This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2828088, IEEE Transactions on Automatic Control

Design and Analysis of State-feedback Optimal Strategies for the Differential Game of Active Defense

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***Abstract*—This paper is concerned with a scenario of active target defense modeled as a zero-sum differential game. Differential game theory as developed by Isaacs provides the correct framework for the analysis of pursuit-evasion conflicts and the design of optimal strategies for the players involved in the game. This paper considers an Attacker missile pursuing a Target aircraft protected by a Defender missile which aims at intercepting the Attacker before the latter reaches the Target aircraft. A differential game is formulated where two opposing players/teams try to minimize/maximize the distance between the Target and the Attacker at the time of interception of the Attacker by the Defender and such time indicates the termination of the game. The Attacker aims to minimize the terminal distance between itself and the Target at the moment of its interception by the Defender. The opposing player/team consists of two cooperating agents: the Target and the Defender. These two agents cooperate in order to accomplish the two objectives: guarantee interception of the Attacker by the Defender and maximize the terminal Target-Attacker separation. In this paper we provide a complete, closed form solution of the active target defense differential game: we synthesize closed-loop state feedback optimal strategies for the agents and we obtain the Value function of the game. We characterize the Target's escape set and show that the Value function is continuous and continuously differentiable over the Target's escape set, and that it satisfies the Hamilton-Jacobi-Isaacs equation everywhere in this set.**

I. Introduction

Multi-agent pursuit-evasion scenarios present important and interesting, but also challenging, problems in aerospace guidance and control. In these types of problems one or more pursuers try to maneuver and reach a relatively small distance with respect to one or more evaders, which strive to escape the pursuers. These scenarios are typically considered in the context of dynamic games [1], [2]. Pursuit-evasion problems also arise in Unmanned Aerial Vehicles (UAVs) operations. For instance, in [3] a receding-horizon formulation is employed that provides evasive maneuvers for a UAV assuming a *known* pursuer's input. In [4], a multi-agent scenario is considered where a number of pursuers are assigned to intercept a group of evaders and where the goals of the evaders are assumed to be known.

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The interesting work by Breakwell and Hagedorn [5] studied the dynamic game of a fast pursuer trying to capture in minimal time two slower evaders in succession. Reference [6] is a take on the work in [5] and considered the case where the fast pursuer tries to capture several evaders. The slow evaders cooperate in order to maximize the total time from the beginning of the game until the last evader is captured. The obtained numerical solution shows that the optimal strategies of every agent, pursuer and evaders, consist of constant headings (the pursuer's heading is piecewise constant and it changes at time instants when an evader is captured).

The scenario of Active Target Defense addressed in this paper involves three agents: the Target (T), the Attacker (A), and the Defender (D). It was previously analyzed for the cases where A implements the known guidance laws of Pure Pursuit (PP) [7] or Proportional Navigation (PN) [8]. Under those assumptions on A's strategy, T and D cooperate and solve a one-sided optimal control problem that returns the optimal control laws for the T and D team so that A is intercepted by D and the separation between T and A at the interception time instant is maximized. In this respect, cooperative missile strategies have been considered by different researchers. For instance, the work in [9] describes multi-missile cooperative attacks on a single stationary target (ship) and in [10] for moving targets. Cooperation to control the impact time in order to simultaneously hit the ship is implemented as an outer loop around the typical Proportional Navigation (PN) guidance law.

Active defense of a non-maneuverable aircraft was investigated in [11]. The problem of aircraft active defense was studied by Li and Cruz [12]. Reference [12] considered the game of defending an asset from an attacking intruder using an interceptor. The differential game of active target defense with obstacles was analyzed in [13] where the problem was also approached by separating it into two smaller two-player games in order to reduce computational complexity. In [14] a differential game with multiple attackers, multiple defenders, and a stationary target in a bounded domain is studied. Due to numerical intractability the authors of this reference use the solution of the single attacker-single defender case in order to determine pairwise outcomes favorable to the defender team.

The authors of [15] provided a game formulation to solve reach and avoid problems involving nonlinear systems. The paper [16] considers a group of cooperative pursuers that try to capture a single evader within a bounded domain. The domain may also contain obstacles and the solution employs Voronoi partitions of the plane.

This paper addresses the Active Target Defense Differential Game (ATDDG) introduced in [17] and further analyzed in [18]. In these works the ATDDG is studied in a reduced state space of dimension three; however, formal proof was not provided in order to conclude that the proposed solution is in fact the complete solution of the differential game.

The possible non-differentiability of the solution of the HJI PDE is a concern in differential games and the viscosity solution of the Hamilton-Jacobi-Isaacs (HJI) Partial Differential Equation (PDE) [19], [20] provides a generalized solution concept. For instance, on an equivocal singular surface the Value function is not differentiable so it does not satisfy the HJI PDE in the classical sense, but it does in the viscosity sense. Hence, viscosity solutions provide an important theoretical foundation which is useful in differential games where a classical solution does not exist. The non-differentiability of solutions of the Hamilton-Jacobi equation in optimal control with state constraints is addressed in [21] and [22].

In this paper non-differentiability is not an issue. A thorough analysis and design of the players optimal strategies following Isaacs' method [23] is provided and it is shown that no singular surfaces exist, and the Value function is C^1. As stated in [14] this is the ideal situation in differential games, if it is attainable. A disadvantage of Isaacs' method is that it does not scale well as the dimension of the state increases. Many games may still have a classical solution in closed-form but it is very difficult to obtain it. Therefore, the results in this paper represent an important contribution where, for a differential game with higher dimensional state space, a (closed-form) continuous and continuously differentiable Value function in the Target's escape set is obtained which is the classical solution of the HJI PDE.

In the present paper the complete solution of the ATDDG in the original fixed Cartesian frame of reference, where the dimension of the state space is six instead of dimension three as in [17], is provided. This approach avoids the need for a change of coordinates to obtain the solution in the reduced state space and the necessary coordinate transformation to return this solution to the original frame and implement the obtained strategies in the fixed Cartesian frame. More importantly, we offer a formal synthesis of the saddle point strategies and we provide a complete analysis of the proposed solution in order to verify that it is in fact the complete solution of the differential game. Note that the term reduced state space is not used for approximation but for a fully equivalent state space representation where the state space dimension is reduced from six to three, but the dynamics become nonlinear.

The present paper provides a complete solution of the ATDDG. The main contributions with respect to [17] are as follows. A formal synthesis of the regular solution of the ATDDG is provided and it is proved that the regular solution covers the state space region where the Target is guaranteed to survive. In addition, the optimal Target heading for the interesting case where it is located on the orthogonal bisector of the segment connecting the instantaneous positions of A and D is obtained. Also, the solution of the Game of Kind is provided, that is, the state space where the Target's escape is guaranteed is characterized.

The paper is organized as follows. Section II states the Active Target Defense Differential Game. The ATDDG is solved in Section III for the case where the Target aircraft is closer to the Defender missile than to the Attacker missile. The opposite case is addressed in Section IV, subject to the condition that the Target is located in the escape region. The solution to the Game of Kind is presented in Section V and concluding remarks are included in Section VI.

II. The Game of Active Target Defense

The Target (T), the Attacker (A), and the Defender (D) have "simple motion" à la Isaacs, that is, are holonomic and the game is played in the Euclidean plane. Thus, the controls of T, A, and D are their respective instantaneous headings ϕ, χ, and ψ, see Fig. 1. In addition, T, A and D have constant speeds of V_T, V_A, and V_D, respectively. The states of the Target, the Attacker, and the Defender are specified by their Cartesian coordinates $\mathbf{x}_T = (x_T, y_T)$, $\mathbf{x}_A = (x_A, y_A)$, and $\mathbf{x}_D = (x_D, y_D)$, respectively; the dimension of the state space is six. We assume that the Attacker and Defender have similar capabilities, so $V_A = V_D$, while the Target/Attacker speed ratio $\alpha = V_T/V_A < 1$. Without loss of generality, the players' speeds are normalized so that $V_A = 1$.

The complete state of the ATDDG is specified by $\mathbf{x} := (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$. The game set is the entire space $\mathbb{R}^6$. We consider the ATDDG in the fixed Cartesian frame in order to not introduce nonlinear dynamics. The Attacker's control is his instantaneous heading angle, $\mathbf{u}_A = \{\chi\}$. The $T-D$ team affects the state of the game by cooperatively choosing the instantaneous respective headings, ϕ and ψ, of both the Target and the Defender, so the $T-D$ team's control is $\mathbf{u}_B = \{\phi, \psi\}$. The dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_A, \mathbf{u}_B)$ are specified by the system of ordinary differential equations

$$\begin{aligned}
\dot{x}_T &= \alpha\cos\phi, & x_T(0) &= x_{T_0}, \\
\dot{y}_T &= \alpha\sin\phi, & y_T(0) &= y_{T_0}, \\
\dot{x}_A &= \cos\chi, & x_A(0) &= x_{A_0}, \\
\dot{y}_A &= \sin\chi, & y_A(0) &= y_{A_0}, \\
\dot{x}_D &= \cos\psi, & x_D(0) &= x_{D_0}, \\
\dot{y}_D &= \sin\psi, & y_D(0) &= y_{D_0}
\end{aligned} \tag{1}$$

where the speed ratio $\alpha = V_T/V_A < 1$ is the problem parameter and the admissible controls are given by $\chi, \phi, \psi \in [-\pi, \pi]$. Both, the state and the controls, are unconstrained. The initial state of the system is

$$\mathbf{x}_0 := (x_{T_0}, y_{T_0}, x_{A_0}, y_{A_0}, x_{D_0}, y_{D_0}) = \mathbf{x}(t_0).$$

In this paper, we confine our attention to point capture, that is, the $A-D$ separation has to become zero in order for the Defender to intercept the Attacker. Another condition for termination is when the Attacker captures the Target, that

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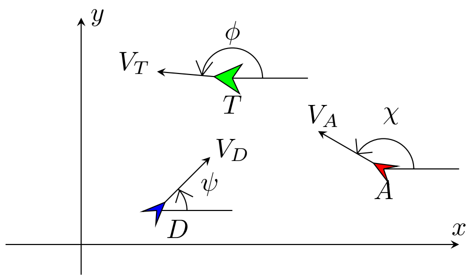

Fig. 1. Target, Attacker, and Defender scenario

is, the $A-T$ separation becomes zero. Thus, the target (or termination) set is

$$\mathcal{C} := \mathcal{C}_1 \bigcup \mathcal{C}_2 \tag{2}$$

where

$$\mathcal{C}_1 := \left\{ \mathbf{x} \mid \sqrt{(x_A - x_D)^2 + (y_A - y_D)^2} = 0 \right\}$$

represents interception of the Attacker by the Defender (and the Target escapes) and

$$\mathcal{C}_2 := \left\{ \mathbf{x} \mid \sqrt{(x_A - x_T)^2 + (y_A - y_T)^2} = 0 \right\}$$

represents the opposite outcome where the Attacker wins by capturing the Target. Conceptually, the ATDDG with termination set given in (2) belongs to the class of two-target differential games [24]–[26]. The two-target differential game concept was introduced in order to extend classical pursuit-evasion games where only one target, or termination, set exists. The pursuer tries to minimize the time to reach the termination set whereas the evader wants to maximize the time to reach the termination set or, when possible, to avoid reaching that set at all. Naturally, the two-target differential game is useful in the analysis of combat games [26] where the roles of pursuer and evader are not designated ahead of time; instead each player wants to defeat the opponent by terminating the game in its own terminal set.

The two termination conditions in (2) give rise to the Game of Kind within the framework of the ATDDG. The Game of Kind is solved in Section V—it is a prerequisite for addressing the Game of Degree, the focus of this paper. If the initial state is in a part of the state space where a strategy for the T/D team exists such that, under optimal play, the Defender intercepts the Attacker and the Target escapes then the ATDDG Game of Degree is played where the opposing teams try to min/max the terminal $A-T$ separation. On the other hand, if the initial state is in the state space region where the solution of the Game of Kind indicates that, under optimal play, the Attacker will capture the Target without being intercepted by the Defender, then, the ATDDG Game of Degree is not played in this part of the state space. A different Game of Degree could then be played in this region where the Target is going to be captured: The solution of this differential game will return the optimal strategy for the Attacker to capture the Target unmolested by the Defender. This part of the state space and this differential game will not be discussed in this paper.

Let us now define the order of preference in the ATDDG which will help in defining the admissible strategies of each player. Let m_t and n_t be numerical outcomes of terminating plays within the ATDDG, where $m_t < n_t$. Let $P = A$, that is, the Attacker is a pursuer (pursues the Target). Also, let $E = T/D$, that is, the Target evades the Attacker (aided by the Defender).

Definition 1: Define Υ as the outcome of any non-terminating play. Then, the ATDDG has an order of preference of Type P.

In a Type P order of preference [27] the following holds:

$$m_t \overset{P}{\succ} n_t \overset{P}{\succ} \Upsilon \quad \text{and} \quad \Upsilon \overset{E}{\succ} n_t \overset{E}{\succ} m_t.$$

This means that player P, the Attacker, who aims at minimizing the numerical outcome of the game, considers non-termination to be inferior to all other outcomes. Player E, the Target/Defender team, who aims at maximizing the numerical outcome of the game, considers non-termination to be superior to all other outcomes.

Definition 1 is well suited to the ATDDG. This definition not only establishes the order of preferences and types of outcomes of the game but agrees with the conditions of the real-life problem. The ATDDG will not terminate in separating or grazing contact [27] and the game will terminate in a penetrating type of contact, where either the Target is captured, or the Attacker is intercepted before it can capture the Target. The Attacker having the same speed as the Defender would, otherwise, separate from D and employ a non-terminating play. D can follow A, but, more importantly, T will increase the $A-T$ separation which is detrimental to the Attacker and beneficial to the Target-Defender team. Hence, non-termination is an inferior outcome for A compared to any other outcome. Also, non-termination is a superior outcome for the T/D team compared to any other outcome.

With respect to the information pattern, every agent knows the dynamics (1) and the speed ratio parameter α. It is assumed that the agents use causal strategies and that every agent has access to the state $\mathbf{x}$ at the current time t, that is, the ATDDG is a perfect information differential game; the optimal strategies will be state feedback strategies. Finally, and most importantly, it is assumed that the agents do not know the opponent's current decision: no discriminatory/stroboscopic strategies are needed in the ATDDG. Considering the ATDDG Game of Degree, the termination condition is

$$x_A = x_D, \qquad y_A = y_D. \tag{3}$$

The terminal manifold (3) is a four dimensional hyperplane in $\mathbb{R}^6$. The terminal time t_f is defined as the time instant when the state of the system satisfies (3), at which time the terminal state is $\mathbf{x}_f := (x_{T_f}, y_{T_f}, x_{A_f}, y_{A_f}, x_{D_f}, y_{D_f}) = \mathbf{x}(t_f)$. The terminal cost/payoff functional is

$$J(\mathbf{u}_A(t), \mathbf{u}_B(t), \mathbf{x}_0) = \Phi(\mathbf{x}_f) \tag{4}$$

where

$$\Phi(\mathbf{x}_f) := \sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}. \tag{5}$$

The cost/payoff functional depends only on the terminal state - the ATDDG is a terminal cost/Mayer type game. Its Value is given by

$$V(\mathbf{x}_0) := \min_{\mathbf{u}_A(\cdot)} \max_{\mathbf{u}_B(\cdot)} J(\mathbf{u}_A(\cdot), \mathbf{u}_B(\cdot); \mathbf{x}_0) \tag{6}$$

subject to (1) and (3), where $\mathbf{u}_A(\cdot)$ and $\mathbf{u}_B(\cdot)$ are the teams' state feedback strategies.

In the ATDDG Game of Degree A strives to close in on T while T and D form a team to defend from A: T and D maneuver such that D intercepts A before the latter reaches T and the A - T separation at interception time is maximized, while A strives to minimize the separation between A and T at the instant of interception. The $T-D$ team employs a cooperative optimal state feedback strategy $\phi^* = \phi^*(\mathbf{x})$, $\psi^* = \psi^*(\mathbf{x})$ to maximize the separation between the Target and the Attacker at the time instant of the Defender-Attacker contact while the Attacker devises his optimal state feedback strategy, that is, his instantaneous heading $\chi_A^* = \chi^*(\mathbf{x})$, to minimize the terminal $A-T$ separation/miss distance. The active target defense engagement is illustrated in Fig. 1. The objective is to synthesize closed-loop state feedback strategies and to obtain the complete solution of the differential game in the winning region of the $T\&D$ team.

III. Solution of the Differential Game

The "Two-sided" Pontryagin Maximum Principle (PMP) will be applied to the Target Defense Differential Game of degree - the method is also referred to as the "Two-person" extension of the PMP [28]. The state is $\mathbf{x} = (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$ where the dynamics are given by (1). The co-state is naturally $\lambda := (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}) \in \mathbb{R}^6$. Concerning the terminal condition, the cost/payoff function is evaluated when D intercepts A at time t_f. At the time instant where A is intercepted by D, the terminal time t_f is free and the terminal manifold $\mathcal{C}_1$ is the hyperplane in $\mathbb{R}^6$

$$\mathcal{C}_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ x_D \\ y_D \\ x_T \\ y_T \end{bmatrix} = 0_{2\times 1} \tag{7}$$

The Hamiltonian of the differential game is

$$\begin{aligned}\mathcal{H} &= \lambda_{x_A}\cos\chi + \lambda_{y_A}\sin\chi + \lambda_{x_D}\cos\psi \\ &\quad + \lambda_{y_D}\sin\psi + \alpha\lambda_{x_T}\cos\phi + \alpha\lambda_{y_T}\sin\phi.\end{aligned} \tag{8}$$

We note that the Hamiltonian and the dynamics are separable (or decoupled) in the controls ϕ, ψ and χ. Hence, $\min_\chi \max_{\phi,\psi} \mathcal{H} = \max_{\phi,\psi}\min_\chi \mathcal{H}$ and Isaacs' condition holds.

The solution of the game of degree is given in the following Theorem. The proof of the theorem follows the method for constructing regular solutions of differential games using the "Two-sided" or "Two-person" PMP to synthesize state feedback strategies [28], [27]. In the process, the Value function V is obtained and then, it is shown that V and $\frac{\partial V}{\partial \mathbf{x}}$ are continuous in the part of the state space where the Target is guaranteed to escape. It is also shown that the Value function globally satisfies the HJI equation in the T/D team's winning region of the state space, which is delineated by the solution of the Game of Kind, and where the ATDDG is played.

The midpoint O between A and D is given by

$$x_0(\mathbf{x}) = \tfrac{1}{2}(x_A + x_D), \quad y_0(\mathbf{x}) = \tfrac{1}{2}(y_A + y_D). \tag{9}$$

Define

$$m(\mathbf{x}) = -\tfrac{x_A - x_D}{y_A - y_D}, \quad n(\mathbf{x}) = y_0(\mathbf{x}) - m(\mathbf{x})x_0(\mathbf{x}). \tag{10}$$

Also, $\overline{DT} = \sqrt{(x_D - x_T)^2 + (y_D - y_T)^2}$ and $\overline{AT} = \sqrt{(x_A - x_T)^2 + (y_A - y_T)^2}$. We first consider the case where the initial state is such that $\overline{DT} \le \overline{AT}$, the distance between the Target and Defender is less or equal than the distance between the Target and the Attacker. In this case the Target is guaranteed to be able to escape for any speed ratio $\alpha \ge 0$.

The case where the initial state is such that $\overline{DT} > \overline{AT}$ will be discussed in Section IV. In such a case there is a subset of the state space, determined by the speed ratio parameter $0 \le \alpha < 1$ and characterized by the solution of the Game of Kind, where the Target is able to escape.

Theorem 1: Consider the Active Target Defense Differential Game (1)-(7) where the state is such that $\overline{DT} \le \overline{AT}$. The problem parameter is the speed ratio $0 \le \alpha < 1$. The optimal state feedback strategies of the Target, the Defender, and the Attacker are given by

$$\begin{aligned}\cos\phi^* &= \tfrac{x_T - x}{\sqrt{(x_T-x)^2+(y_T-y)^2}}, \\ \sin\phi^* &= \tfrac{y_T - y}{\sqrt{(x_T-x)^2+(y_T-y)^2}},\end{aligned} \tag{11}$$

$$\begin{aligned}\cos\psi^* &= \tfrac{x - x_D}{\sqrt{(x-x_D)^2+(y-y_D)^2}}, \\ \sin\psi^* &= \tfrac{y - y_D}{\sqrt{(x-x_D)^2+(y-y_D)^2}},\end{aligned} \tag{12}$$

$$\begin{aligned}\cos\chi^* &= \tfrac{x - x_A}{\sqrt{(x-x_A)^2+(y-y_A)^2}}, \\ \sin\chi^* &= \tfrac{y - y_A}{\sqrt{(x-x_A)^2+(y-y_A)^2}}\end{aligned} \tag{13}$$

where the coordinates x and y of the point of interception of A by D satisfy

$$y = mx + n \tag{14}$$

and m and n are given by (10). The coordinate x is a real solution of the quartic equation

$$\begin{aligned}&(1-\alpha^2)(m^2+1)^3x^4 + (1-\alpha^2)(m^2+1)^2(k_1+2k_2)x^3 \\ &+\big[(m^2+1)^2(k_3-\alpha^2k_4)+2(1-\alpha^2)(m^2+1)k_1k_2 \\ &+(m^2+1)(k_2^2-\tfrac{\alpha^2}{4}k_1^2)\big]x^2 \\ &+\big[(m^2+1)(2k_2k_3-\alpha^2k_1k_4)+k_1k_2(k_2-\tfrac{\alpha^2}{2}k_1)\big]x \\ &+k_2^2k_3-\tfrac{\alpha^2}{4}k_1^2k_4 = 0\end{aligned} \tag{15}$$

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which is parameterized by the speed ratio α and where

$$\begin{aligned} k_1 &= 2mn - [x_A + x_D + m(y_A + y_D)],\\ k_2 &= mn - x_T - my_T,\\ k_3 &= \tfrac{1}{2}(x_A^2 + x_D^2 + y_A^2 + y_D^2) + n^2 - n(y_A + y_D),\\ k_4 &= x_T^2 + (y_T - n)^2. \end{aligned} \tag{16}$$

The Value function is C^1, it satisfies the HJI PDE, and is explicitly given by

$$\begin{aligned} V(\mathbf{x}) = \alpha\Big[&\tfrac{1}{4}\big((x_A - x_D)^2 + (y_A - y_D)^2\big)\\ &+ \big(x - \tfrac{1}{2}(x_A{+}x_D)\big)^2 + \big(y - \tfrac{1}{2}(y_A{+}y_D)\big)^2\Big]^{1/2}\\ &+\sqrt{(x_T - x)^2 + (y_T - y)^2}. \end{aligned} \tag{17}$$

Proof. The optimal control inputs in terms of the co-state variables are obtained from Isaacs' Main Equation 1 (ME 1)

$$\min_{\chi}\max_{\phi,\psi}\mathcal{H} = 0 \tag{18}$$

and they are characterized by the relationships

$$\cos\chi^* = -\frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2+\lambda_{y_A}^2}},\quad \sin\chi^* = -\frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2+\lambda_{y_A}^2}}, \tag{19}$$

$$\cos\psi^* = \frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2+\lambda_{y_D}^2}},\quad \sin\psi^* = \frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2+\lambda_{y_D}^2}}, \tag{20}$$

$$\cos\phi^* = \frac{\lambda_{x_T}}{\sqrt{\lambda_{x_T}^2+\lambda_{y_T}^2}},\quad \sin\phi^* = \frac{\lambda_{y_T}}{\sqrt{\lambda_{x_T}^2+\lambda_{y_T}^2}}. \tag{21}$$

The co-state dynamics are obtained from $\dot{\lambda} = -\frac{\partial H}{\partial \mathbf{x}}$ which results in: $\dot{\lambda}_{x_A} = \dot{\lambda}_{y_A} = \dot{\lambda}_{x_D} = \dot{\lambda}_{y_D} = \dot{\lambda}_{x_T} = \dot{\lambda}_{y_T} = 0$; hence, all co-states are constant and we have that the optimal headings χ^*, ψ^* and ϕ^* are constant as well.

Concerning the solution of the attendant TPBVP in $\mathbb{R}^{12}$ on $0 \le t \le t_f$, we have 6 initial states specified by (1) and we need 6 more conditions at the terminal time t_f. In this respect, define the augmented terminal value function $\Phi_a : \mathbb{R}^6 \to \mathbb{R}^1$

$$\begin{aligned} \Phi_a(\mathbf{x}_f) := &\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}\\ &+ \nu_1(x_{A_f} - x_{D_f}) + \nu_2(y_{A_f} - y_{D_f}) \end{aligned} \tag{22}$$

where ν_1 and ν_2 are Lagrange multipliers. The PMP, or Dynamic Programming, directly yields the transversality/terminal co-state conditions

$$\lambda(t_f) = \frac{\partial}{\partial \mathbf{x}}\Phi_a(\mathbf{x}_f) \tag{23}$$

that is,

$$\lambda_{x_A} = \frac{x_{A_f} - x_{T_f}}{\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}} + \nu_1, \tag{24}$$

$$\lambda_{y_A} = \frac{y_{A_f} - y_{T_f}}{\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}} + \nu_2, \tag{25}$$

$$\lambda_{x_D} = -\nu_1, \tag{26}$$

$$\lambda_{y_D} = -\nu_2, \tag{27}$$

$$\lambda_{x_T} = \frac{x_{T_f} - x_{A_f}}{\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}}, \tag{28}$$

$$\lambda_{y_T} = \frac{y_{T_f} - y_{A_f}}{\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}}. \tag{29}$$

At this point, we have that (24)-(29) together with (3) yield 8 conditions. Since only 6 conditions are needed we eliminate the introduced Lagrange multipliers ν_1 and ν_2 from (24)-(27) and we obtain

$$\lambda_{x_A} + \lambda_{x_D} = \frac{x_{A_f} - x_{T_f}}{\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}}, \tag{30}$$

$$\lambda_{y_A} + \lambda_{y_D} = \frac{y_{A_f} - y_{T_f}}{\sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}}. \tag{31}$$

Thus, we have 6 relationships at the terminal time t_f: equations (3) and (28)-(31). Finally, the time t_f is specified by the PMP requirement that the Hamiltonian $\mathcal{H}(\mathbf{x}(t), \lambda(t), \chi^*, \psi^*, \phi^*)|_{t_f} \equiv 0$ which, for this problem, takes the form of Isaacs' ME 1

$$\begin{aligned} \lambda_{x_A}\cos\chi^* + \lambda_{y_A}\sin\chi^* + \lambda_{x_D}\cos\psi^* + \lambda_{y_D}\sin\psi^*\\ +\alpha\lambda_{x_T}\cos\phi^* + \alpha\lambda_{y_T}\sin\phi^* \equiv 0. \end{aligned} \tag{32}$$

Let $x_T = x_T(t')$, $y_T = y_T(t')$, $x_A = x_A(t')$, $y_A = y_A(t')$, $x_D = x_D(t')$, and $y_D = y_D(t')$ be the instantaneous positions at some time $t' < t_f$. From (1), and since the optimal headings of A, D, and T are constant, we obtain the following

$$x_{T_f} = x_T + \alpha(t_f - t')\cos\phi, \tag{33}$$

$$y_{T_f} = y_T + \alpha(t_f - t')\sin\phi, \tag{34}$$

$$x_{A_f} = x_A + (t_f - t')\cos\chi, \tag{35}$$

$$y_{A_f} = y_A + (t_f - t')\sin\chi, \tag{36}$$

$$x_{D_f} = x_D + (t_f - t')\cos\psi, \tag{37}$$

$$y_{D_f} = y_D + (t_f - t')\sin\psi. \tag{38}$$

From the terminal condition (3) we define

$$x \equiv x_{A_f} = x_{D_f}, \qquad y \equiv y_{A_f} = y_{D_f}. \tag{39}$$

Because $V_A = V_D$ the interception point $I : (x, y)$ is on the orthogonal bisector of the segment $\overline{AD}$ which is characterized, in terms of the state $\mathbf{x}$, by equation (14) where the functions $m(\mathbf{x})$ and $n(\mathbf{x})$ are given by (10).

In addition, (28)-(31) can be written as follows

$$\lambda_{x_T} = \frac{x_{T_f} - x}{\sqrt{(x - x_{T_f})^2 + (y - y_{T_f})^2}}, \tag{40}$$

$$\lambda_{y_T} = \frac{y_{T_f} - y}{\sqrt{(x - x_{T_f})^2 + (y - y_{T_f})^2}}, \tag{41}$$

$$\lambda_{x_A} + \lambda_{x_D} = \frac{x - x_{T_f}}{\sqrt{(x - x_{T_f})^2 + (y - y_{T_f})^2}}, \tag{42}$$

$$\lambda_{y_A} + \lambda_{y_D} = \frac{y - y_{T_f}}{\sqrt{(x - x_{T_f})^2 + (y - y_{T_f})^2}}. \tag{43}$$

Substituting (40) and (41) into (21) we obtain the optimal Target heading in terms of the terminal state (x_{T_f}, y_{T_f})

$$\begin{aligned} \cos\phi^* &= \frac{x_{T_f} - x}{\sqrt{(x_{T_f} - x)^2 + (y_{T_f} - y)^2}},\\ \sin\phi^* &= \frac{y_{T_f} - y}{\sqrt{(x_{T_f} - x)^2 + (y_{T_f} - y)^2}}. \end{aligned} \tag{44}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2828088, IEEE Transactions on Automatic Control

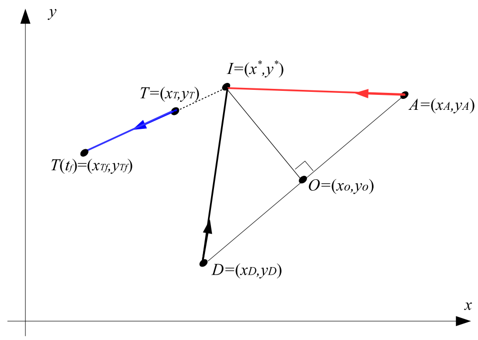

Fig. 2. Optimal play. The coordinates of the $\overline{AD}$ midpoint O are: $x_0 = \frac{1}{2}(x_A + x_D)$ and $y_0 = \frac{1}{2}(y_A + y_D)$

Since we normalized the dynamical equations using V_A, we have that $t_f - t' = \overline{AI} = \sqrt{(\frac{1}{2}\overline{AD})^2 + \overline{OI}^2}$. Without loss of generality, assume that $t' = 0$, whereupon

$$t_f(\mathbf{x}; x) = \Big[\tfrac{1}{4}\big((x_A - x_D)^2 + (y_A - y_D)^2\big) + \big(x - \tfrac{1}{2}(x_A{+}x_D)\big)^2 + \big(y - \tfrac{1}{2}(y_A{+}y_D)\big)^2\Big]^{1/2} \quad (45)$$

where $y = mx{+}n$. Additionally, because the optimal headings are constant, (44) can be equivalently written in terms of the state (x_T, y_T) as follows

$$\cos\phi^* = \frac{x_T - x}{\sqrt{(x_T-x)^2+(y_T-y)^2}}, \quad \sin\phi^* = \frac{y_T - y}{\sqrt{(x_T-x)^2+(y_T-y)^2}}. \quad (46)$$

Similarly, the A and D optimal headings are given, respectively, by

$$\cos\chi^* = \frac{x - x_A}{\sqrt{(x-x_A)^2+(y-y_A)^2}}, \quad \sin\chi^* = \frac{y - y_A}{\sqrt{(x-x_A)^2+(y-y_A)^2}} \quad (47)$$

and

$$\cos\psi^* = \frac{x - x_D}{\sqrt{(x-x_D)^2+(y-y_D)^2}}, \quad \sin\psi^* = \frac{y - y_D}{\sqrt{(x-x_D)^2+(y-y_D)^2}}. \quad (48)$$

The optimal play when initially the Target was closer to the Defender than to the Attacker is illustrated in Fig. 2.

From (19) and (47) we obtain

$$-\frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2+\lambda_{y_A}^2}} = \frac{x - x_A}{\sqrt{(x-x_A)^2+(y-y_A)^2}}, \quad -\frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2+\lambda_{y_A}^2}} = \frac{y - y_A}{\sqrt{(x-x_A)^2+(y-y_A)^2}}. \quad (49)$$

Also, use (20) and (48) to obtain the following relationships

$$\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2+\lambda_{y_D}^2}} = \frac{x - x_D}{\sqrt{(x-x_D)^2+(y-y_D)^2}}, \quad \frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2+\lambda_{y_D}^2}} = \frac{y - y_D}{\sqrt{(x-x_D)^2+(y-y_D)^2}}. \quad (50)$$

We have four equations (42), (43), (49), and (50) in the four unknowns λ_{x_A}, λ_{y_A}, λ_{x_D}, and λ_{y_D}. The solution is given by

$$\begin{aligned}
\lambda_{x_A} &= \frac{x_A - x}{\sqrt{(x-x_{T_f})^2+(y-y_{T_f})^2}} \cdot \frac{x_{T_f} - x - \frac{x-x_D}{y-y_D}(y_{T_f}-y)}{x - x_A - \frac{x-x_D}{y-y_D}(y-y_A)}, \\
\lambda_{y_A} &= \frac{y_A - y}{\sqrt{(x-x_{T_f})^2+(y-y_{T_f})^2}} \cdot \frac{x_{T_f} - x - \frac{x-x_D}{y-y_D}(y_{T_f}-y)}{x - x_A - \frac{x-x_D}{y-y_D}(y-y_A)}, \\
\lambda_{x_D} &= \frac{x - x_D}{\sqrt{(x-x_{T_f})^2+(y-y_{T_f})^2}} \cdot \frac{x_{T_f} - x - \frac{x-x_A}{y-y_A}(y_{T_f}-y)}{\frac{x-x_A}{y-y_A}(y-y_D)-(x-x_D)}, \\
\lambda_{y_D} &= \frac{y - y_D}{\sqrt{(x-x_{T_f})^2+(y-y_{T_f})^2}} \cdot \frac{x_{T_f} - x - \frac{x-x_A}{y-y_A}(y_{T_f}-y)}{\frac{x-x_A}{y-y_A}(y-y_D)-(x-x_D)}
\end{aligned} \quad (51)$$

which together with (40) and (41) specify the co-states in terms of the state components (x_A, y_A, x_D, y_D) but also in terms of the Target's terminal position (x_{T_f}, y_{T_f}). In order to eliminate the dependence of (51) on the Target terminal position and introduce instead the current Target state (x_T, y_T) we use (33), (34), (45), and (46) to obtain

$$\begin{aligned}
x_T(t_f) &= x_T + \alpha t_f(\mathbf{x}, x)\frac{x_T - x}{\sqrt{(x_T-x)^2+(y_T-y)^2}}, \\
y_T(t_f) &= y_T + \alpha t_f(\mathbf{x}, x)\frac{y_T - y}{\sqrt{(x_T-x)^2+(y_T-y)^2}}
\end{aligned} \quad (52)$$

where $t_f(\mathbf{x}, x)$ is given by (45). Inserting (52) into (40)-(41) and (51) allows us to express the co-states in terms of the state $\mathbf{x}$ and the point of interception coordinates (x, y):

$$\begin{aligned}
\lambda_{x_T} &= \frac{x_T - x}{\sqrt{(x_T-x)^2+(y_T-y)^2}}, \\
\lambda_{y_T} &= \frac{y_T - y}{\sqrt{(x_T-x)^2+(y_T-y)^2}}, \\
\lambda_{x_A} &= \frac{x_A - x}{\sqrt{(x_T-x)^2+(y_T-y)^2}} \cdot \frac{x_T - x - \frac{x-x_D}{y-y_D}(y_T-y)}{x - x_A - \frac{x-x_D}{y-y_D}(y-y_A)}, \\
\lambda_{y_A} &= \frac{y_A - y}{\sqrt{(x_T-x)^2+(y_T-y)^2}} \cdot \frac{x_T - x - \frac{x-x_D}{y-y_D}(y_T-y)}{x - x_A - \frac{x-x_D}{y-y_D}(y-y_A)}, \\
\lambda_{x_D} &= \frac{x - x_D}{\sqrt{(x_T-x)^2+(y_T-y)^2}} \cdot \frac{x_T - x - \frac{x-x_A}{y-y_A}(y_T-y)}{\frac{x-x_A}{y-y_A}(y-y_D)-(x-x_D)}, \\
\lambda_{y_D} &= \frac{y - y_D}{\sqrt{(x_T-x)^2+(y_T-y)^2}} \cdot \frac{x_T - x - \frac{x-x_A}{y-y_A}(y_T-y)}{\frac{x-x_A}{y-y_A}(y-y_D)-(x-x_D)}.
\end{aligned} \quad (53)$$

So far we have not used (32), which in view of (19)-(21) is equivalent to

$$\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2} - \sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2} - \alpha\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2} = 0. \quad (54)$$

It is however more convenient to use (32) in conjunction with (46)-(48) for the optimal controls and (53) for the co-states. Doing so we obtain the following

$$\begin{aligned}
&\alpha\frac{(x_T-x)^2+(y_T-y)^2}{\sqrt{(x_T-x)^2+(y_T-y)^2}} \\
&- \frac{x_T - x + \frac{x-x_D}{y-y_D}(y-y_T)}{x - x_A - \frac{x-x_D}{y-y_D}(y-y_A)} \cdot \frac{(x-x_A)^2+(y-y_A)^2}{\sqrt{(x-x_A)^2+(y-y_A)^2}} \\
&+ \frac{x_T - x + \frac{x-x_A}{y-y_A}(y-y_T)}{\frac{x-x_A}{y-y_A}(y-y_D)-(x-x_D)} \cdot \frac{(x-x_D)^2+(y-y_D)^2}{\sqrt{(x-x_D)^2+(y-y_D)^2}} = 0
\end{aligned}$$

which can also be written as

$$\begin{aligned}
&\alpha\sqrt{(x_T - x)^2 + (y_T - y)^2} \\
&- \frac{(x_T-x)(y-y_D)+(x-x_D)(y-y_T)}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)}\sqrt{(x - x_A)^2 + (y - y_A)^2} \\
&+ \frac{(x_T-x)(y-y_A)+(x-x_A)(y-y_T)}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)}\sqrt{(x - x_D)^2 + (y - y_D)^2} \\
&= 0
\end{aligned}$$

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where the common term $\frac{1}{\sqrt{(x_T-x)^2+(y_T-y)^2}}$ has been canceled. Note that $\overline{AI} = \sqrt{(x-x_A)^2+(y-y_A)^2} = \overline{DI} = \sqrt{(x-x_D)^2+(y-y_D)^2} = t_f(\mathbf{x}, x)$. Then, the above equation can be written as follows

$$\begin{aligned} & t_f\big[(x_T-x)(y-y_D)+(x-x_D)(y-y_T) \\ & \quad -(x_T-x)(y-y_A)-(x-x_A)(y-y_T)\big] \\ = & \ \alpha\sqrt{(x_T-x)^2+(y_T-y)^2}\big[(x-x_A)(y-y_D) \\ & \quad -(x-x_D)(y-y_A)\big]. \end{aligned}$$

Substituting the value of t_f given by (45) into the equation above and simplifying the terms inside the brackets we obtain

$$\begin{aligned} & \Big[\tfrac{1}{4}\big((x_A-x_D)^2+(y_A-y_D)^2\big) \\ & \quad +\big(x-\tfrac{1}{2}(x_A+x_D)\big)^2+\big(y-\tfrac{1}{2}(y_A+y_D)\big)^2\Big]^{1/2} \\ & \times\big[(x_T-x)(y_A-y_D)+(y-y_T)(x_A-x_D)\big] \\ = & \ \alpha\sqrt{(x_T-x)^2+(y_T-y)^2}\big[x(y_A-y_D) \\ & \quad +y(x_D-x_A)+x_Ay_D-x_Dy_A\big]. \end{aligned}$$

Using the definitions of the functions m and n from (10) and dividing both sides of the equation by (y_A-y_D) we can write the following

$$\begin{aligned} & \Big[\tfrac{1}{4}\big((x_A-x_D)^2+(y_A-y_D)^2\big) \\ & \quad +\big(x-\tfrac{1}{2}(x_A+x_D)\big)^2+\big(y-\tfrac{1}{2}(y_A+y_D)\big)^2\Big]^{1/2} \\ & \times\big[x_T-x+m(y_T-y)\big] \\ = & \ \alpha\sqrt{(x_T-x)^2+(y_T-y)^2}\big[x+my \\ & \quad -\tfrac{1}{2}\big(x_A+x_D+m(y_A+y_D)\big)\big]. \end{aligned} \tag{55}$$

We have two equations, (14) and (55), in two variables, x and y. We substitute $y = mx+n$ into (55) and obtain the square of both sides of the resulting expression. Doing so we have that

$$\begin{aligned} \big[(m^2+1)x^2+k_1x+k_3\big]\big[(m^2+1)x+k_2\big]^2 = \\ \alpha^2\big[(m^2+1)x^2+2k_2x+k_4\big]\big[(m^2+1)x+\tfrac{k_1}{2}\big]^2 \end{aligned} \tag{56}$$

where k_1, k_2, k_3, and k_4 were defined in (16). Expanding and rearranging the terms in (56) we obtain the quartic equation (15) in x. Hence, the optimal interception point I^* is given by (x^*, y^*) where x^* is a real solution of the quartic equation (15) and then $y^* = mx^*+n$.

To provide a complete solution of the differential game, including when the Target is located on the orthogonal bisector of the segment $\overline{AD}$ and whose equation is given by (14), we will write (32) in terms of the Target heading $\cos\phi$ and $\sin\phi$. This is the case where the Target is located at the same distance from the Attacker and from the Defender, $\overline{DT} = \overline{AT}$.

Let us first write the co-states (53) in terms of $\cos\phi$ and $\sin\phi$ as follows

$$\begin{aligned} \lambda_{x_T} &= \cos\phi, \\ \lambda_{y_T} &= \sin\phi, \\ \lambda_{x_A} &= \tfrac{x_A-x}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)} \\ & \quad \times\big[(y-y_D)\cos\phi-(x-x_D)\sin\phi\big], \\ \lambda_{y_A} &= \tfrac{y_A-y}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)} \\ & \quad \times\big[(y-y_D)\cos\phi-(x-x_D)\sin\phi\big], \\ \lambda_{x_D} &= \tfrac{x-x_D}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)} \\ & \quad \times\big[(y-y_A)\cos\phi-(x-x_A)\sin\phi\big], \\ \lambda_{y_D} &= \tfrac{y-y_D}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)} \\ & \quad \times\big[(y-y_A)\cos\phi-(x-x_A)\sin\phi\big]. \end{aligned} \tag{57}$$

Then, using the co-states in the form (57), (32) can be written as follows

$$\begin{aligned} & \alpha\cos^2\phi+\sin^2\phi \\ & -\tfrac{(y-y_D)\cos\phi-(x-x_D)\sin\phi}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)}\cdot\tfrac{(x-x_A)^2+(y-y_A)^2}{\sqrt{(x-x_A)^2+(y-y_A)^2}} \\ & +\tfrac{(y-y_A)\cos\phi-(x-x_A)\sin\phi}{(x-x_A)(y-y_D)-(x-x_D)(y-y_A)}\cdot\tfrac{(x-x_D)^2+(y-y_D)^2}{\sqrt{(x-x_D)^2+(y-y_D)^2}} = 0. \end{aligned}$$

Simplifying terms in the above equation we obtain

$$\begin{aligned} & t_f[(y_A-y_D)\cos\phi-(x_A-x_D)\sin\phi] \\ & = \alpha[(x-x_A)(y-y_D)-(x-x_D)(y-y_A)]. \end{aligned} \tag{58}$$

We now divide both sides of (58) by (y_A-y_D)

$$\begin{aligned} & t_f[\cos\phi-\frac{x_A-x_D}{y_A-y_D}\sin\phi] \\ & = \alpha[x-\frac{x_A-x_D}{y_A-y_D}y+\frac{x_Ay_D-x_Dy_A}{y_A-y_D}] \\ \Rightarrow\ & t_f[\cos\phi+m\sin\phi] \\ & = \alpha[x+my-\frac{1}{2}(x_A+x_D)-\frac{1}{2}m(y_A+y_D)] \\ \Rightarrow\ & t_f[\cos\phi+m\sin\phi] \\ & = \alpha[x-x_0+m(y-y_0)]. \end{aligned}$$

Finally, using the relationship $m = \frac{y-y_0}{x-x_0}$ we obtain

$$m\sin\phi+\cos\phi = \alpha(m^2+1)\frac{x-x_0}{t_f}. \tag{59}$$

Equation (59) can be written only in terms of $\sin\phi$ or only in terms $\cos\phi$ in order to determine the optimal Target heading. Using $\cos^2\phi = 1-\sin^2\phi$ we obtain the quadratic equation in $\sin\phi$

$$\begin{aligned} (m^2+1)\sin^2\phi-2\alpha m(m^2+1)\tfrac{x-x_0}{t_f}\sin\phi \\ +\alpha^2(m^2+1)^2\tfrac{(x-x_0)^2}{t_f^2}-1=0 \end{aligned}$$

and the solution is given by

$$\sin\phi^* = \alpha m\tfrac{x^*-x_0}{t_f} \pm \tfrac{\sqrt{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x^*-x_0)^2}}{t_f\sqrt{m^2+1}}. \tag{60}$$

We obtain $\cos\phi$ in a similar way

$$\cos\phi^* = \alpha\tfrac{x^*-x_0}{t_f} \mp m\tfrac{\sqrt{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x^*-x_0)^2}}{t_f\sqrt{m^2+1}}. \tag{61}$$

In general, we still need to solve the quartic equation (15) to determine the optimal interception point coordinate x^*, this in order to find the optimal Target heading (60)-(61). One can see that the optimal value of x^* is needed in these equations and to determine $t_f(\mathbf{x}, x^*)$. At this point we have two equivalent ways to determine $\sin\phi$ and $\cos\phi$: (46) or (60)-(61); both need knowledge of the optimal interception coordinate x^*.

However, when the Target is on the orthogonal bisector, the expressions in (46) cannot be evaluated and the optimal Target heading cannot be determined through this formula. This is so, because the optimal interception point coordinates are given by $x^* = x_T$ and $y^* = y_T$. In the particular case where $\overline{DT} = \overline{AT}$, we have that the Target's coordinates (x_T, y_T) satisfy the linear equation $y_T = mx_T + n$. From (55) we can see that the solution is $x^* = x_T$ and $y^* = y_T$; the Attacker and the Defender aim at the Target's position (x_T, y_T), as expected. In such a case, the Target heading cannot be evaluated when using (46). However, (60)-(61) provide the optimal Target heading for any initial configuration, including the special case when $\overline{DT} = \overline{AT}$. In this case we have that, by substituting $x^* = x_T$, the expressions in (60)-(61) simplify to

$$\sin\phi^* = \alpha m \frac{x_T - x_0}{t_f(\mathbf{x}, x_T)} \pm \frac{\sqrt{\overline{AD}^2/4 + (1-\alpha^2)(m^2+1)(x_T-x_0)^2}}{t_f(\mathbf{x}, x_T)\sqrt{m^2+1}}, \tag{62}$$

$$\cos\phi^* = \alpha \frac{x_T - x_0}{t_f(\mathbf{x}, x_T)} \mp m \frac{\sqrt{\overline{AD}^2/4 + (1-\alpha^2)(m^2+1)(x_T-x_0)^2}}{t_f(\mathbf{x}, x_T)\sqrt{m^2+1}} \tag{63}$$

where

$$t_f(\mathbf{x}; x_T) = \Big[\tfrac{1}{4}\big((x_A - x_D)^2 + (y_A - y_D)^2\big) + \big(x_T - \tfrac{1}{2}(x_A + x_D)\big)^2 + \big(y_T - \tfrac{1}{2}(y_A + y_D)\big)^2\Big]^{1/2}$$

$y_T = mx_T + n$, and m and n are given by (10).

Using Isaacs' method of solving differential games [23], the original problem was simplified to calculating $x^*(\mathbf{x})$, providing a state-feedback solution of the differential game. The solution $x^*(\mathbf{x})$ hinges on the rooting in real time of the quartic equation (15); but a quartic equation can be solved in closed form.

The value function is C^1 and is explicitly given by (17). The gradient of the Value function, which is shown in (66) below, is well defined in the Target's escape region where the Game of Degree is played.

Finally, we show that the Value function satisfies the HJI equation $-\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, \chi^*, \psi^*, \phi^*) + g(t, \mathbf{x}, \chi^*, \psi^*, \phi^*)$. Note that in our problem $\frac{\partial V}{\partial t} = 0$ and $g(t, \mathbf{x}, \chi^*, \psi^*, \phi^*) = 0$.

The Value function is explicitly given by

$$\begin{aligned} V(\mathbf{x}) &= \alpha\big[\tfrac{1}{4}[(x_A - x_D)^2 + (y_A - y_D)^2] \\ &\quad + [x^*(\mathbf{x}) - \tfrac{1}{2}(x_A + x_D)]^2 \\ &\quad + [y^*(\mathbf{x}) - \tfrac{1}{2}(y_A + y_D)]^2\big]^{1/2} \\ &\quad + \sqrt{[x_T - x^*(\mathbf{x})]^2 + [y_T - y^*(\mathbf{x})]^2} \\ &= \alpha\overline{AI} + \overline{TI}. \end{aligned}$$

In the Value function the expression $y^* = mx^* + n$ holds where x^* is a function of the state $\mathbf{x}$ because it is a solution of (15).

The gradient of $V(\mathbf{x})$ is $\frac{\partial V}{\partial \mathbf{x}} = [\frac{\partial V}{\partial x_i} + \frac{dV}{dx^*} \cdot \frac{dx^*}{dx_i}, \quad \frac{\partial V}{\partial y_i} + \frac{dV}{dx^*} \cdot \frac{dx^*}{dy_i}]^T$ for $i = A, D, T$. Let us compute the term

$$\frac{dV}{dx^*} = \frac{\alpha\Big(x^* - \frac{1}{2}(x_A + x_D) + [y^* - \frac{1}{2}(y_A + y_D)]\frac{dy^*}{dx^*}\Big)}{\sqrt{\frac{1}{4}[(x_A - x_D)^2 + (y_A - y_D)^2] + [x^* - \frac{1}{2}(x_A + x_D)]^2 + [y^* - \frac{1}{2}(y_A + y_D)]^2}} - \frac{x_T - x^* + (y_T - y^*)\frac{dy^*}{dx^*}}{\sqrt{(x_T - x^*)^2 + (y_T - y^*)^2}} \tag{64}$$

where $\frac{dy^*}{dx^*} = m$. Also, write (55) as follows

$$\alpha \frac{x + my - \frac{1}{2}[x_A + x_D + m(y_A + y_D)]}{\sqrt{\frac{1}{4}[(x_A - x_D)^2 + (y_A - y_D)^2] + [x - \frac{1}{2}(x_A + x_D)]^2 + [y - \frac{1}{2}(y_A + y_D)]^2}} - \frac{x_T - x + m(y_T - y)}{\sqrt{(x_T - x)^2 + (y_T - y)^2}} = 0. \tag{65}$$

We have that $\frac{dV}{dx^*}$ is equal to the left hand side of (65) where x^* is a solution of (15), or equivalently, of (65). Hence,

$$\frac{dV}{dx^*} = 0.$$

Therefore, the partial derivatives of the Value function with respect to each component of the state $\mathbf{x}$ are given by the following

$$\begin{aligned} \frac{\partial V}{\partial x_A} &= \frac{\alpha}{2\overline{AI}}(x_A - x^*) = -\frac{\alpha}{2}\cos\chi^*, \\ \frac{\partial V}{\partial y_A} &= \frac{\alpha}{2\overline{AI}}(y_A - y^*) = -\frac{\alpha}{2}\sin\chi^*, \\ \frac{\partial V}{\partial x_D} &= \frac{\alpha}{2\overline{AI}}(x_D - x^*) = -\frac{\alpha}{2}\cos\psi^*, \\ \frac{\partial V}{\partial y_D} &= \frac{\alpha}{2\overline{AI}}(y_D - y^*) = -\frac{\alpha}{2}\sin\psi^*, \\ \frac{\partial V}{\partial x_T} &= \frac{1}{\overline{TI}}(x_T - x^*) = \cos\phi^*, \\ \frac{\partial V}{\partial y_T} &= \frac{1}{\overline{TI}}(y_T - y^*) = \sin\phi^* \end{aligned} \tag{66}$$

where $\cos\phi^*$ and $\sin\phi^*$ are given, in general, by (60)-(61). Then, the gradient of $V(\mathbf{x})$ for any state satisfying $\overline{DT} \leq \overline{AT}$ is given by $\frac{\partial V}{\partial \mathbf{x}} = [-\frac{\alpha}{2}\cos\chi^* \quad -\frac{\alpha}{2}\sin\chi^* \quad -\frac{\alpha}{2}\cos\psi^* \quad -\frac{\alpha}{2}\sin\psi^* \quad \cos\phi^* \quad \sin\phi^*]^T$.

The HJI equation for the ATDDG is then given by

$$\begin{aligned} &\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, \chi^*, \psi^*, \phi^*) \\ &\quad = -\frac{\alpha}{2}\cos^2\chi^* - \frac{\alpha}{2}\sin^2\chi^* - \frac{\alpha}{2}\cos^2\psi^* - \frac{\alpha}{2}\sin^2\psi^* \\ &\qquad + \alpha\cos^2\phi^* + \alpha\sin^2\phi^* \\ &\quad = -\frac{\alpha}{2} - \frac{\alpha}{2} + \alpha \\ &\quad = 0. \end{aligned} \tag{67}$$

In summary, state feedback optimal strategies of the three agents in the differential game were synthesized and the Value function was obtained. It was also shown that when $\overline{DT} \leq \overline{AT}$ the Value function is C^1 and that it satisfies the HJI equation. □

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2828088, IEEE Transactions on Automatic Control

In the proof of Theorem 1 and for the particular case where $\overline{DT} = \overline{AT}$, several sets of possible solutions were provided in (62)-(63). We will now explicitly obtain the optimal Target heading in this case. Define the pairs of solutions of (62)-(63)

$$\begin{aligned}(\sin\phi_1,\ \cos\phi_1) &= (mp+q,\ p-mq),\\ (\sin\phi_2,\ \cos\phi_2) &= (mp-q,\ p+mq)\end{aligned} \tag{68}$$

where $q = \frac{\sqrt{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}}{t_f(\mathbf{x},x_T)\sqrt{m^2+1}}$ and $p = \alpha\frac{x_T-x_0}{t_f(\mathbf{x},x_T)}$. Also define $\Delta_1 = \sqrt{(x_D-x_{T_1})^2+(y_D-y_{T_1})^2}$ and $\Delta_2 = \sqrt{(x_D-x_{T_2})^2+(y_D-y_{T_2})^2}$ where $x_{T_i} = x_T + \alpha t_f(\mathbf{x},x_T)\cos\phi_i$ and $y_{T_i} = y_T + \alpha t_f(\mathbf{x},x_T)\sin\phi_i$, for $i = 1,2$. The next result yields the optimal Target heading between the two feasible pair of solutions in (62)-(63).

Corollary 1: When $\overline{DT} = \overline{AT}$ the optimal Target heading is given by

$$\cos\phi^* = \cos\phi_\kappa, \qquad \sin\phi^* = \sin\phi_\kappa \tag{69}$$

where

$$\kappa = \arg\min_i \{\Delta_i\} \tag{70}$$

for $i = 1,2$.

Proof. When $\overline{DT} = \overline{AT}$ the optimal Target heading is obtained from (62)-(63) which offer four possible combinations. We will first show that only two combinations are feasible headings. Let us evaluate the expression

$$\begin{aligned}&\sin\phi_1^2+\cos\phi_1^2\\ &\quad= (mp+q)^2+(p-mq)^2\\ &\quad= \frac{\alpha^2m^2(x_T-x_0)^2}{t_f^2} + \frac{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}{t_f^2(m^2+1)}\\ &\qquad+ \frac{2\alpha m(x_T-x_0)\sqrt{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}}{t_f^2\sqrt{m^2+1}}\\ &\qquad+ \frac{\alpha^2(x_T-x_0)^2}{t_f^2} + m^2\frac{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}{t_f^2(m^2+1)}\\ &\qquad- \frac{2\alpha m(x_T-x_0)\sqrt{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}}{t_f^2\sqrt{m^2+1}}\\ &\quad= (m^2+1)\\ &\qquad\times\frac{\alpha^2(m^2+1)(x_T-x_0)^2+\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}{t_f^2(m^2+1)}\\ &\quad= \frac{(m^2+1)(x_T-x_0)^2+\overline{AD}^2/4}{t_f^2}\\ &\quad= \frac{t_f^2}{t_f^2}\\ &\quad= 1\end{aligned}$$

where $t_f = t_f(\mathbf{x},x_T)$. For the combinations $(\sin\phi_1,\cos\phi_2)$ and $(\sin\phi_2,\cos\phi_1)$, an additional term is obtained when evaluating the expression $\sin\phi^2+\cos\phi^2$. For instance,

$$\begin{aligned}&\sin\phi_1^2+\cos\phi_2^2\\ &\quad= (mp+q)^2+(p+mq)^2\\ &\quad= 1+4\alpha m(x_T-x_0)\frac{\sqrt{\overline{AD}^2/4+(1-\alpha^2)(m^2+1)(x_T-x_0)^2}}{t_f^2\sqrt{m^2+1}}\end{aligned}$$

where the second term on the right-hand side is not equal to zero in general and, therefore, the combination is not a valid heading angle. The same situation occurs when the combination $\sin\phi_2^2+\cos\phi_1^2$ is evaluated. Thus, only the two headings in (68) are feasible heading angles.

Alternatively, the conclusion in the previous paragraph is obtained by noting that only the two headings in (68) satisfy (59) while the other two combinations do not satisfy this equation. We will now determine how to choose the optimal Target heading from the two feasible headings in (68). To accomplish this we use the following definition.

Definition 2: (Dual Problem). Given an ATDDG configuration, which we will refer to as the original problem, we define a dual problem as the configuration where the positions of the Attacker and the Defender are interchanged, $x_A \leftrightarrow x_D$ and $y_A \leftrightarrow y_D$, while both the speed ratio parameter α and the Target coordinates (x_T, y_T) remain the same.

Given an original and dual pair of problems, when $\overline{DT} = \overline{AT}$, the variables used to compute (62)-(63) remain the same for the original and the dual problem. These variables are the separation $\overline{AD}$, the midpoint (x_0, y_0), the slope m, and the terminal time t_f. Hence, the sets of solutions given by (62)-(63) are the same for the original and for the dual problem. Since only the two combinations given by (68) are feasible, then, each one corresponds to the solution of the original and the dual problem. Because the Target is headed away from the point on the orthogonal bisector of $\overline{AD}$ into the reachable side of D, then, the correct solution is the heading that moves the Target into the reachable side of D. The other solution will make the Target go into the reachable side of A, i.e., into the side of D in the dual problem. Therefore the optimal Target heading is obtained from (69). $\square$

The choice of deriving the quartic equation in terms of x instead of y is not the only option. One can obtain an equivalent equation in terms of y. Let $x = m'y + n'$, where $m' = \frac{1}{m}$, $n' = -\frac{n}{m}$, and m and n are given by (10). Then, substitute $x = m'y + n'$ into (55) to obtain

$$\begin{aligned}&(1-\alpha^2)(m'^2+1)^3y^4+(1-\alpha^2)(m'^2+1)^2(k_1'+2k_2')y^3\\ &+\big[(m'^2+1)^2(k_3'-\alpha^2k_4')+2(1-\alpha^2)(m'^2+1)k_1'k_2'\\ &\quad+(m'^2+1)(k_2'^2-\tfrac{\alpha^2}{4}k_1'^2)\big]y^2\\ &+\big[(m'^2+1)(2k_2'k_3'-\alpha^2k_1'k_4')+k_1'k_2'(k_2'-\tfrac{\alpha^2}{2}k_1')\big]y\\ &+k_2'^2k_3'-\tfrac{\alpha^2}{4}k_1'^2k_4'=0\end{aligned} \tag{71}$$

where

$$\begin{aligned}k_1' &= 2m'n'-[m'(x_A+x_D)+(y_A+y_D)],\\ k_2' &= m'n'-m'x_T-y_T,\\ k_3' &= \tfrac{1}{2}(x_A^2+x_D^2+y_A^2+y_D^2)+n'^2-n'(x_A+x_D),\\ k_4' &= y_T^2+(x_T-n')^2.\end{aligned} \tag{72}$$

Either quartic equation, (15) or (71), can be used to determine the optimal interception point coordinates $I^* = (x^*, y^*)$. Only two particular cases require the selection of only one of these equations. When the Defender and Attacker relative orientation is such that $m = 0$, then only (15) can be used. On the other hand, when the Defender and Attacker relative orientation is such that $m' = 0$, then only (71) can be used.

It is also possible to obtain expressions equivalent to (60)-(61) written in terms of m' and n'. In this case (58) is divided

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2828088, IEEE Transactions on Automatic Control

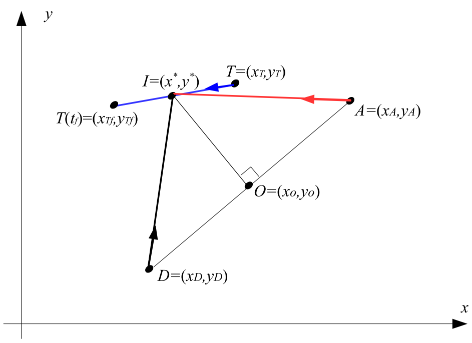

Fig. 3. Optimal play when initially $\overline{DT} > \overline{AT}$

by $-(x_A - x_D)$ resulting in

$$\begin{aligned}
& t_f[-\frac{y_A - y_D}{x_A - x_D}\cos\phi + \sin\phi] \\
&= \alpha[-\frac{y_A - y_D}{x_A - x_D}x + y - \frac{x_A y_D - x_D y_A}{x_A - x_D}] \\
&\Rightarrow \; t_f[m'\cos\phi + \sin\phi] \\
&= \alpha[m'x + y - \frac{1}{2}m'(x_A + x_D) - \frac{1}{2}(y_A + y_D)] \\
&\Rightarrow \; t_f[m'\cos\phi + \sin\phi] = \alpha[m'(x - x_0) + y - y_0] \\
&\Rightarrow \; m'\cos\phi + \sin\phi = \alpha(m'^2 + 1)\frac{y - y_0}{t_f}
\end{aligned}$$

where $m' = \frac{x - x_0}{y - y_0}$. The solution of the previous equation is then given by

$$\begin{aligned}
\sin\phi^* &= \alpha\frac{y^* - y_0}{t_f} \pm m'\frac{\sqrt{\overline{AD}^2/4 + (1-\alpha^2)(m'^2+1)(y^* - y_0)^2}}{t_f\sqrt{m'^2+1}}, \\
\cos\phi^* &= \alpha m'\frac{y^* - y_0}{t_f} \mp \frac{\sqrt{\overline{AD}^2/4 + (1-\alpha^2)(m'^2+1)(y^* - y_0)^2}}{t_f\sqrt{m'^2+1}}.
\end{aligned}$$

Note that by making the substitutions $m' = \frac{1}{m}$ and $y - y_0 = m(x - x_0)$, the previous solution for the Target optimal heading, ϕ^*, is exactly equal to the solution (60)-(61).

Finally, in the special case where $\overline{DT} = \overline{AT}$, the Target's optimal heading is given by

$$\begin{aligned}
\sin\phi^* &= \alpha\frac{y_T - y_0}{t_f(\mathbf{x}, x_T)} \pm m'\frac{\sqrt{\overline{AD}^2/4 + (1-\alpha^2)(m'^2+1)(y_T - y_0)^2}}{t_f(\mathbf{x}, x_T)\sqrt{m'^2+1}}, \\
\cos\phi^* &= \alpha m'\frac{y_T - y_0}{t_f(\mathbf{x}, x_T)} \mp \frac{\sqrt{\overline{AD}^2/4 + (1-\alpha^2)(m'^2+1)(y_T - y_0)^2}}{t_f(\mathbf{x}, x_T)\sqrt{m'^2+1}}
\end{aligned}$$

since $y^* = y_T$ in this case.

IV. ATDDG WHEN $\overline{DT} > \overline{AT}$

We now consider the case where the Target is closer to the Attacker than to the Defender but the Target is in the escape region $\mathcal{R}_e \subset \mathbb{R}^6$, so a solution to the ATDDG exists. This means that the Game of Kind has been solved and the state space region $\mathcal{R}_e$ where the Game of Degree is played has been established. The Game of Kind is solved in Section V in the sequel.

Theorem 2: Consider the Active Target Defense Differential Game (1)-(7) where $\overline{DT} > \overline{AT}$ and the state $\mathbf{x} \in \mathcal{R}_e$. The problem parameter is the speed ratio $0 < \alpha < 1$. The optimal headings of the Attacker and the Defender are given by the state feedback control laws (13)-(12) and the optimal Target heading is given by the state feedback control law

$$\begin{aligned}
\cos\phi^* &= \frac{x - x_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}}, \\
\sin\phi^* &= \frac{y - y_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}}
\end{aligned} \tag{73}$$

where x and y satisfy (14), m and n are given by (10), and the coordinate x is a real solution of the quartic equation (15).

The Value function is C^1, it satisfies the HJI PDE, and is explicitly given by

$$\begin{aligned}
V(\mathbf{x}) = \alpha\big[&\tfrac{1}{4}\big((x_A - x_D)^2 + (y_A - y_D)^2\big) \\
&+ \big(x - \tfrac{1}{2}(x_A + x_D)\big)^2 + \big(y - \tfrac{1}{2}(y_A + y_D)\big)^2\big]^{1/2} \\
&- \sqrt{(x - x_T)^2 + (y - y_T)^2}.
\end{aligned} \tag{74}$$

Proof. As in Theorem 1 the optimal control inputs are constant and given by (19)-(21). The co-states are given, in terms of the states (x_A, y_A, x_D, y_D) and of the Target's terminal position (x_{T_f}, y_{T_f}), by (40)-(41) and (51). Substituting (40)-(41) into (21) we obtain the optimal Target heading (44) in terms of its own terminal position (x_{T_f}, y_{T_f}). Because the optimal Target heading ϕ^* is constant we can write the optimal Target heading in terms of the state (x_T, y_T) as shown in (73). An optimal play is illustrated in Fig. 3.

The Target's terminal position is as follows

$$\begin{aligned}
x_T(t_f) &= x_T + \alpha t_f(\mathbf{x}, x)\frac{x - x_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}}, \\
y_T(t_f) &= y_T + \alpha t_f(\mathbf{x}, x)\frac{y - y_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}}.
\end{aligned} \tag{75}$$

The co-states in terms of the state $\mathbf{x}$ and the interception point's coordinates (x, y) are obtained by substituting (75) into (40)-(41) and (51). Doing so we obtain

$$\begin{aligned}
\lambda_{x_T} &= \frac{x - x_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}}, \\
\lambda_{y_T} &= \frac{y - y_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}}, \\
\lambda_{x_A} &= \frac{x_A - x}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \cdot \frac{x - x_T - \frac{x - x_D}{y - y_D}(y - y_T)}{x - x_A - \frac{x - x_D}{y - y_D}(y - y_A)}, \\
\lambda_{y_A} &= \frac{y_A - y}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \cdot \frac{x - x_T - \frac{x - x_D}{y - y_D}(y - y_T)}{x - x_A - \frac{x - x_D}{y - y_D}(y - y_A)}, \\
\lambda_{x_D} &= \frac{x - x_D}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \cdot \frac{x - x_T - \frac{x - x_A}{y - y_A}(y - y_T)}{\frac{x - x_A}{y - y_A}(y - y_D) - (x - x_D)}, \\
\lambda_{y_D} &= \frac{y - y_D}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \cdot \frac{x - x_T - \frac{x - x_A}{y - y_A}(y - y_T)}{\frac{x - x_A}{y - y_A}(y - y_D) - (x - x_D)}.
\end{aligned} \tag{76}$$

Inserting the optimal headings (13)-(12) and (73) and the co-states (76) into (32) we obtain the following

$$\begin{aligned}
&\alpha\frac{(x - x_T)^2 + (y - y_T)^2}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \\
&- \frac{x - x_T - \frac{x - x_D}{y - y_D}(y - y_T)}{x - x_A - \frac{x - x_D}{y - y_D}(y - y_A)} \cdot \frac{(x - x_A)^2 + (y - y_A)^2}{\sqrt{(x - x_A)^2 + (y - y_A)^2}} \\
&+ \frac{x - x_T - \frac{x - x_A}{y - y_A}(y - y_T)}{\frac{x - x_A}{y - y_A}(y - y_D) - (x - x_D)} \cdot \frac{(x - x_D)^2 + (y - y_D)^2}{\sqrt{(x - x_D)^2 + (y - y_D)^2}} = 0
\end{aligned} \tag{77}$$

which can be written as the quartic equation (15). The optimal interception point coordinate x^* is obtained by rooting the same quartic equation, and $y^* = mx^* + n$.

Alternatively, the Target heading can be obtained by writing the co-states (76) in terms of $\cos\phi$ and $\sin\phi$. Doing so we obtain the same expressions shown in (57), where $\cos\phi$ and $\sin\phi$ are given by (73). Hence, by substituting the co-state equations (57) into (32) we obtain (59) and the optimal Target heading is given by (60)-(61). By following this procedure it is determined that the optimal Target's heading equations (60)-(61) hold for any state in the Target's escape region $\mathcal{R}_e$, regardless of whether $\overline{DT}$ is less than, greater than, or equal to $\overline{AT}$.

Finally, in order to show that the Value function satisfies the HJI equation, we evaluate the derivative of the Value function (74) with respect to x^*

$$\frac{dV}{dx^*} = \frac{\alpha\left(x^* - \frac{1}{2}(x_A+x_D) + [y^* - \frac{1}{2}(y_A+y_D)]\frac{dy^*}{dx^*}\right)}{\sqrt{\frac{1}{4}[(x_A-x_D)^2+(y_A-y_D)^2]+[x^*-\frac{1}{2}(x_A+x_D)]^2+[y^*-\frac{1}{2}(y_A+y_D)]^2}} - \frac{x^*-x_T+(y^*-y_T)\frac{dy^*}{dx^*}}{\sqrt{(x^*-x_T)^2+(y^*-y_T)^2}} \tag{78}$$

where $\frac{dy^*}{dx^*} = m$. By writing (77) in the following form

$$\alpha\frac{x+my-\frac{1}{2}[x_A+x_D+m(y_A+y_D)]}{\sqrt{\frac{1}{4}[(x_A-x_D)^2+(y_A-y_D)^2]+[x-\frac{1}{2}(x_A+x_D)]^2+[y-\frac{1}{2}(y_A+y_D)]^2}} - \frac{x-x_T+m(y-y_T)}{\sqrt{(x-x_T)^2+(y-y_T)^2}} = 0 \tag{79}$$

we conclude that $\frac{dV}{dx^*}$ is equal to the left hand side of (79) where x^* is a solution of the quartic equation (15), or equivalently, of (79). Hence,

$$\frac{dV}{dx^*} = 0.$$

Then, the gradient of the Value function $V(\mathbf{x})$ is $\frac{\partial V}{\partial \mathbf{x}} = [-\frac{\alpha}{2}\cos\chi^* \ -\frac{\alpha}{2}\sin\chi^* \ -\frac{\alpha}{2}\cos\psi^* \ -\frac{\alpha}{2}\sin\psi^* \ \cos\phi^* \ \sin\phi^*]^T$ which is the same expression obtained in (66). Finally, it can be shown that the HJI equation is satisfied in the same way as in (67). □

The design and analysis of optimal strategies for the ATDDG was separated into two parts, depending on whether the state is such that $\overline{DT} \leq \overline{AT}$ or $\overline{DT} > \overline{AT}$. This brings about a sign difference in the respective Value function expressions and in the Target's optimal heading characterization. The Value function in the entire escape region is

$$V(\mathbf{x}) = \begin{cases} V_s & \text{if } \overline{DT} \leq \overline{AT} \\ V_g & \text{if } \overline{DT} > \overline{AT} \text{ and } \mathbf{x} \in \mathcal{R}_e \end{cases}$$

where V_s and V_g are explicitly given by (17) and (74), respectively. Note that, as expected, $V_s = V_g$ when $\overline{DT} = \overline{AT}$ since $x^* = x_T$ and $y^* = y_T$ in this case. Hence, the Value function $V(\mathbf{x})$ is continuous. The gradient of the Value function $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}$ is given by the same expression as before, (66), regardless of whether $\overline{DT}$ is less than, greater than, or equal to $\overline{AT}$. Equation (66) together with (60)-(61) provide continuity of $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}$, including in the case where $\overline{DT} = \overline{AT}$, where the Target optimal heading cannot be evaluated using (11) but the alternative solution (60)-(61) holds for any $\mathbf{x} \in \mathcal{R}_e$. Hence, no discontinuity is present in the Target's optimal state feedback strategy. The Value function is C^1 and there are no singular surfaces in the ATDDG. Finally, it was shown that the Value function satisfies the HJI equation for any $\mathbf{x} \in \mathcal{R}_e$.

In general, regardless of wether T is closer to D or to A, it is possible to show existence of solutions, that is, to show that the quartic equation (15) has real solutions. Let $F(x)$ be the polynomial in the left hand side of (15). Note that $F(x=-\infty) > 0$ and $F(x=+\infty) > 0$. Now consider $T_p = (x_{T_p}, y_{T_p})$, the closest point on the orthogonal bisector to the Target position T, which satisfies $\frac{y_{T_p}-y_T}{x_{T_p}-x_T} = -\frac{1}{m}$. Calculate the following

$$\begin{aligned}(m^2+1)x_{T_p}+k_2 &= (m^2+1)x_{T_p} + mn - x_T - my_T \\ &= m(mx_{T_p} + n - y_T) + x_{T_p} - x_T \\ &= x_{T_p} - x_T + m(y_{T_p} - y_T) \\ &= x_{T_p} - x_T - (x_{T_p} - x_T) \\ &= 0\end{aligned} \tag{80}$$

Let us now compute $F(x_{T_p})$ using the equivalent expression in (56)

$$\begin{aligned}&\left[(m^2+1)x_{T_p}^2 + k_1x_{T_p} + k_3\right]\left[(m^2+1)x_{T_p} + k_2\right]^2 \\ &\quad - \alpha^2\left[(m^2+1)x_{T_p}^2 + 2k_2x_{T_p} + k_4\right]\left[(m^2+1)x_{T_p} + \tfrac{k_1}{2}\right]^2 \\ &= -\left(\alpha\overline{TT_p}\left[(m^2+1)x_{T_p} + \tfrac{k_1}{2}\right]\right)^2 \ < \ 0\end{aligned}$$

where the result in (80) was used. Because $F(x=-\infty) > 0$, $F(x_{T_p}) < 0$, and $F(x=+\infty) > 0$, the quartic equation (15) has at least two real roots. Uniqueness of the optimal solution is shown in the Appendix.

V. Game of Kind

In the previous sections we were concerned with the ATDDG Game of Degree: The Target and Defender not only envision interception of the Attacker (and survival of the Target) but they also strive to maximize a payoff, the terminal separation between Target and Attacker, where the Attacker works to minimize the same. The Game of Degree is played for initial conditions where, under optimal play, survival of the Target is guaranteed and it returns a real number, the Value of the game.

The solution of the Game of Kind provides a binary answer (yes or no) to the question: If A plays optimally, can the Target be captured given the initial state? The notions of Game of Degree and Game of Kind are central in pursuit and evasion games [23]. The solution of the Game of Kind provides the subset of the state space in which it is actually possible to play the Game of Degree.

In the region of the state space where $\overline{DT} > \overline{AT}$, that is, when the Target is closer to the Attacker than to the Defender, and for a given speed ratio parameter α this state space region is partitioned into two sets: $\mathcal{R}_e$ and $\mathcal{R}_c$. The ATDDG is played

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2828088, IEEE Transactions on Automatic Control

in $\mathcal{R}_e$. We characterize the set $\mathcal{R}_e$ as follows

$$\mathcal{R}_e := \{\ \mathbf{x}\ |H(\mathbf{x};\alpha) < 0\}. \tag{81}$$

We provide a visual representation of the function $H(\mathbf{x};\alpha)$ which specifies the surface that separates $\mathcal{R}_e$ from $\mathcal{R}_c$. By fixing the Attacker and Defender coordinates (x_A, y_A) and (x_D, y_D) we obtain the closed form of the Barrier surface $H(x, y; x_A, y_A, x_D, y_D, \alpha) = 0$. In other words, we characterize the coordinate pairs (x, y) of the possible Target position with respect to the Attacker and Defender coordinates which guarantee that the Target will escape the Attacker if the Target and the Defender team implements its optimal strategy. Fixing the A and D coordinates provides a clear illustration of the Target's escape region, as a function of these coordinates. Note that A and D, and even α can be varied in order to obtain families of curves (Barrier surface cross sections) $H(x, y; x_A, y_A, x_D, y_D, \alpha) = 0$ which yield the Target's escape region for different initial conditions and speed ratio values. Let $R_e \subset \mathbb{R}^2$ denote the escape region in the Cartesian plane for fixed A and D coordinates.

When $\overline{DT} \le \overline{AT}$ the Target is always able to escape for any $0 < \alpha < 1$. In other words, when the Target is closer to the Defender than to the Attacker, this part of the plane is located in the region R_e. Hence, the Defender's side of the plane, as given by the orthogonal bisector of $\overline{AD}$, is part of R_e and the boundary between regions R_e and R_c is located on the Attacker's side of the plane.

Theorem 3: For a given speed ratio parameter $0 < \alpha < 1$ and Attacker and Defender coordinates (x_A, y_A) and (x_D, y_D) the Barrier surface cross section that separates the Cartesian plane into the two regions R_e and R_c is given by the Attacker's side branch of the hyperbola

$$h_{xx}x^2 + h_{yy}y^2 + 2h_{xy}xy + 2h_x x + 2h_y y + h = 0 \tag{82}$$

where

$$\begin{aligned}
h_{xx} &= \cos^2\sigma - \alpha^2,\\
h_{yy} &= \sin^2\sigma - \alpha^2,\\
h_{xy} &= \sin\sigma\cos\sigma,\\
h_x &= \alpha^2 x_A \sin^2\sigma - (1-\alpha^2)x_0\cos^2\sigma\\
&\quad - [\alpha^2 y_A + (1-\alpha^2)y_0]\sin\sigma\cos\sigma,\\
h_y &= \alpha^2 y_A \cos^2\sigma - (1-\alpha^2)y_0\sin^2\sigma\\
&\quad - [\alpha^2 x_A + (1-\alpha^2)x_0]\sin\sigma\cos\sigma,\\
h &= [\alpha^2 y_A + (1-\alpha^2)y_0]^2\sin^2\sigma\\
&\quad + [\alpha^2 x_A + (1-\alpha^2)x_0]^2\cos^2\sigma - \alpha^2(x_A^2 + y_A^2)\\
&\quad + 2[\alpha^2 y_A + (1-\alpha^2)y_0][\alpha^2 x_A + (1-\alpha^2)x_0]\\
&\qquad \times \sin\sigma\cos\sigma
\end{aligned} \tag{83}$$

$\cos\sigma = \frac{x_A - x_D}{\sqrt{(x_A-x_D)^2+(y_A-y_D)^2}}$, $\sin\sigma = \frac{y_A - y_D}{\sqrt{(x_A-x_D)^2+(y_A-y_D)^2}}$, and x_0 and y_0 are given by (9).

Additionally, the center of the hyperbola, (x_{oh}, y_{oh}), is given by $x_{oh} = x_0$ and $y_{oh} = y_0$. Finally, the angle θ of the principal axes of the hyperbola with respect to the x-axis is $\theta = \sigma = \arctan(\frac{y_A - y_D}{x_A - x_D})$.

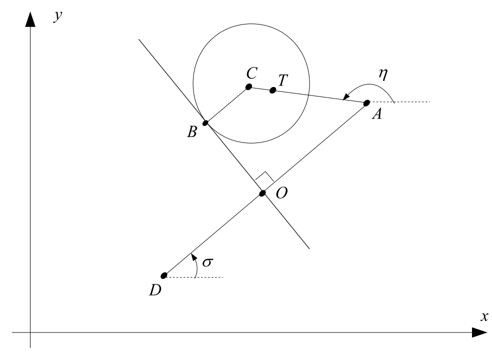

Fig. 4. Critical case: Apollonius circle is tangent to the orthogonal bisector of $\overline{AD}$ given by (14)

Proof. When the Target is initially closer to the Attacker than it is to its Defender, the Target needs to be able to break into the Defender's side without being intercepted by the Attacker. Then, the Defender will be able to assist the Target to escape, by intercepting, on the orthogonal bisector of $\overline{AD}$, the Attacker who is pursuing the Target.

Before we proceed we introduce the Apollonius circle concept. It is the locus of points P that have a constant ratio α of distances to two given points (also called foci), in our case, A and T, *i.e.*, $\alpha = \frac{\overline{TP}}{\overline{AP}}$. This circle is referred to as the Apollonius circle and it is an important tool in the analysis of pursuit problems. A strives to intercept T. A and T travel in straight lines at constant speeds V_A and V_T, respectively, where the parameter $\alpha = \frac{V_T}{V_A}$ is the speed ratio. A intercepts T at a point on the circumference of the Apollonius circle and at that point the distance traveled by T is equal to α times the distance traveled by A. Hence, an Apollonius circle can be constructed based on the speed ratio α and on the (initial) distance between the Attacker and the Target. The radius of the circle is denoted by r_A and its center is denoted by C. The points A, T, and C are collinear and T is between A and C. The Apollonius circle has the following properties: Its radius $r_A = \frac{\alpha}{1-\alpha^2}d$ and its center, denoted by C, is located at a distance $\frac{\alpha^2}{1-\alpha^2}d$ from T on the line

$$y = \frac{y_A - y_T}{x_A - x_T}x + \frac{x_A y_T - y_A x_T}{x_A - x_T} \tag{84}$$

where the distance $d = \sqrt{(x_A - x_T)^2 + (y_A - y_T)^2}$. The three points A, T, and C are collinear. The linear equation (84) is obtained using the two points A and T.

Let $\cos\eta = \frac{x_T - x_A}{d}$ and $\sin\eta = \frac{y_T - y_A}{d}$. Then, one can obtain the y-coordinate of the center of the Apollonius circle, y_c, by noting that

$$\begin{aligned}
y_c - y_T &= \frac{\alpha^2}{1-\alpha^2}d\sin\eta\\
&= \frac{\alpha^2}{1-\alpha^2}d\cdot\frac{y_T - y_A}{d}\\
&= \frac{\alpha^2}{1-\alpha^2}(y_T - y_A)\\
\Rightarrow\quad y_c &= \frac{1}{1-\alpha^2}y_T - \frac{\alpha^2}{1-\alpha^2}y_A.
\end{aligned} \tag{85}$$

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Similarly, the x-coordinate of the center of the Apollonius circle, x_c, is given by

$$x_c = \tfrac{1}{1-\alpha^2} x_T - \tfrac{\alpha^2}{1-\alpha^2} x_A. \tag{86}$$

Equations (84)-(86) hold for any speed ratio α and any Target and Attacker coordinates. The critical case (that provides the boundary between regions R_e and R_c) corresponds to the case where the Apollonius circle is tangent to the line (14), which is the orthogonal bisector of the segment $\overline{AD}$; such a case is illustrated in Fig. 4.

Returning to the ATDDG, the Target can escape from the Attacker and the Defender has a role to play in the target defense differential game if and only if the Apollonius circle, which is based on the segment $\overline{AT}$ and the speed ratio α, intersects the orthogonal bisector of the segment $\overline{AD}$. Recall that the equation of the orthogonal bisector of the segment $\overline{AD}$ is given by (14) where m and n are given by (10).

For given T and A positions such that $\overline{AT} < \overline{DT}$ assume that $\bar{\alpha}$ corresponds to the critical Apollonius circle. Let B be the point of tangency of the critical Apollonius circle with the line (14). Note that the line $\overline{BC}$ is parallel to the line $\overline{AD}$. Hence, using the slope given by A and D, the equation of the line joining the points B and C is given by

$$y = \tfrac{y_A - y_D}{x_A - x_D}(x - \bar{x}_c) + \bar{y}_c \tag{87}$$

where $\bar{x}_c$ and $\bar{y}_c$ are the coordinates of the center C of the critical Apollonius circle. The coordinates of point $B : (x_B, y_B)$ can then be obtained by determining the intersection of two lines, (14) and (87). For instance, the coordinate x_B can be obtained by solving the following equation

$$\begin{aligned} & \tfrac{y_A - y_D}{x_A - x_D}(x_B - \bar{x}_c) + \bar{y}_c = m x_B + n \\ \Rightarrow\ & \tfrac{(y_A - y_D)^2 + (x_A - x_D)^2}{(x_A - x_D)(y_A - y_D)} x_B = y_0 - \bar{y}_c \\ & \qquad + \tfrac{x_A - x_D}{y_A - y_D} x_0 + \tfrac{y_A - y_D}{x_A - x_D} \bar{x}_c \\ \Rightarrow\ & x_B = (y_0 - \bar{y}_c)\sin\sigma\cos\sigma + x_0\cos^2\sigma + \bar{x}_c \sin^2\sigma. \end{aligned}$$

The coordinate y_B can then be written in terms of x_B as $y_B = m x_B + n$.

We can now establish the condition for the Target to escape:

$$(\bar{y}_c - y_B)^2 + (\bar{x}_c - x_B)^2 < \big(\tfrac{\alpha}{1-\alpha^2} d\big)^2 \tag{88}$$

that is, for the Target to escape, the radius of the Target Apollonius circle needs to be larger than the distance $\overline{BC}$ given by the critical Apollonius circle. This condition ensures that the Target Apollonius circle intersects the line (14) instead of just being tangent to it as in the critical case.

We will now substitute the 'less than' sign in (88) by an 'equal' sign and simplify the resulting equation to obtain a characterization of the curve which is a cross section of the surface that separates the regions R_e and R_c where the $^-$sign has been removed from the center coordinates for simplicity. Doing so we have

$$\begin{aligned} & \big(y_c - m[(y_0 - y_c)\sin\sigma\cos\sigma + x_0\cos^2\sigma + x_c\sin^2\sigma] \\ & \quad - y_0 + m x_0\big)^2 \\ & + \big(x_c - (y_0 - y_c)\sin\sigma\cos\sigma - x_0\cos^2\sigma - x_c\sin^2\sigma\big)^2 \\ = \ & \big(\tfrac{\alpha}{1-\alpha^2} d\big)^2. \end{aligned} \tag{89}$$

Note that

$$\begin{aligned} m\sin\sigma\cos\sigma &= -\tfrac{x_A - x_D}{y_A - y_D} \times \tfrac{y_A - y_D}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}} \\ & \quad \times \tfrac{x_A - x_D}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}} \\ &= -\cos^2\sigma. \end{aligned}$$

Then, we can write (89) as follows

$$\begin{aligned} & \big(y_c(1 - \cos^2\sigma) + y_0\cos^2\sigma - m x_0\cos^2\sigma \\ & \quad - m x_c \sin^2\sigma - y_0 + m x_0\big)^2 \\ & + \big(x_c(1 - \sin^2\sigma) - (y_0 - y_c)\sin\sigma\cos\sigma - x_0\cos^2\sigma\big)^2 \\ = \ & \big(\tfrac{\alpha}{1-\alpha^2} d\big)^2. \end{aligned}$$

Using the relations $-m\sin\sigma = \cos\sigma$ and $\sin^2\sigma + \cos^2\sigma = 1$ we obtain

$$\begin{aligned} & \big([y_c - y_0 - m(x_c - x_0)]\sin^2\sigma\big)^2 \\ & \quad + \big((x_c - x_0)\cos^2\sigma + (y_c - y_0)\sin\sigma\cos\sigma\big)^2 \\ & = \big(\tfrac{\alpha}{1-\alpha^2} d\big)^2 \\ \Rightarrow\ & \big(y_c - y_0 - m(x_c - x_0)\big)^2 \sin^4\sigma \\ & \quad + \big(-m(x_c - x_0) + y_c - y_0\big)^2 \cos^2\sigma\sin^2\sigma \\ & = \big(\tfrac{\alpha}{1-\alpha^2} d\big)^2 \\ \Rightarrow\ & \big(y_c - y_0 - m(x_c - x_0)\big)^2 \sin^2\sigma = \big(\tfrac{\alpha}{1-\alpha^2} d\big)^2. \end{aligned} \tag{90}$$

Substituting the expressions for y_c and x_c given by (85) and (86) into (90) we have

$$\begin{aligned} & \big(\tfrac{1}{1-\alpha^2}(y_T - \alpha^2 y_A) - y_0 \\ & \quad - m[\tfrac{1}{1-\alpha^2}(x_T - \alpha^2 x_A) - x_0]\big)^2 \sin^2\sigma \\ & = \tfrac{\alpha^2}{(1-\alpha^2)^2}[(x_A - x_T)^2 + (y_A - y_T)^2] \\ \Rightarrow\ & \big(y_T - \alpha^2 y_A - m(x_T - \alpha^2 x_A) \\ & \quad - (1-\alpha^2)(y_0 - m x_0)\big)^2 \sin^2\sigma \\ & = \alpha^2[(x_A - x_T)^2 + (y_A - y_T)^2] \\ \Rightarrow\ & \big([y_T - \alpha^2 y_A - (1-\alpha^2) y_0]\sin\sigma \\ & \quad + [x_T - \alpha^2 x_A - (1-\alpha^2) x_0]\cos\sigma\big)^2 \\ & = \alpha^2[(x_A - x_T)^2 + (y_A - y_T)^2]. \end{aligned} \tag{91}$$

Since we aim at describing the curve (the cross section of the surface that separates the winning region of the Target/Defender team from the winning region of the Attacker) in terms of generic Cartesian coordinates (x, y) we make the change of variables $x_T \to x$ and $y_T \to y$ in (91) and expand

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the resulting equation

$$
\begin{aligned}
&\big(y^2 + [\alpha^2 y_A + (1-\alpha^2)y_0]^2 \\
&\quad - 2[\alpha^2 y_A + (1-\alpha^2)y_0]y\big)\sin^2\sigma \\
&+\big(x^2 + [\alpha^2 x_A + (1-\alpha^2)x_0]^2 \\
&\quad - 2[\alpha^2 x_A + (1-\alpha^2)x_0]x\big)\cos^2\sigma \\
&+2[y-\alpha^2 y_A - (1-\alpha^2)y_0] \\
&\quad \times [x - \alpha^2 x_A - (1-\alpha^2)x_0]\sin\sigma\cos\sigma \\
= \; & \alpha^2[x_A^2 + x^2 - 2x_A x + y_A^2 + y^2 - 2y_A y].
\end{aligned} \tag{92}
$$

Grouping together common terms from both sides of the above equation we obtain the hyperbola equation (82) where the discriminant is given by

$$
\begin{aligned}
D &= \begin{vmatrix} \cos^2\sigma - \alpha^2 & \sin\sigma\cos\sigma \\ \sin\sigma\cos\sigma & \cos^2\sigma-\alpha^2 \end{vmatrix} \\
&= (\cos^2\sigma - \alpha^2)(\sin^2\sigma - \alpha^2) - \sin^2\sigma\cos^2\sigma \\
&= \sin^2\sigma\cos^2\sigma - \alpha^2(\sin^2\sigma + \cos^2\sigma) + \alpha^4 \\
&\quad - \sin^2\sigma\cos^2\sigma \\
&= -\alpha^2(1-\alpha^2) \\
&< 0.
\end{aligned} \tag{93}
$$

The coordinates of the center of the hyperbola, (x_{oh}, y_{oh}), can be obtained by evaluating the following determinants

$$
x_{oh} = -\frac{1}{D}\begin{vmatrix} h_x & h_{xy} \\ h_y & h_{yy} \end{vmatrix},
$$
$$
y_{oh} = -\frac{1}{D}\begin{vmatrix} h_{xx} & h_x \\ h_{xy} & h_y \end{vmatrix}.
$$

Evaluating the determinant corresponding to x_{oh} we obtain the following expression

$$
\begin{aligned}
x_{oh} = \; & \tfrac{1}{\alpha^2(1-\alpha^2)}[\alpha^2\sin\sigma(x_A\sin\sigma - y_A\cos\sigma)(\sin^2\sigma-\alpha^2) \\
& - (1-\alpha^2)\cos\sigma(x_0\cos\sigma + y_0\sin\sigma)(\sin^2\sigma - \alpha^2) \\
& - \alpha^2\cos\sigma(y_A\cos\sigma - x_A\sin\sigma)\sin\sigma\cos\sigma \\
& + (1-\alpha^2)\sin\sigma(x_0\cos\sigma + y_0\sin\sigma)\sin\sigma\cos\sigma].
\end{aligned}
$$

Expanding and canceling common terms and factors we obtain the following

$$
\begin{aligned}
x_{oh} &= \tfrac{1}{1-\alpha^2}[\sin^2\sigma(x_A\sin^2\sigma + x_A\cos^2\sigma - \alpha^2 x_A) \\
&\quad + \sin\sigma\cos\sigma(\alpha^2 y_A - y_A\sin^2\sigma - y_A\cos^2\sigma) \\
&\quad + (1-\alpha^2)\cos\sigma(x_0\cos\sigma + y_0\sin\sigma)] \\
&= \tfrac{1}{1-\alpha^2}[(1-\alpha^2)x_A\sin^2\sigma - (1-\alpha^2)y_A\sin\sigma\cos\sigma \\
&\quad + (1-\alpha^2)\cos\sigma(x_0\cos\sigma + y_0\sin\sigma)] \\
&= x_A\sin^2\sigma - y_A\sin\sigma\cos\sigma \\
&\quad + x_0\cos^2\sigma + y_0\sin\sigma\cos\sigma.
\end{aligned}
$$

Writing $\sin\sigma$ and $\cos\sigma$ in terms of A and D we have

$$
\begin{aligned}
x_{oh} &= \frac{\sin\sigma}{\sqrt{(x_A-x_D)^2+(y_A-y_D)^2}}\big[(y_A - y_D)x_A \\
&\quad - (x_A - x_D)y_A - \tfrac{1}{2}(x_A + x_D)(y_A - y_D) \\
&\quad + \tfrac{1}{2}(y_A + y_D)(x_A - x_D)\big] + x_0 \\
&= \frac{\sin\sigma}{\sqrt{(x_A-x_D)^2+(y_A-y_D)^2}}\big(x_D y_A - x_A y_D \\
&\quad - x_D y_A + x_A y_D\big) + x_0 \\
&= x_0.
\end{aligned}
$$

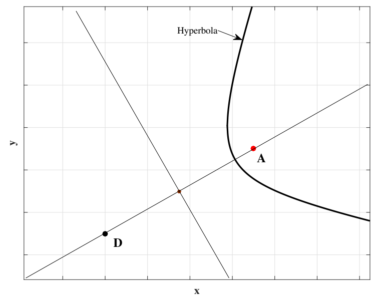

Fig. 5. Cross section of Barrier surface that separates the Target's escape and capture regions R_e and R_c

By similar evaluation of the determinant corresponding to y_{oh} we have that $y_{oh} = y_0$ and the center of the hyperbola is located at the midpoint between the Attacker and the Defender.

Finally, the angle between the principal axes of the hyperbola and the x-axis can be obtained using the formula

$$
\begin{aligned}
\tan 2\theta &= \tfrac{2h_{xy}}{h_{xx}-h_{yy}} \\
&= \tfrac{2\sin\sigma\cos\sigma}{\cos^2\sigma - \alpha^2 - \sin^2\sigma + \alpha^2}.
\end{aligned}
$$

Multiplying and dividing by $\frac{1}{\cos^2\sigma}$ we obtain

$$
\tan 2\theta = 2\tfrac{\tan\sigma}{1-\tan^2\sigma}
$$

which is the trigonometric identity for the tangent of a double-angle. Hence, we have that $\theta = \sigma$ and the hyperbola is as shown in Fig. 5 where only the branch on the Attacker's side is shown. All of the Defender's side of the plane is in R_e, so, the hyperbola branch on the Defender's side is irrelevant to the differential game and is not shown in the Figure. □

Remark. The saddle point solution of the ATDDG has been obtained. Also, the Value function in the region of win of the $T\&D$ team has been explicitly derived. The ATDDG has been successfully solved using Isaacs' method. The Value function is C^1 and there is no need for invoking the viscosity solution concept. This would have been the case irrespective of whether, by employing Isaacs' method, the solution had been analytically or numerically obtained and this is so even if the Value function would have turned out not to be C^1, or even continuous. In general, the application of Isaacs' method to the solution of differential games is successful, that is, the whole part of the state space where the solution of the Game of Degree applies is covered, there is no problem with the interpretation of the results, or the meaning of the Value function/state feedback strategies. Also, there is no need to interpret the Value function associated with the solution of the HJI hyperbolic PDE as the limit of a sequence of solutions of modified PDEs which are slightly elliptic. A case in point is the Homicidal Chauffeur differential game where its complete solution, that is, the Value function is not continuous across the Barrier line and it is not differentiable on the Equivocal line, and yet we have a clear understanding of the time-to-capture Value function across the entire state space and we have a good understanding of the players maneuvers.

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VI. Conclusions

The differential game where a Target aircraft evades an Attacker missile being helped by a Defender missile who is tasked to intercept the Attacker before the latter captures the Target has been solved in closed form. The synthesis of the players' optimal state feedback strategies and verification of the closed-loop solution of the differential game of active target defense have been formally presented. The cooperative strategies of the Target and the Defender are highlighted. These strategies enable interception of the Attacker and allow successful Target evasion, for otherwise, without a Defender, the slower Target would invariably be captured by the incoming Attacker. The ATDDG pursuit-evasion scenario and its analysis lay the cornerstone for future and more challenging pursuit-evasion games with non-holonomic dynamics, or scenarios where multiple Attackers and Defenders may be present. There are few pursuit-evasion differential games whose state space dimension is more than two which have been solved in closed form. Our solution of the ATDDG is a contribution to this limited repertoire.

VII. Appendix

Additional analysis is provided in this Appendix in order to determine the optimal interception coordinate x^* from the possible candidates which are the real roots of the critical quartic equation (15). It was shown that the optimal interception point I^* satisfies (14) and it is the instantaneous aimpoint of each agent. Then, the Value function can be expressed in terms of the interception point's coordinates $I=(x,y)$ and we can search for extrema conditions. Consider first the case $\overline{DT}<\overline{AT}$ - refer to Fig. 6.a. We can write

$$\begin{aligned} V(\overline{OI})) &= \alpha\overline{AI}+\overline{TI} \\ &= \alpha\sqrt{\overline{AO}^2+\overline{OI}^2}+\sqrt{\overline{TT_p}^2+(\overline{OT_p}-\overline{OI})^2}. \end{aligned}$$

Compute first and second derivative with respect to $\overline{OI}$

$$\begin{aligned} \frac{dV(\overline{OI})}{d\overline{OI}} &= \alpha\frac{\overline{OI}}{\overline{AI}}-\frac{\overline{OT_p}-\overline{OI}}{\overline{TI}} \\ \frac{d^2V(\overline{OI})}{d\overline{OI}^2} &= \alpha\frac{\overline{AI}^2-\overline{OI}^2}{\overline{AI}^3}+\frac{\overline{TI}^2-(\overline{OT_p}-\overline{OI})^2}{\overline{TI}^3} \\ &= \alpha\frac{\overline{AI}^2-\overline{AI}^2\sin^2\hat{\chi}}{\overline{AI}^3}+\frac{\overline{TI}^2-\overline{TI}^2\sin^2\hat{\phi}}{\overline{TI}^3} \\ &= \frac{\alpha}{\overline{AI}}\cos^2\hat{\chi}+\frac{1}{\overline{TI}}\cos^2\hat{\phi}>0. \end{aligned}$$

The function $V(\overline{OI})=V(x)$ is a strictly convex function of $\overline{OI}$, that is, it is a strictly convex function of x, where x satisfies $y=mx+n$. It was shown in (64)-(65) that if we set $\frac{dV}{dx}$ equal to zero we obtain the quartic equation (15). Hence, although the quartic equation (15) may have more than one real root, one and only one of the real roots is the optimal solution. Because $V(x)$ is strictly convex the optimal solution is $x^*=\arg\min V(x)$.

It can be shown in general that in the limit case $\alpha=0$ the optimal root of (15) is $x^*=x_{T_p}$ and $y^*=y_{T_p}=mx_{T_p}+n$ such that T_p is the closest point on the orthogonal bisector to the Target position T. Also, it can be shown that in the limit case $\alpha=1$ the optimal solution of (15) constitutes Line-Of-Sight (LOS) evasion where the Target runs directly away from the Attacker, that is, $\phi=\chi$; the corresponding point on the orthogonal bisector is denoted as $L=(x_L,y_L)$.

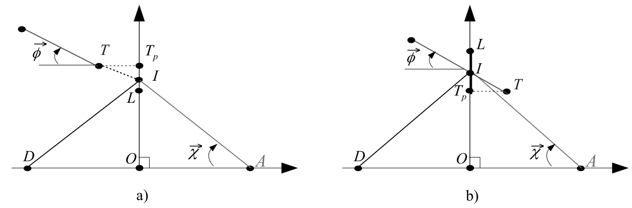

Fig. 6. Optimal interception point

We now analyze the case where $\overline{DT}>\overline{AT}$. Because of the continuous dependence of (15) on the speed ratio parameter α, the optimal interception point for $0<\alpha<1$ lies on the segment of the orthogonal bisector $\overline{LT_p}$ shown in bold face in Fig. 6.b. When $\overline{DT}>\overline{AT}$ it is not straightforward to show concavity over the entire interval. However, it is sufficient to analyze the possible set of solutions $\overline{LT_p}$. In this case

$$\begin{aligned} V(\overline{OI})) &= \alpha\overline{AI}-\overline{TI} \\ &= \alpha\sqrt{\overline{AO}^2+\overline{OI}^2}-\sqrt{\overline{TT_p}^2+(\overline{OI}-\overline{OT_p})^2}. \end{aligned}$$

The first derivative with respect to $\overline{OI}$ is

$$V'(\overline{OI})=\frac{dV(\overline{OI})}{d\overline{OI}} \quad =\alpha\frac{\overline{OI}}{\overline{AI}}-\frac{\overline{OI}-\overline{OT_p}}{\overline{TI}}=\alpha\sin\hat{\chi}-\sin\hat{\phi}.$$

Analyzing relevant values of $V'(\overline{OI})=V'(x)$ we have that $V'(x=x_{T_p})=\alpha\sin\hat{\chi}>0$ and $V'(x=x_L)=(\alpha-1)\sin\hat{\phi}<0$. Since both, $0<\hat{\phi}<\pi/2$ and $0<\hat{\chi}<\pi/2$ increase as x goes from $x=x_{T_p}$ to $x=x_L$ and the trigonometric function $\sin(\cdot)$ is a monotonically increasing function in the interval $(0,\pi/2)$, we conclude that $V(x)=V(\overline{OI}))=\alpha\overline{AI}-\overline{TI}$ is monotonically increasing in the interval $x\in[x_{T_p},x^*)$ and it is monotonically decreasing in the interval $x\in(x^*,x_L]$ where x^* is such that $V'(x)=0$. Therefore, $V(x)$ has only one maximum in the interval of interest $x\in[x_{T_p},x_L]$. If more than one real root of the quartic equation (15) lies in $[x_{T_p},x_L]$ then the optimal solution is uniquely determined by obtaining $x^*=\arg\max V(x)$.

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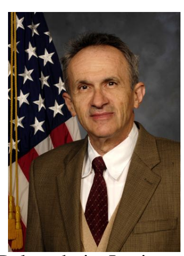

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