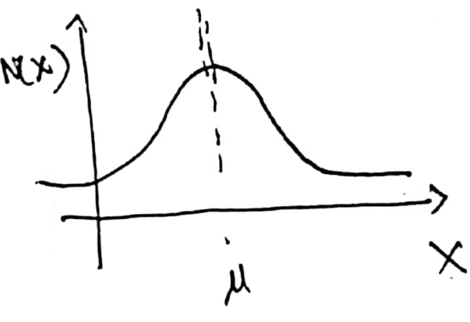
Normal distribution :-

Let μ and σ be two arbitrary real constants such that $-\infty < \mu < \infty$ and $\sigma > 0$. Then the probability distribution for which $N(\mu, \sigma; x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is called the normal distribution.

is the density function

μ is the Mean and σ is the standard deviation.

The graph of $N(\mu, \sigma; x)$ is called as the normal curve.

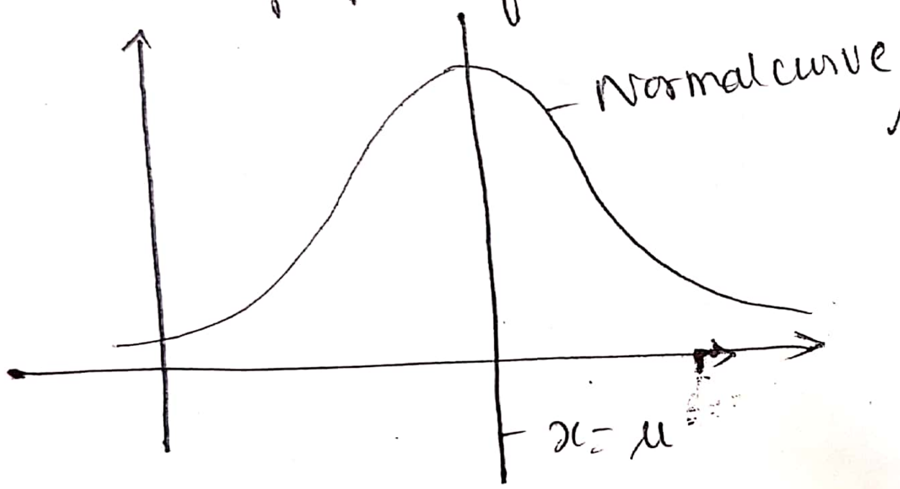


The curve is symmetric about the line $x = \mu$

Evaluation of $P(a \le x \le b)$.

$$P(a \le x \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_a^b N(x)\,dx.$$

The graph of $N(x)$ is as follows.



The curve is symmetric about $x = \mu$.

put $\frac{x-\mu}{\sigma} = z$

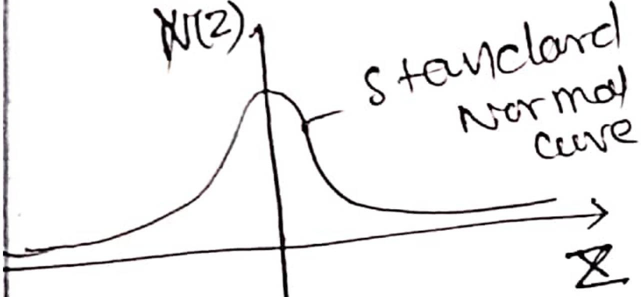
$x - \mu = \sigma z$

$dx = \sigma\, dz$

let the limits of z be z_1 and z_2, then.

$$P(a \le x \le b) = \int_{z_1}^{z_2} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cancel{\sigma}\, dz = P(z_1 \le z \le z_2)$$

$$N(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$ graph is as follows.



which is symmetric about $N(z)$ axis.

We can find $P(z_1 \le z \le z_2)$ by using the Normal Probability table.

which gives value of $A(z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$.

$$A(\infty) = P(0 < z < \infty) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0.5.$$

$$P(z_1 \le z \le z_2) = A(z_2) - A(z_1)$$

$$P(-z_1 \le z \le -z_2) = P(z_2 \le z \le z_1) = A(z_1) - A(z_2)$$

$$P(-z_1 \le z \le z_2) = P(-z_1 \le z \le 0) + P(0 \le z \le z_2)$$

$$= P(0 \le z \le z_1) + P(0 \le z \le z_2)$$

$$= A(z_1) + A(z_2) = A(z_2) + A(z_1)$$

$$P(z \ge z_1) = P(z_1 \le z < \infty) = A(\infty) - A(z_1) = 0.5 - A(z_1)$$

$$P(z \le z_1) = P(-\infty < z \le z_1) = A(z_1) + A(\infty)$$

$$= A(z_1) + 0.5$$

Normal Probability Table

z	0	1	2	3	4	5	6	7	8	9
	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0:09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	0.319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4278	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4430	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4648	.4656	.4664	.4671	.4678	.4686	.4693	.4700	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4762	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4874	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.5966	.5967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4990	.4993	.4995	.4997	.4998	.4998	.4999	.4999	.5000

Note: For values in between those provided, take proportionate increments.

K

(iv) $P(Z \leq K) = P(-\infty < Z < K)$

$= P(-\infty < Z < 0) + P(0 < Z < K)$

$= P(0 < Z < \infty) + A(K) = A(\infty) + A(K) = 0.5 + A(K)$

(v) $P(Z \leq -K) = P(-\infty < Z \leq -K)$

$= P(K < Z < \infty) = A(\infty) - A(K) = 0.5 - A(K)$

(vi) $P(Z \geq K) = P(K < Z < \infty) = A(\infty) - A(K)$

$= 0.5 - A(K)$

(vii) $P(Z \geq -K) = P(-K < Z < \infty)$

$= P(-K < Z < 0) + P(0 < Z < \infty)$

$= P(0 < Z < K) + A(\infty) = A(K) + 0.5$

① Evaluate the following probabilites with the help of normal probability tables.

(i) $P(Z \geq 0.85)$ (ii) $P(-1.64 \leq Z \leq -0.88)$

(iii) $P(Z \leq -2.43)$ (iv) $P(|Z| \leq 1.94)$

Sol:- $P(Z \geq 0.85) = P(0.85 \leq Z < \infty)$

$= A(\infty) - A(0.85) = 0.5 - 0.3023 = 0.1977$

$P(-1.64 \leq Z \leq -0.88) = P(0.88 \leq Z \leq 1.64)$

$= A(1.64) - A(0.88) = 0.4495 - 0.3106 = 0.1389$.

$P(Z \leq -2.43) = P(-\infty \leq Z \leq -2.43)$

$= P(2.43 \leq Z \leq \infty) = A(\infty) - A(2.43)$

$= 0.5 - 0.4925 = 0.0075$.

$$P(|z| \le 1.94) = P(-1.94 \le z \le 1.94)$$
$$= A(1.94) + A(1.94) = 2A(1.94)$$
$$= 2 \times 0.4738 = 0.9476.$$

② If x is a normally distributed variable with mean 30 and S.D 4 find
(i) $P(x > 40)$ (ii) $P(x < 21)$ (iii) $P(30 < x < 35)$

Sol: Here $\mu = 30$, $\sigma = 4$. X - normal variable.

Put $\frac{x-\mu}{\sigma} = z$, where z is standard normal variable.

i.e. $\frac{x-30}{4} = z$.

(i) $P(x > 40)$

for $x = 40$, $z = \frac{40-30}{4} = \frac{10}{4} = 2.5$

$\therefore P(x > 40) = P(z > 2.5) = P(2.5 < z < \infty)$
$$= \phi(\infty) - \phi(2.5) = 0.5 - 0.4938 = 0.0062$$

(ii) $P(x < 21)$

for $x = 21$, $z = \frac{21-30}{4} = \frac{-9}{4} = -2.25$

$\therefore P(x < 21) = P(z < -2.25) = P(-\infty < z < -2.25)$
$$= P(2.25 < z < \infty) = A(\infty) - A(2.25)$$
$$= 0.5 - 0.4878 = 0.0122$$

(iii) $P(30 < x < 35)$

for $x = 30$, $z = \frac{30-30}{4} = 0$, for $x = 35$, $z = \frac{35-30}{4} = \frac{5}{4}$
$= 1.25$

$\therefore\ P(30 < x < 35) = P(0 < z < 1.25) = A(1.25) - A(0)$

$= A(1.25) = 0.3944$ [Because $A(0) = 0$

(3) The annual salaries of employees in a large company distributed with a mean of 5 lakhs and a standard deviation of 2 lakhs. If a person is selected at random who is employ of that company what is the probability that (i) Annual Salary is less than 4 lakhs.

(ii) between 4.5 lakhs and 6.5 lakhs.

(iii) more than 7 lakhs.

Solution:- Here the normal variable 'x' is annual salary of an employee.

$\mu = 5$, $\sigma = 2$.

put $\frac{x-\mu}{\sigma} = z$ ie. $\frac{x-5}{2} = z$.

(i) probability that annual salary of an employee is less than 4 lakhs

$= P(x < 4)$

for $x = 4$, $z = \frac{4-5}{2} = -\frac{1}{2} = -0.5$.

$= P(z < -0.5) = P(-\infty < z < -0.5)$

$= P(0.5 \le z < \infty) = A(\infty) - A(0.5)$

$= 0.5 - 0.1915 = 0.3085$

(ii) Prob. that Salary is between 4.5 l and 6.5 l

$= P(4.5 < x < 6.5)$

for $x = 4.5$, $z = \frac{4.5-5}{2} = \frac{-0.5}{2} = -0.25$

for $x = 6.5$, $z = \frac{6.15-5}{2} = \frac{1.5}{2} = 0.75$

$= P(-0.25 < z < 0.75) = A(0.75) + A(0.25)$

$= 0.2734 + 0.0987 = \underline{0.3721}$

(iii) prob. that Salary is more than 7 lakhs

$= P(x > 7)$

for $x = 7$, $z = \frac{7-5}{2} = \frac{2}{2} = 1$

$= P(z > 1) = P(1 < z < \infty) = \phi(\infty) - \phi(1)$

$= 0.5 - 0.3413 = \underline{0.1587}$

(4) The marks of 1000 students in an examination follow a normal distribution with mean 70 and variance 25. Find no. of students whose marks will be (i) less than 65 (ii) more than 75 (iii) between 65 and 75

Solution:- Here normal variable x is marks of a student. $\mu = 70$, $\sigma^2 = 25$ $\therefore \sigma = 5$.

put $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$

(i) Prob. that a student has scored less than 65 marks $= P(x<65)$.

for $x=65$, $z = \frac{65-70}{5} = -\frac{5}{5} = -1$

$= P(z<-1) = P(-\infty < z < -1) = P(1<z<\infty)$

$= A(\infty) - A(1) = 0.5 - 0.3413 = 0.1587$

No. of students who have scored less than 65 marks $= 1000 \times P(x<65) = 1000 \times 0.1587$

$= 158.7$

Rounding off to nearest integer $= 159$ students.

(ii) No. of students who have scored more than 75 marks $= 1000 \times P(x>75)$

for $x=75$, $z = \frac{75-70}{5} = \frac{5}{5} = 1$.

$= 1000 \times P(z>1) = 1000 \times P(1<z<\infty)$

$= 1000 \times [\phi(\infty) - \phi(1)] = 1000 \times [0.5 - 0.3413]$

$= 1000 \times 0.1587 = 158.7 \simeq 159$ Students

(iii) Number of students who have scored between 65 and 75 marks $= 1000 \times P(65<x<75)$

for $x=65$, $z = \frac{65-70}{5} = \frac{-5}{5} = -1$

for $x=75$, $z = \frac{75-70}{5} = \frac{5}{5} = 1$

$= 1000 \times P(-1<Z<1) = 1000 \times [\phi(1)+\phi(1)] = 1000 \times 2 \times \phi(1)$

$= 2000 \times 0.3413 = 682.6 \approx \underline{683}$ Students.

(5) Find the mean and standard deviation of a normal distribution in which 7% of items are under 35 and 89% are under 63.

<u>Solution</u>:: let x be normal variable and z be the standard normal variable then $z = \frac{x-\mu}{\sigma}$,

7% of items are under 35

$\therefore P(x<35) = 7\% = 0.07$

for $x=35$, $z = \frac{35-\mu}{\sigma}$.

$\therefore P\left(z < \frac{35-\mu}{\sigma}\right) = 0.07$.

i.e. $P\left(-\infty < z < \frac{35-\mu}{\sigma}\right) = 0.07$

Since R.H.S is <0.5 and lower limit is $-\infty$ upper limit is also $-\underline{ve}$.

$\therefore P\left(\frac{\mu-35}{\sigma} < z < \infty\right) = 0.07$.

[+ve value of $\frac{35-\mu}{\sigma}$ is $\frac{\mu-35}{\sigma}$.]

i.e. $A(\infty) - A\left(\frac{\mu-35}{\sigma}\right) = 0.07$.

i.e. $0.5 - A\left(\frac{\mu-35}{\sigma}\right) = 0.07$

i.e. $-A\left(\frac{\mu-35}{\sigma}\right) = 0.07 - 0.5 = -0.4300$.

From table, $\phi(1.48) = 0.4306 \approx 0.4300$

$\therefore A\left(\frac{\mu-35}{\sigma}\right) = A(1.48)$

i.e. $\frac{\mu-35}{\sigma} = 1.48$

$\therefore \mu - 35 = 1.48\sigma$.

$\mu - 1.48\sigma = 35$ — ①.

89% are under 63.

i.e. $P(x < 63) = 89\% = 0.89$

for $x = 63$, $z = \frac{63-\mu}{\sigma}$.

$P\left(z < \frac{63-\mu}{\sigma}\right) = 0.89$

$P\left(-\infty < z < \frac{63-\mu}{\sigma}\right) = 0.89$.

Since R.HS is > 0.5, the interval hase to be -ve to +ve $\therefore \frac{63-\mu}{\sigma}$ is +ve.

i.e. $A\left(\frac{63-\mu}{\sigma}\right) + A(\infty) = 0.89$

$A\left(\frac{63-\mu}{\sigma}\right) = 0.89 - 0.5 = 0.39 \simeq A(1.23)$
$\simeq 0.03907$

$\therefore \frac{63-\mu}{\sigma} = 1.23$.

$63 - \mu = 1.23\sigma$.

$-\mu - 1.23\sigma = -63$ $\times -1$

i.e. $\mu + 1.23\sigma = 63$ — ②

Solving ① and ② we get

$\mu = 50.2915$ and $\sigma = 10.3321$