# Generating Functions

$x_1 + x_2 + x_3 = a$

While shopping one Saturday, Mildred buys 12 oranges for her children, Grace, Mary, and Frank. In how many ways can she distribute the oranges so that Grace gets at least four, and Mary and Frank get at least two, but Frank gets no more than five?

$g \geq 4$

$m \geq 2$

$\boxed{5 \leq} f \geq 2$

$c_1 + c_2 + c_3 = 12$ where $4 \leq c_1$, $2 \leq c_2$, and $2 \leq c_3 \leq 5$

$$(x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5).$$

$x^{12}$

$x^4 \cdot x^3 \cdot x^5 = x^{12}$

$x^5 \cdot x^4 \cdot x^3 = x^{12}$

| G | M | F | G | M | F |
|---|---|---|---|---|---|
| 4 | 3 | 5 | 6 | 2 | 4 |
| 4 | 4 | 4 | 6 | 3 | 3 |
| 4 | 5 | 3 | 6 | 4 | 2 |
| 4 | 6 | 2 | 7 | 2 | 3 |
| 5 | 2 | 5 | 7 | 3 | 2 |
| 5 | 3 | 4 | 8 | 2 | 2 |
| 5 | 4 | 3 | | | |
| 5 | 5 | 2 | | | |

If there is an unlimited number (or at least 24 of each color) of red, green, white, and black jelly beans, in how many ways can Douglas select 24 of these candies so that he has an even number of white beans and at least six black ones?

$$f(x) = (1 + x + x^2 + \cdots + x^{24})^2(1 + x^2 + x^4 + \cdots + x^{24})(x^6 + x^7 + \cdots + x^{24})$$

How many integer solutions are there for the equation $c_1 + c_2 + c_3 + c_4 = 25$ if $0 \leq c_i$ for all $1 \leq i \leq 4$?

$$f(x) = (1 + x + x^2 + \dots + x^{25})^4$$

co-efficient $x^{25}$.

$$g(x) = (1 + x + x^2 + \dots + x^{25} + x^{26} + \dots)^4$$

# Definition

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function

$a_0\, H_0(x) + a_1\, H_1(x) + a_2\, H_2(x)$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

For any $n \in \mathbf{Z}^+$,

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

so $(1 + x)^n$ is the generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots.$$

$a$ , $b$ & $c$

$a$ $b$ $c$ $a+b+c$

$ab+ac+bc$

$abc$

$$(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+bc+ca)x^2+(abc)x^3$$

$\Rightarrow a_0, a_1, a_2, a_3 \ldots\ldots$ $H_0(x)$ $H_1(x)$ $H_2(x) \ldots\ldots$ $E_1$

$E_2$

$b_0, b_1, b_2, b_3 \ldots\ldots$

For $n \in \mathbf{Z}^+$,

$$x^{n-1}$$

$$(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + x^3 + \cdots + x^n).$$

$$1 + \cancel{x} + x^2 + x^3 + \ldots + x^n - \cancel{x} - x^2 - x^3 - \ldots - x^n - x^{n+1}$$

$$= 1 - x^{n+1}$$

So

$$\frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n,$$

and $(1 - x^{n+1})/(1 - x)$ is the generating function for the sequence 1, 1, 1, . . . , 1, 0, 0, 0, . . . , where the first $n + 1$ terms are 1.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$$

valid for all real $x$ where $|x| < 1$

$\frac{1}{1-x}$ is the generating function for the sequence 1, 1, 1, 1, . . .

taking the derivative yields

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$\frac{1}{(1-x)^2}$ is the generating function for the sequence 1, 2, 3, 4, . . .

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

is the generating function for the sequence 0, 1, 2, 3, . . . .

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + \cdots)$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots$$

$\frac{x+1}{(1-x)^3}$ generates $1^2, 2^2, 3^2, \ldots$

$\frac{x(x+1)}{(1-x)^3}$ generates $0^2, 1^2, 2^2, 3^2, \ldots$ .

$$
\begin{aligned}
f_0(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \\
f_1(x) &= x\frac{d}{dx} f_0(x) = \frac{x}{(1-x)^2} \\
&= 0 + x + 2x^2 + 3x^3 + \cdots \\
f_2(x) &= x\frac{d}{dx} f_1(x) = \frac{x^2+x}{(1-x)^3} \\
&= 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots \\
f_3(x) &= x\frac{d}{dx} f_2(x) = \frac{x^3+4x^2+x}{(1-x)^4} \\
&= 0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \cdots \\
f_4(x) &= x\frac{d}{dx} f_3(x) = \frac{x^4+11x^3+11x^2+x}{(1-x)^5} \\
&= 0^4 + 1^4x + 2^4x^2 + 3^4x^3 + \cdots
\end{aligned}
$$

$1/(1-2x)$ is the generating function for

$$\frac{1}{1-\boxed{x}} \qquad \frac{1}{1-y} \qquad \frac{1}{1-2x} \qquad \boxed{x} = 2x \qquad y = 2x$$

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots \Rightarrow 1, 2, 2^2, 2^3, 2^4, \ldots$$

$$= 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots$$

for each $a \in \mathbf{R}$, it follows that $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots = 1 + ax + a^2x^2 + a^3x^3 + \cdots$, so $1/(1-ax)$ is the generating function for the sequence $1\ (= a^0), a\ (= a^1), a^2, a^3, \ldots$.

$g(x) = f(x) - x^2$ is the generating function for

$f(x) = 1/1-x$

$= 1+x+x^2+x^3+ \cdots \quad -x^2$

$= 1+x+0x^2+x^3+ \cdots$

1, 1, 0, 1, 1, . . . . . .

$h(x) = f(x) + 2x^3$

1, 1, 1, 3, 1, 1, . . . .

find a generating function for the sequence 0, 2, 6, 12, 20, 30, 42, . . .

$a_0 = 0 \quad = 0^2 + 0 \qquad a_n = n^2 + n$

$a_1 = 2 \quad = 1^2 + 1$

$a_2 = 6 \quad = 2^2 + 2$

$a_3 = 12 \quad = 3^2 + 3$

$$\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(x+1) + x(1-x)}{(1-x)^3} = \underline{\underline{\frac{2x}{(1-x)^3}}}$$

With $n, r \in \mathbf{Z}^+$ and $n \geq r > 0$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

If $n \in \mathbf{R}$, we use

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

as the definition of $\binom{n}{r}$.

for each *real* number $n$, we define $\binom{n}{0} = 1$.

$$
\begin{aligned}
\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} \\
&= \frac{(-1)^r(n)(n+1)(n+2)\cdots(n+r-1)}{r!} \\
&= \frac{(-1)^r(n+r-1)!}{(n-1)!\,r!} = (-1)^r\binom{n+r-1}{r}.
\end{aligned}
$$

For $n \in \mathbf{Z}^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is given by

$$\begin{aligned}(1+x)^{-n} &= 1 + (-n)x + (-n)(-n-1)x^2/2! \\ &\quad + (-n)(-n-1)(-n-2)x^3/3! + \cdots \\ &= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}x^r \\ &= \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r.\end{aligned}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = \sum_{r=0}^{\infty} \binom{-n}{r}x^r$$

$(1+x)^{-n}$ is the generating function for the sequence $\binom{-n}{0}, \binom{-n}{1}, \binom{-n}{2}, \binom{-n}{3}, \ldots$.

Find the coefficient of $x^5$ in $(1-2x)^{-7}$.

$(1-\underset{\sim}{2x})^{-7}$ $\qquad x^5$ $\qquad y = -2x$

$(1+y)^{-7}$ $\qquad n = -7$ $\qquad r = 5$

$\binom{-7}{5} y^5 = (-1)^5 \binom{7+5-1}{5} (-2)^5$

$= \underline{\underline{\binom{11}{5} 32}}$

For each real number $n$, the Maclaurin series expansion for $(1+x)^n$ is

$$1 + nx + n(n-1)x^2/2! + n(n-1)(n-2)x^3/3! + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r.$$

$(1+3x)^{1/3}$

$$(1+3x)^{-1/3} = 1 + \sum_{r=1}^{\infty} \frac{(-1/3)\overset{-4/3}{(-1/3-1)}\overset{-7/3}{(-1/3-2)}\cdots}{r!}(3x)^r.$$

Determine the coefficient of $x^{15}$ in $f(x) = (x^2 + x^3 + x^4 + \cdots)^4$.

$$(x^2+x^3+x^4+\cdots\cdots) = x^2\underline{(1+x+x^2+x^3+\cdots\cdots)}$$

$$f(x) = [x^2(1+x+x^2+x^3+\cdots\cdots)]^4 = \underline{x^8}\cdot(1+x+x^2+x^3+\cdots\cdots)^4$$

$$15-8 = \underline{7}$$

$$x^7 \text{ in } (1+x+x^2+\cdots\cdots)^4 \Rightarrow \underline{(1-x)^{-4}}$$

$$\Rightarrow 1/1-x$$

$$\binom{-4}{7}(-1)^7 = (-1)^7\binom{4+7-1}{7}(-1)^7 = \underline{{}^{10}C_7}$$

# Identities

For all $m, n \in \mathbf{Z}^+$, $a \in \mathbf{R}$,

**1)** $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

**2)** $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$

**3)** $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$

**4)** $(1-x^{n+1})/(1-x) = 1 + x + x^2 + \cdots + x^n$

**5)** $1/(1-x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$

**6)** $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$

$$= \sum_{i=0}^{\infty}(ax)^i = \sum_{i=0}^{\infty} a^i x^i$$

$$= 1 + ax + a^2x^2 + a^3x^3 + \cdots$$

7) $$1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$$
$$= \sum_{i=0}^{\infty} \binom{-n}{i} x^i$$
$$= 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i$$

8) $$1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots$$
$$= \sum_{i=0}^{\infty} \binom{-n}{i} (-x)^i$$
$$= 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$$
$$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, and $h(x) = f(x)g(x)$, then $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where for all $k \geq 0$,

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 = \sum_{j=0}^{k} a_j b_{k-j}.$$

In how many ways can we select, with repetitions allowed, $r$ objects from $n$ distinct objects?

$$(1 + x + x^2 + x^3 + \cdots)^n$$

co-efficient of $x^r$.

$$\left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$

$$\boxed{\binom{n+r-1}{r}}$$

In how many ways can a police captain distribute 24 rifle shells to four police officers so that each officer gets at least three shells, but not more than eight?

$$f(x) = (x^3 + x^4 + x^5 + \cdots + x^8)^4$$

Co-efficient of $x^{24}$ in $f(x)$

$$f(x) = \left[x^3(1 + x + x^2 + \cdots + x^5)\right]^4 = x^{12} \cdot (1 + x + x^2 + \cdots x^5)^4$$

$$= x^{12} \cdot \left[\frac{1 - x^6}{(1-x)}\right]^4$$

co-efficient of $x^{12}$ in $(1-x^6)^4 \cdot (1-x)^{-4}$ $\qquad \binom{-4}{12}(-x)^{12}$

$$= \left[1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + x^{24}\right] \cdot \left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \cdots\right]$$

$$\binom{-4}{12}(-1)^{12} + \binom{4}{1}(-1)\binom{-4}{6}(-1)^6 + \binom{4}{2}\binom{-4}{0} = \binom{4+12-1}{12} - \binom{4}{1}\binom{4+6-1}{6} + \binom{4}{2}$$

$$= 125$$

Determine the coefficient of $x^8$ in $\dfrac{1}{(x-3)(x-2)^2}$.

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$= \frac{A(x-2)^2 + B(x-3)(x-2) + C(x-3)}{(x-3)(x-2)^2}$$

$$1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$$

$$= A(x^2-4x+4) + B(x^2-5x+6) + C(x-3)$$

$$0.x^2 + 0.x + 1 = (A+B)x^2 + (-4A-5B+C)x + (4A+6B-3C)$$

$A+B=0$ $\quad -4A-5B+C=0$ $\quad 4A+6B-3C=1$

$A=1,\ B=-1,\ C=-1$

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= \left(\frac{-1}{3}\right)\frac{1}{1-x/3} + \frac{1}{2}\frac{1}{1-x/2} + \left(\frac{-1}{4}\right)\frac{1}{(1-x/2)^2}$$

$1 - x$

$x - 3$

$$\frac{1}{x-a} = \frac{-1}{a}\frac{1}{(1-x/a)}$$

$\to 1/a$

$$\frac{1}{x-3} = \frac{-1}{3}\cdot\frac{1}{(1-x/3)}$$

$x^2$

$y = 2x$

$y = 1/3\, x$

$$= \left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^{i} + \left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^{i} + \left(\frac{-1}{4}\right)\left[\binom{-2}{0} + \binom{-2}{1}\frac{-x}{2} + \binom{-2}{2}\left(\frac{-x}{2}\right)^{2} + \cdots\right]$$

co-efficient of $x^8$ is

$$\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)^{8} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{8} + \left(\frac{-1}{4}\right)\binom{-2}{8}\left(\frac{-1}{2}\right)^{8} =$$

$$= -\left(\frac{1}{3}\right)^{9} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{8} + \frac{-1}{4}\binom{2+8-1}{8}(-1)^{8}\left(\frac{-1}{2}\right)^{8}$$

$$= -\left(\frac{1}{3}\right)^{9} + \left(\frac{1}{2}\right)^{9} + \frac{-1}{4}\cdot 9\cdot\left(\frac{1}{2}\right)^{8} = -\left(\frac{1}{3}\right)^{9} + \left(\frac{1}{2}\right)^{9}\left[1 - \frac{9}{2}\right]$$

$$= -\left[\left(\frac{1}{3}\right)^{9} + 7\left(\frac{1}{2}\right)^{10}\right]$$