Discrete Joint probability distribution.

let S be a Sample Space and X and Y be two discrete random variables.

let $X = f(s)$, $Y = g(s)$, $s \in S$; where f and g are real valued functions defined on S, and $f(S) = \{x_1, x_2, x_3 \dots\}$ and $g(S) = \{y_1, y_2, y_3, \dots\}$. be the image sets of S under the rules f and g.

let $f(x_i, y_j)$ $p_{ij} = P(x_i, y_j) = P\{s \in S \mid f(s) = x_i \text{ and } g(s) = y_j\}$

then p_{ij} satisfies the conditions

(i) $p_{ij} \geq 0$ (ii) $\sum_i \sum_j p_{ij} = 1$

p_{ij} is called Discrete Joint probability density function. The corresponding distribution is called as Discrete Joint probability density distribution.

We can express it in the form of probability contingency table.

$X \backslash Y$	y_1	y_2	y_3 ...
x_1	p_{11}	p_{12}	p_{13} ...
x_2	p_{21}	p_{22}	p_{23} .
x_3	p_{31}	p_{32}	p_{33} ...

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Marginal probability distributions :-

let $P_i = p_{i1} + p_{i2} + p_{i3} + \dots = \sum_j p_{ij}$ (for fixed i)

the sum of the entries in the i^{th} row.

$\bar{P}_j = p_{1j} + p_{2j} + p_{3j} + \dots = \sum_i p_{ij}$ (for fixed j)

the sum of the entries in the j^{th} column.

Then the sets $\{P_i\}_{i=1,2,3\dots}$ and

$\{\bar{P}_j\}_{j=1,2,3,\dots}$ are called the marginal probability distributions of X and Y respectively.

They are identical with the individual probability distributions of X and Y respectively.

Stochastic independence :-

The two variables X and Y are said to be independent or statistically independent or stochastically independent if $p_{ij} = P_i \bar{P}_j$

Covariance and correlation :-

Expectation of $\phi(x, y) = \sum_i \sum_j \phi(x_i, y_j) p_{ij} = E[\phi(x, y)]$

$(r, s)^{th}$ moment

$\mu_{rs} = E[(X-\mu_x)^r (Y-\mu_y)^s] = \sum_i \sum_j (x_i - \mu_x)^r (y_j - \mu_y)^s p_{ij}$

Where μ_x and μ_y are mean values of the X and Y distributions respectively.

μ_{11} is called as Covariance of X and Y and it is denoted by $cov(X,Y) = \mu_{11} = \sum_i \sum_j p_{ij}(x_i - \mu_x)(y_j - \mu_y)$

$$cov(X,X) = E[(X-\mu_x)(X-\mu_x)] = E[(X-\mu_x)^2]$$

$$= Var(X)$$

We can S.T. $\underline{cov(X,Y)} = E[XY] - \mu_x \mu_y$.

The correlation coefficient of the joint distribution of X and Y is $\rho(X,Y) = \dfrac{cov(X,Y)}{\sigma_x \sigma_y}$, where σ_x and σ_y are the standard deviations of the distributions X and Y respectively.

① The joint distribution of two random variables X and Y is given by the following table.

X \ Y	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Determine the individual distributions of X and Y. Also verify that X and Y are stochastically independent

Sol:- We are given $p_{11} = 0.06$, $p_{12} = 0.15$, $p_{13} = 0.09$

$p_{21} = 0.14$, $p_{22} = 0.35$, $p_{23} = 0.21$

$P_1 = p_{11} + p_{12} + p_{13} = 0.06 + 0.15 + 0.09 = 0.3$.

$P_2 = p_{21} + p_{22} + p_{23} = 0.14 + 0.35 + 0.21 = 0.7$

$\overline{P_1} = p_{11} + p_{21} = 0.06 + 0.14 = 0.2$, $\overline{P_2} = p_{12} + p_{22} = 0.15 + 0.35$

$= 0.5$, $\overline{P_3} = p_{13} + p_{23} = 0.09 + 0.21 = 0.3$

The distribution of X is

x_i	x_1	x_2
P_i	P_1	P_2

ie.

x_i	1	2
P_i	0.3	0.7

The distribution of Y is

y_j	y_1	y_2	y_3
$\bar{P}_j$	$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$

ie.

y_j	2	3	4
$\bar{P}_j$	0.2	0.5	0.3

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$P_1 \cdot \bar{P}_1 = 0.3 \times 0.2 = 0.06 = p_{11}$

$P_1 \bar{P}_2 = 0.3 \times 0.5 = 0.15 = p_{12}$

$P_1 \bar{P}_3 = 0.3 \times 0.3 = 0.09 = p_{13}$

$P_2 \bar{P}_1 = 0.7 \times 0.2 = 0.14 = p_{21}$

$P_2 \bar{P}_2 = 0.7 \times 0.5 = 0.35 = p_{22}$.

$P_2 \bar{P}_3 = 0.7 \times 0.3 = 0.21 = p_{23}$

$\therefore p_{ij} = P_i \bar{P}_j$

for all $i = 1, 2$,

$j = 1, 2, 3$

$\therefore$ X and Y are independan

2) A coin is tossed three times. Let X denote 0 or 1 according as a tail or a head occurs on the first toss. Let Y denote the no. of tails which occur. Determine the distributions of X and Y and (ii) the joint distribution of X and Y

Sol:- If a coin is tossed 3 times, then the outcome set is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$n(S) = 8$. let $X : f(s) = \begin{cases} 0 & \text{if first toss is a tail} \\ 1 & \text{if first toss is a Head.} \end{cases}$

Then X takes two values $x_1 = 0$, $x_2 = 1$

$p(x_1) = p(0) = p\{s \in S / f(s) = x_1 = 0\}$

$= P\{THH, THT, TTH, TTT\}$ $\therefore p(x_1) = \frac{4}{8} = \frac{1}{2}$.

$p(x_2) = p(1) = p\{s \in S / f(s) = x_2 = 1\}$

$= p\{HHH, HHT, HTH, HTT\}$ $\therefore p(x_2) = \frac{4}{8} = \frac{1}{2}$

$\therefore$ The distribution of X is

X	0	1
P(X)	0.5	0.5

$\therefore g(s) =$ no. of tails which occur.

$\therefore$ Then $Y=0$ for the outcome HHH,

$Y=1$ for the outcomes HHT, HTH, THH.

$Y=2$ for the outcomes HTT, THT, TTH.

$Y=3$ for the outcome TTT.

$\therefore P(Y=0) = \frac{1}{8}$, $P(Y=1) = \frac{3}{8}$, $P(Y=2)=\frac{3}{8}$, $P(Y=3)=\frac{1}{8}$

$\therefore$ The Distribution of Y is

y	0	1	2	3
$p(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$p_{ij} = P(x_i, y_j)$ $\quad i=1,2, \quad j=1,2,3,4.$

$P_{11} = P(x_1, y_1) = P(X=0, Y=0) = P\{\phi\} = 0.$

$P_{12} = P(x_1, y_2) = P(x=0, y=1) = P\{THH\} = \frac{1}{8}.$

$P_{23} = P(x_1, y_3) = P(x=0, y=2) = P\{THT, TTH\} = \frac{2}{8}$

$P_{14} = P(x_1, y_4) = P(x=0, y=3) = P\{TTT\} = \frac{1}{8}.$

$P_{21} = P(x=1, y=0) = P\{HHH\} = \frac{1}{8}$

$P_{22} = P(x=1, y=1) = P\{HHT, HTH\} = \frac{2}{8}$

$P_{23} = P(x=1, y=2) = P(HTT) = \frac{1}{8}$

$P_{24} = P(x=1, y=4) = P\{\phi\} = 0$

∴ The joint probability distribution is

X \ Y	0	1	2	3
0	0	$\frac{1}{8}$	$2/8$	$1/8$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0

3) The distribution of two stochastically independent random variables X and Y are given in the following tables.

X	0	1
P(X)	0.2	0.8

Y	1	2	3
P(X)	0.1	0.4	0.5

Find the Joint Probability distribution of X and Y

Sol:- $p_i = \sum_j p_{ij}$, $\bar{p}_j = \sum_i p_{ij}$

Since X and Y are stochastically independent, we have $p_{ij} = p_i \bar{p}_j$.

From the given data we have

$P_1 = 0.2$, $P_2 = 0.8$, $\bar{P}_1 = 0.1$, $\bar{P}_2 = 0.4$, $\bar{P}_3 = 0.5$

$p_{11} = P_1 \bar{P}_1 = 0.2 \times 0.1 = 0.02$, $P_{12} = P_1 \bar{P}_2 = 0.2 \times 0.4 = 0.08$

$P_{13} = P_1 \bar{P}_3 = 0.2 \times 0.5 = 0.1$, $P_{21} = P_2 \bar{P}_1 = 0.8 \times 0.1 = 0.08$

$p_{22} = P_2 \bar{P}_2 = 0.8 \times 0.4 = 0.32$, $P_{23} = P_2 \bar{P}_3 = 0.8 \times 0.5 = 0.4$

The joint probability distribution is

X \ Y	1	2	3
0	0.02	0.08	0.1
1	0.08	0.32	0.4

A joint probability distribution is given by the table

Y / X	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

find cov(X, Y) and $\rho(X,Y)$

Sol:- $P_1 = p_{11}+p_{12}+p_{13} = 0.1+0.2+0.2 = 0.5$

$P_2 = p_{21}+p_{22}+p_{23} = 0.3+0.1+0.1 = 0.5$

$\overline{P_1} = p_{11}+p_{12} = 0.1+0.3 = 0.4$, $\overline{P_2} = p_{12}+p_{22} = 0.2+0.1 = 0.3$

$\overline{P_3} = p_{13}+p_{23} = 0.2+0.1 = 0.3$.

The Individual distributions are.

X	1	3
P(X)	0.5	0.5

Y	-3	2	4
P(Y)	0.4	0.3	0.3

$\mu_X = \sum_i x_i\, p(x_i) = 1\times0.5 + 3\times0.5 = 2$

$\mu_Y = \sum_j y_j\, P(y_j) = -3\times0.4 + 2\times0.3 + 4\times0.3 = 0.6.$

$E[X^2] = \sum_i x_i^2\, p(x_i) = 1^2\times0.5 + 3^2\times0.5 = 5$

$E[Y^2] = \sum_j y_j^2\, p(y_j) = (-3)^2\times0.4 + 2^2\times(0.3) + 4^2\times0.3 = 9.6.$

$\sigma_X^2 = E[X^2] - \mu_X^2 = 5 - 2^2 = 1 \quad \therefore \sigma_X = 1$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = 9.6 - (0.6)^2 = 9.24 \quad \therefore \sigma_Y = 3.04.$

$E[XY] = \sum_i \sum_j p_{ij} x_i y_j$

$= p_{11}x_1y_1 + p_{12}x_1y_2 + p_{13}x_1y_3 + p_{21}x_2y_1 + p_{22}x_2y_2 + p_{23}x_3y_3$

by

$= (0.1)(1)(-3) + (0.2)(1)(2) + (0.2)(1)4 + (0.3)(3)(-3)$ (4)

$+ (0.1)(3)(2) + (0.1)(3)(4) = 0.$

Covariance $= cov(X,Y) = E[XY] - \mu_X \mu_Y$

$= 0 - 2(0.6) = -1.2.$

Correlation coefficient $= \rho(X,Y) = \dfrac{cov(X,Y)}{\sigma_X \sigma_Y}$

3 $= \dfrac{-1.2}{1 \times 3.04} = -0.3947$

5) For the following distribution find the (i) individual (marginal) distributions (ii) cov(X,Y) (iii) $\rho(X,Y)$.

X \ Y	-4	2	7
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$

Ans:-

X	1	5
P(X)	$\frac{1}{2}$	$\frac{1}{2}$

Y	-4	2	7
P(Y)	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$cov(X,Y) = \frac{3}{2}$, $\rho(X,Y) = 0.17$