Continuous Probability distribution

If X is a continuous variable and $p(x)$ or $f(x)$ is a function in that variable satisfying the conditions (i) $p(x)$ or $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} p(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$, then $p(x)$ or $f(x)$ is called as continuous probability function or P.D.F. and corresponding probability distribution is called as continuous probability distribution.

Probability over an interval.

If X is a continuous variable, $P(x)$ or $f(x)$ is its P.D.F then Probability over the interval is $P(a \leq x \leq b) = P(a < x < b)$

$$= P(a \leq x < b) = P(a < x \leq b) = \int_a^b f(x)\,dx$$

Note: (i) From the definition it is clear that Probability over any kind of intervals between a and b i.e. closed interval or open interval or Half one are same.

(ii) The P.D.F of continuous distribution is denoted either as $p(x)$ or $f(x)$.

Cumulative distribution function (C.D.F.) ②

It is defined as $F(t) = P(x \leq t)$

$$= P(-\infty < x \leq t) = \int_{-\infty}^{t} f(x)\, dx.$$

We can show that

$$P(a \leq x \leq b) = P(a < x < b) = F(b) - F(a).$$

Expectation of a function

If $\phi(X)$ is a function in the continuous variable X, then its expectation is defined as $E[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx.$

Mean:

It is defined as $\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$

Standard deviation

Variance is defined as $\sigma^2 = E[(X-\mu)^2]$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx.$$

Standard deviation is the positive square root of variance i.e. 'σ'

It can be S.T. $\boxed{\sigma^2 = E[X^2] - \mu^2}$

Note:- Problems on continuous probability distribution are problems on Evaluation of definite integrals.

Problems:-

1) Find K such that $p(x) = Ke^{-|x|}$, $-\infty < x < \infty$ is a P.D.F of a continuous probability distribution. Also find its mean and S.D.

Solution:- If $p(x)$ is P.D.F. then

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

ie $$\int_{-\infty}^{\infty} K e^{-|x|}\,dx = 1$$

But $e^{-|x|}$ is an even function and therefore we get $2K\int_{0}^{\infty} e^{-x}\,dx = 1$ $\left[\because e^{-|x|} = e^{-x} \text{ in } 0 < x < \infty\right]$

ie. $$2K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

ie. $-2K\left[e^{-\infty} - e^{0}\right] = 1$ But $e^{-\infty} = 0$.

$$-2K[0-1] = 1$$

$$-2K(-1) = 1$$

$$2K = 1$$

$$\boxed{K = \frac{1}{2}.}$$

mean $= \mu = E[X] = \int_{-\infty}^{\infty} x\, p(x)\, dx$

$= \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|} dx$ But the function $x e^{-|x|}$ is an odd function

$= \underline{0}$.

$E[X^2] = \int_{-\infty}^{\infty} x^2 p(x) dx = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|} dx$

$x^2 e^{-|x|}$ is an even function.

$\therefore E[X^2] = 2 \cdot \frac{1}{2} \int_0^{\infty} x^2 e^{-x} dx$.

using Bernoulli's rule.

$= \cancel{x^2 \left(\frac{e^{-x}}{-1}\right)}\Big|_0^{\infty} - \cancel{(2x)\left(\frac{e^{-x}}{(-1)^2}\right)}\Big|_0^{\infty} + 2\left(\frac{e^{-x}}{(-1)^3}\right)\Big|_0^{\infty}$

$= -2 \left(e^{-x}\right)_0^{\infty} = -2\left(e^{-\infty} - e^0\right)$

$= -2(0-1)$

$= \underline{2}$.

Variance $= \sigma^2 = E[X^2] - \mu^2 = 2 - 0^2$

$\therefore \sigma^2 = 2$

Standard deviation $= \sigma = \underline{\sqrt{2}}$

(2) If P.D.F $f(x) = K(x+3)$ in $(2,8)$, determine.
(i) K (ii) $P(3<x<5)$ (iii) $P(x \geq 4)$ (iV) $P(|x-5|<0.5)$

Solution :- The given P.D.F of Continuous probability distribution is

$$f(x) = \begin{cases} 0 & , -\infty < x \leq 2 \\ K(x+3) & , 2 < x < 8 \\ 0 & , 8 < x \not< \infty \end{cases}$$

For $f(x)$ to be P.D.F it has to satisfy the conditions $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $f(x) \geq 0$

i.e. $\int_{-\infty}^{2} f(x)\,dx + \int_{2}^{8} f(x)\,dx + \int_{8}^{\infty} f(x)\,dx = 1$

$\cancel{\int_{-\infty}^{2} 0\,dx} \to 0 + \int_{2}^{8} K(x+3)\,dx + \cancel{\int_{8}^{\infty} 0\,dx} \to 0 = 1$

i.e. $K \left.\frac{(x+3)^2}{2}\right|_2^8 = 1$

$\frac{K}{2}\left[(8+3)^2 - (2+3)^2\right] = 1$

$\frac{K}{2}\left[11^2 - 5^2\right] = 1$

$\frac{K}{2}\left[121 - 25\right] = 1$

48

$\frac{K}{\cancel{2}}(\cancel{96}) = 1$

$\therefore \boxed{K = \frac{1}{48}}$

$$P(3<x<5) = \int_3^5 f(x)\,dx = \int_3^5 \frac{1}{48}(x+3)\,dx$$ ⑥

$$= \frac{1}{48}\left[\frac{(x+3)^2}{2}\right]_3^5$$

$$= \frac{1}{48}\left[\frac{8^2-6^2}{2}\right] = \frac{1}{48}\left[\frac{64-36}{2}\right] = \frac{\cancel{28}^{14}}{48\times\cancel{2}}$$

$$= \frac{7}{24}$$

$$P(x\geq 4) = P(4\leq x<\infty) \quad \left[x\geq K \text{ means } K\leq x<\infty\right]$$

$$= \int_4^\infty f(x)\,dx = \int_4^8 f(x)\,dx + \int_8^\infty f(x)\,dx$$

$$= \int_4^8 \frac{1}{48}(x+3)\,dx + \cancel{\int_8^\infty 0\,dx}^{\,0} = \frac{1}{48}\frac{(x+3)^2}{2}\Big|_4^8$$

$$= \frac{1}{48\times 2}\left[11^2-7^2\right] = \frac{1}{48\times 2}\left[121-49\right] = \frac{\cancel{72}^{36}}{48\times\cancel{2}} = \frac{3}{4}$$

$$P(|x-5|<0.5) = P(-0.5<x-5<0.5)$$

$$= P(5-0.5<5+x-5<5+0.5)$$

$$= P(4.5<x<5.5) = \int_{4.5}^{5.5} f(x)\,dx.$$

$$= \int_{4.5}^{5.5} \frac{1}{48}(x+3)\,dx = \frac{1}{48}\frac{(x+3)^2}{2}\Big|_{4.5}^{5.5}$$

$$= \frac{1}{96}(8.5^2-4.5^2) = \frac{1}{6}.$$

③ For the continuous probability distribution ⑦

having $p(x) = \begin{cases} x & , 0 < x < 1 \\ 2-x & , 1 \leq x < 2 \\ 0 & , \text{otherwise} \end{cases}$ find C.D.F $F(t)$

and hence evaluate (i) $P(x < 0.5)$ (ii) $P(0.75 < x < 1)$

(iii) $P(x > 1.5)$

Solution:- P.D.F is $p(x) = \begin{cases} 0 & , -\infty < x < 0 \\ x & , 0 < x < 1 \\ 2-x & , 1 \leq x < 2 \\ 0 & , 2 < x < \infty \end{cases}$

C.D.F is $F(t) = \int_{-\infty}^{t} p(x)\,dx$.

Since $p(x)$ is a splitting function, $F(t)$ is also a splitting function. $\therefore$ we have to find $F(t)$, for t belonging to 4 diff intervals here.

For $-\infty < t < 0$, $F(t) = \int_{-\infty}^{t} p(x)\,dx = \int_{-\infty}^{t} 0\,dx = 0$.

For $0 < t < 1$, $F(t) = \int_{-\infty}^{t} p(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{t} x\,dx$

$= 0 + \frac{x^2}{2}\Big|_0^t = \frac{t^2}{2} - 0 = \frac{t^2}{2}$.

For $1 < t < 2$, $F(t) = \int_{-\infty}^{t} p(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx$

$+ \int_{1}^{t} (2-x)\,dx = 0 + \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)_1^t = \frac{1}{2} - 0 + 2t - \frac{t^2}{2}$

$-\left(2 - \frac{1^2}{2}\right) = \frac{1}{2} + 2t - \frac{t^2}{2} - 2 + \frac{1}{2} = -\frac{t^2}{2} + 2t - 1$

For $2<t<\infty$, $F(t)=\int_{-\infty}^{t} P(x)\,dx$ (8)

$$F(t)=\int_{-\infty}^{0} 0\,dx+\int_{0}^{1} x\,dx+\int_{1}^{2}(2-x)\,dx+\int_{2}^{t} 0\cdot dx$$

$$=0+\frac{x^2}{2}\Big|_0^1+\left(2x-\frac{x^2}{2}\right)\Big|_1^2+0=\frac{1}{2}+\left(2\cdot 2-\frac{2^2}{2}\right)-\left(2-\frac{1^2}{2}\right)$$

$$=\frac{1}{2}+4-2-2+\frac{1}{2}=\underline{1}.$$

C.D.F is $F(t)=\begin{cases} 0, & -\infty<t<0 \\ \frac{t^2}{2} & 0<t<1 \\ -\frac{t^2}{2}+2t\cancel{+}-1, & 1\le t<2 \\ 1, & 2<t<\infty \end{cases}$

(i) $P(x<0.5)=F(0.5)=\left(\frac{t^2}{2}\right)_{t=0.5}=\frac{(0.5)^2}{2}=$

(iii) $P(0.75<x<1)=F(1)-F(0.5)$

$$=-\frac{1^2}{2}+2(1)-1-(0.75)^2=$$

(iv) $P(x>1.5)=1-P(x\le 1.5)=1-F(1.5)$

$$=1-1=\underline{0}.$$

(4) Find K such that $f(x)=\begin{cases} Ke^{-\frac{x}{4}} & \text{for } x>0 \\ 0 & \text{elsewere} \end{cases}$

is probability function of continuous probability distribution. Determine (i) K (ii) mean (iii) S.D.

Ans:- $K=\frac{1}{4}$, $\mu=4$, $\sigma=\sqrt{80}$