Combinatorics and Graph Theory

Dr. Sitara K.

sitara@nitt.edu

Objective

- To introduce basic concepts of combinatorics and graph theory
- To study graphs, trees and networks
- To discuss Euler formula, Hamilton paths, planar graphs and coloring problem
- To practice useful algorithms on networks such as shortest path algorithm, minimal spanning tree algorithm and min-flow max-cut algorithm

Course content

- Unit – I
  - Introduction to combinatorics, permutation of multisets. Combinations of Multisets, distribution of distinct objects into distinct cells, distribution of non-distinct objects into distinct cells, Shamire secret sharing. Catalan number. Principle of inclusion and exclusion, Derangement.
- Unit – II
  - Generating functions, Partitions of integer, Ferrer graph. Solving recurrence relations using generating functions, Generating permutations and combinations. Pigeonhole principle: simple and strong Form, A THEOREM OF RAMSEY
- Unit – III
  - Graph, simple graph, graph isomorphism, incidence and adjacency matrices, Haveli-Hakimi criterion. Subgraphs Tree, minimum spanning tree, Kruskal, Prims algorithm, Caleys’ formula, Kirchoff-Matrix- tree Theorem, Fundamental circuits, Algorithms for fundamental circuits, Cut-sets and Cut-vertices, fundamental cut-sets.

Course content

- Unit – IV
  - Eular graph, Fleury’s algorithm Hamiltonian graph, Planar and Dual Graphs, Kuratowski's graphs. Coloring, Greedy coloring algorithm, chromatic polynomial.
- Unit – V
  - Mycielski’s theorem, Matching, halls marriage problem. Independent set, Dominating set, Vertex cover, clique, approximation algorithms

Course Outcomes

- Comprehend the fundamentals of combinatorics and apply combinatorial ideas in mathematical arguments in analysis of algorithms, queuing theory, etc.
- Comprehend graph theory fundamentals and tackle problems in dynamic programming, network flows,etc.
- Design and develop real time application using graph theory
- Construct and communicate proofs of theorems

Text Books

- Ralph P. Grimaldi, “Discrete and Combinatorial Mathematics”, 5^{th} Edition, PHI/Pearson Education,2004
- G. Chartrand and P. Zhang, “Introduction to Graph Theory”, McGraw-Hill,2006
- Narsingh Deo, “Graph Theory with Applications to Engineering and Computer Science”, PHI

Reference Books

- Kenneth H. Rosen, “Discrete Mathematics and its Applications”, 7th edition, McGraw- Hill, 2012
- John Harris, Jeffry L. Hirst, Michael Mossinghoff, “Combinatorics and Graph Theory”, 2nd edition, Springer Science & Business Media, 2008
- J. H. Van Lint and R. M. Wilson, “A course in Combinatorics”, 2nd edition, Cambridge Univ. Press,2001
- Dr. D. S. Chandrasekharaiah, "Graph Theory and Combinatorics", Prism,2005.

Combinatorics has emerged as a new subject standing at the crossroads between pure and applied mathematics, the center of bustling activity, a simmering pot of new problems and exciting speculations.

-- Gian-Carlo Rota

Combinatorics

5, 6, 10, 12, 15, 18, 20

- is the mathematics of counting
- the study of arrangements: pairings and groupings, rankings and orderings, selections and allocations
- Three principal branches
  - Enumerative combinatorics is the science of counting.
  - Existential combinatorics studies problems concerning the existence of arrangements that possess some specified property.
  - Constructive combinatorics is the design and study of algorithms for creating arrangements with special properties.

Basic principles of counting

- Rules of Sum
- Rules of Product

- A college library has 40 textbooks on Data structures and 50 text books on algorithms. If a student can take only one book at a time, how many choices he/she has?

$40 + 50 = 90$

- If CS faculty has 2 colleagues, one of them has 3 textbooks on algorithms and other has 5 such textbooks. If *n* denotes the number of different books on this topic that the faculty can borrow from them, what is the range of n?

$1 \leq n \leq 8$

$5 \leq n \leq 8$

Basic principles of counting …

- Rules of Sum:
  - If a first task can be performed in *m* ways, while a second task can be performed in *n* ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of *m* + *n* ways.

$$T_1 \quad T_2 \quad T_3 \quad \cdots \quad T_n$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \quad \downarrow$$

$$n_1 + n_2 + n_3 + \cdots + n_m$$

- How many eight-character passwords are possible if each character is either an uppercase letter A–Z, a lowercase letter a–z, or a digit 0–9?

$(26 + 26 + 10) = 62$ $\quad$ 62^8

a $\quad$ a

$\underline{62}\ \underline{62}\ \underline{62}\ \_\ \_\ \_$

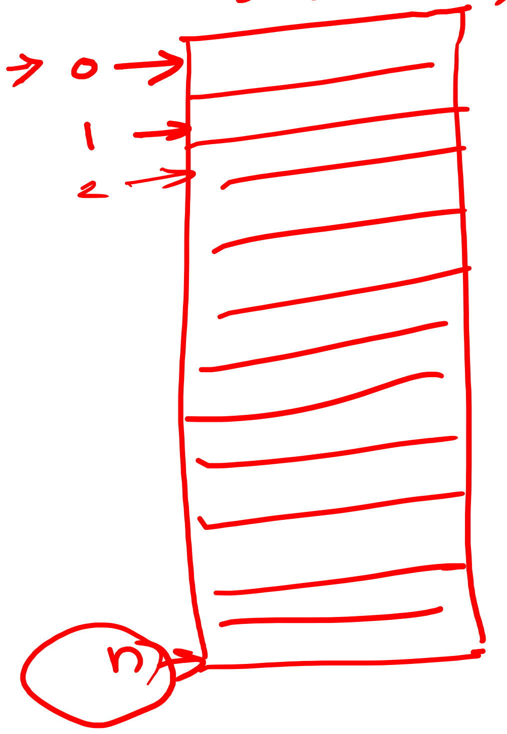
1 $\quad$ ↑

$\underline{\underline{(62)^8}}$

Memory addressing

8-bit

8 bits ⇒ 1 byte



0/1

32-bit

2^{32}

2^{64}

Basic principles of counting…

- Rules of Product
  - If a procedure can be broken down into first and second stages, and if there are *m* possible outcomes for the first stage and if for each of these outcomes, there are *n* possible outcomes for the second stage, then the total procedure can be carried out in the designated order, in *mn* ways.

$$\begin{array}{ccccc} S_1 & & S_2 & \cdots & S_n \\ \downarrow & & \downarrow & & \downarrow \\ n_1 & \times & n_2 & \times \cdots \times & n_m \end{array}$$

Basic principles of counting…

- At times it is necessary to combine different counting principles to solve a problem
  - For example: A coffee shop menu is limited to: 6 kinds of muffins, 8 kinds of sandwiches and 5 beverages (hot coffee, hot tea, iced tea, cola, and orange juice). A manager sends his assistant to the shop to get his breakfast – either a muffin and a hot beverage or a sandwich and a cold beverage.

$$(6 \times 2) + (8 \times 3) = 36$$

- Given nine players, in how many different ways can a manager write out a batting lineup?

$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times \ldots 1 \Rightarrow 9!$

$=$

- License plates consisting of 2 letter words followed by 4 digits

Permutation

- Given a collection of *n* distinct objects, any (linear) arrangement of these objects – permutation
- If there are *n* distinct objects and *r* is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size *r* for *n* objects is

$$P(n, r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r-1)$$

nPr

$$= n \times (n-1) \times (n-2) \times \cdots \times (n-r-1) \times \frac{(n-r) \times (n-r-1) \times \cdots 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \cdots 3 \times 2 \times 1}$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

- If repetitions are allowed then, there are n^r possible arrangements.

COMPUTER

$^{8}P_{5}$

$n = 8$

$r = 5$

$8 \times 7 \times 6 \times 5 \times 4 = {}^{8}P_{5}$

BALL

$n = 4 \qquad r = 4$

$BAL_1L_2 \Rightarrow {}^4P_4 \Rightarrow 4!$

$L_1, L_2 \to L$

$\boxed{4!/2!}$

$BL_1AL_2 \Rightarrow BLAL$

$BL_2AL_1 \Rightarrow BLAL$

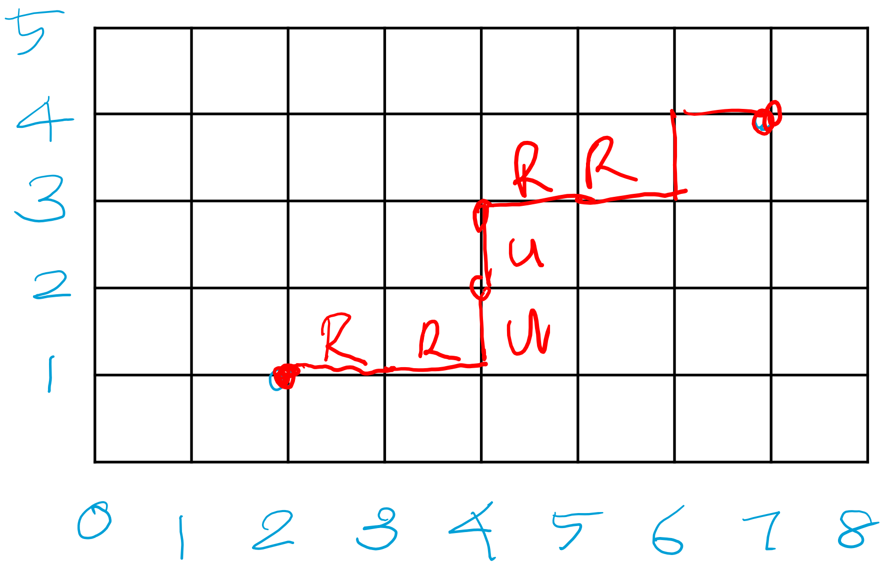
DATABASES

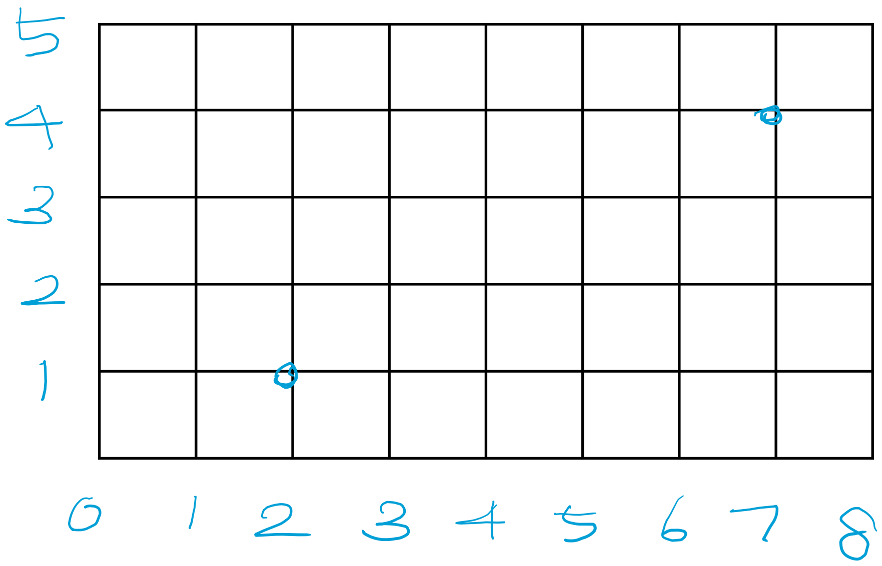
$9! / (3! \times 2!)$

- If there are n distinct objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable of second type, … and n_r indistinguishable of an r^{th} type, where $n_1 + n_2 + \ldots + n_r = n$, then there are $\frac{n!}{n_1!n_2!\ldots n_r!}$ (linear) arrangements of the given n objects

- Determine the number of paths in the xy-plane from (2,1) to (7,4) where each path is made up of individual steps going one unit to the right (R) or one unit upward (U).

RRUURRUR





$$\frac{8!}{5!\,3!}$$

- If n and k are positive integers, and n=2k, the number of ways in which we can arrange all of these n symbols is

k - distinct

$$\frac{n!}{2!\cdot 2!\cdot 2!\ldots 2!}$$

a a

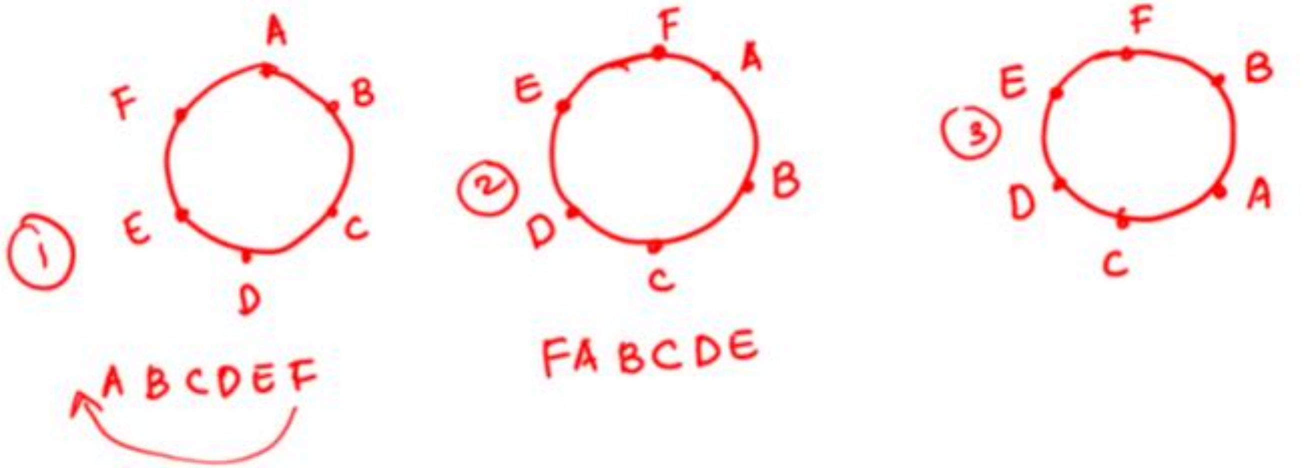
b b

$n = 4$

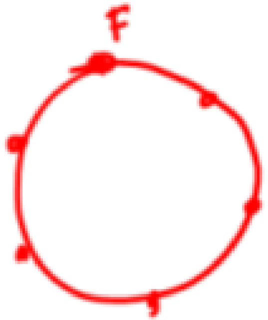
$k = 2$

$n!/(2!)^k$

If 6 people, designated as A, B, C, D, E, and F are seated in a round table, how many circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

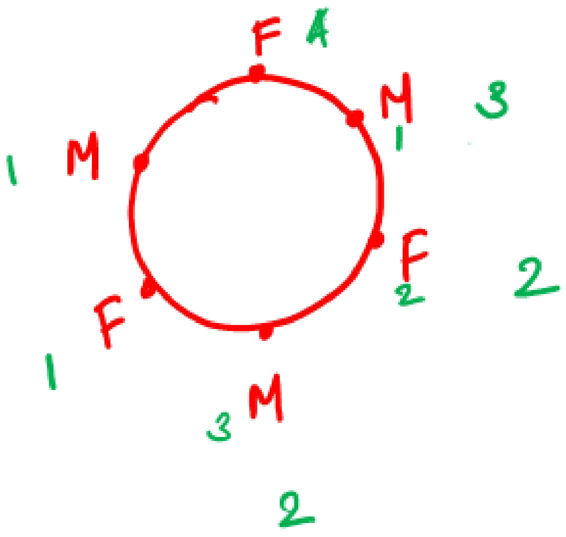


$$6!/6 = 5!$$



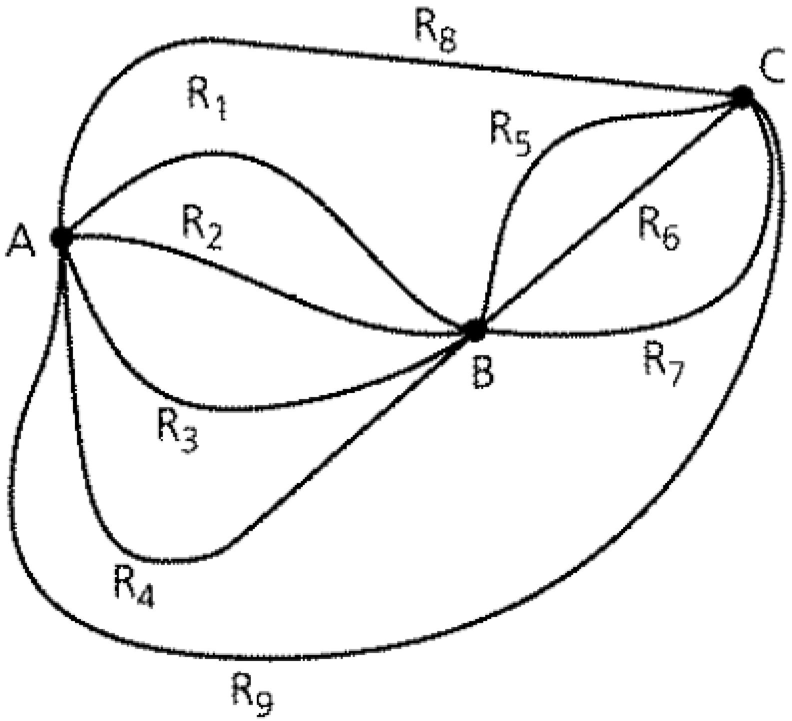
$\Rightarrow 5!$

Suppose in that 6 people, A, B, and C are female and D, E, and F are male. If we want to arrange people such that sexes alternate, how many ways can we arrange them?



$$\boxed{3! \times 2!}$$

Three small towns, designated by A, B and C are interconnected by a system of two-way roads, as shown in the figure



a) In how many ways can Linda travel from town A to town C? $2 + (4 \times 3) = 14$ ways

b) How many different round trips can Linda travel from town A to town C and back to town A? 14×14

c) How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.) 14×13

a) Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, and k are integer variables.)

```
counter := 0
for i := 1 to 12 do
   counter := counter + 1
for j := 5 to 10 do
   counter := counter + 2
for k := 15 downto 8 do
   counter := counter + 3
```

12×1

$+$

6×2

$+$

8×3

b) Which counting principle is at play in part (a)?

Combinations

- If we start with n distinct objects, each selection or combination of r of these objects with no reference to order, corresponds to r! permutations of size r from the n objects. Thus, the number of combinations of size r from a collection of size n is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}\ , \qquad 0 \le r \le n$$

$nC_r \qquad \binom{n}{r}$

- How many n-digit binary numbers have exactly k 1s?

^nC_k

5- digit 2

11000

$\boxed{01001}$

10001

$^5C_2$

- Lynn and Patt decide to buy a powerball ticket. To win the grand prize, one must match 5 numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the 5 numbers and Patt selects the powerball. How many ways can they select the numbers for powerball ticket?

$$_{49}C_5 \times {}_{42}C_1$$

- A student taking an examination is directed to answer 7 of the 10 questions where at least 3 are selected from the first 5. How many ways can the students answer?

$$5C_3 \cdot 5C_4 + 5C_4 \cdot 5C_3 + 5C_5 \cdot 5C_2$$

7 10

	5	5
$\Rightarrow$ 1	3	4
2	4	3
3	5	2

- A gym teacher must make up for 4 volleyball teams of nine girls each from the 36 girls in her P.E class. In how many ways can she select these four teams?

36 4

A A A A . . . . . A 9

B B B B 9

C 9

D 9

$$\frac{36!}{9!\,9!\,9!\,9!}$$

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}$$

- How many number of arrangements of the letters in TALLAHASSEE is possible without adjacent A’s?

3 A's

T L L H S S E E

$$\frac{8!}{(2!)^3} \times {}^{9}C_{3}$$

Concise way of writing the sum of a list of n+1 terms

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i$$

Results related to Combinations

- For integers n and r with $n \geq r \geq 0, C(n,r) = C(n, n-r)$
- Binomial Theorem: If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

In view of this theorem, $\binom{n}{k}$ is often referred to as a *binomial coefficient*.

For each integer $n > 0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, and

b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof: Part (a) follows from the binomial theorem when we set $x = y = 1$. When $x = -1$ and $y = 1$, part (b) results.

Proof: In the expansion of the product

$$\underset{\substack{\text{1st}\\\text{factor}}}{(x+y)}\ \underset{\substack{\text{2nd}\\\text{factor}}}{(x+y)}\ \underset{\substack{\text{3rd}\\\text{factor}}}{(x+y)}\cdots\underset{\substack{n\text{th}\\\text{factor}}}{(x+y)}$$

the coefficient of $x^k y^{n-k}$, where $0 \leq k \leq n$, is the number of different ways in which we can select k x's [and consequently $(n-k)$ y's] from the n available factors. (One way, for example, is to choose x from the first k factors and y from the last $n-k$ factors.) The total number of such selections of size k from a collection of size n is $C(n, k) = \binom{n}{k}$, and from this the binomial theorem follows.

- Multinomial Theorem: generalization of binomial theorem

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_t!},$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, . . . , and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.22, in

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_3-\cdots-n_{t-1}}{n_t}$$

ways.

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient.* (When $t = 2$ this reduces to a binomial coefficient.)

Seven high school students stop at a restaurant, where each of them has one of the following: a cheeseburger, hot dog, taco and fish burger. How many different purchases are possible?

c h t f   7

1.	c, c, h, h, t, t, f	1.	x x | x x | x x | x
2.	c, c, c, c, h, t, f	2.	x x x x | x | x | x
3.	c, c, c, c, c, c, f	3.	x x x x x x | | | x
4.	h, t, t, f, f, f, f	4.	| x | x x | x x x x
5.	t, t, t, t, t, f, f	5.	| | x x x x x | x x
6.	t, t, t, t, t, t, t	6.	| | x x x x x x x |
7.	f, f, f, f, f, f, f	7.	| | | x x x x x x x

$n=4, \; r=7$

$(n+r-1)C_r$

$10C_7$

Combinations with repetitions

- The number of combinations of n objects taken r at a time, with repetition is

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!\,(n-1)!}$$

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$n+r-1-r$

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them \$1000 in Christmas bonus checks, where each check will be written for a multiple of \$100.

- Allowing the situation in which one or more of the vice presidents get nothing, then in how many ways can President Helen distribute the bonus checks?

$n = 4$ $\qquad$ \$1000 $\qquad$ \$100

$C(n+r-1, r) = C(4+10-1, 10)$ $\quad \hookrightarrow 10 \leftarrow r$

$= C(13, 10)$

- If each vice president must get at least \$100, then in how many ways can President Helen distribute the bonus checks?

$n = 4$

$C(4+6-1, 6) = {}^{9}C_{6}$

$\$1000 - \$400$

$= \$600$

$r = 6$

- If each vice president must get at least \$100 and Mona gets at least \$500, then in how many ways can President Helen distribute the bonus checks?

$\$100$

$\Downarrow$

$\$400$

Mona $\Leftarrow$ $\$500$

$\$400$

$\$1000 - \$800 = \$200$

$n = 4$ $\quad r = 2$

$\boxed{{}_5C_2}$

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

$n = 4 \quad r_b = 3$

$C(4+3-1, 3) = {}^6C_3$

$r_o = 6 \quad n = 4$

$C(4+6-1, 6)$

$= C(9,6)$

$\Rightarrow \boxed{{}^6C_3 \times {}^9C_6}$

Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \qquad \text{where } x_i \geq 0 \quad \text{for all } 1 \leq i \leq 4.$$

$n = 4$ $r = 7$

$C(10, 7)$

The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

The number of selections, with repetition, of size r from a collection of size n.

The number of ways r identical objects can be distributed among n distinct containers.

How many solutions are there for

$$x_1 + x_2 + \cdots + x_6 \underset{=}{<} 10$$

$x_i \geq 0$

$1 \leq i \leq 6$

$$x_1 + x_2 + \cdots + x_6 + x_7 = 10$$

$x_7 > 0$

$n = 7$ $\quad r = 9$

$$C(7+9-1, 9) = {}^{15}C_9$$

Total number of terms in a binomial expansion

$(x+y)^{\underline{\underline{n}}}$

$\binom{n}{k} x^{k} y^{n-k}$

$k+n-k \Rightarrow n$

$x^{n_1} y^{n_2} \Rightarrow \boxed{n_1+n_2 = \underline{\underline{n}}}$

$n_\alpha = 2 \qquad r = n$

$C(n_\alpha+r-1, r) = C(2+n-1, n) = C(n+1, n) = \frac{(n+1)!}{n!\,(\cancel{n}+1-\cancel{n})!}$

$= \frac{n!\,(n+1)}{n!}$

$= \underline{\underline{n+1}}$

Total number of terms in a multinomial expansion

$(x+y+z+w)^{10}$

$(x+y+z+\cdots)^n$

$\boxed{p+q+r+s=10}$

$x^p y^q z^r w^s$

$n=4 \qquad r=10$

$C(n+r-1, r)$

$C(4+10-1, 10) = \underline{C(13, 10)}$

Determine the number of compositions for the number 4

$\boxed{C(n+r-1,\ r)}$

T_0 1. 4

T_1 2. $1+3$

3. $3+1$

4. $2+2$

T_2 5. $1+1+2$

6. $1+2+1$

7. $2+1+1$

T_3 8. $1+1+1+1$

$x_1+x_2=4$

$\boxed{x_1'+x_2'=2}$ $n=2, r=2 \Rightarrow {}^3C_2$

$y_1+y_2+y_3=4$

$\boxed{y_1'+y_2'+y_3'=1}$ $n=3, r=1 \Rightarrow {}^3C_1$

$z_1+z_2+z_3+z_4=4$

$\boxed{z_1'+z_2'+z_3'+z_4'=0}$ $n=4, r=0 \Rightarrow {}^3C_0$

${}^3C_3+{}^3C_2+{}^3C_1+{}^3C_0=\sum_{k=0}^{3}{}^3C_k = 2^3$

Determine the number of compositions for the number 7

The counter at Patti and Terri's Bar has 15 bar stools. Upon entering the bar Darrell finds the stools occupied as follows:

$$\underbrace{O\ O}_{1}\ \underbrace{E}_{2}\ \underbrace{O\ O\ O\ O}_{3}\ \underbrace{E\ E\ E}_{4}\ \underbrace{O\ O\ O}_{5}\ \underbrace{E}_{6}\ \underbrace{O}_{7},$$

where O indicates an occupied stool and E an empty one. (Here we are not concerned with the occupants of the stools, just whether or not a stool is occupied.) In this case we say that the occupancy of the 15 stools determines seven runs.

find the total number of ways five E's and 10 O's can determine seven runs.

Case 1: Starting with O

4 runs Os

3 runs Es

$$x_1 + x_3 + x_5 + x_7 = 10$$

$$x_2 + x_4 + x_6 = 5$$

$$\boxed{\begin{aligned} y_1 + y_3 + y_5 + y_7 &= 6 \\ y_2 + y_4 + y_6 &= 2 \end{aligned}}$$

Case 2: Starting with E

4 runs E

3 runs O

$$\underbrace{E}_{1}\ \underbrace{OOO}_{2}\ \underbrace{EE}_{3}\ \underbrace{OO}_{4}\qquad \underbrace{E}_{5}\ \underbrace{OOOOO}_{6}\ \underbrace{E}_{7}$$

$$x_1 + x_3 + x_5 + x_7 = 5$$

$$x_2 + x_4 + x_6 = 10$$

$$\boxed{\begin{aligned} y_1 + y_3 + y_5 + y_7 &= 1 \\ y_2 + y_4 + y_6 &= 7 \end{aligned}}$$