ASSIGNMENT NO.-1

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Course- B. Tech. (III Sem)
Section- CS BDA
Sub- Graph Theory
Roll No - 51

1) i) Informally graph is a pictorial representation of points called vertices (vertex) and lines called edges. Each edge joins two vertices exactly.

OR

A graph $G(V, E)$ consists of

a) A set of object $V = \{V_1, V_2, V_3 \dots V_n\}$ whose elements are the vertices, nodes or the points in graph G.

b) A set of edges $E = \{E_1, E_2 \dots E_n\}$ is the unordered pair of elements (distinct vertices) which form lines, limbs or edges of the graph G such that E_r is identified with an order pair.

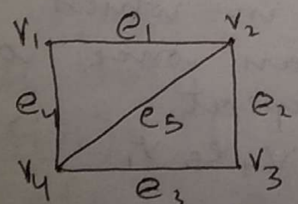
$$E_r = (v_i, v_j) \text{ of vertices.}$$

Subgraph-

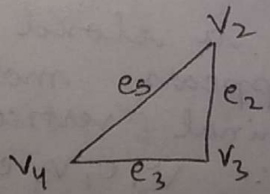
A graph $G_1(V_1, E_1)$ is called to be subgraph of $G(V, E)$ if

$$E_1(G_1) \subseteq E(G) \text{ and}$$
$$V_1(G_1) \subseteq V(G)$$

such that end vertices of edges in G_1 are same as that in G.

ii 

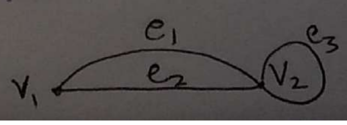
G (Graph)



G_1 (Sub-graph)

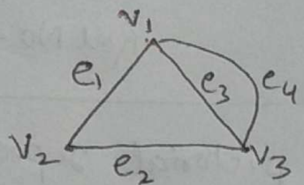
iii) Self-Loop →

An edge associated with pair of vertices (v_i, v_j) such that $i = j$.



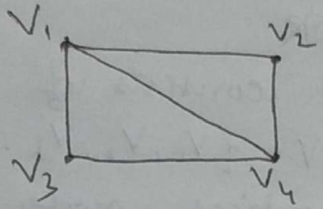
Multi edge -

In a graph G, multiple edges connecting same pairs of vertices are called parallel edges.



Adjacent vertex -

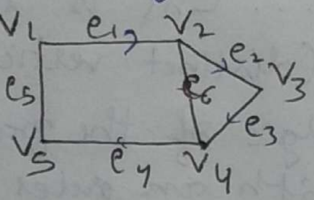
Two vertices joint with an edge are called adjacent vertex.



$deg(V_1) = 3$

V_2, V_4, V_3 are adjacent to V_1.

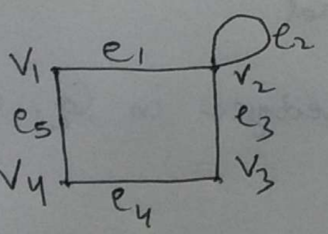
-> Walk in a graph G is an alternative finite sequence of vertices and edges.



$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5$

Trail is a walk in which no edge reappears more than once and vertices may be repeated.

Path is an open walk in which no vertex appears more than once. Total no. of edges appearing in the path are called length of a path.



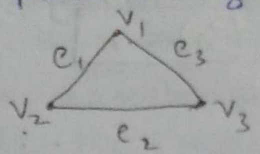
$V_1 e_1 V_2 e_3 V_3 e_4 V_4$

Circuit is a closed walk in which may the vertex reappears more than once and ~~only~~ terminal vertices repeat.

Eg: $V_1 e_1 V_2 e_3 V_3 e_4 V_4 e_5 V_1$

Cycle is a closed walk in which no vertices reappears more than once and only terminal vertices repeat.

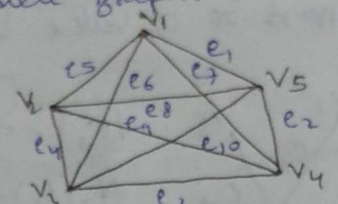
Simple graph - A graph which has neither self-loop nor parallel-edges is called simple graph.



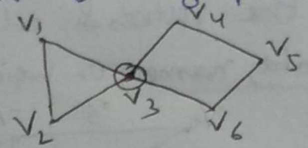
Finite, Simple, graph

Finite path graph - A graph containing finite numbers of vertices and edges is called finite graph.

Complete graph - A Simple graph with n-vertices consisting of each vertex having (n-1) degree is called complete graph.

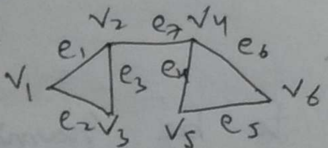


v) Cut vertex is a vertex in graph G which when removed makes the graph disconnected.



$V_3 \rightarrow$ Cut Vertex

Bridge is an edge in graph G in which when the edge is removed, the graph becomes disconnected.



$e_7 \rightarrow$ bridge

vi) Isomorphic graph - A graph G is said to be isomorphic with $G'(V', E')$ then $\exists$ a function $f: V(G) \rightarrow V'(G')$ and $g: E(G) \rightarrow E'(G')$

a) such that,

$$f(v_i) = f(v_j) \Rightarrow v_i = v_j \ \forall v_i, v_j \in G$$

$$\& \ g(e_i) = g(e_j) \Rightarrow e_i = e_j \ \forall e_i, e_j \in G$$

b) For each vertex, $u \in V(G') \ \exists$ a vertex $v \in V(G)$

c) $(v_i, v_j) \in E(G) \Leftrightarrow (v_i', v_j') \in E'(G')$

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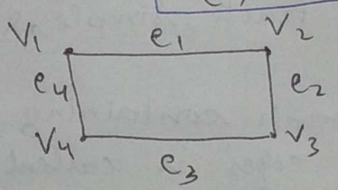
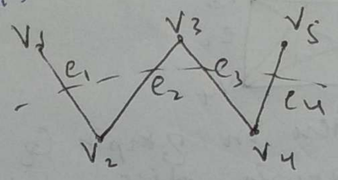
vii) Regular graph – A simple graph is to be regular if each of the vertices have ~~degree of~~ equal degree

$$\boxed{\Delta(G) = \delta(G) = r}$$

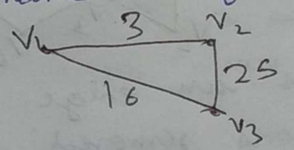


viii) Bipartite graph – A graph G is said to be a bipartite graph if vertex set of G ; V (G) can be partitioned into two sets $V_1(G)$ and $V_2(G)$ such that each edge ~~has be~~ a vertex in $V_1(G)$ and another in $V_2(G)$



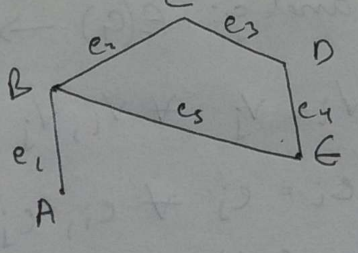
$V_1(G) = \{V_1, V_3, V_5\}$
$V_2(G) = (V_2, V_4\}$

ix) Weighted graph – If the vertices & edges of graph G are labelled with data, name or weight then it is called weighted graph



x) Hamiltonian path –

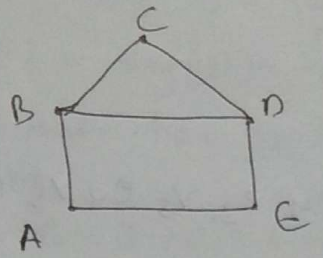
A path is said to be hamiltonian if the path passes through all the vertex of graph and only once.



$A e_1 B e_2 C e_3 D e_4 E$

Hamiltonian Graph -

A graph is said to be Hamiltonian if it contains a hamiltonian ~~graphe~~ i.e. it passes through all the vertices of graph G exactly once except terminal vertices.

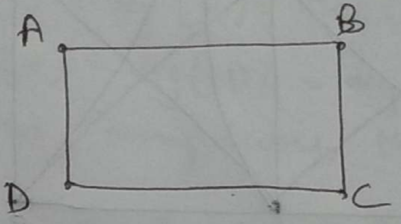


=) A B C D E A

xi) Arbitrary Traceable Graph -

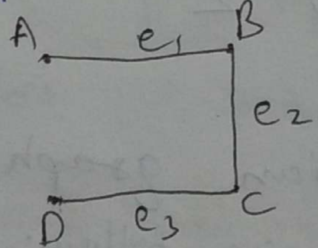
It is a graph in which each ~~vertex~~ edge can be ~~stra~~traced arbitrary starting from any vertex.

Eg:



xii) Eulerian Trail -

A trail is said to be Eulerian if the trail passes through all the edges of the graph and only once.

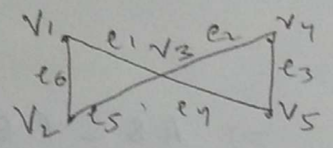


Eg: $A e_1 B e_2 C e_3 D$

Eulerian graph -

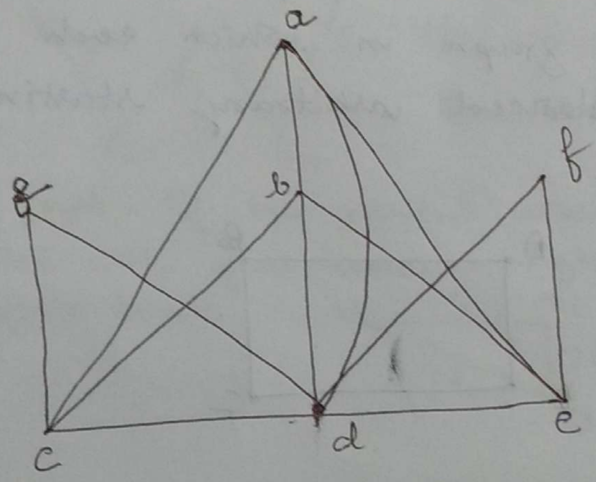
A circuit which passes through every edge of the graph G exactly once, though vertices may repeat ~~and~~ is called an Eulerian graph.

Eg:



$\Rightarrow$ $v_1\ e_1\ v_3\ e_2\ v_4\ e_3\ v_5\ e_4\ v_3\ e_5\ v_2\ e_6\ v_1$

Eg:



$\therefore$ All vertices are of even degree.

$\deg(c) = \deg(e) = \deg(a)$
$= \deg(b) = 4$

$\deg(g) = \deg(f) = 2$

$\deg(d) = 6$

$\therefore$ It is an eulerian graph.

and the circuit is as follows:

$c \to b \to e \to f \to d \to g \to c \to d \to e$
$\to a \to b \to d \to a \to c$

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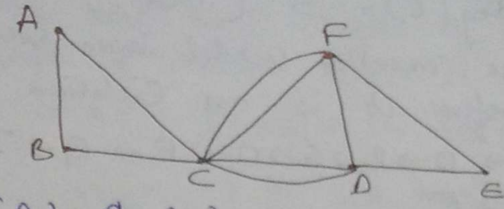
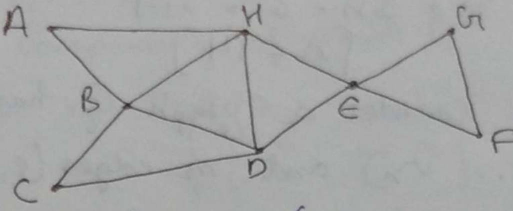
2)



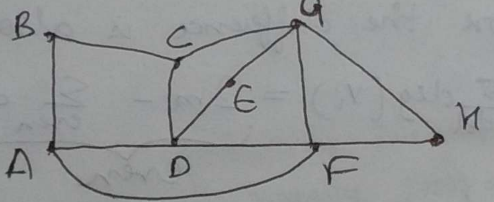
deg (A) = deg (B) = deg (E) = 2
deg (F) = deg (D) = 4 ; deg (C) = 6
∴ Each vertex has even degree, this is Eulerian graph
Eul. ckt →
B→C→D→E→F→C→D→F→C→A→B

3)



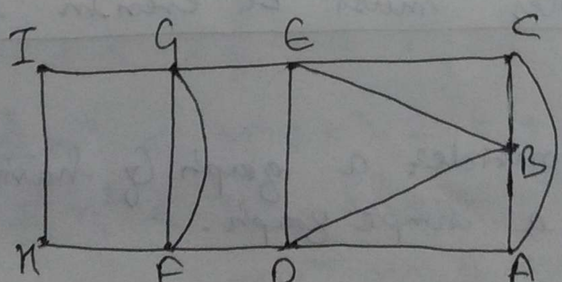
deg (A) = deg (C) = deg (G) = deg (F) = 2
deg (B) = deg (H) = deg (D) = deg (E) = 4
∴ Each vertex has even degree, this is an Eulerian graph.
Elu. ckt. →
A→B→D→E→G→F→E→H→B→C→D→H→A
It also contains Hamiltonian path,
D→C→B→A→H→E→G→F

4)



No, it does not contains Hamiltonian cycle

5)



deg (I) = deg (H) = 2
deg (G) = deg (F) = deg (E) = deg (D) = deg (B) = 4

$deg(A) = deg(C) = 3$ (odd

and since exactly 2 odd degree vertecies are present. Therefore it is an Eulerian Trail

$A \to B \to C \to A \to D \to B \to E \to D \to F \to G \to I \to H \to F$
$\to G \to E \to C.$

6) No. of edges = 22

Let vertices be 'n' in number

& w.k.t $\left[\sum_{i=1}^{n} deg(v_i) = 2e\right]$

$\Rightarrow 2 \times 4 + 3(n-2) = 2 \times 22$

$8 + 3n - 6 = 44$

$[n = 14]$

7) Proof: Let us consider a graph G having n-vertices $(v_1, v_2, v_3 \ldots v_n)$ and m-edges $(e_1, e_2 \ldots e_m)$

and w.k.t $\sum_{i=1}^{n} deg(v_i) = 2 \times m$ (no. of edges)

Let us separate the vertices of odd degree and even degrees.

$$\sum_{odd} deg(v_i) + \sum_{even} deg(v_j) = 2 \times m$$

$\because \sum_{even} deg(v_j)$ and $2 \times m$ both are even numbers therefore the difference is also even

$$\sum_{odd} deg(v_i) = \underbrace{2m - \sum_{even} deg(v_j)}_{even}$$

$\therefore$ For $\sum_{odd} deg(v_i)$ to be even, the no. of odd-degree vertices must be even in the gr

8) (10) Proof: Let us consider a graph G having n-vertices and is a simple graph.

Therefore, the max^m degree of each vertex ca be $(n-1)$

Now, the total max^m degree of the n- vertices in graph G can be $n(n-1)$.

and w.k.t $\left[\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{edges} \right]$

$\therefore \quad n(n-1) = 2 \times$ max^m no. of edges

$\left\{ \therefore \text{ max}^m \text{ edges } = \frac{n(n-1)}{2} \right\}$

9) (ii) Let G be a disconnected graph without self loops and parallel edges and n- vertices.

Let $n_1, n_2, n_3, n_4 \ldots n_k$ be the no. of vertices of k- components of graph G.

and $n_1 + n_2 + n_3 \ldots + n_k = n$

$$\Rightarrow \left[\sum_{i=1}^{k} n_i = n \right]$$

also, $(n_1 - 1) + (n_2 - 1) + \ldots (n_k - 1) = n - k$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = n - k$$

Squaring both sides, we get

$$\left[\sum_{i=1}^{k} (n_i - 1) \right]^2 = (n-k)^2$$

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \ldots (n_k - 1)^2 + \text{non-negative cross-terms} = (n-k)^2$$

$$\Rightarrow (n_1 - 1)^2 + (n_2 - 1)^2 + \ldots (n_k - 1)^2 \leq (n-k)^2$$

[Taking inequality by removing non-negative cross-terms)

$$\Rightarrow n_1^2 + n_2^2 + n_3^2 \ldots n_k^2 + (1 + 1 + \ldots k) - 2(n_1 + n_2 + \ldots n_k) \leq (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2 \sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 + 2n - 2nk - k \quad \text{—— (1)}$$

and w.k.t max^m no. of edges in a simple graph G is given by $\frac{n(n-1)}{2}$

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Therefore for an 'i' vertex max^m edges can be $\frac{n_i(n_i-1)}{2}$.

$\therefore$ Max^m edges in graph $G = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$= \sum_{i=1}^{k} \frac{n_i^2}{2} - \sum_{i=1}^{k} \frac{n_i}{2}$

(1) $\Rightarrow$

$\leq \frac{1}{2}[n^2 + k^2 + 2n - 2nk - k - n]$

$\leq \frac{1}{2}[n^2 + k^2 + n - 2nk - k]$

$\leq \frac{1}{2}[n^2 + k^2 - 2nk + (n-k)]$

$\leq \frac{1}{2}[(n-k)^2 + (n-k)]$

$\left\{ \leq \frac{(n-k)(n-k+1)}{2} \right\}$

10) (9) Proof:

Suppose G can be decomposed into circuits $C_1, C_2, C_3 \ldots C_n$ such that

$G = C_1 \cup C_2 \cup C_3 \ldots \cup C_n$

Since the degree of every vertex in a circuit is two, therefore the degree of every vertex in the graph G will be even. Hence degree of every vertex will be even. Hence graph is even.

Conversely:

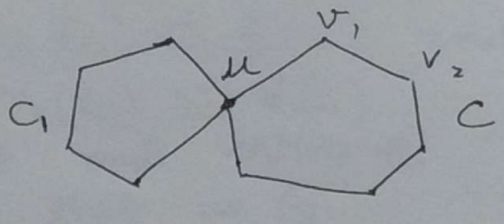
Let G be an Eulerian graph. Consider a vertex 'v_1' having atleast two incident edges on it, so it is even. Let one of the edges be b/w the vertex v_1 and its adjacent vertex v_2, But since v_2 is also of even degree therefore it must have other edge with vertex v_3. In this manner we move on to get a circuit until we reach the vertex v_1, where we started forming circuit C_1.

Let us remove ckt C_1 from the graph G and now all vertices remaining must also be of even degree, in same way we get another circuit C_2 and so on and we keep removing them until we get no edges left

Hence we can decompose Eulerian graph into circuits and A connected graph G is an Eulerian graph if & only if it can be decomposed into circuits.

11) (a) Necessary Condition :

Let $G(V, E)$ be an Euler graph. Thus G contains an Euler ckt (say c) which is a closed walk.
Let this walk start and end at vertex $u \in V$
Since each visit to intermediate vertex 'v' contributes two to the degree of 'v' and since ckt 'c' traverses each edge once, deg(V) is even for every vertex. Each intermediate visit to vertex u contributes two to the degree of u and also the initial and end edges contribute one each to the deg (u). Therefore degree, {deg(u)} is also even.



Sufficient Condition :

Let G be a connected graph and let degree of each vertex of G be an even number.

Suppose G is not Eulerian graph and let G contains least number of edges. Since (min^m degree) $\delta \geqslant 2$, G has a cycle.

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Let Z be a closed walk in G of max^m length ∴
Clearly, $G - E(Z)$ is an even degree graph. Let
G_1 be one of the component of $G - E(Z)$. As
G_1 has less number of edges than G, it is
Eulerian and has a vertex v in common with
Z. Let Z' be an Euler circuit in G_1.
Then $Z' \cup Z$ is closed in G starting & ending
at v, since it is longer than Z, the choice of
Z is contradicted.
Hence graph G is an Eulerian graph
Hence Proved

Hence proved that a connected graph G is an
Euler graph if and only if all the vertices of
graph G are of even degree.

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