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Matching-based capture strategies for 3D heterogeneous multiplayer reach-avoid differential games ☆

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ABSTRACT

This paper studies a 3D multiplayer reach-avoid differential game with a goal region and a play region. Multiple pursuers defend the goal region by consecutively capturing multiple evaders in the play region. The players have heterogeneous moving speeds and the pursuers have heterogeneous capture radii. Since this game is hard to analyze directly, we decompose the whole game as many subgames which involve multiple pursuers and only one evader. Then, these subgames are used as a building block for the pursuer–evader matching. First, for multiple pursuers and one evader, we introduce an evasion space (ES) method characterized by a potential function to construct a guaranteed pursuer winning strategy. Then, based on this strategy, we develop conditions to determine whether a pursuit team can guard the goal region against one evader. It is shown that in 3D, if a pursuit team is able to defend the goal region against an evader, then at most three pursuers in the team are necessarily needed. We also compute the value function of the Hamilton–Jacobi–Isaacs (HJI) equation for a special subgame of degree. To capture the maximum number of evaders in the open-loop sense, we formulate a maximum bipartite matching problem with conflict graph (MBMC). We show that the MBMC is NP-hard and design a polynomial-time constant-factor approximation algorithm to solve it. Finally, we propose a receding horizon strategy for the pursuit team where in each horizon an MBMC is solved and the strategies of the pursuers are given. We also extend our results to the case of a bounded convex play region where the evaders escape through an exit. Two numerical examples are provided to demonstrate the results.

1. Introduction

Problem description and motivation: Consider a multiplayer reach-avoid differential game in 3D space where a plane divides the space into two disjoint regions, i.e., the play region and the goal region. An evasion team with multiple evaders of different speeds, initially lying in the play region, tries to send as many of its team members as possible into the goal region. Meanwhile, a pursuit team with multiple pursuers of different speeds and capture radii, initially spreading over the space, aims to guard the goal region by capturing the evaders. From a different point of view, this is also equivalent to a game where the evaders try to escape from the play region and avoid adversaries and dynamic obstacles formulated as a pursuit team. As an extension, we also consider a game played in a bounded convex region in 3D with a planar exit. This game is hard to solve directly because of the high dimension and multiple stages (Agharkar et al., 2015; Chen et al., 2017; Liu et al., 2013; Yan et al., 2020). This paper provides a matching-based capture strategy for the pursuit team to capture the maximum number of evaders, partially dealing with this game.

This problem is motivated by robotic applications, including the robot competition, dynamic collision avoidance and region surveillance (Bullo et al., 2011; Duan et al., 2020; Oyler et al., 2016). For example, in region protection games, multiple pursuers are used to intercept multiple adversarial intruders in Chen et al. (2017). In path planning, a group of vehicles aim to get into some target region or escape from a bounded region through an

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exit, while avoiding dangerous situations, such as collisions with moving obstacles, as in Yan et al. (2019).

Literature review: The problem in this paper is related to games such as lifeline games, two-target differential games, reach-avoid differential games and target guarding differential games.

Two-player lifeline games were introduced in Isaacs (1965, Section 9.5) in which the evader aims to reach a so-called lifeline (a straight line) prior to capture, and then were revisited by Yan et al. (2019), where only one evader and two equal-speed pursuers with zero capture radii were considered. As for two-target differential games, the first quantitative and qualitative results appeared in Blaquière et al. (1969), where the two-player case was analyzed. In Getz and Pachter (1981), Olsder and Breakwell (1974) and Pachter and Getz (1980), several variations of two-player games were considered such as complex dynamics and targets of different shapes. However, it is hard to extend these results to multiplayer cases due to the difficulty in solving the Hamilton–Jacobi–Isaacs (HJI) equation.

Reach-avoid differential games were first discussed by Margellos and Lygeros (2011), Mitchell et al. (2005) and Zhou et al. (2012), and then extensive studies including many variations and practical applications appeared (Huang et al., 2015). The methods in these works involve solving an HJI equation, whose analytical solution is hard to derive if not impossible and thus numerical tools are often resorted to. The common numerical tool is the level set method by means of gridding. This method suffers from the curse of dimensionality, so various techniques have been proposed, including approximation function (Chipade & Panagou, 2019), system decomposition (Chen et al., 2018), and cone programming (Lorenzetti et al., 2018). Most of these works focus on approximation, two-player or open loop games (Liu et al., 2013) due to the exponential growth of the computation as the size of the states increases. For multiplayer cases, Chen et al. (2017) greatly reduced the computation burden by creating a number of straight lines in 2D to output matching pairs. Our work will provide an analytical method for a class of multiplayer reach-avoid differential games.

The two-player target guarding differential games were also first studied in Isaacs (1965), and revisited by Mohanan et al. (2019) with the goal of real-time implementation, where the one-to-one case and zero capture radii were considered. Recently, multiplayer cases have received increasing attention from algorithms to game setups. For example, Shishika and Paley (2019) used the swarming behavior of male mosquitoes to design the motion strategy for multiple guardians against fast intruders in area protection, where the goal region is a point and the strategies of the intruders are predefined. Multiplayer scenarios of special setup were explored by Agharkar et al. (2015) and Shishika and Kumar (2018), where Shishika and Kumar (2018) restrict the motion of pursuers on the boundary of target set, and Agharkar et al. (2015) study the escape from a circular disk; all these works focused on zero capture radii.

To the best of our knowledge, there are only a few works concerning multiplayer reach-avoid differential games. Chen et al. in Chen et al. (2017) considered a multiplayer reach-avoid differential game where no cooperation except the matching among pursuers exists and thus provided a suboptimal solution, where the basic one-to-one solution is numerical and approximate. In Yan et al. (2020), the authors focused on the case of zero capture radius and homogeneous players, and they solved the task assignment problem by a 0–1 integer programming without complexity analysis and polynomial-time algorithms. Agharkar et al. (2015) and Shishika and Kumar (2018) limited their attention to the case of zero capture radius and non-fully competitive strategies. It is desirable to develop an analytical and efficient method to analyze multiplayer pursuit-evasion games with heterogeneous capture radii, as discussed in Chen et al. (2017), Garcia, Casbeer et al. (2019), Hayoun and Shima (2017) and Sun et al. (2019). Our work will consider pursuit cooperation more deeply, and allow heterogeneous nonzero capture radii and speeds.

Another focus of this work is on the *constrained matching problems* (Itai et al., 1978) or *maximum matching problems with conflicts (MMPC)*. Compared with the traditional matching problem, the MMPC introduces new constraints specifying that some particular edges cannot coexist in the resulting matching. These constraints are usually defined by a conflict graph. Conflict graphs have been considered in many combinatorial optimization problems such as knapsack problem (Pferschy & Schauer, 2017) and minimum spanning tree problems (Zhang et al., 2011). The first work introducing conflicts into matching problems was presented by Itai et al. (1978), where a constrained bipartite matching problem was considered. Then, Thomas (Thomas, 2016, Chapter 4) revisited the constrained matching problem, and summarized the complexity of different variations based on fixed and variable parameters. An important work was by Darmann et al. (2011), where they established primary complexity results for several variations. Then, these results were extended later on in Öncan et al. (2013) where the authors proposed additional complexity results, identified special polynomially solvable cases, and also designed several heuristic algorithms. However, none of the aforementioned results addresses our problems. Thus, to the best of our knowledge, this paper is the first that attempts to prove the NP-hardness and design approximation algorithms for the MBMC.

Contributions: In this paper, we study the cooperative strategies for multiple pursuers to guard a 3D region against multiple evaders. The main contributions are as follows. First, heterogeneous players are considered, where heterogeneity refers to different speeds for players and different capture radii for pursuers. Second, we decompose the whole game into many subgames which involve multiple pursuers and one evader. For each subgame under some initial configurations, a guaranteed pursuer winning strategy is designed. Third, by this pursuer winning strategy, we find the conditions to determine the game winner between a pursuit team and one evader. We further prove that in 3D if a pursuit team can guard the goal region against one evader, then at most three pursuers in the team are necessarily needed. Fourth, we provide a new perspective on solving HJI equations within the challenging context of multiplayer games (Fisac et al., 2015; Mylvaganam et al., 2017) by showing that for a special subgame the associated HJI equation can be solved via convex programming. Fifth, for the whole game, we piece together the outcomes of all subgames. Then, in order to capture the maximum number of evaders in an open-loop manner based on the derived results, we propose a new class of constrained matching problems, i.e., *maximum bipartite matching with conflict graph*. We prove that this class of constrained matching problem is NP-hard. Finally, we design the polynomial-time constant-factor Sequential Matching Algorithm to approximately solve the matching problem and show its APX-completeness. Based on the matching, a receding horizon capture strategy is proposed. An extension where all players play in a bounded convex region is also investigated. Compared with Agharkar et al. (2015), Chen et al. (2017), Shishika and Kumar (2018), Shishika and Paley (2019) and Yan et al. (2020), we consider more practical cases when the pursuers have different capture radii and speeds, can closely cooperate with each other, and have no knowledge of the strategies of the evaders. The existing techniques cannot be applied directly to these cases.

Paper organization: We introduce the multiplayer reach-avoid differential games in Section 2. Section 3 presents the main results of the case where multiple pursuers defend against one evader. In

Section 4, by solving a constrained matching problem, we design a receding horizon strategy for the pursuers. An extension to the case of a bounded convex play region with an exit is discussed in Section 5. Numerical results are presented in Section 6, and we conclude the paper in Section 7.

Notation: Let $\mathbb{0}_{m\times n}$ be an $m \times n$ zero matrix. For any finite set S, the cardinality of S is given by $|S|$, the set of non-empty subsets is given by $[S]^+$, and the set of non-empty subsets with cardinality less than or equal to i for $1 \le i \le |S|$ is denoted by $[S]^i$. For any subset S of a topological space X, denote its boundary by ∂S. Let $\mathbb{R}$ and $\mathbb{R}^+$ be the set of reals and positive reals, respectively. Let $\mathbb{R}^n$ be the set of n-dimensional real column vectors and $\|\cdot\|_2$ be the Euclidean norm. Denote the unit sphere in $\mathbb{R}^3$ by $\mathbb{S}^2$. Let $\mathbf{x} = [x \ \ y \ \ z]^\top \in \mathbb{R}^3$.

2. Problem description

2.1. Multiplayer reach-avoid differential games

Consider a reach-avoid differential game with $N_p + N_e$ players, where there are N_p pursuers $\mathcal{P} = \{P_1, \dots, P_{N_p}\}$ and N_e evaders $\mathcal{E} = \{E_1, \dots, E_{N_e}\}$. The players are assumed to be mass points and they have simple motion as Isaacs states (Isaacs, 1965), i.e., they are holonomic. The game is played in the 3D Euclidean space $\mathbb{R}^3$, where a plane $\mathcal{T}$ divides the game space $\mathbb{R}^3$ into two disjoint subregions Ω_{goal} and Ω_{play}. The mathematical descriptions of $\mathcal{T}$, Ω_{goal} and Ω_{play} are given by $\{\mathbf{x} \in \mathbb{R}^3 \mid z = 0\}$, $\{\mathbf{x} \in \mathbb{R}^3 \mid z \le 0\}$ and $\{\mathbf{x} \in \mathbb{R}^3 \mid z > 0\}$, respectively. Let $\mathbf{x}_{P_i}(t) = [x_{P_i}(t) \ y_{P_i}(t) \ z_{P_i}(t)]^\top \in \mathbb{R}^3$ and $\mathbf{x}_{E_j}(t) = [x_{E_j}(t) \ y_{E_j}(t) \ z_{E_j}(t)]^\top \in \mathbb{R}^3$ be the positions of P_i and E_j at time t, respectively. The dynamics of the players are described by the system for $t \ge 0$:

$$\begin{aligned} \dot{\mathbf{x}}_{P_i}(t) &= v_{P_i}\mathbf{u}_{P_i}(t), \quad \mathbf{x}_{P_i}(0) = \mathbf{x}_{P_i}^0, \quad P_i \in \mathcal{P}, \\ \dot{\mathbf{x}}_{E_j}(t) &= v_{E_j}\mathbf{u}_{E_j}(t), \quad \mathbf{x}_{E_j}(0) = \mathbf{x}_{E_j}^0, \quad E_j \in \mathcal{E}, \end{aligned} \tag{1}$$

where $\mathbf{x}_{P_i}^0$ and $\mathbf{x}_{E_j}^0$ are the initial positions of P_i and E_j, and $v_{P_i} \in \mathbb{R}^+$ and $v_{E_j} \in \mathbb{R}^+$ denote the speeds of P_i and E_j, respectively. The control inputs at time t for P_i and E_j are their respective instantaneous headings $\mathbf{u}_{P_i}(t)$ and $\mathbf{u}_{E_j}(t)$, which satisfy the constraint $\mathbf{u}_{P_i}(t), \mathbf{u}_{E_j}(t) \in \mathbb{S}^2$. Note that all players are allowed to change their orientations instantaneously. For notational simplicity, the time t will be omitted hereafter.

Suppose that the pursuer P_i has capture radius $r_i \ge 0$. The evader E_i is captured as soon as his distance from at least one of pursuers becomes equal to the corresponding capture radius. The capture set of the pursuit team is defined by $\mathcal{C} := \cup_{i=1}^{N_p}\mathcal{C}_i$, where $\mathcal{C}_i$ is the capture set of pursuer P_i and is given by $\{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x} - \mathbf{x}_{P_i}\|_2 \le r_i\}$. Assume that the number of pursuers remains constant, and the pursuers chase the evaders until all evaders in the play region Ω_{play} are captured.

The evasion team tries to send as many evaders as possible into Ω_{goal}, while the pursuit team aims at capturing as many evaders as possible before they enter Ω_{goal}. This paper presents a receding horizon capture strategy for the pursuit team to capture as many evaders as possible. The game setups are shown in Fig. 1. Next, we present the adopted information structure and summarize the assumptions we shall need throughout this paper.

2.2. Information structure and assumptions

As is the usual convention in differential game theory, the information available to each player plays an important role in generating optimal strategies (Başar & Olsder, 1999). The non-anticipative information structure is often employed to guarantee that the game admits a value (see for example, Elliott and Kalton (1972), Margellos and Lygeros (2011), Mitchell et al. (2005) and Yan et al. (2020)). However, under this information structure, one player is allowed to make decisions about its current input with all the information of current states and the opposing player' current input, leading to an instantiation of the Stackelberg game (Başar & Olsder, 1999). For our game, it is more likely that the pursuer or the evader has no access to the opponent's current input, due to their adversarial goals. Thus, in this paper, we adopt the state feedback information structure, where each player chooses its current input, $\mathbf{u}_{P_i}$ or $\mathbf{u}_{E_j}$, based on the current value of the information set $\{\mathbf{x}_{P_i}, \mathbf{x}_{E_j}\}_{P_i\in\mathcal{P}, E_j\in\mathcal{E}}$. It is worth noting that under feedback strategies the differential game may not necessarily admit a value (Cardaliaguet, 1996, 1997; Krasovski et al., 1987, Section 3). In the following, we are able to construct a class of feedback strategies which are sufficient to guarantee the pursuit winning for certain initial configurations.

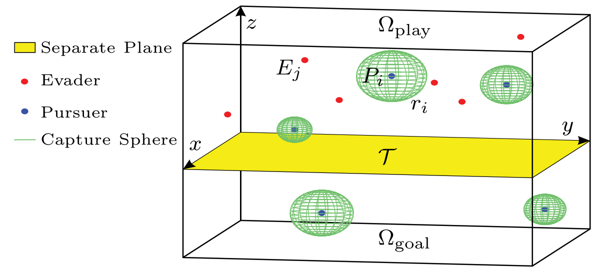

Fig. 1. The 3D multiplayer reach-avoid differential games. The pursuit team with multiple pursuers (blue circles) aims to capture as many evaders (red circles) as possible before these evaders penetrate a separating plane $\mathcal{T}$ (yellow) and enter the goal region Ω_{goal}. Each player is allowed to have different speed and each pursuer has a possibly different capture radius (green sphere). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Assume that the initial configurations satisfy the following conditions, which can focus our attention on the main situations and remove some technical problems (eg., two pursuers initially lie at the same position).

Assumption 2.1 (*Initial Deployment*)**.** The initial positions of all players satisfy the following four conditions:

(1) $\|\mathbf{x}_{P_i}^0 - \mathbf{x}_{P_j}^0\|_2 > 0$ for all $P_i, P_j \in \mathcal{P}, P_i \neq P_j$;
(2) $\|\mathbf{x}_{E_i}^0 - \mathbf{x}_{E_j}^0\|_2 > 0$ for all $E_i, E_j \in \mathcal{E}, E_i \neq E_j$;
(3) $\|\mathbf{x}_{E_j}^0 - \mathbf{x}_{P_i}^0\|_2 > r_i$ for all $P_i \in \mathcal{P}, E_j \in \mathcal{E}$;
(4) $\mathbf{x}_{P_i}^0 \in \mathbb{R}^3$ for all $P_i \in \mathcal{P}$ and $\mathbf{x}_{E_j}^0 \in \Omega_{\text{play}}$ for all $E_j \in \mathcal{E}$.

In Assumption 2.1, (1) and (2) guarantee that all players play the game from different initial positions, (3) ensures that evaders are not captured by the pursuers initially, and (4) says that every evader initially lies in Ω_{play} while every pursuer may start from any position.

Most of current works on multiplayer reach-avoid differential games focus on homogeneous players in both teams (Agharkar et al., 2015; Yan et al., 2019, 2020). We instead consider heterogeneous players, i.e., different speeds and capture radii. We focus on faster pursuers.

Assumption 2.2 (*Speed Ratio*)**.** Suppose the speed ratio $\alpha_{ij} = v_{P_i}/v_{E_j} > 1$ for all $P_i \in \mathcal{P}$ and $E_j \in \mathcal{E}$.

Note that if $v_{P_i} < v_{E_j}$, and also if the point capture is considered, i.e., $r_i = 0$, then the evader E_i can always win against P_i. Thus, most of the existing works on reach-avoid differential games focus on the case $v_{P_i} \ge v_{E_j}$ and $r_i = 0$, such as Garcia et al. (2020), Garcia, Moll et al. (2019), Liu et al. (2013), Mohanan et al. (2019), Shishika and Kumar (2018) and Yan et al. (2019). As for the case $v_{P_i} < v_{E_j}$ and $r_i > 0$ or the case $v_{P_i} = v_{E_j}$, it is possible for the pursuer P_i to capture E_j; however, this requires the separate analysis and we leave it for future work.

3. Multiple pursuers versus one evader

3.1. Problem statement

It is hard to analyze the whole game directly (Chen et al., 2017), so we will decompose the whole game as many subgames which involve multiple pursuers and one evader. Then, these subgames are used as a building block for the pursuer–evader matching in the next section. In Chen et al. (2017), the authors only consider the subgames between one pursuer and one evader before the matching. Next, these subgames will be discussed.

We first give several critical definitions. For any $s \in [\{1, \ldots, N_p\}]^+$, let $P_s = \{P_i \in \mathcal{P} \mid i \in s\}$ be an element of $[\mathcal{P}]^+$, and we refer to P_s as a pursuit coalition containing pursuer P_i if the subscript satisfies $i \in s$. In other word, P_s is a pursuit coalition with its members specified by the index set s. For $P_s \in [\mathcal{P}]^+$, we stack the states and control inputs of all pursuers in P_s into $\mathbf{x}_s$ and $\mathbf{u}_s$ respectively. Let $\mathbf{x}_s^0$ be the initial state of $\mathbf{x}_s$.

For the subgame between a pursuit coalition P_s and an evader E_j, the winning conditions are defined as follows. The pursuit coalition P_s wins if it can capture E_j before the latter reaches Ω_{goal}; otherwise, the evader E_j wins. The capture means $\mathbf{x}_{E_j} \in \mathcal{C}_s$, where $\mathcal{C}_s := \cup_{i\in s}\mathcal{C}_i$.

This section will address the following problems.

Problem 3.1. Consider a pursuit coalition P_s and an evader E_j. Given $\mathbf{x}_s$ and $\mathbf{x}_{E_j}$, find the conditions to determine which one can win the game. If P_s can win the game, what are the strategies to be adopted by the pursuers in P_s to guarantee this winning?

Problem 3.2. Consider a pursuit coalition P_s and an evader E_j. If P_s can win against E_j, how many pursuers at most in P_s are necessary to guarantee this winning?

3.2. Evasion space

Next, we introduce the tool we will use to deal with the above problems, and then discuss its properties for later proofs. We consider the concept introduced in Section 6.7 in Isaacs' book (Isaacs, 1965) as follows.

Definition 3.1 (*Evasion Space*). Given any $P_s \in [\mathcal{P}]^+$ and $E_j \in \mathcal{E}$, the evasion space (ES) $\mathbb{E}(s,j)$ is the set of positions in $\mathbb{R}^3$ that E_j can reach without being captured by P_s, regardless of P_s' control input, and let the surface $\partial\mathbb{E}(s,j)$ which bounds the space $\mathbb{E}(s,j)$ be designated by the BES (boundary of evasion space).

We define a class of potential functions, which will be used to characterize the ES.

Definition 3.2 (*Potential Function*). Given $\mathbf{x}_{P_i}$ and $\mathbf{x}_{E_j}$ satisfying $\|\mathbf{x}_{E_j} - \mathbf{x}_{P_i}\|_2 > r_i$, define the potential function $f_{ij}(\mathbf{x}) : \mathbb{R}^3 \mapsto \mathbb{R}$ associated with P_i and E_j with speed ratio α_{ij}, as follows

$$f_{ij}(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_{P_i}\|_2 - \alpha_{ij}\|\mathbf{x} - \mathbf{x}_{E_j}\|_2 - r_i, \tag{2}$$

whose gradient with respect to $\mathbf{x}$ is denoted by $\nabla f_{ij}(\mathbf{x}) \in \mathbb{R}^3$, and given by

$$\nabla f_{ij}(\mathbf{x}) = \frac{\mathbf{x} - \mathbf{x}_{P_i}}{\|\mathbf{x} - \mathbf{x}_{P_i}\|_2} - \alpha_{ij}\frac{\mathbf{x} - \mathbf{x}_{E_j}}{\|\mathbf{x} - \mathbf{x}_{E_j}\|_2}, \tag{3}$$

when $\mathbf{x} \neq \mathbf{x}_{P_i}$ and $\mathbf{x} \neq \mathbf{x}_{E_j}$.

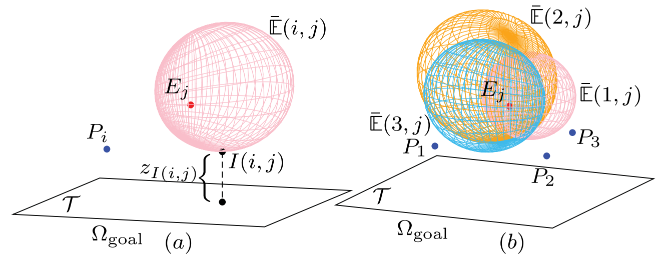

Fig. 2. Evasion space (ES) and interception point. (*a*) Single-pursuer case: The closure of ES $\bar{\mathbb{E}}(i,j)$ (pink) associated with a pursuer P_i and an evader E_j is a strictly convex set, and the related interception point $I(i,j)$ is the unique point in $\bar{\mathbb{E}}(i,j)$ that is closest to Ω_{goal}. (*b*) Multiple-pursuer case: The closure of ES $\bar{\mathbb{E}}(s,j)$ associated with a pursuit coalition $P_s = \{P_1, P_2, P_3\}$ and an evader E_j is the intersection set of three one-to-one ESs, i.e., $\cap_{i\in s}\bar{\mathbb{E}}(i,j)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Thus, according to Section 6.7 in Isaacs' book (Isaacs, 1965), the ES $\mathbb{E}(i,j)$ and the BES $\partial\mathbb{E}(i,j)$ for P_i and E_j can be respectively given by

$$\begin{aligned}\mathbb{E}(i,j) &= \left\{\mathbf{x} \in \mathbb{R}^3 \mid f_{ij}(\mathbf{x}) > 0\right\},\\ \partial\mathbb{E}(i,j) &= \left\{\mathbf{x} \in \mathbb{R}^3 \mid f_{ij}(\mathbf{x}) = 0\right\}.\end{aligned} \tag{4}$$

Thus, the closure of the ES $\mathbb{E}(i,j)$, denoted by $\bar{\mathbb{E}}(i,j)$, is given by $\bar{\mathbb{E}}(i,j) = \left\{\mathbf{x} \in \mathbb{R}^3 \mid f_{ij}(\mathbf{x}) \geq 0\right\}$, shown in Fig. 2(a). Next, we prove that this closure has good features.

Lemma 3.1 (*ES for One Pursuer*)**.** *For the ES $\mathbb{E}(i,j)$ with respect to $P_i \in \mathcal{P}$ and $E_j \in \mathcal{E}$, its closure $\bar{\mathbb{E}}(i,j)$ is bounded and strictly convex.*

Proof. We build a new polar coordinate system with $\mathbf{x}_{E_j}$ as the origin, and let $\mathbf{x} = \mathbf{x}_{E_j} + \rho\mathbf{e}$, where $\rho \in \mathbb{R}^+$ and $\mathbf{e} \in \mathbb{S}^2$. We parameterize $\mathbf{e}$ by

$$\begin{aligned}\mathbf{e} =& \big(\cos(\psi+\psi_0)\cos(\theta+\theta_0), \cos(\psi+\psi_0)\sin(\theta+\theta_0),\\ &\sin(\psi+\psi_0)\big),\\ &\theta\in[0,\pi], \theta_0\in[0,\pi], \psi\in[0,2\pi], \psi_0\in[0,2\pi],\end{aligned}$$

where θ and ψ are rotations with respect to positive x-axis and x–y-plane respectively, and θ_0 and ψ_0 are initial rotations with respect to the original coordinates. By (2) and (4), the boundary $\partial\mathbb{E}(i,j)$, i.e., $f_{ij}(\mathbf{x}) = 0$, in this polar coordinates becomes

$$\begin{aligned}&\|\mathbf{x}_{E_j} + \rho\mathbf{e} - \mathbf{x}_{P_i}\|_2 - \alpha_{ij}\rho - r_i = 0\\ &\Rightarrow \rho = \frac{1}{\alpha_{ij}^2 - 1}\big(h_1(\theta,\psi) + h_2(\theta,\psi)\big),\end{aligned} \tag{5}$$

where two scalar functions $h_1(\theta,\psi)$ and $h_2(\theta,\psi)$ are

$$\begin{aligned}h_1(\theta,\psi) &= (\mathbf{x}_{E_j} - \mathbf{x}_{P_i})^\top\mathbf{e} - \alpha_{ij}r_i,\\ h_2(\theta,\psi) &= \sqrt{h_1^2(\theta,\psi) + (\alpha_{ij}^2 - 1)(\|\mathbf{x}_{E_j} - \mathbf{x}_{P_i}\|_2^2 - r_i^2)}.\end{aligned} \tag{6}$$

In deriving (5) and (6), we have used the fact that $\alpha_{ij} > 1$ and $\|\mathbf{x}_{E_j} - \mathbf{x}_{P_i}\|_2 > r_i$, which also implies that $h_2 > 0$. From (5), we have that ρ is bounded and $\rho > 0$, and thus $\bar{\mathbb{E}}(i,j)$ is bounded.

Regarding the strict convexity of $\bar{\mathbb{E}}(i,j)$, according to the definition of $h_2(\theta,\psi)$ in (6), we have

$$\begin{aligned}\frac{\partial h_2}{\partial\psi} &= \frac{h_1}{\sqrt{h_1^2 + (\alpha_{ij}^2 - 1)(\|\mathbf{x}_{E_j} - \mathbf{x}_{P_i}\|_2^2 - r_i^2)}}\frac{\partial h_1}{\partial\psi}\\ &= \frac{h_1}{h_2}\frac{\partial h_1}{\partial\psi}.\end{aligned} \tag{7}$$

It follows from (5) that by fixing θ, the first and second order partial derivatives of ρ with respect to ψ are

$$\begin{aligned}\frac{\partial\rho}{\partial\psi}&=\frac{1}{\alpha_{ij}^2-1}\frac{h_2+h_1}{h_2}\frac{\partial h_1}{\partial\psi},\\ \frac{\partial^2\rho}{\partial\psi^2}&=\frac{1}{\alpha_{ij}^2-1}\Big(\frac{h_2+h_1}{h_2}\frac{\partial^2 h_1}{\partial\psi^2}+\frac{h_2^2-h_1^2}{h_2^3}\big(\frac{\partial h_1}{\partial\psi}\big)^2\Big),\end{aligned}\tag{8}$$

where (7) is used. Then, by (5) and (8), we have

$$\begin{aligned}&\rho^2+2\big(\frac{\partial\rho}{\partial\psi}\big)^2-\rho\frac{\partial^2\rho}{\partial\psi^2}\\ &=\frac{(h_1+h_2)^2}{(\alpha_{ij}^2-1)^2}+2\frac{1}{(\alpha_{ij}^2-1)^2}\Big(\frac{h_2+h_1}{h_2}\frac{\partial h_1}{\partial\psi}\Big)^2\\ &\quad-\frac{h_1+h_2}{\alpha_{ij}^2-1}\frac{1}{\alpha_{ij}^2-1}\Big(\frac{h_2+h_1}{h_2}\frac{\partial^2 h_1}{\partial\psi^2}+\frac{h_2^2-h_1^2}{h_2^3}\big(\frac{\partial h_1}{\partial\psi}\big)^2\Big)\\ &=\frac{(h_2+h_1)^2}{(\alpha_{ij}^2-1)^2}\Big(1-\frac{1}{h_2}\frac{\partial^2 h_1}{\partial\psi^2}+\frac{h_2+h_1}{h_2^3}\big(\frac{\partial h_1}{\partial\psi}\big)^2\Big)\\ &\geq\frac{(h_2+h_1)^2}{(\alpha_{ij}^2-1)^2}\Big(1-\frac{1}{h_2}\frac{\partial^2 h_1}{\partial\psi^2}\Big).\end{aligned}$$

Next, we prove that $h_2>\frac{\partial^2 h_1}{\partial\psi^2}$. Note that by (6), $h_2>0$ and $\frac{\partial^2 h_1}{\partial\psi^2}=-(\mathbf{x}_{E_j}-\mathbf{x}_{P_i})^\top\mathbf{e}=-h_1-\alpha_{ij}r_i$. On the one hand, if $h_1>-\alpha_{ij}r_i$, then we have $h_2>0>\frac{\partial^2 h_1}{\partial\psi^2}$. On the other hand, if $h_1\leq-\alpha_{ij}r_i$, then we have

$$\begin{aligned}&h_2^2-\big(\frac{\partial^2 h_1}{\partial\psi^2}\big)^2\\ &\quad=(\alpha_{ij}^2-1)(\|\mathbf{x}_{E_j}-\mathbf{x}_{P_i}\|_2^2-r_i^2)-\alpha_{ij}r_i(2h_1+\alpha_{ij}r_i)\\ &\quad>-\alpha_{ij}r_i(2h_1+\alpha_{ij}r_i)\geq\alpha_{ij}^2r_i^2\geq 0.\end{aligned}$$

Thus, we have $\rho^2+2\big(\frac{\partial\rho}{\partial\psi}\big)^2-\rho\frac{\partial^2\rho}{\partial\psi^2}>0$ for all ψ. By Lemma A.1, $\bar{\mathbb{E}}(i,j)$ is strictly convex for any fixed θ. Also note that we can take any $\theta_0\in[0,\pi]$ and $\psi_0\in[0,2\pi]$ as the initial rotations. Thus, by taking any admissible θ_0 and ψ_0 and then considering all θ in $[0,\pi]$, we obtain that $\bar{\mathbb{E}}(i,j)$ is strictly convex. Here, it is worth emphasizing again that the strict convexity is correct because any admissible θ_0 and ψ_0 can be taken as initial rotations. $\square$

It can be seen that if $r_i=0$, the BES $\partial\mathbb{E}(i,j)$ is actually the Apollonius circle in 3D (Isaacs, 1965) proposed for the case of zero capture radius. However, for the case of non-zero capture radius, the analysis is far beyond the scope of Apollonius circle. Most of current works on multiplayer reach-avoid differential games focus on zero capture radius in terms of analytical results, because the non-zero capture radius greatly increases the complexity of explicit computation (Agharkar et al., 2015; Garcia, Casbeer et al., 2019; Shishika & Kumar, 2018; Yan et al., 2020).

Next we compute the ES when multiple pursuers are involved, as Fig. 2(b) shows. According to Section 6.8 in Isaacs' book (Isaacs, 1965) and Definition 3.1, the ES $\mathbb{E}(s,j)$ with respect to a pursuit team P_s and an evader E_j can be computed by

$$\mathbb{E}(s,j)=\big\{\mathbf{x}\in\mathbb{R}^3\mid f_{ij}(\mathbf{x})>0,i\in s\big\}.$$

Thus, the closure of $\mathbb{E}(s,j)$, denoted $\bar{\mathbb{E}}(s,j)$, is given by $\bar{\mathbb{E}}(s,j)=\big\{\mathbf{x}\in\mathbb{R}^3\mid f_{ij}(\mathbf{x})\geq 0,i\in s\big\}$. Note that $\mathbb{E}(s,j)=\cap_{i\in s}\mathbb{E}(i,j)$ and $\partial\mathbb{E}(s,j)\subseteq\cup_{i\in s}\partial\mathbb{E}(i,j)$. From Lemma 3.1, $\bar{\mathbb{E}}(s,j)$ is also bounded and strictly convex.

3.3. Guaranteed pursuer winning strategies

Now, we employ the ES introduced in Section 3.2 to address Problems 3.1 and 3.2. This subsection considers the case when $\bar{\mathbb{E}}(s,j)\cap\Omega_{\text{goal}}$ is empty, implying that there is no point in Ω_{goal} that E_j can reach without being captured by P_s. We will propose a feedback strategy for the pursuit coalition P_s to guarantee its winning in this case. Note that the closure of evasion space $\bar{\mathbb{E}}(s,j)$ is bounded and strictly convex. Thus, we define a critical point in $\bar{\mathbb{E}}(s,j)$.

Definition 3.3 (*Interception Point*)**.** For $P_s\in[\mathcal{P}]^+$ and $E_j\in\mathcal{E}$, if $\bar{\mathbb{E}}(s,j)\cap\Omega_{\text{goal}}$ is empty, let the interception point $I(s,j)=\big[x_{I(s,j)}\ y_{I(s,j)}\ z_{I(s,j)}\big]^\top\in\bar{\mathbb{E}}(s,j)$ be the unique point that is closest to Ω_{goal}.

The geometrical meaning of the interception point $I(s,j)$ is shown in Fig. 2. The interception point has the following properties, which is required in solving Problem 3.2.

Lemma 3.2 (*Properties of the Interception Point*)**.** *Given any $P_s\in[\mathcal{P}]^+$ and $E_j\in\mathcal{E}$, suppose that $\bar{\mathbb{E}}(s,j)\cap\Omega_{\text{goal}}$ is empty. The interception point $I(s,j)$ has the following properties:*

(i) $I(s,j)$ lies on $\partial\mathbb{E}(s,j)$;
(ii) for any s with $|s|=3$, if E_j and the pursuers in P_s are not coplanar and $I(s,j)\in\bigcap_{i\in s}\partial\mathbb{E}(i,j)$, then there exists a plane such that $I(s,j)$ is an intersection point of two strictly convex closed curves in the plane;
(iii) for any s with $|s|=3$, if E_j and the pursuers in P_s are not coplanar, then $\bigcap_{i\in s}\partial\mathbb{E}(i,j)$ contains at most 4 intersection points.

Proof. Regarding *(i)*, it follows from the strict convexity of $\bar{\mathbb{E}}(s,j)$ and the definition of $I(s,j)$.

Regarding *(ii)*, let $P_s=\{P_1,P_2,P_3\}$, and then it follows from $I(s,j)\in\bigcap_{i\in s}\partial\mathbb{E}(i,j)$ and (4) that the interception point $I(s,j)=\mathbf{x}_{E_j}+\rho\mathbf{e}$ satisfies

$$\|\mathbf{x}_{E_j}-\mathbf{x}_{P_i}+\rho\mathbf{e}\|_2=\alpha_{ij}\rho+r_i,\qquad\forall i\in s,\tag{9}$$

where $\rho\in\mathbb{R}^+$ and $\mathbf{e}\in\mathbb{S}^2$. Equivalently, we have

$$\rho^2=c_i+\rho(\mathbf{m}_i^\top\mathbf{e}-b_i),\quad\forall i\in s,\tag{10}$$

where $\mathbf{m}_i=2(\mathbf{x}_{E_j}-\mathbf{x}_{P_i})/(\alpha_{ij}^2-1)$, $b_i=2\alpha_{ij}r_i/(\alpha_{ij}^2-1)$, and $c_i=(\|\mathbf{x}_{E_j}-\mathbf{x}_{P_i}\|_2^2-r_i^2)/(\alpha_{ij}^2-1)$. When we eliminate the term ρ^2 in (10), we then have

$$\begin{cases}\big((\mathbf{m}_1-\mathbf{m}_2)^\top\mathbf{e}+b_2-b_1\big)\rho=c_2-c_1\\ \big((\mathbf{m}_2-\mathbf{m}_3)^\top\mathbf{e}+b_3-b_2\big)\rho=c_3-c_2\end{cases}$$

which implies that

$$\begin{aligned}&\big((c_2-c_3)\mathbf{m}_1+(c_3-c_1)\mathbf{m}_2+(c_1-c_2)\mathbf{m}_3\big)^\top\mathbf{e}\\ &\quad=(c_3-c_2)(b_2-b_1)-(c_2-c_1)(b_3-b_2).\end{aligned}\tag{11}$$

Since the four players are not coplanar, the vectors $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ are linearly independent. Hence, by (11), the vector $\mathbf{e}$ lies in a plane, and thus the same for $I(s,j)$. To solve (9), we could replace the case of $i=3$ in (9) with (11). Note that the intersection of (11) and $\partial\mathbb{E}(1,j)$ is a strictly convex closed curve, and the same for that of (11) and $\partial\mathbb{E}(2,j)$. Thus, $I(s,j)$ is an intersection point of two strictly convex closed curves in the plane given by (11).

Regarding *(iii)*, we show that there are at most four solutions to (9), i.e., (10). We rewrite (10) as $\mathbf{m}_i^\top\mathbf{e}=(\rho^2-c_i)/\rho+b_i$ for $i=1,2,3$. Then, given ρ, the vector $\mathbf{e}$ is uniquely given by

$$\begin{aligned}\mathbf{e}&=[\mathbf{m}_1\ \ \mathbf{m}_2\ \ \mathbf{m}_3]^{-\top}\\ &\quad\cdot[(\rho^2-c_1)/\rho+b_1,(\rho^2-c_2)/\rho+b_2,(\rho^2-c_3)/\rho+b_3],\end{aligned}\tag{12}$$

where the matrix inversion is well-defined because $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ are linearly independent as the four players are not coplanar. By $\|\mathbf{e}\|_2=1$, (12) becomes a quartic equation of ρ, which has at most four solutions. $\square$

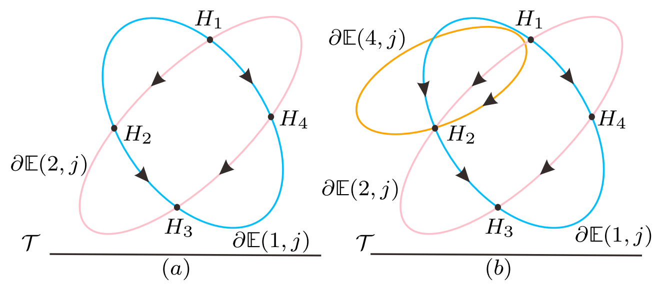

Fig. 3. Degeneration of the interception point, showing that at most three pursuers are needed to compute the interception point for a pursuit coalition, as in Lemma 3.3. (a) Consider $P_{s_1}=\{P_1,P_2,P_3\}$ and E_j, and four players are not coplanar. Note that $\partial\mathbb{E}(i,j)$ is the boundary of evasion space (BES) associated with P_i and E_j, which is the surface of a ball-like shape as in Fig. 2. If $I(s_1,j)\in\cap_{i\in s_1}\partial\mathbb{E}(i,j)$, then there exists a plane intersecting with two BESs $\partial\mathbb{E}(1,j)$ and $\partial\mathbb{E}(2,j)$ by two strictly convex curves (in blue and pink) respectively as depicted, which are proved to allow at most four intersection points H_1,H_2,H_3 and H_4. The black line is the intersection between this plane and $\mathcal{T}$. The interception point occurs at one of these four interception points and the z-coordinate of the points on these two curves decreases along the arrow. Thus, H_3 is the interception point. (b) When adding another pursuer P_4, an associated strictly convex curve (in orange) which is the intersection between this plane and $\partial\mathbb{E}(4,j)$ can be obtained. If the interception point also depends on P_4, it is proved that at least one pursuer in P_{s_1} is redundant for computing the interception point as illustrated in the proof of Lemma 3.3. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Before discussing the strategies, the following lemma is presented, which reduces the complexity of the strategy selection and the matching in the next section.

Lemma 3.3 (*Degeneration of the Interception Point*). *For any $P_s\in[\mathcal{P}]^+$ and $E_j\in\mathcal{E}$, suppose that $\bar{\mathbb{E}}(s,j)\cap\Omega_{\rm goal}$ is empty. Then, there must exist a pursuit subcoalition $s_1\subseteq s$ such that $|s_1|\le 3$ and $I(s_1,j)=I(s,j)$.*

Proof. The statement holds trivially if $|s|\le 3$. Therefore, we focus on the case when $|s|\ge 4$. Consider $|s_1|=3$ and assume that $I(s_1,j)$ depends on all pursuers in P_{s_1}. Thus, $I(s_1,j)\in\cap_{i\in s_1}\partial\mathbb{E}(i,j)$. The case when E_j and the three pursuers in P_{s_1} are coplanar will be discussed separately below. Let $P_{s_1}=\{P_1,P_2,P_3\}$.

Case 1: E_j,P_1,P_2 and P_3 are not coplanar. It follows from the properties *(ii)* and *(iii)* in Lemma 3.2 that $I(s_1,j)$ is one of the intersection points of two strictly convex closed curves in a plane, which have at most four intersection points, as Fig. 3(a) shows. Note that we also replace the condition $I(s_1,j)\in\partial\mathbb{E}(3,j)$ by the fact that $I(s_1,j)$ lies in an associated plane, as the proof of the property *(ii)* in Lemma 3.2 shows. The z-coordinate of the point on these two curves decreases along the arrow. Thus, H_3 is the interception point.

Then, we add the pursuer P_4 which corresponds to another strictly convex closed curve in the same plane, as Fig. 3(b) shows. If $I(s,j)$ depends on all four pursuers, then it must be one of the four points H_1,H_2,H_3 and H_4. If $I(s,j)=H_3$, then P_4 is redundant. If $I(s,j)=H_2$, as Fig. 3(b) indicates, then the arrow of the new curve must decrease from inside to outside of $\cap_{i\in s_1}\bar{\mathbb{E}}(i,j)$ at H_2; conversely, if the new curve increases from inside to outside, then H_2 cannot be the interception point as the point on the new curve can continue to decrease along the new curve after H_2 and also lies in the closure of the ES $\bar{\mathbb{E}}(s_1\cup 4,j)$. Thus, P_2 is redundant, because we can still conclude that H_2 is the interception point only by P_1,P_3 and P_4. Similarly, if $I(s,j)=H_4$, then P_1 is redundant. Since two curves go down at H_1, $I(s,j)$ cannot be H_1. Thus, adding a new pursuer does not increase the number of pursuers which the interception point necessarily depends on. Therefore, at most three pursuers are needed to locate the interception point.

Case 2: E_j,P_1,P_2 and P_3 are coplanar. Thus, the vectors $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ are linearly dependent. Then, by following the argument of the property *(ii)* in Lemma 3.2, we can obtain (10) and write (10) in the matrix form

$$\begin{aligned}&[\mathbf{m}_1\ \ \mathbf{m}_2\ \ \mathbf{m}_3]^\top\mathbf{e}\\&=[(\rho^2-c_1)/\rho+b_1,(\rho^2-c_2)/\rho+b_2,(\rho^2-c_3)/\rho+b_3],\end{aligned}\tag{13}$$

where $[\mathbf{m}_1\ \ \mathbf{m}_2\ \ \mathbf{m}_3]^\top$ is singular. If (13) admits solutions, then we can obtain that the first two equalities can induce the third equality directly, that is, all intersection points between $\partial\mathbb{E}(1,j)$ and $\partial\mathbb{E}(2,j)$ must belong to $\partial\mathbb{E}(3,j)$. Thus, we can ignore P_3 and continue to consider the remaining pursuers in P_s. If (13) admits no solution, then there exists pursuer P_i in P_{s_1} such that $I(s_1,j)\notin\partial\mathbb{E}(i,j)$. Thus, we can ignore P_i and continue to consider the remaining pursuers in P_s. □

Thus, according to Lemma 3.3, for any P_s and E_j, if $\bar{\mathbb{E}}(s,j)\cap\Omega_{\rm goal}$ is empty, then we only need at most three pursuers in s to compute the interception point $I(s,j)$. We next present ES-based strategies. The ES-based strategies are feedback strategies, while the strategies in Chen et al. (2017) are semi-open-loop strategies.

Theorem 3.1 (*ES-Based Strategies for Multiple Pursuers*). *For any $P_s\in[\mathcal{P}]^+$ and $E_j\in\mathcal{E}$, suppose that $\bar{\mathbb{E}}(s,j)\cap\Omega_{\rm goal}$ is empty, and let s_1 be a subset of s such that $I(s_1,j)=I(s,j)$ and $|s_1|\le 3$. If every pursuer P_i in P_{s_1} adopts the feedback strategy $\mathbf{u}_{P_i}=\frac{I(s_1,j)-\mathbf{x}_{P_i}}{\|I(s_1,j)-\mathbf{x}_{P_i}\|_2}$, then the pursuit subcoalition P_{s_1} guarantees that $\bar{\mathbb{E}}(s,j)$ does not approach $\Omega_{\rm goal}$, i.e., $\dot z_{I(s,j)}\ge 0$ for any $\mathbf{u}_{E_j}\in\mathbb{S}^2$. Moreover, $\dot z_{I(s,j)}=0$ if and only if E_j adopts the feedback strategy $\mathbf{u}_{E_j}=\frac{I(s_1,j)-\mathbf{x}_{E_j}}{\|I(s_1,j)-\mathbf{x}_{E_j}\|_2}$.*

Proof. Note that Lemma 3.3 guarantees the existence of a subcoalition s_1 satisfying $I(s_1,j)=I(s,j)$ and $|s_1|\le 3$. For notational convenience, let $\mathbf{x}_I=[x_I\ \ y_I\ \ z_I]^\top$ be the coordinate of $I(s_1,j)$. According to Definition 3.3 and the expression of $\bar{\mathbb{E}}(s_1,j)$ given in Section 3.2, $\mathbf{x}_I$ is the unique solution of the following convex problem

$$\begin{aligned}&\underset{\mathbf{x}\in\mathbb{R}^3}{\text{minimize}}\quad z\\&\text{subject to}\quad f_{ij}(\mathbf{x})\ge 0,\quad \forall i\in s_1.\end{aligned}$$

The interception point $\mathbf{x}_I$ should satisfy the Karush–Kuhn–Tucker (KKT) conditions as follows:

$$\begin{aligned}&[0\ \ 0\ \ -1]^\top=\sum_{i\in s_1}\lambda_i\nabla f_{ij}(\mathbf{x}_I),\\&f_{ij}(\mathbf{x}_I)\ge 0,\ \lambda_i\le 0,\ \lambda_i f_{ij}(\mathbf{x}_I)=0,\quad \forall i\in s_1,\end{aligned}\tag{14}$$

where $\lambda_i\in\mathbb{R}$ is the Lagrange multiplier. The slack conditions in (14) imply that s_1 can be classified into two disjoint index sets $s_1^{=0}$ and $s_1^{>0}$ ($s_1^{>0}$ may be empty) satisfying

$$\begin{cases}f_{ij}(\mathbf{x}_I)=0,\ \lambda_i\le 0, & \text{if } i\in s_1^{=0},\\ f_{ij}(\mathbf{x}_I)>0,\ \lambda_i=0, & \text{if } i\in s_1^{>0}.\end{cases}\tag{15}$$

For any $i\in s_1^{=0}$, according to (1)–(3), taking derivative of $f_{ij}(\mathbf{x}_I)=0$ at both sides with respect to t, we have

$$\begin{aligned}\frac{df_{ij}(\mathbf{x}_I)}{dt}=0\Rightarrow&\frac{(\mathbf{x}_I-\mathbf{x}_{P_i})^\top(\dot{\mathbf{x}}_I-v_{P_i}\mathbf{u}_{P_i})}{\|\mathbf{x}_I-\mathbf{x}_{P_i}\|_2}\\&=\frac{\alpha_{ij}(\mathbf{x}_I-\mathbf{x}_{E_j})^\top(\dot{\mathbf{x}}_I-v_{E_j}\mathbf{u}_{E_j})}{\|\mathbf{x}_I-\mathbf{x}_{E_j}\|_2},\end{aligned}\tag{16}$$

namely,

$$\begin{aligned}
&\nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I \\
&= \left(\frac{\mathbf{x}_I - \mathbf{x}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} - \frac{\alpha_{ij}(\mathbf{x}_I - \mathbf{x}_{E_j})}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2}\right)^\top \dot{\mathbf{x}}_I \\
&= \frac{v_{P_i}(\mathbf{x}_I - \mathbf{x}_{P_i})^\top \mathbf{u}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} - \frac{\alpha_{ij} v_{E_j}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2} \\
&= v_{P_i} - \frac{v_{P_i}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2} \geq v_{P_i} - v_{P_i} = 0
\end{aligned} \tag{17}$$

We emphasize that in deriving (16), $\mathbf{x}_{P_i}$ and $\mathbf{x}_{E_j}$ in $f_{ij}(\mathbf{x}_I)$ are also functions of time t as in (1).

By (3) and (17), the sign of the following satisfies

$$\begin{aligned}
&\nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I \\
&= \left(\frac{\mathbf{x}_I - \mathbf{x}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} - \frac{\alpha_{ij}(\mathbf{x}_I - \mathbf{x}_{E_j})}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2}\right)^\top \dot{\mathbf{x}}_I \\
&= \frac{v_{P_i}(\mathbf{x}_I - \mathbf{x}_{P_i})^\top \mathbf{u}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} - \frac{\alpha_{ij} v_{E_j}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2} \\
&= v_{P_i} - \frac{v_{P_i}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2} \geq v_{P_i} - v_{P_i} = 0,
\end{aligned} \tag{18}$$

where P_i adopts the feedback strategy $\mathbf{u}_{P_i} = \frac{\mathbf{x}_I - \mathbf{x}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2}$ in the second equation. Also note that (18) holds for any $\mathbf{u}_{E_j} \in \mathbb{S}^2$. Moreover, the inequality in (18) becomes an equality if and only if $\mathbf{u}_{E_j} = \frac{\mathbf{x}_I - \mathbf{x}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2}$.

Thus, it follows from (14), (15) and (18) that

$$\begin{aligned}
-\dot{z}_I = [0\ \ 0\ \ -1]\dot{\mathbf{x}}_I &= \sum_{i \in s_1} \lambda_i \nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I \\
&= \sum_{i \in s_1^{=0}} \lambda_i \nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I + \sum_{i \in s_1^{>0}} \lambda_i \nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I \\
&= \sum_{i \in s_1^{=0}} \lambda_i \left(\nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I\right) \leq 0,
\end{aligned}$$

holds for any $\mathbf{u}_{E_j} \in \mathbb{S}^2$ when P_i adopts the strategy $\mathbf{u}_{P_i} = \frac{\mathbf{x}_I - \mathbf{x}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2}$. Moreover, $\dot{z}_I = 0$ if and only if E_j adopts the feedback strategy $\mathbf{u}_{E_j} = \frac{\mathbf{x}_I - \mathbf{x}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2}$. □

Therefore, according to the definition of ES and the ES-based strategies for multiple pursuers, we have the following results for Problems 3.1 and 3.2.

Corollary 1 (*Game of Kind*)**.** *The game winner between a pursuit coalition P_s and an evader E_j can be determined as follows: If $\bar{\mathbb{E}}(s,j) \cap \Omega_{\mathrm{goal}}$ is empty, then the pursuit team P_s wins; if $\bar{\mathbb{E}}(s,j) \cap \Omega_{\mathrm{goal}}$ has more than one element, then E_j wins; if $\bar{\mathbb{E}}(s,j) \cap \Omega_{\mathrm{goal}}$ has a unique element, then two teams are tied. The maximum number of pursuers required to capture an evader before the evader reaches the goal region, is three.*

Proof. This corollary directly follows from Definition 3.1 and Theorem 3.1. □

3.4. The Hamilton–Jacobi–Isaacs equation

Before continuing to analyze the whole game, we consider a game of degree and solve its associated HJI equation by a convex program. These results provide new perspectives on solving HJI equations. This subsection revisits the case when $\bar{\mathbb{E}}(s,j) \cap \Omega_{\mathrm{goal}} = \emptyset$, i.e., the pursuit coalition P_s wins the game. We consider such a special subgame of degree (Isaacs, 1965): Although the pursuit coalition P_s can capture the evader E_j, the evader tries to be captured at the closest point to the goal region and the pursuit coalition seeks the opposite. Formally, the terminal set Ψ and payoff function J respectively are

$$\begin{aligned}
&\Psi := \left\{(\mathbf{x}_s, \mathbf{x}_{E_j}) \mid \exists i \in s,\ \text{s.t. } \mathbf{x}_{E_j} \in \mathcal{C}_i\right\}, \\
&J(\mathbf{u}_s, \mathbf{u}_{E_j}; \mathbf{x}_s^0, \mathbf{x}_{E_j}^0) := z_{E_j}(t_f),
\end{aligned} \tag{19}$$

where the terminal time t_f is defined as the time instant when the system state enters Ψ. The goals of P_s and E_j lead to the following value function

$$V(\mathbf{x}_s^0, \mathbf{x}_{E_j}^0) := \max_{\mathbf{u_s}} \min_{\mathbf{u}_{E_j}} J(\mathbf{u}_s, \mathbf{u}_{E_j}; \mathbf{x}_s^0, \mathbf{x}_{E_j}^0). \tag{20}$$

The following theorem shows that the value function V for this special subgame, can be computed via a convex program for the states such that V is differentiable, because in this case V is the solution of an associated HJI equation (Isaacs, 1965). It is worth noting that for some differential games, there exist states at which the value function V is not differentiable. For example, V is discontinuous across the barrier surfaces (Lewin, 1994, Section 5.3) and the gradient of V has a jump discontinuity across the singular surfaces (Lewin, 1994, Section 9.1). Since there exist no systematic analysis methods for singular surfaces and barrier surfaces (Lewin, 1994), finding the set of states at which V is differentiable is quite hard and beyond the scope of this paper. Additionally, several works on pursuit-evasion differential games (Garcia, Casbeer et al., 2019; Garcia et al., 2020; Garcia, Moll et al., 2019) have shown that the value function is highly likely to be differentiable over the winning space of one player/team, similar to our case in which we only consider the states for which the pursuit coalition wins the game.

Theorem 3.2 (*Value Function*)**.** *Consider the differential game (1), (19) and (20) where $\bar{\mathbb{E}}(s,j) \cap \Omega_{\mathrm{goal}}$ is empty. For the states $(\mathbf{x}_s, \mathbf{x}_{E_j})$ such that the value function $V(\mathbf{x}_s, \mathbf{x}_{E_j})$ is differentiable, then $V(\mathbf{x}_s, \mathbf{x}_{E_j})$ can be computed by the convex optimization problem*

$$\begin{aligned}
V(\mathbf{x}_s, \mathbf{x}_{E_j}) = \ &\underset{\mathbf{x} \in \mathbb{R}^3}{\text{minimize}} && z \\
&\text{subject to} && f_{ij}(\mathbf{x}) \geq 0, \qquad \forall i \in s.
\end{aligned} \tag{21}$$

Proof. Since we only consider the states $(\mathbf{x}_s, \mathbf{x}_{E_j})$ such that the value function $V(\mathbf{x}_s, \mathbf{x}_{E_j})$ is differentiable, the value function satisfies the HJI equation (Isaacs, 1965, Chapter 4) with respect to this subgame as follows:

$$\begin{aligned}
-\frac{\partial V(\mathbf{x}_s, \mathbf{x}_{E_j})}{\partial t} = \max_{\mathbf{u}_s} \min_{\mathbf{u}_{E_j}} \Bigg\{ &\sum_{i \in s} \frac{\partial V(\mathbf{x}_s, \mathbf{x}_{E_j})}{\partial \mathbf{x}_{P_i}}^\top v_{P_i} \mathbf{u}_{P_i} \\
&+ \frac{\partial V(\mathbf{x}_s, \mathbf{x}_{E_j})}{\partial \mathbf{x}_{E_j}}^\top v_{E_j} \mathbf{u}_{E_j} \Bigg\},
\end{aligned} \tag{22}$$

where (1) is employed. In our problem, we have $\frac{\partial V}{\partial t} = 0$ which is obtained from Isaacs' main equation 1 (Isaacs, 1965, equation 4.2.1 with several vital formulas in equations 2.1.1, 2.3.1, and 2.4.1), because the terminal set Ψ and payoff function J are both independent of time. In other words, for any two distinct time instants $t_1, t_2 \geq 0$, if $\mathbf{x}_s(t_1) = \mathbf{x}_s(t_2)$ and $\mathbf{x}_{E_j}(t_1) = \mathbf{x}_{E_j}(t_2)$, then $V(\mathbf{x}_s(t_1), \mathbf{x}_{E_j}(t_1)) = V(\mathbf{x}_s(t_2), \mathbf{x}_{E_j}(t_2))$. Therefore, (22) can also be equivalently rewritten as

$$0 = \max_{\mathbf{u}_s} \min_{\mathbf{u}_{E_j}} \frac{dV(\mathbf{x}_s, \mathbf{x}_{E_j})}{dt} = \max_{\mathbf{u}_s} \min_{\mathbf{u}_{E_j}} \dot{V}(\mathbf{x}_s, \mathbf{x}_{E_j}). \tag{23}$$

Next, we show that the unique optimal value to the convex optimization problem (21) satisfies (23). Let $\mathbf{x}_I$ be the solution of

(21) and suppose that $V(\mathbf{x}_s, \mathbf{x}_{E_j}) = z_I$. Since $\mathbf{x}_I$ satisfies the KKT conditions of (21)

$$[0 \;\; 0 \;\; -1]^\top = \sum_{i\in s} \lambda_i \nabla f_{ij}(\mathbf{x}_I),$$
$$f_{ij}(\mathbf{x}_I) \geq 0,\ \lambda_i \leq 0,\ \lambda_i f_{ij}(\mathbf{x}_I) = 0, \quad \forall i \in s, \tag{24}$$

we have

$$\dot{V}(\mathbf{x}_s, \mathbf{x}_{E_j}) = \dot{z}_I = [0 \;\; 0 \;\; 1]\dot{\mathbf{x}}_I = -\sum_{i\in s} \lambda_i \nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I. \tag{25}$$

The slackness conditions in (24) imply that s can be classified into two disjoint index sets $s^{=0}$ and $s^{>0}$ satisfying

$$\begin{cases} f_{ij}(\mathbf{x}_I) = 0,\ \lambda_i \leq 0, & \text{if } i \in s^{=0}, \\ f_{ij}(\mathbf{x}_I) > 0,\ \lambda_i = 0, & \text{if } i \in s^{>0}. \end{cases} \tag{26}$$

Moreover, for any $i \in s^{=0}$, similar to the argument (16) and (17), we can obtain

$$\nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I$$
$$\frac{v_{P_i}(\mathbf{x}_I - \mathbf{x}_{P_i})^\top \mathbf{u}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} - \frac{\alpha_{ij} v_{E_j}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2}. \tag{27}$$

where (3) is used in the left side of (27). By (25)–(27), we compute

$$\begin{aligned}
&\max_{\mathbf{u}_s} \min_{\mathbf{u}_{E_j}} \dot{V}(\mathbf{x}_s, \mathbf{x}_{E_j}) = \max_{\mathbf{u}_s} \min_{\mathbf{u}_{E_j}} -\sum_{i\in s^{=0}} \lambda_i \nabla f_{ij}(\mathbf{x}_I)^\top \dot{\mathbf{x}}_I \\
&= \max_{\mathbf{u}_s} \min_{\mathbf{u}_{E_j}} \sum_{i\in s^{=0}} \lambda_i \Bigg(\frac{-v_{P_i}(\mathbf{x}_I - \mathbf{x}_{P_i})^\top \mathbf{u}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} \\
&\quad + \frac{\alpha_{ij} v_{E_j}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2} \Bigg) \\
&= \max_{\mathbf{u}_s} \sum_{i\in s^{=0}} \lambda_i \frac{-v_{P_i}(\mathbf{x}_I - \mathbf{x}_{P_i})^\top \mathbf{u}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2} \\
&\quad + \min_{\mathbf{u}_{E_j}} \sum_{i\in s^{=0}} \lambda_i \frac{\alpha_{ij} v_{E_j}(\mathbf{x}_I - \mathbf{x}_{E_j})^\top \mathbf{u}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2} \\
&= -\sum_{i\in s^{=0}} \lambda_i v_{P_i} + \sum_{i\in s^{=0}} \lambda_i \alpha_{ij} v_{E_j} = 0,
\end{aligned}$$

where in the max and min operations, we take $\mathbf{u}_{P_i} = \frac{\mathbf{x}_I - \mathbf{x}_{P_i}}{\|\mathbf{x}_I - \mathbf{x}_{P_i}\|_2}$ for all $i \in s^{=0}$ and $\mathbf{u}_{E_j} = \frac{\mathbf{x}_I - \mathbf{x}_{E_j}}{\|\mathbf{x}_I - \mathbf{x}_{E_j}\|_2}$ by noting that $\lambda_i \leq 0$. Thus, the value function $V(\mathbf{x}_s, \mathbf{x}_{E_j}) = z_I$ satisfies the HJI equation (23), i.e., (22).

Finally, we prove that the terminal condition $V(\mathbf{x}_s, \mathbf{x}_{E_j}) = z_{E_j}$ from (19) is satisfied. By the definition of Ψ, when the game ends, there exists one pursuer P_i in P_s such that $\|\mathbf{x}_{E_j} - \mathbf{x}_{P_i}\|_2 = r_i$, which implies that $\bar{\mathbb{E}}(i, j)$ contains a unique point $\mathbf{x}_{E_j}$. In other words, we have that $f_{ij}(\mathbf{x}) \geq 0$ leads to $\mathbf{x} = \mathbf{x}_{E_j}$. Note that for the other pursuers in P_s, the constraints in (21) are feasible. Thus, the convex optimization problem (21) admits the unique solution $\mathbf{x}_I = \mathbf{x}_{E_j}$. Therefore, the value function satisfies $V(\mathbf{x}_s, \mathbf{x}_{E_j}) = z_I = z_{E_j}$. □

Remark 3.1. Theorem 3.2 shows that the HJI equation of a special subgame between a pursuit coalition and an evader, which describes the value function and often is hard to solve, can be transformed into a convex optimization problem with greatly reduced computational complexity. For the strategies of the players, they are the gradients of the value function with respect to states, and in our case they can be obtained through the optimal solution to a convex optimization problem, as the proof of Theorem 3.2 indicates. We hope that the method of connecting KKT conditions with HJI equations, as in the proof of Theorem 3.2, can be useful for solving more general differential games.

Remark 3.2. For a pursuit coalition P_s with three pursuers, the differential game (1), (19) and (20) has twelve states. However, if we solve the associated HJI equation (22) by the level set methods commonly adopted in the differential games (Margellos & Lygeros, 2011; Mitchell, 2002; Mitchell et al., 2005), then the number of grid points is 10^{12} even if we only adopt 10 grid points per dimension. In general, when a problem has twelve or greater system dimensions and the computation time is limited analogous to this game, a more accurate solution can be obtained by Theorem 3.2 than level set methods. This distinction stems from the fact that our results reply on solving a convex problem, while the level set methods crucially depend on the grid resolution.

4. Maximum-matching capture strategies

4.1. Maximum matching

We piece together the outcomes of all pursuit coalitions and evader pairs using maximum matching (Chen et al., 2017; Yan et al., 2020). Thanks to Corollary 1, the matching problem is simplified greatly as we only need to consider all pursuit coalitions of size less than or equal to three. The pursuit team $\mathcal{P}$ consists of $N_p(N_p^2 + 5)/6$ possible coalitions: N_p one-pursuer coalitions, $N_p(N_p - 1)/2$ two-pursuer coalitions, and $N_p(N_p - 1)(N_p - 2)/6$ three-pursuer coalitions. For notational convenience, we define the number of possible vertices for $\mathcal{P}$ in the bipartite graph by $N_o = N_p(N_p^2 + 5)/6$.

Let $G = (U \cup V, E)$ be an undirected bipartite graph consisting of two independent vertex sets U and V, where E is the set of edges. We denote the edge connecting vertex $P_s \in U$ and vertex $E_j \in V$ by e_{sj}. In our problem, the vertex set U consists of all pursuit coalitions of size no more than three, and V represents the set of evaders. The bipartite graph G is formally defined as follows:

$$\begin{aligned}
&U = [\mathcal{P}]^3, \quad V = \mathcal{E}, \\
&E = \Big\{ e_{sj} \mid P_s \in U, E_j \in V, |\bar{\mathbb{E}}(s, j) \cap \Omega_{\text{goal}}| \leq 1, \\
&\qquad\qquad \forall s_1 \subsetneq s, |\bar{\mathbb{E}}(s_1, j) \cap \Omega_{\text{goal}}| > 1 \Big\}.
\end{aligned} \tag{28}$$

Note that $|U| = N_o$ and $|V| = N_e$. An edge $e_{sj} \in E$ if and only if P_s is able to capture E_j in Ω_{play} or at $\mathcal{T}$, while any subcoalition s_1 of s except s itself cannot.

We aim to find a matching in the bipartite graph G that contains a maximum number of evaders. However, since each pursuer can only appear in at most one pursuit coalition, the pursuit coalitions containing at least one common pursuer cannot coexist in the matching. As a result, our problem becomes a constrained maximum bipartite matching problem. We can also interpret the problem as an assignment problem with N_p workers and N_e jobs. In this assignment problem, some jobs are easy in the sense that they each can be finished by one individual worker, and some jobs are hard in the sense that they require cooperation among multiple workers.[1] The goal is to find an assignment of workers to jobs such that as many jobs as possible are finished.

The conflicts among the pursuit coalitions can be represented by a conflict graph $C = (E, \bar{E})$ as in Darmann et al. (2011), Itai et al. (1978), Öncan et al. (2013) and Thomas (2016). Each vertex in C corresponds uniquely to an edge $e \in E$ of G. An edge $\bar{e} \in \bar{E}$ implies that the two vertices connected by $\bar{e}$ (two

[1] Even though we consider at most three workers for each job, the derived results could be extended to the cases with any given number of workers for one job rather routinely.

edges in G) cannot coexist in the maximum matching of G. The conflict graph C may contain isolated vertices, which means that the corresponding edges in G do not conflict with others. In our case, edges of G incident to the vertices with at least one common pursuer are conflicting, and thus the conflict graph C is

$$\bar{E} = \left\{(e_{sj}, e_{pq}) \mid e_{sj} \in E, e_{pq} \in E, s \neq p, s \cap p \neq \emptyset\right\}. \tag{29}$$

Given the bipartite graph (28) and the conflict graph (29), we define the binary integer programming (BIP) formulation for the MBMC as follows:

$$\begin{aligned} &\text{maximize} \sum_{e_{sj}\in E} a_{sj} \\ &\text{subject to} \sum_{s\in U} a_{sj} \leq 1, \qquad \forall E_j \in V, \\ &\qquad \sum_{j\in V} a_{sj} \leq 1 \qquad forall P_s \in U, \\ &\qquad a_{sj} + a_{pq} \leq 1, \qquad \forall (e_{sj}, e_{pq}) \in \bar{E}, \\ &\qquad a_{sj} \in \{0, 1\}, \qquad \forall e_{sj} \in E, \\ &\qquad a_{sj} = 0, \qquad \forall e_{sj} \notin E, \end{aligned} \tag{30}$$

where $a_{sj} = 1$ indicates the assignment of pursuit coalition P_s to capture E_j, and $a_{sj} = 0$ means no assignment.

Remark 4.1. We construct a bipartite graph G and a conflict graph C by (28) and (29) respectively at time t. Let N^* and $\boldsymbol{a}^*$ be an exact solution and the corresponding optimal value of (30). If the pursuit team adopts the matching $\boldsymbol{a}^*$ until the game ends, then the maximum number of captured evaders after time t is N^*. This control strategy where the pursuit team does not recompute a possibly better matching during the pursuit is open-loop.

Next, we prove the complexity of MBMC, which is relevant when the number of players is large.

Theorem 4.1 (*Hardness of the Matching*)**.** *The MBMC problem (30) is NP-hard.*

Proof. We polynomially reduce the well-known NP-complete *3-dimensional matching problem* (Karp, 1972) to special instances of the MBMC problem. Let $\mathcal{I} = (X, Y, Z, T)$ be an arbitrary instance of 3-dimensional matching, where X, Y and Z are finite, disjoint sets with $|X| = |Y| = |Z| = m$, and T, a subset of $X\times Y\times Z$, consists of triples (i, j, k) such that $i \in X$, $j \in Y$, and $k \in Z$. The problem is to determine whether there is a set $M \subseteq T$ such that $|M| = m$ and no two elements of M agree in any coordinate. If so, the set M is called a 3-dimensional matching of $\mathcal{I}$. We define the bipartite graph $G = (U \cup V, E)$ as follows:

$$U = X \times Y,\ \ V = Z,\ \ E = \left\{((i, j), k) \mid (i, j, k) \in T\right\}.$$

Let M be a complete matching of G. If M does not contain two edges $((i, j_1), k_1), ((i, j_2), k_2)$ or $((i_1, j), k_1), ((i_2, j), k_2)$, then M corresponds to a 3-dimensional matching of $\mathcal{I}$. Thus, the restrictions are imposed:

$$\begin{aligned} \bar{E} =\Big\{ &\big(((i_1, j_1), k_1), ((i_2, j_2), k_2)\big) \mid i_1 = i_2 \text{ or } j_1 = j_2, \\ &((i_1, j_1), k_1) \in E, ((i_2, j_2), k_2) \in E \Big\}. \end{aligned}$$

Let the conflict graph be $C = (E, \bar{E})$. The matching problem in graph G with conflict graph C can be interpreted as a matching problem for the game with $2m$ (i.e., $|X| + |Y|$) pursuers and m (i.e., $|Z|$) evaders. Each evader $k \in Z$ can be captured by two cooperative pursuers $i \in X$ and $j \in Y$ if $(i, j, k) \in T$. The pursuit coalitions with one or three pursuers do not exist in this case.

We now show that the MBMC problem G with conflict graph C has a complete matching if and only if $\mathcal{I}$ has a 3-dimensional matching.

(1) Assume that G with conflict graph C has a complete matching M. Then, $|M| = m$ and M is a subset of E. The conflict graph C ensures that no two elements of M agree in X or Y coordinate. Since M is a matching, no two elements of M agree in Z coordinate. Therefore, M is a 3-dimensional matching of $\mathcal{I}$ when we write $((i, j), k)$ as (i, j, k).
(2) Let M be a 3-dimensional matching of $\mathcal{I}$. For all $(i, j, k) \in M$, by definition, the edges $((i, j), k)$ constitute a complete matching for graph G with conflict graph C.

Note that the decision problem of whether G with conflict graph C has a complete matching can be solved by computing the maximum matching of G with conflict graph C. Thus, the 3-dimensional matching of $\mathcal{I}$ can be polynomially reduced to the MBMC problem, and the MBMC problem is NP-hard. □

Algorithm 1 Sequential Matching Algorithm

Input: A bipartite graph $G = (U \cup V, E)$ with $U = [\mathcal{P}]^3$ and $V = \mathcal{E}$, where $e_{sj} \in E$ if the pursuit coalition P_s in U can defeat the evader E_j in V
Output: An approximation matching M in G
1: $U_1 \leftarrow \mathcal{P}$, $V_1 \leftarrow \mathcal{E}$, $E_1 \leftarrow \left\{e_{sj} \in E \mid P_s \in U_1, E_j \in V_1\right\}$
2: Compute the maximum matching M_1 in the subgraph $G_1 = (U_1 \cup V_1, E_1)$ by maximum network flow;
3: $A_1 \leftarrow \{P_i \mid i \in s, e_{sj} \in M_1\}$, $B_1 \leftarrow \{E_j \mid e_{sj} \in M_1\}$
4: $U_2 \leftarrow [\mathcal{P} \setminus A_1]^2$, $V_2 \leftarrow \mathcal{E} \setminus B_1$
5: $E_2 \leftarrow \{e_{sj} \in E \mid P_s \in U_2, E_j \in V_2\}$
6: Compute an approximation matching M_2 in the subgraph $G_2 = (U_2 \cup V_2, E_2)$ by local search maximum with size 2 swaps;
7: $A_2 \leftarrow \{P_i \mid i \in s, e_{sj} \in M_2\}$, $B_2 \leftarrow \{E_j \mid e_{sj} \in M_2\}$
8: $U_3 \leftarrow \left[\mathcal{P} \setminus \left(A_1 \cup A_2\right)\right]^3$, $V_3 \leftarrow \mathcal{E} \setminus \left(B_1 \cup B_2\right)$
9: $E_3 \leftarrow \{e_{sj} \in E \mid P_s \in U_3, E_j \in V_3\}$
10: Compute an approximation matching M_3 in the subgraph $G_3 = (U_3 \cup V_3, E_3)$ by local search maximum with size 2 swaps;
11: **Return** $M = M_1 \cup M_2 \cup M_3$.

Next, we give an approximation algorithm called Sequential Matching Algorithm stated in Algorithm 1 for MBMC. We sketch out the main idea of the Sequential Matching Algorithm as follows:

Step 1 (from line 1 to 2): Use maximum network flow (Ford, Jr & Fulkerson, 1962) to compute the maximum matching M_1 of the subgraph G_1 which only considers the pursuit vertices containing one pursuer.

Step 2 (from line 3 to 6): Let A_1 and B_1 be the sets of pursuers and evaders in M_1 respectively. Remove the vertices of G containing at least one player occurring in $A_1 \cup B_1$, and for the remainder, construct the subgraph G_2 which only considers the pursuit vertices containing at most two pursuers. Use local search maximum with size 2 swaps (Cygan, 2013; Hurkens & Schrijver, 1989) to compute the maximum matching M_2 of G_2.

Step 3 (from line 7 to 10): Let A_2 and B_2 be the sets of pursuers and evaders in M_2 respectively. Remove the vertices of G containing at least one player occurring in $A_1 \cup A_2 \cup B_1 \cup B_2$, and for the remainder, obtain the subgraph G_3. Then, use local search maximum with size 2 swaps to compute the maximum matching M_3 of G_3.

Step 4: The output is the union of these three matchings.

Algorithm 1 has the following properties.

Theorem 4.2 (*Constant-Factor Approximation Algorithm*)**.** *The Sequential Matching Algorithm is of polynomial time and*

(i) a 1/3-approximation algorithm for MBMC;
(ii) a 1/2-approximation algorithm if the solution of MBMC does not contain pursuit coalitions with three pursuers;
(iii) an exact algorithm if the solution of MBMC does not contain pursuit coalitions with two or three pursuers.

Proof. We postpone the proof to Appendix B. □

Theorem 4.2 guarantees that the Sequential Matching Algorithm can find an approximate solution to the MBMC in polynomial time, and the ratio between the performance of this approximate solution and that of the optimal is bounded by a known constant. These properties are particularly important when the computational efficiency is critical and the number of players involved is large. Furthermore, the MBMC has the following conclusion on its class of complexity.

Corollary 2 (*Class of Complexity on Approximation Algorithm*)**.** *The MBMC is APX-complete.*

Proof. The APX-complete (Kann, 1991) 3-dimensional matching can be reduced to the MBMC in polynomial time, Theorem 4.2 implies that the MBMC is in APX, the statement follows. □

Algorithm 2 Receding Horizon Strategy

Input: $\mathcal{P}, \mathcal{E}, \{\mathbf{x}_{P_i}\}_{P_i\in\mathcal{P}}, \{\mathbf{x}_{E_j}\}_{E_j\in\mathcal{E}}, M_a \leftarrow \emptyset, L_c \leftarrow 0$

1: **Repeat**
2: $M \leftarrow \text{Max}\big(\mathcal{P}, \mathcal{E}, \{\mathbf{x}_{P_i}\}_{P_i\in\mathcal{P}}, \{\mathbf{x}_{E_j}\}_{E_j\in\mathcal{E}}\big)$
3: **if** $|M| > |M_a|$ or $L_c == 1$ **then**
4: $M_a \leftarrow M, L_c \leftarrow 0$
5: **end if**
6: Assign a pursuit coalition to each evader that is part of the matching M_a;
7: For a short duration Δ, apply the ES-based strategy for each pursuer that is part of the matching M_a. For the rest of the pursuers and for all evaders in $\mathcal{E}$, apply some (any) strategy;
8: Update the player positions after the duration Δ;
9: **for** every evader E_j in $\mathcal{E}$ **do**
10: **if** E_j is captured or enters Ω_{goal} **then**
11: $L_c \leftarrow 1$ if E_j is captured
12: $\mathcal{E} \leftarrow \mathcal{E} \setminus E_j$
13: **end if**
14: **end for**
15: **until** $\mathcal{E} = \emptyset$;

4.2. Receding horizon strategy

Next we design a receding horizon pursuit strategy. This strategy is useful because a better matching may occur as the game evolves, and a rematching should be performed when an evader is captured. This strategy is necessary, because the current matching (if the exact solution to (30) is obtained) only corresponds to the maximum number of captured evaders in the open-loop sense. With Algorithm 2, the bipartite graph and the corresponding (approximate) maximum matching can be updated, as players change their positions. M_a and L_c denote the adopted matching and label for the capture of evaders, respectively. $\text{Max}\big(\mathcal{P}, \mathcal{E}, \{\mathbf{x}_{P_i}\}_{P_i\in\mathcal{P}}, \{\mathbf{x}_{E_j}\}_{E_j\in\mathcal{E}}\big)$ computes the maximum matching by solving (30) exactly with algorithms such as cutting plane methods, or computes the approximate maximum matching by our Sequential Matching Algorithm.

This paper considers a constant time step $\Delta > 0$. In general, the solution to the maximum matching is not unique. Besides, as the players play out the game in real time, more alternative solutions may generate. Thus, to avoid unnecessary change of the interception scheme, assume that the pursuit team would never change the former matching unless more evaders can be captured in the new matching as lines 3–5 show.

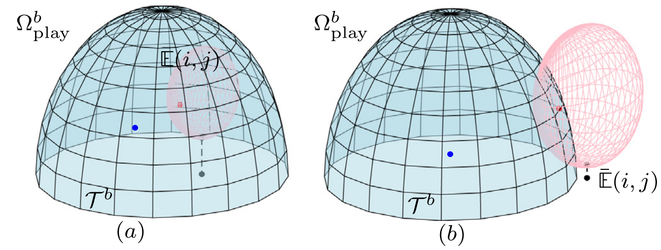

Fig. 4. Bounded convex play region with a planar exit, where the interception point $I^b(i,j)$ lies (a) in the interior of the play region; (b) at the boundary of the play region.

Remark 4.2. Our ultimate goal is to capture the most evaders over the *entire horizon* of the game. The matching problem, solved (exactly or approximately) at any instant, gives the number of evaders that can be captured if the pursuer team sticks to the matching result. However, as the game evolves, a better matching that leads to a larger number of captured evaders may emerge. At this point, switching to this better matching result guarantees that more evaders can be captured. This monotonicity property makes the receding horizon strategy appealing. On the other hand, such a rolling one-step greedy strategy does not come with global optimality guarantees, i.e., it is not clear that the total number of evaders captured is maximum over the entire time horizon. The above argument applies to both cases when the matching problem is solved exactly or approximately at each step.

5. Bounded convex play region with an exit

In this section, we extend the previous analysis to the case when the game is played in a 3D bounded convex region. We consider a bounded convex play region with an exit through which the evaders escape from the play region. The exit is assumed to be a part of a plane. The play region Ω^b_{play} is a closed convex region in $\mathbb{R}^3$ and the exit $\mathcal{T}^b$ is a part of its boundary. Formally,

$$\begin{aligned} \Omega^b_{\text{play}} &= \big\{\mathbf{x} \in \mathbb{R}^3 \mid z \geq 0, g(\mathbf{x}) \geq 0\big\}, \\ \mathcal{T}^b &= \big\{\mathbf{x} \in \Omega^b_{\text{play}} \mid z = 0\big\}, \end{aligned} \tag{31}$$

where $g : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that the set $\Omega^b_1 = \{\mathbf{x} \in \mathbb{R}^3 \mid g(\mathbf{x}) \geq 0\}$ is convex. Additionally, Assume that Ω^b_{play} is non-empty and contains more than one point. Condition 4) in Assumption 2.1 becomes that $\mathbf{x}^0_{P_i} \in \Omega^b_{\text{play}}$ for all $P_i \in \mathcal{P}$ and $\mathbf{x}^0_{E_j} \in \Omega^b_{\text{play}}$ for all $E_j \in \mathcal{E}$. An example of the play region is given in Fig. 4.

For $P_s \in [\mathcal{P}]^+$ and $E_j \in \mathcal{E}$, if $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b = \emptyset$, then we define the interception point $I^b(s,j) \in \bar{\mathbb{E}}(s,j) \cap \Omega^b_{\text{play}}$ as the closest point to the plane containing $\mathcal{T}^b$, and $I^b(s,j)$ is the solution to the following convex problem

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^3}{\text{minimize}} \quad & z \\ \text{subject to} \quad & f_{ij}(\mathbf{x}) \geq 0, \qquad \forall i \in s, \\ & g(\mathbf{x}) \geq 0, \end{aligned} \tag{32}$$

where the constraint $z \geq 0$ in Ω^b_{play} is not involved because it holds naturally when $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b = \emptyset$.

Lemma 5.1 (*Uniqueness of the Interception Point*)**.** *For any $P_s \in [\mathcal{P}]^+$ and $E_j \in \mathcal{E}$, if $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b = \emptyset$, the interception point $I^b(s,j)$ is unique.*

Proof. Since $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b = \emptyset$, similar to Definition 3.3, we denote $I(s,j)$ as the unique point in $\bar{\mathbb{E}}(s,j)$ that is closest to the plane containing $\mathcal{T}^b$. If $I(s,j)$ lies in Ω^b_{play} as Fig. 4(a) shows, then $I^b(s,j) = I(s,j)$ and thus $I^b(s,j)$ is unique. If $I(s,j)$ lies out of Ω^b_{play} as Fig. 4(b) indicates, we consider a plane $\mathcal{T}_1$ parallel to $\mathcal{T}^b$ and move it from $\mathcal{T}^b$ towards Ω^b_{play}. At the beginning, $\mathcal{T}_1$'s intersection sets with $\bar{\mathbb{E}}(s,j)$ and Ω^b_{play} are two disjoint sets: a strictly convex set and a convex set respectively. As $\mathcal{T}_1$ moves, these two intersection sets are tangent when intersecting at the first time. The tangent point is $I^b(s,j)$ and the statement follows from the uniqueness of $I^b(s,j)$. □

Next we can still construct a similar ES-based strategy to guarantee the winning of the pursuer coalitions.

Lemma 5.2 (*ES-Based Strategy*)**.** *For any $P_s \in [\mathcal{P}]^+$ and $E_j \in \mathcal{E}$, suppose that $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b = \emptyset$. If every pursuer P_i in P_s adopts the feedback strategy $\mathbf{u}_{P_i} = \frac{I^b(s,j)-\mathbf{x}_{P_i}}{\|I^b(s,j)-\mathbf{x}_{P_i}\|_2}$, then it is guaranteed that $\bar{\mathbb{E}}(s,j)$ does not approach $\mathcal{T}^b$, i.e., $\dot{z}_{I^b(s,j)} \geq 0$ for any $\mathbf{u}_{E_j} \in \mathbb{S}^2$. Moreover, $\dot{z}_{I^b(s,j)} = 0$ if and only if E_j adopts the feedback strategy $\mathbf{u}_{E_j} = \frac{I^b(s,j)-\mathbf{x}_{E_j}}{\|I^b(s,j)-\mathbf{x}_{E_j}\|_2}$.*

Proof. Since $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b = \emptyset$, by Lemma 5.1, the interception point $I^b(s,j)$ is unique. If $I(s,j)$ lies inside of Ω^b_{play} as Fig. 4(a) shows, then $I^b(s,j) = I(s,j)$ and the statement follows from Theorem 3.1. In the following, we focus on the case when $I(s,j)$ lies outside of Ω^b_{play} as in Fig. 4(b). For simplicity, we denote $I(s,j)$ and $I^b(s,j)$ by $\mathbf{x}_I$ and $\mathbf{x}_I^b$, respectively.

Since $\mathbf{x}_I^b$ moves on $\partial\Omega_1^b$, i.e., $g(\mathbf{x}_I^b) \equiv 0$, thus we have

$$\frac{dg(\mathbf{x}_I^b)}{dt} = 0 \Rightarrow \nabla g(\mathbf{x}_I^b)^\top \dot{\mathbf{x}}_I^b = 0. \tag{33}$$

The KKT conditions for (32) are as follows:

$$\begin{aligned} &[0 \;\; 0 \;\; -1]^\top = \sum_{i\in s} \lambda_i \nabla f_{ij}(\mathbf{x}_I^b) + \lambda_g \nabla g(\mathbf{x}_I^b), \\ &f_{ij}(\mathbf{x}_I^b) \geq 0,\ \lambda_i \leq 0,\ \lambda_i f_{ij}(\mathbf{x}_I^b) = 0, \quad \forall i \in s, \\ &\qquad \lambda_g \leq 0,\ g(\mathbf{x}_I^b) \geq 0,\ \lambda_g g(\mathbf{x}_I^b) = 0, \end{aligned} \tag{34}$$

where $\lambda_g \in \mathbb{R}$ is the Lagrange multiplier related to $g(\mathbf{x}) \geq 0$. Additionally, similar to (18), we obtain

$$\nabla f_{ij}(\mathbf{x}_I^b)^\top \dot{\mathbf{x}}_I^b \geq 0 \text{ for all } i \in s \text{ with } f_{ij}(\mathbf{x}_I^b) = 0, \tag{35}$$

when P_i adopts the feedback strategy $\mathbf{u}_{P_i} = \frac{\mathbf{x}_I^b-\mathbf{x}_{P_i}}{\|\mathbf{x}_I^b-\mathbf{x}_{P_i}\|_2}$. It follows from (15), (33), (34) and (35) that z_I^b satisfies

$$\begin{aligned} -\dot{z}_I^b &= [0 \;\; 0 \;\; -1]\dot{\mathbf{x}}_I^b \\ &= \sum_{i\in s} \lambda_i \nabla f_{ij}(\mathbf{x}_I^b)^\top \dot{\mathbf{x}}_I^b + \lambda_g \nabla g(\mathbf{x}_I^b)^\top \dot{\mathbf{x}}_I^b \leq 0, \end{aligned}$$

which leads to the similar conclusion as Theorem 3.1. □

Then, the results about the game of kind are straightforward, formally stated below.

Remark 5.1. Similarly, the game winner between a pursuit team P_s and an evader E_j for the bounded convex play region is determined as follows: If $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b$ is empty, then the pursuit team P_s wins; if $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b$ has more than one element, then E_j wins; if $\bar{\mathbb{E}}(s,j) \cap \mathcal{T}^b$ has a unique element, then two teams are tied.

Since $\bar{\mathbb{E}}(s,j) \cap \Omega^b_{\text{play}} \subseteq \bar{\mathbb{E}}(s,j)$, similar to the case of unbounded play region, we need at most three pursuers to capture one evader before the latter escapes. Thus, the results of maximum matching can also be applied.

6. Numerical results

This section presents numerical results to illustrate the previous theoretical developments for the cases of unbounded and bounded convex play regions. Numerical studies are performed in Matlab R2017b on a laptop with a Core i7-8550U processor with 16 GB of memory. The Sequential Matching Algorithm is used to compute an approximate maximum matching.

6.1. Unbounded play region

We first consider the unbounded play region with $N_p = 8$ and $N_e = 9$. Initially, as Fig. 5(a) shows, the maximum-matching strategy indicates that four evaders are matched by seven pursuers, including two 1-to-1, one 2-to-1 and one 3-to-1 matchings. The matched evaders will be captured, unless the pursuit team changes its matching when a matching of greater size occurs as the game runs. A snapshot of the game is presented in Fig. 5(b) where the pursuit team changes its matching because a better matching with six matched evaders occurs. In the end, as shown in Fig. 5(c), one evader reaches the goal region successfully and eight evaders are captured in the play region. We also consider another similar pursuit strategy with the following different setups: The pursuers stick to their matched evaders until the capture happens, and a reassignment takes place only at the capture moment. As shown in Fig. 5(d), two evaders reach the goal region successfully and seven evaders are captured.

6.2. Bounded convex play region

In this section, we consider a bounded convex play region with a planar exit, where $N_p = 7$ and $N_e = 7$. As Fig. 6(a) shows, four evaders are matched by six pursuers at the beginning, including two 1-to-1 and two 2-to-1 matchings. Finally, the pursuit team captures six evaders successfully and one evader escapes.

7. Conclusion

We studied a 3D reach-avoid game where multiple pursuers defend a goal region against multiple evaders. For multiple pursuers and one evader, we showed that the evasion space corresponding to a pursuit coalition and an evader is strictly convex and the associated interception point is unique and can be computed via a convex program. We further revealed that the pursuit coalition can always defend the goal region by moving towards the interception point if the initial condition allows. We also found that in 3D if a pursuit coalition can defend the goal region against an evader, then at most three pursuers in the coalition are necessarily needed. We solved the HJI equation associated with a special subgame of degree by a convex program. For multiple pursuers and multiple evaders, the matching is considered. We have shown that our matching between pursuit coalitions and evaders is an instance of a class of constrained matching problems, i.e., MBMC. We analyzed the complexity of MBMC and designed a constant-factor approximation algorithm with polynomial computation time to solve it. We also demonstrated that our results can be applied to the case of a bounded convex play region by slightly modifying the interception point.

Appendix A. Proof of Lemma A.1

Lemma A.1 (*Convexity of Sets in Polar Coordinates*)**.** *For a twice differentiable simple 2D closed curve $\rho : [0, 2\pi] \mapsto \mathbb{R}^+$ with $\rho(0) = \rho(2\pi)$, the set of points consisting of this curve and its interior is strictly convex if for all $\psi \in [0, 2\pi]$, $\rho(\psi)$ satisfies*

$$\rho^2 + 2\left(\frac{d\rho}{d\psi}\right)^2 - \rho\frac{d^2\rho}{d\psi^2} > 0. \tag{36}$$

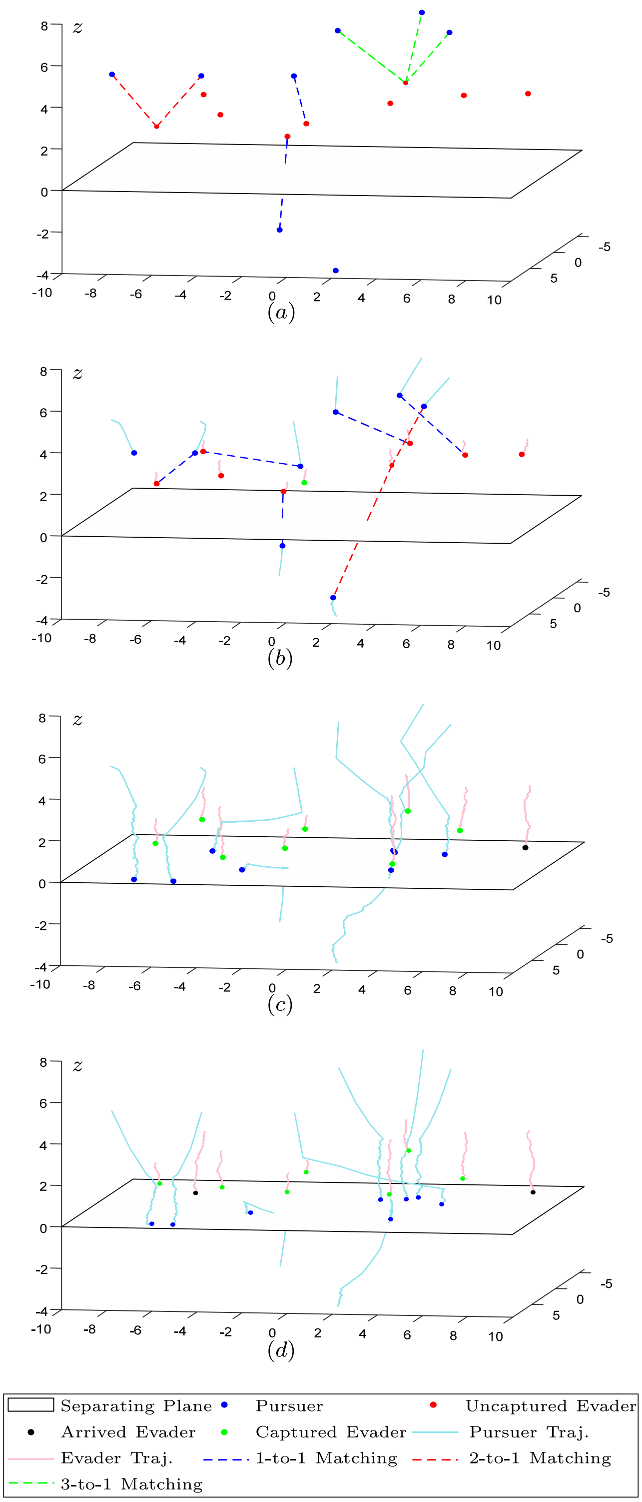

Fig. 5. Simulation of a game with eight pursuers and nine evaders in the unbounded play region.

Proof. The curvature κ of the curve $\rho = \rho(\psi)$ in polar coordinates is given by Abbena et al. (2017, Lemma 3.7)

$$\kappa = \frac{\rho^2 + 2(\frac{d\rho}{d\psi})^2 - \rho\frac{d^2\rho}{d\psi^2}}{\left(\rho^2 + (\frac{d\rho}{d\psi})^2\right)^{3/2}}.$$

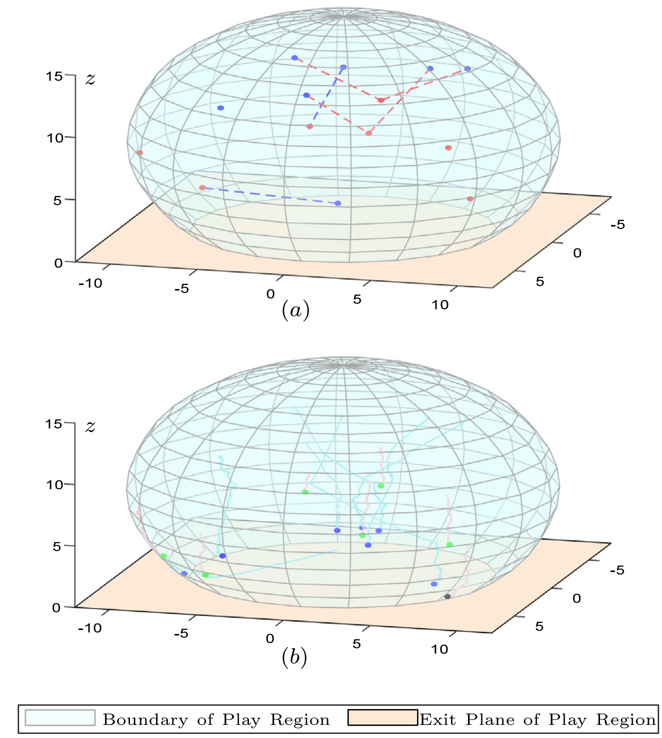

Fig. 6. Simulation of a game with seven pursuers and seven evaders in the bounded convex play region.

Therefore, if (36) holds, then we have $\kappa > 0$ for all $\psi \in [0, 2\pi]$, and by Toponogov (2006, Problem 1.7.6), the curve is convex. Moreover, since κ is strictly positive, the curve does not contain any line segments and thus is strictly convex.

Since the curve is convex, by definition, the set consisting of the curve and its interior has a supporting hyperplane at every point on the boundary. Along with the fact that the set is closed and has nonempty interior, we have that the set is convex. The strict convexity of the set follows from that of the curve.

Appendix B. Proof of Theorem 4.2

Let $G = (U \cup V, E)$ with conflict graph C be an instance of MBMC, where G and C are given by (28) and (29), respectively. Assume that the Sequential Matching Algorithm is applied on this instance, and returns a matching $M = M_1 \cup M_2 \cup M_3$, where M_1, M_2 and M_3 may be empty. As lines 4 and 8 in Algorithm 1 show, we remove the matched pursuers and evaders when computing the next matching. Thus, M satisfies the conflict graph C. Since this algorithm involves solving three local search maximum with size 2 swaps which can be solvable in polynomial time, it is also of polynomial time.

Let $M^* = M_1^* \cup M_2^* \cup M_3^*$ be an optimal solution of MBMC on the given instance, where M_i^* $(i = 1, 2, 3)$ is the set of edges incident to vertices in U with i pursuers. Next, we give an upper bound of $|M_i^*|$.

(i) Since M_1 is the maximum matching of G_1, we have

$$|M_1^*| \le |M_1| = |A_1| = |B_1| \le |M|. \tag{37}$$

(ii) For G_2, we add a pursuer $P_i \in A_1$ into U_2, and obtain a new subgraph $G_2' = (U_2' \cup V_2', E_2')$ of G with

$$U_2' = [\mathcal{P}/A_1 \cup P_i]^2, \quad V_2' = V_2,$$
$$E_2' = \left\{e_{sj} \in E \mid P_s \in U_2', E_j \in V_2'\right\}.$$

From now on, we omit the formulation of the conflict graph for any graph we construct, because it can be obtained routinely by at most one appearance of each pursuer in the matching.

Let M_2' be the maximum matching of G_2', and it is easy to see that $|M_2'| \geq |M_2|$. Note that pursuer P_i can capture at most one evader by itself or cooperation with another pursuer. If we remove pursuer P_i from U_2', $|M_2'|$ decreases by at most 1 and G_2' is reduced to G_2. Moreover, since each edge in E_2 involves three players, then the local search maximum with size 2 swaps is a 1/2-approximation algorithm (Cygan, 2013; Hurkens & Schrijver, 1989). Thus, we have $|M_2'| \leq 2|M_2| + 1$.

The similar results can be obtained when we add an evader $E_j \in B_1$ into V_2. Thus, by adding all pursuers in A_1 and all evaders in B_1 into the graph G_2, we can obtain the subgraph $G_2'' = (U_2'' \cup V_2'', E_2'')$ of G with

$$U_2'' = [\mathcal{P}]^2,\ V_2'' = \mathcal{E},\ E_2'' = \left\{e_{sj} \in E \mid P_s \in U_2'', E_j \in V_2''\right\},$$

and the maximum matching M_2'' of G_2'' satisfies

$$|M_2''| \leq 2|M_2| + |A_1| + |B_1|.$$

The graph G_2'' consists of all edges of G incident to vertices in U containing one or two pursuers. Thus, $|M_1^*| + |M_2^*|$ is bounded by

$$|M_1^*| + |M_2^*| \leq |M_2''| \leq 2|M_2| + 2|M_1| \leq 2|M|. \tag{38}$$

(*iii*) For the subgraph G_3, we add a pursuer $P_i \in A_1 \cup A_2$ into U_3 and obtain a new subgraph $G_3' = (U_3' \cup V_3', E_3')$ of G with

$$U_3' = \left[\mathcal{P} \setminus \left(A_1 \cup A_2\right) \cup P_i\right]^3, \quad V_3' = V_3,$$
$$E_3' = \left\{e_{sj} \in E \mid P_s \in U_3', E_j \in V_3'\right\}.$$

Analogously, the maximum matching M_3' of G_3' satisfies $|M_3'| \leq |M_3| + 1$. By putting all pursuers in $A_1 \cup A_2$ and all evaders in $B_1 \cup B_2$ into G_3, then G_3 becomes G and its maximum matching M^* satisfies

$$\begin{aligned} |M^*| &\leq 3|M_3| + |A_1| + |A_2| + |B_1| + |B_2| \\ &= 3|M_3| + 2|M_1| + 3|M_2| \leq 3|M|, \end{aligned} \tag{39}$$

where $|A_1| = |B_1| = |M_1|$ and $|A_2| = 2|B_2| = 2|M_2|$. Thus, the statement follows from (37)–(39).

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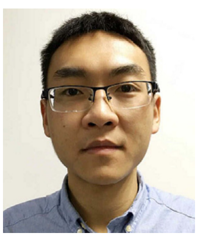

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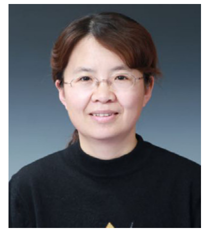

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