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Nonsmooth Semipermeable Barriers, Isaacs' Equation, and Application to a Differential Game with One Target and Two Players

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Abstract. We study the target problem which is a differential game where one of the players aims at reaching a target while the other player aims at avoiding this target forever. We characterize the victory domains of the players by means of geometric conditions and prove that the boundary of the victory domains is a nonsmooth semipermeable surface, i.e., is a solution (in a weak sense) of the Isaacs equation: $\sup_u \inf_v \langle f(x,u,v), p\rangle = 0$, where f is the dynamic of the system, u and v are the respective controls of the players, and p is a normal to the boundary of the victory domains at the point x.

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Introduction

The target problem, in two-players differential game theory, is a game where the first player aims at reaching a target while his opponent aims at avoiding this target forever. To fix the ideas, let

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)) & \text{for almost every} \quad t \geq 0, \\ u(t) \in U \quad \text{and} \quad v(t) \in V, & \end{cases}$$

be the dynamic of the game. The first player plays with u while the second player plays with v.

In such a qualitative game, where there is no value function, the interest is in the victory domains of the players. The victory domain of a player is the set of initial conditions starting from which this player may win against any action of his opponent. In this paper we characterize the victory domains of the players by means of geometric conditions. We also prove that the boundary of these domains are semipermeable barriers, i.e., satisfies Isaacs' equation in a weak sense.

We now briefly recall what are the semipermeable barriers and Isaacs' equation. In his pioneering work [16], Isaacs proves that if the common part of the boundary of the two victory domains is smooth, then it is a smooth semipermeable barrier. A smooth semipermeable barrier is a $\mathcal{C}^2$ manifold $\{x \mid g(x) = 0\}$ such that

$$g(x) = 0 \quad \Rightarrow \quad \sup_u \inf_v \langle f(x, u, v), g'(x) \rangle = 0. \tag{1}$$

Semipermeable barriers are oriented surfaces which enjoy the "semipermeability property." Namely, each player can avoid the state of the system $x(\cdot)$ to cross a semipermeable barrier in one sense. Indeed, if the initial state of the system belongs to a semipermeable barrier $\{x \mid g(x) = 0\}$, then the player which plays with u can avoid the state of the system to enter in the set $\{x \mid g(x) < 0\}$ by "playing the strategy"[1]

$$x \to \operatorname*{Argmax}_u \inf_v \langle f(x, u, v), g'(x) \rangle.$$

If [2] the player which plays with v "plays the strategy"

$$x \to \operatorname*{Argmin}_v \sup_u \langle f(x, u, v), g'(x) \rangle,$$

then the state of the system cannot enter in the set $\{x \mid g(x) > 0\}$. Conversely, a smooth set which enjoys this "semipermeability property" is necessarily a semipermeable barrier (see [16]).

Unfortunately, the boundary of the victory domains is seldom smooth, even in very simple examples. There are several interesting extensions of Isaacs' results to cases where the smoothness of the boundary is weakened (see, for instance, [9] and [5]). However, the general case has not been treated yet.

The aim of this paper is to prove that the victory domains of the players can be characterized by means of geometric conditions and that their boundary satisfies Isaacs' equation in a weak sense. Namely, the victory domain of the second player is the largest closed subset D of the complement of the target satisfying the following condition:

$$\forall x \in D, \quad \forall p \in NP_D(x), \qquad H(x, p) \leq 0,$$

where $H(x, p) := \sup_u \inf_v \langle f(x, u, v), p \rangle$ is the hamiltonian of the problem and where

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[1] Providing satisfying definitions of "strategies" is a main problem in game theory. We recall one of the possible definitions in the last section of this paper. For simplicity, we just mean here the regular maps from $\mathbb{R}^N$ to U (or V). For that purpose, we assume that the maps $x \to \operatorname{Argmin}_v \sup_u \langle f(x, u, v), g'(x) \rangle$, etc., are well defined and smooth.

[2] We assume here that Isaacs' condition is fulfilled:

$$\forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad \sup_u \inf_v \langle f(x, u, v), p \rangle = \inf_v \sup_u \langle f(x, u, v), p \rangle.$$

$NP_D(x)$ is the set of (outward) proximal normals to D at x (see Definition 1.1 below). This largest closed subset is called the *discriminating kernel* or the complement of the target for the hamiltonian H. Discriminating kernels have been introduced by Aubin in [1] and [2] with a slightly different definition.

The boundary of the discriminating kernel satisfies a generalized Isaacs' equation. Namely, if x belongs to the boundary of this set but not to the boundary of the target, then:

- If p is an (exterior) proximal normal at x, then $H(x, p) \leq 0$.
- If q is an interior proximal normal at x, then $H(x, -q) \geq 0$.

Putting these inequalities together yields that $H(x, p) = 0$, if p is an (outward) proximal normal and $q := -p$ is an interior proximal normal. Thus we have here a definition of (nonsmooth) semipermeable barriers. This definition fits with the one introduced by Quincampoix [19] in control theory.

The results of this paper have been announced and used in the joint work [12] where numerical schemes for computing the discriminating kernel of a closed set K are given. The results of this paper are also used in [11], where the target problem is studied for a more accurate framework of strategies (Elliot and Kalton nonanticipative strategies—see [14]).

Notation. Throughout this paper, we denote by B the closed unit ball of the state-space $\mathbb{R}^N$. If K is a closed set, the (euclidian) distance from a point x to K is denoted by $d_K(x)$:

$$d_K(x) := \min_{y \in K} \|x - y\|.$$

For any $r > 0$, we denote by $K + rB$ the closed set of points x such that $d_K(x) \leq r$.

1. Discriminating Domains and Kernels

1.1. *Definition and Existence of the Discriminating Kernel*

We first introduce the definition of the outward normals and state some of their basic properties.

Definition 1.1. Let K be a closed subset of $\mathbb{R}^N$ and let x belong to K. A vector $\nu \in \mathbb{R}^N$ is a proximal normal to K at x if and only if

$$d_K(x + \nu) = \|\nu\|. \tag{2}$$

We denote by $NP_K(x)$ the set of proximal normals to K at x.

A similar definition of the proximal normals to a closed set can be found in [8] and [13]. More recently it has been used by Veliov [21] in the context of viability theory.

Remarks. (1) A vector $\nu \neq 0$ is a proximal normal to a closed set K at a point $x \in K$ if and only if the open ball of center $x + \nu$ and of radius $\|\nu\|$ has an empty intersection with K. Roughly speaking, the closed ball of center $x + \nu$ and of radius $\|\nu\|$ is "tangent" to the set K.

(2) Let K be a $\mathcal{C}^2$ manifold and let $\bar{x}$ be long to ∂K. Assume that there is a neighborhood $\mathcal{O}$ of $\bar{x}$ such that

$$K \cap \mathcal{O} = \{x \in \mathcal{O} \mid g(x) \leq 0\},$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is a $\mathcal{C}^2$ map and $\nabla g(x) \neq 0$ if $g(x) = 0$.

Then there is some $\lambda > 0$ such that $\lambda \nabla g(\bar{x})$ belongs to $NP_K(\bar{x})$.

Proposition 1.1. *Let K be a closed subset of $\mathbb{R}^N$.*

(1) *If $x \in K$ and $\nu \in NP_K(x)$, then x belongs to the projection of $x + \nu$ onto K.*
(2) *If y does not belong to K and if x belongs to the projection of y onto K, then $y - x$ belongs to $NP_K(x)$.*
(3) *If $x \in K$ and $\nu \in NP_K(x)$, then, for any $\lambda \in [0, 1)$, x is the unique projection of $x + \lambda\nu$ onto K.*
(4) *If $K_1 \subset K_2$ are closed subset of $\mathbb{R}^N$ and if x belongs to K_1, then $NP_{K_2}(x) \subset NP_{K_1}(x)$.*
(5) *If K is the lower-limit*[3] *of a sequence (K_p) of closed subsets of $\mathbb{R}^N$ (in the Painlevé–Kuratowski sense), then the lower-limit of the graphs Graph(NP_{K_p}) is contained in the graph Graph(NP_K).*
(6) *If K is the upper-limit*[4] *of a sequence (K_p) of closed subsets of $\mathbb{R}^N$ (in the Kuratowski sense), then the upper-limit of the graphs Graph(NP_{K_p}) contains the graph Graph(NP_K).*
(7) *The set-valued map $NP_K(\cdot)$ has a closed graph.*
(8) *For any x of K, the set $NP_K(x)$ is a closed convex set and* 0 *belongs to this set.*

We give an outline of the proof of these results in the Appendix.

We are now ready to state the main definitions of this paper:

Definition 1.2. Let $H\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a hamiltonian. A closed set D is a discriminating domain for H if

$$\forall x \in D, \quad \forall \nu \in NP_D(x), \qquad H(x, \nu) \leq 0. \tag{3}$$

Theorem 1.1 (Existence of the Discriminating Kernel). *Let K be a closed subset of $\mathbb{R}^N$ and let $H\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a lower semicontinuous hamiltonian. There is a (may*

[3] The lower-limit of a sequence of closed sets (K_p) is the closed set of points x such that $\lim d_{K_p}(x) = 0$ (see [3]).

[4] The upper-limit of a sequence of closed sets (K_p) is the closed set of points x such that $\liminf d_{K_p}(x) = 0$ (see [3]).

be empty) closed discriminating domain for H contained in K which contains any other discriminating domain for H contained in K. This set is called the discriminating kernel of K for H and is denoted by $Disc_H(K)$.

Proof of Theorem 1.1. Let D be the closure of the union of all closed discriminating domains for H contained in K. We have to show that D is a discriminating domain for H. Note that D is obviously contained in K.

Let x belong to D and ν belong to $NP_D(x)$. We have to prove that $H(x, \nu) \leq 0$.

Let $\lambda \in (0, 1)$. We first show that the normal condition $H(x, \nu_\lambda) \leq 0$ is fulfilled for $\nu_\lambda := \lambda\nu$. According to Proposition 1.1, x is the unique projection of $x + \nu_\lambda$ onto D.

From the very definition of D, for any $p \in \mathbb{N}^*$, there is a discriminating domain D_p for H contained in K such that $d_{D_p}(x)$ is smaller than $1/p$. Let x_p be a projection of $x + \nu_\lambda$ onto D_p. Lemma 3.1 yields that a subsequence of the sequence (x_p) converges to x because the upper-limit of the sets D_p is contained in D.

The vector $x + \nu_\lambda - x_p$ is a proximal normal to D_p at x_p. Since D_p is a discriminating domain for H, $H(x_p, x + \nu_\lambda - x_p)$ is nonpositive. When $p \to +\infty$, we get $H(x, \nu_\lambda) \leq 0$ because H is lower semicontinuous. Now let $\lambda \to 1^-$. We obtain $H(x, \nu) \leq 0$ still because H is lower semicontinuous.

This proves that D is a discriminating domain for H because normal condition (3) is everywhere fulfilled. Since D contains any other discriminating domain for H contained in K, D is the discriminating kernel of K for H. □

1.2. *Generalized Isaacs' Equation*

Since $Disc_H(K)$ is a maximal discriminating domain for H contained in K, $Disc_H(K)$ enjoys a particular property. To state this property, we have to provide the following definition:

Definition 1.3. Let K be a closed subset of $\mathbb{R}^N$ and let x belong to ∂K. A vector ν is an interior proximal normal to K at x if $x + \nu + \|\nu\| B$ is contained in K.

In another words, ν is an interior proximal normal to K at x if ν belongs to $NP_{\overline{\mathbb{R}^N \setminus K}}(x)$.

Theorem 1.2. *Let K be a closed subset of $\mathbb{R}^N$ and let H be a lower semicontinuous hamiltonian which is positively homogeneous in the second variable. Let x belong to the boundary of $Disc_H(K)$ and to the interior of K. Assume that H is continuous at (x, p) for any $p \in \mathbb{R}^N$. Then, for any interior proximal normal ν to $Disc_H(K)$ at x, we have*

$$H(x, -\nu) \geq 0. \tag{4}$$

Remarks. (1) Since $Disc_H(K)$ is a discriminating domain for H, the following inequalities also hold true:

$$\forall x \in Disc_H(K), \quad \forall \nu \in NP_{Disc_H(K)}(x), \qquad H(x, \nu) \leq 0.$$

In particular, if $Disc_H(K)$ is a smooth (say $\mathcal{C}^2$) manifold in the neighborhood of some point x of $\partial Disc_H(K)\backslash\partial K$, the normal ν (in the usual sense) of the boundary of $Disc_H(K)$ satisfies

$$H(x, \nu) = 0, \tag{5}$$

thanks to (3) and (4).

(2) Similar result have been obtained, in the framework of control theory, by Quincampoix [19]. We discuss his results after the statement of the Viability Theorem.

We now assume that H is of the form

$$H(x, p) := \sup_u \inf_v \langle f(x, u, v), p\rangle, \tag{6}$$

where $f\colon \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ (where U and V are compact metric space) is a continuous map. Then (5) is actually Isaacs' equation (1).

Combining (3) and (4) provides a generalized solution to Isaacs' equation (1).

Definition 1.4 (Nonsmooth Semipermeable Barrier). Let H be defined from a dynamics f as in (6). The boundary of a closed set $D \subset \mathbb{R}^N$ is a semipermeable barrier for f in a neighborhood $\mathcal{W}$ of a point $\hat{x} \in \partial D$ if it satisfies Isaacs equation (1) in the following sense:

$$\forall x \in \partial D \cap \mathcal{W}, \quad \forall \nu \in NP_D(x), \qquad H(x, \nu) \leq 0,$$

and

$$\forall x \in \partial D \cap \mathcal{W}, \quad \forall \nu \in NP_{\overline{\mathbb{R}^N\backslash D}}(x), \qquad H(x, -\nu) \geq 0.$$

Corollary 1.1. *If the previous conditions on f are satisfied and if H is defined as in* (6), *then, for any closed set K, the set $Disc_H(K)$ is a semipermeable barrier in a neighborhood of any point $\hat{x} \in \partial Disc_H(K)\backslash\partial K$.*

Proof of Theorem 1.2. Assume for a while that the normal condition (4) is not fulfilled at a point x which belongs to $\partial Disc_H(K)$, but not to ∂K. Then there is ν interior proximal normal to K at x such that $H(x, -\nu) < 0$.

Since x does not belong to ∂K, there is some radius $r > 0$ such that $x + rB$ is contained in K. The hamiltonian H is continuous at $(x, -\nu)$. So there is some positive $\varepsilon \leq r$ such that

$$\forall (y, \mu) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad \left.\begin{array}{rl} & \|y - x\| \leq \varepsilon \\ \text{and} & \|\mu - \nu\| \leq \varepsilon \end{array}\right\} \Rightarrow \quad H(y, -\mu) \leq 0. \tag{7}$$

In what follows we fix b and c such that $0 < c < b < 1$ and such that

$$\frac{b^2 - c^2}{c^2(1 - c)}\|\nu\|^2 \leq \varepsilon^2.$$

We are going to prove that the set

$$D := Disc_H(K) \cup (x + cv + b\|v\|B)$$

is still a discriminating domain for H contained in K. Since x belongs to the interior of D, there is a contradiction because x is supposed to belong to the boundary of the largest closed discriminating domain for H contained in K. We first prove that D is contained in K and then that D is a discriminating domain for H.

(1) *D is contained in K.* Let y belong to ∂D. If y belongs to $Disc_H(K)$, then y belongs to K because $Disc_H(K)$ is contained in K. Assume now that y belongs to D but not to $Disc_H(K)$. Then y does not belong to the interior of $x + v + \|v\|B$, because $x + v + \|v\|B$ is contained in $Disc_H(K)$. Thus

$$\|y - (x + v)\| \geq \|v\|$$

and so

$$\|y - x\|^2 - 2\langle y - x, v\rangle \geq 0. \tag{8}$$

Since y belongs to $(x + cv + b\|v\|B)$, we have

$$\|y - x\|^2 - 2c\langle y - x, v\rangle + c^2\|v\|^2 \leq b^2\|v\|^2. \tag{9}$$

From (8) and (9) we deduce that

$$\|y - x\|^2 \leq \frac{b^2 - c^2}{1 - c}\|v\|^2 \leq \varepsilon^2 c^2. \tag{10}$$

Since $c < 1$, the right-hand side is not larger than ε^2 and so neither than r^2. Thus y belongs to $x + rB$ which is contained in K. We have finally proved that D is contained in K.

(2) *D is a discriminating domain for H.* We have to prove that the conditions $\forall y \in D$, $\forall \mu \in NP_D(y)$,

$$\sup_u \inf_v \langle f(y), u, v), \mu\rangle \leq 0, \tag{11}$$

is fulfilled any y of D.

If y belongs to $Disc_H(K)$, then any proximal normal μ to D at x is a proximal normal to $Disc_H(K)$ at x, because D contains $Disc_H(K)$ (see Proposition 1.1(4)). Thus (11) holds true because $Disc_H(K)$ is a discriminating domain for H.

It remains to show that if a point y belongs to ∂D but not to $Disc_H(K)$, then (11) is still satisfied.

Let μ be a proximal normal to D at y. Since y belongs to the boundary of $(x + cv + b\|v\|B)$, which is contained in D, μ is also a proximal normal to $(x + cv + b\|v\|B)$ at y. Thus μ is equal to $t(y - x - cv)$ for some $t > 0$.

Then

$$H(y, t(y - x - c\nu)) = ctH\left(y, \frac{y - x}{c} - \nu\right)$$

because H is positively homogeneous in the second variable. From (10),

$$\|y - x\| \leq \varepsilon \quad \text{and} \quad \left\| \frac{y - x}{c} - \nu - (-\nu) \right\| \leq \varepsilon.$$

Inequality (7) states then that $H(y, t(y - x - c\nu)) \leq 0$, i.e., $H(y, \nu) \leq 0$.

Thus normal condition (11) is fulfilled everywhere in D, so that D is a discriminating domain for H. We have proved moreover that D is contained in K and strictly contains $Disc_H(K)$. This is in contradiction with the very definition of $Disc_H(K)$. □

1.3. *Some Properties of the Discriminating Domains of a Hamiltonian*

We first point out two monotonicity properties:

(1) Assume that $H_1: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $H_2: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are lower semicontinuous hamiltonians and that

$$\forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad H_1(x, p) \leq 0 \quad \Rightarrow \quad H_2(x, p) \leq 0.$$

Then any discriminating domain for H_1 is a discriminating domain for H_2. Moreover, for any closed set K, the discriminating kernel of K for H_1 is contained in the discriminating kernel of K for H_2.

(2) Assume now that H is a lower semicontinuous hamiltonian and that $K \subset K'$ are closed subsets of $\mathbb{R}^N$. Then $Disc_H(K)$ is contained in $Disc_H(K')$.

The discriminating domains for a hamiltonian are stable in the following sense:

Proposition 1.2 (Stability of Discriminating Domains). *Let K_p be a sequence of closed subsets of $\mathbb{R}^N$ and let $H_p: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a sequence of lower semicontinuous hamiltonians. Assume that H is a lower semicontinuous hamiltonian such that*

$$(x, \nu) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad \liminf_{\substack{x' \to x, \nu' \to \nu, \\ \text{and } p \to +\infty}} H_p(x', \nu') \leq 0 \quad \Rightarrow \quad H(x, \nu) \leq 0. \tag{12}$$

Let K be the upper-limit of the sequence K_p. If, for any p, K_p is a discriminating domain for H_p, then K is a discriminating domain for H.

In particular, condition (12) is fulfilled if the sequence (H_p) converges to H for the compact topology.

Proof of Proposition 1.2. Fix $x \in K$, $\nu \in NP_K(x)$. We have to prove that $H(x, \nu) \leq 0$.

From Proposition 1.1, the upper-limit of *Graph*(NP_{K_p}) contains *Graph*(NP_K). So there are $x_p \in K_p$ and $\nu_p \in NP_{K_p}(x_p)$ which converge respectively (up to a subsequence) to x and ν. Since K_p is a discriminating domain for H_p,

$$\forall p \in IN^*, \qquad H_p(x_p, \nu_p) \leq 0.$$

Thus $\liminf_p H_p(x_p, \nu_p) \leq 0$ and so, from assumption, $H(x, \nu) \leq 0$. So K is a discriminating domain for H. □

An application of Proposition 1.2 is the following:

Corollary 1.2. *Assume that K is a closed subset of $\mathbb{R}^N$, that H is a lower semicontinuous hamiltonian, and that a point x belongs to K but not to $Disc_H(K)$. Then there is some positive ε such that x does not belong to $Disc_H(K + \varepsilon B)$.*

Proof. For any positive ε, we have

$$Disc_H(K) \subset Disc_H(K + \varepsilon B) \subset K + \varepsilon B.$$

Thus

$$\bigcap_{\varepsilon>0} Disc_H(K + \varepsilon B) \subset K.$$

Proposition 1.2 states that the decreasing intersection of $Disc_H(K + \varepsilon B)$ for $\varepsilon > 0$ is still a discriminating domain for H. So it is contained in $Disc_H(K)$, from the very definition of the discriminating kernel of K. Thus the intersection of $Disc_H(K + \varepsilon B)$ for $\varepsilon > 0$ is equal to $Disc_H(K)$. Since x does not belong to $Disc_H(K)$, there is some positive ε such that x does not belong to $Disc_H(K + \varepsilon B)$, which is the desired conclusion. □

1.4. *Characterizations of the Discriminating Domains*

Our purpose is to characterize, by means of viability theory, the discriminating domains and kernels for hamiltonians of the form

$$\forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad H(x, p) := \sup_u \inf_v \langle f(x, u, v), p \rangle, \tag{13}$$

where $f\colon \mathbb{R}^N \times U \times V \to \mathbb{R}^N$. These characterizations are very useful below.

For simplicity, we introduce the following definitions:

Definition 1.5. Let $f\colon \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ be some dynamic. A closed set D is a discriminating domain for f if it is a discriminating domain for the associated hamiltonian H defined by (13). If the hamiltonian H defined by (13) is lower semicontinuous, the discriminating kernel of a closed set K for f is the discriminating kernel of K for H. We denote it $Disc_f(K)$.

For convenience for the reader, we briefly recall some basic results of viability theory. Viability theory deals with differential inclusions under constraints:

Theorem 1.3 (Viability Theorem). *Let K be a closed subset of $\mathbb{R}^N$ and let $F\colon \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be an upper semicontinuous set-valued map with closed convex values and with a linear growth.*[5] *There is an equivalence between:*

(a) *The set K enjoys the viability property, i.e., for any $x \in K$, there is a solution of the differential inclusion:*

$$\begin{cases} x'(t) \in F(x(t)) & \textit{for almost all} \quad t \geq 0, \\ x(0) = x, \end{cases} \tag{14}$$

which remains in K (i.e., $x(t) \in K$ for any $t \geq 0$).

(b) *The set K satisfies the normal condition:*

$$\forall x \in K, \quad \forall \nu \in NP_K(x), \qquad \inf_{v \in F(x)} \langle v, \nu \rangle \leq 0. \tag{15}$$

When K is not a viability domain, there is a largest closed viability domain for F contained in K: The viability kernel of K for F. The viability kernel of K for F is actually the set of initial positions x_0 of K for which there is a solution of (14) *which remains in K forever. This set is denoted by $Viab_F(K)$.*

Results on viability theory can be found in the monograph [2], whenever equivalence between (a) and (b) is not the classical one.[6]

The existence of a viability kernel can be obtained as a corollary of Theorem 1.1 by setting $H(x, p) := \inf_{v \in F(x)} \langle v, p \rangle$. Theorem 1.2 for this particular hamiltonian H is a corollary of Quincampoix' result in control theory [19]. Quincampoix' results are more precise than ours because they are concerned with the behavior of the trajectories on the boundary of the viability kernel. The reader may also refer to Saint-Pierre [20] for the case where F is only upper semicontinuous.

We now characterize the discriminating domains for f. We summarize the assumptions on f:

$$\begin{cases} (1) & U \text{ and } V \text{ are metric compact spaces.} \\ (2) & f\colon \mathbb{R}^N \times U \times V \to \mathbb{R}^N \text{ is continuous.} \\ (3) & f(\cdot, u, v) \text{ is an } \ell\text{-Lipschitz map for any } u \text{ and } v. \end{cases} \tag{16}$$

Proposition 1.3. *Let f satisfy* (16). *A closed set D is a discriminating domain for f if and only if D is a viability domain for the set-valued maps $x \rightsquigarrow \overline{\text{co}}[\bigcup_v f(x, u, v)]$ for any $u \in U$.*

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[5] Such a set-valued map is called a Marchaud set-valued map. For more details, see the monograph [2].

[6] A proof of this equivalence can be found in [21] or [10]. In fact, this result is already contained in Aubin and Frankowska's proof [4] of the equivalence between (a) and

$$\forall x \in K, \qquad F(x) \cap \overline{\text{co}}\, T_K(x) \neq \emptyset,$$

where $T_K(x)$ denotes the contingent cone to K at x.

The proof is obvious. Note that if

$$\forall x \in \mathbb{R}^N, \quad \forall u \in U, \qquad \bigcup_{v\in V} f(x, u, v) \text{ is convex,} \tag{17}$$

the discriminating domains for f are actually the discriminating domains defined in [2] and also studied in [18].

The following result also appeared in [2]:

Proposition 1.4. *Assume that f satisfies* (16) *and* (17). *A closed set D is a discriminating domain for f if and only if, for any initial position $x_0 \in D$, for any continuous feedback $\tilde{u}(\cdot)\colon \mathbb{R}^N \to U$, there exist a time-measurable control $v(\cdot)\colon \mathbb{R}^N \to V$ and a solution $x(\cdot)$ to*

$$\begin{cases} x'(t) = f(x(t), \tilde{u}(x(t)), v(t)) & \text{for almost every} \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{18}$$

which remains in D forever (i.e., $x(t) \in D$ for any $t \geq 0$).

Let now K be any closed subset of $\mathbb{R}^N$ and let f satisfy (16) and (17). Since $Disc_f(K)$ is a discriminating domain for f, for any initial position $x_0 \in Disc_f(K)$, for any continuous feedback $\tilde{u}(\cdot)\colon \mathbb{R}^N \to U$, there exist a control $v(\cdot)$ and a solution of (18) which remains in K forever, because it remains in $Disc_f(K)$. Does this latter property characterize the discriminating kernel?

This characterization would be very interesting because it would mean that, for any $x_0 \in K \setminus Disc_f(K)$, there is a continuous feedback $\tilde{u}(\cdot)$ such that, for any control $v(\cdot)$, any solution of (18) leaves K in finite time.

Unfortunately, this characterization is generally not true.[7] This is the reason why it is necessary to introduce the difficult notion of *strategy* is two player differential games. We do so in Section 2.

We now show that the discriminating kernel of a closed set K can be computed as intersections of viability domains:

Proposition 1.5 (Algorithm for the Discriminating Kernel). *Let K be a closed set and let f satisfy* (16) *and* (17). *Define the following decreasing sequence of closed sets*:

$$\begin{cases} K_1 := K, \\ K_{i+1} := \bigcap_{u\in U} Viab_{f(\cdot,u,V)}(K_i). \end{cases} \tag{19}$$

Then the intersection of the K_i is equal to $Disc_f(K)$.

Proposition 1.5 means that if the initial state of the system is in the "slice" $K_{i_1} \setminus K_{i_1+1}$, then a constant control $u \in U$ can be played so that the state of the system leaves K_{i_1} in finite time, i.e., reaches either $\mathbb{R}^N \setminus K$ or some "slice" $K_{i_2} \setminus K_{i_2+1}$ with $i_2 < i_1$. So, roughly speaking, one should be able to define by induction a "strategy" $\tilde{u}(\cdot)$ which makes the

[7] The reader can find a counterexample in [10]. We do not provide it here because it is rather long.

state of the system leave K in finite time for any control $v(\cdot)$. This result is proved in the framework of the feedback strategies below and in the framework of the nonanticipative strategies [11].

Proof of Theorem 1.5. Let K_∞ be the intersection of the K_i. We first prove that K_∞ is a discriminating domain for f. From Proposition 1.3, it is sufficient to show that K_∞ is a viability domain for the set-valued map $f(\cdot, u, V)$, for any $u \in U$.

Fix $u \in U$. Then

$$K_{i+1} \subset \mathit{Viab}_{f(\cdot,u,V)}(K_i) \subset K_i.$$

So K_∞ is the decreasing limit of the closed sets $\mathit{Viab}_{f(\cdot,u,V)}(K_i)$, which are viable for $f(\cdot, u, V)$. This implies that K_∞ is viable for any $f(\cdot, u, V)$ from Stability Theorem 3.6.2 of [2]. Since this holds true for any $u \in U$, K_∞ is a discriminating domain for f contained in K, and thus K_∞ is contained in $\mathit{Disc}_f(K)$.

We now show that K_∞ contains the discriminating kernel for f of K. It is sufficient to prove by induction that $\mathit{Disc}_f(K)$ is contained in any K_i. It is obvious for $K_1 = K$. Assume that $\mathit{Disc}_f(K)$ is contained in K_i. Then, for any $u \in U$, $\mathit{Viab}_{f(\cdot,u,V)}(\mathit{Disc}_f(K))$ is contained in $\mathit{Viab}_{f(\cdot,u,V)}(K_i)$. However, $\mathit{Disc}_f(K)$ is already viable for $f(\cdot, u, V)$ and thus $\mathit{Disc}_f(K)$ is equal to $\mathit{Viab}_{f(\cdot,u,V)}(\mathit{Disc}_f(K))$. This proves that $\mathit{Disc}_f(K)$ is contained in K_{i+1}, and recursively that $\mathit{Disc}_f(K)$ is contained in K_∞. So K_∞ is the discriminating kernel of K for f. □

2. The Target Problem in Differential Game Theory

2.1. *Characterization of the Victory Domains*

We now investigate a differential game whose dynamics is described by the differential equation:

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)), \\ u(t) \in U,\ v(t) \in V, \end{cases}$$

where $f\colon \mathbb{R}^N \times U \times V \to \mathbb{R}^N$.

Let Ω be an open subset of the state-space $\mathbb{R}^N$: Ω is called the target of the game. We are interested by the game where the first player—Ursula, playing with u—aims the state of the system at reaching Ω in finite time. The second player—Victor, playing with v—aims the state of the system at avoiding Ω forever. This game is called the target problem.

We study this game if the framework of the *feedback strategies*. If we denote by

$$\begin{cases} \mathcal{M} = \{u(\cdot)\colon\ [0, +\infty[\ \to U,\ \text{(Lebesgue) measurable application}\}, \\ \mathcal{N} = \{v(\cdot)\colon\ [0, +\infty[\ \to V,\ \text{measurable application}\} \end{cases} \tag{20}$$

the sets of time-measurable controls, feedback strategies are defined in the following way:

Definition 2.1. The sets of maps $\mathcal{U}$ and $\mathcal{V}$ are sets of feedback strategies if:

(1) The elements of $\mathcal{U}$ and $\mathcal{V}$ are (Lebesgue–Borel) measurable maps from $\mathbb{R}^+ \times \mathbb{R}^N$ to respectively U and V.
(2) $\mathcal{M} \subset \mathcal{U}$ and $\mathcal{N} \subset \mathcal{V}$.
(3) For any $x_0 \in \mathbb{R}^N$, for any $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$ and $\tilde{v}(\cdot,\cdot) \in \mathcal{V}$, there is one and only one solution of the differential equation

$$\begin{cases} x'(t) = f(x(t), \tilde{u}(t, x(t)), \tilde{v}(t, x(t))) & \text{for almost all} \quad t \geq 0, \\ x(0) = x_0. \end{cases} \tag{21}$$

This solution is denoted by $x[x_0, \tilde{u}(\cdot,\cdot), \tilde{v}(\cdot,\cdot)]$.

Feedback strategies are also often called Isaacs–Breakwell–Bernard strategies from [16], [9], [6], and [7]. See also [15].

We are interested in characterizing the *victory domains* of each player. The victory domain of a player is the set of initial conditions from where this player may win whatever his adversary plays. More precisely:

Definition 2.2 (Victory Domains).

- Victor's victory domain is the set of initial conditions $x_0 \notin \Omega$ for which Victor can find a feedback strategy $\tilde{v}(\cdot,\cdot) \in \mathcal{V}$ such that, for any strategy $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$, the solution $x[x_0, \tilde{u}(\cdot,\cdot), \tilde{v}(\cdot,\cdot)]$ avoids Ω, i.e.,

  $$\forall t \geq 0, \qquad x[x_0, \tilde{u}(\cdot,\cdot), \tilde{v}(\cdot,\cdot)](t) \notin \Omega.$$

- Ursula's victory domain is the set of initial conditions $x_0 \notin \Omega$ for which Ursula can find a feedback strategy $\tilde{u}(\cdot,\cdot)$, positive ε, and T such that, for any strategy $\tilde{v}(\cdot,\cdot) \in \mathcal{V}$, the solution $x[x_0, \tilde{u}(\cdot,\cdot), \tilde{v}(\cdot,\cdot)]$ reaches the set $\Omega_\varepsilon := \{x \mid d_{\Omega^c}(x) \geq \varepsilon\}$ before T (and thus enters into Ω). Namely,

  $$\exists t \leq T \quad \text{such that} \quad d_{\mathbb{R}^N \backslash \Omega}(x[x_0, \tilde{u}(\cdot,\cdot), \tilde{v}(\cdot,\cdot)](t)) \geq \varepsilon.$$

Notations. Victor's victory domain is denoted by $\mathcal{W}_{\mathrm{V}}$, while Ursula's victory domain is denoted by $\mathcal{W}_{\mathrm{U}}$.

The victory domains $\mathcal{W}_{\mathrm{V}}$ and $\mathcal{W}_{\mathrm{U}}$ depend on the sets of feedback strategies $(\mathcal{U}, \mathcal{V})$. In fact, we are only interested in the case when the game *has a solution*, i.e.,

Definition 2.3. We say that the game has a solution if

$$\mathcal{W}_{\mathrm{V}} \cup \mathcal{W}_{\mathrm{V}} = \mathbb{R}^N \backslash \Omega \quad \text{and} \quad \mathcal{W}_{\mathrm{U}} \cap \mathcal{W}_{\mathrm{V}} = \emptyset.$$

Even under Isaacs' condition

$$\forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad \sup_u \inf_v \langle f(x, u, v), p \rangle = \inf_v \sup_u \langle f(x, u, v), p \rangle, \tag{22}$$

a game may have no solution. We give an example below for which there is no couple

$(\mathcal{U}, \mathcal{V})$ such that the game has a solution. No sufficient condition for a target problem to have a solution is known up to now. It seems to be an interesting (but difficult) question.

However, recall that Krasovskii and Subbotin have proved in [17] that the game always has a solution if it is played in the context of positional strategies (for which, unfortunately, (21) has no Carathéodory solution). We proved in [11] that the game also has a solution if it is played in the context of Elliot and Kalton's nonanticipative strategies.

The main result of this section is the following characterization:

Theorem 2.1. *Assume that f satisfies* (16) *and* (17), *and that Ω is an open target. Then, if Isaacs' condition* (22) *is fulfilled*:

(1) *Victor's victory domain is contained in $Disc_f(\mathbb{R}^N \backslash \Omega)$, while Ursula's victory domain is contained in the complement of $Disc_f(\mathbb{R}^N \backslash \Omega)$.*
(2) *Suppose moreover that the game has a solution. Then Victor's victory domain is equal to $Disc_f(\mathbb{R}^N \backslash \Omega)$, while Ursula's victory domain is equal the complement of $Disc_f(\mathbb{R}^N \backslash \Omega)$.*

We point out that, if the game has a solution, then the victory domains do not depend of the couple $(\mathcal{U}, \mathcal{V})$. Moreover, we have characterized the victory domains by the mean of geometrical conditions.

An example below (see Section 4.2) shows that the conclusion of the theorem is false if we do not require in the definition of Ursula's victory domain that any solution leaves $K + \varepsilon B$ before T, with ε and T which do not depend on the strategy $\tilde{v}(\cdot, \cdot)$.

The proof of Theorem 2.1 divides naturally in two lemmas:

Lemma 2.1. *Under the hypothesis of Theorem* 2.1, *the set $\mathcal{W}_V$ is contained in $Disc_f(K)$.*

The proof of Lemma 2.1 given below, while next subsection is devoted to the proof of the following:

Lemma 2.2. *Let D be a closed subset of $\mathbb{R}^N$ and let f satisfy* (16). *If D is a discriminating domain for f, then, for any $x_0 \in D$, for any $T \geq 0$, and for any $\varepsilon > 0$, for any feedback strategy $\tilde{u}(\cdot, \cdot)$ of $\mathcal{U}$, there is $v(\cdot) \in \mathcal{N}$ such that $x[x_0, \tilde{u}(\cdot, \cdot), v(\cdot)]$ remains in $D + \varepsilon B$ on $[0, T]$.*

Remarks. (1) We do not know if it is possible to set $\varepsilon = 0$ and $T = +\infty$ in the previous lemma. If $\tilde{u}(\cdot)$ is Lipschitz and if f satisfies (17), then Proposition 1.4 states that, for any x_0 of D, there is a control $v(\cdot)$ such that the solution $x[x_0, \tilde{u}(\cdot), v(\cdot)]$ remains in D.

(2) Lemma 2.2 remains true if we only assume that D is a locally compact subset of $\mathbb{R}^N$. In this case, we have to replace "for any $T \geq 0$" by "there is some positive T such that ...."

We are now ready to prove Theorem 2.1:

Proof of Theorem 2.1. Set $K := \mathbb{R}^N \backslash \Omega$. From Lemma 2.1, $\mathcal{W}_V$ is contained in $Disc_f(K)$ Lemma 2.2 states that the set $\mathcal{W}_U$ has an empty intersection with $Disc_f(K)$. So $\mathcal{W}_U \subset K \backslash Disc_f(K)$.

Victor's victory domain is contained in $Disc_f(K)$ which is contained in the complement of Ursula's victory domain. Since Victor's victory domain and Ursula's victory domain form a partition of K, the three previous inclusions are in fact equalities. □

Proof of Lemma 2.1. Let x_0 belong to $K \backslash Disc_f(K)$. We are going to prove that $x_0 \notin \mathcal{W}_V$. Fix $\tilde{v}(\cdot,\cdot) \in \mathcal{V}$. We have to define some time-measurable control $u(\cdot) \in \mathcal{M}$ such that $x[x_0, u(\cdot), \tilde{v}(\cdot,\cdot)]$ leaves K in finite time.

From Proposition 1.5, the following decreasing sequence of closed sets,

$$\begin{cases} K_0 := \mathbb{R}^N \quad \text{and} \quad K_1 := K, \\ K_{i+1} := \bigcap_{u \in U} Viab_{f(\cdot,u,V)}(K_i), \end{cases}$$

converges to $Disc_f(K)$. Thus there is some index i_0 such that $x_0 \in K_{i_0} \backslash K_{i_0+1}$. From the very definition of K_{i_0+1}, some $u_0 \in U$ can be found such that x_0 does not belong to $Viab_{f(\cdot,u_0,V)}(K_{i_0})$. Thus any solution of the differential inclusion for $f(\cdot, u_0, V)$, starting from x_0, leaves K_{i_0} in finite time. This is in particular the case for $x[x_0, u_0, \tilde{v}(\cdot,\cdot)]$, for which a time t_0 exists, with $x_1 := x[x_0, u_0, \tilde{v}(\cdot,\cdot)](t_0) \notin K_{i_0}$.

If x_1 does not belong to K, then the time-measurable control $u(\cdot) \equiv u_0$ satisfies our requirement. Otherwise, let $i_1 < i_0$ be the index such that x_1 belongs to $K_{i_1} \backslash K_{i_1+1}$. As for x_0 some $u_1 \in U$ can be found such that any solution of the differential inclusion for $f(\cdot, u_1, V)$ leaves K_1 in finite time. This is the case for $x[x_1, u_1, \tilde{v}(\cdot + t_0, \cdot)]$ for which there is some time t_1 with $x_2 := x[x_1, u_1, \tilde{v}(\cdot + t_0, \cdot)](t_1) \notin K_{i_1}$.

Recursively, it is possible to construct sequences (u_j), (t_j), (x_j), and (i_j) in the same way such that:

(1) $t_j \le t_{j+1}$ and $i_{j+1} < i_j$,
(2) $x_{j+1} := x[x_j, u_j, \tilde{v}(\cdot + t_j, \cdot)](t_{j+1} - t_j)$,
(3) $x_j \in K_j \backslash K_{j+1}$.

So there is some index j_0 such that $x_{j_0} \notin K$. Define the control $u(\cdot)$ in the following way: $u(\cdot) \equiv u_j$ on $[t_j, t_{j+1})$. Then the solution $x(\cdot) := x[x_0, u(\cdot), \tilde{v}(\cdot,\cdot)]$ satisfies obviously $x(t_j) = x_j$ for any j, so that this solution leaves K before t_{j_0}. □

2.2. *Proof of Lemma* 2.2

Let x_0 belong to a closed set D which is a discriminating domain for f. Fix $\tilde{u}(\cdot,\cdot)$ a strategy of $\mathcal{U}$, ε and T positive. We have to define a time-measurable control $v(\cdot) \in \mathcal{N}$ such that the solution $x[x_0, \tilde{u}(\cdot,\cdot), v(\cdot)]$ remains in $D + \varepsilon B$ on $[0, T]$.

Construction of $v(\cdot)$. Fix some (small) positive a that will be chosen later. With any a, we can associate a partition $([t_p, t_{p+1}))$ of $[0, T]$: Set $(t_p = ap)_{p \le T \backslash a}$. We define recursively the time-measurable control $v(\cdot)$, in such a way that $v(\cdot)$ is constant on any interval $[t_p, t_{p+1})$. Assume we have defined $v(\cdot)$ on $[0, t_p)$. For simplicity, we set $x_p := x[x_0, \tilde{u}(\cdot,\cdot), v(\cdot)](t_p)$.

- ⋆ If x_p belongs to D, we set $v(t) := v_{p+1}$ on $[t_p, t_{p+1})$, where v_{p+1} is any element of V.
- ⋆ If x_p does not belong to D, then let y_p be a projection of x_p onto D. Since D is a discriminating domain for f and $x_p - y_p$ is a proximal normal to D at y_p, there exists v_{p+1} such that $\sup_u \langle f(y_p, u, v_{p+1}), x_p - y_p\rangle \leq 0$. Then we set $v(\cdot) := v_{p+1}$ on $[t_p, t_{p+1})$.

From now on, we set $x(t) := x[x_0, \tilde{u}(\cdot,\cdot), v(\cdot)](t)$. We have to define a sufficiently small so that $d_D(x(t)) \leq \varepsilon$ for $t \in [0, T]$. For that purpose, we have to estimate (for any p) $d_D(x_{p+1})$ in the function of $d_D(x_p)$. This estimate comes from two lemmas: The first one explains that we can proceed as if the dynamics f is bounded by some M. The second provides the desired estimates.

Throughout the proof, we keep the notations of the following lemma:

Lemma 2.3. *Under assumption* (16) *on* f, *there is some radius* R *such that, for any* $\tilde{u}(\cdot,\cdot)$ *of* $\mathcal{U}$ *and any* $v(\cdot)$ *of* $\mathcal{N}$, *the solution* $x[x_0, \tilde{u}(\cdot,\cdot), v(\cdot)]$ *remains in* $x_0 + RB$ *on* $[0, 2T]$. *We denote by* M *a bound of* $\|f(\cdot,\cdot,\cdot)\|$ *on* $(x_0 + RB) \times U \times V$.

In particular, Lemma 2.3 states that we can proceed as if f is bounded by M, because we study the solutions only on $[0, T]$.

Proof of Lemma 2.3. For simplicity, we set $x(\cdot) := x[x_0, \tilde{u}(\cdot,\cdot), v(\cdot)]$ and $\lambda := \sup_u \sup_v \|f(x_0, u, v)\|$. The derivative of $\|x(t) - x_0\|$ is not larger than $\|x'(t)\|$ for almost every t. Thus, for almost every t, we have

$$\begin{aligned}\frac{d}{dt}\|x(t) - x_0\| &\leq \|x'(t)\|\\ &\leq \|f(x(t), \tilde{u}(t, x(t)), v(t))\|\\ &\leq \lambda + \ell\|x(t) - x_0\|\end{aligned}$$

because f is ℓ-Lipschitz. Gronwall's lemma yields

$$\|x(t) - x_0\| \leq \frac{\lambda}{\ell}[e^{\ell t} - 1].$$

Thus, if we set $R := \lambda/\ell[e^{\ell 2T} - 1]$, the proof of Lemma 2.3 follows. □

The estimate of $d_D(x_{p+1})$ in the function of $d_D(x_p)$ comes from the following:

Lemma 2.4. *Assume that the map* f *satisfies* (16) *and is bounded by* M. *Let* D *be a discriminating domain for* f. *Let* $\bar{x} \notin D$ *and* y *belong to the projection of* $\bar{x}$ *onto* D. *Choose* $\bar{v} \in V$ *such that*

$$\sup_u \langle f(\bar{x}, u, \bar{v}), \bar{x} - y\rangle \leq 0.$$

There are positive constants c and τ (which only depends on ℓ and M) such that, for any $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$, the following estimate holds true:

$$\forall t \leq \tau, \qquad d_D(x[\bar{x}, \tilde{u}(\cdot,\cdot), \bar{v}](t))^2 \leq ct^2 + d_D^2(\bar{x})e^{2\ell t}.$$

Proof of Lemma 2.4. For simplicity, set $x(t) := x[x, \tilde{u}(\cdot,\cdot), \bar{v}](t)$. Let $t \leq T$ for which the derivative of $x(\cdot)$ exists and is equal to $f(x(t), \tilde{u}(t, x(t)), \bar{v})$. Recall also that f is ℓ-Lipschitz and bounded by M. So we have the following estimate:

$$\begin{aligned}(\tfrac{1}{2}\|x(t) - y\|^2)' &= \langle f(x(t), \tilde{u}(t, x(t)), \bar{v}), x(t) - y\rangle \\ &\leq \langle f(y, \tilde{u}(t, x(t)), \bar{v}), x(t) - y\rangle + \ell\|x(t) - y\|^2 \\ &\leq \langle f(y, \tilde{u}(t, x(t)), \bar{v}), \bar{x} - y\rangle + M\|x(t) - \bar{x}\| + \ell\|x(t) - y\|^2\end{aligned}$$

for almost every $t \leq T$.

Recall that $\langle f(y, u, \bar{v}), \bar{x} - y\rangle \leq 0$ for any u of U. Since $\|f(\cdot,\cdot,\cdot)\|$ is bounded by M, the distance between $x(t)$ and $\bar{x}$ is not larger than Mt. Thus

$$(\tfrac{1}{2}\|x(t) - y\|^2)' \leq M^2 t + \ell\|x(t) - y\|^2.$$

This inequality is fulfilled for almost every $t \geq 0$. Gronwall's lemma yields

$$\forall t \geq 0, \qquad \|x(t) - y\|^2 \leq \|\bar{x} - y\|^2 e^{2\ell t} - \frac{M^2}{\ell}t + \frac{M^2}{2\ell^2}[e^{2\ell t} - 1]. \tag{23}$$

Note that the map φ: $t \to -(M^2/\ell)t + (M^2/2\ell^2)[e^{2\ell t} - 1]$ vanishes at $t = 0$, and $\varphi'(0) = 0$. Thus there are positive constants c and τ (which only depend on M and ℓ) such that

$$\forall t \in [0, \tau], \qquad -\frac{M^2}{\ell}t + \frac{M^2}{2\ell^2}[e^{2\ell t} - 1] \leq ct^2.$$

In particular, (23) yields

$$\forall t \in [0, \tau), \qquad d_D^2(x(t)) \leq \|x(t) - y\|^2 \leq d_D^2(\bar{x})e^{2\ell t} + ct^2.$$

So we have proved Lemma 2.4. □

From now on, we assume that a belongs to $(0, \tau)$.

Note that is the worst case (i.e., the case when $\sup_{t\in[0,T]} d_D(x(t))$ is maximum) the x_p do not belong to D as soon as $p \geq 1$.

If t belongs to $[t_p, t_{p+1})$, then $v(t) = v_{p+1}$, where v_{p+1} satisfies

$$\sup_u \langle f(y_p, u, v_{p+1}), x_p - y_p\rangle \leq 0,$$

where y_p is a projection of x_p onto D. So we can apply Lemma 2.4:

$$d_D^2(x(t)) \leq c(t-t_p)^2 + d_D^2(x_p)e^{2\ell a}$$

because $a \leq \tau$.

Applying Lemma 2.4 again yields

$$d_D^2(x_p) \leq ca^2 + d_D^2(x_{p-1})e^{2\ell a}.$$

We obtain by induction

$$d_D(x_p)^2 \leq d_D^2(x_1)e^{2p\ell a} + ca^2 \sum_{j=0}^{i-1} e^{2j\ell a}.$$

Moreover,

$$d_D^2(x_1) \leq \|x_0 - x_1\|^2 \leq M^2a^2.$$

So we have finally proved that

$$\forall t \leq T, \qquad d_D(x(t))^2 \leq M^2a^2e^{2T\ell} + ca^2\frac{e^{2T\ell}-1}{e^{2\ell a}-1}.$$

If we choose $a \in (0, \tau)$ small enough such that

$$M^2a^2e^{2T\ell} + ca^2\frac{e^{2T\ell}-1}{e^{2\ell a}-1} < \varepsilon^2,$$

then we have constructed a control $v(\cdot)$ such that

$$\forall t \in [0, T], \qquad d_D(x[x_0, \tilde{u}(\cdot,\cdot), v(\cdot)](t)) \leq \varepsilon.$$

This is the desired conclusion. □

2.3. *An Example of a Game Without Solution*

Let Ω be the open target:

$$\Omega := \{x \in \mathbb{R}^N \mid \|x\| > 1\}$$

and let $f\colon \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ be defined by

$$f(x, u, v) := (\|x\| + 1)u + v,$$

where $U = V = B$. Set $K := \mathbb{R}^N \backslash \Omega$. Isaacs' condition is obviously satisfied for that game. It is not difficult to check that $Disc_f(K) = \{0\}$. We are going to prove that 0 does not belong to Victor's victory domain:

Let $\tilde{v}(\cdot,\cdot) \in \mathcal{V}$ be a feedback strategy. We are going to construct a control $u(\cdot)$ such that the solution $x[,0,u(\cdot),\tilde{v}(\cdot,\cdot)]$ reaches Ω in finite time.

Fix $u_0 \in U\backslash\{0\}$. Let $\bar{u}(t) := \tilde{v}(t,0)$ if $\tilde{v}(t,0) \neq 0$ and $\bar{u}(t) := u_0$ if $\tilde{v}(t,0) = 0$. Then $\bar{u}(\cdot)$ is measurable. Since $f(0,\bar{u}(t),\tilde{v}(t,0))$ is never equal to 0, the solution $x[0,\bar{u}(\cdot),\tilde{v}(\cdot,\cdot)]$ leaves $\{0\}$ in finite time. Thus there is some time $\tau > 0$ such that $x_0 := x[0,\bar{u}(\cdot),\tilde{v}(\cdot,\cdot)](\tau)$ is not equal to 0.

We now define the control $u(\cdot)$: $u(t) := \bar{u}(t)$ on $[0,\tau]$, and $u(t) := x_0/\|x_0\|$ on $(\tau,+\infty)$. It is not difficult to check that the solution $x[0,u(\cdot),\tilde{v}(\cdot,\cdot)]$ leaves K in finite time.

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Appendix

In this section we give an outline of the proof of Proposition 1.1.

Since $d_K(x+\nu) \le \|x+\nu-x\| = \|\nu\|$, property (1) follows from the very definition of the proximal normals. Property (2) is also obvious because if x belongs to the projection of y onto K, and if $\nu := y-x$, then the distance from $x+\nu = y$ to K is equal to $\|y-x\| = \|\nu\|$. Thus $d_K(x+\nu) = \|\nu\|$ and ν is a proximal normal to K at x.

(3) Let $\nu \in NP_K(x)$, $\lambda \in (0,1)$, and x' be a projection of $x+\lambda_\nu$ onto K. We have to prove that $x = x'$.

Note first that

$$\|x+\lambda\nu-x'\| \le \|x+\lambda\nu-x\| = \lambda\|\nu\| \tag{24}$$

because x belongs to K and x' is a projection of $x+\lambda\nu$ onto K. Writing $\|x+\nu-x'\| = \|(x+\lambda\nu-x')+(1-\lambda)\nu\|$ and combining with (24) gives

$$\|x+\nu-x'\|^2 \le [\lambda^2+(1-\lambda)^2]\|\nu\|^2 + 2(1-\lambda\langle\nu, x+\lambda\nu-x'\rangle. \tag{25}$$

Since $\nu \in NP_K(x)$, $\|x+\nu-x'\| \ge \|\nu\|$ because $\|\nu\| = d_K(x+\nu)$. Thus, if we write $\|\nu\|^2 = \|\lambda\nu+(1-\lambda)\nu\|^2$, (25) implies

$$\lambda(1-\lambda)\|\nu\|^2 \le (1-\lambda)\langle\nu, x+\lambda\nu-x'\rangle. \tag{26}$$

Since $\lambda \neq 1$, we have

$$\begin{aligned}\lambda\|\nu\|^2 &\le \langle\nu, x+\lambda\nu-x'\rangle\\ &\le \|\nu\|\,\|x+\lambda\nu-x'\|\\ &\le \lambda\|\nu\|^2\end{aligned}$$

from (25).

Thus $\|\nu\|\,\|x+\lambda\nu-x'\| = \langle\nu, x+\lambda\nu-x'\rangle$, and there is some nonnegative t such that $t\nu = x+\lambda\nu-x'$. From (26), t equals λ. Hence, $x = x'$ and property (3) is proved.

(4) Let ν belong to $NP_{K_2}(x)$. Then $\|\nu\| = d_{K_2}(x+\nu)$ from the very definition of the proximal normals. Since $K_1 \subset K_2$, $d_{K_2}(x+\nu) \le d_{K_1}(x+\nu)$. The point x belongs to K_1, thus $d_{K_1}(x+\nu) \le \|\nu\|$. So we have proved finally that $d_{K_1}(x+\nu) = \|\nu\|$, i.e., ν belongs to $NP_{K_1}(x)$.

(5) Assume that a sequence (x_p, ν_p) of elements of $Graph(NP_{K_p})$ converge to some (x, ν). We have to prove that (x, ν) belongs to the graph of $NP_K(\cdot)$. Note first that, since K equals the lower-limit of the K_p, x belongs to K. Thus $d_K(x+\nu) \le \|\nu\|$. Assume for a while that ν does not belong to $NP_K(x)$, i.e., there is some $y \in K$ such that $\|x+\nu-y\| < \|\nu\|$. A sequence y_p of K_p converging to y can be found, because K equals the lower-limit of the K_p. Since $\|\nu_p\|$ and $\|x_p+\nu_p-y_p\|$ converge respectively to $\|\nu\|$ and $\|x+\nu-y\|$, there is some p for which $\|x_p+\nu_p-y_p\|$ which is impossible because ν_p is supposed to be a proximal normal to K_p at x_p. Thus ν belongs to $NP_K(x)$ and property (5) holds.

(6) Assume now that (x, ν) belongs to $Graph(NP_K)$. We have to find a sequence (x_p, ν_p) of $Graph(NP_{K_p})$ which converges, up to a subsequence, to (x, ν). From property (3), for any $k \in \mathbb{N}^*$, x is the unique projection of $x + (1-1/k)\nu$ onto K. For any k, we denote by x_p^k a projection of $x+(1-1/k)\nu$ onto K_p. The following useful lemma implies that, up to a subsequence, the sequence (x_p^k) converges to x if $p \to +\infty$:

Lemma A.1. *Suppose that a closed set K contains the upper-limit of a sequence of closed sets K_p. Assume that x belongs to the upper-limit of the K_p. Let ν be a proximal normal to K at x such that x is the unique projection of $x+\nu$ onto K. If x_p belongs to the projection of $x+\nu$ onto K_p, then a subsequence of the sequence (x_p) converge to x.*

Lemma A.1 is proved below.

From Lemma A.1, x_p^k converges, up to a subsequence, to x when $p \to +\infty$ and the proximal normals $x+(1-1/k)\nu - x_p^k$ to K_p at x_p^k converge, up to a subsequence, to $(1-1/k)\nu$ when $p \to +\infty$.

The sequence $((1-1/k)\nu)$ converges to ν when $k \to +\infty$. Thus it is possible to construct[8] a sequence $x_{p_k}^k$ which converge to x.

In particular, the proximal normals $(x+(1-1/k)\nu - x_{p_k}^k)$ to K_{p_k} at $x_{p_k}^k$, converge to ν while the sequence $(x_{p_k}^k)$ converges to x. Thus property (5) holds.

Property (7) is a consequence of properties (5) and (6): Take $K_p = K$, then $Graph(K)$ is equal to its closure.

(8) We have already proved that the graph of NP_K is closed, thus NP_K has closed values. It is obvious that 0 belongs to $NP_K(x)$ for any x because the projection of $x+0$ onto K equals x. To prove property (8), it remains to prove that $NP_K(x)$ is convex for any x of K.

Assume now that ν_1 and ν_2 belong to $NP_K(x)$, for some x. We have to show that, for any $\lambda \in [0,1]$, $\lambda\nu_1 + (1-\lambda)\nu_2$ belongs to $NP_K(x)$. Note that there is an equivalence between ν_i $(i=1,2)$ belongs to $NP_K(x)$, and

$$\forall y \in K, \qquad \|y-x\|^2 - 2\langle y-x, \nu_i\rangle \ge 0. \tag{27}$$

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[8] The construction of the subsequence can be done in the following way: Since x_p^k converge, up to a subsequence, to x when $p \to +\infty$, there is, for any k, some p_k sufficiently large, such that $\|x_{p_k}^k - x\| \le 1/k$. Then $x_{p_k}^k \to x$.

Indeed, equation (2) yields that, for any y of K, $\|y-(x+\nu_i)\|$ is not smaller than $\|\nu_i\|$, which give the previous inequality.

If we add (27) for ν_1 multiplied by λ and that for ν_2 multiplied by $(1-\lambda)$, we get

$$\forall y \in K, \qquad \|y-x\|^2 - 2\langle y-x, \lambda\nu_1 + (1-\lambda)\nu_2\rangle \geq 0.$$

So $\lambda\nu_1 + (1-\lambda)\nu_2$ is a proximal normal and property (8) holds. □

Proof of Lemma A.1. We have to prove that a subsequence of the sequence x_p converges to x. For any p, $d_{K_p}(x+\nu)$ is equal to $\|x+\nu-x_p\|$, from the very definition of x_p. Moreover, since x belongs to the upper-limit of the K_p, there is $y_p \in K_p$ which converges, up to a subsequence (again denoted by (y_p)), to x, so that

$$\|x+\nu-x_p\| \leq \|x+\nu-y_p\| \leq \|x-y_p\| + \|\nu\|. \tag{28}$$

In particular this inequality yields that

$$\|x_p\| \leq \|x+\nu\| + \|x-y_p\| + \|\nu\|,$$

so that the sequence (x_p) is bounded. Thus there is a subsequence of the x_p which converge to some z. Since the upper-limit of the K_p is contained in K, z belongs to K. From (28), $\|x+\nu-z\| \leq \|\nu\|$. So the distance from z to $(x+\nu)$ is equal to $d_K(x+\nu) = \|\nu\|$. In particular, z belongs to the projection of $x+\nu$ onto K, which is equal to x. So we have proved that there is a subsequence of the sequence (x_p) which converge to x. □

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