnected.

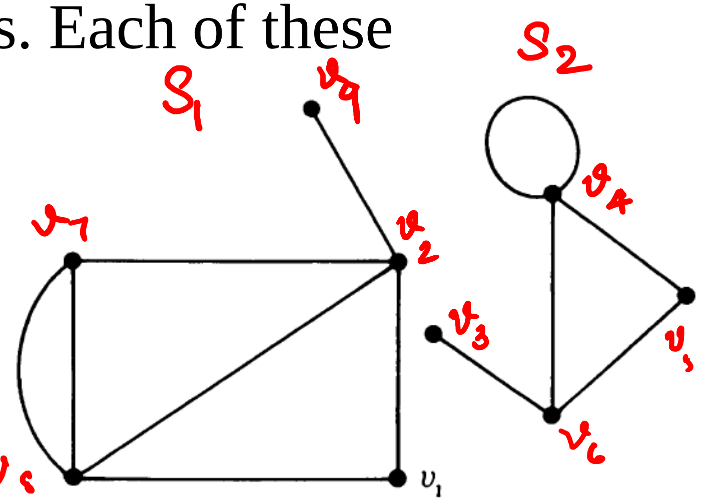
A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

$V(G) = \{v_1, \dots v_9\}$

- A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

$V(S_1) = \{v_1, v_2, v_7, v_8, v_9\}$

$V(S_2) = \{v_3, v_4, v_5, v_6\}$



Theorem 1-2

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

- Proof:
  - Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.
  - Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition.

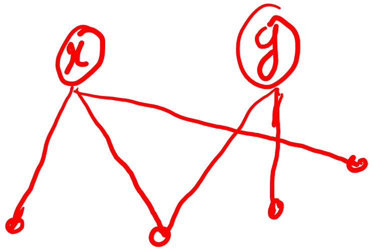
THEOREM 1-3

If a graph (connected or disconnected) has exactıy two vertices of odd degree, there must be a path joining these two vertices.

- *Proof:* Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd. From Theorem 1-1, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, $v1$ and $v2$ must belong to the same component, and hence must have a path between them.

Theorem 1-4

- Let G be a graph of order n. If d(u)+d(v) >= n-1 for every two non-adjacent vertices u and v of G, then G is connected.
- Proof – We need to prove that every two vertices of G are connected by a path
  - Let x,y $\in V$ of G. If (x,y) $\in E$, x and y are adjacent. Assume that (x,y)$\notin E$.
  - d(x) + d(y) >= n-1 implies that there must be a vertex, v_i adjacent to x and y. When x and y are not adjacent, there is a path between x and y always.
  - Hence, G is connected



$n=6$

$d(x)+d(y) \geq 5$

Theorem 1-5

- Let G be a graph of order n with d(G) >= (n-1)/2, then G is connected.
- Proof –
  - For every two non-adjacent vertices u and v of G
  - d(x) + d(y) >= (n-1)/2 + (n-1)/2 = n-1
  - Hence, according to Theorem 1-4, G is connected

THEOREM 1-6

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

$n_1 + n_2 + n_3 + \dots n_k = n$

$\frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2}$

${}^nC_2$

$\frac{1}{2} n_1(n_1 - 1)$

$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1)$

Complement of graph

- Has the same set of vertices as G, but 2 vertices are adjacent in G' if they are adjacent in G.

Theorem 1-7

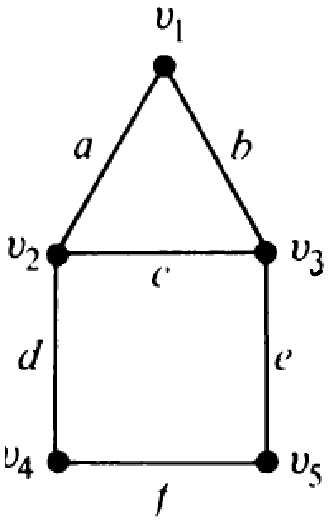
- If G is disconnected then G’ is connected.

Operations on Graphs

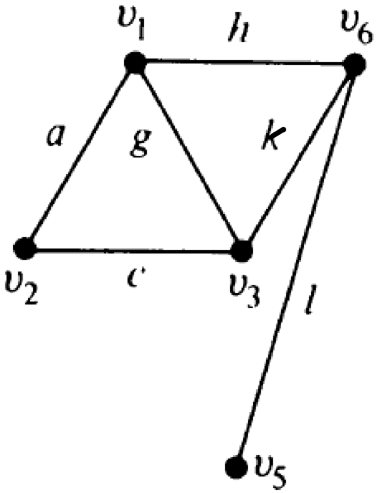
The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3 (written as $G_3 = G_1 \cup G_2$) whose vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$.

The *intersection* $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2.

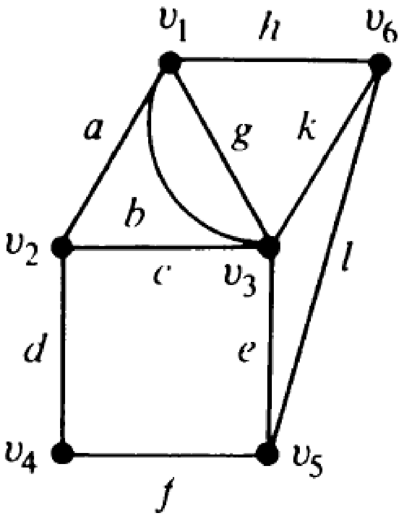
The *ring sum* of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but *not* in both.



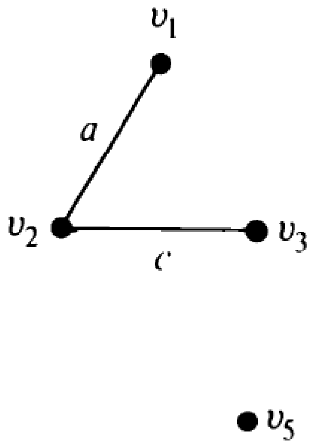
G_1



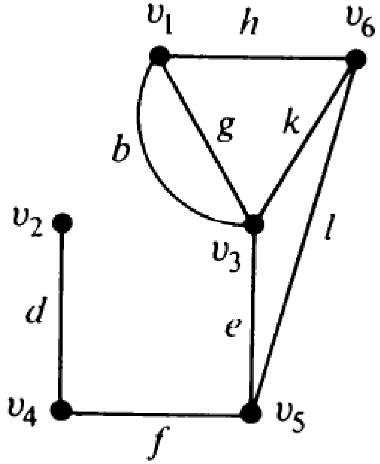
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \oplus G_2$

It is obvious from their definitions that the three operations just mentioned are commutative. That is,

$$G_1 \cup G_2 = G_2 \cup G_1, \qquad G_1 \cap G_2 = G_2 \cap G_1,$$

$$G_1 \oplus G_2 = G_2 \oplus G_1.$$

If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$. If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.

For any graph G,

$$G \cup G = G \cap G = G,$$

and

$$G \oplus G = \text{a null graph.}$$

If g is a subgraph of G, then $G \oplus g$ is, by definition, that subgraph of G which remains after all the edges in g have been removed from G. Therefore, $G \oplus g$ is written as $G - g$, whenever $g \subseteq G$. Because of this complementary nature, $G \oplus g = G - g$ is often called the complement of g in G.

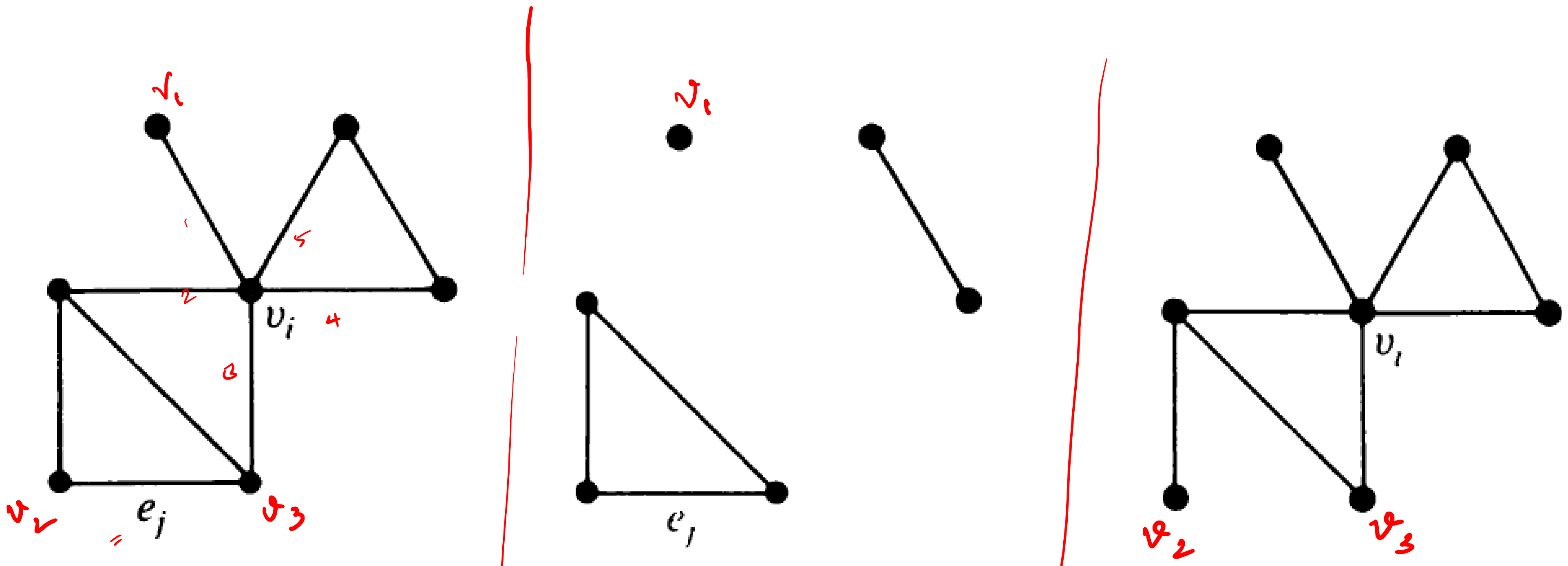
Decomposition: A graph G is said to have been *decomposed* into two subgraphs g_1 and g_2 if

$$g_1 \cup g_2 = G,$$

and

$$g_1 \cap g_2 = \text{a null graph.}$$

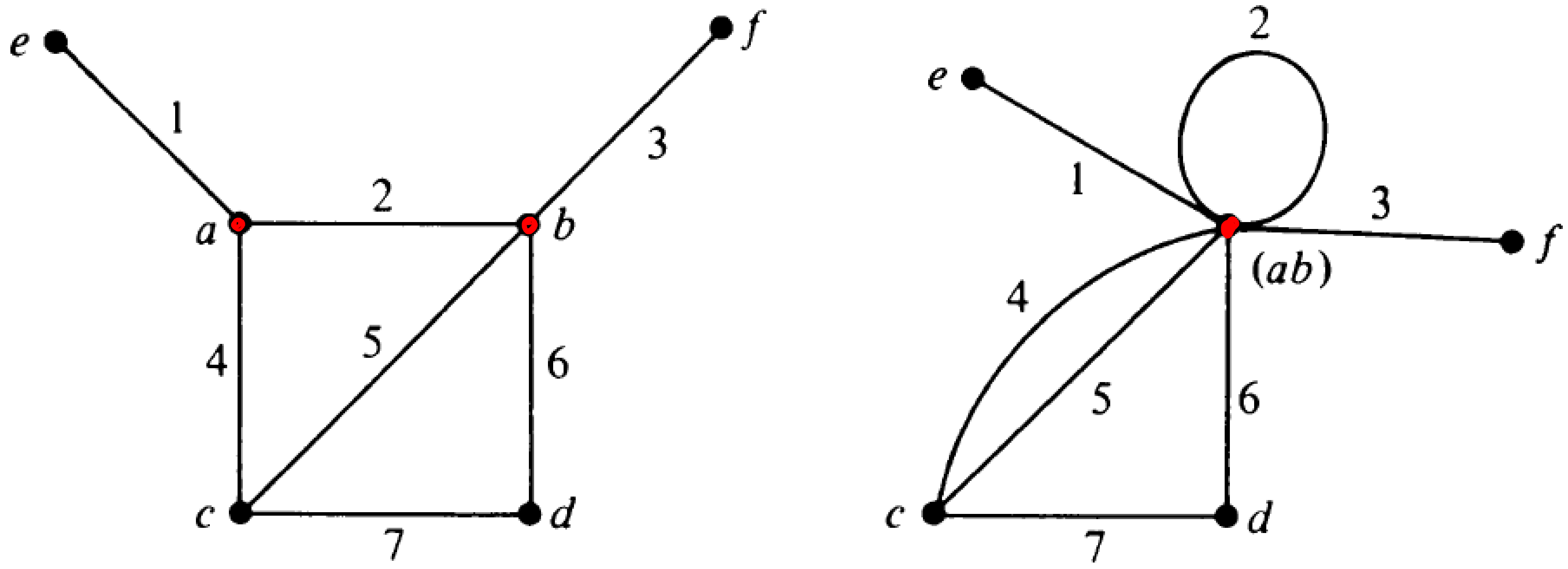
Deletion: If v_i is a vertex in graph G, then $G - v_i$ denotes a subgraph of G obtained by deleting (i.e., removing) v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex. If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices. Therefore $G - e_j = G \oplus e_j$.



G $(G - v_i)$ $(G - e_j)$

Vertex deletion and edge deletion.

Fusion: A pair of vertices a, b in a graph are said to be *fused* (*merged* or *identified*) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. Thus fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one.



G Fusion of vertices a and b. G'

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