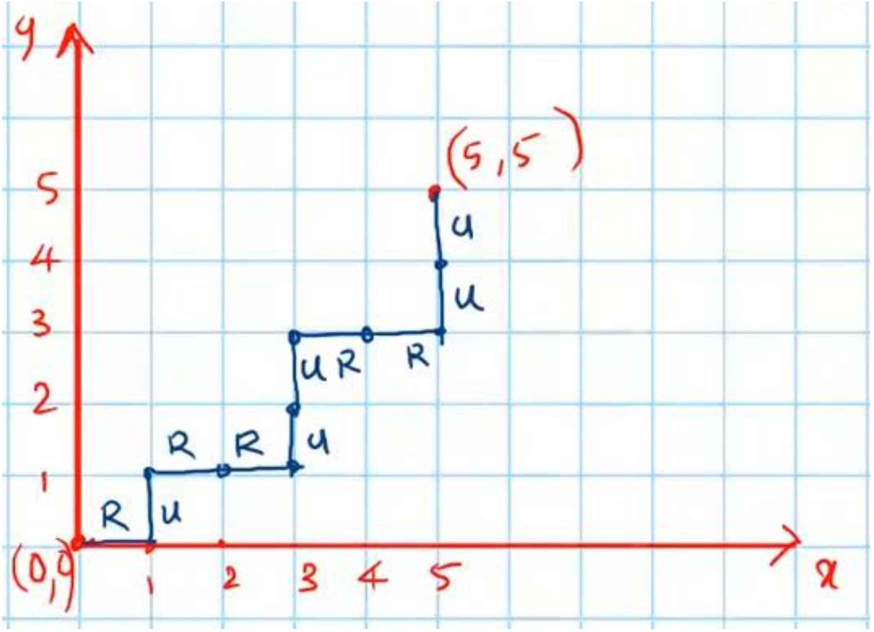
Catalan Numbers

CATALAN NUMBERS

$(0,0)$ to $(5,5)$

$R \Rightarrow (x,y)$ to $(x+1,y)$

$U \Rightarrow (x,y)$ to $(x,y+1)$



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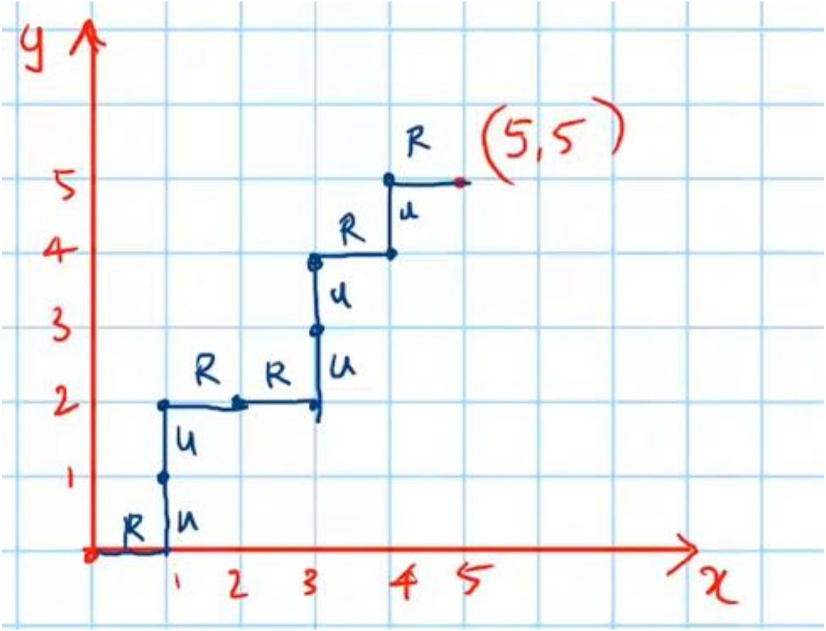

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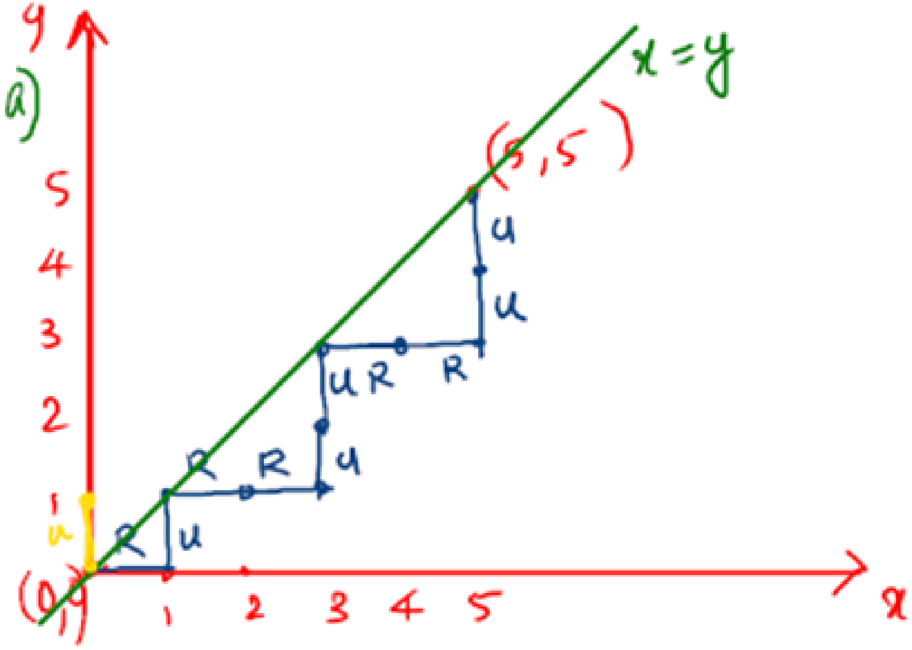
CATALAN NUMBERS

(0,0) to (5,5)

$^{10}C_5$

R ⇒ (x,y) to (x+1,y)

U ⇒ (x,y) to (x,y+1)



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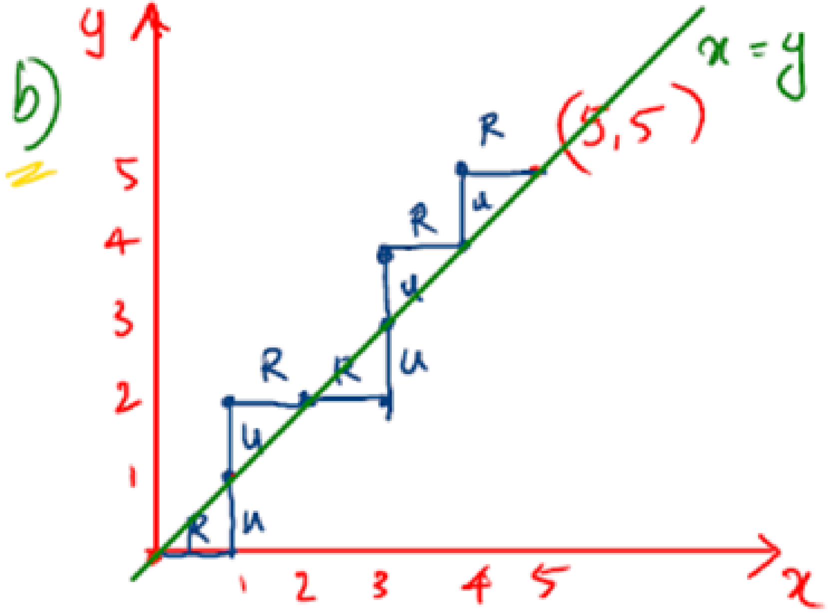

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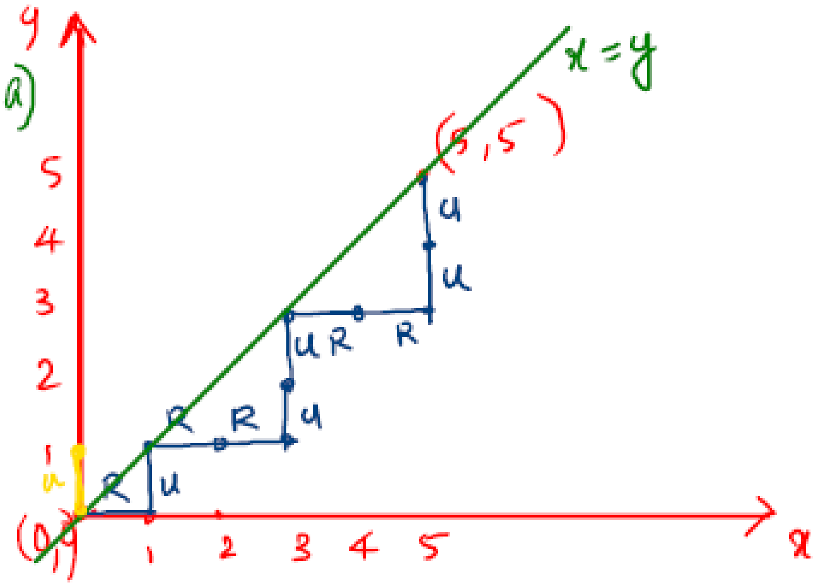
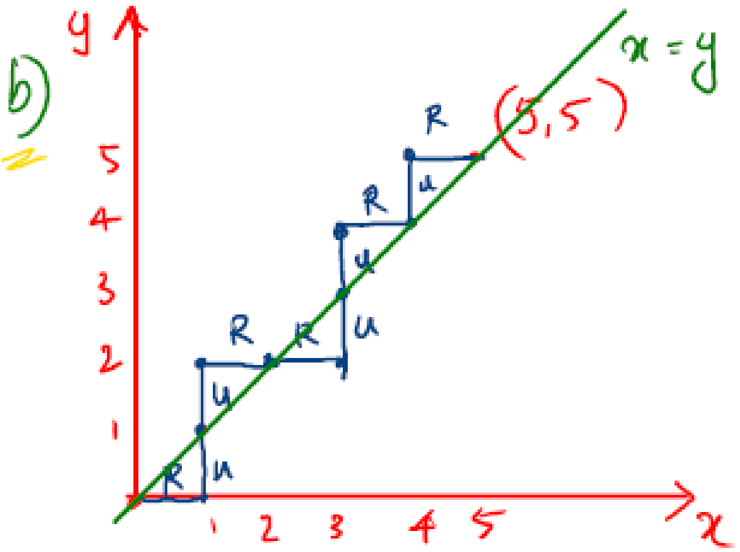
CATALAN NUMBERS

$R \Rightarrow (x,y)$ to $(x+1,y)$

$U \Rightarrow (x,y)$ to $(x,y+1)$

$(0,0)$ to $(5,5)$ to ${}^{10}C_5$





$R=5 \quad U=5$ $\qquad {}^{10}C_5 - N$

a) R U R R U U R R U U

b) R U U R R U U R U R $\qquad R=5,\ U=5$

R U U U U R R U R U $\qquad R=4,\ U=6$ $\qquad \boxed{{}^{10}C_4}$

$${}^{10}C_5 - {}^{10}C_4$$

R U R U U U U U R R

R U R U U R R R U U $\Rightarrow$ R's 5 U's 5

$$ {}^{10}C_5 - {}^{10}C_4 = \frac{10!}{5!\,5!} - \frac{10!}{4!\,6!} $$

$$ {}^{2n}C_n - {}^{2n}C_{n-1} $$

$$ = \frac{6 \cdot 10!}{5!\,6!} - \frac{5 \cdot 10!}{5!\,6!} = \frac{10!}{5!\,6!} $$

$$ = \frac{1}{6} \cdot \frac{10!}{5!\,5!} = \frac{1}{(5+1)}\ {}^{10}C_5 $$

$$ b_5 = \frac{1}{(5+1)}\ {}^{(2\times 5)}C_5 $$

$$ \boxed{b_n = \frac{1}{n+1}\binom{2n}{n}} $$

$(0,0)$ to (n,n)

Catalan Numbers

$$b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 1, \qquad b_0 = 1.$$

- b_0, b_1, b_2, … are called the Catalan numbers

How many ways one can parenthesize the product abcd ?

S_1	S_2	S_3
1. (((ab)c)d)	(((abc	111000
2. ((a(bc))d)	((a(bc	110100
3. ((ab)(cd))	((ab(c	110010
4. (a((bc)d))	(a((bc	101100
5. (a(b(cd)))	(a(b(c	101010

$a_1 a_2 a_3 a_4 \cdots a_n$

b_{n-1}

((ab(c $\Rightarrow$ ((ab)(cd))

(((abc $\Rightarrow$ (((ab)c)d)

$b_3 = 5$

Let us arrange the integers 1, 2, 3, 4, 5, 6 in two rows of three so that (1) the integers increase in value as each row is read, from left to right, and (2) in any column the smaller integer is on top. For example, one way to do this is

$$\text{1} \quad \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}$$

How many ways can you arrange this?

$$\text{2} \quad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 6 \end{array}$$

$$\cancel{\begin{array}{ccc} 1 & 5 & 6 \\ 2 & 3 & 4 \end{array}}$$

$$\text{3} \quad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}$$

	1	2	3	4	5	6
1 ⇒	1	1	0	1	0	0
2 ⇒	1	0	1	1	0	0
3 ⇒	1	1	0	0	1	0

$b_3 = 5$ ways

Twelve patrons, six each with a \$5 bill and the other six each with a \$10 bill, are the first to arrive at a movie theater, where the price of admission is five dollars. In how many ways can these 12 individuals (all loners) line up so that the number with a \$5 bill is never exceeded by the number with a \$10 bill (and, as a result, the ticket seller is always able to make any necessary change from the bills taken in from the first 11 of these 12 patrons)?

\$5   \$10

1   0

~

6   6   ⇒ 12 length

b_6