Multiple Pursuer Multiple Evader Differential Games

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***Abstract*—In this article an N-pursuer versus M-evader team conflict is studied. This article extends classical differential game theory to simultaneously address weapon assignments and multiplayer pursuit-evasion scenarios. Saddle-point strategies that provide guaranteed performance for each team regardless of the actual strategies implemented by the opponent are devised. The players' optimal strategies require the codesign of cooperative optimal assignments and optimal guidance laws. A representative measure of performance is employed and the Value function of the attendant game is obtained. It is shown that the Value function is continuously differentiable and that it satisfies the Hamilton–Jacobi–Isaacs equation—the curse of dimensionality is overcome and the optimal strategies are obtained. The cases of $N = M$ and $N > M$ are considered. In the latter case, cooperative guidance strategies are also developed in order for the pursuers to exploit their numerical advantage. This article provides a foundation to formally analyze complex and high-dimensional conflicts between teams of N pursuers and M evaders by means of differential game theory.**

***Index Terms*—Autonomous systems, intelligent control, optimal control.**

I. Introduction

In this article, a multiplayer border-defense problem is considered where the players are divided into two opposing teams: The pursuer team and the evader team. The evaders aim is to reach the border. In the case where the evaders are captured before reaching the border, the evaders, as a team, try to minimize their cumulative terminal distance from the border. The pursuers strive to capture the evaders while maximizing the same metric. The problem is correctly posed as a pursuit-evasion differential game.

Differential game theory provides the right framework to analyze pursuit-evasion problems and, as a corollary, combat games. Pursuit-evasion problems were first formulated in the seminal work [1]. Games with many players have also been considered [2]–[4]. Reach and avoid differential games which include time-varying dynamics, targets, and constraints were addressed in [5] by means of a modified Hamilton–Jacobi–Isaacs (HJI) equation. Other approaches and applications regarding reach and avoid games are found in [6]–[8]. The paper [9] considered a group of cooperative pursuers that try to capture a single evader within a bounded domain. The domain may also contain obstacles and the solution employs Voronoi partitions of the plane. Similar games concerning multiple pursuers that try to capture an evader have been addressed in [10]–[16]. In [17]–[21], cooperative behaviors within pursuit-evasion games are analyzed in order to protect or rescue teammates in the presence of adversarial entities. Several papers have addressed pursuit-evasion scenarios and missile interception problems posed as differential games (see, e.g., [22]–[26]).

In this article, we provide a team cooperative optimal solution of the border defense differential game (BDDG) that can be implemented in real time and is thus able to exploit nonoptimal adversary strategies and/or maneuvers. The agents in each team cooperate to optimize their team's performance. We emphasize that, while the members within a given team cooperate among themselves, the game is noncooperative since the two opposing teams play a zero-sum game. Members of the pursuer team are tasked to intercept the members of the evader team and capture them before they can reach the border. Hence, the solution of the game should provide state feedback, optimal strategies as well as the optimal assignments of N pursuers to M evaders, which is a discrete decision problem with combinatorial overtones. In other words, the players (pursuers and evaders) need to dynamically determine their optimal headings/guidance/maneuvers over the continuum of space and time. Simultaneously, the team has to obtain the optimal assignments over a discrete space of possibilities.

The article is organized as follows. The two-team multiagent BDDG is formally stated in Section II. Section III addresses the case of two pursuers against two evaders. The more general case of N pursuers versus M evaders is considered in Section IV. Section V discusses examples and extensions, followed by concluding remarks in Section VI.

II. Game

We consider a multiagent pursuit-evasion differential game where each agent belongs to either one of two opposing teams. This multiagent pursuit-evasion scenario presents unique challenges within classical differential game theory. In addition to computing state feedback optimal strategies, this game also requires the optimal assignment of pursuers to evaders to determine which pursuer captures which evader. In other words, we need to codesign the optimal guidance strategies and the optimal assignments which are represented by discrete variables. The hybrid nature of the problem has rarely been tackled within the theory of differential games [27]–[29].

In [27], [28], [30]–[33], similar scenarios to this article are addressed. The authors in [33] focus on the game of kind and do not provide saddle-point strategies. The recent papers [27] and [28] present two of the most related scenarios and approaches to the problem discussed here. In the recent paper [27], the authors address the pursuit-evasion problem where a set of attackers tries to reach a target while avoiding a set of defenders; open-loop strategies are proposed where a given team is assigned to select its strategy first and the opposing team follows with its response. Such a scenario is a Stackelberg game [34]. Furthermore,

Manuscript received November 18, 2019; revised February 21, 2020; accepted June 15, 2020. Date of publication June 22, 2020; date of current version April 26, 2021. This work was supported in part by Air Force Office of Scientific Research LRIR 18RQCOR036. Recommended by Associate Editor R. P. Malhame. *(Corresponding author: Eloy Garcia.)*

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Digital Object Identifier 10.1109/TAC.2020.3003840

due to the open-loop nature of the solution concept and the decomposition approach, the authors focus on computational approaches for solving Hamilton–Jacobi–Bellman local equations (to avoid the curse of dimensionality) as an approximation of the HJI equation of the overall game over the high-dimensional state space of all player states. The authors in [28] focus on approximating the solution of the HJI equation and for the players to implement "semi-open-loop" control strategies.

The solution presented in this article is not an approximation but the optimal solution of the game over the complete state space. In fact, we are able to obtain the (closed-form) optimal solution of the operationally relevant multiplayer BDDG. We provide the complete solution of the BDDG: We derive state feedback optimal strategies for each player, the Value function $V(\mathbf{x})$ is obtained, and it is shown that $V(\mathbf{x})$ is continuously differentiable and it satisfies the HJI partial differential equation.

An N versus M team differential game is considered with N pursuers and M evaders. The players are holonomic and it is assumed that $N \geq M$. The players in the pursuer team are denoted by P_i, $i = 1,\ldots,N$ and their speeds are $v_{P_i} \in [\underline{v}_{P_i}, \bar{v}_{P_i}]$, where $\underline{v}_{P_i}$ and $\bar{v}_{P_i}$ denote the minimum and maximum speed of player P_i. Similarly, the evaders are denoted by E_j, $j = 1,\ldots,M$ and their speeds are denoted by $v_{E_j} \in [\underline{v}_{E_j}, \bar{v}_{E_j}]$. Optimal strategies demand maximum speed by each holonomic player, hence, we denote $v_{P_i} = v^*_{P_i} = \bar{v}_{P_i}$, for $i = 1,\ldots,N$ and $v_{E_j} = v^*_{E_j} = \bar{v}_{E_j}$ for $j = 1,\ldots,M$. It is assumed that the pursuers are faster than the evaders, so the speed ratios satisfy $\alpha_{ij} = v_{E_j}/v_{P_i} < 1$, for $i = 1,\ldots,N$ and $j = 1,\ldots,M$. The obtained results can be extended to the case where a subset of pursuers are slower than a subset of evaders by imposing a constraint on the assignments where slow pursuers cannot be assigned to intercept faster evaders.

The states of P_i and E_j are given by their Cartesian coordinates $\mathbf{x}_{P_i} = (x_{P_i}, y_{P_i})$ and $\mathbf{x}_{E_j} = (x_{E_j}, y_{E_j})$. The complete state of the differential game is defined by $\mathbf{x} := (x_{P_i}, y_{P_i}, x_{E_j}, y_{E_j}) \in \mathbb{R}^{2(N+M)}$, for $i = 1,\ldots,N$, $j = 1,\ldots,M$. The players have simple motion, so the control variables of the pursuer team are given by the cooperative instantaneous heading angles of each player P_i, that is, $\mathbf{u}_P = \{\psi_i\}$ for $i = 1,\ldots,N$. The evader team controls the state of the system by cooperatively choosing the instantaneous headings of each evader E_j, that is, $\mathbf{u}_E = \{\phi_j\}$ for $j = 1,\ldots,M$. The dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_E, \mathbf{u}_P)$ are specified by the system of $2(N+M)$ differential equations

$$\begin{aligned}
\dot{x}_{P_i} &= v_{P_i}\cos\psi_i, \quad x_{P_i}(0) = x_{P_{i_0}}\\
\dot{y}_{P_i} &= v_{P_i}\sin\psi_i, \quad y_{P_i}(0) = y_{P_{i_0}}\\
\dot{x}_{E_j} &= v_{E_j}\cos\phi_j, \quad x_{E_j}(0) = x_{E_{j_0}}\\
\dot{y}_{E_j} &= v_{E_j}\sin\phi_j, \quad y_{E_j}(0) = y_{E_{j_0}}
\end{aligned} \tag{1}$$

where the admissible controls are the players' headings $\psi_i, \phi_j \in [-\pi, \pi)$. The initial state of the system is $\mathbf{x}_0 := (x_{P_{i_0}}, y_{P_{i_0}}, x_{E_{j_0}}, y_{E_{j_0}}) = \mathbf{x}(t_0) \in \mathbb{R}^{2(N+M)}$. We consider the scenario of border defense where the border line is the x-axis of the Euclidean plane and the game is played in the half plane $y \geq 0$. Define the binary variable μ_{ij} such that $\mu_{ij} = 1$ if pursuer i is assigned to capture evader j and $\mu_{ij} = 0$ otherwise. For any pursuer–evader pair, i, j, such that $\mu_{ij} = 1$, the game will terminate in two possible ways. The first criterion is $y_{E_j} = 0$, meaning that the evader reaches the border before being captured by the assigned pursuer. Otherwise the game will terminate when the pursuer captures the evader. We consider the case where a pursuer can be assigned to at most one evader, that is, $\sum_{j=1}^{M} \mu_{ij} \leq 1$.

In this article, we consider point capture and we focus on the game of degree in the state space region $\mathcal{R}_P \subset \mathbb{R}^{2(N+M)}$ where capture of all evaders is guaranteed and thus the pursuers' team is the winner. However, the obtained strategies also provide the solution to the game of kind; this is discussed at the end of Section IV-B. In the winning region of the pursuers, the terminal set is given by

$$\begin{aligned}
\mathcal{T} := \{\mathbf{x}\,|\forall j \in 1,\ldots,M, \exists i \in 1,\ldots,N, \mu_{ij} = 1,\\
x_{P_i} = x_{E_j}, y_{P_i} = y_{E_j}\}.
\end{aligned} \tag{2}$$

Note that (2) includes the case $N > M$ where more than one pursuer could be assigned to an evader. In such a case, the pursuers assigned to the same evader will also need to determine a cooperative pursuit strategy. This will be clarified in Section IV-B. The terminal time t_f is defined as the time instant when the state of the system satisfies (2), that is, the time instant when the last evader is captured, at which time the terminal state is $\mathbf{x}_f := (x_{E_f}, y_{E_f}, x_{1_f}, y_{1_f}, x_{2_f}, y_{2_f}) = \mathbf{x}(t_f)$. We define the individual terminal times $t_{f_{ij}}$ corresponding to the interception of E_j by P_i. In order to guarantee regularity of the solutions, we define $\dot{x}_{P_i} = \dot{y}_{P_i} = \dot{x}_{E_j} = \dot{y}_{E_j} = 0$ for $t \geq t_{f_{ij}}$. These definitions allow for the game to continue until all evaders are captured. The terminal cost/payoff functional is

$$J(\mathbf{u}_P(t), \mathbf{u}_E(t); \mathbf{x}_0) = \Phi(\mathbf{x}_f) \tag{3}$$

where

$$\Phi(\mathbf{x}_f) := \sum_{j=1}^{M} y_{E_j}(t_f). \tag{4}$$

The cost/payoff functional depends only on the terminal state. The game's Value is

$$V(\mathbf{x}_0) := \max_{\mathbf{u}_P(\cdot)} \min_{\mathbf{u}_E(\cdot)} J(\mathbf{u}_P(\cdot), \mathbf{u}_E(\cdot); \mathbf{x}_0) \tag{5}$$

subject to (1) and (2), where $\mathbf{u}_P(\cdot)$ and $\mathbf{u}_E(\cdot)$ are the teams' state feedback strategies.

With respect to the information pattern, every agent knows the dynamics (1) and the speed ratio parameter α. It is assumed that the agents use causal strategies and that every agent has access to the state $\mathbf{x}$ at the current time t, that is, the capture game is a perfect information differential game; the optimal strategies will be state feedback strategies. Finally, and most importantly, it is assumed that the agents do not know the opponent's current decision. Most importantly, we assume firm commitment to the initial assignment by the pursuers; this means that $\mu_{ij}(t) = \mu_{ij}(t_0)$, that is, the pursuers do not switch assignments during the engagement. In addition to providing the foundation for a framework to analyze more complex scenarios, which include switching assignments and capture in succession, the case of firm commitment is useful by itself in several interception applications.

Let the costate be represented by $\lambda^T = (\lambda_{x_{P_1}}, \lambda_{y_{P_1}}, \ldots, \lambda_{x_{P_N}}, \lambda_{y_{P_N}}, \lambda_{x_{E_1}}, \lambda_{y_{E_1}}, \ldots, \lambda_{x_{E_M}}, \lambda_{y_{E_M}}) \in \mathbb{R}^{2(N+M)}$. The Hamiltonian of the differential game is

$$\begin{aligned}
\mathcal{H} = &\sum_{i=1}^{N} v_{P_i}(\lambda_{x_{P_i}}\cos\psi_i + \lambda_{y_{P_i}}\sin\psi_i)\\
&+ \sum_{j=1}^{M} v_{E_j}(\lambda_{x_{E_j}}\cos\phi_j + \lambda_{y_{E_j}}\sin\phi_j).
\end{aligned} \tag{6}$$

Theorem 1: Consider the cooperative differential game (1)–(5). The headings of the players E_j and P_i are constant under optimal play and the optimal trajectories are straight lines.

Proof: This follows from the fact that the agents have simple motion and the cost is of Mayer type. ■

III. 2 Versus 2 Differential Game

In this section, we will address the case of two pursuers versus two evaders. In the 2 versus 2 BDDG, the state is given by $\mathbf{x} := (x_{E_1}, y_{E_1}, x_{E_2}, y_{E_2}, x_{P_1}, y_{P_1}, x_{P_2}, y_{P_2}) \in \mathbb{R}^8$. Let us define in general

$$\underline{y}_{ij}(\mathbf{x}) = \frac{y_{E_j} - \alpha_{ij}^2 y_{P_i} - \alpha_{ij}\sqrt{(x_{E_j} - x_{P_i})^2 + (y_{E_j} - y_{P_i})^2}}{1 - \alpha_{ij}^2}. \tag{7}$$

For the case of two pursuers and two evaders, let

$$\begin{aligned} y_{s_1}(\mathbf{x}) &= \underline{y}_{11}(\mathbf{x}) + \underline{y}_{22}(\mathbf{x}) \\ y_{s_2}(\mathbf{x}) &= \underline{y}_{12}(\mathbf{x}) + \underline{y}_{21}(\mathbf{x}). \end{aligned} \tag{8}$$

The following theorem provides the solution of the differential game: It dictates what the optimal assignment it provides the state feedback optimal headings for each one of the four players.

Theorem 2: Consider the 2 versus 2 BDDG (1)–(5) with $\alpha_{ij} = v_{E_j}/v_{P_i} < 1$, and where $\mathbf{x} \in \mathcal{R}_P$. The Value function is continuously differentiable (except at the dispersal surface $y_{s_1} = y_{s_2}$) and it satisfies the HJI equation. The Value function is explicitly given by $V(\mathbf{x}) = y_{s_1}(\mathbf{x})$ if $y_{s_1} > y_{s_2}$ and $V(\mathbf{x}) = y_{s_2}(\mathbf{x})$ if $y_{s_2} > y_{s_1}$. The optimal state feedback strategies are

$$\begin{aligned}
\cos\phi_1^* &= \frac{x_{E_1}^* - x_{E_1}}{\sqrt{(x_{E_1}^* - x_{E_1})^2 + (y_{E_1}^* - y_{E_1})^2}} \\
\sin\phi_1^* &= \frac{y_{E_1}^* - y_{E_1}}{\sqrt{(x_{E_1}^* - x_{E_1})^2 + (y_{E_1}^* - y_{E_1})^2}} \\
\cos\phi_2^* &= \frac{x_{E_2}^* - x_{E_2}}{\sqrt{(x_{E_2}^* - x_{E_2})^2 + (y_{E_2}^* - y_{E_2})^2}} \\
\sin\phi_2^* &= \frac{y_{E_2}^* - y_{E_2}}{\sqrt{(x_{E_2}^* - x_{E_2})^2 + (y_{E_2}^* - y_{E_2})^2}} \\
\cos\psi_1^* &= \frac{x_{P_1}^* - x_{P_1}}{\sqrt{(x_{P_1}^* - x_{P_1})^2 + (y_{P_1}^* - y_{P_1})^2}} \\
\sin\psi_1^* &= \frac{y_{P_1}^* - y_{P_1}}{\sqrt{(x_{P_1}^* - x_{P_1})^2 + (y_{P_1}^* - y_{P_1})^2}} \\
\cos\psi_2^* &= \frac{x_{P_2}^* - x_{P_2}}{\sqrt{(x_{P_2}^* - x_{P_2})^2 + (y_{P_2}^* - y_{P_2})^2}} \\
\sin\psi_2^* &= \frac{y_{P_2}^* - y_{P_2}}{\sqrt{(x_{P_2}^* - x_{P_2})^2 + (y_{P_2}^* - y_{P_2})^2}}
\end{aligned} \tag{9}$$

where the players' optimal aimpoints are

$$\begin{aligned}
x_{E_1}^* = x_{P_1}^* &= \frac{x_{E_1} - \alpha_{11}^2 x_{P_1}}{1 - \alpha_{11}^2} \\
y_{E_1}^* = y_{P_1}^* &= \frac{y_{E_1} - \alpha_{11}^2 y_{P_1} - \alpha_{11} d_{11}}{1 - \alpha_{11}^2} \\
x_{E_2}^* = x_{P_2}^* &= \frac{x_{E_2} - \alpha_{22}^2 x_{P_2}}{1 - \alpha_{22}^2}
\end{aligned}$$

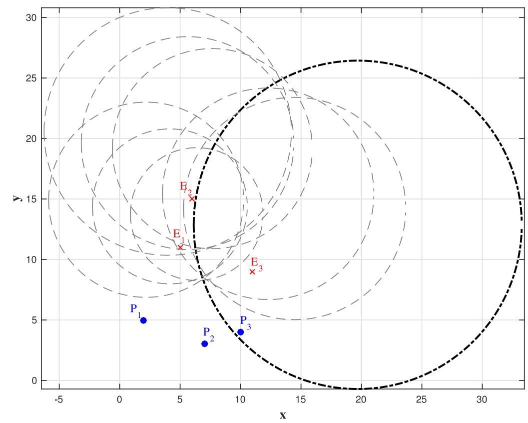

Fig. 1. BDDG example: Three pursuers versus three evaders.

$$y_{E_2}^* = y_{P_2}^* = \frac{y_{E_2} - \alpha_{22}^2 y_{P_2} - \alpha_{22} d_{22}}{1 - \alpha_{22}^2} \tag{10}$$

if $y_{s_1} > y_{s_2}$, and

$$\begin{aligned}
x_{E_1}^* = x_{P_2}^* &= \frac{x_{E_1} - \alpha_{21}^2 x_{P_2}}{1 - \alpha_{21}^2} \\
y_{E_1}^* = y_{P_2}^* &= \frac{y_{E_1} - \alpha_{21}^2 y_{P_2} - \alpha_{21} d_{21}}{1 - \alpha_{21}^2} \\
x_{E_2}^* = x_{P_1}^* &= \frac{x_{E_2} - \alpha_{12}^2 x_{P_1}}{1 - \alpha_{12}^2} \\
y_{E_2}^* = y_{P_1}^* &= \frac{y_{E_2} - \alpha_{12}^2 y_{P_1} - \alpha_{12} d_{12}}{1 - \alpha_{12}^2}
\end{aligned} \tag{11}$$

if $y_{s_2} > y_{s_1}$, where

$$d_{ij} = \sqrt{(x_{E_j} - x_{P_i})^2 + (y_{E_j} - y_{P_i})^2} \text{ for } i, j = 1, 2. \tag{12}$$

Proof: Proof of this theorem is given in the extended version of this article [35]. ■

Remark 1: The optimal headings in (9) are state feedback policies. As such, the pursuers are able to react to nonoptimal strategies by the evaders by continuously recomputing its optimal heading given by (9). When an evader does not follow its prescribed optimal strategy, not only is it captured by the assigned pursuer but the terminal cost/payoff increases with respect to the Value of the game. This is of benefit to the pursuers.

IV. Multiagent Differential Game

In this section, we extend the BDDG to address the case of N pursuers and M evaders, for $N = M$ and for $N > M$. The case $N = M$ is presented first in order to introduce the enumeration of feasible assignments. Next, the more general case $N > M$ is addressed which involves cooperative guidance between two pursuers in order to intercept an evader.

A. Case: $N = M$

We start by enumerating all feasible assignments $\mathcal{A}_\iota$. Feasible assignments mean those assignments where all evaders can be potentially captured. For instance, in Fig. 1, the Apollonius circle between P_1 and E_3 (shown by the bold dot-dashed line) intersects the x-axis; hence, any assignment matching P_1 with E_3 is not feasible. Thus, the feasible assignments in Fig. 1 are $\mathcal{A}_1 : \mu_{11} = \mu_{22} = \mu_{33} = 1$; $\mathcal{A}_2 : \mu_{11} = \mu_{23} = \mu_{32} = 1$; $\mathcal{A}_3 : \mu_{12} = \mu_{21} = \mu_{33} = 1$; and $\mathcal{A}_4 : \mu_{12} = \mu_{23} =$

$\mu_{32} = 1$. In general, the number of feasible assignments is denoted by $\bar{\iota}$ so the assignment index $\iota = 1, \ldots, \bar{\iota}$. Define

$$y_{s_\iota}(\mathbf{x}) = \sum_{j=1}^{M} \mu_{ij}^{\iota} \underline{y}_{ij}(\mathbf{x}). \tag{13}$$

for $\iota = 1, \ldots, \bar{\iota}$, where the assignment variables μ_{ij}^{ι} are specified by the corresponding assignment $\mathcal{A}_\iota$. The optimal assignment variables are denoted by μ_{ij}^{*}.

Theorem 3: Consider the N versus M BDDG where $N = M$, $\alpha_{ij} = v_{E_j}/v_{P_i} < 1$, and $\mathbf{x} \in \mathcal{R}_P$. The Value function is continuously differentiable (except at dispersal surfaces $y_{s_\iota} = y_{s_{\iota'}}$ for any $\iota, \iota' = 1, \ldots, \bar{\iota}$) and it satisfies the HJI equation. The Value function is explicitly given by $V(\mathbf{x}) = \max_\iota y_{s_\iota}(\mathbf{x})$. The corresponding optimal assignment is $\iota^* = \arg\max_\iota y_{s_\iota}(\mathbf{x})$. The optimal state feedback strategies are given by

$$\begin{aligned}
\cos\phi_j^* &= \frac{x_{E_j}^* - x_{E_j}}{\sqrt{(x_{E_j}^* - x_{E_j})^2 + (y_{E_j}^* - y_{E_j})^2}} \\
\sin\phi_j^* &= \frac{y_{E_j}^* - y_{E_j}}{\sqrt{(x_{E_j}^* - x_{E_j})^2 + (y_{E_j}^* - y_{E_j})^2}} \\
\cos\psi_i^* &= \frac{x_{P_i}^* - x_{P_i}}{\sqrt{(x_{P_i}^* - x_{P_i})^2 + (y_{P_i}^* - y_{P_i})^2}} \\
\sin\psi_i^* &= \frac{y_{P_i}^* - y_{P_i}}{\sqrt{(x_{P_i}^* - x_{P_i})^2 + (y_{P_i}^* - y_{P_i})^2}}
\end{aligned} \tag{14}$$

where the optimal aimpoints are

$$\begin{aligned}
x_{E_j}^* &= x_{P_i}^* = \frac{x_{E_j} - \alpha_{ij}^2 x_{P_i}}{1 - \alpha_{ij}^2} \\
y_{E_j}^* &= y_{P_i}^* = \frac{y_{E_j} - \alpha_{ij}^2 y_{P_i} - \alpha_{ij} d_{ij}}{1 - \alpha_{ij}^2}
\end{aligned} \tag{15}$$

for a pair E_j/P_i such that $\mu_{ij}^* = 1$, where

$$d_{ij} = \sqrt{(x_{E_j} - x_{P_i})^2 + (y_{E_j} - y_{P_i})^2} \text{ for } i, j = 1, \ldots, N. \tag{16}$$

B. Case: $N > M$

We now consider the multiagent BDDG with N pursuers and M evaders with $N > M$. Cooperative pursuit by two pursuers against one evader is beneficial for the pursuers because in most cases it will cause capture to occur farther away from the border than in the noncooperative single pursuer single evader case (the assignment of three pursuers to one evader does not bring any extra benefit [35]). Cooperation is also important to prevent escape of an evader compared to one-on-one assignments.

We now apply the cooperative guidance concept in order to obtain the saddle point solution to the multiagent BDDG: When the pursuers outnumber the evaders, cooperation among a group of N pursuers entails the best cooperative assignment together with the cooperatively designed heading strategy in order to maximize the team's payoff. The best strategy by the outnumbered evaders in order to minimize their combined cost is for each one to head to the lowest point in its dominance region which is determined by the optimal assignment of pursuers to evaders. As expected, the solution of the game provides the optimal strategies for each agent.

In order to address simultaneous capture, we consider the following. If an evader E_j can be potentially captured simultaneously by two pursuers P_i and $P_{i'}$, we use E_jP_i and $E_jP_{i'}$ to denote the corresponding Apollonius circles. They are given, respectively, by

$$\begin{aligned}
(x - x_{c_{ij}})^2 + (y - y_{c_{ij}})^2 &= r_{ij}^2 \\
(x - x_{c_{i'j}})^2 + (y - y_{c_{i'j}})^2 &= r_{i'j}^2
\end{aligned} \tag{17}$$

where $x_{c_{ij}} = \frac{1}{1-\alpha_{ij}^2}(x_{E_j} - \alpha_{ij}^2 x_{P_i})$, $y_{c_{ij}} = \frac{1}{1-\alpha_{ij}^2}(y_{E_j} - \alpha_{ij}^2 y_{P_i})$, $r_{ij} = \frac{\alpha_{ij}}{1-\alpha_{ij}^2} d_{ij}$, for i, i'.

In general, the pursuers have different speeds and a third Apollonius circle can be constructed in terms of the positions of P_i and $P_{i'}$, and in terms of their corresponding speed ratio $\alpha_{i'i}$. Without loss of generality, we consider $P_{i'}$ to be the faster of the two pursuers and we define the speed ratio $\alpha_{i'i} = v_{P_i}/v_{P_{i'}} < 1$. The $P_{i'}P_i$ Apollonius circle is given by

$$(x - x_{c_{i'i}})^2 + (y - y_{c_{i'i}})^2 = r_{i'i}^2 \tag{18}$$

where $x_{c_{i'i}} = \frac{1}{1-\alpha_{i'i}^2}(x_{P_i} - \alpha_{i'i}^2 x_{P_{i'}})$, $y_{c_{i'i}} = \frac{1}{1-\alpha_{i'i}^2}(y_{P_i} - \alpha_{i'i}^2 y_{P_{i'}})$, $r_{i'i} = \frac{\alpha_{i'i}}{1-\alpha_{i'i}^2} d_{i'i}$, and $d_{i'i} = \sqrt{(x_{P_i} - x_{P_{i'}})^2 + (y_{P_i} - y_{P_{i'}})^2}$.

Similar to the case $N = M$, $\mathcal{A}_\iota$ for $\iota = 1, \ldots, \bar{\iota}$ denotes the feasible assignments. In order to enumerate the feasible assignments, we consider the choices where simultaneous capture helps the pursuers to increase their payoff. In the simple example in Fig. 2, with $N = 3$, $M = 2$, the feasible assignments are shown in Table I. In this table, the first column represents the assignment index, the second column shows the potential matching to be analyzed in the assignment, and the third column provides the resulting assignment variables.

We will now provide the solution of the N versus M BDDG, for the case $N > M$. Let us define

$$y_{s_\iota}(\mathbf{x}) = \sum_{j=1}^{M} \mu_{ij}^{\iota} \underline{y}_{ij} \tag{19}$$

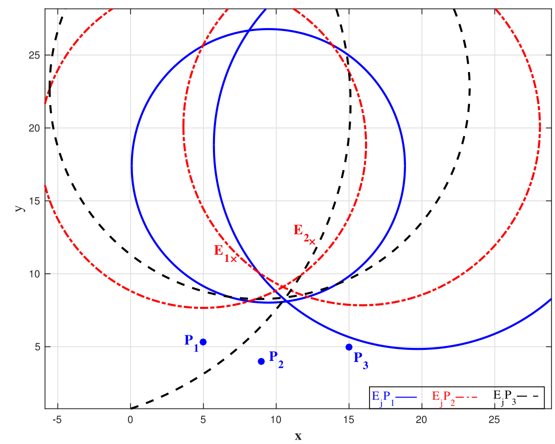

Fig. 2. Three pursuers and two evaders—feasible assignments.

TABLE I
FEASIBLE ASSIGNMENTS FOR $3P$ VERSUS $2E$ EXAMPLE

$\mathcal{A}_\iota$	Potential match	μ_{ij}
$\mathcal{A}_1$	$P_1P_2 \Rightarrow E_1,\ P_3 \Rightarrow E_2$	$\mu_{11} = \mu_{21} = \mu_{32} = 1$
$\mathcal{A}_2$	$P_1 \Rightarrow E_1,\ P_2P_3 \Rightarrow E_2$	$\mu_{11} = \mu_{22} = \mu_{32} = 1$
$\mathcal{A}_3$	$P_1P_2 \Rightarrow E_2,\ P_3 \Rightarrow E_1$	$\mu_{22} = \mu_{31} = 1$
$\mathcal{A}_4$	$P_1 \Rightarrow E_2,\ P_2P_3 \Rightarrow E_1$	$\mu_{12} = \mu_{21} = 1$
$\mathcal{A}_5$	$P_1P_3 \Rightarrow E_1,\ P_2 \Rightarrow E_2$	$\mu_{11} = \mu_{22} = 1$
$\mathcal{A}_6$	$P_1P_3 \Rightarrow E_2,\ P_2 \Rightarrow E_1$	$\mu_{12} = \mu_{32} = \mu_{21} = 1$

where $\underline{y}_{ij}(\mathbf{x})$ is given by (7) if, in assignment $\mathcal{A}_\iota$, $\mu^\iota_{ij} = 1$ holds only for one pursuer i, that is, E_j is captured by only one pursuer. Also define

$$\underline{y}_{ij}(\mathbf{x}) = \frac{F_{ij} - \sqrt{(x_{c_{ij}} - x_{c_{i'j}})^2 G_{ij}}}{D_{ij}} = V_s(\mathbf{x}) \tag{20}$$

if, in assignment $\mathcal{A}_\iota$, $\mu^\iota_{ij} = 1$ holds for two pursuers i, i', that is, E_j is captured simultaneously by two pursuers where

$$\begin{aligned} F_{ij} &= y_{c_{i'i}}(x_{c_{ij}} - x_{c_{i'j}})^2 \\ &\quad - (y_{c_{ij}} - y_{c_{i'j}}) \left(\frac{R_{ij}}{2} - x_{c_{i'i}}(x_{c_{i'j}} - x_{c_{ij}}) \right) \\ G_{ij} &= r^2_{i'i} D_{ij} - \left(\frac{R_{ij}}{2} + x_{c_{i'i}}(x_{c_{ij}} - x_{c_{i'j}}) \right. \\ &\quad \left. + y_{c_{i'i}}(y_{c_{ij}} - y_{c_{i'j}}) \right)^2 \\ D_{ij} &= (x_{c_{ij}} - x_{c_{i'j}})^2 + (y_{c_{ij}} - y_{c_{i'j}})^2 \\ R_{ij} &= r^2_{ij} - r^2_{i'j} - x^2_{c_{ij}} + x^2_{c_{i'j}} - y^2_{c_{ij}} + y^2_{c_{i'j}}. \end{aligned} \tag{21}$$

Theorem 4: Consider the N versus M BDDG where $N > M$, $\alpha_{ij} = v_{E_j}/v_{P_i} < 1$, and $\mathbf{x} \in \mathcal{R}_P$. The Value function is continuously differentiable (except at dispersal surfaces $y_{s_\iota} = y_{s_{\iota'}}$ for any $\iota, \iota' = 1, \ldots, \bar{\iota}$) and it satisfies the HJI equation. The Value function is explicitly given by $V(\mathbf{x}) = \max_\iota y_{s_\iota}(\mathbf{x})$. The corresponding optimal assignment is $\iota^* = \arg\max_\iota y_{s_\iota}(\mathbf{x})$. The optimal state feedback strategies are given by (14). The optimal aimpoints are given by (15) if E_j is captured by only one pursuer and they are given by

$$x^* = x^*_{E_j} = x^*_{P_i} = x^*_{P_{i'}} = \frac{R_{ij} - 2(y_{c_{i'j}} - y_{c_{ij}})V_s(\mathbf{x})}{2(x_{c_{i'j}} - x_{c_{ij}})} \tag{22}$$

and $y^* = y^*_{E_j} = y^*_{P_i} = y^*_{P_{i'}} = V_s(\mathbf{x})$ as defined in (20) if E_j is captured simultaneously by two pursuers.

Proof: Proof of this theorem can be found in the extended version of this article [35]. ∎

Remark 2: In the case where not all evaders can be intercepted, the ideas presented in this article can still be used by the pursuers in order to minimize the damage. This could be in the form of intercepting as many evaders as possible. This is directly related to the solution to the game of kind, that is, whether the border can be protected. Complete protection is automatically given by the solution of the initial assignment if $V > 0$. In more detail, if each $\underline{y}_{ij} > 0$ in $y_{s^*_\iota}$ then all evaders can be captured before reaching the border. If some $\underline{y}_{ij} < 0$ then the best assignment is the one that minimizes the number of evaders reaching the border and the border is only partially protected in such a case.

V. Examples and Extensions

Example. We consider the computation of assignments for relatively large number of agents in this example. Several examples focusing on robustness under uncertain, nonoptimal guidance can be found in [35]. Simulations were run in MATLAB 2017 and a desktop computer with a CPU at 3.60 GHz and 32 GB of memory. In order to obtain the optimal assignment, we compute the list of feasible assignments according to Section IV-A. This requires the computation of all individual and two-on-one cooperative interception coordinates and the associated individual distances (7) and cooperative distances (20). Note that a given distance is used in possibly many different assignments. After unfeasible assignments are detected and discarded, the optimal assignment is obtained from the list of feasible assignments. We consider 18 agents, 12 pursuers, and 6 evaders. For each run, the initial conditions of each pursuer is drawn from a uniform distribution $x \in [0, 20]$ and $y \in [0, 5]$; and for the evaders $x \in [0, 20]$ and $y \in [2, 10]$. The tabulated results are shown in Table II.

TABLE II
Assignments for 12 Pursuers versus 6 Evaders

Type of computation	Coordinates/distance	assignments
Average time (sec)	2.183×10^{-3}	0.957
Minimum time (sec)	1.546×10^{-3}	0.360
Maximum time (sec)	4.803×10^{-3}	1.819

The computation of all possible interception coordinates is fast and the computation time is relatively constant between runs, i.e., a feasible assignment can be computed almost instantaneously. However, the computation of the optimal assignment takes longer in some instances due to the variability of number of feasible assignments under random initial conditions. Future research will focus on approaches and heuristics to decrease the computation time for larger number of agents while obtaining the optimal or a suboptimal assignment among the many feasible options. For instance, greedy algorithms will significantly reduce computation of feasible assignments and associated payoffs, importantly, we are able to obtain a guaranteed payoff for a suboptimal assignment; regardless of the evader's controls, pursuers are guaranteed the obtained payoff.

The saddle-point strategies derived in this article are robust against unknown, nonoptimal strategies implemented by the opponent. For instance, if the evaders do not play optimally, the pursuers can guarantee themselves a better payoff by keeping the initial optimal assignment and updating the optimal state-feedback strategies obtained in Theorems 3 and 4. In the case where the pursuers do not play optimally, the evaders can also take advantage and reduce their cost. Additional details and examples can be found in [35].

Extensions. The differential game with two teams and multiple players could be extended to consider additional facets of combat scenarios: Decoys, players willing to sacrifice to benefit their teammates, and players with different levels of importance will be analyzed in future research. An important extension addressed in this section is to analyze the same BDDG but without the prior commitment restriction. We will focus on the particular case considered in Section III of two pursuers versus two evaders. The following is a corollary to Theorem 2.

Corollary 1: Consider the 2 versus 2 BDDG (1)–(5) with $\alpha_{ij} = v_{E_j}/v_{P_i} < 1$, and where $\mathbf{x} \in \mathcal{R}_P$ and the pursuers do not commit to their initial assignment. The pursuers' strategies with commitment given by

$$\begin{aligned} \cos\psi^*_1 &= \frac{x^*_{P_1} - x_{P_1}}{\sqrt{(x^*_{P_1} - x_{P_1})^2 + (y^*_{P_1} - y_{P_1})^2}} \\ \sin\psi^*_1 &= \frac{y^*_{P_1} - y_{P_1}}{\sqrt{(x^*_{P_1} - x_{P_1})^2 + (y^*_{P_1} - y_{P_1})^2}} \\ \cos\psi^*_2 &= \frac{x^*_{P_2} - x_{P_2}}{\sqrt{(x^*_{P_2} - x_{P_2})^2 + (y^*_{P_2} - y_{P_2})^2}} \\ \sin\psi^*_2 &= \frac{y^*_{P_2} - y_{P_2}}{\sqrt{(x^*_{P_2} - x_{P_2})^2 + (y^*_{P_2} - y_{P_2})^2}} \end{aligned} \tag{23}$$

are robust state-feedback strategies for the game without commitment, where

$$\begin{aligned} x^*_{P_1} &= \frac{x_{E_1} - \alpha^2_{11} x_{P_1}}{1 - \alpha^2_{11}} \\ y^*_{P_1} &= \frac{y_{E_1} - \alpha^2_{11} y_{P_1} - \alpha_{11} d_{11}}{1 - \alpha^2_{11}} \end{aligned}$$

$$x^*_{P_2} = \frac{x_{E_2} - \alpha^2_{22} x_{P_2}}{1 - \alpha^2_{22}}$$
$$y^*_{P_2} = \frac{y_{E_2} - \alpha^2_{22} y_{P_2} - \alpha_{22} d_{22}}{1 - \alpha^2_{22}}$$

if $y_{s_1} > y_{s_2}$, and

$$x^*_{P_2} = \frac{x_{E_1} - \alpha^2_{21} x_{P_2}}{1 - \alpha^2_{21}}$$
$$y^*_{P_2} = \frac{y_{E_1} - \alpha^2_{21} y_{P_2} - \alpha_{21} d_{21}}{1 - \alpha^2_{21}}$$
$$x^*_{P_1} = \frac{x_{E_2} - \alpha^2_{12} x_{P_1}}{1 - \alpha^2_{12}}$$
$$y^*_{P_1} = \frac{y_{E_2} - \alpha^2_{12} y_{P_1} - \alpha_{12} d_{12}}{1 - \alpha^2_{12}}$$

if $y_{s_2} > y_{s_1}$, where d_{ij} is given by (12). The pursuers' guaranteed payoff is $y_s(\mathbf{x}) = y_{s_1}(\mathbf{x})$ if $y_{s_1} > y_{s_2}$ and $y_s(\mathbf{x}) = y_{s_2}(\mathbf{x})$ if $y_{s_2} > y_{s_1}$ where y_{s_1} and y_{s_2} are given by (8).

Remark 3: By relaxing the restriction regarding initial commitment, it is also possible to extend the region $\mathcal{R}_P$ by cooperation and switch. For example, when initially an evader is able to reach the border if only one pursuer is assigned to it, then two pursuers cooperate in order to decrease the region of dominance of the evader so that he is intercepted farther away from the border. This makes possible for one of them to eventually single-handedly capture the evader while the other one is free to switch its assignment and to pursue a different opponent.

VI. Conclusion

In this article, N versus M pursuit-evasion games were considered and the joint optimal assignment of pursuers to evaders and optimal pursuit and evasion strategies in a multiplayer engagement has been analyzed. The two-team multiplayer scenario of border defense was correctly posed as a pursuit-evasion differential game. Unlike classical differential games, where only state feedback strategies are sought, the results of this article show how to solve this hybrid differential game and provide the complete solution over the joint set of continuous time state feedback strategies and discrete (binary) assignment variables. This challenge was addressed by employing a simple motion kinematic model of the agents. In addition to considering more realistic models, future work will address other shortcomings and limitations of this work such as switching assignments, containment, and capture in sequence.

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