Partitions of Integers

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- Partitioning a positive integer n into positive summands and seeking the number of such partitions without regard to the order

$$
\begin{aligned}
p(1) &= 1: \quad 1 \\
p(2) &= 2: \quad 2 = 1 + 1 \\
p(3) &= 3: \quad 3 = 2 + 1 = 1 + 1 + 1 \\
p(4) &= 5: \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \\
p(5) &= 7: \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\
&\qquad\quad = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

determine $p(10)$

$$f(x) = \overset{1}{(1 + x + x^2 + x^3 + x^4 + \cdots)}\overset{2}{(1 + x^2 + x^4 + x^6 + \cdots)}\overset{3}{(1 + x^3 + x^6 + x^9 + \cdots)}\cdots$$

$$\cdots(1 + x^{10} + x^{20} + \cdots)$$

$$f(x) = \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\cdots\frac{1}{1-x^{10}} = \prod_{i=1}^{10}\frac{1}{(1-x^i)}$$

$$P(x) = \prod_{i=1}^{\infty}\left[1/(1-x^i)\right]$$

$$\prod_{i=1}^{r}\frac{1}{1-x^i}$$

x^n

Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into *distinct* summands.

let us consider the 11 partitions of 6:

1) $1+1+1+1+1+1$	**2)** $1+1+1+1+2$
3) $1+1+1+3$	**4)** $1+1+4$
5) $1+1+2+2$	**6)** $1+5$
7) $1+2+3$	**8)** $2+2+2$
9) $2+4$	**10)** $3+3$
11) 6	

Partitions (6), (7), (9), and (11) have distinct summands, so $p_d(6) = 4$.

In calculating $p_d(n)$, for each $k \in \mathbf{Z}^+$ there are two choices: Either k is not used as one of the summands of n, or it is. This can be accounted for by the polynomial $1 + x^k$, and consequently, the generating function for these partitions is

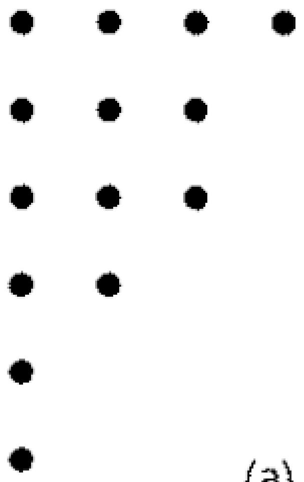
$$P_d(x) = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{i=1}^{\infty}(1+x^i).$$

For each $n \in \mathbf{Z}^+$, $p_d(n)$ is the coefficient of x^n in $(1+x)(1+x^2)\cdots(1+x^n)$.

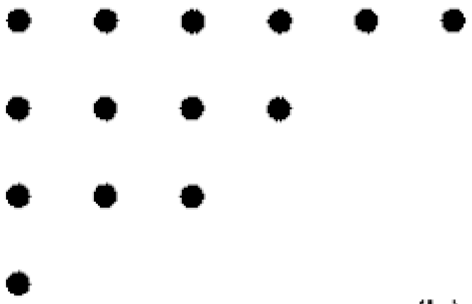
Ferrers graph

- This graph uses rows of dots to represent a partition of an integer where the number of dots per row does not increase as we go from any row to the one below it

Ferrers graphs for two partitions of 14: (a) $4 + 3 + 3 + 2 + 1 + 1$ and (b) $6 + 4 + 3 + 1$. The graph in part (b) is said to be the *transposition* of the graph in part (a), and vice versa, because one graph can be obtained from the other by interchanging rows and columns.



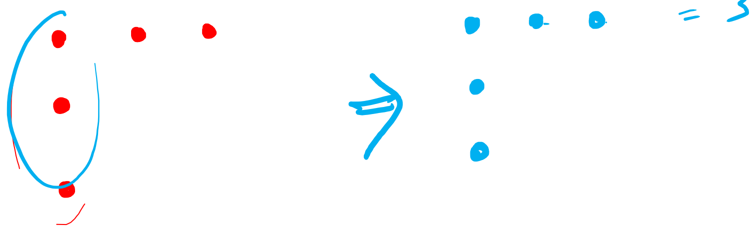
(a)



(b)

There is a one-to-one correspondence between a Ferrers graph and its transposition, so this example demonstrates a particular instance of the general result: The number of partitions of an integer n into m summands is equal to the number of partitions of n into summands where m is the largest summand.

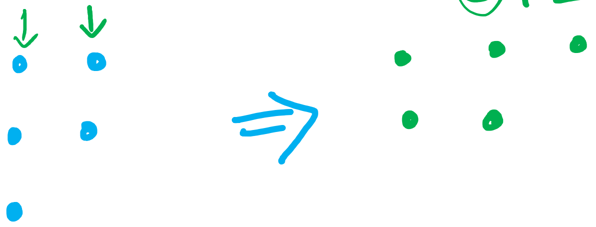
$P(5) = \overset{1}{3}+\overset{2}{1}+\overset{3}{1}$



$\Rightarrow$ self conjugate

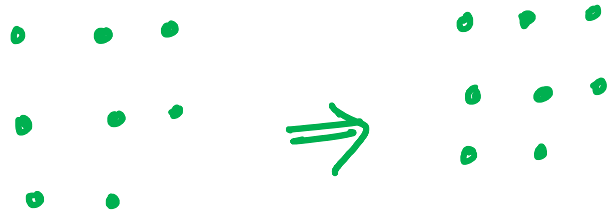
$P(5) = \overset{1}{2}+\overset{2}{2}+\overset{3}{1}$ $m=3$

$\overset{1}{3}+\overset{2}{2}$



$P(4) = 2+2$

$P(8) = 3+3+2$



Exponential Generating Functions

- Ordinary generating functions are used in selection problems where order was not relevant

Now for all $0 \le r \le n$,

$$C(n, r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n, r),$$

where $P(n, r)$ denotes the number of permutations of n objects taken r at a time. So

$$\begin{aligned}(1+x)^n &= C(n, 0) + C(n, 1)x + C(n, 2)x^2 + C(n, 3)x^3 + \cdots + C(n, n)x^n \\ &= P(n, 0) + P(n, 1)x + P(n, 2)\frac{x^2}{2!} + P(n, 3)\frac{x^3}{3!} + \cdots + P(n, n)\frac{x^n}{n!}.\end{aligned}$$

For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the *exponential generating function* for the given sequence.

Examining the Maclaurin series expansion for e^x, we find

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

so e^x is the exponential generating function for the sequence $1, 1, 1, \ldots$. (The function e^x is the ordinary generating function for the sequence $1, 1, 1/2!, 1/3!, 1/4!, \ldots$.)

In how many ways can four of the letters in ENGINE be arranged?

E & N $\left(1 + x + x^2/2!\right)$

G & I $(1+x)$

E	E	N	N	4!/(2! 2!)
E	E	G	N	4!/2!
E	E	I	N	4!/2!
E	E	G	I	4!/2!
E	G	N	N	4!/2!
E	I	N	N	4!/2!
G	I	N	N	4!/2!
E	I	G	N	4!

$$f(x) = [1 + x + (x^2/2!)][1 + x + (x^2/2!)](1 + x)(1 + x)$$

$(x^2/2!)(x^2/2!)(1)(1) \Rightarrow x^4/2!.2! \Rightarrow \frac{4!}{2!.2!} \cdot \frac{x^4}{4!}$

$(x^2/2!)(1)(x)(x) \Rightarrow x^4/2! \Rightarrow \frac{4!}{2!} \cdot \frac{x^4}{4!}$

In the complete expansion of $f(x)$, the term involving x^4 is

$$\left(\frac{x^4}{2!\,2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + x^4\right)$$

$$= \left[\left(\frac{4!}{2!\,2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + 4!\right]\left(\frac{x^4}{4!}\right)$$

Consider the Maclaurin series expansions of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Adding these series together, we find that

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right),$$

or

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

Subtracting e^{-x} from e^x yields

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

A ship carries 48 flags, 12 each of the colors red, white, blue, and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.

a) How many of these signals use an even number of blue flags and an odd number of black flags?

The exponential generating function

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$f(x) = (e^x)^2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right)(e^{2x})(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{4x} - 1)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty}\frac{(4x)^i}{i!} - 1\right) = \left(\frac{1}{4}\right)\sum_{i=1}^{\infty}\frac{(4x)^i}{i!},$$

coefficient of $x^{12}/12!$ in $f(x)$ yields $(1/4)(4^{12}) = 4^{11}$

b) How many of the signals have at least three white flags or no white flags at all? In this situation we use the exponential generating function

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$= e^x\left(e^x - x - \frac{x^2}{2!}\right)(e^x)^2 = e^{3x}\left(e^x - x - \frac{x^2}{2!}\right) = e^{4x} - xe^{3x} - \left(\frac{1}{2}\right)x^2e^{3x}$$

$$= \sum_{i=0}^{\infty}\frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty}\frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right).$$

$\sum_{i=0}^{\infty}\frac{(4x)^i}{i!}$ — Here we have the term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$, so the coefficient of $x^{12}/12!$ is 4^{12}

$x\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$ — to get $x^{12}/12!$ consider the term

$x[(3x)^{11}/11!] = 3^{11}(x^{12}/11!) = (12)(3^{11})(x^{12}/12!)$, and here the coefficient of $x^{12}/12!$ is $(12)(3^{11})$

$$(x^2/2)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$$

$(x^2/2)[(3x)^{10}/10!] = (1/2)(3^{10})(x^{12}/10!) = (1/2)(12)(11)(3^{10})(x^{12}/12!)$,
where this time the coefficient of $x^{12}/12!$ is $(1/2)(12)(11)(3^{10})$.

Consequently, the number of 12 flag signals with at least three white flags, or none at all, is

$$4^{12} - 12(3^{11}) - (1/2)(12)(11)(3^{10}) = 10{,}754{,}218.$$