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LPP :-

Min / Max $f(\vec{x}) = c^T \vec{x}$

Objective function

s.t. $\left[A\vec{x} = b \right]$

Constraints eq^n s

$\vec{x} \geq 0$

$$\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad c = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix}$$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Min / Max

$$c^T \vec{x} = c_1x_1 + c_2x_2 + \cdots + c_nx_n = f(x_1, x_2, \ldots, x_n)$$

s.t the constraints.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$x_1, x_2, \ldots, x_n \geq 0.$$

$$b = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{Bmatrix}$$

Linear Programming Problem

Constraints eq^n s :-

$Ax = b$

(i) $AX \leq b$ $\quad\to\quad$ $AX + s = b$ $\to$ slack variables.

(ii) $Ax = b$ $\quad\to\quad$ $AX + a = b$ $\to$ artificial variable

(iii) $Ax \geq b$

$\downarrow$

$Ax - s + a = b$

$$Ax - S + a = b$$

$S \rightarrow$ Surplus variable.

Method 1 :- Graphical method

Max $50x_1 + 100x_2$

s.t. $10x_1 + 5x_2 \le 2500$

$4x_1 + 10x_2 \le 2000$

$x_1 + \frac{3}{2}x_2 \le 450$

$x_1, x_2 \geq 0$

$$\frac{x_1}{250} + \frac{x_2}{500} \le 1$$

$$\frac{x_1}{500} + \frac{x_2}{200} \le 1$$

$$\frac{x_1}{450} + \frac{x_2}{300} \le 1$$

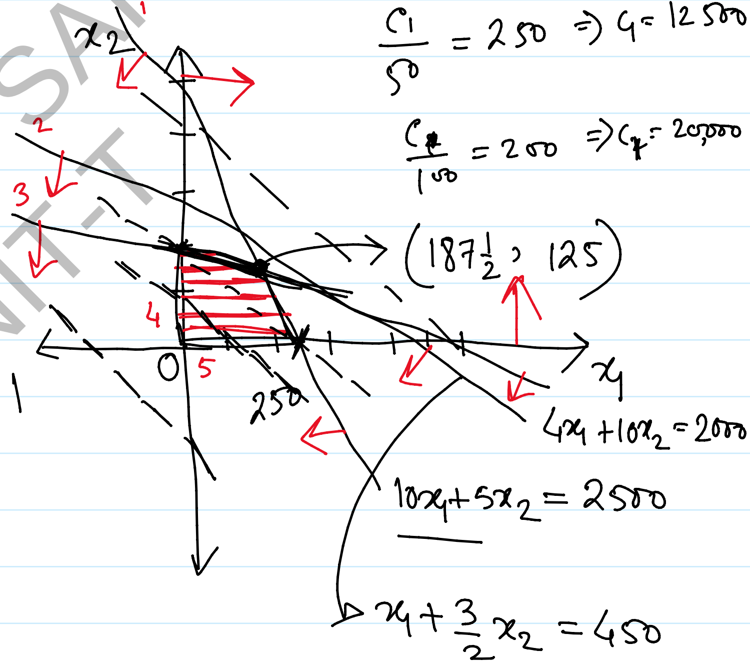
$$x_1, x_2 \geq 0$$

$$50x_1 + 100x_2 = c_1 = 20{,}000$$

$$\frac{x_1}{\frac{c_1}{50}} + \frac{x_2}{\frac{c_1}{100}} = 1$$

$$\frac{c_1}{50} = 250 \Rightarrow c_1 = 12500$$

$$\frac{c_1}{100} = 200 \Rightarrow c_1 = 20{,}000$$



$$50x_1 + 100x_2 = c_1$$

$$\Rightarrow \frac{x_1}{c_1/50} + \frac{x_2}{c_1/100} = 1$$

$c_1/50 = 100$, $c_1/100 = 100$

$\Rightarrow c_1 = 5000$

$c_1 = 10000$

Max $Z = 6x_1 + 8x_2$

s.t. . . . . .

The slopes of constraint eq^ns are not equal to the slope of obj. fn. Therefore opt.

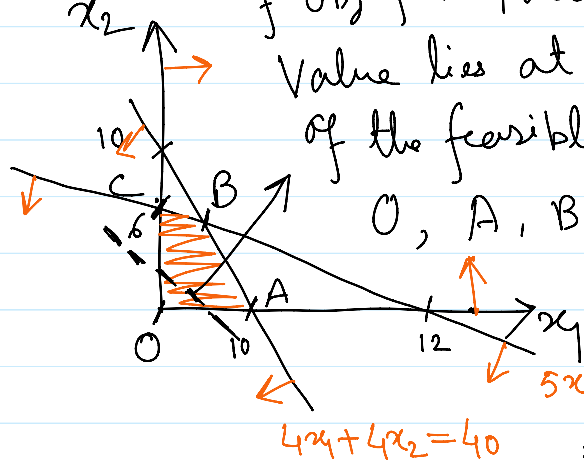
Home Work

Home Work

1. Max $Z = 6x_1 + 8x_2$

Min

st $\begin{cases} 5x_1 + 10x_2 \le 60 \\ 4x_1 + 4x_2 \le 40 \\ x_1, x_2 \geq 0 \end{cases}$



of obj. fnc. Therefore opt. Value lies at the vertices of the feasible region.

O, A, B, C

$Z(O) =$

$Z(A) =$

$Z(B) =$

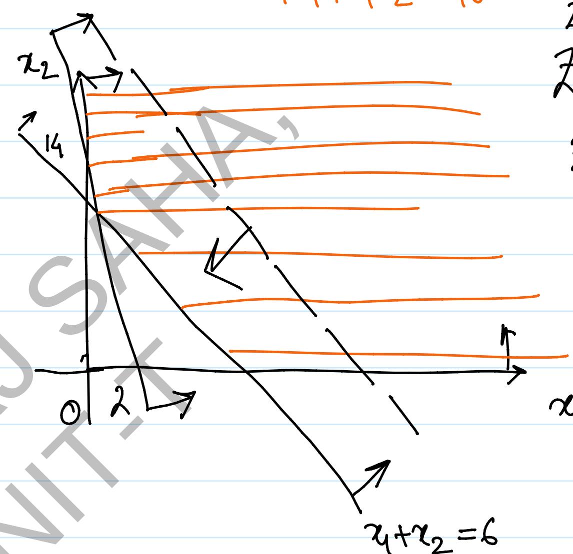
$Z(C) =$

2. Min $Z = 2x_1 + 3x_2$

St. $x_1 + x_2 \geq 6$

$7x_1 + x_2 \geq 14$

$x_1, x_2 \geq 0$



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Simplex Method

$$\text{Max } Z = 6x_1 + 8x_2$$

s.t.
$$\begin{cases} 5x_1 + 10x_2 \le 60 \\ 4x_1 + 4x_2 \le 40 \\ x_1, x_2 \ge 0 \end{cases}$$

$Ax \le b$

$Z = 64$, $x_1 = 8$, $x_2 = 2$ [By graphical method]

$$\text{Max } Z = 6x_1 + 8x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

Slack Variables

Modified System
$$\begin{cases} 5x_1 + 10x_2 + 1 \cdot s_1 + 0 \cdot s_2 = 60 \\ 4x_1 + 4x_2 + 0 \cdot s_1 + 1 \cdot s_2 = 40 \\ x_1, x_2, s_1, s_2 \ge 0 \end{cases}$$

x_1, x_2, s_1, s_2

$x_1 = 0 = x_2$

$\hat{x} = \hat{A}^{-1} b$

$\hat{A}\hat{x} = b$

$\hat{A} = [A | I]$

$\hat{A} = I$

$$\text{Max } Z = c^T x$$
$$\text{st } Ax = b$$
$$x \ge 0$$

$s_1 = 60$, $s_2 = 40$ } Initial basic sol^n

$A_{10 \times 8}$ $x_1, x_2, \ldots$??

$$\left.\begin{aligned} 5x_1 + 10x_2 &= 60 \\ 4x_1 + 4x_2 &= 40 \end{aligned}\right\} \quad x_1 = 8, \; x_2 = 2$$

$A_{2 \times 2}$

$$\text{Max } Z = 6x_1 + 8x_2$$
s.t.
$$5x_1 + 10x_2 \le 60$$
$$4x_1 + 4x_2 \le 40$$
$$x_1, x_2 \ge 0$$

$$\text{Max } Z = c^T x \; ; \; c = \begin{Bmatrix} 6 \\ 8 \end{Bmatrix}$$
$$\text{st } Ax \le b$$
$$x \ge 0 \qquad A = \begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix}$$
$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\text{Max } Z = 6x_1 + 8x_2 + 0 \cdot s_1 + 0 \cdot s_2$$
$$5x_1 + 10x_2 + 1 \cdot s_1 + 0 \cdot s_2 = 60$$

$\text{Max } Z = \hat{c}^T \hat{x}$

$c = \begin{Bmatrix} 6 \\ 8 \end{Bmatrix}$, $\hat{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

$$\left\{\begin{array}{l} \text{Max } Z = 6x_1 + 8x_2 + 0\cdot s_1 + 0\cdot s_2 \\ 5x_1 + 10x_2 + 1\cdot s_1 + 0\cdot s_2 = 60 \\ 4x_1 + 4x_2 + 0\cdot s_1 + 1\cdot s_2 = 40 \\ \\ x_1, x_2, s_1, s_2 \geq 0 \end{array}\right.$$

$$\text{Max } Z = \tilde{c}^T \tilde{x}$$

$$\text{st } \tilde{A}\tilde{x} = b$$

$$c = \begin{Bmatrix} 6 \\ 8 \\ 0 \\ 0 \end{Bmatrix} \quad \tilde{x} = \begin{Bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{Bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix}$$

$$= [A \mid I]$$

$\underline{X = 0}$ $\quad x_1 = 0 = x_2$

$$s_1 = 60, \quad s_2 = 40$$

Basic variables.

$$\tilde{A}\tilde{x} = b \Rightarrow \tilde{x} = \tilde{A}^{-1} b$$

$$I\tilde{x} = b \Rightarrow \tilde{x} = I^{-1} b = b$$

CB_j	Basic variable	$C_j \rightarrow$ Sol^n	6 x_1	8 x_2	0 s_1	0 s_2	Ratio
0	s_1	60	5	$\boxed{10}$	1	0	$60/10 = 6 \rightarrow$
0	s_2	40	4	4	0	1	$40/4 = 10$
		$Z_j = 0$	0	0	0	0	$\rightarrow Z_j$
		$Z_j - C_j$	-6	-8	0	0	

Pivot

leaving variable

entering variable

Table-1

For Max find min $(Z_j - C_j)$

For Max when all $(Z_j - C_j) \geq 0$ Then optimality is achieved

Otherwise we need to proceed for next iteration.

$$0 - \frac{1 \times 4}{10} = -2/5$$

$$4 - \frac{5 \times 4}{10} = 2$$

$$4 - \frac{10 \times 4}{10} = 0$$

$$1 - \frac{0 \times 4}{10} = 1$$

Table-2

Table-2

$1 - \frac{0\times 4}{10} = 1$

		C_j	6	8	0	0	
CB_j	Basic var.	Solⁿ	x_1	x_2	s_1	s_2	Ratio
8	x_2	6	$1/2$	1	$1/10$	0	$6/\frac{1}{2} = 12$
0	s_2	16	$\boxed{2}$	0	$-2/5$	1	$16/2 = 8 \rightarrow$
		$Z_j = 48$	4	8	$4/5$	0	
		$Z_j - C_j$	$-2 \uparrow$	0	$4/5$	0	

$$\text{New value} = \text{old value} - \frac{(\text{Pivot row value}) \times (\text{Pivot col. value})}{\text{Pivot}}$$

$40 - \frac{60 \times 4}{10} = 16$

$6 - \frac{16 \times 1/2}{2} = 2$

$\frac{1}{2} - \frac{2 \times \frac{1}{2}}{2} = 0$

$1 - 0$

$\frac{1}{10} - \frac{-2/5 \times 1/2}{2}$

$0 - \frac{1 \times 1/2}{2} = -1/4$

$-2+3$

		C_j	6	8	0	0	
CB_j	Basic var.	Solⁿ	x_1	x_2	s_1	s_2	Ratio
8	x_2	2	0	1	$1/5$	$-1/4$	
6	x_1	8	1	0	$-1/5$	$1/2$	
		$Z_j = 64$	6	8	$2/5$	1	
		$Z_j - C_j$	0	0	$2/5$	1	

For Max. prob.
all $Z_j - C_j \geq 0$
∴ here we have optimality and opt. value is $Z = 64$
Corresponding to $x_1 = 8$, $x_2 = 2$ as

all x_j ... $z = 64$

Therefore we have achieved optimality. Corresponding to $x_1 = 8, x_2 = 2$ as basic variable

and $s_1 = 0 = s_2$ as non-basic variables.

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Unbounded Sol^n :-

$$\text{Max } Z = 3x_1 + 2x_2$$
$$\text{st} \quad x_1 - x_2 \le 1$$
$$3x_1 - 2x_2 \le 6, \quad x_1, x_2 \ge 0$$

Standard form. $\text{Max } Z = 3x_1 + 2x_2 + 0\cdot s_1 + 0\cdot s_2$

$$\text{st} \quad x_1 - x_2 + s_1 = 1$$
$$3x_1 - 2x_2 + s_2 = 6$$
$$x_1, x_2, s_1, s_2 \ge 0$$

s_1, s_2 are slack Variables.

Table-1

C_{B_i}	Basic Var.	Sol^n	x_1 ($c_j = 3$)	x_2 ($c_j = 2$)	s_1 ($c_j = 0$)	s_2 ($c_j = 0$)	Ratio
0	s_1	1	$\boxed{1}$	-1	1	0	$1/1 = 1$ →
0	s_2	6	3	-2	0	1	$6/3 = 2$
	Z_j	0	0	0	0	0	
	$Z_j - c_j$		-3 ↑	-2	0	0	

C_{B_i}	Basic Var.	Sol^n	x_1 ($c_j = 3$)	x_2 ($c_j = 2$)	s_1 ($c_j = 0$)	s_2 ($c_j = 0$)	Ratio
3	x_1	1	1	-1	1	0	—
0	s_2	3	0	$\boxed{1}$	-3	1	$3/1 = 3$ →
	$Z_j = 3$		3	-3	3	0	
	$Z_j - c_j$		0	-5 ↑	3	0	

Table-3

C_{B_j}	Basic Var.	Sol^n	x_1 (3)	x_2 (2)	s_1 (0)	s_2 (0)	Ratio
3	x_1	4	1	0	-2	1	—
2	x_2	3	0	1	-3	1	—
		$Z_j = 18$	3	2	-12	5	
		$Z_j - C_j$	0	0	-12	5	

The ratio is undefined. Therefore the problem has unbounded sol^n.

Infinite number of sol^{n}s :-

$$\text{Max } Z = 40x_1 + 100x_2$$
$$\text{s.t. } 10x_1 + 5x_2 \le 2500$$
$$4x_1 + 10x_2 \le 2000$$
$$2x_1 + 3x_2 \le 900$$
$$x_1, x_2 \ge 0$$

Standard form

$$\text{Max } Z = 40x_1 + 100x_2 + 0\cdot s_1 + 0\cdot s_2 + 0\cdot s_3$$
$$\text{s.t. } 10x_1 + 5x_2 + s_1 = 2500$$
$$4x_1 + 10x_2 + s_2 = 2000$$
$$2x_1 + 3x_2 + s_3 = 900$$
$$x_1, x_2, x_3, s_1, s_2, s_2 \ge 0$$

s_1, s_2, s_3 are slack variables.

C_{B_j}	Basic Var.	Sol^n	x_1 (40)	x_2 (100)	s_1 (0)	s_2 (0)	s_3 (0)	Ratio
0	s_1	2500	10	5	1	0	0	500
0	s_2	2000	4	$\boxed{10}$	0	1	0	200 →
0	s_3	900	2	3	0	0	1	300
		$Z_j = 0$	0	0	0	0	0	
		$Z_j - C_j$	-40	-100	0	0	0	

$Z_j - c_j$	-40	-100	0	0	0

		c_j	40	100	0	0	0	
CB_j	Basic Var.	Solⁿ	x_1	x_2	s_1	s_2	s_3	Ratio
0	s_1	1500	$\boxed{8}$	0	1	$-1/2$	0	$1500/8 = \frac{375}{2}$ →
100	x_2	200	$2/5$	1	0	$1/10$	0	$200/_{2/5} = 1000$
0	s_3	300	$8/10$	0	0	$-3/10$	1	$300/_8 = 750/2$
		$Z_j = 20{,}000$	40	100	0	10	0	
		$Z_j - c_j$	0 ↑	0	0	10	0	

All $Z_j - c_j \geq 0$ $\therefore$ Optimality is reached. ✓

Opt. values are $Z = 20{,}000$, $x_2 = 200$, $s_1 = 1500$, $s_3 = 300$ as basic variables and $x_1 = 0$, $s_2 = 0$ as non-basic variables.

		c_j	40	100	0	0	0
CB_j	Basic var.	Solⁿ	x_1	x_2	s_1	s_2	s_3
40	x_1	$375/2$	1	0	$1/8$	$-1/16$	0
100	x_2	125	0	1	$-1/20$	$1/8$	0
0	s_3	150	0	0	$-1/10$	$-1/4$	1
		$Z = 20{,}000$	40	100	0	10	0
		$Z_j - c_j$	0	0	0	10	0

x_1, x_2, s_3 are basic variables.

$Z_{max} = 20000$, $x_1 = 375/2$, $x_2 = 125$, $s_3 = 150$, $s_1 = 0 = s_2$

s_1, s_2 are non-basic variables, but $Z_j - c_j$ corresponding to s_1 is 0 $\Rightarrow$ alternate set of solⁿ exists

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Big-M method :-

Simplex method   Max / Min $Z = c^T x$
st $Ax \le b$ $=$ $\geq$
$x \geq 0$

$$\left.\begin{array}{l} \text{Max / Min } Z = c^T x \\ \text{st } \quad Ax = \geq b \\ \qquad x \geq 0 \end{array}\right\} \text{ Big-M method} \quad \text{— (A)}$$

(1) If the obj function is Max, then artificial variables in Z will associate with $(-M)$ sign.

(2) Min then artificial variable will be associated with $(+M)$ sign.

Ex 1 : Min $Z = 2x_1 + 3x_2$

st
$$x_1 + x_2 \geq 6$$
$$7x_1 + x_2 \geq 14$$
$$x_1, x_2 \geq 0$$

$\left\{\begin{array}{l} Ax \le b \\ Ax + S_1 = b \end{array}\right.$ S_1 → Slack var.

$\left\{\begin{array}{l} Ax \geq b \\ Ax - S_2 + A_1 = b \end{array}\right.$ S_2 → surplus var.

Standard form   Min $Z = 2x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2 + Ma_1 + Ma_2$

s.t
$$x_1 + x_2 - s_1 + a_1 = 6$$
$$7x_1 + x_2 - s_2 + a_2 = 14$$

$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

s_1, s_2 are surplus variables, and
a_1, a_2 " artificial "

$X \geq 0$
$X_1 \rightarrow X \geq 0$

$x_1 = 0 = x_2$, $s_1 = -6$, $s_2 = -14$ ✗

$x_1 = 0 = x_2 = s_1 = s_2$

$a_1 = 6$, $a_2 = 14$ ≥ 0 ✓

M = Very large no.

$Z_1 = 20M$

$Z_1 = 20M \longrightarrow Z_{min}$

$$\text{Max } Z = 2x_1 + 3x_2 + 0s_1 + 0 \cdot s_2 - Ma_1 - Ma_2$$

M = Very large non-negative no

$a_1 = 6, a_2 = 14$

$$Z_1 = -20M \longrightarrow Z_{max}$$

$$\text{Min } Z = 2x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2 + Ma_1 + Ma_2$$

$$\left.\begin{array}{ll} \text{s.t} & x_1 + x_2 - s_1 + a_1 = 6 \\ & 7x_1 + x_2 - s_2 + a_2 = 14 \\ & x_1, x_2, s_1, s_2, a_1, a_2 \geq 0 \end{array}\right\} \text{Standard form}$$

		c_j	2	3	0	0	M	M	
c_{B_j}	Basic var.	Solⁿ	x_1	x_2	s_1	s_2	a_1	a_2	Ratio
M	a_1	6	1	1	−1	0	1	0	6/1 = 6
M	a_2	14	7	1	0	−1	0	1	14/7 = 2 →
	Z_j =	20M	8M	2M	−M	−M	M	M	
	$Z_j - c_j$		8M-2 ↑	2M-3	−M	−M	0	0	

(Max , $Z_j - c_j \geq 0$ algo. stops

Max, $z_j - c_j \geq 0$ algo. stops

If $z_j - c_j \leq 0$ find min $z_j - c_j$ as entering variable

Min, $z_j - c_j \leq 0$ algo stops.

If $z_j - c_j \geq 0$, find max $z_j - c_j$ as entering variable

		c_j	2	3	0	0	M	M	
c_{B_j}	Basic var	solⁿ	x_1	x_2	s_1	s_2	a_1	a_2	Ratio
M	a_1	4	0	$6/7$	-1	$1/7$	1	$-1/7$	$28/6$ →
2	x_1	2	1	$1/7$	0	$-1/7$	0	$1/7$	$14/1$
		$z_j = 4M+4$	2	$\frac{6}{7}M + 2/7$	$-M$	$\frac{1}{7}M - 2/7$	M	$\left(-\frac{1}{7}M + 2/7\right)$	
		$z_j - c_j$	0	$\frac{6}{7}M - \frac{19}{7}$ ↑	$-M$	$\frac{1}{7}(M-2)$	0	$\left(-\frac{8}{7}M + 2/7\right)$	

		c_j	2	3	0	0	M	M	
c_{B_j}	Basic var.	solⁿ	x_1	x_2	s_1	s_2	a_1	a_2	Ratio
3	x_2	$14/3$	0	1	$-7/6$	$1/6$	$7/6$	$-1/6$	28 →
2	x_1	$4/3$	1	0	$1/6$	$-1/6$	$-1/6$	$1/6$	—
		$z_j = 50/3$	2	3	$-19/6$	$1/6$	$19/6$	$-1/6$	
		$z_j - c_j$	0	0	$-19/6$	$1/6$ ↑	$\frac{19}{6} - M$	$-\frac{1}{6} - M$	

$\frac{14/3}{1/6} = 28$

$4/3 - \frac{14/3 \times (-1/6)}{1/6}$

$= \frac{4}{3} + \frac{14}{3}$

$= 6$

0	s_2	28	0	6	-7	1	7	-1
2	x_1	6	1	1	-1	0	1	0
		$z_j = 12$	2	2	-2	0	2	0
		$z_j - c_j$	0	1	-2	0	$2-M$	$-M$

$z_j = 12$	2	2	-2	0	2	0
$z_j - c_j$	0	-1	-2	0	$2-M$	$-M$

All $z_j - c_j \le 0$ as it is a Min. problem. So we have reached optimality. The opt. sol^n is $Z = 12$ s.t

$x_1 = 6,\ s_2 = 28$ as basic var.

and $x_2 = 0 = s_1 = a_1 = a_2$ as non-basic var.

Artificial Variable at non-zero level :-

$$\left.\begin{aligned} \text{Max } Z &= x_1 + 2x_2 \\ \text{s.t } 2x_1 + x_2 &\le 2 \\ 3x_1 + 4x_2 &\ge 12 \\ x_1, x_2 &\ge 0 \end{aligned}\right\}$$

Final Table

			1	2	0	0	$-M$
			x_1	x_2	s_1	S_1	a_1
2	x_2	2	2	1	1	0	0
$-M$	a_1	4	-5	0	-4	-1	1
		$z_j = 4-4M$	$4+5M$	2	$2+4M$	M	$-M$
		$z_j - c_j$	$5M+3$	0	$4M+2$	M	0

Max. and all $z_j - c_j \ge 0$, $x_2 = 2$, $a_1 = 4$ (basic variable)

The problem has no feasible sol^n.

The problem has no feasible sol^n.

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Artificial Variable is present (in optimal table) at zero-level:-

$$\text{Max } Z = 2x_1 + 2x_2 + 4x_3$$

$$\text{s.t.} \quad 2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Final table

c_{B_j}	Basic var.	Sol^n \ c_j	x_1 (2)	x_2 (2)	x_3 (4)	s_1 (0)	S_1 (0)	a_1 (−M)
2	x_2	2	2	1	1	1	0	0
−M	a_1	0	−5	0	−2	−4	−1	1
		$Z_j = 4$	5M+4	2	2M+2	4M+2	M	−M
		$Z_j - c_j$	5M+2	0	2M−2	4M+2	M	0

$Z_{max} = 4$, when $x_2 = 2$, $a_1 = 0$ as basic variables.
and $x_1 = x_3 = s_1 = S_1 = 0$ as non-basic var.

The problem has a degenerate sol^n.

c_{B_j}	Basic var.	Sol^n \ c_j	x_1 (2)	x_2 (2)	x_3 (4)	s_1 (0)	S_1 (0)	Ratio
2	x_2	2	0	1	1/5	13/5	−2/5	$2/\frac{1}{5} = 10$
2	x_1	0	1	0	2/5	4/5	1/5	$0/\frac{2}{5} = 0$
		$Z_j = 4$	2	2	1	34/5	−2/5	
		$Z_j - c_j$	0	0	3	34/5	−2/[illegible]	

		Z_j-c_j	0	0	-3	$34/5$	$-2/5$	

		c_j	2	2	4	0	0	
C_{B_j}	Basic var.	Solⁿ	x_1	x_2	x_3	s_1	S_1	Ratio
2	x_2	2	$-1/2$	1	0	$11/5$	-1	
4	x_3	0	$5/2$	0	1	2	$1/2$	
		$Z_j = 4$	9	2	4	$62/5$	0	
		$Z_j - c_j$	7	0	0	$62/5$	0	

$Z_{max} = 4$, $\quad x_2 = 2,\ x_3 = 0$ (as basic var.)

$x_1 = 0 = s_1 = S_1$ (non-basic var.)

[infinitely many solⁿs.]

Dual-Simplex Method:-

Ex 1: Min $Z = 2x_1 + 4x_2$

st $2x_1 + x_2 \geq 4$

$x_1 + 2x_2 \geq 3$

$2x_1 + 2x_2 \leq 12$

$x_1, x_2 \geq 0$

Theory say $\geq$ to $\leq$

$$\left\{\begin{array}{l} -2x_1 - x_2 \leq -4 \\ -x_1 - 2x_2 \leq -3 \\ 2x_1 + 2x_2 \leq 12 \end{array}\right.$$

Min $Z = 2x_1 + 4x_2$

$$\text{st} \quad \begin{array}{l} -2x_1 - x_2 + s_1 = -4 \\ -x_1 - 2x_2 + s_2 = -3 \\ 2x_1 + 2x_2 + s_3 = 12 \end{array} \quad , x_1, x_2, x_3 \geq 0 \quad s_1, s_2, s_3$$

$$2x_1 + 2x_2 \qquad + s_3 = 12 \qquad\qquad s_1, s_2, s_3$$

Intial solⁿ :- $s_1 = -4, \; s_2 = -3, \; s_3 = 12$

Simplex	Dual-Simplex
Solⁿ is always feasible. We search for optimality	Optimality is present at all the tables. but solⁿ is not feasible. Our task is to find feasible solⁿ

		c_j	2	4	0	0	0
Obj	Basic var.	solⁿ	x_1	x_2	s_1	s_2	s_3
0	s_1	-4	-2	-1	1	0	0
0	s_2	-3	-1	-2	0	1	0
0	s_3	12	2	2	0	0	1
	$Z_j =$	0	0	0	0	0	0
	$Z_j - c_j$		-2	-4	0	0	0
	Ratio		$-2/-2 = 1$	$-4/-1 = 4$	—	—	—

i) How to choose leaving variable :-

Variable with most $-$ve solⁿ.

ii) How to choose entering variable :-

Find $(Z_j - c_j)$ / (Corresponding row value of leaving variable)

* If row-value of leaving variable is 0 or +ve.
No ratio can be found

Then find min ratio and variable associated with min ratio act as entering variable.

feasibility not present

C_{B_i}	Basic	C_j / Sol^n	2	4	0	0	0
			x_1	x_2	s_1	s_2	s_3
2	x_1	2	1	$1/2$	$-1/2$	0	0
0	s_2	-1	0	$-3/2$	$-1/2$	1	0
0	s_3	8	0	1	1	0	1
		$Z_j = 4$	2	1	-1	0	0
		$Z_j - c_j$	0	-3	-1	0	0
		Ratio	—	2	2	—	—

optimality present

Pl. Complete the problem.

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Duality of LPP :-

Ex 1 :- Primal Problem

$$\text{Max } Z = 4x_1 + 10x_2 + 25x_3$$
$$\text{s.t } 2x_1 + 4x_2 + 8x_3 \le 25$$
$$4x_1 + 9x_2 + 8x_3 \le 30$$
$$6x_1 + 8x_2 + 2x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

$$\text{Max } Z = C^T X$$
$$\text{st } AX \le b$$
$$X \ge 0$$

Dual Problem

$$\text{Min } \bar{Z} = 25y_1 + 30y_2 + 40y_3$$
$$\text{st. } 2y_1 + 4y_2 + 6y_3 \ge 4$$
$$4y_1 + 9y_2 + 8y_3 \ge 10$$
$$8y_1 + 8y_2 + 2y_3 \ge 25$$
$$y_1, y_2, y_3 \ge 0$$

$$\text{Min } \bar{Z} = b^T Y$$
$$\text{st } A^T Y \ge C$$
$$Y \ge 0$$

Ex 2. Primal

$$\text{Max } Z = 4x_1 + 10x_2 + 25x_3$$
$$\text{st } 2x_1 + 4x_2 + 8x_3 = 25 \quad (C1)$$
$$4x_1 + 9x_2 + 8x_3 \le 30 \quad (C2)$$
$$6x_1 + 8x_2 + 2x_3 \le 40 \quad (C3)$$
$$x_1, x_2, x_3 \ge 0$$

Dual

$$\text{Min } \bar{Z} = 25y_1 + 30y_2 + 40y_3$$
$$\text{s.t } 2y_1 + 4y_2 + 6y_3 \ge 4$$
$$4y_1 + 9y_2 + 8y_3 \ge 10$$
$$8y_1 + 8y_2 + 2y_3 \ge 25$$

y_1 is unrestricted in sign

$$y_2, y_3 \ge 0.$$

$$2x_1 + 4x_2 + 8x_3 \ge 25$$
$$2x_1 + 4x_2 + 8x_3 \le 25$$
$$-2x_1 - 4x_2 - 8x_3 \le -25$$

$$\text{Max } Z = 4x_1 + 10x_2 + 25x_3$$
$$(C1) \Rightarrow -2x_1 - 4x_2 - 8x_3 \le -25 \quad \text{— P1}$$
$$+2x_1 + 4x_2 + 8x_3 \le 25 \quad \text{— P2}$$
$$4x_1 + 9x_2 + 8x_3 \le 30 \quad \text{— P3}$$

$$\text{Min } \bar{Z} = -25y_1' + 25y_2' + 30y_3' + 40y_4'$$
$$\text{s.t } -2y_1' + 2y_2' + 4y_3' + 6y_4' \ge 4$$
$$-4y_1' + 4y_2' + 9y_3' + 8y_4' \ge 10$$

$+2x_1+4x_2+8x_3 \le 25$ | – P2

$4x_1+9x_2+8x_3 \le 30$ | – P3

$6x_1+8x_2+2x_3 \le 40$ | – P4

$x_1, x_2, x_3 \geq 0$

s.t. $(-2y_1'+2y_2') + 4y_3' + 6y_4' \geq 4$

$-4y_1'+4y_2' + 9y_3' + 8y_4' \geq 10$

$-8y_1'+8y_2' + 8y_3' + 2y_4' \geq 25$

$\underline{y_1'}, \underline{y_2'}, \underline{y_3'}, \underline{y_4'} \geq 0$

$25(-y_1'+y_2')$ Let $w_1 = -y_1'+y_2'$

$w_2 = y_3'$

$w_3 = y_4'$

$\therefore$ Dual Min $\bar{Z} = 25w_1 + 30w_2 + 40w_3$

s.t. $2w_1 + 4w_2 + 6w_3 \geq 4$

$4w_1 + 9w_2 + 8w_3 \geq 10$

$8w_1 + 8w_2 + 2w_3 \geq 25$

$w_2, w_3 \geq 0$ $\quad w_1 = -y_1' + y_2'$

$\therefore w_1$ is unrestricted in sign.

Ex 3 :- Min $Z = 5x_1 + 8x_2$

s.t. $4x_1 + 9x_2 \geq 100$

$2x_1 + x_2 \leq 20$

Primal $\quad 2x_1 + 5x_2 \geq 120$

$x_1, x_2 \geq 0$

Dual.

Max $\bar{Z} = 100y_1 - 20y_2 + 120y_3$

s.t. $4y_1 - 2y_2 + 2y_3 \leq 5$

$9y_1 - y_2 + 5y_3 \leq 8$

$y_1, y_2, y_3 \geq 0$.

Reformulation of Primal

Min $Z = 5x_1 + 8x_2$

s.t. $4x_1 + 9x_2 \geq 100$

$-2x_1 - x_2 \geq -20$

$2x_1 + 5x_2 \geq 120$

$x_1, x_2 \geq 0$

$$2x_1 + 5x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

<u>Ex 4</u> :- Primal

$$\text{Max } Z = 2x_1 + 3x_2$$

st
$$5x_1 + 2x_2 \leq 40$$
$$6x_1 + 12x_2 \leq 80$$

x_1, x_2 are Unrestricted in sign.

Let $x_1 = x_1' - x_1''$, $x_2 = x_2' - x_2''$
$x_1', x_1'', x_2', x_2'' \geq 0$

$$\text{Max } Z = 2x_1' - 2x_1'' + 3x_2' - 3x_2''$$

s.t
$$5x_1' - 5x_1'' + 2x_2' - 2x_2'' \leq 40$$
$$6x_1' - 6x_1'' + 12x_2' - 12x_2'' \leq 80$$
$$x_1', x_1'', x_2', x_2'' \geq 0$$

$$\text{Min } \bar{Z} = 40w_1 + 80w_2$$

st
$$5w_1 + 6w_2 \geq 2$$
$$-5w_1 - 6w_2 \geq -2$$
$$2w_1 + 12w_2 \geq 3$$
$$-2w_1 - 12w_2 \geq -3$$
$$w_1, w_2 \geq 0$$

$$\text{Min } \bar{Z} = 40w_1 + 80w_2$$

st
$$5w_1 + 6w_2 = 2$$
$$2w_1 + 12w_2 = 3$$
$$w_1, w_2 \geq 0$$

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Duality and Simplex method :-

Primal

Simplex Method (P)

Optimal soln → Dual has opt. soln

Unbounded soln / infeasible → Dual has infeasible soln

Big-M method (Primal)

Optimal soln → Dual has opt. soln

Unbounded or infeasible soln → Dual has infeasible soln

Note :-

(1) Let $w = (w_1, w_2, \ldots, w_n)$, $w_i \geq 0$ soln of the dual Problem.

$w_i \geq 0 \Rightarrow$ The constraint eqn's in primal contain $\leq$ sign

$\Rightarrow$ Standard form of primal contains only slack variables $s_1, s_2, \ldots, s_m$ (say)

$\Rightarrow$ Consider $(z_j - c_j)$ col. corresponding to s_i's.

Then, $w_i = z_j - c_j$

(2) Let $w_i \leq 0$. Then the constraint eq^{n}s in primal are associated with $\geq$ type inequality

$\Rightarrow$ Primal in standard form contains surplus variables S_i and artificial variables α_i

$\Rightarrow$ Simply consider the $(Z_j - C_j)$ col. corresponding to S_i and $w_i = -(Z_j - C_j) = C_j - Z_j$

There is no need to consider the values associated with artificial variables α_i.

(3) w_i is unrestricted in sign.

$\Rightarrow$ Primal is associated with $=$ sign.

$\Rightarrow$ Primal in standard form contains artifical variables α_i.

$\Rightarrow$ Consider $Z_j - C_j$ value corresponding to the col^n of α_i. and $\boxed{w_i = Z_j}$.

Ex :- Primal

$$\begin{cases} \text{Min } Z = 40x_1 + 30x_2 + 25x_3 \\ \text{s.t. } 4x_1 + 2x_2 + 5x_3 \geq 30 \\ \quad 3x_1 + 6x_2 + x_3 \geq 20 \\ \quad x_1 + 3x_2 + 6x_3 \geq 36, \quad x_1, x_2, x_3 \geq 0 \end{cases}$$

Dual $\quad \text{Max } W = 30\omega_1 + 20\omega_2 + 36\omega_3$

$$\text{s.t.} \quad 4\omega_1 + 3\omega_2 + \omega_3 \leq 40$$
$$2\omega_1 + 6\omega_2 + 3\omega_3 \leq 30$$
$$5\omega_1 + \omega_2 + 6\omega_3 \leq 25, \quad \omega_i \geq 0, i = 1,2,3$$

We are going solve the dual using Simplex.

		c_j	30	20	36	0	0	0	
C_{B_i}	Basic var.	Sol^n	ω_1	ω_2	ω_3	s_1	s_2	s_3	Ratio
0	s_1	85/7	0	0	$-113/28$	1	$-11/28$	$-9/14$	
20	ω_2	275/77	0	1	3/28	0	55/308	$-1/14$	
30	ω_1	30/7	1	0	33/28	0	$-1/28$	3/14	
		$Z = 200$	30	20	525/14	0	5/2	5	
		$Z_j - c_j$	0	0	3/2	0	5/2	5	
						x_1	x_2	x_3	

Dual has opt. sol^n $\Rightarrow$ Primal has opt. sol^n

$$\left[\, x_1 = 0, \; x_2 = 5/2, \; x_3 = 5, \quad Z = 200 \,\right]$$

Ex 2: $\quad \text{Max } Z = 3x_1 + 2x_2$

$$\text{s.t.} \quad x_1 - x_2 \leq 1$$
$$-x_1 - x_2 \leq -3, \quad x_1, x_2 \geq 0$$

$$\text{s.t.} \quad x_1 - x_2 \le 1$$
$$-x_1 - x_2 \le -3, \quad x_1, x_2 \geq 0$$

Dual is $\quad \text{Min } W = w_1 - 3w_2$

$$\text{s.t.} \quad w_1 - w_2 \geq 3$$
$$-w_1 - w_2 \geq 2, \quad w_1, w_2 \geq 0$$

		c_j	1	-3	0	0	M	M
CB_j	Basic	sol^n	w_1	w_2	s_1	s_2	α_1	α_2
M	α_1	3	1	-1	-1	0	1	0
M	α_2	2	-1	-1	0	-1	0	1
		$Z_j = 5M$	0	-2M	-M	-M	M	M
		$Z_j - c_j$	-1	-2M+3	-M	-M	0	0

opt. reached

$\alpha_1 = 3, \ \alpha_2 = 2$

Infeasible sol^n

Primal has either unbdd or infeasible sol^n.

Ex 3: $\quad \text{Min } Z = 24x_1 + 30x_2$

$$\text{s.t.} \quad 2x_1 + 3x_2 \geq 10$$
$$4x_1 + 9x_2 \geq 15$$
$$6x_1 + 6x_2 \geq 20 \quad x_1, x_2 \geq 0$$

$\text{Max } W = 10w_1 + 15w_2 + 20w_3$

$$\text{s.t.} \quad 2w_1 + 4w_2 + 6w_3 \le 24$$
$$3w_1 + 9w_2 + 6w_3 \le 30, \quad w_i \geq 0.$$

		c_j	10	15	20	0	0
CB_j	Basic	Sol^n	w_1	w_2	w_3	s_1	s_2
20	w_3	2	0	-1	1	1/2	-1/3
10	w_1	6	1	5	0	-1	1
		$Z_j = 100$	10	30	20	0	+10/3

$Z = 100,$

$w_1 = 6, \ w_3 = 2$

$w_2 = 0 = s_1 = s_2$

10	w_1	6	1	2	0	1	1	
		$z_j = 100$	10	30	20	0	$+10/3$	
		$z_j - c_j$	0	15	0	0	$+10/3$	
						x_1	x_2	

$w_2 = 0 = s_1 = s_2$

Primal $Z = 100$

$x_1 = 0, \ x_2 = 10/3$

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Sensitivity Analysis :-

$$\left.\begin{aligned} \text{Max } & Z = C^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}\right\}$$

(i) Change in RHS of Constraint eqⁿs

(ii) Change in coeff of obj fnc.

(1) Change in RHS Of Constraint eqⁿs :

$$\begin{cases} \text{Max } Z = 6x_1 + 8x_2 \\ \text{s.t } 5x_1 + 10x_2 \leq 60 \\ \quad 4x_1 + 4x_2 \leq 40 \\ \quad x_1, x_2 \geq 0 \end{cases}$$

Case I

$$b = \underbrace{\begin{bmatrix} 60 \\ 40 \end{bmatrix}}_{\text{old value}} \longrightarrow \underbrace{\begin{bmatrix} 40 \\ 20 \end{bmatrix}}_{\text{New value}}$$

		c_j	6	8	0	0	
C_{B_j}	Basic	Solⁿ	x_1	x_2	s_1	s_2	Ratio
8	x_2	2	0	1	$1/5$	$-1/4$	
6	x_1	8	1	0	$-1/5$	$1/2$	
		$Z_j = 64$	6	8	$2/5$	1	
		$Z_j - C_j$	0	0	$2/5$	1	

$$\begin{bmatrix} \text{Basic Variables in} \\ \text{opt. table} \end{bmatrix} = \begin{bmatrix} \text{Coeff. columns in} \\ \text{opt. table wrt} \\ \text{basic variables at} \\ \text{initial table} \end{bmatrix} \times \begin{bmatrix} \text{New value} \\ \text{of RHS} \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ \end{bmatrix} = \begin{bmatrix} 1/5 & -1/4 \\ \end{bmatrix} \times \begin{bmatrix} 40 \\ \end{bmatrix} \quad \begin{bmatrix} 3 \\ \end{bmatrix} > 0$$

$$\Rightarrow \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{bmatrix} \times \begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \geq 0$$

$\therefore$ New solⁿ wrt $b = \begin{bmatrix} 40 \\ 20 \end{bmatrix}$ is $Z_{max} = 36$, st $x_1 = 2$ and $x_2 = 3$

Case II

$b = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$ changes to $\begin{bmatrix} 20 \\ 40 \end{bmatrix}$

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{bmatrix} \times \begin{bmatrix} 20 \\ 40 \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \end{bmatrix}, \quad x_2 \leq 0 \quad \therefore \text{ New sol}^n \text{ is not feasible}$$

Use dual-simplex method to obtain feasible.

		c_j	6	8	0	0
CB_j	Basic	Solⁿ	x_1	x_2	s_1	s_2
8	x_2	-6	0	1	$1/5$	$-1/4$
6	x_1	16	1	0	$-1/5$	$1/2$
		$Z_j = 48$	6	8	$2/5$	1
		$Z_j - c_j$	0	0	$2/5$	1
		Ratio	–	–	–	4

(ii) Change in coeff. obj function :-

Max $Z = 10x_1 + 15x_2 + 20x_3$

s.t. $2x_1 + 4x_2 + 6x_3 \leq 24$ $\qquad x_1, x_2, x_3 \geq 0$

$3x_1 + 9x_2 + 6x_3 \leq 30$

a) Find range of c_1 and $\underline{c_2}$ so that optimality is unchanged.

b) If $c^T = (10, 15, 20)$ changes to $(7, 14, 15)$ then check whether optimality is affected or not?

		c_j	10 c_1	15	20	0	0
C_{B_j}	Basic	Sol^n	x_1	x_2	x_3	s_1	s_2
20	x_3	2	0	-1	1	$1/2$	$-1/3$
c_1 10	x_1	6	1	5	0	-1	1
		$Z = 100$	10 c_1	30	20	0	$10/3$
		$Z_j - c_j$	0	15	0	0	$10/3$
		$Z_j - c_j$	0	$(5c_1 - 35)$	0	$(-c_1 + 10)$	$(c_1 - 20/3)$

Then for optimality we must have
$$\left.\begin{aligned} 5c_1 - 35 \geq 0 &\Rightarrow c_1 \geq 7 \\ -c_1 + 10 \geq 0 &\Rightarrow c_1 \leq 10 \\ c_1 - 20/3 \geq 0 &\Rightarrow c_1 \geq 6\tfrac{2}{3} \end{aligned}\right\}$$

All the eq^n's hold good when $7 \leq c_1 \leq 10$

b)

			7	14	15		
		c_j	10	15	20	0	0
C_{B_j}	Basic	Sol^r	x_1	x_2	x_3	s_1	s_2
15 20	x_3	2	0	-1	1	$1/2$	$-1/3$

	C_{B_j}	Basic	Soln	x_1	x_2	x_3	s_1	s_2
15	20	x_3	2	0	-1	1	$1/2$	$-1/3$
7	10	x_1	6	1	5	0	-1	1
			$Z=100$	10	30	20	0	$10/3$
			Z_j-c_j	0	15	0	0	$10/3$
				0	6	0	$1/2$	2

All $Z_j - c_j \geq 0$ $\therefore$ Opt. remains unaffected.

$x_1 = 6,\ x_3 = 2$ (as basic variables)

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Transportation Problems :-

Destination : (Showroom)

Source (Warehouse)

$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & \cdots j \cdots & & n & \\
\hline
1 & c_{11} & c_{12} & \cdots c_{ij} \cdots & & c_{1n} & a_1 \\
2 & c_{21} & c_{22} & \cdots c_{2j} \cdots & & c_{2n} & a_2 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots \\
i & c_{i1} & c_{i2} & \cdots c_{ij} \cdots & & c_{in} & a_i \\
\vdots & \vdots & \vdots & & & \vdots & \vdots \\
m & c_{m1} & c_{m2} & \cdots c_{mj} \cdots & & c_{mn} & a_m \\
\hline
 & b_1 & b_2 & \cdots b_j \cdots & & b_n &
\end{array}
$$

Supply

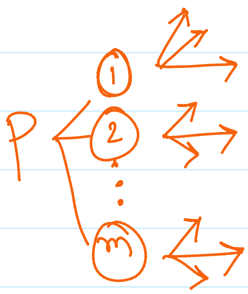
Demand

$10 \rightarrow 11$, $10 \rightarrow 15$

$a_1 > b_1$

$a_1 < b_1$

$a_1 - b_1$



(i) Balanced transportation problem

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(ii) Unbalanced transportation problem :-

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

How to solve :-

(i) Find initial basic feasible sol^n → (i) North-West corner method; (ii) Vogel's approx. method.

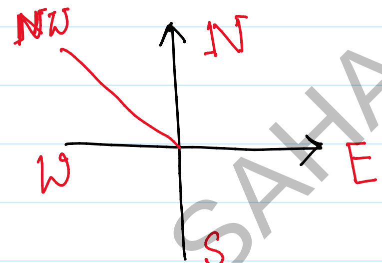
(ii) Optimize the initial sol^n

Ex1 :

3	1	7	4	300
2	6	5	9	400
8	3	3	2	500
250	350	400	200	= 1200

balanced transportation problem

NW - corner method



$\boxed{250}$ 3	$\boxed{50}$ 1	7	4	~~300~~ ~~50~~
2	$\boxed{300}$ 6	$\boxed{100}$ 5	9	~~400~~ ~~100~~
8	3	$\boxed{300}$ 3	$\boxed{200}$ 2	~~500~~ ~~200~~
~~250~~	~~350~~ ~~300~~	~~400~~ ~~300~~	~~200~~	

$\boxed{250}$ 3	$\boxed{50}$ 1	7	4	300
2	$\boxed{300}$ 6	$\boxed{100}$ 5	9	400
8	3	$\boxed{300}$ 3	$\boxed{200}$ 2	500
250	350	400	200	

The total initial cost is

$$250 \times 3 + 50 \times 1 + 300 \times 6 + 100 \times 5 + 300 \times 3 + 200 \times 2 = 4400$$

Initial solⁿ by Vogel's Approximation method (VAM) :-

Penalty1 Penalty2 Penalty3 Penalty4

	D_1	D_2	D_3	D_4	Supply	Penalty1	Penalty2	Penalty3	Penalty4
	3	$\boxed{300}$ 1	7	4	~~300~~	2	3*		
	$\boxed{250}$ 2	6	$\boxed{150}$ 5	9	~~400~~ ~~150~~	3*	1	1	1
	8	$\boxed{50}$ 3	$\boxed{250}$ 3	$\boxed{200}$ 2	~~500~~ ~~300~~ 250	1	1	1	0
Demand	~~250~~	~~350~~ 50	400	~~200~~					
Penalty 1	1	2	2	2					
Penalty2		2	2	2					
Penalty3		3	2	7*					
Penalty 4		3*	2						

	3	$\boxed{300}$ 1	7	4	300
	$\boxed{250}$ 2	6	$\boxed{150}$ 5	9	400
	8	$\boxed{50}$ 3	$\boxed{250}$ 3	$\boxed{200}$ 2	500
	250	350	400	200	

The total initial cost is $300 \times 1 + 250 \times 2 + 150 \times 5 + 50 \times 3 + 250 \times 3 + 200 \times 2$

$= 2850$

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Ex 2.

10	2	16	14	10	300
6	18	12	13	16	500
8	4	14	12	10	825
14	22	20	8	18	375
350	400	250	150	400	2000 / 1550

The problem is unbalanced.

10	2	16	14	10	$\boxed{300}$ 0	~~300~~
$\boxed{350}$ 6	18	12	13	16	$\boxed{150}$ 0	~~500~~ ~~350~~
8	$\boxed{400}$ 4	14	$\boxed{25}$ 12	$\boxed{400}$ 10	0	~~825~~ ~~425~~ ~~25~~
14	22	$\boxed{250}$ 20	$\boxed{125}$ 8	18	0	~~375~~ 125
~~350~~	~~400~~	~~250~~	~~150~~ 125	~~400~~	~~450~~ ~~150~~	2000

Least cell allocation method :-

The total initial cost is $300 \times 0 + 350 \times 6 + 150 \times 0 + 400 \times 4$

$+ 25 \times 12 + 400 \times 10 + 250 \times 20 + 125 \times 8$

$=$

\# Find initial basic solⁿ using

Homework :- (i) N-W Corner method, and
(ii) Vogels approximation method.

Ex 3 :-

Homework :

10	2	3	15	9	25
5	10	15	2	4	30
15	5	14	7	15	20
20	15	13	—	8	30
20	20	30	10	25	

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250 3	50 1	7	4	300
2	300 6	100 5	9	400
8	3	300 3	200 2	500
250	350	400	200	

(3×4)

(i) Check whether there are $(m+n-1)$ no. of basic cells.

$A_{m\times n}$

(ii) Compute $u_i + v_j = c_{ij}$ (assuming $u_1 = 0$) for all basic cells.

non-basic cell

	$v_1=3$	$v_2=1$	$v_3=0$	$v_4=-1$	
$u_1=0$	250 -ve 3	50 300 +ve 1	-ve 7	-ve 4	300
$u_2=5$	250 (6) +ve 2	300 50 -ve 6	100 5	-ve 9	400
$u_3=3$	-ve 8	1 3	300 3	200 2	500
	250	350	400	200	

(iii) Now calculate penalty for non-basic cells by $p_{ij} = u_i + v_j - c_{ij}$

(iv) Check if all $p_{ij} \le 0$

* If yes then we have reached optimality.

* If no, then search for the non-basic cell with max +ve penalty and call it as a new basic cell

	$v_1=-3$	$v_2=1$	$v_3=0$	$v_4=-1$	
$u_1=0$	-ve 3	300 1	-ve 7	-ve 4	300
$u_2=5$	250 2	50 -ve 6	100 150 +ve 5	-ve 9	400
	-ve	50 1	300 250	200	

non basic cell

(i) Calculate u_i, v_j using $(u_i + v_j - c_{ij})$ from the basic cells only.

$u_2 = 5$	2	−ve 6	+ve 5	9	400
$u_3 = 3$	−ve 8	50 +ve 3	1 300	250 3 −ve	200 2	500
	250	350	400	200	

cells only.

	$v_1 = -2$	$v_2 = 1$	$v_3 = 1$	$v_4 = 0$	
$u_1 = 0$	−ve 3	300 1	−ve 7	−ve 4	300
$u_2 = 4$	250 2	−ve 6	150 5	−ve 9	400
$u_3 = 2$	−ve 8	50 3	250 3	200 2	500
	250	350	400	200	

$\therefore$ Optimality is reached.

Opt. solⁿ is

$$300 \times 1 + 250 \times 2 + 150 \times 5 + 50 \times 3 + 250 \times 3 + 200 \times 2$$

$$= 2850$$

Home Work :- Find optimal solⁿs using u-v method for the example problems given on 14th September, 2021 Classwork.

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Assignment Problems :-

$$
\begin{array}{c|ccccccc}
 & \multicolumn{7}{c}{\text{Operator}} \\
 & 1 & 2 & \cdots & j & \cdots & & m \\
\hline
1 & t_{11} & t_{12} & \cdots & t_{1j} & \cdots & & t_{1m} \\
2 & t_{21} & t_{22} & \cdots & t_{2j} & \cdots & & t_{2m} \\
\vdots & \vdots & & & & & & \\
i & t_{i1} & t_{i2} & \cdots & t_{ij} & \cdots & & t_{im} \\
\vdots & \vdots & & & & & & \\
m & t_{m1} & t_{m2} & \cdots & t_{mj} & \cdots & & t_{mm}
\end{array}
$$

jobs

$A_{m \times m}$ ↓ job ↓ 5 ; → operators ↓ 5

1, 2, 3, 4, 5

a – 5
b – 4
c – 1
d – 3
e – 2

→ Operators

Ex 1 :- job ↓

$$
\begin{bmatrix}
10 & 12 & 15 & 12 & 8 \\
7 & 16 & 14 & 14 & 11 \\
13 & 14 & 7 & 9 & 9 \\
12 & 10 & 11 & 13 & 10 \\
8 & 13 & 15 & 11 & 15
\end{bmatrix}
\quad 5 \times 5
$$

Row scan :-

$$
\begin{bmatrix}
2 & 4 & 7 & 4 & 0 \\
0 & 9 & 7 & 7 & 4 \\
6 & 7 & 0 & 2 & 2 \\
2 & 0 & 1 & 3 & 0 \\
0 & 5 & 7 & 3 & 7
\end{bmatrix}
$$

Column Scan :- $2 \quad 4 \quad 7_{-1} \quad 2_{-1} \quad \boxed{0}$ ✓ $m = m = 5$

Column Scan :-

$$\begin{array}{ccccc} 2 & 4 & 7_{-1} & 2_{-1} & \boxed{0} \\ \boxed{0} & 9 & 7_{-1} & 5_{-1} & 4 \\ 6_{+1} & 7_{+1} & \boxed{0} & 0 & 2_{+1} \\ 2 & \boxed{0} & 1_{-1} & 1_{-1} & 0 \\ 0 & 5 & 7_{-1} & 1_{-1} & 7 \end{array}$$

& $m = n = 5$

Find no. of marked zeros.

If no. of marked zeros $= m/n$

Then we have attained optimality

If not then proceed in the following manner.

(i) Identify min value of the undeleted cells. (say r)

(ii) Copy the entries which are deleted by horizontal/vertical lines.

(iii) Add the min. value r, to the entries appearing in the junction of horizontal and vertical lines.

(iv) Substract min. value r from the undeleted cells.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 4 & 6 & 1 & \boxed{0} \\ 2 & \boxed{0} & 9 & 6 & 4 & 4 \\ 3 & 7 & 8 & \boxed{0} & 0 & 3 \\ 4 & 2 & \boxed{0} & 0 & 0 & 0 \\ 5 & 0 & 5 & 6 & \boxed{0} & 7 \end{array}$$

$m = n = 5 = \#$ of marked zeros

$\therefore$ optimality is reached

Optimal sol^{n} is

Assign 1st job $\longrightarrow$ 5th operater

2nd job $\longrightarrow$ 1st operater

3rd job $\longrightarrow$ 3rd operater

4th job $\longrightarrow$ 2nd operater

5th job $\longrightarrow$ 4th operater

Total time or cost incurred $= 8 + 7 + 7 + 10 + 11$

$= 43$ units.

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Project Management :-

(i) Node :- The starting / ending point of an activity

(ii) Branch/ Arc :- The branch represents the actual activity which consumes some kind of resources.

(iii) Precedence relation among activities :-

The precedence relation provides information on the activities that preceed the other activity and also the activities that follows it.

Rules for network construction :-

(i) Starting and ending events of an activity are called tail and head, respectively.

(ii) The network must have a unique tail event.

(iii) The network must have a unique head event

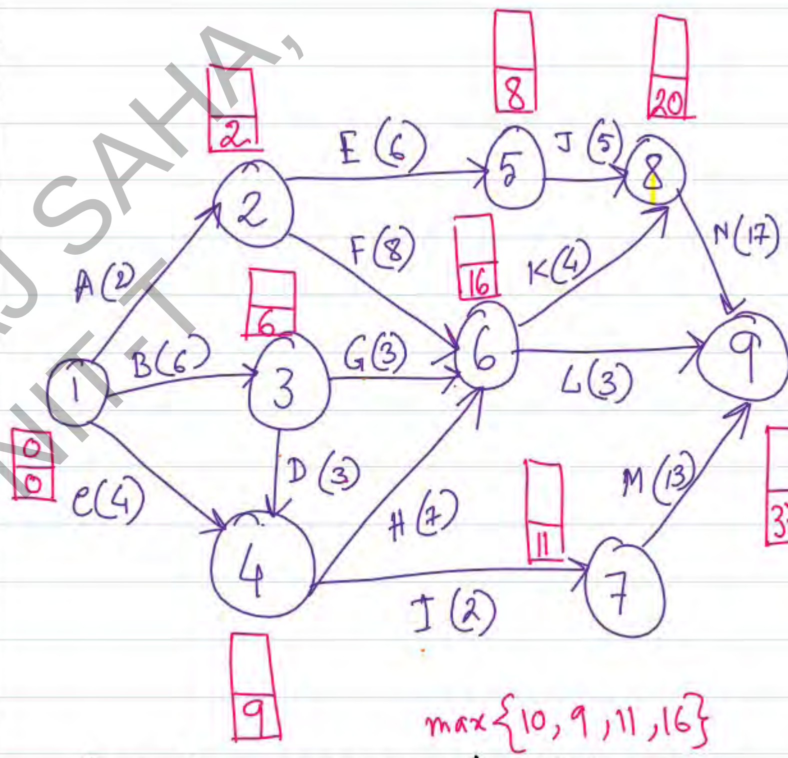
(iv) No network should be represented by more than one arc in the same network.

(v) No two activities should have same starting and ending node.

(vi) Dummy activity is an imaginary activity indicating precedence relations only. The duration of such activities are zero.

CRITICAL PATH METHOD (CPM) :-

Activity	immediate precedence	Duration
A	—	2
B	—	6
C	—	4
D	B	3
E	A	6
F	A	8
G	B	3
H	C, D	7
I	C, D	2
J	E	5
K	F, G, H	4
L	F, G, H	3
M	I	13
N	J, K	17



max{10, 9, 11, 16}

Project management Network

max{6, 11}

max{13, 14, 13, 15, 20}

Determination of critical path :-

Phase I :- Determine earliest starting time (ES) for all nodes.

Phase II :- Determine latest completion time (LC) for all nodes.

⇒ D_{ij} : duration of the activity (i,j)

⇒ ES_j : earliest starting time of all activities originating from node j and $ES_j = \max_i \left(ES_i + D_{ij} \right)$

⇒ LC_i : latest completion time of all activities ending at node i, and $LC_i = \min_j \left(LC_j - D_{ij} \right)$

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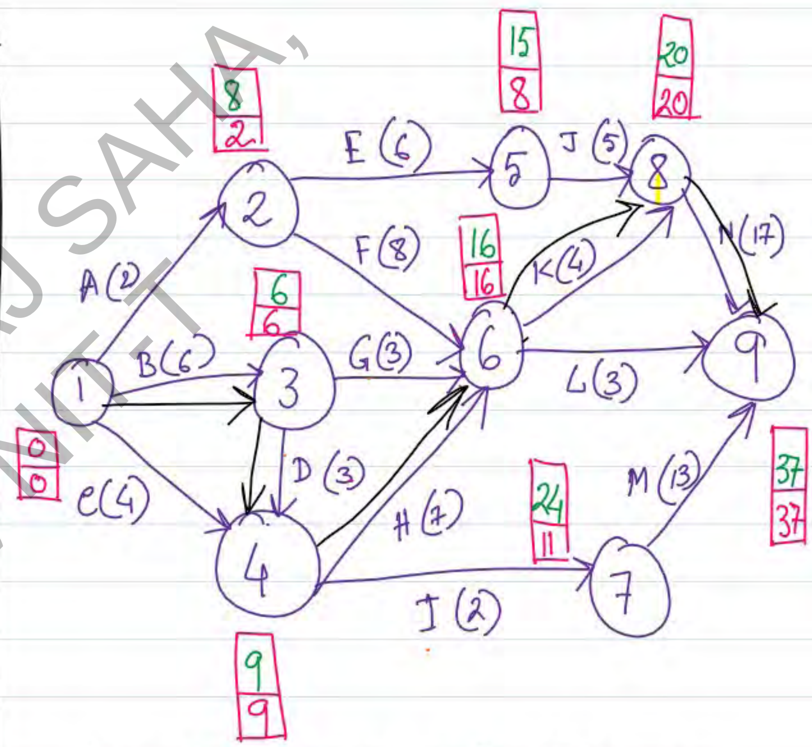
Activity	immediate precedence	Duration
A	—	2
B	—	6
C	—	4
D	B	3
E	A	6
F	A	8
G	B	3
H	C, D	7
I	C, D	2
J	E	5
K	F, G, H	4
L	F, G, H	3
M	I	13
N	J, K	17



Project management Network

Critical path is

1 → 3 → 4 → 6 → 8 → 9

$\boxed{B \to D \to H \to K \to N}$

Determination of critical path :-

Phase I :- Determine earliest starting time (ES) for all nodes.

Phase II :- Determine latest completion time (LC) for all nodes.

⇒ D_{ij} : duration of the activity (i,j)

⇒ ES_j : earliest starting time of all activities originating from node j and $ES_j = \max\limits_{i} \left(ES_i + D_{ij} \right)$

⇒ LC_i : latest completion time of all activities ending at node i, and $LC_i = \min\limits_{j} \left(LC_j - D_{ij} \right)$

Total floats :-

The amount of time taken to complete an activity can be delayed without hampering the project completion time and total floats are defined as

$$TF_{ij} = LC_j - ES_i - D_{ij} = \left(LC_j - D_{ij} \right) - ES_i$$

Free floats :-

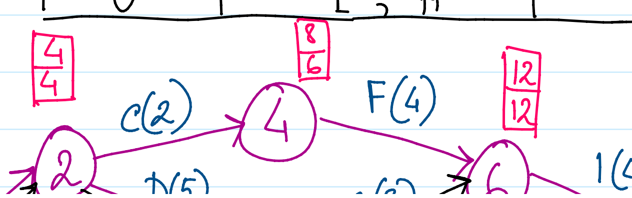
The amount of time that the activity completion time can be delayed without affecting the earliest starting time of immediate successor activities in the network.

$$FF_{ij} = ES_j - ES_i - D_{ij}$$

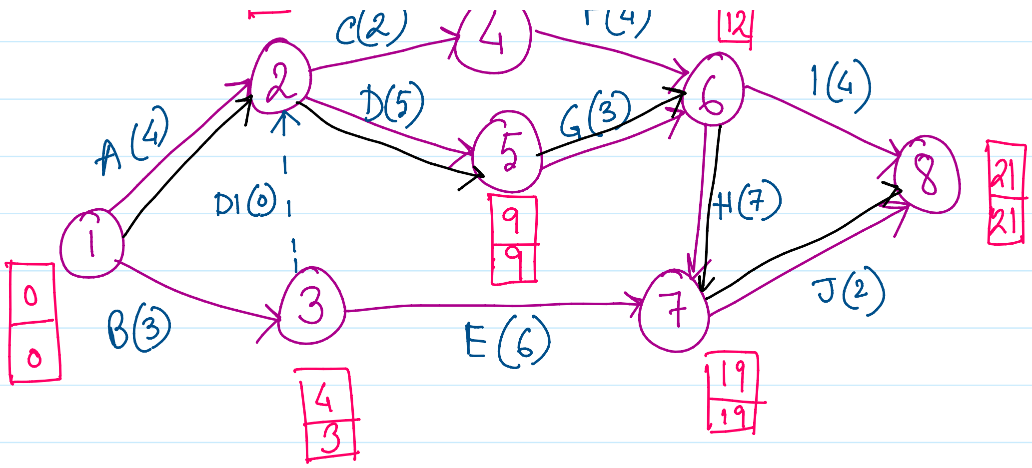
Activity (i,j)	Duration (D_{ij})	TF_{ij}	FF_{ij}
1-2	2	6	0
1-3	6	$\boxed{0}$	$\boxed{0}$
1-4	4	5	5
2-5	6	7	0
2-6	8	6	6
3-4	3	$\boxed{0}$	$\boxed{0}$
3-6	3	7	7
4-6	7	$\boxed{0}$	$\boxed{0}$
4-7	2	3	0
5-8	5	7	7
6-8	4	$\boxed{0}$	$\boxed{0}$
6-9	3	8	8
7-9	13	3	3
8-9	17	$\boxed{0}$	$\boxed{0}$

Ex 2 :-

Activity	Imm. Predecessor	Duration
A	—	4
B	—	3
C	A, B	2
D	A, B	5
E	B	6
F	C	4
G	D	3
H	F, G	7
I	F, G	4
J	E, H	2



Critical path is



Critical path is

$A \to D \to G \to H \to J$

Home task to break the tie by finding total floats and free floats.

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Project Evaluation and Review Technique :- (PERT)

Here we are considering an activity in probabilistic sense.

(i) a – Optimistic time [The time estimate when the execution of the project goes extremely good]

(ii) b – pessimistic time [The time estimate when the execution of the project goes very badly]

(iii) m – most likely time [If the execution is normal]

The probabilistic data for the project activities generally follows the Beta distribution.

The formula for mean, $\mu = \dfrac{a+4m+b}{6}$

" " " Varience, $\sigma^2 = \left(\dfrac{b-a}{6}\right)^2$

The time estimate ranges from $a \to b$

* The most likely time will be anywhere between a and b.

* The expected project completion time is $\sum_i \mu_i$, where μ_i denotes the expected duration of the i^{th} critical activities.

the expected duration of the i^{th} critical activities.

* The variance of the project completion time is $\sum_i \sigma_i^2$, where σ_i^2 is the variance of i^{th} critical activity.

* As a part of statistical analysis, we may be interested in knowing the probability of completion of a project on or before due date. For this, the β-distribution is approximated to standard normal distribution whose statistics are

$Z = \dfrac{x-\mu}{\sigma}$, x = actual project completion time

μ = expected project completion time

σ = standard deviation of expected project completion time.

Therefore $p(x \le c)$ represents the probability that the project will be completed on or before time 'c'. This can also be expressed in terms of standard normal statistics as

$$\boxed{P\left[\frac{x-\mu}{\sigma} \le \frac{c-\mu}{\sigma}\right] = P\left[Z \le \frac{c-\mu}{\sigma}\right]}$$

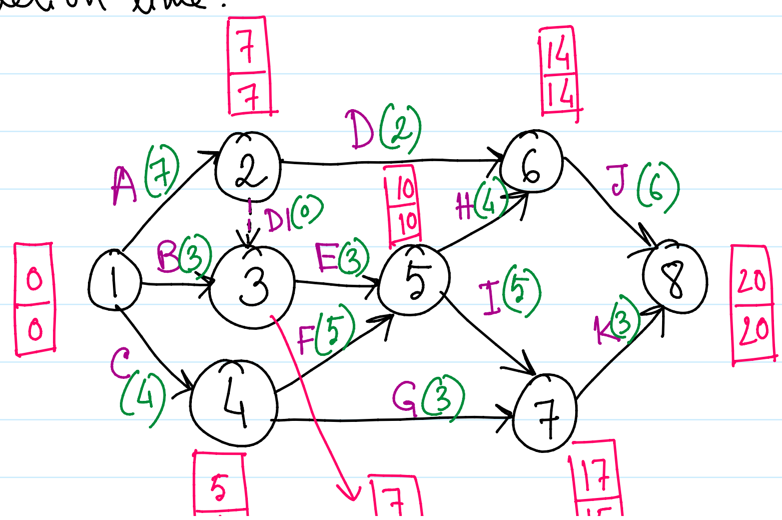
$\left(\frac{a+4m+b}{6}\right) \quad \left(\frac{b-a}{6}\right)^2$

Ex 1:

Activity	Immediate Predecessor	Duration in weeks: a	m	b	Mean (μ_i) Duration	Variance (σ_i)
A	—	6	7	8	7	0.11
B	—	1	2	9	3	1.78
C	—	1	4	7	4	1.00
D	A	1	2	3	2	0.11
E	A, B	1	2	9	3	1.78
F	C	1	5	9	5	1.78
G	C	2	2	8	3	1.00
H	E, F	4	4	4	4	0.00
I	E, F	4	4	10	5	1.00
J	D, H	2	5	14	6	4.00
K	I, G	2	2	8	3	1.00

(i) Construct a project network.
(ii) Find expected duration and variance of each activity (μ_i and σ_i)
(iii) Find critical path and expected project completion time.
(iv) What is the probability of completion a project on or before 25 weeks
(v) If the prob. of completing the project is 0.84, then find project completion time.

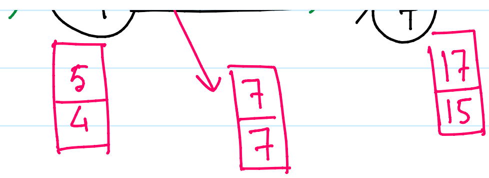
Ans (i)



The critical path is

$A \to D1 \to E \to H \to J$

and total completion time is 20 weeks.



is 20 weeks.

(iv) $\sigma = \sum_i \sigma_i = \sqrt{0.11 + 1.78 + 4.00} = \sqrt{5.89} = 2.43$

Sum of all activities lying only on the critical path

$$P(x \le 25) = P\left[\frac{x-\mu}{\sigma} \le \frac{25-20}{2.43}\right],$$

$\mu \to$ sum of the mean of all activities lying only on critical path

$= 7+3+4+6 = 20$

$$= P[z \le 2.06]$$

[You have to find using standard normal table]

(v) $P[x \le c] = 0.84$

$$\Rightarrow P\left[\frac{x-\mu}{\sigma} \le \frac{c-\mu}{\sigma}\right] = 0.84$$

$$\Rightarrow P\left[z \le \frac{c-20}{2.43}\right] = 0.84$$ [From standard normal table find c]

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Probability :-

Basic terminologies :-

(1) Exhaustive events :- A set of events is said to be exhaustive, if it includes all possible events.

$\{H, T\}$

(ii) Mutually exclusive :- If the occurrence of one of the events precludes the occurrence of all other events, then it called mutually exclusive.

(iii) Equally likely ; If one of the events cannot be expected to happen in preference to another then such events are called equally likely.

Ex 1 :- What is the chance that a leap year selected at random will contain 53 Sundays.

Solⁿ :- No. of days in a leap year $= 366$

No. of weeks $= 366/7 = 52$ weeks and 2 days

$\downarrow$ 52 Sundays

2 days $= \{MoTu, TuWe, WeTh, ThFr, FrSa, SaSu, SuMo\} = 7$

2days = {MoTu, TuWe, WeTh, ThFr, FrSa, SaSu, SuMo} = 7

$= 2/7$

$\therefore$ The chance of having 53^{th} day as Sunday is $2/7$.

Ex2 :- A 5 digit number is formed by the digits 0, 1, 2, 3, 4 without repetation. Find the prob. that the no. formed is divisible by 4.

Solⁿ :- 5 digits can be arranged in $5!$ ways.

There will be $4!$ no. of digits starting with 0.

Total no. of possible 5 digit no. is $5! - 4! = 96$

Nos. that are divisible by 4 ends with 04, 12, 20, 24, 32 and 40,

Now. nos. ending with $04 = 3! = 6$

" " " $20 = 3! = 6$

" " " $32 = 3! - 2! = 4$

" " " $12 = 3! - 2! = 4$

" " " $24 = 3! - 2! = 4$

" " " $40 = 3! = 6$

Total no. of possible cases $= 6 \times 3 + 4 \times 3 = 30$

The reqd. prob. is $30/96 = 10/32$.

The reqd. prob. is $30/96 = 10/32$.

Ex 3 :- A bag contains 40 tickets numbering $1, 2, 3, \ldots, 40$, four tickets are drawn at random and arranged in ascending order ie $(t_1 < \underline{t_2} < t_3 < t_4)$. Find the probability that $\underline{t_3 = 25}$

Solⁿ :- Total no. of possible cases ${}^{40}C_4$.

If $t_3 = 25$, then t_1 and t_2 must be less or equal to 24

This can be done in ${}^{24}C_2$ ways.

If $t_3 = 25$, then $t_4 \geq 26$. This can be done in ${}^{15}C_1$ ways.

$\therefore$ Favorable no. of cases are ${}^{24}C_2 \times {}^{15}C_1$

$\therefore$ Reqd. probability is ${}^{24}C_2 \times {}^{15}C_1 \Big/ {}^{40}C_4$.

Defⁿ :-

(i) Random Experiment :-

The experiments which are performed essentially under the same conditions and whose results cannot be predicted are called random experiments.

(ii) Sample space :-

The set of all possible outcomes of a random experiment is called Sample space and in general it is denoted by S.

(iii) Event :-

The outcome of a random experiment is called an event.

(iv) Null set :-

The set for which probability is zero is called a null set.

Axioms :-

(i) If $p(A)$ denotes the probability of an event A, then

$$0 \leq p(A) \leq 1$$

(ii) The sum of prob. of all sample events is 1, that is,

$$p(S) = 1$$

(iii) Prob. of an event made two or more sample events is sum of their probabilities.

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Notations :-

(i) The probability of occurance of the events A or B is written as $P(A+B)$ or $P(A\cup B)$

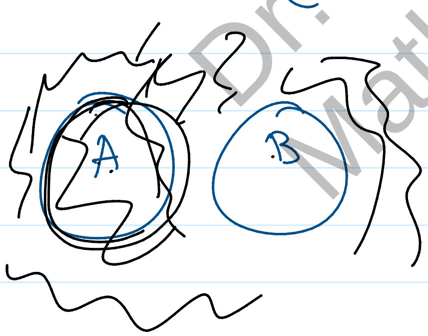
(ii) The probability of the occurence of both the events A and B is written as $P(AB)$ or $P(A\cap B)$

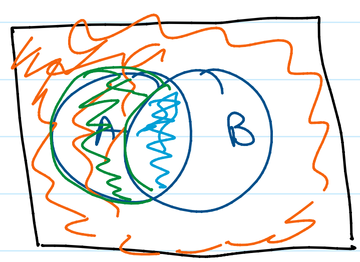
(iii) Event A $\Rightarrow$ Event B that is event $A \subset$ event B

(iv) Events A and B are mutually exclusive is expressed as
$$A\cap B=\phi.$$

(v) For any two events A and B

$$P(A\cap B') = P(A) - P(A\cap B)$$





Theorem of total Probability :-

(1) If the prob. of the occurence of an event A is $P(A)$ and prob. of occurence of a mutually exclusive event B is $P(B)$

then prob. of the occurrence of either of the events is

$$P(A+B) \text{ or } P(A\cup B) = P(A) + P(B)$$

(ii) If the events are not mutually exclusive then

$$P(A+B) = P(A)+P(B) - P(AB)$$

$$P(A\cup B) = P(A)+P(B) - P(A\cap B)$$

(iii) $P(A+B+C) = P(A)+P(B)+P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$.

Independent events :-

Two events are called independent if the occurence / failure of any one event does not affect the other event.

If so, then the events are dependent.

Defn :- If the prob. of occurence of an event A is $p(A)$ and after the occurence of A another event B occurs as a result of another trial [that means conditional probability of the occurence of B subject to A] is $P(B/A)$ and in this case prob. of the occurrence of both the events together is

$$P(AB) = p(A) \cdot p(B/A)$$

Ex 1:- Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first card is a King and the second card is a queen if the first card is

(i) replaced and (ii) not replaced.

Solⁿ :- (i) Chance of drawing a Kings $4/52 = 1/13$

If the card gets replaced, the we will again have 52 cards back in the deck.

$\therefore$ prob. of drawing a queen is $4/52 = 1/13$.

The two events are independent, then prob. of drawing both the cards in succession is $1/13 \times 1/13 = 1/169$.

(ii) prob. of drawing a Kind $1/13$.

Now, card is not replaced $\Rightarrow$ deck has 51 cards.

$\therefore$ prob. of drawing a queen is $4/51$

Hence prob. of drawing both King and queen in succession is

$$1/13 \times 4/51 = 4/13\times 51$$

$\frac{1}{13} \times \frac{4}{51} = \frac{4}{13 \times 51}$

Ex 2 :- A pair of dice is tossed twice. Find the prob. of scoring 7 points (i) once (ii) at least once (iii) twice

Solⁿ :- Out Come is 7

Possible choices (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) are 6 ways.

$\therefore$ prob. of getting 7 once is $\frac{6}{36} = \frac{1}{6}$. ✓

$\therefore$ prob. of not getting 7 once is $1 - \frac{1}{6} = \frac{5}{6}$ ✓

Prob. of getting 7 in first toss and not getting 7 in 2nd toss

$= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$ → Event A

prob. of not getting 7 in first toss and 7 in 2nd toss $= \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

— Event B

mutually exclusive events →

$\therefore$ Prob. of $p(A+B) = p(A) + p(B) = \frac{5}{18}$.

getting 7 once.

(b) prob. of not getting 7 in both the toss $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

prob. of getting 7 atleast once $1 - \frac{25}{36} = \frac{11}{36}$

prob. of getting 7 atleast onece $1 - 25/36 = 11/36$

(c) prob. of getting 7 twice $1/6 \times 1/6 = 1/36$.

Homework :-

1. Two cards are selected at random from 10 cards having numbers 1, 2, 3, ..., 10. Find prob. 'p' that the sum is odd, if

(i) two cards are drawn together, $5\times5 / {}^{10}C_2$

(ii) two cards are drawn one after another without replacement.

$5\times5 + 5\times5 \;/\; 10\times9$

(iii) two cards are drawn one after another with replacement.

$5\times5 + 5\times5 / 10\times10$

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Random Variables :-

If a real variable X associated with the outcome of some random experiment. Then X is called random variable.

Discrete random Variable :-

(i) Discrete Probability distribution :-

Suppose X is a discrete variable associated with some exp.

If the prob. that X takes the value x_i is p_i, then

$$P(X=x_i) = p_i \quad \text{for } i=1,2,3\ldots$$

(i) $p_i \geq 0$ for all i

(ii) $\sum_i p_i = 1$

The set of values x_i, with their prob. p_i constitute discrete prob. distribution of the discrete random variable X.

For example :- We throw a pair of dice and find the sum of the nos. appearing.

$X=x_i$	2	3	4	5	6	7	8	9	10	11	12
p_i	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$
		(1,2)	(1,3)					(4,5)	(4,6)	(5,6)	(6,6)

(1,2) (2,1)

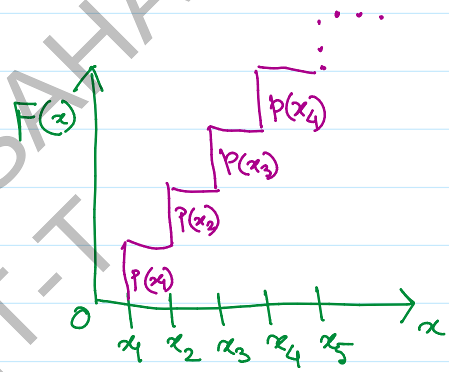
(1,3) (2,2) (3,1)

(4,5) (4,6) (5,6) (6,6) (3,6) (5,5) (6,5) (6,3) (6,4) (5,4)

 # Distribution function :- The distribution function $F(x)$ of the discrete variate X is defined by

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i) \quad \text{where } x \text{ is any integer.}$$

The graph of $F(x)$ looks like steps and it is also called cumulative distribution function.



Ex 1 :- A dice is thrown 3 times. The success of getting 1 or 6 in a toss.

Solⁿ :- Prob. of success is $2/6 = 1/3$

Prob. of failure is $1 - 1/3 = 2/3$

$\therefore$ Prob. of no success at all = Prob. of all 3 failures $= \left(\frac{2}{3}\right)^3 = 8/27$

Now. Prob. of 1 success and 2 failures $= {}^3C_1 \times 1/3 \times 2/3 \times 2/3 = 4/9$

Prob. of 2 successes and 1 failure $= {}^3C_2 \times \left(\frac{1}{3}\right)^2 \times 2/3 = 2/9$

Prob. of 3 successes $= \left(\frac{1}{3}\right)^3 = 1/27$

Prob. of 3 successes $= \left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

Ex2:- The prob. density function of a variate X is

X :	0	1	2	3	4	5	6
$p(X)$:	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

(i) Find $P(X<4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$

(ii) Find min. value of K such that $P(X \leq 2) > 0.3$

Sol^{n}s :- (i) X is a random variable, then

$$\sum_{i=0}^{6} p(x_i) = 1 \Rightarrow (K+3K+5K+7K+9K+11K+13K) = 1 \Rightarrow K = \frac{1}{49}$$

$$P(X<4) = K + 3K + 5K + 7K = \frac{16}{49}$$

$$P(X \geq 5) = \underbrace{1 - P(X<5)}_{} = 1 - [K + 3K + 5K + 7K + 9K]$$

$$\overset{\parallel}{(11+13)K} = \frac{24}{49} \qquad = 1 - \frac{25}{49} = \frac{24}{49}$$

$$P(3 < X \leq 6) = 9K + 11K + 13K = \frac{33}{49}$$

(ii) $P(X \leq 2) = K + 3K + 5K = 9K > 0.3 \Rightarrow K > \frac{1}{30}$

K should be more than $\frac{1}{30}$.

(ii) Continuous Prob. distribution :-

When a random variate X takes up every values in an interval, then we get a continuous prob. distribution.

Defⁿ : Distribution function

If $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

then $F(x)$ is called cumulative dist. function or simple distribution fnc. of the continuous random variable X.

Properties :- (i) $F'(x) = f(x) \geq 0$ so that $F(x)$ is a nondecreasing function.

(ii) $F(-\infty) = 0$

(iii) $F(\infty) = 1$

(iv) $P(a \le x \le b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx = F(b) - F(a)$.

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Expectation :-

The mean value of the prob. distribution of a random variable X is called expectation,

$$E(X) = \sum_i x_i f(x_i) \rightarrow \text{discrete case}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \rightarrow \text{Continuous case}$$

Varience :- $\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i)$, μ = mean

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

σ = is called standard deviation.

Moment :- The r^{th} moment is defined as

$$\mu_r = \sum_i (x_i - \mu)^r f(x_i)$$

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx$$

$$\mu_2 = \sigma^2$$

→ Mean deviation is given by $\sum |x_i - \mu| f(x_i)$

↳ Mean deviation is given by $\sum_i |x_i - \mu| f(x_i)$

$$\int_{-\infty}^{\infty} (x-\mu) f(x)\, dx$$

Ex 1 :- X is a cont. rand. var. with prob. density as

$$f(x) = \begin{cases} Kx, & 0 \le x \le 2 \\ 2K, & 2 \le x \le 4 \\ -Kx + 6K, & 4 \le x \le 6 \end{cases}$$

Find K and mean value of X.

Solⁿ:- $\int_{-\infty}^{\infty} f(x)\, dx = 1 \Rightarrow \left[\int_0^2 + \int_2^4 + \int_4^6 \right] f(x)\, dx = 1$

$\therefore K = 1/8$

Mean value of $X = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^6 x f(x)\, dx$

$$= \int_0^2 \frac{x^2}{8} dx + \int_2^4 \frac{x}{4} dx + \int_4^6 x\left(-\frac{x}{8} + 3/4\right) dx$$

$= 3.$

Ex 2 :- A variate X has prob. distribution

x	-3	6	9
$P(X=x)$	$1/6$	$1/2$	[illegible]

x	-3	6	9
$P(X=x)$	$1/6$	$1/2$	$1/3$

Find $E(X)$ and $E(X^2)$. Hence evaluate $E(2X+1)^2$.

Solⁿ :- $E(X) = \sum_i x_i\, p(x_i) = -3\times 1/6 + 6\times 1/2 + 9\times 1/3$

$$= -1/2 + 3 + 3 = 5\tfrac{1}{2}.$$

$$E(X^2) = \sum_i x_i^2\, p(x_i) = 9\times 1/6 + 36\times 1/2 + 81/3 = 3/2 + 18 + 27 = 46\tfrac{1}{2}$$

$$E(2X+1)^2 = \sum_i (2x_i+1)^2\, p(x_i) = \sum_i (4x_i^2 + 4x_i + 1)\, p(x_i)$$

$$= 4\sum_i x_i^2\, p(x_i) + 4\sum x_i\, p(x_i) + \sum p(x_i)$$

$$= 4E(X^2) + 4E(X) + 1$$

$$= 4\times\left(46\tfrac{1}{2}\right) + 4\times\left(5\tfrac{1}{2}\right) + 1$$

$$= 186 + 22 + 1 = 209.$$

Moment Generating Function :- (MGF)

The mgf of a discrete prob. distribution of a variate X about $x=a$ is defined by

$$M_a(t) = \sum_i p(x_i)\, e^{t(x_i-a)}$$

$$M_a(t) = \sum_i f(x_i) e^{\ldots}$$

$$= \sum_i e^{t(x_i - a)} p(x_i)$$

$$= E\left(e^{t(x-a)}\right)$$

$$M_a(t) = \sum_i p(x_i)\left[1 + t(x_i - a) + \frac{t^2 (x_i - a)^2}{2!} + \ldots\right]$$

moment genering function.

$$= 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \ldots \quad \text{--- (1)}$$

$\mu_r = \frac{t^r}{r!}$ is the expansion of $M_a(t)$

$$\frac{d}{dt} M_a(t) = \mu_1 + t[\ldots]$$

$$\left.\frac{d}{dt} M_a(t)\right|_{t=0} = \mu_1 \qquad \text{lly} \quad \left.\frac{d^2 M_a(t)}{dt^2}\right|_{t=0} = \mu_2$$

Likewise, $\mu_r = \left.\frac{d^r}{dt^r} M_a(t)\right|_{t=0}$

Ex 1: Find MGF of the distribution $f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x < \infty$, $c > 0$.

Also find its mean and σ.

Solⁿ:- $M_0(t) = \int_0^\infty e^{tx} \frac{1}{c} e^{-x/c}\, dx = (1-ct)^{-1}$

$= 1 + ct + c^2t^2 + c^3t^3 + \cdots$

mean — $\mu_1 = \frac{d}{dt} M_0(t) \Big|_{t=0} = c$

Var. $= \sigma^2 = \mu_2 = \frac{d}{d} M_0(t) \Big|_{t=0} = c^2$ $\therefore$ SD $= \sigma = c$.

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Binomial Distribution :-

Consider p as the prob. of success

q " " " " failure

We are performing n-trials and prob. of getting 'r' successes is ${}^nC_r\, p^r q^{(n-r)}$

The prob. of r successes out of n trials is given by Binomial distribution ${}^nC_r\, p^r q^{n-r} = {}^nC_r\, p^r (1-p)^{n-r}$

Constants of Binomial Distribution :-

$$M_0(t) = E(e^{tx}) = \sum {}^nC_x\, p^x q^{n-x} e^{tx}$$

$$= \sum {}^nC_x\, (pe^t)^x q^{n-x}$$

$$= (q+pe^t)^x$$

$\mu_1 = np.$

$$M_m(t) = e^{-npt}(q+pe^t)^n = \left[q\underline{e^{-pt}} + p\underline{\underline{e^{qt}}}\right]^n.$$

$$= \left[1 + pq\frac{t^2}{2!} + pq(q^2-p^2)\frac{t^3}{3!} + pq(q^3+p^3)\frac{t^4}{4!}\right.$$

$$= \left[1 + pq\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + pq(q^3 + p^3)\frac{t^4}{4!} + \cdots\right]^n$$

$$\Rightarrow 1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \mu_4 \frac{t^4}{4!} + \cdots$$

$$= 1 + npq\frac{t^2}{2!} + npq(q-p)\frac{t^3}{3!} + npq\left[1 + 3(n-2)pq\right]\frac{t^4}{4!} + \cdots$$

Equate the coeffs of like powers of t.

$\mu_2 = npq$

$\mu_3 = npq(q-p)$

$\mu_4 = npq\left(1 + 3(n-2)pq\right)$

mean $= \mu_1 = np$

Standard deviation $= \sqrt{\mu_2}$

$= \sqrt{npq}$

Ex 1 :- The prob. of manufacturing a defective pen is $1/10$. If 12 such pens are manufactured, then find prob.

(i) Exactly 2 pens are defective

(ii) at least 2 pens are defective

(ii) at least 2 pens are defective ✓

(iii) none are defective.

Solⁿ :- prob. of defective pen $1/10$. ✓

prob. of good pen $= 1 - 1/10 = 9/10$.

(i) $^{12}C_2 \left(1/10\right)^2 \left(9/10\right)^{10} =$

(ii) $1 - [$ prob. that either no or one is defective $]$

$$= 1 - \left[{}^{12}C_0 \left(1/10\right)^0 \left(9/10\right)^{12} + {}^{12}C_1 \left(1/10\right)^1 \left(9/10\right)^{11} \right]$$

(iii) $^{12}C_{12} \left(1/10\right)^0 \left(9/10\right)^{12} =$

Ex 2:- In sampling, a large no. of parts produced by a machine. the mean no. of defective in the sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.

Solⁿ :- Mean no. of defectives $= np = 2$

$$\Rightarrow 20p = 2 \Rightarrow p = 2/20 = 1/10$$

Prob. of non-defective is $1 - 1/10 = 9/10$.

Prob. of non-defective is $1 - 1/10 = 9/10$.

The prob of at least 3 defectives in a samples of size 20.

$$= 1 - [\text{prob. of either no or one or 2 defectives}]$$

$$= 1 - \left[{}^{20}C_0 \left(\tfrac{1}{10}\right)^0 \left(\tfrac{9}{10}\right)^{20} + {}^{20}C_1 \left(\tfrac{1}{10}\right)^1 \left(\tfrac{9}{10}\right)^{19} + {}^{20}C_2 \left(\tfrac{1}{10}\right)^2 \left(\tfrac{9}{10}\right)^{18}\right]$$

$$= a$$

Out of 20 prob. of getting 3 defectives is a

$\therefore$ out of 1000 the no. of defectives is $1000a$.

Home Works:

1. Determine BD for which mean $= 2 \times$ Variance and mean + Variance $= 3$. Also find $P(X \leq 3)$.

2. If the chance that one out of 10 phone lines is busy at an instant 0.2.

   (i) What is the chance that 5 phone lines are busy.

   (ii) What is the most probable no. of busy lines and what is the prob. of this no.

is the prob. of this no.

(iii) What is the prb. that all lines are busy.

Poisson Distribution :-

The prob. of r successes in BD is

$$P(r) = {}^{n}C_r\, p^r q^{n-r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\, p^r q^{n-r}$$

$$= \frac{\overset{m}{np}\,(\overset{m}{np}-p)(\overset{m}{np}-2p)\cdots[\overset{m}{np}-(r-1)p]}{r!}\,(1-p)^{n-r}$$

$p = m/n$

Now, $n \to \infty$, $m \overset{\shortparallel}{=} np$, $p \to 0$

$$p(r) = \frac{m^r}{r!} \lim_{n\to\infty} \frac{[1-m/n]^n}{[1-m/n]^r} = \frac{m^r}{r!} e^{-m}$$

The prob. of r successes in Poisson distribution is

$$\boxed{p(r) = \frac{m^r}{r!} e^{-m}}$$

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Constants of Poisson distribution :-

$np = m$ (Mean)

$n \to \infty$, $p \to 0$, $q \to 1$

Standard deviation $\sqrt{\mu_2} = \sqrt{m}$

Ex1 :- If the prob. of a bad reaction from a certain injection is 0.001. Then determine the chance that out of 2000 individuals more than 2 will get a bad reaction?

Solⁿ :- $n = 2000$, $p = 0.001$, mean, $m = np = 2$.

We will follow Poisson distribution :-

prob. that more than two will get a bad reaction

= 1 − [prob. that noone gets infection + prob. that 1 only get infection + prob. that 2 get infection]

$$= 1 - [p(0) + p(1) + p(2)]$$

$$= 1 - \left[e^{-2} + \frac{2}{1}e^{-2} + \frac{2^2}{2!}e^{-2}\right]$$

$$p(r) = \frac{m^r}{r!}e^{-m}$$

Ex2 :- Fit a Poisson distribution to the set of observations :-

Ex 2 :- Fit a Poisson distribution to the set of observations :-

x	0	1	2	3	4
f	122	60	15	2	1

Solⁿ :- Mean $\dfrac{\sum x_i f_i}{\sum f_i} = \dfrac{60+30+6+4}{122+60+15+2+1} = \dfrac{100}{200} = \frac{1}{2}. \; (=m)$

Now, frequency of r successes $N \dfrac{e^{-m}(m)^r}{r!}$ where $r = 0,1,2,3,4$

$N = \sum f_i$

$$= 200 . \frac{e^{-1/2}}{r!} \left(\frac{1}{2}\right)^r$$

x	0	1	2	3	4
f	$200e^{-1/2}$	$200e^{-1/2}\left(\frac{1}{2}\right)$	$\dfrac{200e^{-1/2}\left(\frac{1}{2}\right)^2}{2!}$	$\dfrac{200e^{-1/2}\left(\frac{1}{2}\right)^3}{3!}$	$\dfrac{200e^{-1/2}\left(\frac{1}{2}\right)^4}{4!}$

Home Work :-

1. X is a Poisson variable and the prob. that $X=2$ is $2/3$rd of the prob. that $X=1$. $\left[P(X=2) = \frac{2}{3} P(X=1) \right]$. Find the prob. that $X=0$ and prob. of $X=3$. What is the prob. that X exceeds 3.

2. A Company knows that the product Contains 1% defectives. The

2. A company knows that its product contains 1% defectives. The company packs all products in boxes of 100 pieces. What is the prob. that a randomly picked box will contain 3 or more defective items.

3. A metered-car company has 2 cars which hires out day by day. The no. of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand,

(ii) on which demand is refused.

NORMAL Distribution :-

Let us define $Z = \dfrac{x - np}{\sqrt{npq}}$ where x is a Binomial variate

- np is the mean, and
- $\sqrt{npq}$ is the SD.

for Z, mean is 0 and variance is 1.

When $n \to \infty$, the distribution Z becomes a continuous quantity ranging from $(-\infty, \infty)$.

ranging from $(-\infty, \infty)$.

$\therefore$ for large n when neither p or q are very small, we get the normal distribution as

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]} , \quad \mu \text{ is the mean. } \sigma \text{ is the SD.}$$

Properties :-

I. The normal curve is bell shaped as is symm. about the mean.

II. Deviation from the mean :-

$$\int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad \text{put } z = \frac{x-\mu}{\sigma}$$

$$= \int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{2\pi}} |z| e^{-z^2/2} dz = \dots = \frac{4\sigma}{5}$$

Pl. do the calc.

III. Moments about the mean :-

odd moment $$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx , \quad z = \frac{x-\mu}{\sigma}$$

= find the value

even moment μ_{2n} = Pl. find out.

IV. Prob. that x lies between x_1 and x_2 is given by the area under the normal curve

$$P(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

$$= P_2(z) - P_1(z), \text{ where } \underbrace{P(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}\, dz.}_{\text{Probability integral or error function.}}$$

* Area between $z=0$ to $z=1$ $\Rightarrow$ $x=\mu$ to $x=\mu+\sigma$

[0.3413] Verify.

* Area between $x=\mu-\sigma$, $x=\mu+\sigma$
* Area between $x=\mu-2\sigma$, $x=\mu+2\sigma$
* Area " $x=\mu-3\sigma$, $x=\mu+3\sigma$

Home Works

The standardised normal curve is $y = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ where

$$z = \frac{x-\mu}{\sigma}$$

z is called standard normal variate.

Ex1 :- X is a normal variate with mean 30 and SD 5. Find the Prob. that (i) $26 \le X \le 40$, (ii) $X \geq 45$, and (iii) $|X-30| > 5$

Sol^n :- $m = 30$, $\sigma = 5$

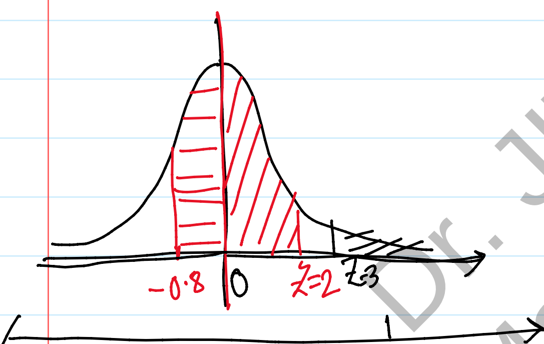
$$\therefore Z = \frac{X-m}{\sigma} = \frac{X-30}{5}$$

(i) $P(26 \le X \le 40) = P\left(\frac{26-30}{5} \le \frac{X-30}{5} \le \frac{40-30}{5}\right)$

$= P(-0.8 \le Z \le 2)$

$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$

$= 0.2881 + 0.4772$



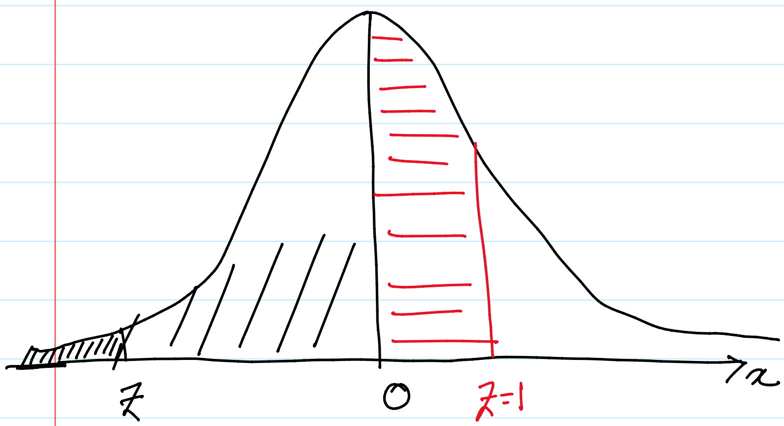
(ii) $X \geq 45$, $P(X \geq 45) = P(Z \geq 3) = \frac{1}{2} - P(0 \le Z \le 3)$

(iii) $P(|X-30| \le 5) \Rightarrow P(-5 \le X-30 \le 5)$

$= P(25 \le X \le 35)$

$= P(-1 \le Z \le 1) = 2\underline{P(0 \le Z \le 1)}$

$= 2\left[P(Z=1) - P(Z=0)\right]$



$$= 2\left[P(z=1) - P(z=0)\right]$$

$$= 2\left[0.8413 - 0.5\right]$$

$$= 2 \times \left[0.3413\right]$$

$$= 0.6826.$$

Ex 2:- In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find mean and SD of the distribution.

Solⁿ:- Let μ be the mean

σ be the SD.

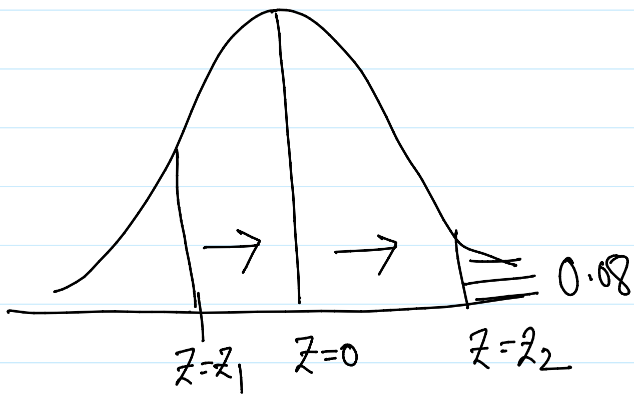
31% are under 45 $\Rightarrow$ area to the left of the ordinate $x=45$.

$\therefore$ When $x = 45$, let $z = z_1$ and $z_1 = \dfrac{45-\mu}{\sigma}$

$$\therefore \int_{-\infty}^{z_1} \phi(z)\,dz = 0.31$$

$$\Rightarrow \left[\int_{-\infty}^{0} + \int_{0}^{z_1}\right] \phi(z)\,dz = 0.31$$

$$\Rightarrow \int_{0}^{z_1} \phi(z)\,dz = -\underbrace{\int_{-\infty}^{0} \phi(z)\,dz}_{} + 0.31 = -0.5 + 0.31 = -0.19$$



$$\int_{z_1}^{0} \phi(z)\,dz = 0.19$$

Now, we have to find z_1 (using SN table).

When $x=64$, let $z=z_2$ so that $z_2 = \dfrac{64-\mu}{\sigma}$

$$\therefore \int_{z_2}^{\infty} \phi(z)\,dz = 0.08 \quad \text{or,} \quad \int_{0}^{\infty} \phi(z)\,dz - \int_{0}^{z_2} \phi(z)\,dz = 0.08$$

$$\Rightarrow \int_{0}^{z_2} \phi(z)\,dz = 0.5 - 0.08 = 0.42$$

Find z_2 using standard normal table.

2-dimensional Random Variables :-

X and Y are two random variables taken from a set S.

Let $X(s), Y(s)$, $s \in S$ are assigned with real numbers.

Then the two-dimensional random variate is denoted as (X, Y).

Discrete form

(x_i, y_j) $\quad i = 1, 2, \ldots, m$
$\qquad\qquad j = 1, 2, \ldots, n$

Continuous form.

can assume all possible values in the xy-plane.

Joint Prob. function / Mass function of (X, Y).

Let (X, Y) be 2-dim. discrete random variables and also let

$$P(X = x_i, Y = y_j) = P(X = x_i \cap Y = y_j) = p(x_i, y_j) = p_{ij}$$

Then p_{ij} is called joint prob. function of (X, Y) if the following conditions are satisfied;

(i) $p_{ij} \geq 0 \quad \forall\, i, j$

(ii) $\sum\limits_{j} \sum\limits_{i} p_{ij} = 1$

$j \quad i$

The set of triplets $\{x_i, y_j, p_{ij}\}$ $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$

is called joint prob. distribution.

X \ Y	y_1	y_2	$\cdots$	y_j	$\cdots$	y_n	Total.
x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$	p_{1n}	p_{1*}
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$	p_{2n}	p_{2*}
$\vdots$	$\vdots$						$\vdots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$	p_{in}	p_{i*}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_m	p_{m1}	p_{m2}	$\cdots$	p_{mj}	$\cdots$	p_{mn}	p_{m*}
Total	p_{*1}	p_{*2}	$\cdots$	p_{*j}	$\cdots$	p_{*n}	

Marginal Prob. function:-

(X, Y) is a 2-dim discrete random Variate. Then prob. dist. of X or marginal prob. function of X is defined as

$$P(X = x_i) = P\left[(X = x_i \cap Y = y_1) \text{ or } (X = x_i \cap Y = y_2) \text{ or } \cdots \text{ or } (X = x_i \cap Y = y_n)\right]$$

$$= p_{i1} + p_{i2} + \cdots + p_{in} = p_{i*}$$

$$\Rightarrow \sum_{j=1}^{n} p_{ij} = \sum_{j=1}^{n} P(x_i, y_j) = p_{i*}$$

* Marginal prob. function of Y is defined as.

$$P(Y = y_j) = P\left[(X = x_1 \cap Y = y_j) \text{ or } (X = x_2 \cap Y = y_j) \text{ or } \cdots \text{ or } (X = x_m \cap Y = y_j)\right]$$

$$= p_{1j} + p_{2j} + \cdots + p_{mj}$$

$$= \sum_{i=1}^{m} p_{ij} = p_{*j}$$

Condinal prob. function of X for a given $Y = y_j$ is

$$P(X = x_i \mid Y = y_j) = \frac{P[X = x_i \cap Y = y_j]}{P[Y = y_j]} = \frac{p_{ij}}{p_{*j}}$$

Similarly. Cond. prob fnc. of Y given $X = x_i$ is

$$P(Y = y_j \mid X = x_i) = \frac{P[Y = y_j \cap X = x_i]}{P(X = x_i)} = \frac{p_{ij}}{p_{i*}}$$

$$P(Y = y_j \mid X = x_i) = \frac{\quad}{P(X = x_i)} = p_{i*}$$

Note :- If X and Y are independent then

$$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j).$$

Cumulative dist function :-

Let (X, Y) be a 2-dim. discrete random variable.

Then cdf (X, Y) defined by.

$$F(x, y) = P(X \le x \text{ and } Y \le y) = \sum_i \sum_j p_{ij}, \quad y_j \le y, \; x_i \le x.$$

The cdf satisfies

(i) $F(-\infty, y) = 0$, $F(x, -\infty) = 0$, $F(-\infty, \infty) = 1$

(ii) F is monotonic non-decreasing function.

(iii) $P(a \le X \le b, \; Y \le y) = F(b, y) - F(a, y)$

(iv) $P(X \le x, \; c < Y < d) = F(x, d) - F(x, c)$

(v) At all points of continuity of the density function $f(x, y)$,

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y).$$

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Joint Probability density function:-

(X,Y) continuous random variables.

dx, dy, The area enclosed is $dx\,dy$.

$$\left[x-\frac{dx}{2}\le X\le x+\frac{dx}{2}\right],\left[y-\frac{dy}{2}\le Y\le y+\frac{dy}{2}\right]$$

The prob. that (x,y) lies in the area is given by

$$P\left[\left(x-\frac{dx}{2}\le X\le x+\frac{dx}{2}\right) \text{ and } \left(y-\frac{dy}{2}\le Y\le y+\frac{dy}{2}\right)\right]=f(x,y)\,dx\,dy$$

The function $f(x,y)$ is called joint pdf of (X,Y) if

(i) $f(x,y)\geq 0$ for $(x,y)\in R$ where R is the range space

(ii) $\displaystyle\iint_R f(x,y)\,dx\,dy=1$

If $D\in R$, then $P[(X,Y)\in D]=\displaystyle\iint_D f(x,y)\,dx\,dy$

$D:=\{a\le x\le b,\ c\le y\le d\}$ $\quad\downarrow\quad \displaystyle\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx.$

The cdf of (X,Y) is defined by $\displaystyle\int_{-\infty}^{y}\int_{x_1=-\infty}^{x} f(x_1,y_1)\,dx_1\,dy_1$

$$\int_{-\infty} \int_{x_1=-\infty} f(x_1,y_1)\,dx_1\,dy_1$$

Now, $P\left[\left(x-\frac{dx}{2} \le X \le x+\frac{dx}{2}\right), -\infty < Y < \infty\right]$

$$= \int_{-\infty}^{\infty} \int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} f(x,y)\,dx\,dy$$

$$= \int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} \left[\underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}\right] dx$$

$$= f_X(x)\,dx$$

$f_X(x)$ is called marginal density w.r.t X.

|||y, marginal density w.r.t Y is $\int_{-\infty}^{\infty} f(x,y)\,dx$.

The marginal distribution functions are

$$F_X(x) = \int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] dx$$

$$F_Y(y) = \int_{-\infty}^{y}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right] dy.$$

In particular, $P(a \le X \le b) = P(a \le X \le b, -\infty < Y < \infty)$

$$= \int^{\infty}\left[\int^{b} f(x,y)\,dx\right] dy = \int^{b}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] dx.$$

$$= \int_{-\infty} \left[\int_a f(x,y)\, dx \right] dy = \int_a \left[\int_{-\infty} f(x,y)\, dy \right] dx.$$

$$= \int_a^b f_X(x)\, dx.$$

Similarly, $P(c \le Y \le d) = \int_c^d f_Y(y)\, dy.$

Remark :-

If the joint pdf $f(x,y)$ is given, then we can determine the marginal pdfs. However, the converse is not true.

The Conditional prob. is defined by

$$P\left[\left(x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \right) \Big| Y = y \right]$$

$$= P\left[\left(x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \right) \Big| \left(y - \frac{dy}{2} \le Y \le y + \frac{dy}{2} \right) \right]$$

$$= \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy} = \left[\frac{f(x,y)}{f_Y(y)} \right] dx.$$

Similarly, $\frac{f(x,y)}{f_Y(y)}$ defines the Conditional density of X for a given Y.

and $\frac{f(x,y)}{f_X(x)}$ defines the Conditional density of Y for a given X.

Independent Random Variables :-

$$f(x,y) = \underbrace{f_X(x)}_{\text{marginal pdf of } X} \; f_Y(y) \rightarrow \text{marginal pdf of } Y.$$

Expected Values :-

$$(X,Y) \rightarrow E[g(X,Y)] = \sum_i \sum_j g(x_i, y_j) \underbrace{p_{ij}}_{\text{prob. mass function.}}$$

$$(X,Y) \rightarrow E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \underbrace{f(x,y)}_{\text{joint pdf.}} \, dx\, dy.$$

Ex 1 :-

X \ Y	0	1	2	3
0	1/32	2/32	3/32	2/32
1	2/32	1/8	1/4	1/32
2	3/32	2/16	0	1/16

(i) $P(X \le 1, Y = 3)$, (ii) $P(Y \le 2)$, (iii) $P(X \le 1 \mid Y \le 2)$

Sol^n :- $P(X \le 1, Y = 3) = P(X = 0, Y = 3) + P(X = 1, Y = 3)$

Sol^n:- $P(X\le 1, Y=3) = P(X=0, Y=3) + P(X=1, Y=3)$

$$= 2/32 + 1/32 = 3/32$$

$P(Y\le 2) = P(Y=0) + P(Y=1) + P(Y=2)$

$$= 6/32 + 10/32 + 11/32 = 27/32$$

$$P(X\le 1 \mid Y\le 2) = \frac{P(X\le 1, Y\le 2)}{P(Y\le 2)} = \left[P(X\le 1, Y=0) + P(X\le 1, Y=1) + P(X\le 1, Y=2)\right] \Big/ \frac{27}{32}$$

$$= \left[3/32 + 6/32 + 11/32\right]\frac{32}{27}$$

$$= 20/27$$

Ex 2:- joint prob. mass function (X, Y) is given by

$$P(x,y) = K(3x+5y), \quad x = 1,2,3, \quad y = 0,1,2.$$

Find marginal and conditional prob. distribution of X,

$P(X=x_i \mid Y=2)$, $P(X\le 2 \mid Y\le 1)$.

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Ex 1 :- The joint pdf of the random variables X and Y is given by $f(x,y) = K(xy+y^2)$, $0 \le x \le 1$, $0 \le y \le 2$

Find $P(Y>1)$, $P(X>\frac{1}{2}, Y<1)$ and $P(X+Y \le 1)$.

Solⁿ :- $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(xy+y^2)\,dx\,dy = 1 \Rightarrow \int_0^2\int_0^1 K(xy+y^2)\,dx\,dy = 1$$

$$\Rightarrow K\int_0^2 \left[\frac{x^2y}{2} + xy^2\right]_0^1 dy = 1$$

$$\Rightarrow K\int_0^2\left(\frac{1}{2}y + y^2\right)dy = 1 \Rightarrow K\left[\frac{y^2}{4} + \frac{y^3}{3}\right]_0^2 = 1 \Rightarrow K\left(1+\frac{8}{3}\right) = 1$$

$$K = \frac{3}{11}$$

$$\therefore f(x,y) = \frac{3}{11}(xy+y^2)$$

$$P(Y>1) = \iint_R \frac{3}{11}(xy+y^2)\,dx\,dy$$

$$= \int^2 \int^1 \frac{3}{11}(xy+y^2)\,dx\,dy = \text{find}$$

$$= \int_{y=1} \int_{x=0} \frac{3}{11}\left(xy+y^2\right) dx\, dy = \text{find}$$

$$* \; P\left(X > 1/2, Y < 1\right) = \frac{3}{11} \int_{y=0}^{1} \int_{x=1/2}^{1} \left(xy+y^2\right) dx\, dy = \text{find}$$

$$* \; P\left(X+Y \leq 1\right) = \iint_{R_3} f(x,y)\, dx\, dy$$

$x+y \leq 1$

$x \leq 1-y$

$$= \int_{y=0}^{2} \int_{x=0}^{1-y} f(x,y)\, dx\, dy = \text{find}.$$

Functions of one random variable :-

X : random variable, S : sample space

g be a function s.t $Y = g(X)$

Then Y is also a random variable.

Let $f_X(x)$ is the pdf of X. and we assume $y = g(x)$ is strictly monotonic function of x, and $g(x)$ is diff for all x

Then the pdf of there new random Varable Y is defined by

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x).$$

Remark :- If x takes the values $x_1, x_2, \ldots, x_m$, then

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_m)\left|\frac{dx_m}{dy}\right|.$$

Ex 1 :- Let $F(x)$: distribution function

$f(x)$: density function of random var. X.

Then find the distribution and density fnc. of $Y = aX + b$, $a \neq 0$, b real.

Sol :- $G(Y)$: distribution $g(y)$: density of rand. var. Y.

When $a > 0$, $G(y) = P(Y \le y) = P(aX + b \le y)$

$$= P\left(X \le \frac{y-b}{a}\right) = F\left(\frac{y-b}{a}\right)$$

When $a < 0$: $G(y) = P(Y \le y) = P(aX + b \le y)$

$$= P\left(X \geq \frac{y-b}{a}\right) =$$

$$= 1 - P\left(X \le \frac{y-b}{a}\right) = 1 - F\left(\frac{y-b}{a}\right).$$

Now, $g_Y(y) = \dfrac{dG(y)}{dy}$

Now, $g_Y(y) = \dfrac{dG(y)}{dy}$

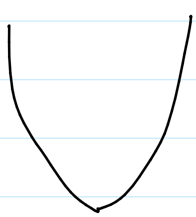
Hence, $g_Y(y) = \begin{cases} \dfrac{1}{a} f\left(\dfrac{y-b}{a}\right) & \text{when } a>0 \\ -\dfrac{1}{a} f\left(\dfrac{y-b}{a}\right) & \text{when } a<0. \end{cases}$

$$\therefore \; g_Y(y) = \frac{1}{|a|} f\left(\frac{y-b}{a}\right)$$

Ex 2 :- X has pdf $f_X(x) = \begin{cases} \dfrac{2}{25}(x+2), & -2<x<3 \\ 0, & \text{elsewhere.} \end{cases}$

Find pdf of $Y = X^2$.

Solⁿ :- $y = x^2 \geq 0$ and it is not monotonic in $(-2,3)$



Therefore, we divide the interval in $(-2,2)$ due to Symm nature of x^2 and $(2,3)$.

$G_Y(y)$, $g_Y(y)$ are dist and density fnc of Y resp.

When $-2<x<2$, $\Rightarrow 0 \leq y < 4$

$$G(y) = P(Y \leq y) = P(X^2 < y)$$

$$= P(-\sqrt{y} < X < \sqrt{y}) = F(\sqrt{y}) - F(-\sqrt{y}).$$

Diff. wrt y.

$$g_Y(y) = \frac{1}{2\sqrt{y}}\left[f(\sqrt{y}) + f(-\sqrt{y})\right]$$

$$= \frac{1}{2\sqrt{y}}\left(\frac{2}{25}\right)\left[\left(2+\sqrt{y}\right)\right] = \frac{2+\sqrt{y}}{25\sqrt{y}}.$$

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Two dimensional random Variables and their transformations :-

X, Y are random Variables

U, V are our new random Variables

Transformation associated is defined as

$$u = u(x,y) \quad \text{and} \quad v = v(x,y)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

J^{-1} exists $\Rightarrow x = x(u,v)$, $y = y(u,v)$

Result 1: X, Y are random variables
U, V are transformed rand. variable

joint pdf of X, Y is $f_{XY}(x,y)$

joint pdf of U, V is $g_{UV}(u,v)$

Then the joint pdf $\boxed{g_{UV}(u,v) = |J| \, f_{XY}(x,y)}$

Result 2:- If X and Y are random variables, independent and

Result 2 :- If X and Y are random variables, independent and continuous, then pdf of $U = X+Y$; $V = X$ is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_X(v) f_Y(u-v)\, dv$$

Let $f_{XY}(x,y)$ is the joint pdf of X, Y.

Consider the transformation $u = x+y$, $v = x$

$$\Rightarrow y = u-x$$
$$= u-v$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

joint pdf of U, V is $g_{UV}(u,v) = |J| f_{XY}(x,y)$

$$= f_X(x) f_Y(y)$$
$$= f_X(v) f_Y(u-v)$$

$\therefore$ Marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} g_{UV}(u,v)\, dv = \int_{-\infty}^{\infty} f_X(v) f_Y(u-v)\, dv$$

Ex 1:- The pdf of rand. variables (X, Y) is given by

Ex 1 :- The pdf of rand. variables (X, Y) is given by

$$f(x,y) \begin{cases} = \frac{1}{4} e^{-(x+y)/2} & , x \geq 0, \ y > 0 \\ = 0 & , \text{elsewhere} \end{cases}$$

Find distribution of $X-Y/4$.

Solⁿ :- $u = \frac{1}{4}(x-y)$, $v = y$

$\Rightarrow x = 4u + y$
$= 4u + v$, $y = v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} = 4$$

$x = 4u + v > 0$
$\Rightarrow v > -4u$

$y > 0 \Rightarrow v > 0$

joint pdf of random variables (U, V) is

$$g_{UV}(u,v) = |J| f_{XY}(x,y)$$
$$= 4 \cdot \frac{1}{4} e^{-(4u+v+v)/2} \quad , -\infty < u < \infty$$
$$= e^{-(2u+v)}$$

Marginal densities :- of U

for $u < 0$, $f_U(u) = \int_{-4u}^{\infty} e^{-(2u+v)} dv = e^{2u}$.

for $u > 0$, $\quad f_U(u) = \int_0^\infty \bar{e}^{(2u+v)}\, dv = \bar{e}^{2u}.$

$$\therefore\ f_U(v) = \bar{e}^{|2u|} \quad , -\infty < u < \infty.$$

Ex 2:- X, Y are independent, random variables,

$$f_X(x) = \begin{cases} e^{-x} & , x \geq 0 \\ 0 & , x < 0 \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2\bar{e}^{2y} & , y \geq 0 \\ 0 & , y < 0. \end{cases}$$

Find density function of the sum. $U = X + Y$.

Sol^n:- $\quad U = X + Y \;,\; V = X$

$\Rightarrow u = x + y \;,\; x = v$

$\Rightarrow y = u - v$

Now, $x \geq 0 \Rightarrow v \geq 0$

$\quad\; y \geq 0 \Rightarrow u - v \geq 0 \Rightarrow u \geq v$ $\Big\}\ 0 \leq v \leq u$

$\therefore$ density for U is $f_U(u) = \int_0^u \bar{e}^{v} \left[2\bar{e}^{(u-v)} \right] dv$

$= $ plz Complete.

Ex 3:- joint pdf of Continuous random Variables X, Y

Ex 3 :- joint pdf of continuous random variables X, Y

$$f(x,y) = \begin{cases} 2x\,e^{-y}, & 0<x<1,\ y>0 \\ 0, & \text{elsewhere} \end{cases}$$

Find distributi' of $X+Y$

Solⁿ :- $u = x+y,\quad v = y$

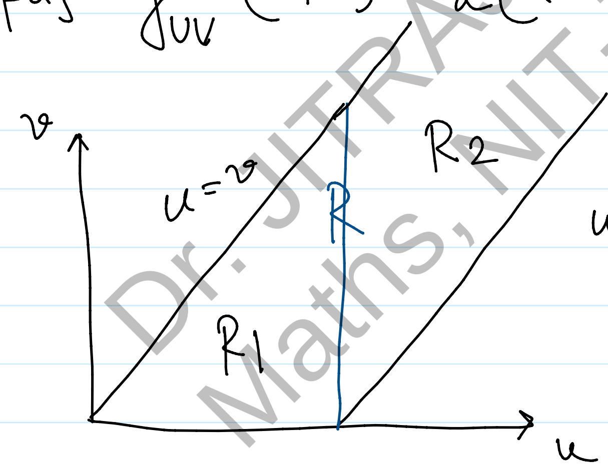
$\Rightarrow x = u - v \qquad \Rightarrow |J| = 1$

Now, $y>0 \Rightarrow v>0$

$0<x<1 \Rightarrow 0<u-v<1 \Rightarrow v<u<1+v$

$\Big\}\ 0<v<u$

joint pdf $g_{UV}(u,v) = 2(u-v)e^{-v}, \quad 0<v<u$



For R_1, $0 \le u \le 1$, $v>0$

$$f_U(u) = \int_0^u g_{UV}(u,v)\,dv = 2\int_0^u (u-v)e^{-v}\,dv$$

$$= 2\left[e^{-u} + u - 1\right]$$

For R_2: $1<u<\infty$, $u-1<v<u$

$$f_U(u) = 2\int_{u-1}^{u} (u-v)e^{-v}\,dv = 2e^{-u}.$$

$$f_U(u) = \begin{cases} 2(e^{-u}+u-1), & 0\le u\le 1 \\ 2e^{-u}, & 1<u<\infty \\ 0, & \text{elsewhere} \end{cases}$$

— x —