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Brief paper

On guaranteeing point capture in linear n-on-1 endgame interception engagements with bounded controls[☆]

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ABSTRACT

A linearized endgame interception scenario along a line between a single evading target and n pursuers is considered, in which the adversaries' controls are bounded and have arbitrary-order dynamics, and the evader's maneuvers are not known a priori to the pursuing team. To determine the merit in utilizing multiple interceptors, in terms of their capability to impose point capture, a capturability analysis is performed, presenting necessary and sufficient conditions for the feasibility of point capture for any admissible evader maneuver. It is shown that the pursuing team is capable of guaranteeing point capture if and only if it consists of at least one pursuer capable of independently imposing point capture. This requirement is independent of the number of pursuers, leading to the conclusion that it cannot be relaxed by increasing the number of interceptors or by any manner of cooperation, in terms of coordinated motion, between the pursuers.

1. Introduction

Cooperative strategies are becoming more and more popular with the continuing evolution and advancement in decision making capabilities of autonomous vehicles. Utilizing multiple agents to perform a given task can be beneficial even in cases when the goal is achievable by a single agent. With regard to interception engagements of an evading target, through shared information and coordinated actions the capability requirements and/or the number of required agents may be relaxed and reduced, respectively. It is therefore a great point of interest, when analyzing interception engagements, to discern under what conditions the evader's capture can be guaranteed, and whether or not these conditions are dependent on the number of pursuers.

Isaacs, in his study of pursuit-evasion games (Isaacs, 1965), was the first to obtain explicit conditions for capture in conflicts between a single pursuer and a single evader. The construction of the capture zone's boundary provided the set of initial conditions from which the pursuer was capable of guaranteeing the evader's capture, given the engagement parameters. An example of a game between adversaries with maneuverability constraints was presented by Isaacs in the form of the so-called game of two cars. This planar engagement includes adversaries which have constant speeds, minimum turn radii and no control dynamics. Additionally, each adversary has knowledge only of its opponent's current position and attitude. In Isaacs' analysis of this game "capture" was defined by the pursuer reaching some non-negative distance from the evader and the pursuer was assumed to be faster than the evader. This was later completed in Meier (1969), in which the case of a slower pursuer was considered. The first to focus on the required capabilities for guaranteeing capture of an evading target was Cockayne (1967). He proved that the pursuer in the game of two cars can capture the evader (achieve position coincidence) from any initial geometry if and only if it has a speed advantage and is at least as maneuverable as the evader. Cockayne stated that these conditions should coincide with Isaacs' results in the game of two cars with the capture radius set to zero. Rublein (1972) later extended Cockayne's work to address motion in three dimensional space. He showed that a sufficient condition for guaranteeing point capture is the pursuer's superiority both in speed and in maneuverability. In Borowko and Rzymowski (1984) an inverse study to Cockayne (1967) was presented, concerning the capabilities required in order to guarantee successful evasion from a pursuer. The author proved that the evader in the game of two cars can avoid capture for any initial conditions if and only if one of the following holds: (a) the evader has a speed advantage and its maximal maneuver capability is greater than or equal to that of the pursuer times the pursuer-to-evader speed ratio, (b) the evader's speed is equal to the pursuer's and it has a maneuverability advantage. These results together with those presented in Cockayne (1967) lead to the conclusion that if the evader has a speed disadvantage,

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but its maximal maneuver capability is greater than that of the pursuer times the pursuer-to-evader speed ratio, then there exist some initial conditions from which the pursuer can guarantee the evader's capture. An extension of this work to a planar case of n pursuers vs. a single evader was presented in Rzymowski (1984). It was shown that if, relative to each pursuer, the evader has a speed advantage and its maximal maneuver capability is greater than or equal to that of any pursuer times the corresponding pursuer-to-evader speed ratio it can avoid capture indefinitely.

There have been various other publications discussing capturability of a single target by multiple pursuers, with varying engagement formulations and capture definitions. In Bopardikar, Bullo, and Hespanha (2009) a cooperative strategy is proposed for the confinement of a more maneuverable but slower evader by a team of identical pursuers. It is shown that, given the non-zero turn radius and capture radius of the pursuers and the evader/pursuer speed ratio, there exists a minimum number of pursuers required in order to guarantee successful confinement. A similar analysis is performed in Chen, Zha, Peng, and Gu (2016) for a different multi-player pursuit-evasion game: a team of pursuers endeavors to capture (impose position coincidence) a single target in the plane, all of which have constant speeds and instantaneous turn capability. In this case the capturability analysis is extended to include the initial conditions from which capture may be guaranteed. Another method used in the development of pursuit strategies for multiple pursuers intercepting a single target is Voronoi partitioning (Bakolas & Tsiotras, 2012; Zhou, Zhang, Ding, Huang, Stipanović, & Tomlin, 2016). In Bakolas and Tsiotras (2012) a capturability condition is derived for a simple planar pursuit-evasion scenario in which each of the adversaries has bounded speed and is capable of instantaneous turns. In Zhou et al. (2016) a cooperative pursuit strategy is developed for a similar problem in which motion is restricted to a convex planar domain. It is shown that under the proposed strategy capture (defined by the evader entering a finite radius) is guaranteed. Generally, as is the case in these presented works, the capturability conditions are dependent on the proposed strategies of the adversaries and may therefore be more stringent than actually needed under optimal play.

In scenarios where during the endgame the adversaries' motion is near their respective collision courses the kinematics of the engagement can be linearized relative to some fixed frame (Zarchan, 1994). For such cases a point of reference with regard to capturability is once again solutions to games of pursuit. Existing solutions to linear pursuit-evasion games of a single pursuer vs. a single evader with bounded controls also include variations on the order of the players' control dynamics as well as the number of control inputs (Gutman, 1979; Gutman & Leitmann, 1976; Qi, Liu, & Tang, 2011; Shima, 2005; Shima & Golan, 2006; Shima & Shinar, 2002; Shinar, 1981; Turetsky & Shinar, 2003). An analysis of a class of linear time-varying feedback pursuit strategies in the same framework was presented in Turetsky (2008), focusing on scenarios in which point capture is guaranteed.

These previous studies have yielded important conclusions with regard to the necessary and sufficient requirements from interceptors in 1-on-1 engagements. Following these works, and considering interception scenarios with multiple pursuers, it is of interest to examine the necessary and sufficient conditions for capture in a general n-on-1 engagement, the results of which have important implications on the merits of utilizing a multiplicity of pursuers.

This paper presents an analytical study of the conditions for the feasibility of exact capture in an n-on-1 linearized endgame engagement along a line in which the adversaries' kinematics and control dynamics are represented together by arbitrary-order time-variant linear systems. Rather than solving a general n-on-1 pursuit-evasion game, we adopt a reachability approach, thereby

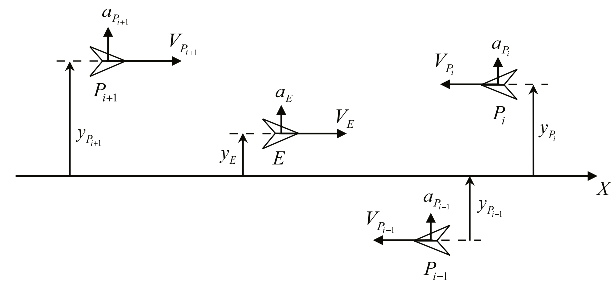


Fig. 1. Linear n-on-1 engagement scheme.

avoiding the need to first define optimal strategies for the adversaries. It is shown that the pursuing team is capable of guaranteeing point capture if and only if it consists of at least one pursuer capable of independently imposing point capture. This requirement is independent of the number of pursuers, leading to the conclusion that it cannot be relaxed by increasing the number of interceptors or by any manner of cooperation, in terms of coordinated motion, between the pursuers.

The remainder of the paper is structured as follows: In Section 2 the formulation of the interception engagement and its mathematical model are presented. Next, general conditions for the existence of a capture zone are derived in Section 3, followed by concluding remarks in Section 4.

2. Linear n-on-1 engagement formulation

Consider the endgame of a planar interception engagement of a single evader by a group of n pursuers, in which it is assumed (as is common in endgame missile interception engagements, see Shima & Shinar, 2002; Zarchan, 1994) that

- the adversaries can be regarded as point masses with linear arbitrary-order control dynamics, having multiple decoupled bounded control inputs, the bounds of which are known functions of time,
- the adversaries' motion is restricted to a plane and their speeds are known functions of time, which are not necessarily equal,
- the adversaries are all near head-on or tail-chase and their motion can be linearized around a common fixed reference line,

as depicted in Fig. 1. V and a denote the speed along X and the acceleration normal to X, respectively, and y denotes the normal displacement relative to X. We define the group of adversaries $G = \{P_1, P_2, \ldots, P_n, E\}$, the control input tags $N_c^j = \{1, 2, \ldots, n_c^j\}$ for each $j \in G$ and the pursuer tags $N = \{1, 2, \ldots, n\}$. Under these assumptions the lateral maneuvers do not affect the horizontal velocities, but only the vertical speeds, relative to the reference line. Such would be the case in realistic scenarios which include multiple pursuers launched from a single platform (e.g. aircraft) and point defense (e.g. ballistic missile defense). As a result, the kinematics along X are solved, yielding

$$r_i(t) = r_i^o - \int_{t_0}^{t} V_{c_i}(\xi)d\xi, \quad i \in N, \tag{1}$$

where r_i is the ith pursuer-to-evader range along X. t_o is the initial time and r_i^o and $V_{c_i}(t)$ are, respectively, the initial ith pursuer-to-evader range and the positive closing speed between the ith pursuer and the evader, both measured along the common reference line (for near head-on $V_{c_i}(t) \approx V_{P_i}(t)+V_E(t)$ and for near tail-chase $V_{c_i}(t) \approx V_{P_i}(t) - V_E(t)$, where for the latter $V_{P_i}(t) > V_E(t\ \forall t \geq$

t_o) is required). Rearranging (1) and setting $t = t_{f_i}$ and $r_i = 0$ the following approximate interception times of the pursuers are obtained

$$t_{f_i} = \underset{t\geq t_o}{arg}\left\{r_i^o - \int_{t_o}^{t} V_{c_i}(\xi)d\xi = 0\right\}, \quad i \in N. \tag{2}$$

It is at this instant that the distance between P_i and E along X is zero and P_i's miss is measured. Also, note that beyond this point pursuer P_i becomes inconsequential, as it is now moving away from the evader and can no longer influence its own miss distance. We assume the pursuers are ordered such that

$$t_{f_1} \leq t_{f_2} \leq \cdots \leq t_{f_n}. \tag{3}$$

The kinematics of the linearized intercept geometry are given by

$$\begin{aligned}\dot{y}_j(t) &= v_j(t)\\ \dot{v}_j(t) &= a_j(t)\end{aligned}, \quad j \in G, \tag{4}$$

where v_j is j's velocity normal to X. Under the stated assumptions the adversaries' control dynamics can be represented by the arbitrary-order, time-variant linear systems

$$\begin{aligned}\dot{\mathbf{z}}_j(t) &= \mathbf{A}_z^j(t)\mathbf{z}_j + \mathbf{B}_z^j(t)\mathbf{u}_j(t)\\ a_j(t) &= \mathbf{c}_z^j(t)\mathbf{z}_j + \mathbf{d}_z^j(t)\mathbf{u}_j(t)\end{aligned}, \quad j \in G, \tag{5}$$

where $\mathbf{A}_z^j(t) \in \mathbb{R}^{n_z^j\times n_z^j}, \mathbf{B}_z^j(t) \in \mathbb{R}^{n_z^j\times n_c^j}, \mathbf{c}_z^j(t) \in \mathbb{R}^{1\times n_z^j}, \mathbf{d}_z^j(t) \in \mathbb{R}^{1\times n_c^j}$. $\mathbf{z}_j$ is j's vector of n_z^j internal dynamic states and $\mathbf{u}_j$ is j's vector of n_c^j dimensionless control inputs, each of which is bounded by

$$|u_i^j(t)| \leq \overline{u}_i^j(t), \quad i \in N_c^j.$$

By defining

$$u_j^{max}(t) = \sum_{i=1}^{n_c^j} \overline{u}_i^j(t), \tag{6}$$

we may rewrite

$$\begin{aligned}&|u_i^j(t)| \leq \gamma_i^j(t) \cdot u_j^{max}(t)\\ &0 \leq \gamma_i^j(t) \leq 1\end{aligned}, \quad i \in N_c^j, \tag{7}$$

where, by definition,

$$\gamma_i^j(t) = \frac{\overline{u}_i^j(t)}{u_j^{max}(t)} \tag{8}$$

and each $\Gamma_j(t) = [\gamma_1^j(t)\ \ \gamma_2^j(t)\ \ \ldots\ \ \gamma_{n_c^j}^j(t)]^T$ satisfies

$$\|\Gamma_j(t)\|_1 = \sum_{i=1}^{n_c^j} \gamma_i^j(t) = 1, \quad j \in G.$$

In essence, each element in $\Gamma_j(t)$ represents the relative level of effectiveness of its corresponding control input at time t. Defining each entity's state vector as follows

$$\mathbf{x}_j(t) = \left[y_j(t)\ \ v_j(t)\ \ \mathbf{z}_j(t)^T\right]^T, \quad j \in G, \tag{9}$$

we obtain the following linear time-variant system

$$\begin{aligned}\dot{\mathbf{x}}_j(t) &= \mathbf{A}_j(t)\mathbf{x}_j(t) + \mathbf{B}_j(t)\mathbf{u}_j(t); \quad \mathbf{x}_j(t_o) = \mathbf{x}_j^o\\ \mathbf{u}_j(t) &\in \mathbf{U}_j(t), \qquad\qquad\qquad\qquad j \in G.\end{aligned} \tag{10}$$

$\mathbf{U}_j(t) = \left\{\mathbf{v} \,\middle|\, \mathbf{v} \in \mathbb{R}^{n_c^j},\ |v_i| \leq \gamma_i^j(t) \cdot u_j^{max}(t)\ \forall i \in N_c^j\right\}$ is j's set of admissible controls at time t and

$$\mathbf{A}_j(t) = \begin{bmatrix} 0 & 1 & \mathbf{0}_{1\times n_z^j}\\ 0 & 0 & \mathbf{c}_z^j(t)\\ \mathbf{0}_{n_z^j\times 1} & \mathbf{0}_{n_z^j\times 1} & \mathbf{A}_z^j(t)\end{bmatrix}, \quad \mathbf{B}_j(t) = \begin{bmatrix}\mathbf{0}_{1\times n_c^j}\\ \mathbf{d}_z^j(t)\\ \mathbf{B}_z^j(t)\end{bmatrix}, \tag{11}$$

$\mathbf{0}_{p\times q}$ denoting a zero matrix of dimensions $p \times q$. We also denote the space of all piecewise continuous functions on the time interval $[t_1, t_2]$ as $\mathcal{PC}([t_1, t_2])$ and define the space of admissible control sequences of each $j \in G$ on the interval $[t_1, t_2]$ as the set of all $n_c^j \times 1$ vectors of piecewise continuous functions on the interval $[t_1, t_2]$ which are bounded at any time in the interval by $u_j^{max}(t)$

$$\begin{aligned}\mathcal{U}_j([t_1, t_2]) = \Big\{&\mathbf{f} : [t_1, t_2] \to \mathbb{R}^{n_c^j\times 1} \mid f_i \in \mathcal{PC}([t_1, t_2])\\ &\forall i \in N_c^j,\ \mathbf{f}(t) \in \mathbf{U}_j(t)\ \forall t \in [t_1, t_2]\Big\}.\end{aligned}$$

The set of all pursuers' control sequences and the set of all pursuers' spaces of admissible control sequences on the interval $[t_1, t_2]$ will be denoted, respectively, by $\mathbf{u}_P(t) \triangleq \{\mathbf{u}_{P_1}(t), \mathbf{u}_{P_2}(t), \ldots \mathbf{u}_{P_n}(t)\}$ and $\mathcal{U}_P([t_1, t_2]) \triangleq \mathcal{U}_{P_1}([t_1, t_2]) \times \mathcal{U}_{P_2}([t_1, t_2]) \times \cdots \times \mathcal{U}_{P_n}([t_1, t_2])$.

We assume a perfect information structure, meaning each of the adversaries has complete information of the entire state vectors of all entities. The pursuers may share their current strategies with other teammates, but the evader's command is unknown to them throughout the engagement. Moreover, since we are not interested in the explicit pursuit strategy, we neglect any communication constraints and assume the pursuers are capable of sharing information instantaneously.

Definition 2.1 (*Positional Reachable Set*). Given its state at time t the positional reachable set of j at instant t_1 from $(t, \mathbf{x}_j(t))$ is the set of all achievable vertical distances, measured from the fixed reference line, at instant t_1 subject to $\mathbf{u}_j(\xi) \in \mathcal{U}_j([t, t_1])$

$$\begin{aligned}&S_j(t_1, t, \mathbf{x}_j(t))\\ &= \left\{y \in \mathbb{R} \,\middle|\, \begin{aligned}&y = \mathbf{D}_j\Phi_j(t_1, t)\mathbf{x}_j(t)\\ &+ \int_t^{t_1} \mathbf{D}_j\Phi_j(t_1, \xi)\mathbf{B}_j(\xi)\mathbf{u}_j(\xi)d\xi,\\ &\mathbf{u}_j(\xi) \in \mathcal{U}_j([t, t_1])\end{aligned}\right\}, \quad j \in G,\end{aligned} \tag{12}$$

where Φ_j is the transition matrix associated with (10) and $\mathbf{D}_j = [1\ \ \mathbf{0}_{1\times(n_z^j+1)}]$.

Since $\mathbf{U}_j(t)$ at any given time is convex $\forall j \in G$, the adversaries' positional reachable sets are convex, as Lemma 2.4 states in Kurzhanski and Pravin (2002). Therefore, since each point in the set is a scalar, it is actually a closed interval

$$S_j(t_1, t, \mathbf{x}_j(t)) = [\underline{y}_j(t_1, t, \mathbf{x}_j(t)), \overline{y}_j(t_1, t, \mathbf{x}_j(t))], \quad j \in G. \tag{13}$$

The limits of the positional reachable sets are defined by

$$\begin{aligned}\underline{y}_j(t_1, t, \mathbf{x}_j(t)) &= \min_{\mathbf{u}_j(\xi)\in\mathcal{U}_j([t,t_1])} y_j(t_1, t, \mathbf{x}_j(t), \mathbf{u}_j(\xi))\\ \overline{y}_j(t_1, t, \mathbf{x}_j(t)) &= \max_{\mathbf{u}_j(\xi)\in\mathcal{U}_j([t,t_1])} y_j(t_1, t, \mathbf{x}_j(t), \mathbf{u}_j(\xi)), \quad j \in G\end{aligned} \tag{14}$$

where

$$\begin{aligned}&y_j(t_1, t, \mathbf{x}_j(t), \mathbf{u}_j(\xi)) = \mathbf{D}_j\Phi_j(t_1, t)\mathbf{x}_j(t)\\ &\quad + \int_t^{t_1} \mathbf{D}_j\Phi_j(t_1, \xi)\mathbf{B}_j(\xi)\mathbf{u}_j(\xi)d\xi, \quad j \in G,\end{aligned} \tag{15}$$

and the size of each set is given by

$$R_j(t_1, t, \mathbf{x}_j(t)) = \overline{y}_j(t_1, t, \mathbf{x}_j(t)) - \underline{y}_j(t_1, t, \mathbf{x}_j(t)), \quad j \in G. \tag{16}$$

Clearly $\forall j \in G$

$$\begin{aligned}&\underline{y}_j(t_1, t, \mathbf{x}_j(t)) = \mathbf{D}_j\Phi_j(t_1, t)\mathbf{x}_j(t)\\ &\quad - \int_t^{t_1} |\mathbf{D}_j\Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)u_j^{max}(\xi)d\xi\\ &\overline{y}_j(t_1, t, \mathbf{x}_j(t)) = \mathbf{D}_j\Phi_j(t_1, t)\mathbf{x}_j(t)\\ &\quad + \int_t^{t_1} |\mathbf{D}_j\Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)u_j^{max}(\xi)d\xi,\end{aligned} \tag{17}$$

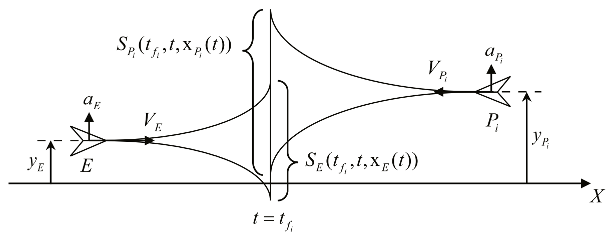


Fig. 2. Positional reachable sets overlap.

where $|\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|$ denotes here the vector with elements equal to the absolute values of the elements in $\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)$, and

$$R_j(t_1, t, \mathbf{x}_j(t)) = R_j(t_1, t) \\ = 2\int_t^{t_1} |\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)u_j^{max}(\xi)d\xi, \quad j \in G. \tag{18}$$

Actually, although not explicitly specified, the positional reachable set of each entity at a specified time t_1 can be placed in the physical space. This results from the solution of the kinematics along X, expressed by the relation in (1), which indicates that time and location along the horizontal reference line are interchangeable.

Remark 2.1. For every pursuer P_i, its and the evader's positional reachable sets at t_{f_i} from any instant t are located at the same point along the horizontal reference line (by definition of t_{f_i}). It is therefore at this instant only that an overlap may exist between the two sets, as demonstrated in Fig. 2. Consequently, t_{f_i} is the only instant at which the positional coincidence of P_i and E is possible.

Let us define the set of all admissible initial conditions

$$\chi_o = \left\{ \{t_o, \mathbf{x}^o_{P_1}, \dots, \mathbf{x}^o_{P_n}, \mathbf{x}^o_E\} \,\middle|\, t_o < t_{f_1},\ \mathbf{x}^o_j \in \mathbb{R}^{n_z^j+2}\ \forall j \in G \right\}.$$

Definition 2.2 (*Independent Capture Zone*)**.** A pursuer's independent capture zone is the set of all initial conditions from which it is capable of guaranteeing point capture, for any admissible evader control sequence, independently

$$\left\{ \chi_o \in \chi_o \,\middle|\, \begin{array}{l} \forall \mathbf{u}_E(t) \in \mathcal{U}_E([t_o, t_{f_i}])\ \exists \mathbf{u}_{P_i}(t) \in \mathcal{U}_{P_i}([t_o, t_{f_i}]) : \\ y_{P_i}(t_{f_i}) = y_E(t_{f_i}) \end{array} \right\}.$$

Definition 2.3 (*Joint Capture Zone*)**.** The set of all initial conditions from which the pursuing team is capable of guaranteeing point capture, for any admissible evader control sequence, is called the joint capture zone

$$\left\{ \chi_o \in \chi_o \,\middle|\, \begin{array}{l} \forall \mathbf{u}_E(t) \in \mathcal{U}_E([t_o, t_{f_i}])\ \exists \mathbf{u}_P(t) \in \mathcal{U}_P([t_o, t_{f_i}]),\ i \in N : \\ y_{P_i}(t_{f_i}) = y_E(t_{f_i}) \end{array} \right\}.$$

Remark 2.2. Should a joint capture zone exist, it is important to point out that from any initial conditions within the capture zone:

- For a given admissible evasion sequence, the evader might not be capturable by each and every one of the pursuers, but only by some subset of the pursuing team.
- For a given admissible evasion sequence, each of the pursuers capable of achieving point capture is not necessarily capable of doing so independently, without the help of its teammates.
- For any two different admissible evasion sequences, positional coincidence with the evader might not be achieved by the same group of pursuers.

Essentially, these observations mean that we do not presume that any initial conditions within the joint capture zone (should it exist) are necessarily included in any of the pursuers' independent capture zones (should they exist). i.e. the existence of a joint capture zone is not initially assumed to be conditional on the existence of independent capture zones.

3. Capture zone existence conditions

In this section we examine whether the addition of pursuers in the defined linear setting is beneficial in terms of the capability of the pursuing team to impose point capture. To do this we derive general necessary and sufficient conditions under which point capture is possible from some set of initial conditions, regardless of the evader's behavior, i.e. conditions for the existence of a capture zone. As opposed to the pursuit-evasion game approach, we obtain these conditions without solving an optimization problem and deriving optimal strategies for the adversaries. Instead, we adopt a reachability approach, through which we define and analyze the capture zone existence conditions.

We begin our analysis with some preliminary results, given in the form of the following propositions and lemma, and conclude with the main result given in Theorem 3.1.

Proposition 3.1. *Any vertical position that is unreachable at t_f by an entity $j \in G$ from its state at some instant $t_1 \in [t_o, t_f)$ remains so throughout the engagement.*

Proof. We will prove this by negation. Assume that there exists some vertical position $\tilde{y} \in \mathbb{R}$ that is unreachable at $t = t_f$ by an entity $j \in G$ from its state at some instant $t_1 \in [t_o, t_f)$, $\mathbf{x}_j(t_1)$. Assume as well that for some instant $t_2 \in (t_1, t_f)$ and admissible control sequence $\tilde{u}_j^{(1)}(t) \in \mathcal{U}_j([t_1, t_2])$ this position becomes reachable from the resulting state $\mathbf{x}_j(t_2)$. This implies that, by definition, $\tilde{y} \in S_j(t_f, t_2, \mathbf{x}_j(t_2))$ and that therefore there exists some admissible control sequence $\tilde{u}_j^{(2)}(t) \in \mathcal{U}_j([t_2, t_f])$ that will yield $y_j(t_f) = \tilde{y}$. In conclusion, if j applies the following admissible control sequence

$$\tilde{u}_j(t) = \begin{cases} \tilde{u}_j^{(1)}(t), & t \in [t_1, t_2] \\ \tilde{u}_j^{(2)}(t), & t \in [t_2, t_f] \end{cases}$$

it will successfully impose $y_j(t_f) = \tilde{y}$, meaning that the exists some admissible control sequence in the interval $[t_1, t_f]$ by which it is possible for j to reach $\tilde{y}$ from state $\mathbf{x}_j(t_1)$. Therefore, $\tilde{y} \in S_j(t_f, t_1, \mathbf{x}_j(t_1))$ by definition, in contradiction with our preliminary assumption. □

Lemma 3.1. *An independent capture zone for pursuer P_i exists if and only if there exists a non-empty set of initial conditions, from which the pursuer is capable of independently maintaining complete coverage of the evader's positional reachable set at t_{f_i} throughout the engagement, for any admissible control sequence of the evader*

$$\begin{array}{l} \exists \mathbf{x}^o_{P_i} \in \mathbb{R}^{n_z^{P_i}+2},\ \mathbf{x}^o_E \in \mathbb{R}^{n_z^E+2},\ t_o \le t_{f_i}, \\ \forall \mathbf{u}_E(t) \in \mathcal{U}_E([t_o, t_{f_i}])\ \exists \mathbf{u}_{P_i}(t) \in \mathcal{U}_{P_i}([t_o, t_{f_i}]) : \\ \quad S_E(t_{f_i}, t, \mathbf{x}_E(t)) \subseteq S_{P_i}(t_{f_i}, t, \mathbf{x}_{P_i}(t))\ \forall t \in [t_o, t_{f_i}]. \end{array} \tag{19}$$

Proof. Assume that this condition is not satisfied, i.e. for any initial condition and for some admissible evader control sequence there is an instant at which there exists a subset of the evader's positional reachable set at t_{f_i} which is not in the pursuer's positional reachable set at t_{f_i}, regardless of the pursuer's control sequence

$$\begin{array}{l} \forall \mathbf{x}^o_{P_i} \in \mathbb{R}^{n_z^{P_i}+2},\ \mathbf{x}^o_E \in \mathbb{R}^{n_z^E+2},\ t_o \le t_{f_i} \\ \exists \tilde{t} \in [t_o, t_{f_i}),\ \mathbf{u}_E(t) \in \mathcal{U}_E([t_o, \tilde{t}]), \\ \quad \tilde{S}_E \subseteq S_E(t_{f_i}, \tilde{t}, \mathbf{x}_E(\tilde{t})) : \\ \quad\quad \tilde{S}_E \bigcap S_{P_i}(t_{f_i}, \tilde{t}, \mathbf{x}_{P_i}(\tilde{t})) = \emptyset\ \forall \mathbf{u}_{P_i}(t) \in \mathcal{U}_{P_i}([t_o, \tilde{t}]). \end{array} \tag{20}$$

Following Proposition 3.1, we know that from $\tilde{t}$ onwards any point in $\tilde{S}_E$ is unreachable at t_{f_i} by P_i. Therefore, it is clear that by applying any admissible control sequence $\tilde{\mathbf{u}}_E(t) \in \mathcal{U}_E([\tilde{t}, t_{f_i}])$ which brings the evader to a final position within the subset $\tilde{S}_E$, the evader can avoid capture by P_i, regardless of the pursuer's actions from $\tilde{t}$ onwards. Thus necessity is proven.

In order to prove sufficiency we merely consider the limit value $\lim_{t\to t_{f_i}} S_j(t_{f_i}, t, \mathbf{x}_j(t))$ of each $j \in \{P_i, E\}$, which obviously converges to a single point $y_j(t_{f_i})$. By definition, if (19) is satisfied it is satisfied particularly for the aforementioned limit values, meaning necessarily that the pursuer's position coincides with that of the evader at the terminal instant t_{f_i}

$$y_{P_i}(t_{f_i}) = y_E(t_{f_i}). \quad \square$$

Remark 3.1. It is important to note that for (19) to hold for pursuer P_i the coverage of the evader's positional reachable set must be maintained up until (and including) t_{f_i}. Therefore, maintaining coverage at a certain instant $t < t_{f_i}$ is, in itself, insufficient.

Proposition 3.2. *The size of the positional reachable set at any fixed instant t_1, $R_j(t_1, t)$, is a continuous decreasing function of time $\forall j \in G,\ t \le t_1$, and $R_j(t_1, t_1) = 0$.*

Proof. Recall that $\forall j \in G$ the size of the positional reachable set at some fixed instant t_1 is given by

$$R_j(t_1, t) = 2\int_t^{t_1} |\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)u_j^{max}(\xi)d\xi \quad \forall t \le t_1$$

and is a continuous function of time, since $\mathbf{D}_j$ is constant and $\mathbf{B}_j(t)$, $\Gamma_j(t)$, $u_j^{max}(t)$ and $\Phi_j(t_1, t)$ are continuous $\forall t \le t_1$ (follows from the fact that $\mathbf{A}_j(t)$ is continuous and $\dot{\Phi}_j(t_1, t) = -\Phi_j(t_1, t)\mathbf{A}_j(t)$; $\Phi(t_1, t_1) = \mathbf{I}$). Since the integrand is non-negative $\forall \xi < t_1$, then for any $\tilde{t} \in [t, t_1]$

$$\begin{aligned}&\int_t^{t_1} |\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)d\xi \\ &= \int_t^{\tilde{t}} |\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)d\xi + \int_{\tilde{t}}^{t_1} |\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)d\xi \\ &\ge \int_{\tilde{t}}^{t_1} |\mathbf{D}_j \Phi_j(t_1, \xi)\mathbf{B}_j(\xi)|\Gamma_j(\xi)d\xi.\end{aligned}$$

Therefore, we have $\forall t \le t_1,\ \tilde{t} \in [t, t_1]$

$$\bar{y}_j(t_1, t, \mathbf{x}_j(t)) - \underline{y}_j(t_1, t, \mathbf{x}_j(t)) \ge \bar{y}_j(t_1, \tilde{t}, \mathbf{x}_j(t)) - \underline{y}_j(t_1, \tilde{t}, \mathbf{x}_j(t)),$$

which leads us to conclude that $\forall j \in G$ the size of the positional reachable set at any given t_1 is a decreasing function of time $\forall t \le t_1$. Furthermore, since $S_j(t_1, t_1, \mathbf{x}_j(t_1))$ is a point, then by definition $R_j(t_1, t_1) = 0$. $\square$

Proposition 3.3. *At any time $t_1 \in [t_o, t_f)$ for any subset of j's positional reachable set at t_f, $\tilde{S}_j \subseteq S_j(t_f, t_1, \mathbf{x}_j(t_1))$, there exist an instant $t_2 \in [t_1, t_f)$ and an admissible control sequence $\mathbf{u}_j(t) \in \mathcal{U}_j([t_1, t_2])$ which, if applied from t_1 to t_2, will yield*

$$S_j(t_f, t_2, \mathbf{x}_j(t_2)) \subseteq \tilde{S}_j.$$

Proof. If $\tilde{S}_j = S_j(t_f, t_1, \mathbf{x}_j(t_1))$ then the desired t_2 is simply equal to t_1.

Suppose that $\tilde{S}_j \subsetneq S_j(t_f, t_1, \mathbf{x}_j(t_1))$. Choose any internal point $y_j^* \in \tilde{S}_j = [\tilde{\underline{y}}_j, \tilde{\bar{y}}_j]$

$$\begin{aligned}&y_j^* = \tilde{\underline{y}}_j + \alpha\tilde{R}_j = \tilde{\bar{y}}_j - \beta\tilde{R}_j \\ &0 < \alpha, \beta < 1,\ \alpha + \beta = 1,\end{aligned}$$

where $\tilde{R}_j = \tilde{\bar{y}}_j - \tilde{\underline{y}}_j$ is the size of $\tilde{S}_j$. Since $y_j^* \in S_j(t_f, t_1, \mathbf{x}_j(t_1))$, there necessarily exists some admissible control sequence $\mathbf{u}_j^*(t) \in \mathcal{U}_j([t_1, t_f])$ that brings j from $y_j(t_1)$ to $y_j(t_f) = y_j^*$. According to Proposition 3.1, this means that if $\mathbf{u}_j^*(t)$ is applied from t_1 to t_f then necessarily $y_j^* \in S_j(t_f, t, \mathbf{x}_j(t))\ \forall t \in [t_1, t_f]$. By Proposition 3.2 $R_j(t_f, t)$ is a continuous decreasing function of time $\forall t \in [t_1, t_f]$ with a final value $R_j(t_f, t_f) = 0$. Therefore, according to the intermediate value theorem, for any arbitrarily small value ρ_j there exists some instant $t_2 \in [t_1, t_f)$ for which the size of $S_j(t_f, t_2, \mathbf{x}_j(t_2))$ is equal to ρ_j. Since $y_j^* \in S_j(t_f, t_2, \mathbf{x}_j(t_2))$ and $R_j(t_f, t_2) = \rho_j$, then by choosing ρ_j such that

$$\begin{aligned}&y_j^* - \rho_j \ge \tilde{\underline{y}}_j \\ &y_j^* + \rho_j \le \tilde{\bar{y}}_j,\end{aligned}$$

we obtain an instant $t_2 \in [t_1, t_f)$ for which $S_j(t_f, t_2, \mathbf{x}_j(t_2)) \subseteq \tilde{S}_j$. $\square$

Theorem 3.1. *In an n-on-1 linear engagement a joint capture zone exists if and only if there exists a pursuer capable of independent capture.*

Proof. We will prove necessity by negation, showing that if none of the pursuers are capable of independent capture then for any initial condition there exists a point capture avoidance strategy for the evader, regardless of the pursuers' control sequences. Recall that if none of the pursuers are capable of independent capture then for any $i \in N$ (20) in the proof of Lemma 3.1 holds. Furthermore, since we can consider the initial conditions to be any combination of states and time, we may generalize this condition, as follows.

$$\begin{aligned}&\forall \mathbf{x}_{P_i}(t) \in \mathbb{R}^{n_z^{P_i}+2},\ \mathbf{x}_E(t) \in \mathbb{R}^{n_z^E+2},\ t \le t_{f_i} \\ &\exists \tilde{t} \in [t, t_{f_i}),\ \mathbf{u}_E(t) \in \mathcal{U}_E([t, \tilde{t}]),\ \tilde{S}_E \subseteq S_E(t_{f_i}, \tilde{t}, \mathbf{x}_E(\tilde{t})) : \\ &\quad \tilde{S}_E \bigcap S_{P_i}(t_{f_i}, \tilde{t}, \mathbf{x}_{P_i}(\tilde{t})) = \emptyset\ \forall \mathbf{u}_{P_i}(t) \in \mathcal{U}_{P_i}([t, \tilde{t}]).\end{aligned} \tag{21}$$

Consider the following evasion strategy for any given initial condition $\chi_o \in \boldsymbol{\chi}_o$:

Step 1:

- Apply control sequence $\mathbf{u}_{E,1}^{(1)}(t) \in \mathcal{U}_E([t_o, t_1^{(1)}])$ from t_o to some $t_1^{(1)} \in [t_o, t_{f_1})$ for which
$$\begin{aligned}&\exists S_E^{(1)} \subseteq S_E(t_{f_1}, t_1^{(1)}, \mathbf{x}_E(t_1^{(1)})) : \\ &\quad S_E^{(1)} \bigcap S_{P_1}(t_{f_1}, t_1^{(1)}, \mathbf{x}_{P_1}(t_1^{(1)})) = \emptyset \\ &\quad\quad \forall \mathbf{u}_{P_1}(t) \in \mathcal{U}_{P_1}([t_o, t_1^{(1)}]).\end{aligned}$$
- Apply some control sequence $\mathbf{u}_{E,1}^{(2)}(t) \in \mathcal{U}_E([t_1^{(1)}, t_1^{(2)}])$ from $t_1^{(1)}$ until some time $t_1^{(2)} \in [t_1^{(1)}, t_{f_1})$ for which
$$S_E(t_{f_1}, t_1^{(2)}, \mathbf{x}_E(t_1^{(2)})) \subseteq S_E^{(1)}.$$

Step 2:

- Apply control sequence $\mathbf{u}_{E,2}^{(1)}(t) \in \mathcal{U}_E([t_1^{(2)}, t_2^{(1)}])$ from $t_1^{(2)}$ to some $t_2^{(1)} \in [t_1^{(2)}, t_{f_2})$ for which
$$\begin{aligned}&\exists S_E^{(2)} \subseteq S_E(t_{f_2}, t_2^{(1)}, \mathbf{x}_E(t_2^{(1)})) : \\ &\quad S_E^{(2)} \bigcap S_{P_2}(t_{f_2}, t_2^{(1)}, \mathbf{x}_{P_2}(t_2^{(1)})) = \emptyset \\ &\quad\quad \forall \mathbf{u}_{P_2}(t) \in \mathcal{U}_{P_2}([t_1^{(2)}, t_2^{(1)}]).\end{aligned}$$
- Apply some control sequence $\mathbf{u}_{E,2}^{(2)}(t) \in \mathcal{U}_E([t_2^{(1)}, t_2^{(2)}])$ from $t_2^{(1)}$ until some time $t_2^{(2)} \in [t_2^{(1)}, t_{f_2})$ for which
$$S_E(t_{f_2}, t_2^{(2)}, \mathbf{x}_E(t_2^{(2)})) \subseteq S_E^{(2)}.$$

⋮

Step k:

- Apply control sequence $\mathbf{u}_{E,k}^{(1)}(t) \in \mathcal{U}_E([t_{k-1}^{(2)}, t_k^{(1)}])$ from $t_{k-1}^{(2)}$ to some $t_k^{(1)} \in [t_{k-1}^{(2)}, t_{f_k})$ for which
$$\begin{aligned}&\exists S_E^{(k)} \subseteq S_E(t_{f_k}, t_k^{(1)}, \mathbf{x}_E(t_k^{(1)})) : \\ &\quad S_E^{(k)} \bigcap S_{P_k}(t_{f_k}, t_k^{(1)}, \mathbf{x}_{P_k}(t_k^{(1)})) = \emptyset \\ &\quad\quad \forall \mathbf{u}_{P_k}(t) \in \mathcal{U}_{P_k}([t_{k-1}^{(2)}, t_k^{(1)}]).\end{aligned}$$

- Apply some control sequence $\mathbf{u}_{E,k}^{(2)}(t) \in \mathcal{U}_E([t_k^{(1)}, t_k^{(2)}])$ from $t_k^{(1)}$ until some time $t_k^{(2)} \in [t_k^{(1)}, t_{f_k})$ for which

  $S_E(t_{f_k}, t_k^{(2)}, \mathbf{x}_E(t_k^{(2)})) \subseteq S_E^{(k)}$.

$\vdots$

Step n:

- Apply control sequence $\mathbf{u}_{E,n}^{(1)}(t) \in \mathcal{U}_E([t_{n-1}^{(2)}, t_n^{(1)}])$ from $t_{n-1}^{(2)}$ to some $t_n^{(1)} \in [t_{n-1}^{(2)}, t_{f_n})$ for which

  $$\exists S_E^{(n)} \subseteq S_E(t_{f_n}, t_n^{(1)}, \mathbf{x}_E(t_n^{(1)})) : \; S_E^{(n)} \bigcap S_{P_n}(t_{f_n}, t_n^{(1)}, \mathbf{x}_{P_n}(t_n^{(1)})) = \emptyset \; \forall \mathbf{u}_{P_n}(t) \in \mathcal{U}_{P_n}([t_{n-1}^{(2)}, t_n^{(1)}]).$$

- Apply some control sequence $\mathbf{u}_{E,n}^{(2)}(t) \in \mathcal{U}_E([t_n^{(1)}, t_n^{(2)}])$ from $t_n^{(1)}$ until some time $t_n^{(2)} \in [t_n^{(1)}, t_{f_n})$ for which

  $S_E(t_{f_n}, t_n^{(2)}, \mathbf{x}_E(t_n^{(2)})) \subseteq S_E^{(n)}$.

Step $n+1$: Apply any admissible control sequence $\mathbf{u}_{E,n+1}(t) \in \mathcal{U}_E([t_n^{(2)}, t_{f_n}])$ from $t_n^{(2)}$ to t_{f_n}.

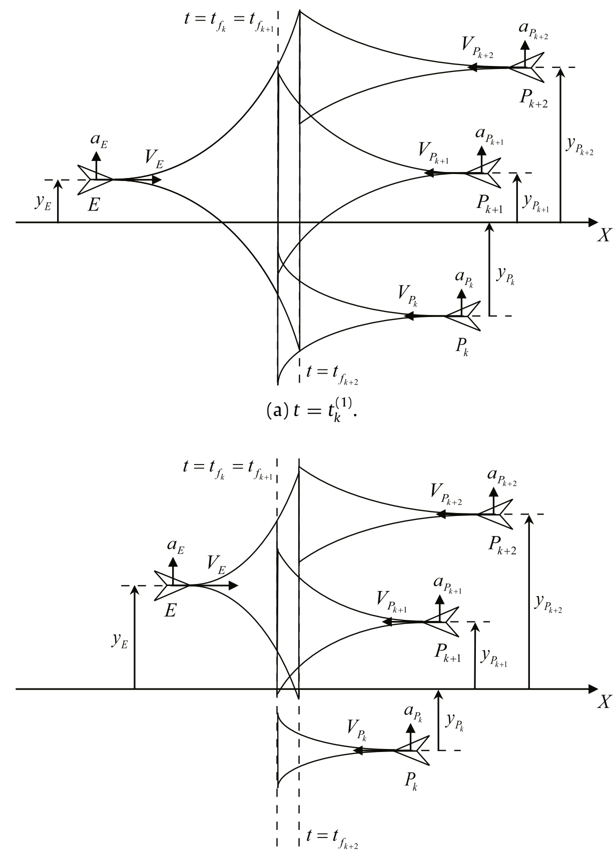
In the first stage of each step $k \in \{1, 2, \ldots, n\}$ the evader executes a maneuver that brings some subset $S_E^{(k)}$ of its positional reachable set at t_{f_k} to become unreachable by the k th pursuer. In the second stage the evader executes a maneuver that brings its entire positional reachable set into $S_E^{(k)}$, as illustrated in Fig. 3. Based on Proposition 3.1 it is evident that at the end of each step $k \in \{1, 2, \ldots, n\}$ the evader becomes and remains entirely unreachable to P_k, regardless of its future maneuvers. After step n ($t = t_n^{(2)}$) the evader's positional reachable set at t_{f_n} is completely unreachable by any of the pursuers. Hence, from this point onwards ($t \in [t_n^{(2)}, t_{f_n}]$) any admissible control of the evader will guarantee point capture avoidance, thereby proving necessity.

Remark 3.2. The existence of $\mathbf{u}_{E,k}^{(1)}(t)$, $t_k^{(1)}$ and $S_E^{(k)}$ in each step $k \in \{1, 2, \ldots, n\}$ is assured according to (21). The existence of $\mathbf{u}_{E,k}^{(2)}(t)$ and of $t_k^{(2)}$ is assured according to Proposition 3.3.

Remark 3.3. The presented evasion strategy is in essence an admissible strategy whereby the evader "eliminates" the pursuers one by one, rendering at each step the presence of another pursuer irrelevant in terms of imposing point capture. It should be pointed out that the order of elimination (according to increasing order of intercept times) is of importance. It is in fact what ensures that the instant $t_k^{(2)}$ at any step $k \in \{1, 2, \ldots, n\}$ maintains $t_k^{(2)} < t_{f_k}$.

Sufficiency is easily proved by simply regarding the n-on-1 engagement as a 1-on-1 interception scenario between pursuer P_i, which is capable of independent capture, and the evader. The same initial conditions which enable pursuer P_i to capture the evader in the 1-on-1 scenario necessarily ensure the capability of exact capture in the n-on-1 scenario. $\square$

Evidently, in a linear interception engagement with bounded controls the addition of pursuers does not relax the conditions for the existence of a capture zone. It should be noted that this result holds regardless of the type of, possibly cooperative, controls the pursuers choose to implement. This is an important result, which in essence means that, in the present framework, no element of cooperation between the pursuers, in terms of coordinated maneuvers, can relax the need for at least one pursuer which is capable of independent capture. Following this result it is deduced that by utilizing a team of pursuers, each incapable of independent capture, it is impossible to "create" a joint capture zone. However, it should be noted that when considering other aspects of capture, such as the capture region size or the capture zones when capture is defined as some maximum allowed non-zero miss, the addition of pursuers is not without influence (Ganebny, Kumkov, Le Ménec, & Patsko, 2012; Le Ménec, 2011).



(a) $t = t_k^{(1)}$.

(b) $t = t_k^{(2)}$.

Fig. 3. Proposed evasion strategy illustration: Step k.

4. Conclusions

A study of the necessary and sufficient conditions for the existence of a capture zone in a linearized n-on-1 endgame interception engagement along a line between adversaries with arbitrary-order control dynamics and bounded controls was presented. The analysis was performed with the use or reachable sets, which enabled us to obtain global results independent of any specific type of pursuit strategies.

It was proved that for a capture zone to exist it is necessary and sufficient to have at least one pursuer capable of independent capture. This points to the fact that, in terms of fulfilling the necessary conditions for point capture, the required combined advantages in maneuver capabilities and dynamic properties of at least one of the pursuers cannot be relaxed by increasing the number of pursuers (to any finite number) or through the use of cooperative pursuit strategies.

This result essentially means that in a linear engagement with multiple pursuers a previously nonexistent capture zone (for any of the pursuers) cannot be "created" by controlling the (finite) number of pursuers or by any means of cooperation. It may only be possible to enlarge an already existing capture zone of a pursuer capable of independent capture.

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