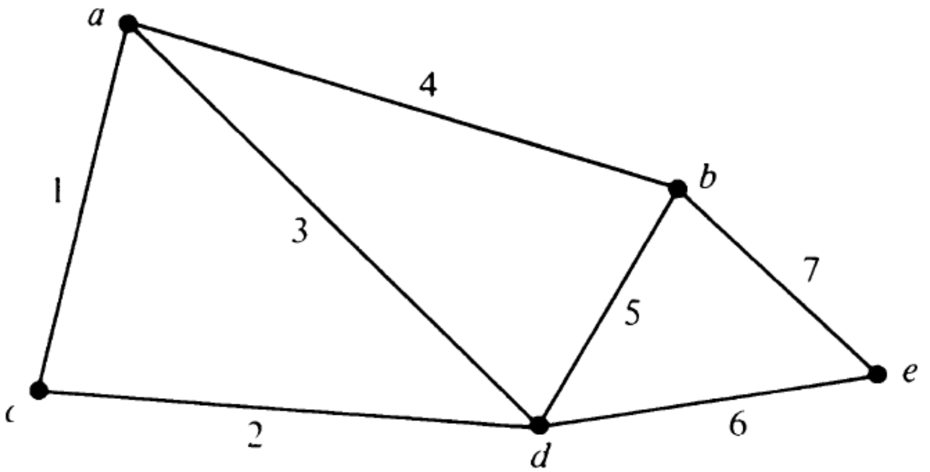
PLANAR AND DUAL GRAPHS

Combinatorial versus Geometric Graphs

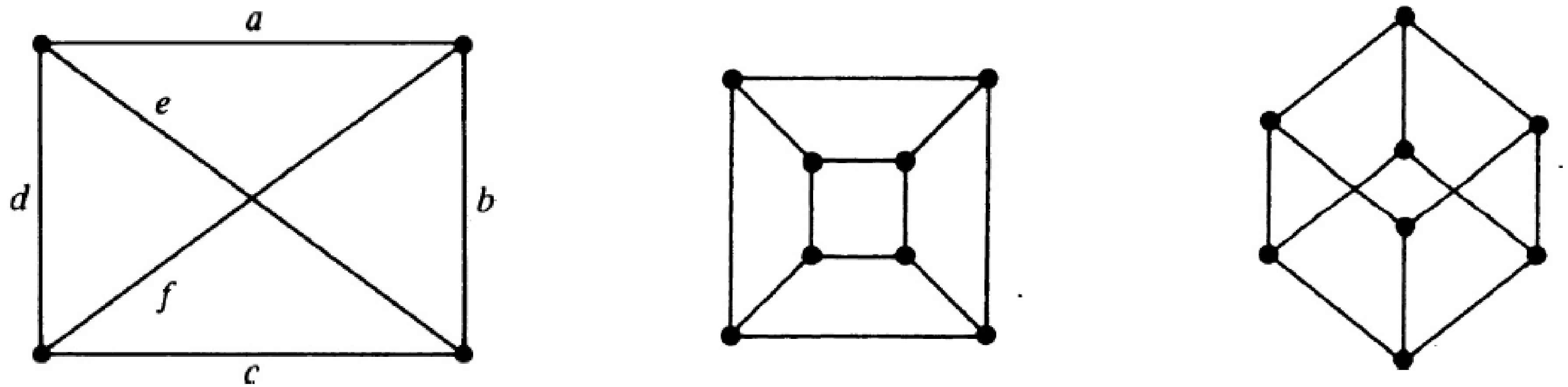
- An abstract graph (or combinatorial) G_1 can be defined as, $G_1 = (V, E, \Psi)$
- where the set V consists of the five objects named a, b, c, d, and e, that is, V = {a, b, c, d, e},
- the set E consists of seven objects (none of which is in set V) named 1, 2, 3, 4, 5, 6, and 7, that is, E = {1, 2, 3, 4, 5, 6, 7},
- and the relationship between the two sets is defined by the mapping Ψ
- Here, the symbol 1 → (a, c) says that object 1 from set E is mapped onto the (unordered) pair (a, c) of objects from set V.

$$\Psi = \begin{bmatrix} 1 \longrightarrow (a, c) \\ 2 \longrightarrow (c, d) \\ 3 \longrightarrow (a, d) \\ 4 \longrightarrow (a, b). \\ 5 \longrightarrow (b, d) \\ 6 \longrightarrow (d, e) \\ 7 \longrightarrow (b, e) \end{bmatrix}$$

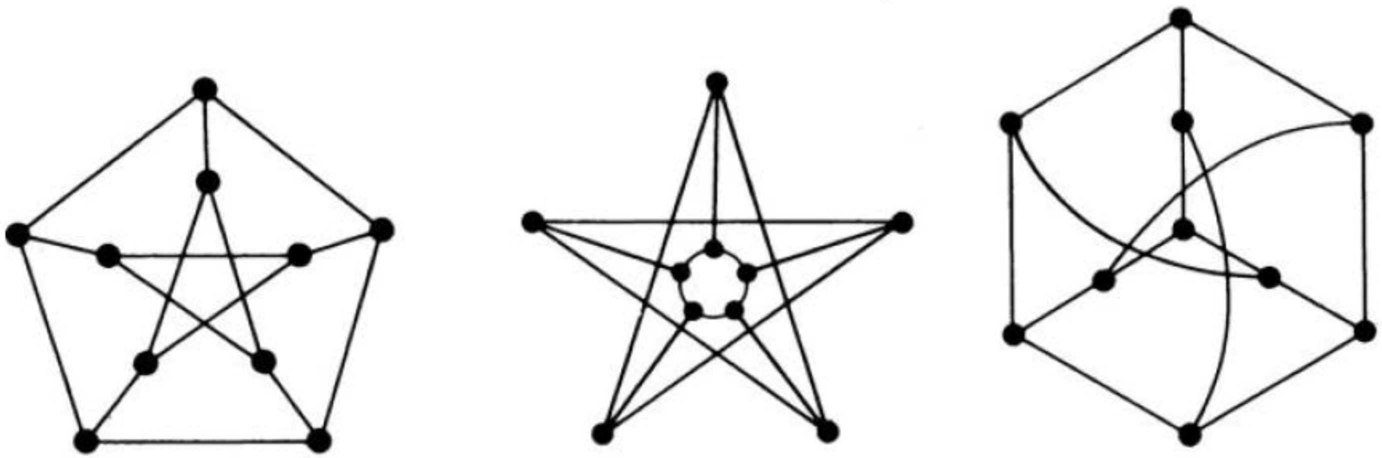


Planar Graphs

- A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.
- A graph that cannot be drawn on a plane without a crossover between its edges is called nonplanar.
- A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.
- Thus, to declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane.
- Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is nonplanar.
- An embedding of a planar graph G on a plane is called a plane representation of G.



Planar graphs

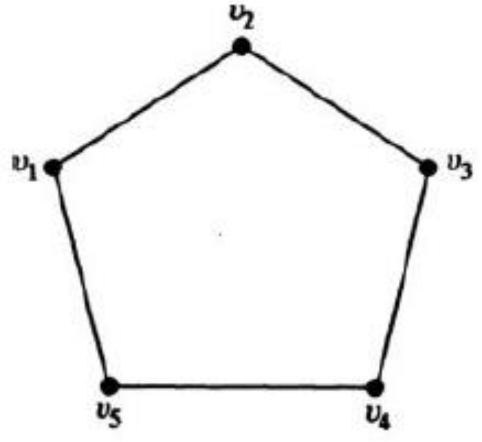


Nonplanar graphs

Kuratowski's Graphs

- Theorem 1: The complete graph of five vertices is nonplanar.
- Theorem 2: $K_{3,3}$ is nonplanar.

- Theorem 1: The complete graph of five vertices is nonplanar.
- Let the five vertices in the complete graph be named v1 v2, v3, v4, and v5.



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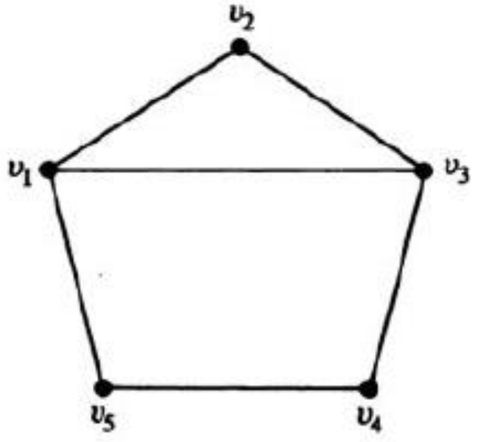

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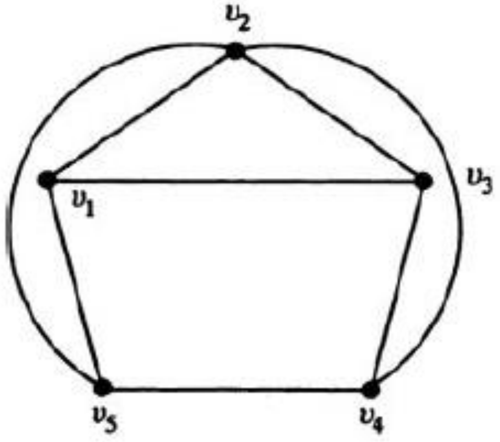

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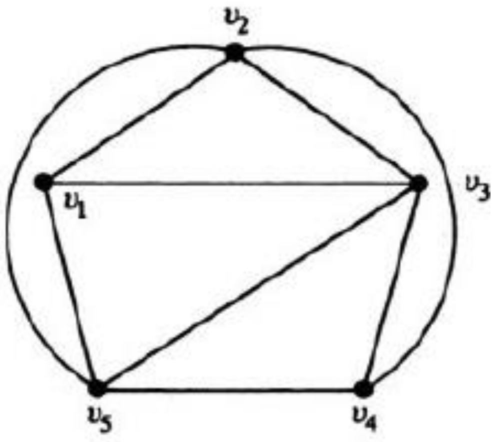

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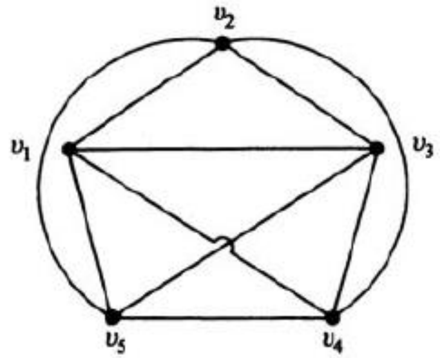
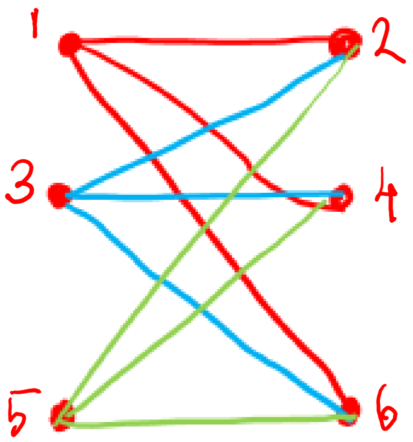
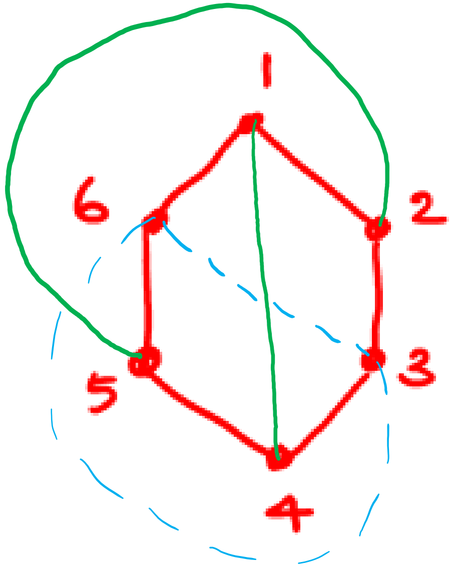


(e)

- Theorem 2: $K_{3,3}$ is nonplanar.

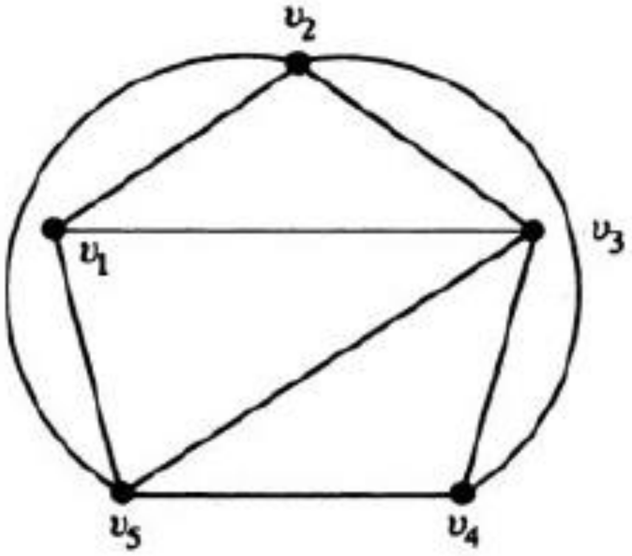


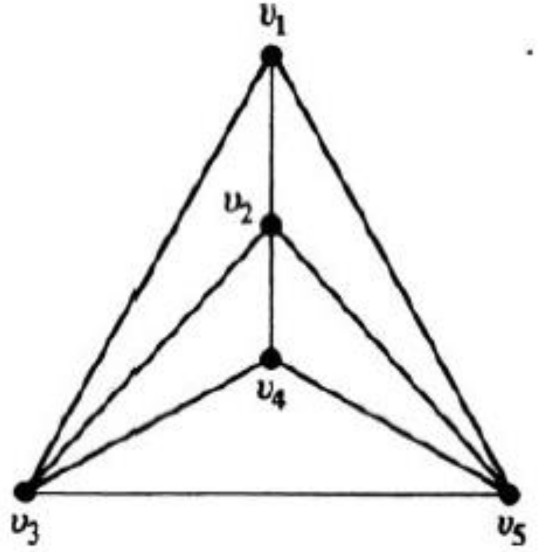


Properties common to the two graphs of Kuratowski

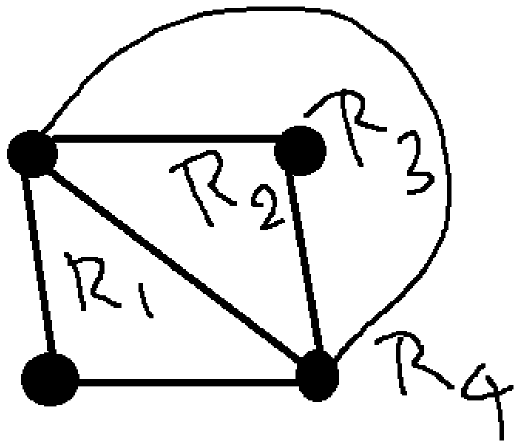
- Both are regular graphs.
- Both are nonplanar.
- Removal of one edge or a vertex makes each a planar graph.
- Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges. Thus both are the simplest nonplanar graphs.

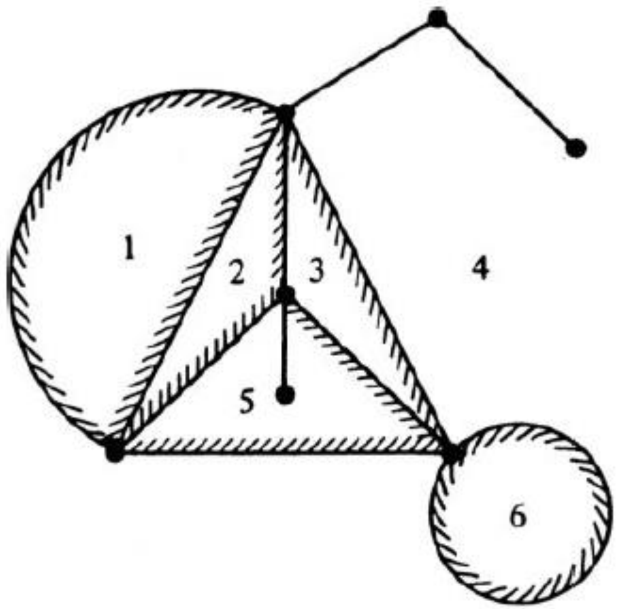
- THEOREM 3: Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.



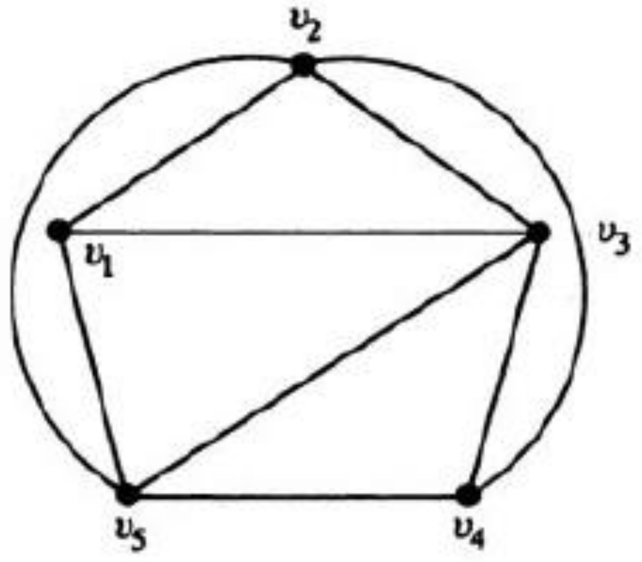


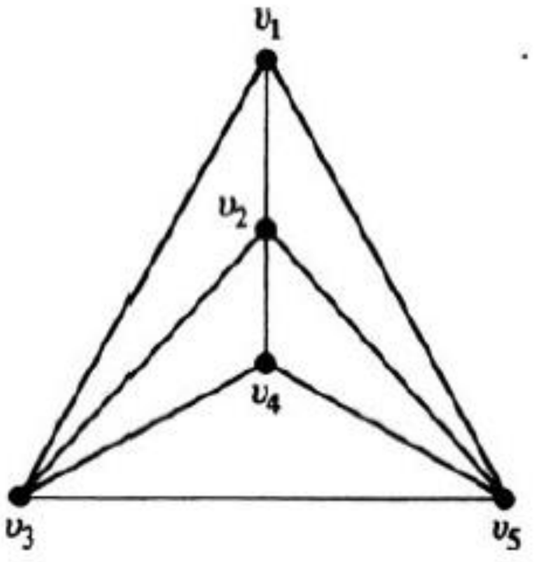
- A plane graph divides the plane into regions (also called windows, faces, or meshes).
- The boundary of a region in a plane graph is characterized by the set of vertices and edges that outine it.
- The portion of the plane lying outside a graph embedded in a plane, such as region 4, is infinite in its extent. Such a region is called the infinite, unbounded, outer, or exterior region for that particular plane representation. Infinite region is also characterized by a set of edges (or vertices).
- Every tree has only region which is the unbounded region.
- Isomorphic graphs can be drawn in different ways in a plane.
- A region is a property of the specific plane representation of a graph and not of an abstract graph per se.





- By changing the embedding of a given planar graph, we can change the infinite region.
- Figures show two different embeddings of the same graph. The finite region v1 v3 v5 in first figure becomes the infinite region in second.
- Any region can be made the infinite region by proper embedding.

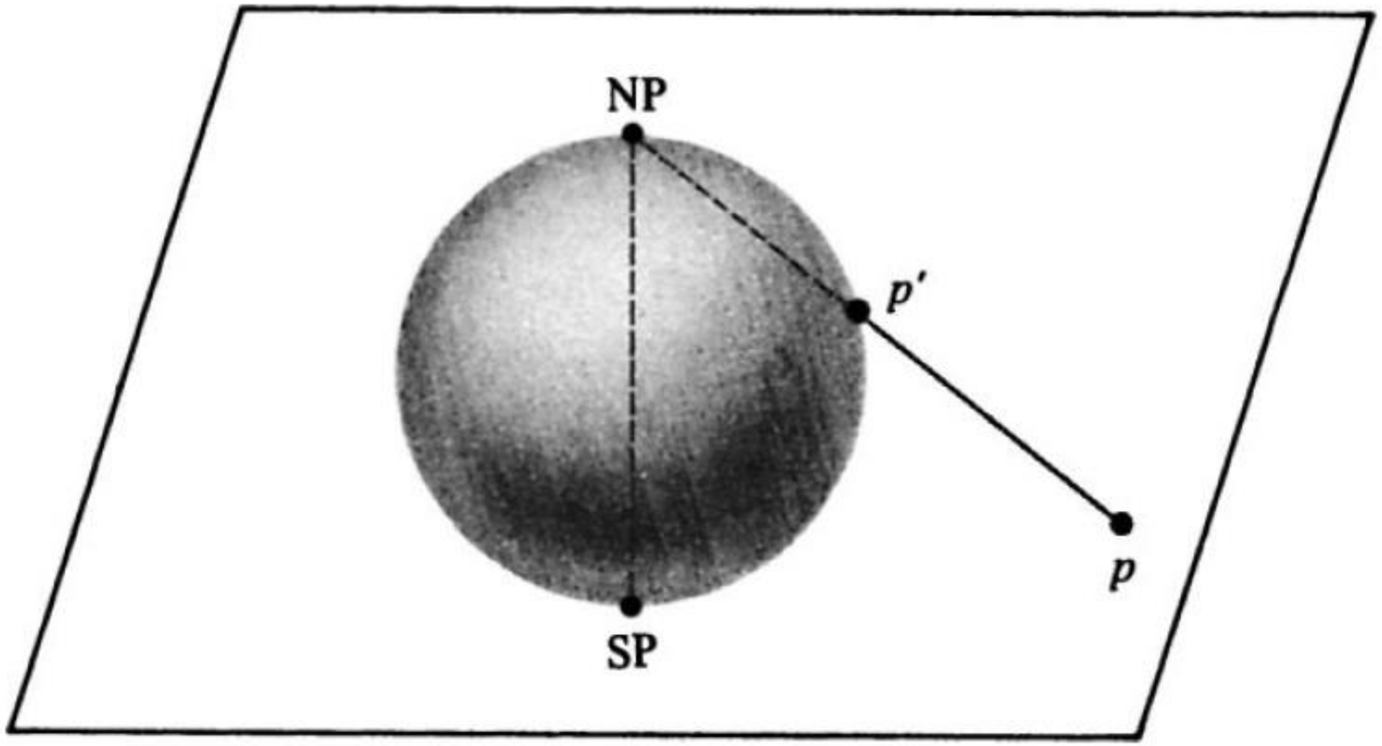




Embedding on a Sphere

- To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere.
- It is accomplished by stereographic projection of a sphere on a plane.
- Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole).
- Corresponding to any point p on the plane, there exists a unique point p′ on the sphere and vice versa, where p′ is the point at which the straight line from point p to point NP intersects the surface of the sphere.
- Thus there is a one-to-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane correspond to the point NP on the sphere.

Stereographic projection



- **THEOREM 4:** A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.
- A planar graph embedded in the surface of a sphere divides the surface into different regions.
- Each region on the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP.
- Now it is clear that by suitably rotating the sphere we can make any specified region map onto the infinite region on the plane.
- **THEOREM 5:** A planar graph may be embedded in a plane such that any specified region (i.e., specified by the edges forming it) can be made the infinite region.

Euler's Formula

- **THEOREM 6:** A connected planar graph with n vertices and e edges has e − n + 2 regions.
- Proof:
  - It will suffice to prove the theorem for a simple graph, because adding a self-loop or a parallel edge simply adds one region to the graph and simultaneously increases the value of e by one. We can also remove all edges that do not form boundaries of any region. Addition (or removal) of any such edge increases (or decreases) e by one and increases (or decreases) n by one, keeping the quantity e − n unaltered.

Proof…

- Since any simple planar graph can have a plane representation such that each edge is a straight line (Theorem 3), any planar graph can be drawn such that each region is a polygon (a polygonal net).
- Let the polygonal net representing the given graph consist of f regions or faces, and let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3 \cdot k_3 + 4 \cdot k_4 + 5 \cdot k_5 + \cdots + r \cdot k_r = 2 \cdot e,$$

- where k_r is the number of polygons, with maximum edges. Also,

$$k_3 + k_4 + k_5 + \cdots + k_r = f.$$

- The sum of all angles subtended at each vertex in the polygonal net is $2\pi n$.

Proof…

- Recalling that the sum of all interior angles of a p-sided polygon is π(p − 2), and the sum of the exterior angles is π(p + 2), let us compute the sum of angles as the grand sum of all interior angles of f − 1 finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\pi(3-2)\cdot k_3 + \pi(4-2)\cdot k_4 + \cdots + \pi(r-2)\cdot k_r + 4\pi$$
$$= \pi(2e - 2f) + 4\pi.$$

- Equating, we get

$$2\pi(e-f) + 4\pi = 2\pi n,$$
$$e - f + 2 = n.$$

- Therefore, the number of regions is $f = e - n + 2.$

Corollary

In any simple, connected planar graph with f regions, n vertices, and e edges ($e > 2$), the following inequalities must hold :

$$e \geq \frac{3}{2}f,$$

$$e \leq 3n - 6.$$

Proof: Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f$$

or

$$e \geq \frac{3}{2}f.$$

Substituting for f from Euler's formula in inequality

$$e \geq \frac{3}{2}(e - n + 2)$$

or

$$e \leq 3n - 6. \quad \blacksquare$$

in the case of K_5, the complete graph of five vertices

$$n = 5, \qquad e = 10, \qquad 3n - 6 = 9 < e.$$

the graph violates inequality and hence it is not planar.

- Kuratowski's second graph, $K_{3,3}$, satisfies the inequality, because
  - $e = 9$
  - $3n - 6 = 3 \cdot 6 - 6 = 12$.
- To prove the nonplanarity of Kuratowski's second graph, we make use of the additional fact that no region in this graph can be bounded with fewer than four edges. Hence, if this graph were planar, we would have

$$2e \geq 4f,$$

and, substituting for f from Euler's formula,

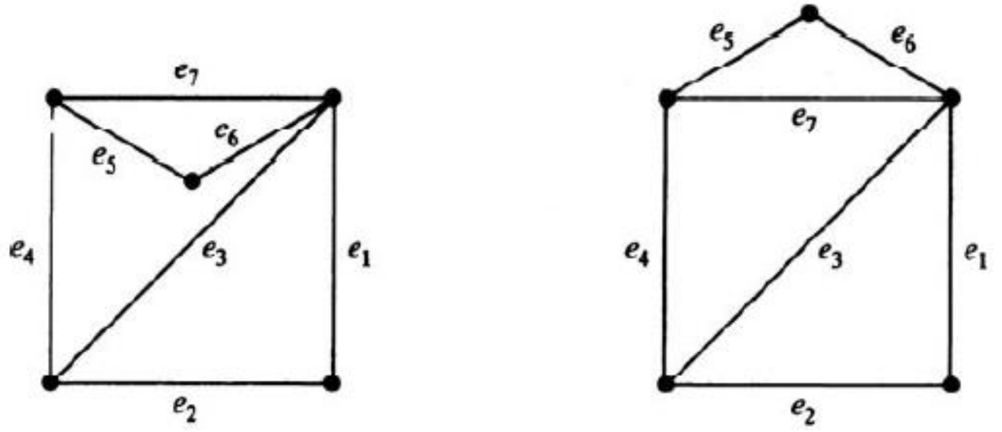
$$2e \geq 4(e - n + 2),$$

or $$2 \cdot 9 \geq 4(9 - 6 + 2),$$

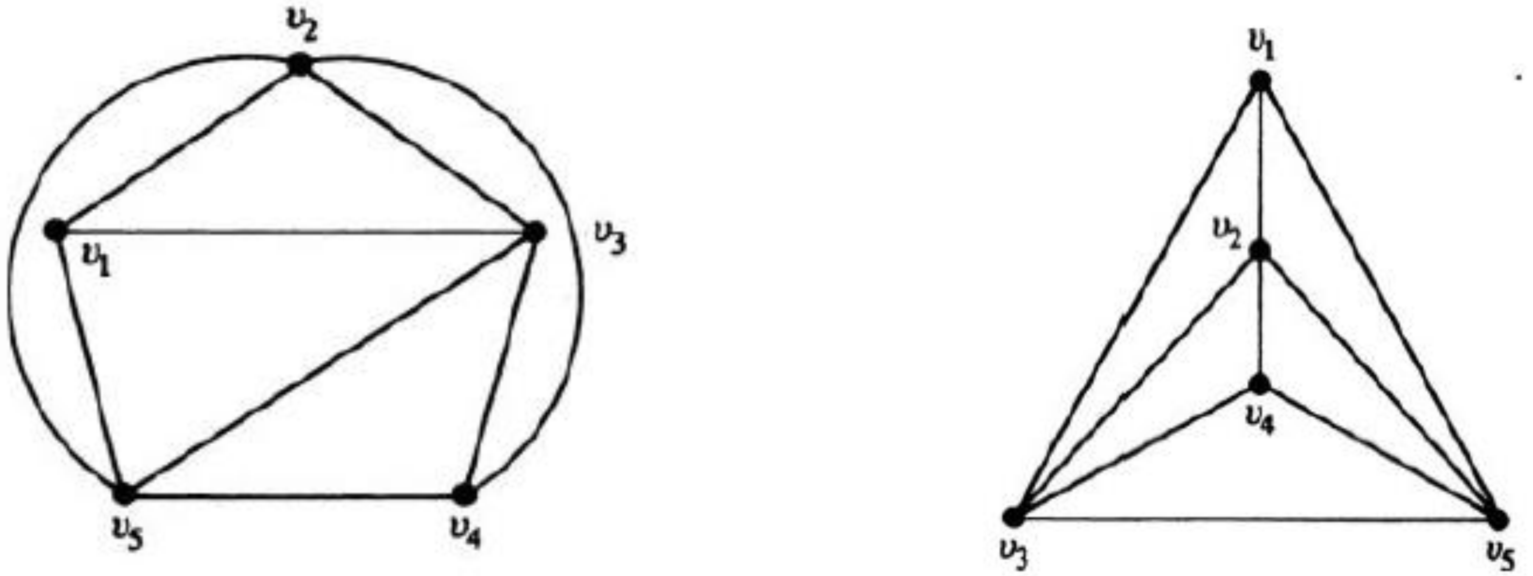
or $$18 \geq 20, \text{ a contradiction.}$$

Plane Representation and Connectivity

- In a disconnected graph the embedding of each component can be considered independently. Therefore, a disconnected graph is planar if and only if each of its components is planar.
- In a separable (or 1-connected) graph the embedding of each block (i.e., maximal non-separable subgraph) can be considered independently.
- Hence a separable graph is planar if and only if each of its blocks is planar.
- Therefore, in questions of embedding or planarity, one need consider only nonseparable graphs.
- Two embeddings of a planar graph on spheres are not distinct if the embeddings can be made to coincide by suitably rotating one sphere with respect to the other and possibly distorting regions.
- If of all possible embeddings on a sphere no two are distinct, the graph is said to have a unique embedding on a sphere (or a unique plane representation).



Two distinct plane representations of the same graph.



Unique plane representations of the same graph.

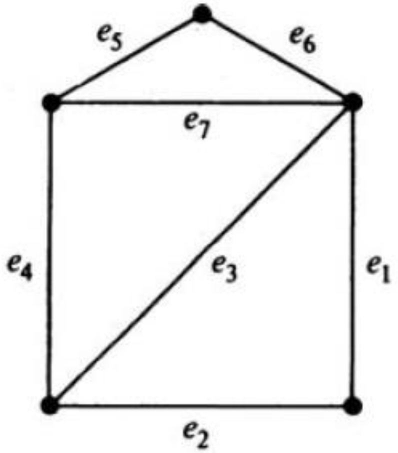
- **THEOREM 7:**
- The spherical embedding of every planar 3-connected graph is unique.

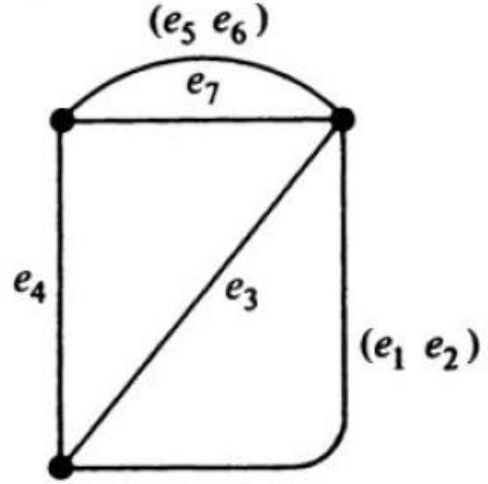
DETECTION OF PLANARITY

- **Elementary Reduction**
- Step 1: Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set $G = \{G_1, G_2, . . . , G_k\}$, where each G_i is a non-separable block of G. Then we have to test each G_i for planarity.
- Step 2: Since addition or removal of self-loops does not affect planarity, remove all self-loops.
- Step 3: Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.
- Step 4: Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series.
- Repeated application of steps 3 and 4 will usually reduce a graph drastically.

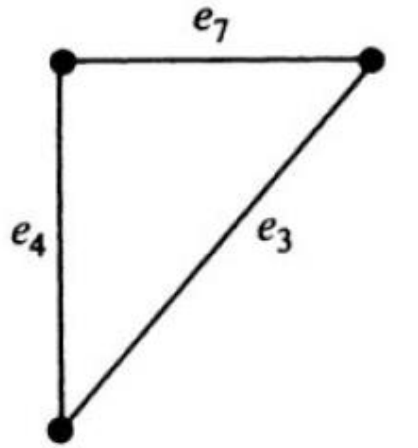
- **THEOREM 8:**
- Graph H_i (graph obtained after elementary reduction) is
    1. A single edge, or
    2. A complete graph of four vertices, or
    3. A non-separable, simple graph with $n \geq 5$ and $e \geq 7$.
- All H_i falling in categories 1 or 2 are planar and need not be checked further.
- Therefore, we need to investigate only simple, connected, non-separable graphs of at least five vertices and with every vertex of degree three or more.
- Next, we can check to see if $e \leq 3n - 6$. If this inequality is not satisfied, the graph Hi is nonplanar. If the inequality is satisfied, we have to test the graph further using Kuratowski's theorem (Theorem 9).

Series-parallel reduction of a graph

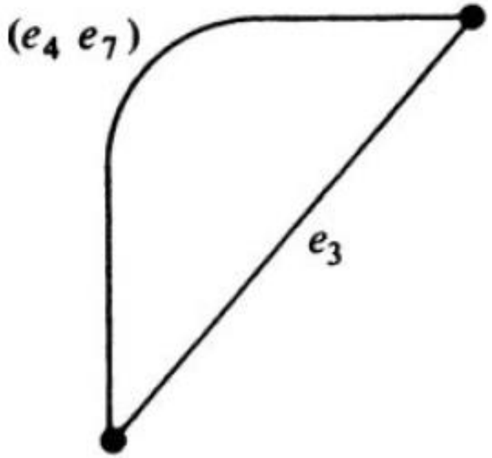




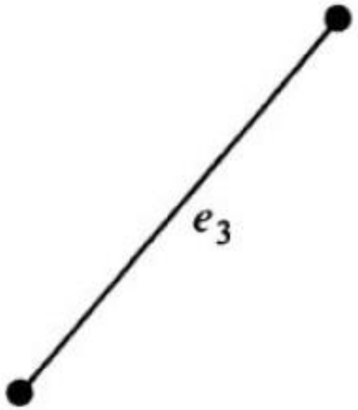
(a) Series Reduced



(b) Parallel Reduced

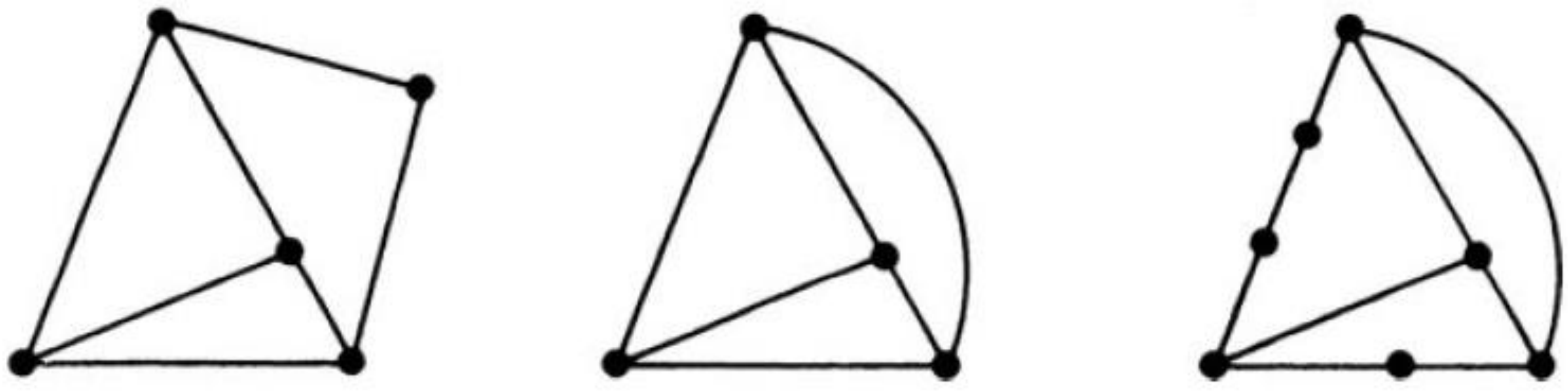


(c) Series Reduced



(d) Parallel Reduced

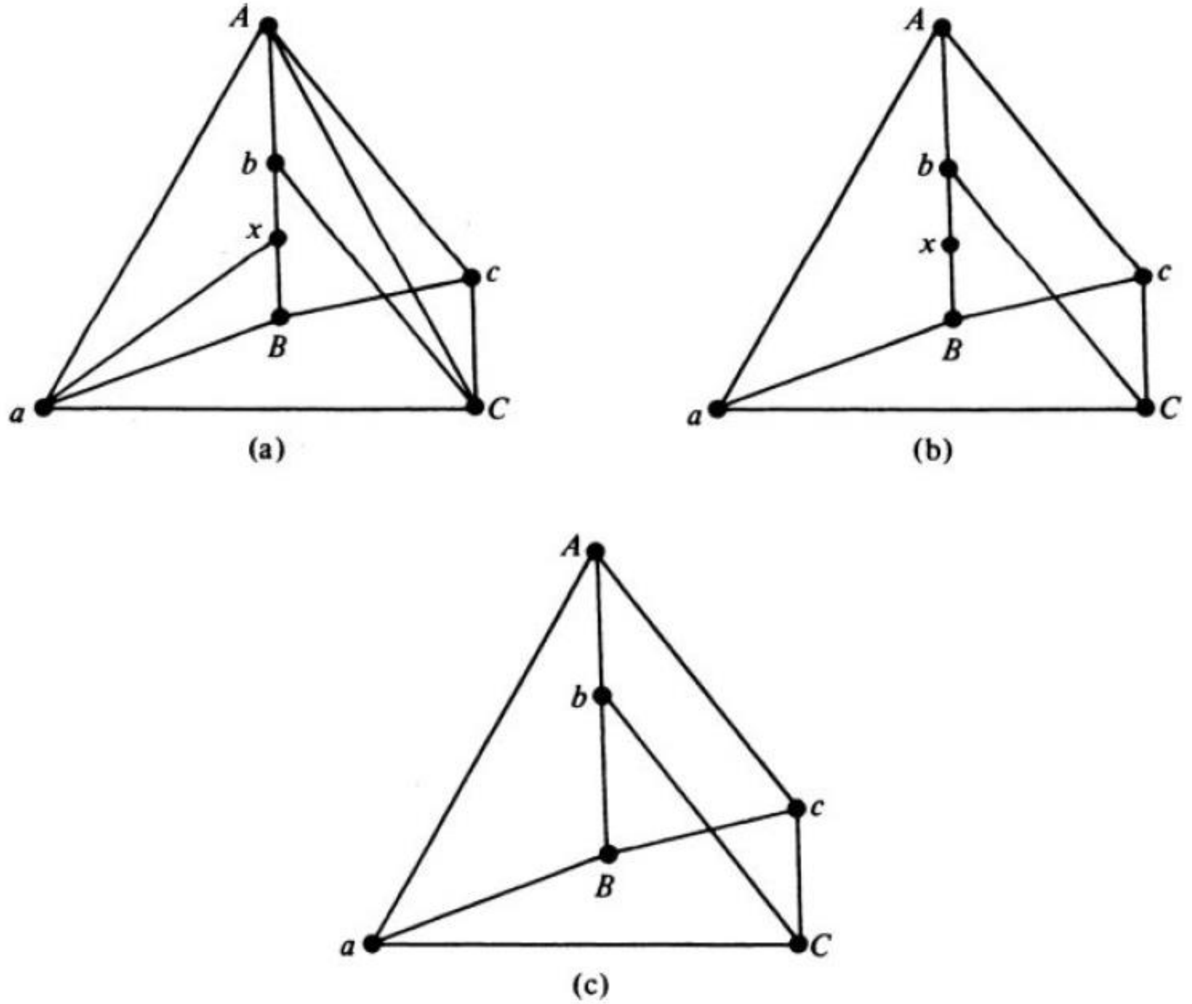
- **Homeomorphic Graphs:**
  - Two graphs are said to be homeomorphic if one graph can be obtained from the other by the creation of edges in series (i.e., by insertion of vertices of degree two) or by the merger of edges in series.
- A graph G is planar if and only if every graph that is homeomorphic to G is planar.



Three graphs homeomorphic to each other.

- **THEOREM 9:**
- A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski’s two graphs or any graph homeomorphic to either of them.

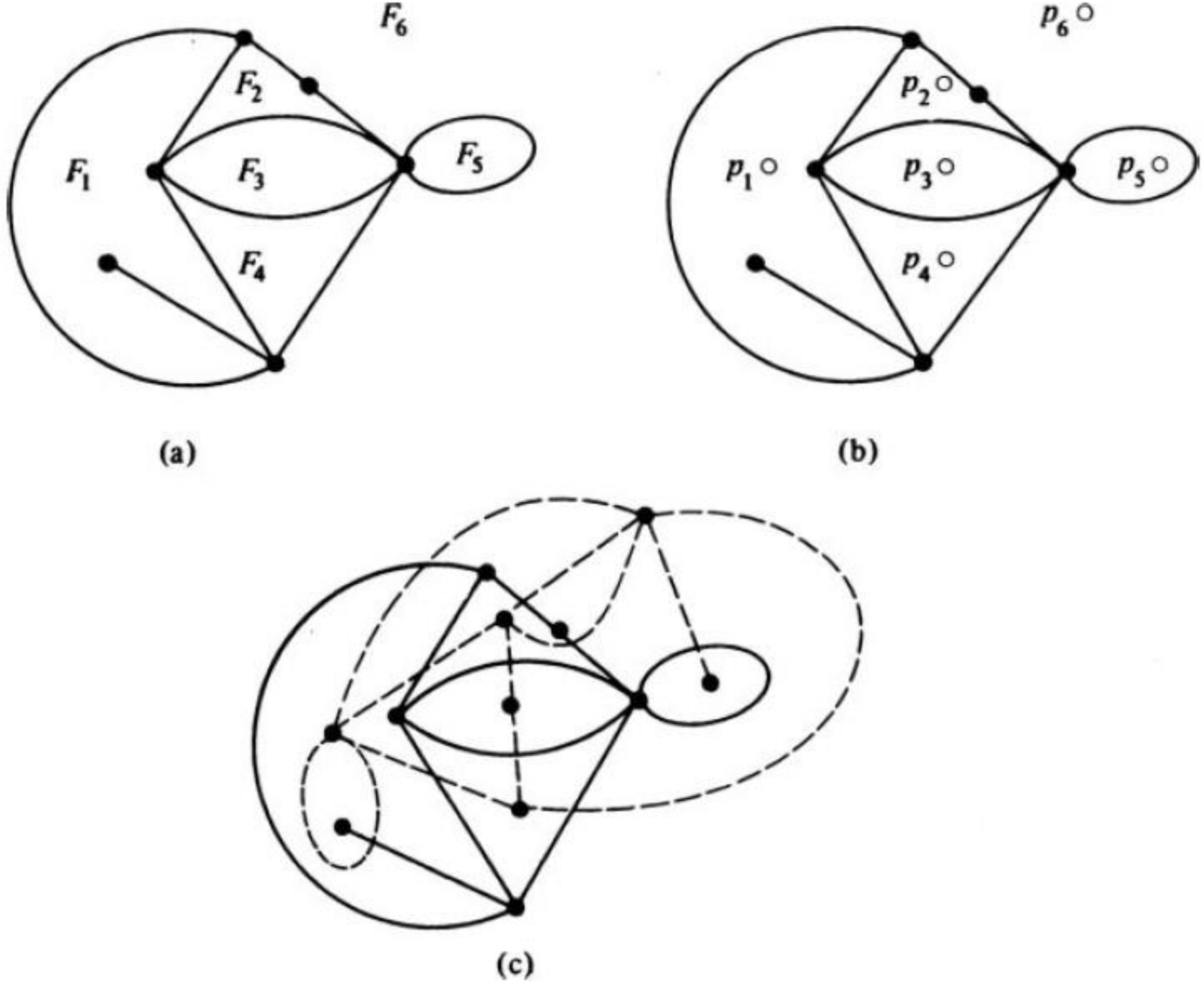
- Note that it is not necessary for a nonplanar graph to have either of the Kuratowski graphs as a subgraph. The nonplanar graph may have a subgraph homeomorphic to a Kuratowski graph.



Nonplanar graph with a subgraph homeomorphic to $K_{3,3}$.

GEOMETRIC DUAL

- Consider the plane representation of a graph G with n regions or faces F_1, F_2, F_3, … F_n. Let us place points $p_1, p_2, \ldots, p_n$, one in each of the regions. Next let us join these points according to the following procedure:
- If two regions F_i and F_j are adjacent (i.e., have a common edge), draw a line joining points p_i and p_j that intersects the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j , draw one line between points p_i and p_j for each of the common edges.
- For an edge e lying entirely in one region, say F_k, draw a self-loop at point p_k intersecting e exactly once.
- Such a graph G* is called a dual (a geometric dual) of G.


F_6
F_2
F_1
F_3
F_5
F_4
(a)
p_6
p_2
p_1
p_3
p_5
p_4
(b)
(c)

- There is a one-to-one correspondence between the edges of graph G and its dual G*−one edge of G* intersecting one edge of G. Some simple observations that can be made about the relationship between a planar graph G and its dual G* are
  - An edge forming a self-loop in G yields a pendant edge in G*.
  - A pendant edge in G yields a self-loop in G*.
  - Edges that are in series in G produce parallel edges in G*.
  - Parallel edges in G produce edges in series in G*.
  - The number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex p_i in G*, and vice versa.
  - Graph G* is also embedded in the plane and is therefore planar.
  - If n, e, f, r, and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n*, e*, f*, r*, and μ* are the corresponding numbers in dual graph G*, then

$$n^* = f, \qquad r^* = \mu,$$
$$e^* = e, \qquad \mu^* = r.$$
$$f^* = n.$$

- **THEOREM 10:**
- All duals of a planar graph G are 2-isomorphic; and every graph 2-isomorphic to a dual of G is also a dual of G.
- **THEOREM 11:**
- A necessary and sufficient condition for two planar graphs G1 and G2 to be duals of each other is as follows: There is a one-to-one correspondence between the edges in G1 and the edges in G2 such that a set of edges in G1 forms a circuit if and only if the corresponding set in G2 forms a cut-set.
- **THEOREM 12:**
- A graph has a dual if and only if it is planar.

Dual of a Subgraph

- Let G be a planar graph and G* be its dual.
- Let a be an edge in G, and the corresponding edge in G* be a*.
- Suppose that we delete edge a from G and then try to find the dual of G − a.
- If edge a was on the boundary of two regions, removal of a would merge these two regions into one. Thus the dual (G − a)* can be obtained from G* by deleting the corresponding edge a* and then fusing the two end vertices of a* in G* − a*.
- On the other hand, if edge a is not on the boundary, a* forms a self-loop. In that case G* − a* is the same as (G − a)*.
- Thus if a graph G has a dual G*, the dual of any subgraph of G can be obtained by successive application of this procedure.

Dual of a Homeomorphic Graph

- Let G be a planar graph and G* be its dual.
- Let a be an edge in G, and the corresponding edge in G* be a*.
- Suppose that we create an additional vertex in G by introducing a vertex of degree two in edge a (i.e., a now becomes two edges in series). It will simply add an edge parallel to a* in G*.
- Likewise, the reverse process of merging two edges in series will simply eliminate one of the corresponding parallel edges in G*.
- Thus if a graph G has a dual G*, the dual of any graph homeomorphic to G can be obtained from G* by the above procedure.