Principle of inclusion and exclusion

Let S represent the set of 100 students enrolled in the freshman engineering program at Central College. Then $|S| = 100$. Now let c_1, c_2 denote the following conditions (or properties) satisfied by some of the elements of S:

c_1: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Freshman Composition.

c_2: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Economics.

Suppose that 35 of these 100 students are enrolled in Freshman Composition and that 30 of them are enrolled in Introduction to Economics.

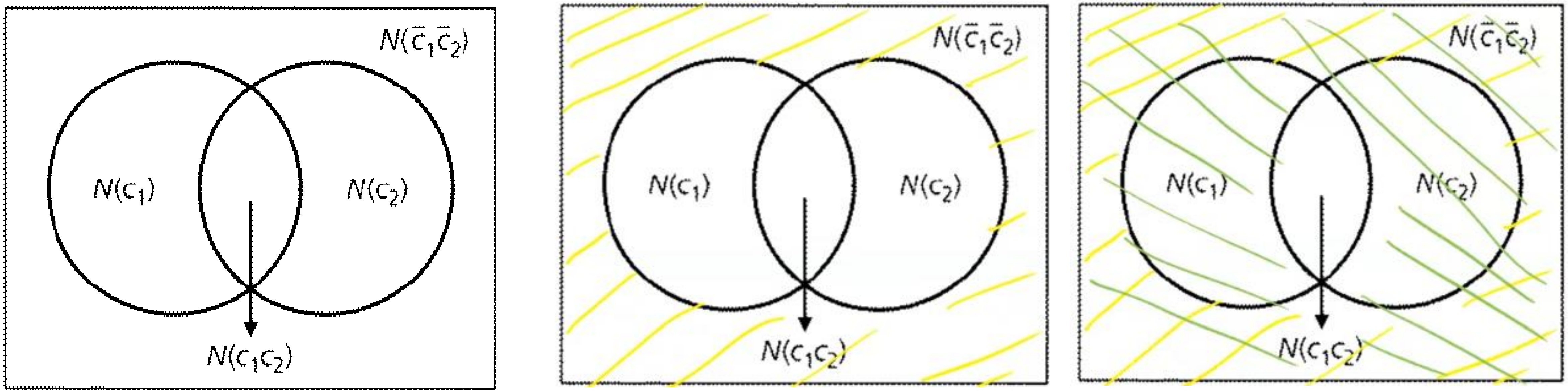
If nine of these 100 students are enrolled in both Freshman Composition and Introduction to Economics then we write $N(c_1c_2) = 9$.

$N = 100 \leftarrow |S|$ $\quad N(C_1) = 35$ $\quad N(C_2) = 30$ $\quad N(C_1C_2) = 9$

$N(\bar{C_1}) = \overset{N}{100} - \overset{N(C_1)}{35} = \underline{\underline{65}}$ $\quad N(\bar{C_2}) = N - N(C_2) = \underline{\underline{70}}$

$N(C_1\bar{C_2}) = N(C_1) - N(C_1C_2) = \underline{\underline{26}}$ $\quad N(\bar{C_1}C_2) = N(C_2) - N(C_1C_2)$

$= \underline{\underline{21}}$



$$N(\bar{c}_1\bar{c}_2) = N - [N(c_1) + N(c_2)] + N(c_1c_2)$$

$$N(\bar{c}_1\,\bar{c}_2) = N(\overline{c_1 c_2})$$

c_3: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Fundamentals of Computer Programming.

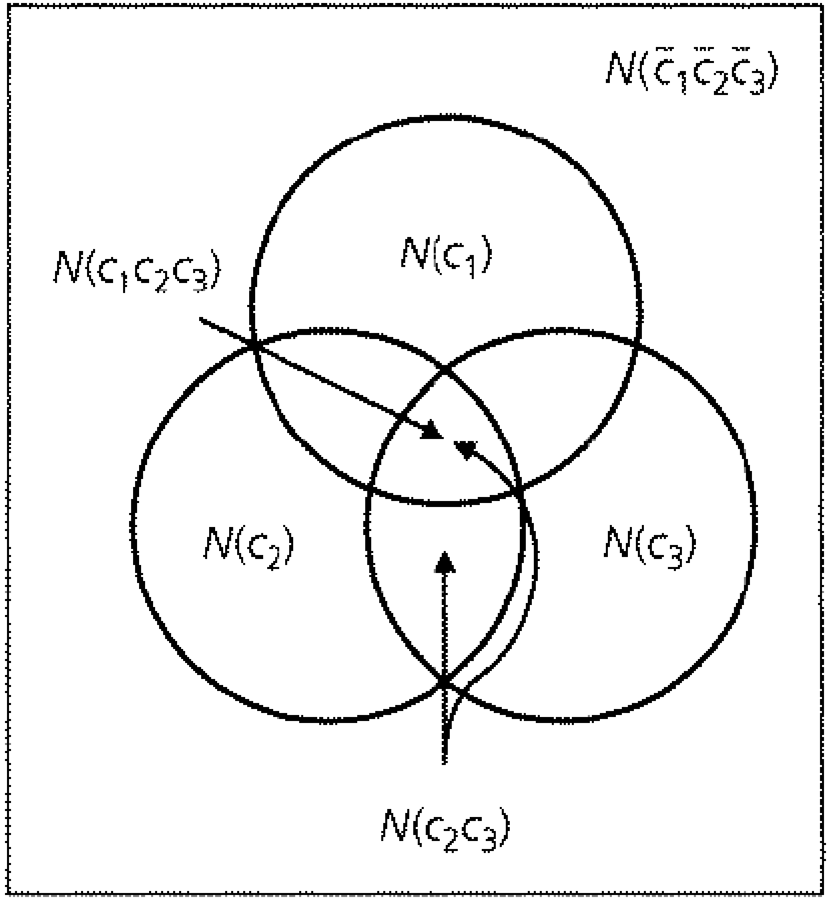
also given that $N(c_3) = 30$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$

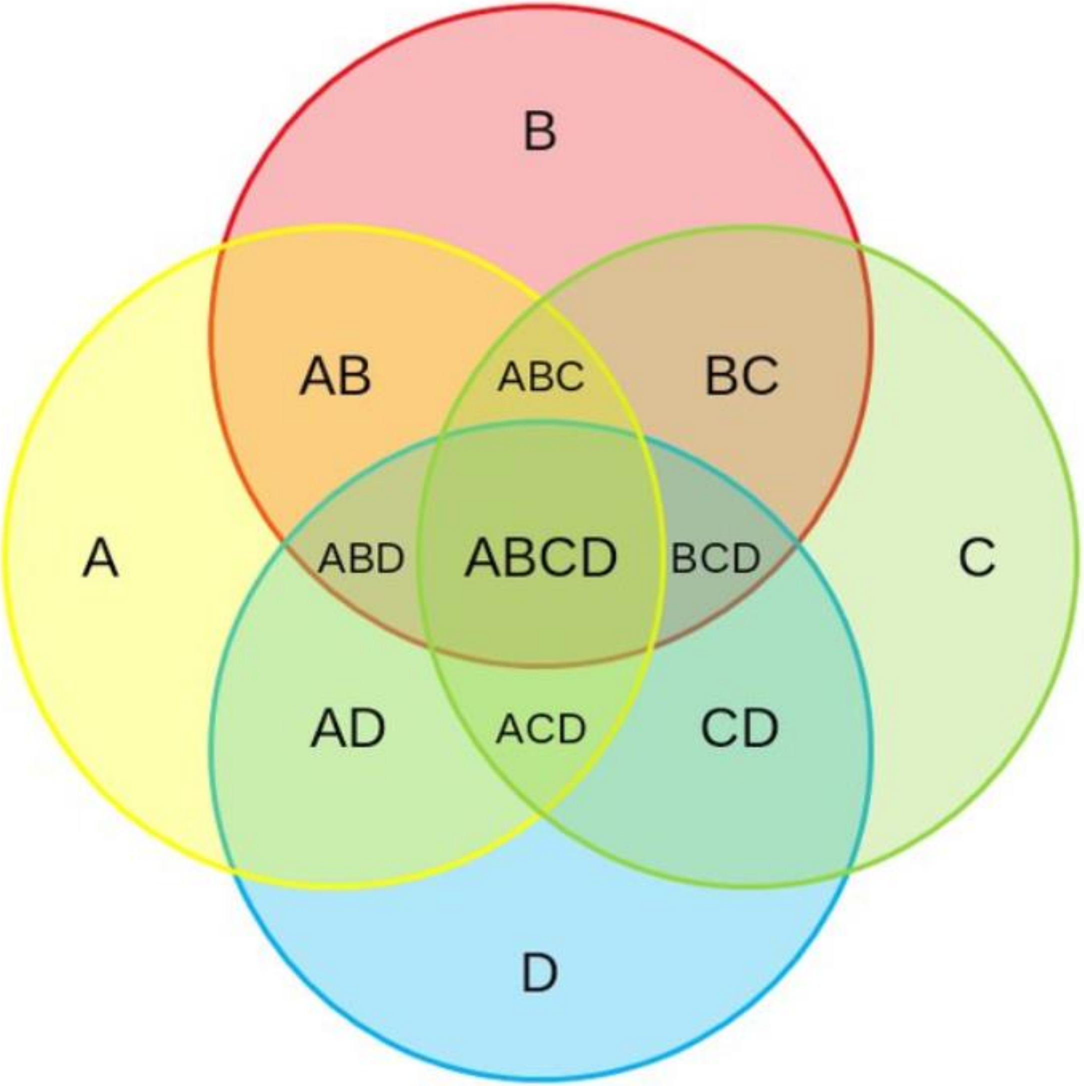
$N(\overline{c}_1\overline{c}_2\overline{c}_3) =$ $N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$

$N(\overline{c}_3) =$ $N - N(c_3) = 70$

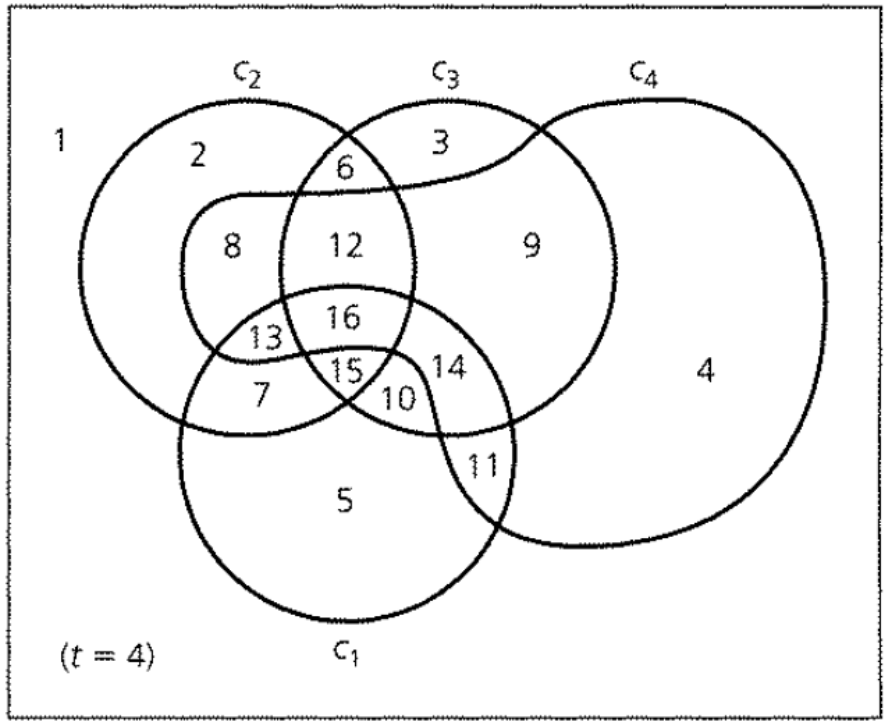
$N(\overline{c}_1\overline{c}_3) =$ $N - [N(c_1) + N(c_3)] + N(c_1c_3) = 46$

$N(\overline{c}_2\overline{c}_3)$




B
AB
ABC
BC
A
ABD
ABCD
BCD
C
AD
ACD
CD
D

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$
$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)]$$
$$+ N(c_1c_2c_3c_4).$$



$x \in S$

0) once in LHS , RHS N

1) $x \in c_1$; LHS 0 times ; RHS $N - N(c_1)$
$1 - 1 = 0$

$1 - \binom{1}{1} = 0$

2) $x \in c_2, x \in c_4$; LHS $= 0$; RHS
$N - [N(c_2) + N(c_4)] + N(c_2c_4)$
$1 - [1 + 1] + 1 = 0$

$1 - \binom{2}{1} + \binom{2}{2}$

3) $x \in C_1, C_2, C_4$; LHS = 0 ; RHS $N - [N(C_1) + N(C_2) + N(C_4)] + [N(C_1C_2) + N(C_1C_4) + N(C_2C_4)] - N(C_1C_2C_4) = 1 - [1+1+1] + [1+1+1] - 1$

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3}$$

4) $x \in C_1C_2C_3C_4$

RHS = $1 - [1+1+1+1] + [1+1+1+1+1+1] - [1+1+1+1] + 1$

$$1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$$

c_4: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Design.

We already know that $N(c_1) = 35$, $N(c_2) = 30$, $N(c_3) = 30$, $N(c_1c_2) = 9$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$. If $N(c_4) = 41$, $N(c_1c_4) = 13$, $N(c_2c_4) = 14$, $N(c_3c_4) = 10$, $N(c_1c_2c_4) = 6$, $N(c_1c_3c_4) = 6$, $N(c_2c_3c_4) = 6$, and $N(c_1c_2c_3c_4) = 4$, then, using

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) =$$

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) =$$

$$N(\overline{c}_2\overline{c}_3\overline{c}_4) = N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) + N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$$

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$$

$$
\begin{aligned}
N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) &= N(\bar{c}_2\bar{c}_3\bar{c}_4) - N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) \\
&= \Big\{N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\
&\quad - N(c_2c_3c_4)\Big\} - \Big\{N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\
&\quad + [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\
&\quad - [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\Big\}, \text{ or}
\end{aligned}
$$

$$
\begin{aligned}
N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) &= N(c_1) - [N(c_1c_2) + N(c_1c_3) + N(c_1c_4)] \\
&\quad + [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4)] - N(c_1c_2c_3c_4).
\end{aligned}
$$

The Principle of Inclusion and Exclusion. Consider a set S, with $|S| = N$, and conditions c_i, $1 \leq i \leq t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy *none* of the conditions c_i, $1 \leq i \leq t$, is denoted by $\overline{N} = N(\overline{c}_1\overline{c}_2\overline{c}_3 \cdots \overline{c}_t)$ where

$$\begin{aligned}\overline{N} = N &- [N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_t)] \\ &+ [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots \\ &+ N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)] + \cdots + (-1)^t N(c_1c_2c_3 \cdots c_t),\end{aligned}$$

or

$$\begin{aligned}\overline{N} = N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i < j \leq t} N(c_ic_j) - \sum_{1 \leq i < j < k \leq t} N(c_ic_jc_k) + \cdots \\ + (-1)^t N(c_1c_2c_3 \cdots c_t).\end{aligned}$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms in Eq. Consequently, x contributes a count of 1 to each side of the equation.

The other possibility is that x satisfies *exactly* r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. x is counted

(1) One time in N.

(2) r times in $\sum_{1 \le i \le t} N(c_i)$. (Once for each of the r conditions.)

(3) $\binom{r}{2}$ times in $\sum_{1 \le i < j \le t} N(c_i c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

(4) $\binom{r}{3}$ times in $\sum_{1 \le i < j < k \le t} N(c_i c_j c_k)$.

. . . .

$\binom{r}{r} = 1$ time in $\sum N(c_{i_1} c_{i_2} \cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq., x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times}$$

the number of elements in S that satisfy at least one of the conditions c_i, where $1 \leq i \leq t$, is given by $N(c_1 \text{ or } c_2 \text{ or } \ldots \text{ or } c_t) = N - \overline{N}$.

Determine the number of positive integers n where $1 \leq n \leq 100$ and n is *not* divisible by 2, 3, or 5.

Here $S = \{1, 2, 3, \ldots, 100\}$ and $N = 100$. For $n \in S$, n satisfies

a) condition c_1 if n is divisible by 2,

b) condition c_2 if n is divisible by 3, and

c) condition c_3 if n is divisible by 5.

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

$N(c_1) = 50$ $\quad N(c_2) = 33$ $\quad N(c_3) = 20$

$N(c_1c_2) = 16$ $\quad N(c_1c_3) = 10$ $\quad N(c_2c_3) = 6$

$N(c_1c_2c_3) = 3$

$$\bar{N} = 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 =$$

the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

with the extra restriction that $x_i \leq 7$, for all $1 \leq i \leq 4$.

$S_0 = N = {}^{21}C_{18}$

4 conditions

$\boxed{C_1 : x_1 > 7 \quad x_1 \geq 8}$ $\quad C_2 : x_2 > 7 \quad x_2 \geq 8$

$C_3 : x_3 \geq 8 \quad C_4 : x_4 \geq 8$

$\bar{N} = N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = S_0 - S_1 + S_2 - \underbrace{S_3 + S_4}_{0}$

$\overset{8}{x_1} + x_2 + x_3 + x_4 = 18$

$y_1 + y_2 + y_3 + y_4 = 10$

$N(C_1) = {}^{13}C_{10} \qquad N(C_2) = {}^{13}C_{10}$

$S_1 = \binom{4}{1} {}^{13}C_{10}$

$N(C_1C_2) = {}^{5}C_{2}$

$S_2 = \binom{4}{2} {}^{5}C_{2}$

$y_1 + y_2 + y_3 + y_4 = 2$

$N(C_1C_2C_3) = 0$

$\bar{N} = {}^{21}C_{18} - {}^{4}C_1 \, {}^{13}C_{10} + {}^{4}C_2 \cdot {}^{5}C_2 - 0 + 0 = 246$

In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns *car*, *dog*, *pun*, or *byte* occurs?

$N = 26!$

$N(c_1) = 24!$ car, b, d, e, ... - 24

$N(c_2) = N(c_3) = 24!$ $N(c_4) = 23!$ $26-4 = 22+1$

$N(c_1 c_2) = 22!$ $N(c_1 c_3) = N(c_2 c_3) = 22!$

$N(c_i c_4) = 21!$ $i \neq 4$

$N(c_1 c_2 c_3) = 20!$ $N(c_i c_j c_4) = 19!$ $1 \leq i < j \leq 3$

$N(c_1 c_2 c_3 c_4) = 17!$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$

=

Six married couples are to be seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband?

$S_0 = N = 11!$

C_1 : couple 1

$10 + 1 = 11$

C_2 : couple 2

$N(C_1) = 2 \cdot 10!$

$S_1 = \binom{6}{1} 2 \cdot 10!$

$N(C_1 C_2) = 2^2 \cdot 9!$

$S_2 = \binom{6}{2} 2^2 \cdot 9!$

$N(C_1 C_2 C_3) = 2^3 \cdot 8!$

$S_3 = \binom{6}{3} \cdot 2^3 \cdot 8!$

$S_4 = \binom{6}{4} 2^4 \cdot 7!$

$S_5 = \binom{6}{5} 2^5 \cdot 6!$

$S_6 = \binom{6}{6} 2^6 \, 5!$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$

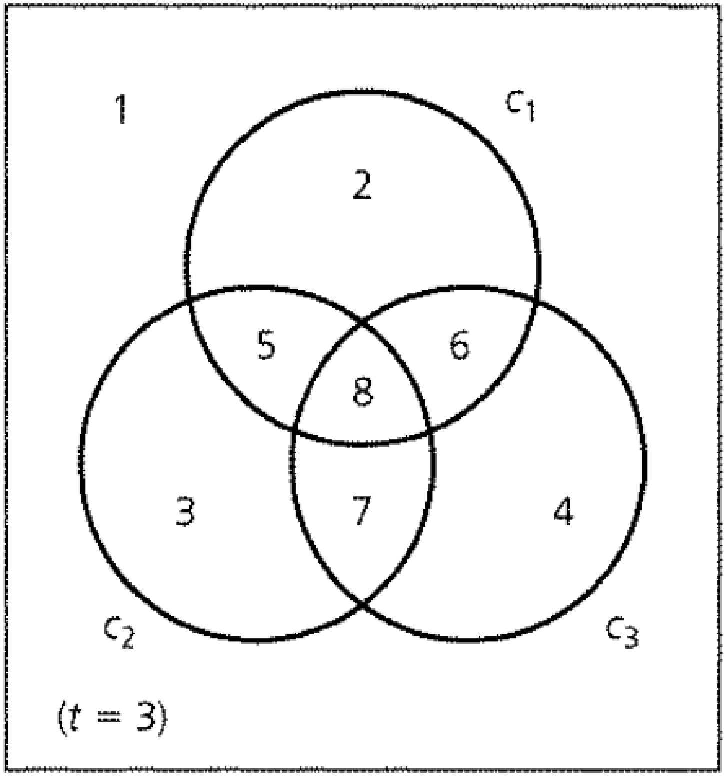
Generalization principle

$$E_1 = N(c_1) + N(c_2) + N(c_3) - 2\,[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3).$$

$$E_1 = S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$$

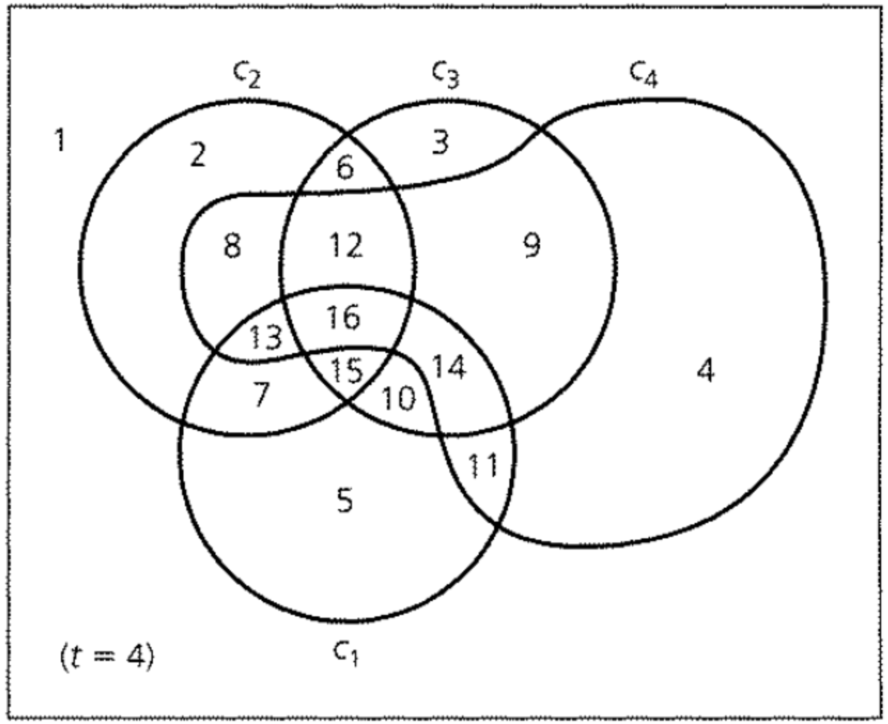
$$E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3) = S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$$

$$E_3 = N(c_1c_2c_3) = S_3$$



E_1 [regions 2, 3, 4, 5]:

$$\begin{aligned} E_1 = {} & [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\ & - 2\,[N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ & + 3\,[N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] \\ & - 4N(c_1c_2c_3c_4) \\ = {} & S_1 - 2S_2 + 3S_3 - 4S_4 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4. \end{aligned}$$



E_2 [regions 6–11]:

$$E_2 = S_2 - 3S_3 + 6S_4 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4.$$

S_2	S_3	S_4
$N(c_1c_2)$: 7, 13, 15, 16 $N(c_1c_3)$: 10, 14, 15, 16 $N(c_1c_4)$: 11, 13, 14, 16 $N(c_2c_3)$: 6, 12, 15, 16 $N(c_2c_4)$: 8, 12, 13, 16 $N(c_3c_4)$: 9, 12, 14, 16	$N(c_1c_2c_3)$: 15, 16 $N(c_1c_2c_4)$: 13, 16 $N(c_1c_3c_4)$: 14, 16 $N(c_2c_3c_4)$: 12, 16	$N(c_1c_2c_3c_4)$: 16

$$E_3 = S_3 - 4S_4 = S_3 - \binom{4}{1}S_4 \qquad E_4 = S_4$$

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t .$$

COROLLARY

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t .$$

a) When x satisfies fewer than m conditions, it contributes a count of 0 to each of the terms E_m, S_m, S_{m+1}, ..., S_t, so it is not counted on either side of the equation.

b) When x satisfies exactly m of the conditions, it is counted once in E_m and once in S_m, but not in S_{m+1}, ..., S_t. Consequently, it is included once in the count for either side of the equation.

c) Suppose x satisfies r of the conditions, where $m < r \leq t$. Then x contributes nothing to E_m. Yet it is counted $\binom{r}{m}$ times in S_m, $\binom{r}{m+1}$ times in S_{m+1}, ..., and $\binom{r}{r}$ times in S_r, but 0 times for any term beyond S_r. So on the right-hand side of the equation, x is counted $\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$ times.
For $0 \leq k \leq r - m$,

$$\binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{k!\,m!} \cdot \frac{r!}{(m+k)!(r-m-k)!}$$

$$= \frac{r!}{m!} \cdot \frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!} \cdot \frac{(r-m)!}{k!(r-m-k)!}$$

$$= \binom{r}{m}\binom{r-m}{k}.$$

Consequently, on the right-hand side of Eq. (1), x is counted

$$\begin{aligned}
&\binom{r}{m}\binom{r-m}{0} - \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m} \\
&\qquad = \binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r-m}{r-m}\right] \\
&\qquad = \binom{r}{m}[1-1]^{r-m} = \binom{r}{m}\cdot 0 = 0 \text{ times},
\end{aligned}$$

Derangement

- A derangement is a permutation with no fixed points
- Example: the derangements of {1,2,3} are {2, 3, 1} and {3, 1, 2}

$\cancel{\{3, 2, 1\}}$

- number of derangements of an n-element set is called the n^{th} derangement number or rencontres number, or the subfactorial of n and is sometimes denoted !n or D_n.

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$n! \cdot e^{-1}$

$$n! - \underline{\binom{n}{1}(n-1)!} + \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)!$$

$$\binom{n}{k}(n-k)! = \frac{n!}{k!\,\cancel{(n-k)!}}\cancel{(n-k)!}$$

$$n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots (-1)^n \frac{n!}{n!} \qquad = \frac{n!}{k!}$$

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \cdot \boxed{\frac{1}{n!}}\right.$$

$$= n!\,\boxed{\sum_{k=0}^{n} (-1)^k \cdot 1/k!} \qquad = \boxed{n!\,e^{-1}}$$

While at the racetrack, Ralph bets on each of the ten horses in a race to come in according to how they are favored. In how many ways can they reach the finish line so that he loses all of his bets?

d_{10}

Peggy has seven books to review for the C–H Company, so she hires seven people to review them. She wants two reviews per book, so the first week she gives each person one book to read and then redistributes the books at the start of the second week. In how many ways can she make these two distributions so that she gets two reviews (by different people) of each book?

7 b   7 p

1^{st} week $= 7!$

2^{nd} week $= d_7 = 7!\,e^{-1}$

$7! \cdot d_7$

$7! \cdot 7!\,e^{-1}$