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$$= \lambda \left[1 - \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu} \right)^n P_0 \right]$$

(i) $$P_0 = \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \sum_{n=s}^{N} \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu} \right)^n \right]^{-1}$$

$$= \left[1 + \frac{15}{12} + \frac{1}{2!\, 2^0} \left(\frac{1/12}{1/15} \right)^2 + \frac{1}{2!\, 2} \left[\frac{1/12}{1/15} \right]^3 \right]^{-1}$$

$$= \left[1 + \frac{5}{4} + \frac{25}{32} + \frac{125}{256} \right]^{-1}$$

$$= \left[\frac{901}{256} \right]^{-1}$$

$$= \frac{256}{901}$$

$$L_q = \frac{1}{12} \left[1 - \frac{2}{2!\, 2^{3-2}} \left(\frac{1/12}{1/15} \right)^3 \cdot \frac{256}{901} \right]$$

$$= \frac{1}{2} \left[1 - 0.71 \right]$$

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(ii) $L_S = ?$

Here,

$$L_q = \frac{(s\rho)^2}{s!\,(1-\rho)^2} \left[1 - \rho^{N-s+1} - (1-\rho)(N-s+1)\rho^{N-s} \right]$$

$$= \frac{(2 \times 5/8)^2}{2!\,(1 - 5/8)^2} \left[1 - \left(\frac{5}{8} \right)^{3-2+1} - \left(1 - \frac{5}{8} \right)(3-2+1) \left(\frac{5}{8} \right)^{3-2} \right]$$

$$= \frac{50}{9} \times \frac{9}{64}$$

$$= \frac{25}{32}$$

$$L_s = L_q + s - P_0 \sum_{n=0}^{s-1} \frac{(s-n)}{n!} \left(\frac{\lambda}{\mu} \right)^n$$

$$= \frac{25}{32} + 2 - \frac{256}{901} \sum_{n=0}^{1} \left[\frac{2-n}{n!} \cdot \left(\frac{15}{12} \right)^n \right]$$

$$= \frac{25}{32} + 2 - \frac{256}{901} \left[2 + 1 \cdot \frac{15}{12} \right]$$

$$= 1.857$$

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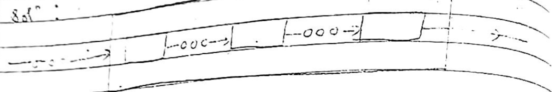
35. Sol^n :



No. of servers (s) = 1
No. of phases (K) = 3

$3 \times 1.5 = 4.5$ min = 1 customer served
$1 \text{ min} = \frac{1}{4.5} = \frac{2}{9}$ customer served

$\therefore$ Service rate $(\mu) = \frac{2}{9}$ customer per minute

Arrival rate $(\lambda) = 6$ customer per hour
$= \frac{6}{60} = \frac{1}{10}$ customer per minute

This problem is related to $M/E_k/1 : (\infty, FIFO)$ queue

$W_s = ?$

Here,

$$W_{E_k, s} = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

$$= \frac{3+1}{2 \times 3} \times \frac{1/10}{2/9 \times (2/9 - 1/10)} + \frac{1}{2/9}$$

$$= \frac{153}{22}$$

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Also,
Most probable time in getting the service $= \frac{k-1}{k\mu}$

$$= \frac{3-1}{3 \times 2/9}$$

$= 3$ min

36. Sol^n :

limited no. of machine (m) = 4
no. of server (mechanic) (s) = 1
5 hours = 1 machine arrive to mechanic
1 hour = $\frac{1}{5}$ machine arrive to mechanic

Arrival time $(\lambda) = \frac{1}{5}$ machine per hour

1 hour = 1 machine served
$\therefore$ Service rate $(\mu) = 1$ machine per ~~pe~~ hour

This problem is related to $(M/M/1 : M/GD)$ queue

i) $P_0 = ?$
ii) $P_n = ?$
iii) $L_s = ?$

Then,

$$\therefore P_0 = \left[\sum_{n=0}^{4} \frac{4!}{(4-n)!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$= \left[\frac{4!}{(4-0)!} \times 1 + \frac{4!}{3!} \left(\frac{1/5}{1}\right)^1 + \frac{4!}{2!} \left(\frac{1}{5}\right)^2 + \frac{4!}{1!} \left(\frac{1}{5}\right)^3 + \frac{4!}{0!} \left(\frac{1}{5}\right)^4 \right]^{-1}$$

$$= \left[1 + 4 \times \frac{1}{5} + 12 \times \frac{1}{25} + 24 \times \frac{1}{125} + \frac{4!}{0!}\left(\frac{1}{5}\right)^4 \right]^{-1}$$

$$= \left(\frac{944}{625}\right)^{-1}$$

$$= \frac{625}{944}$$

6. Soln:

[illegible]

1 message = 176 characters

[illegible] 300 [illegible] min

$$\mu = \frac{300}{176} \times 60 = 102.27 \approx 102 \text{ msg/min}$$

$c = 1$

$N = \infty$

FIFO.

Questions Patterns

PQT Solution

1). Simple Probability
   (a) Rules of Prob
   (b) conditional prob

2) Baye's Theorem

} 1 (a) (b)

3) Random Variable, expectation, variance (1 LQ) 2 or 3

4) Joint Prob Distribution (1 LQ). 2 or 3

5) Theoretical Probability distribution
   (a) Binomial Distribution
   (b) Poisson Distribution
   Imp (c) Normal Distribution
   (d) Exponential "
   (e) Gamma "
   (f) Beta "
   (g) Uniform "

} (2 LQ) Done

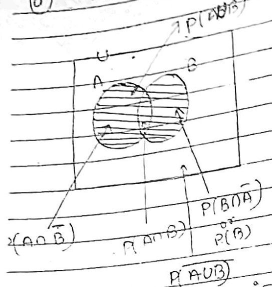
6) Central Limit Theorem, BLLN, Chebyshev's Inequality.
(1 LQ)

7) Markov's Chain (M.C) (1 LQ / 2 LQ.) Done.

8) Theory of Queue (3 LQ.)

① Simple Probability

⑥ Rules of Probability



$P(U) = 1$

① Additional Rule :-

$P(A \cup B)$

↳ Prob of either A or B.

↳ Atleast one of them can happen.

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

② Multiplication Rule.

↳ Prob of $P(A \cap B)$

↳ Prob of A and B.

→ Both of them can happen

→ also known as joint prob of A and B

→ ~~ee~~ formula for independent events

$$P(A \cap B) = P(A) * P(B) \quad \text{or} \quad P(B) * P(A)$$

Selection with replacement is independent.

↳ For dependent events - selection without replacement is dependent.

$$P(A \cap B) = P(A) \times P(B/A)$$

$$P(B \cap A) = P(B) \times P(A/B)$$

Note :

i) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$.

ii) $P(B \cap \bar{A}) = P(B) - P(A \cap B)$

iii) $P(\overline{A \cup B}) = \cancel{P}\ 1 - P(A \cup B)$.

Numericals.

① Mr. A and Mr. B are interested to attend a seminar. The chance of attending seminar by Mr. A is 0.6 and Mr. B is 0.3. They both can also attend the seminar. Attending the seminar by Mr. A and Mr. B are independent to eachother. What is the prob that :

a) Both of them can attend the seminar

b) atleast one of them can attend the seminar.

c) only Mr. A will attend the seminar

d) only Mr. B will attend the seminar

e) they will not have attend ~~the semin~~ in seminar

Soln

$P(A) = 0.6$
$P(B) = 0.3$

a) $P(A \cap B) = P(A) \times P(B)$
$= 0.18$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.6 + 0.3 - 0.18$
$= 0.72$

c) P(only Mr. A will attend)

$= P(A \cap \bar{B})$
$= P(A) - P(A \cap B)$

$= 0.6 - 0.18$
$= 0.42$

d) P(only Mr. B will attend)
$= P(\bar{A} \cap B)$
$= P(B) - P(A \cap B)$
$= 0.3 - 0.18$
$= 0.12$

e) P(they will not have to attend seminar)
$= P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - 0.72$
$= 0.28$

2) A committee of 4 members has to be formed ~~from~~ from 5 IT engineer, 4 civil engineer, 3 software engineer and 2 elx. engineers. Find the prob that committee consists of :-

a) 2 IT and 2 civil engineer

b) No IT engineer $^{4}C_{14}$

c) atleast one IT engineer.

d) atmost one IT engineer.

e) one IT, one civil, one software and one elx engineer

Soln

Total possible no. of committee $= {}^{n}C_{r}$
$= {}^{14}C_{4}$
$= 1001$

a) P(2 IT, 2 civil)

$= \frac{5C_2 \times 4C_2}{14C_4} = 0.05$

(b) P(No IT engineer)

= P(zero IT and 4 others)

$= \frac{5C_0 \times 9C_4}{14C_4} = 0.12$

(c) P(atleast one IT engineer)

= P(1 or more IT)

= 1 − P(zero IT and 4 others)

= 1 − 0.12

= 0.88 #

≥1 IT
(1) 4 IT & 0 others
(2) 3 IT & 1 others
(3) 2 IT & 2 others
(4) 1 IT & 3 others
(5) 0 IT & 4 others
1 − P(0 IT & 4 others)

(d) P(atmost 1 IT)

= P(1 or less 1 IT)

= P(1 IT, 3 others) + P(0 IT, 4 others)

$= \frac{5C_1 \times 9C_3}{14C_4} + \frac{5C_0 \times 9C_4}{14C_4}$

= 0.41 + 0.12 = 0.53

(e) P(1 IT, 1 civil, 1 soft, 1 elx)

$= \frac{5C_1 \times 4C_1 \times 3C_1 \times 2C_1}{14C_4}$

= 0.11 #

2013 fall

1(a) In an org, there are 2 civil engineer, 4 IT engineers and 2 accountants. A committee of 4 member has to be formed. What is the prob that a committee contains:

i) one civil engineer and 3 IT officer

ii) no civil engineer

iii) atleast one IT ~~engine~~ officer.

Soln

$P(C) = 2$
$P(IT) = 4$
$P(\text{accountant}) = 2$

Total no. of committe formed $= {}^{8}C_4$

i) P(one civil, 3 IT)

$= \frac{2C_1 \times 4C_3}{8C_4} = 0.11$

ii) no civil engineer
$=$ P(~~no~~ zero civil, ~~3~~ 4 others)

$= \frac{2C_0 \times 6C_4}{8C_4} = 0.21$

iii) P(atleast one IT officers).

$=$ P(1 or more than 1 IT)

$=$ 1 − P(0 IT and 4 others)

$= 1 - \frac{4C_0 \times 4C_4}{8C_4}$

$= 1 - 0.01$
$= 0.99$ //

2017 (spring)

1(a) A group of five printers in a sample consists of three good, labeled g_1, g_2 and g_3 and two defective, labeled d_1 and d_2. If three printers are selected at random from this group, what is the probability of the even E: "Two of the three selected printers are good"?

Soln

Total possible no of selection of printers $= 5C_3$
$= 10.$

Total no. of good printer $= 3$
Total no of bad printers $= 2$
P(Two of the three selected printers are good)
$= \frac{3C_2 \times 2C_1}{5C_3} = 0.6$ #

OR → $\cup$ either one or both ($A \cup B$)

And → $\cap$ either A & B ($A \cap B$)

⑥ conditional Probability - conditional prob is always dependent

It is of two types -

i) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

ii) $P(B/A) = \frac{P(A \cap B)}{P(A)}$

2015 Spring

1 (a) In a certain grp of computer personnel, 65% have insufficient knowledge of h/w, 45% have inadequate idea of s/w and 70% are either one or both of the two categories. What is the % of people who know s/w among those who have a sufficient knowledge of h/w?

Soln

Let,

A = insufficient knowledge of h/w

B = inadequate " of s/w

Given

$P(A) = 65\% = 0.65$

$P(B) = 45\% = 0.45$

$P(A \cup B) = 70\% = 0.70$

~~Find the prob~~

Here, $\bar{A}$ = sufficient knowledge of h/w = 0.35

$\bar{B}$ = adequate " of s/w = 0.55

$P(\bar{B}/\bar{A}) = ?$

i) $P(\bar{B}/\bar{A}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})}$

$= \frac{P(\overline{A \cup B})}{1 - P(A)}$

$= \frac{1 - P(A \cup B)}{1 - P(A)} = \frac{1 - 0.70}{1 - 0.65}$

$= 0.85$

∴ %

ii) What is the % of people who know software among those who have insufficient knowledge of h/w.

$P(\bar{B}/A) = \frac{P(\bar{B} \cap A)}{P(A)}$

$= \frac{P(\bar{B}) \times P(A)}{P(A)} =$ ~~0.55 × 0.65 / 0.65~~

Here, A = insufficient knowledge of h/w
B = inadequate " " s/w

$\bar{A}$ = sufficient knowledge of h/w
$\bar{B}$ = adequate knowledge of s/w

$$\therefore P(\bar{B}/A) = \frac{P(\bar{B} \cap A)}{P(A)} - \frac{P(A) - P(A \cap B)}{P(A)}$$

$$= \frac{0.65 - [P(A) + P(B) - P(A \cap B)]}{0.65}$$

$$= \frac{0.65 - (0.65 + 0.45 - 0.70)}{0.65}$$

$$= 0.38 \;//$$

ii) what is the prob that personnel have inadequate idea of s/w about s/w given that they have sufficient knowledge about h/w. ?

$$P(B/\bar{A}) = \frac{P(B \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(B) - P(A \cap B)}{P(\bar{A})}$$

$$= \frac{0.45 - [P(A) + P(B) - P(A \cap B)]}{1 - P(A)}$$

$$= \frac{0.45 - (0.65 + 0.45 - 0.70)}{0.35}$$

$$= 0.14 \;//$$

iv) what is the prob that personnel have inadequate knowledge of h/w if they have adequate knowledge about s/w.

$$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{0.65 - 0.4}{0.55}$$

$$= \cancel{0.55} \quad 0.45 \;//$$

(2) Baye's Theorem :-

Note: Likelihood prob निकाल्दा अहिल्यै question मा "कुन" [illegible]

Events	P(Event) prior prob	$P(D/Event)$ likelihood prob	$P(Event \cap \bar{D})$ joint prob	$P(Event/D)$ posterior prob
			$P(Event) \times P(D/Event)$	$P(Event/D) = \frac{P(Event \cap D)}{P(D)}$
A	$P(A)$	$P(D/A) =$	$P(A \cap D) = P(A) \times P(D/A)$	$P(A/D) = \frac{P(A \cap D)}{P(D)}$
B	$P(B)$	$P(D/B) =$	$P(B \cap D) = P(B) \times P(D/B)$	$P(B/D) = \frac{P(B \cap D)}{P(D)}$
C	$P(C)$	$P(D/C) =$	$P(C \cap D) = P(C) \times P(D/C)$	$P(C/D) = \frac{P(C \cap D)}{P(D)}$
	Given in Question (Que मा दिएको हुन्छ)			
Total	1		$P(D) =$	1

2015 (Spring)

2014 Spring 1(b)
2014 Fall 1(a).

1 (b) A factory has three machines A, B and C which produce 40%, 35% and 25% of the products respectively. Out of their o/p, 8%, 10% & 5% are defective respectively. An item is chosen at random from the total o/p and tested. What is the prob that it is good? If it is good, find the prob that it was manufactured by machine

i) A
ii) B
iii) C.

Soln

Let, $\bar{D}$ = good product and D = bad products.

Events	Prior prob P(Event)	Likelihood prob $P(\bar{D}/Event)$	joint prob $P(Event \cap \bar{D}) = P(Event) \times P(\bar{D}/Event)$	posterior prob $P(Event/\bar{D}) = \frac{P(Event \cap \bar{D})}{P(\bar{D})}$
A	$P(A) = 40\% = 0.\cancel{0}4$	$P(\bar{D}/A) = 1 - P(D/A) = 1 - 0.08 = 0.92$	$P(A \cap \bar{D}) = P(A) \times P(\bar{D}/A) = 0.\cancel{0}4 \times 0.92 = 0.\cancel{0}36$	$P(A/\bar{D}) = \frac{P(A \cap \bar{D})}{P(\bar{D})} = \frac{0.\cancel{0}36}{0.581} = 0.61$
B	$P(B) = 35\% = 0.35$	$P(\bar{D}/B) = 1 - P(D/B) = 1 - 0.1 = 0.9$	$P(B \cap \bar{D}) = P(B) \times P(\bar{D}/B) = 0.35 \times 0.9 = 0.315$	$P(B/\bar{D}) = \frac{P(B \cap \bar{D})}{P(\bar{D})} = \frac{0.315}{0.581} = 0.54$
C	$P(C) = 0.25$	$P(\bar{D}/C) = 0.95$	$P(C \cap \bar{D}) = 0.25 \times 0.95 = 0.23$	$P(C/\bar{D}) = \frac{0.23}{0.581} = 0.39$
Total		2.77	$P(\bar{D}) = 0.581$	1.544

i) P(product is good)

= ~~0.00~~ 0.581

ii) $P(A/\bar{D})$ = ~~0.06~~ 0.61

$P(B/\bar{D})$ = 0.54

$P(C/\bar{D})$ = ~~0.08~~ 0.39

2014(fall)

1 (a) an assembly plant receives its voltages regulators from three diffn suppliers, 60% from supplier A, 30% from supplier B and 10% from supplier C. If 95% of the voltages regulators from A, 80% of the voltage regulators from B & 65% of those from C perform accn to the specification given by the suppliers. What is the prob that:

i) assembly plant will receive voltage regulator working accn to the specification.

ii) if a selected voltage regulator is found to be not working accn to the specification, What is the prob that it was supplied by supplier C?

Soln

Let, W = working accn to specification
$\bar{W}$ = notworking accn " "

Events	P(Event)	$P(\bar{W}/\text{Event})$	$\frac{P(\text{Event} \cap \bar{W})}{P(\text{Event})}$
supplier A	P(A) = 60% =	$P(\bar{W}/A) = 1-0.95$ =	$P(A) \times P(\bar{W}/A)$ =
		$P(\bar{W}/B) = 1-0.80$ =	$P(B) \times P(\text{Event}/\bar{W})$
		$P(\bar{W}/C) = 1-0.65$	

2013 (spring)

Note-

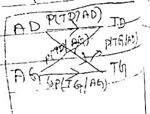
P(AD/TD) = It is the prob chip is actually defective given that tester says defective.

1 ⓐ

Soln

Let D = defective, TD = Tester defective,

Event		Likelihood prob P(TD/Event)		
actually defective	P(AD) = 0.02	P(TD/AD) = 0.94	P(AD∩TD) =	P(AD/TD) =
actually good	P(AG) = 1 − 0.02 = 0.98	P(TD/AG) = 1 − P(TG/AG) = 1 − 0.95 =		
Total	1			



P(AD/TD) = prob of chip ~~that~~ which is actually defective given that the tester is defective.

P(AD/TD) * It is the prob that chip is ~~tester say~~ Actually defective given that tester says defective

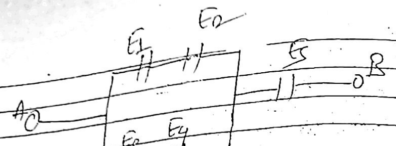
Thus, the prob that chip is actually defective given that tester say it is defective = P(AD/TD)

2014 (spring)

1 ⓐ

2015 (Fall)

1 (a)



prob (current exists bet^n A and B)

$$P\left\{\left[(E_1 \cap E_2) \cup (E_3 \cap E_4)\right] \cap E_5\right\}$$

$$= P\{(E_1 \cdot E_2 \cup E_3 E_4) \cap E_5\}$$

$$= p(E_1 E_2 \cup E_3 E_4) * p(E_5)$$

$$= \{p(E_1 \cdot E_2) + p(E_3 E_4) - p(E_1 E_2 \cap E_3 E_4)\} *$$

$$= \{p(E_1) * p(E_2) + p(E_3) * p(E_4) - p(E_1) * p(E_2) * p(E_3) \times p(E_4)\} * p(E_5)$$

$$= (0.6^2 + 0.6^2 - 0.6^4)\, 0.6$$

Random Variable

Expectation

~~Variable~~ Variance

1) Random Variable (rv) -

① Discrete random variable

② continuous random variable

① Discrete random variable -

It takes integers value. for eg:

$X = 0, 1, 2, 3$

$X = -2, -1, 0, 1, 2$

② continuous random variable.

It takes values within certain range. or certain interval.

for eg: $5 < X < 20$

$0 < X < \infty$

$0 < X < 1$

$-\infty < X < \infty$

Probability Distribution of random variable (rv)

(A) Discrete prob. distribution of random variable

(A) → Eg: coin toss, dice roll

→ count ગણે સમિયાં મળે discrete prob dist^n.

Prob Distribution of rv
prob distn of rv
(A) Discrete prob distn of rv

Q construct prob distribution of no. of head while tossing a coin three times.

Soln

~~No faces~~

Total possible outcomes 8-

HHH

HHT

HTH

TTT

TTH

THT

THH

HTT

Let X be a random variable which denotes number of heads.

X	Prob
0	1/8
1	3/8
2	3/8
3	1/8

This is called discrete prob distribution of no. of head.

Q construct prob distribution of sum of numbers faces when we roll a dice two times

Soln

Total no of possible outcomes = $6^n = 6^2 = 36$

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6).

Let X be a rv which denotes sum of numbers on face.

X	Prob
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

This is called discrete prob distribution of sum of faces when we roll a dice two times.

Range मा आवती गों continuous

(B) continuous prob distribution of rv :-

weight (kg)	No. of stdnts (f)	prob = f/N
30-40	5	5/85
40-50	10	10/85
50-60	20	20/85
60-70	30	30/85
70-80	15	15/85
80-90	5	5/85
Total	N=85	1

This is called continuous prob dist^n of wt.

(II) Expected value / Expectation / Mean of rv / Average of rv.

It is denoted by :- $E(X)$

$$E(X) = \begin{cases} \sum_x x \cdot p(x) \; ; \text{ if } x \text{ is discrete random variable (discrete prob distribution)} \\ \int_{-\infty}^{\infty} x f(x)\,dx \; ; \text{ if } x \text{ is continuous rv.} \end{cases}$$

where
$p(x)$ = pmf = prob mass function
$f(x)$ = pdf = prob density function

pmf $\sum_x^n p(X=x) = 1$

pdf $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

pmf :- It is the prob function which gives the prob of discrete rv. The function value of pmf lies betn 0 and 1 and sum of functional values equal to 1.

For eg:

X	$p(x)$ → This is pmf
0	1/4
1	2/4
2	1/4
Total	1

pdf :- It is the prob function which gives the prob of continuous rv. The function value of pdf lies betn 0 and 1 and sum of functional values equal to 1.

For eg:

Weight (kg)	No. of stdnts (f)	prob = f/N
30-40	5	5/15
40-50	10	10/15
Total	15	1

| Note |

Expected value of x^2 :-

$$E(x^2) = \begin{cases} \sum_{x} x^2 p(x) ; \text{ if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x^2 f(x)\,dx ; \text{ if } x \text{ is continuous} \end{cases}$$

This is called second order raw moment. If x^3 then it will called third order raw moment and so on.

Variance of rv :-

→ Variance measures the deviation from center point (value).

→ It is denoted by $V(X)$ and is given by -

$$\boxed{\therefore V(X) = E(x^2) - [E(x)]^2}$$

Note

$S.D = \sqrt{V(X)}$

S.D = standard deviation.

Properties of Expectation : Roop

Properties of Expectation	properties of variance.
$E(ax) = a\,E(X)$	① $V(ax) = a^2 V(x)$
$E(\text{constant}) = \text{constant}$	② $V(\text{constant}) = 0$
$E(X+Y) = E(X) + E(Y)$	③ $V(X+Y) = V(X) + V(Y) + 2\,cov(X,Y)$ where cov = covariance.
$E(X-Y) = E(X) - E(Y)$	④ $V(X-Y) = V(X) + V(Y) - 2\,cov(X,Y)$

| Note | —

If 'X' and 'Y' are independent rv then :

i) $E(XY) = E(X) \cdot E(Y)$

ii) $Cov(X,Y) = 0$

Q. A rv X has following prob distribution:

X	−2	−1	0	1
p(x)	0.2	K	0.2	2K

Find:

i) value of K

ii) Expected value of X and its variance.

iii) E(3X)

V(2X)

E(2X+5)

V(3X+2).

iv) $P(X<0)$

v) $P(X \le 0)$

vi) $P(-2<X<0)$

vii) $P(X \le 0 / X \geq 1)$

Solution

① We have sum of all p(x) = 1

i.e. $\sum_x p(x) = 1$

or, $P(-2)+P(-1)+P(0)+P(1) = 1$

or, $0.2 + K + 0.2 + 2K = 1$

or, $0.4 + 3K = 1$

or, $\therefore K = 0.2$

ii) $E(X) = \sum_x x\, p(x)$

$= (-2)P(-2) + (-1)P(-1) + 0\cdot P(0) + 1\cdot P(1)$

$= (-2\times 0.2) + (-1\times K) + 0 + (1\times 2K)$

$= -0.4 - K + 2K$

$= -0.4 - 0.2 + (2\times 0.2)$

$=$

and

variance $E(x^2) = \sum x^2 p(x)$.

$= (-2)^2 P(-2) + (-1)^2 P(-1) + 0^2 P(0) + 1^2 P(1)$

$=$

$\therefore$ variance, $V(X) = E(X) - [E(X)]^2$

* $P(X \le 0) \rightarrow$ less than or equal to zero or atmost zero.

iii) $E(3X) = 3E(X) = 3X$

$V(2X) = (2)^2 V(X) = 4X$

$E(2X+5) = E(2X) + E(5) + 2E(X) + 5 =$

$V(3X+2) = V(3X) + V(2) + 2\,cov(3X, 2)$

$= V(3X) + V(2) + 0.$

$= V(3X) + V(2)$

$= (3)^2 V(X) + \cancel{2}\ 0$

$= 9V(X)$

iv) $P(X < 0)$

$= P(-2) + P(-1) = 0.2 + 0.2 = 0.4$

v) $P(X \le 0)$

$= P(-2) + P(-1) + P(0)$

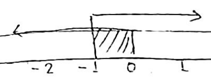
$= 0.2 + 0.2 + 0.2 = 0.6.$

vi) $P(-2 < X < 0)$

$= P(-1) = 0.2$

vii) $P\left(X \le 0 / X \ge -1\right) = \dfrac{P(X \le 0 \cap X \ge -1)}{P(X \ge -1)}$

since, $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$



$= \dfrac{P(-1) + P(0)}{P(-1) + P(0) + P(1)}$

$= \dfrac{0.2 + 0.2}{0.2 + 0.2 + 0.4} = \dfrac{0.4}{0.8} = 0.5$ #

v) $F(0) \rightarrow$ cummulative distribution function prob distribution at $X = 0$

$= P(X \le 0)$

$= P(-2) + P(-1) + P(0)$

Continuous R.V :-

Numerical

~~A r.v. has the f~~

A r.v. 'x' has the following pdf:

$$\boxed{\therefore f(A) = Ae^{-x/5}} \quad ; \quad 0 < x < \infty$$

Find : (i) value of A
(ii) Expected value of 'x' and it's variance
(iii) $P(X < 3)$, $P(X > 5)$
(iv) $P(X < 5 / X > 3)$
(v) $E(3x)$, $V(5x+2)$
(vi) cummulative distribution function.

Solution
(i) We have,

$$\int_x f(x)\, dx = 1$$

$$\int_0^\infty Ae^{-x/5}\, dx = 1$$

$e^{\infty} = \infty$
$e^{-\infty} = 0$
$e^{0} = 1$

we have sum of all pdf = 1

or, $A\left[\dfrac{e^{-x/5}}{-1/5}\right]_0^\infty = 1$

or, $A\left[\dfrac{e^{-\infty}}{-1/5} - \dfrac{e^{-0}}{-1/5}\right] = 1$

or, $A\left[0 + \dfrac{1}{1/5}\right] = 1$

$\therefore A = 1/5$

(ii) $E(x) = \int_0^\infty x \cdot f(x)\, dx$

$$= \int_0^\infty x \cdot A \cdot e^{-x/5}\, dx$$

$\int_0^\infty e^{-ax} x^{n-1}\, dx$
$= \dfrac{\sqrt{n}}{a^n}$

$$= \frac{1}{5}\int_0^\infty x \cdot e^{-\frac{x}{5}}\, dx$$

$$= \frac{1}{5}\left(\frac{x \cdot e^{-x/5}}{-1/5} + e^{-x/5}\right)_0^\infty$$

$$= \frac{1}{5}\left(\frac{-x e^{-x/5}}{5} + e^{-\frac{x}{5}}\right)_0^\infty$$

$$= \frac{1}{5}\left[-\infty e^{-\infty} + e^{-\infty} - (-0 + e^{-0})\right]$$

$$= 1/5\,(-1)$$

$$= \frac{-1}{5} \qquad \text{(correct ans is 5)}$$

$$= \frac{1}{5}\left[\frac{x e^{-x/5}}{-1/5} - \frac{(1)\, e^{-x/5}}{(-1/5)(-1/5)}\right]_0^\infty$$

$$= \frac{1}{5}\left[-5x e^{-x/5} - 25 e^{-x/5}\right]_0^\infty$$

$$= -5\left[x e^{-x/5} - 5e^{-x/5}\right]_0^\infty = -\left[(\infty - 5e^{-\infty}) - (0 - 5e^{0})\right]$$

and,
$$E(x^2) = \int_0^\infty x^2 f(x)\, dx$$
$$= \int_0^\infty x^2 \cdot \tfrac{1}{5} e^{-x/5}\, dx$$
$$= \frac{1}{5}\int_0^\infty e^{-\frac{1}{5}x} x^{3-1}\, dx$$
$$= \frac{1}{5} \frac{\Gamma 3}{\left(\frac{1}{5}\right)^3}$$
$$= \frac{(3-1)!}{(1/5)^2}$$
$$= 25 \cdot (1\times2)$$
$$= 50.$$

also, $V(x) = E(x^2) - [E(x)]^2$
$$= 50 - (5)^2$$
$$= 25$$

iii) we have,
$$P(X < 3)$$
$$= P(0 < X < 3)$$
$$= \int_0^3 f(x)\, dx$$
$$= \int_0^3 \frac{1}{5} e^{-x/5}\, dx$$
$$= \frac{1}{5}\int_0^3 e^{-\frac{1}{5}x}\, dx$$
$$= \frac{1}{5}\left[\frac{e^{-1/5 x}}{-1/5}\right]_0^3 = \frac{1}{5}\times -5 \left[e^{-\frac{x}{5}}\right]_0^3$$
$$= -\left(e^{-\frac{3}{5}} - e^{-0}\right) = -(0.548 - 1)$$
$$= -0.548 + 1 = 0.452$$

also
$$P(X > 5)$$
$$= P(5 < X < \infty)$$
$$= \int_5^\infty f(x)\, dx = \int_5^\infty \frac{1}{5} e^{-\frac{x}{5}} dx$$
$$= \frac{1}{5}\int_5^\infty e^{-x/5} dx$$
$$= \frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-1/5}\right]_5^\infty$$
$$= -\left[e^{-x/5}\right]_5^\infty$$
$$= -\left[e^{-\infty} - e^{-1}\right]$$
$$= -(0 - 0.36)$$
$$= 0.36$$

iv) $P(X<5/X>3)$

$= \frac{P(X<5 \cap X>3)}{P(X>3)}$

$= \frac{\int_3^5 f(x)\,dx}{\int_3^\infty f(x)\,dx}$

For, $\int_3^5 f(x)\,dx$

$= \int_3^5 \frac{1}{5} e^{-\frac{1}{5}x}\,dx$

$= \frac{1}{5}\int_3^5 e^{-\frac{1}{5}x}\,dx$

$= \frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-1/5}\right]_3^5$

$= -\left(e^{-5/5} - e^{-3/5}\right)$

$= -\left(e^{-1} - e^{-0.6}\right)$

$= -(0.36 - 0.54)$

$= -0.36 + 0.54$

$= 0.18$

Again,

for $\int_3^\infty f(x)\,dx$

$= \int_3^\infty \frac{1}{5} e^{-\frac{1}{5}x}\,dx$

$= \frac{1}{5}\int_3^\infty e^{-\frac{1}{5}x}\,dx$

$= \frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-1/5}\right]_3^\infty$

$= -\left(e^{-\frac{x}{5}}\right)_3^\infty$

$= -\left(e^{-\infty} - e^{-\frac{3}{5}}\right)$

$= -\left(0 - e^{-0.6}\right)$

$= 0.54$

Thus, we get

$= \frac{0.18}{0.54}$

$= 0.33$

v) $E(3X)$

$= 3E(X)$

$= 3 * 5$

$= 15$

vi) $V(5X+2)$

$= V(5X) + V(2) + 2\,COV(5X, 2)$

$= 5V(X) + 0 + 0$

$= 5V(X)$

$= 5 \times 5 = 25$

Cummulative distribution function -

$F(x) = P(X \le x)$

$= P(0 < X \le x)$

$= \int_0^x f(x)\,dx$

$= \int_0^x \frac{1}{5} e^{-\frac{x}{5}}\,dx$

$= \frac{1}{5}\int_0^x e^{-\frac{x}{5}}\,dx$

$= \frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-1/5}\right]_0^x$

$= -\left[e^{-1/5x}\right]_0^x$

$= -\left(e^{-\frac{x}{5}} - e^{0}\right)$

$= -e^{-x/5} + e^{0}$

$= -e^{-x/5} + 1$

$= \left(1 - e^{-x/5}\right)$

2013 (spring)

i) atmost 10,000

$= P(X \le 10{,}000)$

$= P(X \le 10)$

$= P(0 < X \le 10)$

$= \int_0^{10} f(x)\, dx$

$=$

Joint Probability Distribution :-

→ 1) Joint continuous probability distribution

Let 'X' & 'Y' be continuous r.v.

(a) jpdf

$f(x, y) =$

i) $0 \le f(x,y) \le 1$

ii) $\int_x \int_y f(x,y)\, dy\, dx = 1$

→ 2) Marginal pdf

i) Marginal pdf of r.v X :-

$$f(x) = \int_y f(x,y)\, dy$$

ii) Marginal pdf of r.v Y :

$$f(y) = \int_x f(x,y)\, dx$$

Note : if X and Y are independent r.v then ;

i) $f(x,y) = f(x) \cdot f(y)$

→ 3) Conditional pdf :

conditional pdf of 'x' given

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

conditional pdf of 'y' given x

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

Note : For independent r.v's :-

i) $f(x/y) = f(x)$

ii) $f(y/x) = f(y)$

→ 4) Covariance betn 'X' and 'Y'

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

where,

$$E(X) = \int_x x\, f(x)\, dx$$

$$E(Y) = \int_y y \cdot f(y)\, dy$$

$$E(X,Y) = \int_x \int_y xy\, f(x,y)\, dy\, dx$$

Note : If 'x' & 'Y' are independent then the covariance betn 'X' & 'Y' is zero.

→ 5) Joint cummulative distribution function:

$$F(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dy\, dx$$

Q. A r.v's 'x' & 'Y' have following joint prob density function.

$$f(x,y) = A(x+y) \; ; \; 0 < x < 1 \; ; \; 1 < y < 2 \; ;$$

Find:

i) value of A

ii) marginal pdf of 'x' and that of 'y'

iii) conditional pdf of $f(x/y)$

iv) conditional pdf of $f(y/x)$

v) Are 'X' & 'Y' independent r.v? give reason

vi) Joint cummulative dist function

vii) co-variance betn 'X' and 'Y'

viii) $P(X < 1/2, Y > 1/2)$

Soln

i) $\int_x \int_y f(x,y)\, dy\, dx = 1$

or, $\int_x \int_y A(x+y)\, dy\, dx = 1$

or, $\int_y \left[Axy + \frac{Ay^2}{} \right]^{2} dx = 1$

or, $\int_x \left[2Ax + A\frac{1}{2} - Ax - A\frac{1}{2} \right] dx = 1$

or, $\int_x \left[Ax + \left(\frac{4A - A}{2} \right) \right] dx = 1$

or, $\int_x \left[\frac{2Ax + 3A}{2} \right] dx = 1$

$\int_x \left[\cancel{\frac{2Ax + 3A}{2}} \right]$ or, $\int_x \left[Ax + \frac{3A}{2} \right] dx = 1$

or, $\left[\frac{Ax^2}{2} + \frac{3A}{2} x \right]_0^1 = 1$

or, $\frac{1}{2} \left(A + 3A - 0 - 0 \right) = 1$

or, $\frac{4A}{2} = 1$

$\therefore A = 1/2$

$\therefore f(x,y) = \frac{1}{2}(x+y) \; ; \; 0 < x < 1 \;\&\; 1 < y < 2$

ii) marginal pdf of X & Y

Marginal pdf of X

$f(x) = \int_y f(x,y)\, dy$

$= \int_1^2 \frac{1}{2} (x+y)\, dy$

$= \frac{1}{2} \left\{ \left(xy + \frac{y^2}{2} \right)_1^2 \right\}$

$= \frac{1}{2} \left[x(2) + \frac{4}{2} - x - \frac{1}{2} \right] = \frac{1}{2}\left(x + 3/2 \right)$

$= \frac{1}{2} \left[x(2-1) + \frac{1}{2}(2^2 - 1^2) \right]$

$= \frac{1}{2}(x + 3/2)$ # ; $0 < x < 1$

Now
marginal pdf of y.

$f(y) = \int_x f(x,y)\,dx$

$= \int_0^1 \frac{1}{2}(x+y)\,dx$

$= \frac{1}{2}\left[\frac{x^2}{2} + xy\right]_0^1$

$= \frac{1}{2}\left[\frac{1}{2}(1^2 - 0^2) + y(1-0)\right]$

$= 1/2\left(\frac{1}{2} + y\right)$

iii) Conditional pdf of $f(x/y)$

$= \frac{f(x,y)}{f(y)}$

$= \frac{\frac{1}{2}(x+y)}{1/2(1/2 + y)}$

$= \frac{1/2(x+y)}{1/2(1/2+y)} = \frac{2(x+y)}{(1+2y)}$ #

iv) conditional pdf of $f(y/x)$

$= \frac{f(x,y)}{f(x)}$

$= \frac{1/2(x+y)}{1/2(x+3/2)} = \frac{2(x+y)}{(2x+3)}$ #

v) $f(x,y) = f(x) \cdot f(y)$.

$= \frac{1}{2}\left(x + \frac{3}{2}\right) \times \frac{1}{2}\left(\frac{1}{2} + y\right)$

$= \left(\frac{x}{2} + \frac{3}{4}\right) \times \left(\frac{1}{4} + \frac{y}{2}\right)$

$f(x,y) \neq f(x) \cdot f(y)$.

Not independent r.v.

vi) Joint cummulative distn function:

$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx$

$= \int_0^x \int_1^y \frac{1}{2}(x+y)\,dy\,dx$

$= \frac{1}{2}\int_0^{\infty}\left(xy + \frac{y^2}{2}\right)_1^y dx$

$= \frac{1}{2}\int_0^x \left[x(y-1) + \frac{1}{2}(y^2 - 1)\right]$

$= \frac{1}{2}\int_0^x \left(xy - x + \frac{y^2}{2} - \frac{1}{2}\right)dx$

$= \frac{1}{2}\left(\frac{x^2 y}{2} - \frac{x^2}{2} + \frac{y^2 x}{2} - \frac{1}{2}x\right)_0^x$

$= \frac{1}{4}\left(y(x^2 - 0^2) - (x^2 - 0^2) + y^2(x-0) - (x-0)\right)$

$= 1/4\left[x^2 y - x^2 + xy^2 - x\right]$ //

vii) $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

Finding $E(X) = \int_x x f(x)\,dx$.

$= \int_x x\left\{\frac{1}{2}(x + 3/2)\right\}dx$

$= \int_0^1 \left(\frac{x^2}{2} + \frac{3x}{4}\right)dx$

$= \frac{1}{2}\int_0^1 \left(x^2 + \frac{3x}{2}\right)dx$

$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{3x^2}{4}\right]_0^1$

$= \frac{1}{2}\left[\frac{1}{3}(1^3 - 0^3) + \frac{3}{4}(1^2 - 0^2)\right]$

$= 1/2\left(\frac{1}{3} + 3/4\right)$

$= 1/2\left(\frac{4+9}{12}\right) = 13/24$

Similarly,

$E(Y) = \int_y y f(y)\,dy$

$= \int_1^2 y \frac{1}{2}\left(\frac{1}{2} + y\right)dy$

$= \frac{1}{2}\int_1^2 \left(\frac{y}{2} + y^2\right)dy$

$= \frac{1}{2}\left[\frac{y^2}{4} + \frac{y^3}{3}\right]_1^2$

Binomial Distn

i) $X = 0, 1, 2, 3, \ldots n$

ii) $P(X=0) \to$ prob of no success

iii) * $P(X=1) \to$ prob of one "

* $P(X=2) \to$ " " two "

* $P(X \geq 1) \to$ prob of atleast one success

* $P(1 \leq X \leq n) \Rightarrow$

* $P(X \leq 2) \to$ prob of atmost two success

$\Downarrow$

which mean $P(0 < X < 2)$

* $P(X = 1 \cup X = 2) \to$ prob of one or two success

$\Downarrow$

$= P(X=1) + P(X=2)$

iv) Pmf :

$$\boxed{P(X=r) = nC_r\, p^r q^{n-r}},$$

$r = 0, 1, 2, 3, \ldots, n$ event

where n = no. of trials ; r = no. of success,

p = prob of success

q = prob of failure

Poisson Distn

i) $X = 0, 1, 2, 3, \ldots \infty$

ii) $P(X \neq 0) \Rightarrow$

iii) pmf :-

$P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}$; $r = 0, 1, 2, \ldots \infty$

where -

λ = avg/mean occurrence of event per unit time

Note :

if $n \geq 20$,
$p \leq 0.05$ then
$\lambda = np$.

where

n = no. of trial

p = prob of success

Imp. Note.

Binomial Distⁿ	Poission Distⁿ
(i) mean and variance	Mean and variance.
mean = $E(X) = n \cdot p$	mean = $E(X) = \lambda$
variance = $V(X) = npq$	variance = $V(X) = \lambda$
standard deviation = $\sqrt{V(X)}$	SD = $\sqrt{V(X)}$

$p(x \geq 1)$

2 (b)

Soln

Given,

$\lambda = 0.6$ calls per minute

where,

$$P(X=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}$$

$$\therefore P(X=r) = \frac{e^{-0.6}(0.6)^{r}}{r!}; \quad r = 0,1,2,\ldots \infty$$

Where,

X = no. of calls per minute

i) P(atleast one call per minute)

$= P(X \geq 1)$; $r = 1, 2, 3, \ldots, \infty$

$= 1 - P(X=0)$

$$= 1 - \frac{e^{-0.6}(0.6)^{0}}{0!}$$

$= 0.45$

ii) Here,

$\lambda = 0.6$

In 1 min, 0.6 calls
In 4 min, 0.6×4 calls
$= 2.4$ calls

$\therefore P(X=r) = \dfrac{e^{-2.4}(2.4)^r}{r!}$; $r = 0, 1, 2, \ldots, \infty$

$\therefore$ P(atleast 3 calls per 4 minutes)

$= P(X \geq 3)$; $r = 3, 4, 5, 6, \ldots \infty$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \dfrac{e^{-2.4}(2.4)^0}{0!} - \dfrac{e^{-2.4}(2.4)^1}{1!} - \dfrac{e^{-2.4}(2.4)^2}{2!}$

$=$

(≥ 4)

iii) atleast 4 calls in 4-min intervals.

2015 (fall)

3 (a)

Soln

Given,

no. of digits $(n) = 10$

prob of having one $(p) = 0.3$

prob of not having one $(q) = 1 - p$ $\quad P(x=3)$

$= 1 - 0.3$

$= 0.7$

${}^{n}C_{r}\, p^{r}\, q^{n-r}$ $\quad P(x = 3)$

${}^{10}C_{3}\, (0.3)^{3}\, (0.7)^{10-3}$

We have,

$P(X=r) = {}^{n}C_r\, p^r q^{n-r}$

$\therefore P(X=r) = {}^{10}C_r\,(0.3)^r\,(0.7)^{10-r}$; $r = 0, 1, 2, \ldots, 10$

where, X = no. of message having one in digit of message

i) P(three ones)

$= P(X=3)$

$= {}^{10}C_3\,(0.3)^3\,(0.7)^{10-3}$

$= 0.26$

ii) P(both 2 and 4 ones) (inclusive)

$= P(2 \le X \le 4)$

$= P(X=2) + P(3) + P(4)$

$= {}^{10}C_2\,(0.3)^2\,(0.7)^{10-2} + {}^{10}C_3\,(0.3)^3\,(0.7)^{10-3} + {}^{10}C_4\,(0.3)^4\,(0.7)^{10-4}$

$= 0.23 + 0.26 + 0.2 = 0.69$

iii) P(no less than 2 zeros)

= P(2 or more zeros)

Here,

$n = 10$

prob of having zeros $(p) = 0.7$

prob of not having zeros $(q) = 0.3$

$\therefore P(X = r) =$

$= P(X \geq 2)$, $r = 2, 3, 4, \ldots, 10$.

$= 1 - P(X=0) - P(X=1)$

$= 1 - {}^{10}C_0\,(0.7)^0\,(0.3)^{10-0} - {}^{10}C_1\,(0.7)^1\,(0.3)^{10-1}$

$= 1 - (5.9 \times 10^{-6}) - (1.3 \times 10^{-4})$

$= 0.99$ #

2013 (Fall)

1 min → 4/10 calls

3
(b)

Soln

Given

avg no. of calls (λ) = 4 calls in 10 min.

∴ In 1 min $= \frac{4}{10}$ calls

$= \frac{2}{5}$ calls per minute.

We know,

$$P(X=r) = \frac{e^{-\lambda} \lambda^{r}}{r!}$$

$$\therefore P(X=r) = \frac{e^{-4} (4)^{r}}{r!}$$

i) P(no calls in 10 min)

$= P(X=0)$

$= \frac{e^{-4}(4)^0}{0!} = 0.01$

ii) P(atmost 2 calls in 5 min interval)

Since,

~~In 10 minutes~~

λ = 4 calls in 10 min

= 1 calls in $\frac{10}{4}$ min

= 2 calls in 5 min

Also

Since,

In 10 min → 4 calls

In 1 min → $\frac{4}{10}$ calls = 0.4 call per min

In 5 min → $\frac{4}{10} \times 5$ calls

= 2 calls in 5 min.

$= P(X \le 2)$, $r = 0, 1, 2$.

$= P(X=0) + P(X=1) + P(X=2)$.

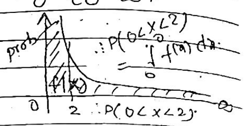
$= P(0) + P(1) + P(2)$.

$= 0.14$ //

195

Exponential distribution (Negative Exponential)

- It is continuous.
- It includes values from 0 to ∞.
  ie $0 < X < \infty$



- Its pdf:

$$\boxed{f(x) = \lambda e^{-\lambda x}} ; \quad x = 0, 1, 2, \ldots, \infty$$

where, λ = positive real number (parameter).

→ mean and variance.

mean $= E(X) = \dfrac{1}{\lambda}$

variance $= V(X) = \dfrac{1}{\lambda^2}$

$SD = \sqrt{V(X)}$

value of (x) increases, prob decreases so it is called negative exponential distribution. Eg: $\boxed{\text{machine life time}}$

2016 (fall)

2 (a) The time required to repair a machine is exponentially distributed with parameter ½. What is the prob that a repair time exceeds 2 hours? What is the conditional prob that a repair time takes atleast 10 hours given that its duration exceeds 9 hours?

Soln

Given, $\lambda = \frac{1}{2}$

We have,

$f(x) = \lambda e^{-\lambda x}$

$f(x) = \frac{1}{2} e^{\left(-\frac{1}{2}\right)x}$; $0 < x < \infty$

i) P(exceeds two hours)

$= P(X > 2)$

$= P(2 < X < \infty)$

$= \int_2^\infty f(x)\,dx$

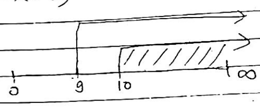
$= \int_0^\infty \frac{1}{2} e^{\left(-\frac{1}{2}\right)x}\,dx$

=

ii) P(atleast 10 hours / exceeds 9 hours)

$= P\left(X \geq 10 / X > 9 \right)$

$= \dfrac{P(X \geq 10 \cap X > 9)}{P(X > 9)}$



$= \dfrac{P(X \geq 10)}{P(X > 9)}$

for $P(X \geq 10) = \int_{10}^{\infty} f(x)\, dx$

$= \int_{10}^{\infty} \frac{1}{2} (e^{-x/2})\, dx$

$= \left[\dfrac{\frac{1}{2} e^{-x/2}}{-1/2} \right]_{10}^{\infty} = \frac{1}{2} \times 2 \left(e^{-x/2} \right)^{\infty}$

$= - \left[e^{-\infty/2} - e^{-10/2} \right] = - (0 - e^{-5})$

$= \left[e^{-5} \right] = 6.7 \times 10^{-3}$

and,

for $P(X > 9) = \int_{9}^{\infty} f(x)\, dx$

$= \int_{9}^{\infty} f(x)\, dx$

$= \int_{9}^{\infty} \frac{1}{2} e^{-\frac{1}{2}x}\, dx$

$= \frac{1}{2} \int_{9}^{\infty} e^{-1/2 x}\, dx$

$= \frac{1}{2} \left[\dfrac{e^{-\frac{1}{2}x}}{-1/2} \right]_{9}^{\infty}$

$= \frac{1}{2} \times -2 \left(e^{-x/2} \right)_{9}^{\infty}$

$= - \left(e^{-\frac{\infty}{2}} - e^{-\frac{9}{2}} \right)$

$= - (0 - e^{-4.5}) = e^{-4.5} = 0.1111$

Thus,

$\dfrac{P(X \geq 10)}{P(X > 9)}$

$= \dfrac{6.7 \times 10^{-3}}{0.111}$

$= 0.6$

Imp

Memoryless property of exponential distribution / Markovian property :-

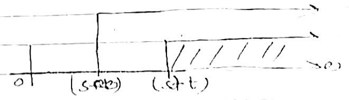
Statements :- For any two real number 's' and 't' we have,

$$P\left[X \geq (s+t) / X \geq s\right] = P(X \geq t).$$

Proof :

$$LHS = P\left[X \geq (s+t) / X \geq s\right]$$

$$= \frac{P[X \geq (s+t) \cap (X \geq s)]}{P(X \geq s)}$$



$$= \frac{P[X \geq (s+t)]}{P(X \geq s)}$$

$$= \frac{\int_{(s+t)}^{\infty} f(x)\,dx}{\int_s^{\infty} f(x)\,dx}$$

$$= \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x}\,dx}{\int_s^{\infty} \lambda e^{-\lambda x}\,dx}$$

$$= \frac{\left[\frac{e^{-\lambda x}}{-\lambda}\right]_{s+t}^{\infty}}{\left[\frac{e^{-\lambda x}}{-\lambda}\right]_s^{\infty}} = \frac{\frac{1}{\lambda}\left[e^{-\lambda x}\right]_{\infty}^{s+t}}{\frac{1}{\lambda}\left[e^{-\lambda x}\right]_{\infty}^{s}}$$

$$= \frac{e^{-\lambda(s+t)} - e^{-\infty}}{e^{-\lambda s} - e^{-\infty}} = \frac{e^{-\lambda(s+t)} - 0}{e^{-\lambda s} - 0} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda s - \lambda t + \lambda s} = e^{-\lambda t}$$

$$RHS = P[X \geq t]$$

$$= \int_t^{\infty} f(x)\,dx$$

$$= \int_t^{\infty} \lambda e^{-\lambda x}\,dx$$

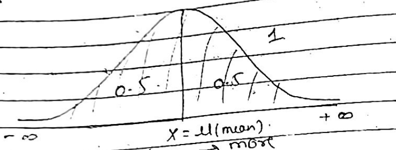
$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_t^{\infty}$$

$$= \left[e^{-\lambda x}\right]_{\infty}^{t}$$

$$= e^{-\lambda t} - e^{-\infty} = e^{-\lambda t}$$

Hence, LHS = RHS proved it.

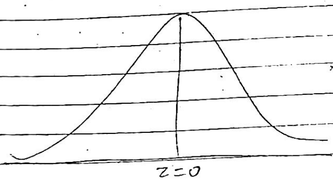
Normal Distribution



$P(X < \mu) = 0.5 \Rightarrow P(-\infty < X < \mu)$

$P(X > \mu) = 0.5 \Rightarrow P(\mu < X < +\infty)$.

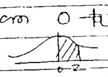
#



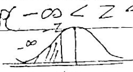
$\boxed{z = \frac{X-\mu}{\sigma}}$

z = SNV (standard normal variate)

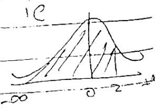
① Table 3a gives prob starts from 0 to z, ie $P(0 < z < z_1)$



② Table 3b give prob from $P(-\infty < z < -1.5)$



③ Table 3c gives prob from $-\infty$ to z ie $P(-\infty < z < 1.25)$

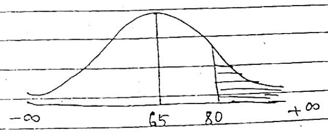


① If mean and S.D of marks obtained by student in pqt are 65 and 12 respectively, what is the prob that a randomly selected stdnt will get marks:-
i) more than 80
ii) less than 50
iii) less than 78
iv) betn 65 to 82.

Soln

Given,
mean(μ) = 65
SD(σ) = 12

(i) more than 80
$= P(X > 80)$



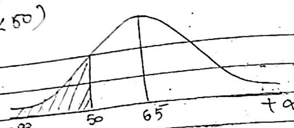
when X=80,

$z = \frac{X-\mu}{\sigma} = \frac{80-65}{12} = 1.25$

$= P(z > 1.25)$
$= P(1.25 < z < +\infty)$
$= 0.10565$

ii) less than 50

$= P(X < 50)$



When $X = 50$,

$Z = \frac{X-\mu}{\sigma} = \frac{50-65}{12} = -1.25$

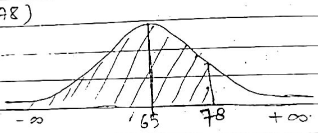
$= P(Z < -1.25)$

$= P(-\infty < Z < -1.25)$

$= 0.10565$

iii) less than 78

$= P(X < 78)$



When $X = 78$,

$Z = \frac{X-\mu}{\sigma} = \frac{78-65}{12} = 1.083$

$= P(X < 1.083)$

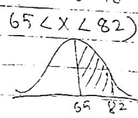
$= P(Z < 1.083)$

$= P(-\infty < Z < 1.083)$

$= 0.85973$

iv) betn 65 to 82

$= P(65 < X < 82)$



When $X = 65$, $Z = \frac{65-65}{12} = 0$

When $X = 82$, $Z = \frac{82-65}{12} = 1.41$

$= P(0 < Z < 1.41)$

$= \cancel{0.420323}$

$= 0.07929$

② If mean and SD of wt in a class are 70 kg and 9 kg respectively, find the prob that a randomly selected student will ~~have~~ has weight:

i) less than 55 kg
ii) more than 82 kg
iii) less than 90 kg
iv) betn 70 to ~~...~~ 95 kg.

Soln

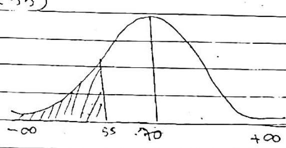
Given

mean $(\mu) = 70$

S.D $(\sigma) = 9$.

(i) less than 55 kg

$= P(X < 55)$



When $X = 55$,

$Z = \frac{X-\mu}{\sigma} = \frac{55-70}{9} = -1.66$

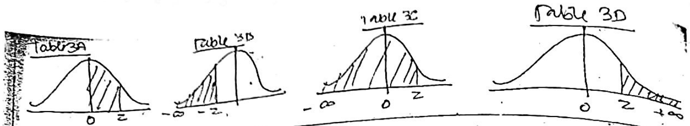
$= P(X < -1.66)$

$= P(Z < -1.66)$

$= P(-\infty < Z < -1.66)$

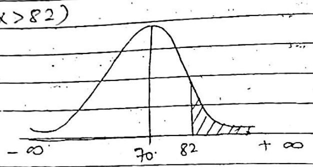
$= 0.04846$ ''

205



ii) more than 82 kg

$= P(X > 82)$



when $X = 82$,

$$Z = \frac{X-\mu}{\sigma} = \frac{82-70}{9} = 1.33$$

$= P(X > 1.33)$

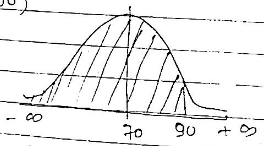
$= P(Z > 1.33)$

$= P(1.33 < Z < +\infty)$

$= 0.09176.$

iii) less than 90 kg.

$= P(X < 90)$



$=$ when $x = 90$,

$$Z = \frac{X-\mu}{\sigma} = \frac{90-70}{9} = 2.22$$

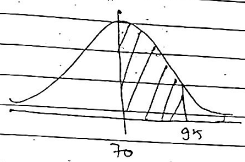
$= P(X < 2.22)$

$= P(Z < 2.22)$

$= P(-\infty < Z < 2.22) = 0.98679$ #

iv) P(betn 70 to 95 kg)

$= P(70 < X < 95)$



When $X = 70$,

$$Z = \frac{70-70}{9} = 0$$

When $X = 95$,

$$Z = \frac{95-70}{9} = 2.77$$

$= P(0 < X < 2.77)$

$= P(0 < Z < 2.77)$

$= 0.4972$ #

2014 (spring)

3 (b)

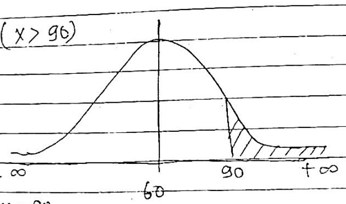
Solution –

Given

mean(μ) = 60

S.D (σ) = 20.

i) $P(X > 90)$



When $X = 90$,

$$Z = \frac{X-\mu}{\sigma} = \frac{90-60}{20} = 1.5$$

$= P(X > 1.5)$

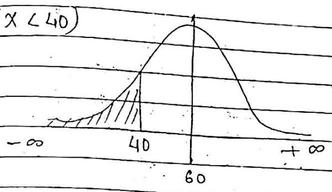
$= P(Z > 1.5)$

$= \cancel{P(90 < X < +\infty)}$ $\quad P(1.5 < Z < \alpha)$

$= P(1.5 < Z < +\infty)$

$= 0.06681$ #

ii) $P(X < 40)$



When $X = 40$,

$$Z = \frac{X-\mu}{\sigma} = \frac{40-60}{20} = -1$$

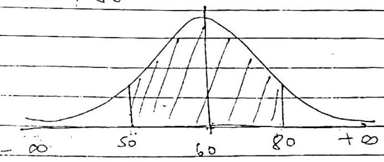
$= P(X < -1)$

$= P(Z < -1)$

$= P(-\infty < Z < -1)$

$= 0.15866.$

iii) Betn 50 & 80



When $X = 50$,

$$Z = \frac{X-\mu}{\sigma} = \frac{50-60}{20} = -0.5$$

When $X = 80$

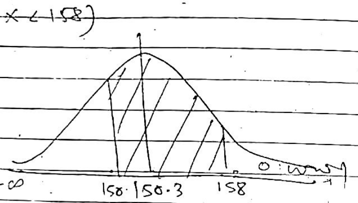
$$Z = \frac{80-60}{20} = 1.$$

$= P(X > 1.54)$

$= P(X > 1.54) =$

(iii) betn 150 & 158

$= P(150 < X < 158)$



When $X = 150$,

$$Z = \frac{X-\mu}{\sigma} = \frac{150-150.3}{5} = -0.06$$

When $X = 158$,

$$Z = \frac{158-150.3}{5} = 1.54$$

$= P(-0.06 < X < 21.54)$

$= P(-0.06 < Z < 21.54)$

$= P(-\infty < Z < -0.06) + P(-\infty < Z < 21.54)$

$= 0.47608 + 0.93822 = 1.414$

If mean and variance of marks obtained by stdnts in a class are 35 and 49 respectively find the prob that a stdnt will get mark:

i) betn 20 and 42

ii) betn 40 to 53

iii) betn 18 to 25

iv) more than 22.

Soln: Assume that distn of mark is normal

Given

mean $(\mu) = 35$

variance $(V(X)) = 49$

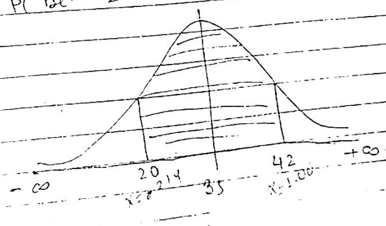
$S.D(\sigma) = \sqrt{V(X)}$

popn mean $(\mu) = 35$

popn variance $(\sigma^2) = 49$

S.D $(\sigma) = \sqrt{\sigma^2} = \sqrt{49} = 7$

i) P(betn 20 and 42)



When $X = 20$,

$Z = \frac{X-\mu}{6}$

$= \frac{20-35}{7} = -2.14$

When $X = 42$,

$Z = \frac{X-\mu}{6}$

$= \frac{40-35}{7} = 1.00$

$= P(-2.14 < \cancel{Z} X < 1.00)$

$= P(-2.14 < Z < 1.00)$

$= \cancel{P(-\infty < Z < -2.14)}$

$= 1 - P(-\infty < Z < -2.14) - P(1.00 < Z < +\infty)$

$= 1 - 0.01618 - 0.15866$

$=$

ii) $P(40 \text{ to } 53)$

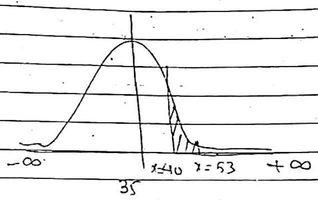
$= P(40 < X < 53)$

When $X = 40$,

$Z = \frac{X-\mu}{6}$

$= \frac{40-35}{7}$

$= 0.71$



When $X = 53$,

$Z = \frac{53-35}{7} = 2.57$

$= P(0.71 < Z < 2.57)$

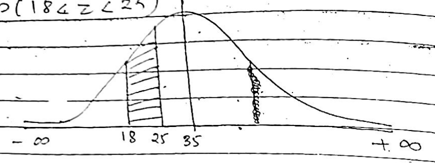
$= 1 - P(-\infty < Z < 0.71) - P(2.57 < Z < \infty)$

$= 1 - 0.76115 - 0.00508$

$= 0.23377$ //

(iii) betn 18 to 25

$= P(18 < z < 25)$



When $X = 18$,

$Z = \frac{X-\mu}{\sigma} = \frac{18-35}{7} = -2.428$

When $X = 25$,

$Z = \frac{25-35}{7} = -1.428$

$= P(-2.428 < X < -1.428)$

$= P(-2.428 < Z < -1.428)$

$= 1 - P(-\infty < Z < -2.438) - P(-1.438 < Z < +\infty)$

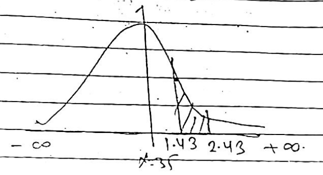
$= 1 - 0.00755 - P(-\infty < Z < 1.43)$ By symmetric property

$= 1 - 0.00755 - 0.92364$

OR −

$= P(-2.43 < Z < -1.43)$

$=$



$= P(1.43 < Z < 2.43)$

$= \cancel{P(0<Z<1.43) - P(0<}$

$= P(0 < Z < 2.43) - P(0 < Z < 1.43)$

(iv) P(more than 22)

$= P(X > 22)$

When $X = 22$

$Z = \frac{22-35}{7}$

$= -1.86$

$= P(Z > -1.86)$

$= 1 - P(-\infty < Z < -1.86)$

$= 1 - 0.03144$

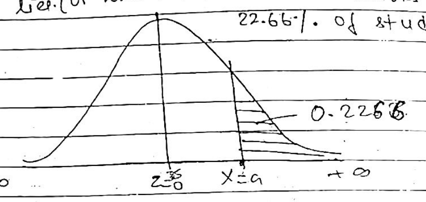
$= 0.96856$ #

Continue question from last question —

If mean and variance of marks obtained by students in a class are 35 and 49 respectively. Find the prob that a student will get marks:-

(i) Find the mark above which 22.66% of student lies. (or what is the minimum marks of top 22.66% of students)

Soln



$P(X > a) = 0.2266$ — (1)

From normal distn table (3D):-

$Z = 0.75$

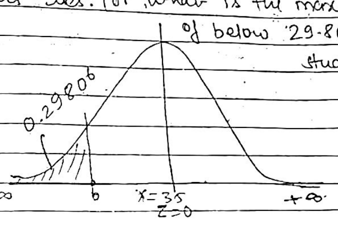
or, $\dfrac{X-\mu}{\sigma} = 0.75$

or, $\dfrac{a-35}{7} = 0.75$

or $a = 35 + 7 \times 0.75$

$\therefore a = 40.25$

(ii) Find the marks below which 29.806% of students lies. (or what is the maximum marks of below 29.806% of students).



$P(X < b) = 0.29806$

or, $P(Z < b) = 0.29806$ — (1)

From normal distn table (3B).

$Z = -0.53$

or, $\dfrac{X-\mu}{\sigma} = -0.53$

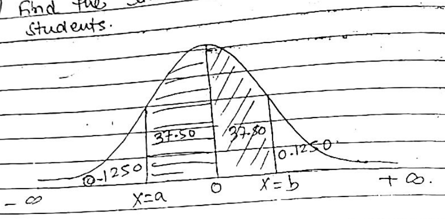
or, $\dfrac{b-35}{7} = -0.53$

or, $\dfrac{b}{7} = 35 + 0.53$

or, $\therefore b = 7(35 + 0.53)$

$\therefore b = 31.29$

iii) Find the limits of middle ~~70~~ ~~75~~ 75% of students.



~~For lower limit at X = a.~~

$P(a < X < b) = 0.75$ — ①

for lower limit at X = a
from normal distn table (3B)
$Z = -1.15$

For upper limit at X = b,
from normal distn table (3D)
$Z = 1.15$.

mean, SD and sample size लिएर गर्ने CLT.

Central Limit Theorem (CLT) :-

Whatever be the distn of popn, sampling distn of sample mean is normal as sample size tends to very large. ($n \geq 30$).

~~normal approximation~~ हो।

$X \sim N$ (follows)

Application of CLT :-

i) To Normal Distn :-

If $X \sim N(\mu, \sigma^2)$ then :

a) $S_n = \Sigma X \sim N(n\mu, n\sigma^2)$ as $n \to \infty$

b) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$

($\bar{X}$: sample mean.)

ii) To Binomial Distribution. (we need to do continuity correction)

If $X \sim B(n, p)$, then

a) $\Sigma X \sim N(np, npq)$ as $n \to \infty$

b) $\bar{X} \sim N(p, \frac{pq}{N})$ as $n \to \infty$

iii) To Poission Distribution - (we need to do continuity correction)

If $X \sim P(\lambda)$, then

a) $\Sigma X \sim N(n\lambda, n\lambda)$ as $n \to \infty$

b) $\bar{X} \sim N(\lambda, \frac{\lambda}{n})$ as $n \to \infty$

4 (a)

Soln

mean(μ) = λ = 3000

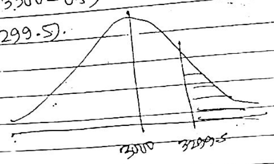
variance($σ^2$) = λ = 3000

SD(σ) = $\sqrt{3000}$ = 54.802

i) P(X ≥ 3300)

= P(X ≥ 3300 − 0.5)

= P(X ≥ 3299.5).



When X = 3299.5,

$Z = \frac{X-\mu}{\sigma}$

$= \frac{3299.5 - 3000}{54.802}$

= 5.465

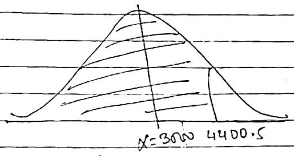
= P(Z ≥ 5.465)

= 0.00005

ii) P(X ≤ 4400)

= P(X ≤ 4400+0.5)

= P(X ≤ 4400.5)



When ~~X=4400~~ X = 4400.5

$Z = \frac{X-\mu}{\sigma}$

$= \frac{4400.5-3000}{54.802}$

= 25.55

= P(Z < 25.55)

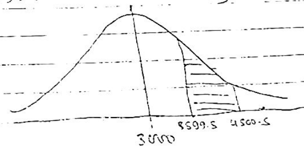
= 0.99995 #

iii) P(3600 ≤ X ≤ 4500)

= P(3600 − 0.5 ≤ X ≤ 4500 + 0.5)

= P(3599.5 ≤ X ≤ 4500.5)

=

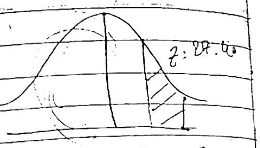


When $X = 3599.5$,

$$Z = \frac{X-\mu}{\sigma}$$

$$= \frac{3599.5 - 3000}{54.802}$$

$$= 10.940$$



When $X = 4500.5$,

$$Z = \frac{X-\mu}{\sigma}$$

$$= \frac{4500.5 - 3000}{54.802}$$

$$= \cancel{27.8}\ 27.405 \quad 1 - [P(-\infty < Z < 10.95) + (27.4 < Z < +\infty)]$$

$$= P(10.940 \le Z \le 27.405)$$

$$= P(0 \le Z \le 27.405) - P(0 \le Z \le 10.940)$$

$$=$$

Chebyshev's Inequality :-

A random variable 'X' has mean = E(X) and variance = V(X) then

$P\{|X - E(X)| < K\} \geq 1 - \frac{V(X)}{K^2}$, It gives lower ~~upper~~bound of prob (It gives atleast prob)

and

$P\{|X - E(X)| \geq K\} \leq \frac{V(X)}{K^2}$, It gives upper bound of prob (It gives atmost prob)

where K = positive real number.

Q Let rv 'X' has mean of 11 and variance of 9 then find:

(i) lower bound of prob. $P(6 < X < 16)$

soln

Given $E(X) = 11$

$V(X) = 9$

We have lower bound of prob from Chebyshev's Inequality:

$$P\{|X - E(X)| < K\} \geq 1 - \frac{V(X)}{K^2}$$

$$\therefore P\{|X - 11| < K\} \geq 1 - \frac{9}{K^2}$$

$$\boxed{\begin{array}{c}|x| < a \\ -a < x < a\end{array}}$$

or, $P\{-K < X-11 < +K\} \geq 1 - \frac{9}{K^2}$

or, $P\{11-K < X < 11+K\} \geq 1 - \frac{9}{K^2}$

Put $K = 5$,

$P\{11-6 < X < 11+6\} \geq 1 - \frac{9}{5^2}$

or, $P(6 < X < 16) \geq \frac{16}{25}$

ii) upperbound of prob $(6 < x < 16)$

We have from upperbound of prob from Chebyshev's inequality,

$P\{|X-E(X)| \geq K\} \leq \frac{V(X)}{K^2}$

or, $P\{|X-11| \geq K\} \leq \frac{9}{K^2}$

or, ~~$P\{-K < (X-11)$~~

or, ~~$P\{-K < (X-11) \geq K\} \leq \frac{9}{K^2}$~~

$|x| < a$

$-a < x < a$

or P Put $K = 5$,

upperbound $= \frac{V(X)}{K^2}$

$= \frac{9}{(5)^2}$

$= \frac{9}{25}$

OR

Note

$K = |X - E(X)|$

$= |16-11| = 5$

lower bound $= 1 - \frac{V(X)}{K^2}$

$= 1 - \frac{9}{25}$

and upperbound $= \frac{V(X)}{K^2}$

→ asked not outcome and ; getting odd or even also BLLN

BLLN

Let 'x' be bernoulli trial with prob of success 'p' and prob of failure 'q'. then

$$P\left\{\left|\frac{x}{n}-p\right|<K\right\} \geq 1-\frac{pq}{nK^2}$$; it gives lowerbound of prob (atleast)

and

$$P\left\{\left|\frac{x}{n}-p\right|\geq K\right\} < \frac{pq}{nK^2}$$; it gives upperbound of prob (atmost)

where, K = positive real number

$\frac{x}{n}$ = proportion of success

To find value of K –

$$K=\left|\underbrace{\frac{x}{n}}_{\text{observed freq}}-\underbrace{p}_{\text{theoretical freq/expected freq}}\right|$$

For coin toss, $p=\frac{1}{2}$, $q=\frac{1}{2}$

For dice roll, $p=\frac{1}{6}$, $q=\frac{5}{6}$

2013(spring), 2016(fall)

3(a) How many times would you have to roll a fair dice in order to be atleast 99% sure that the relative freq of having a six come up is within 0.02 of the theoretical prob $1/6$

soln

Given

$p = 1/6$

$q = 5/6$

$K = \left|\frac{x}{n}-p\right| = 0.02$

lowerbound = 99% = 0.99

we have,

lowerbound $= 1-\frac{pq}{nK^2}$

or, $0.99 = 1-\frac{\frac{1}{6}\times 5/6}{n\times(0.02)^2}$

or, $0.99 = 1-\frac{0.166\times 0.833}{n\times 4\times10^{-4}}$

or $0.99 = \frac{0.862}{n\times4\times10^{-4}}$

$\therefore n =$ ~~[illegible]~~ 34722.22

2015 fall 3b)

2018 (fall)

3 (b)

Soln

X	-1	0	1
$P(X=x)$	$1/8$	$3/4$	$1/8$

Expected value of rv X is given by:-

$$E(X) = \sum_{x=-1,0,1} x \cdot p(x)$$

$$= (-1)(1/8) + 0 \times 3/4 + 1 \times 1/8$$

$$= 0$$

And,

~~variances~~, $E(X^2) = \sum_{x=-1,0,1} x^2 p(x)$

$$= (-1)^2 \times \frac{1}{8} + 0 + (1)^2 \times \frac{1}{8}$$

$$= \frac{1}{4}$$

We have,

$$V(X) = E(X^2) - E(X)$$

$$= \frac{1}{4} - 0 = \frac{1}{4}$$

Now, ~~from~~ ~~[illegible]~~

$$P\{|X - E(X)| \geq 1\}$$

$$= P\{|X - 0| \geq 1\}$$

$$= P\{|X| \geq 1\}$$

$$= 1 - P\{|X| < 1\}$$

$$= 1 - P\{-1 < X < 1\}$$

$$= 1 - P(X=0) = 1 - \frac{3}{4} = \frac{1}{4}$$

From chebyshev's :-

Here, $K = 1$,

$$\text{upperbound} = \frac{V(X)}{K^2}$$

$$= \frac{1/4}{1^2}$$

$$= \frac{1}{4}$$

$\therefore P\{|X - E(X)| \geq 1\} \leq \frac{1}{4}$

The probability obtained by chebyshev's inequality is equal to or is less than the actual proto.

Markov Chain

om distⁿ ... time same path dist^n ... give traffic (cars), arrival of train hint

Uniform Distribution: (Rectangle Distⁿ).

The pdf:

$$\boxed{f(x) = \frac{1}{b-a} \; ; a < x < b}$$

mean:

$$E(x) = \frac{b+a}{2}$$

variance

$$V(x) = \frac{(b-a)^2}{12}$$

2016 (Fall) 4b

Q. If a rv 'x' has uniform distⁿ over [a,b]. Find mean and variance of rv 'x'.

Soln

We know pdf of uniform distⁿ over [a,b]

$$f(x) = \frac{1}{b-a} \; ; a \le x \le b$$

mean:

$$E(x) = \int_a^b x f(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

Continuous → integration
Discrete → summation.

$$= \frac{1}{b-a} \int_a^b x \, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^2}{2} - \frac{a^2}{2} \right]$$

$$= \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{1}{b-a} \times \frac{(b+a)(b-a)}{2} = \frac{b+a}{2}$$

again,

$$E(x^2) = \int_x x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b x^2 \, dx$$

$$= \frac{1}{b-a} \left| \frac{x^3}{3} \right|_a^b$$

$$= \frac{1}{b-a} \left(\frac{b^3 - a^3}{3} \right)$$

$$= \frac{1}{(b-a)} \frac{(b-a)(b^2+ab+a^2)}{3}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\therefore V(X) = E(X^2) - E(X)$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{4(b^2+ab+a^2) - 3(b+a)^2}{12}$$

$$= \frac{4b^2+4ab+4a^2 - 3(b^2+2ab+b^2)}{12}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3b^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

2017 (spring)

3(a)

Soln

8:00	8:12'
↓	↓
$a=0$	$b=12$.

Here $a=0,\ b=12$ minute

$$f(x) = \frac{1}{b-a} = \frac{1}{12-0} = \frac{1}{12};\ 0<x<12$$

Now,

$$P(4<X<8)$$

$$= \int_4^8 f(x)\,dx$$

$$= \int_4^8 \frac{1}{12}\,dx$$

$$= \frac{1}{12}\left[x\right]_4^8$$

$$= \frac{8-4}{12} = \frac{4}{12} = \frac{1}{3}$$

Q How many days he leaves for work i.e betn 8:04 to 8:08.

Soln

No. of days out of 220 days $= \frac{1}{3} \times 220$

$= 73.33$ days.

Gamma Distribution (Question on direct Gamma नोट)

(i) Its pdf :

$$f(x) = \frac{1}{\beta^{\alpha} \Gamma\alpha} e^{-x/\beta} . x^{\alpha-1} \quad ; \ 0 < x < \infty, \ \alpha > 0, \ \beta > 0$$

Note $\Gamma\alpha = (\alpha-1)!$

(ii) mean :

$$E(X) = \alpha\beta$$

(iii) variance : $V(X) = \alpha \beta^2$

3 (c)

Soln

Given

$\alpha = 6, \ \beta = 4$

from Gamma distn

$$f(x) = \frac{1}{\beta^{\alpha} \Gamma\alpha} e^{-x/\beta} x^{\alpha-1}$$

$$= \frac{1}{4^6 (\alpha-1)!} e^{-x/4} x^{6-1}$$

$$= \frac{1}{4^6 (6-1)!} e^{-x/4} x^5$$

$$= \frac{1}{4096 \times 120} e^{-x/4} x^5$$

$$= \frac{1}{491520} e^{-x/4} x^5 \quad ; \ 0 < x < \infty$$

Binomial
Poisson
Normal
Exponential
Uniform

Now,
$P(X \leq 24 \text{ months})$

$= P(0 < X \leq 24)$

$= \int_0^{24} f(x)\,dx$

$= \int_0^{24} \frac{1}{491520} e^{-x/4} x^5\,dx$

=

ii) mean = $E(X) = \alpha\beta$
$= 6 \times 4$
$= 24$ months.

$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}$

Beta Distribution:-

(i) Its pdf:
$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}$; $0<x<1$
$\alpha > 0$
$\beta > 0$.

Note: $B(\alpha,\beta) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}$

(ii) mean
$E(X) = \frac{\alpha}{\alpha+\beta}$

(iii) Variance
$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

2016(spring)

Markov Chain (M.C)

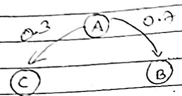


Fig: Trasitional Diagram

Three states {A, B, C}

$$p(1) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.2 & 0 & 0.8 \\ 0.4 & 0.6 & 0 \end{bmatrix} \end{array}$$

It is called tpm (transitional prob matrix)

i) square matrix
ii) row sum is 1.

Notation :

~~P X=B~~

i) $P\left(X_{t+1} = A / X_t = A\right) = 0$

$P\left(X_{t+1} = B / X_t = A\right) = 0.7$

$P\left(X_{t+1} = C / X_t = A\right) = 0.3$

$P\left(X_{t+1} = A / X_t = B\right) = 0.2$

$P\left(X_{t+1} = B / X_t = B\right) = 0$

$P\left(X_{t+1} = C / X_t = B\right) = 0.8$

Two steps tpm :

$p(2) = P \cdot P$

$$= \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.2 & 0 & 0.8 \\ 0.4 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.2 & 0 & 0.8 \\ 0.4 & 0.6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.26 & 0.18 & 0.56 \\ 0.32 & 0.62 & 0.06 \\ 0.12 & 0.28 & 0.60 \end{bmatrix}$$

Notation

$P\left(X_{t+2} = A / X_t = A\right) = 0.26$

prob that X_{t+2} takes the state A when X_t takes the state A.

$P\left(X_{t+2} = B / X_t = A\right) = 0.18$

Three steps tpm :-

$$p(3) = P \cdot P \cdot P$$

$$= \begin{bmatrix} 0.26 & 0.18 & 0.56 \\ 0.32 & 0.62 & 0.06 \\ 0.12 & 0.28 & 0.60 \end{bmatrix} \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.2 & 0.0 & 0.8 \\ 0.4 & 0.6 & 0 \end{bmatrix}$$

$$= \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.26 & 0.518 & 0.222 \\ 0.148 & 0.26 & 0.592 \\ 0.296 & 0.444 & 0.26 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}$$

Two State MC

Theorem : Let a M.C has two states 0,1 ~~then~~ with ~~n-steps~~ tpm is :-

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}$$

then n-step tpm is obtained as:

$$P(n) = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

If $n \to \infty$ (long run/steady state condition)

$$\lim_{n \to \infty} p(n) = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix}$$

2015 (spring)

5 (a)

Soln

Given

$$P = \begin{array}{c|ccc} & A & B & C \\ A & 0.4 & 0.3 & 0.3 \\ B & 0.6 & 0.1 & 0.3 \\ C & 0.2 & 0.3 & 0.5 \end{array}$$

initial prob distribution vector $(P_0) = \begin{array}{ccc} A & B & C \\ [0.4 & 0.4 & 0.2] \end{array}$

three-step tpm = ?

Here 3-steps tpm (after three purchase)

$P(3) = P \cdot P \cdot P$

$= \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} ($

$$\begin{bmatrix} 0.4 & 0.24 & 0.36 \\ 0.36 & 0.28 & 0.36 \\ 0.36 & 0.24 & 0.4 \end{bmatrix}$$

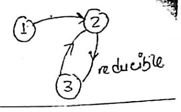
=

Now,

prob distⁿ vector after 3-purchase = $P_0 P(3)$

$[0.3776 \quad 0.2488 \quad 0.3736]$ #

2016 fall 6(b)

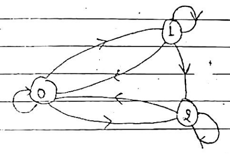


$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{pmatrix} \end{array}$$

Meaning of irreducible MC :- sabai state sabai state ma pugdaino. If all the state are communicable to each other in finite no. of state is said to be MC is irreducible.

To prove :-

Trasitional diagram



Since all state communicate to each other in finite no of states, so the given MC is irreducible.

Long Run

* Steady state probability : (Stable condition)

It is probability of each state in long run ($n \to \infty$)

In steady state condition,

Let, x = steady state prob for state 0.

y = " " " " state 1.

z = " " " " state 2

Step 1 : $[x\ y\ z]_{3\times1} \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix}_{3\times3} = [x\ y\ z]$ — (i)

Step 2 : $x + y + z = 1$ — (ii)

From (i)

$[0.6x + 0.1y + 0.6z \quad 0.2x + 0.8y \quad 0.2x + 0.1y + 0.4z] = [x\ y\ z]$

2013 spring 6(a).

Let $\{X_n, n \geq 0\}$ be MC defined an 3 states 0, 1, 2. Its transition probability matrix is given below.

$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

Let the initial distribution be -

$P_0(X_0 = i) = \frac{1}{3}\ \forall\ ;\ i = 0, 1, 3.$

For all, $P_0 = 1/3,\ P_1 = 1/3,\ P_2 = 1/3$

Obtain (i) $P(X_2 = 1 / X_0 = 2)$

(ii) $P(X_0 = 2, X_2 = 1)$ Note: comma means intersection

(iii) $P(X_0 = 2 / X_2 = 1)$

Note : $P(X_0 = 2 / X_2 = 1)$
→ It is the prob that MC takes state ~~1~~. on 2 initially when MC takes state 1

i) Two steps tpm is given by -

$\therefore P(2) = P.P$

$$= \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.28 & 0.36 & 0.36 \\ 1 & 0.42 & 0.37 & 0.21 \\ 2 & 0.12 & 0.36 & 0.52 \end{array}$$

$\therefore P(X_2 = 1 / X_0 = 2) = 0.36$

(ii) $P(X_0 = 2, X_2 = 1)$

$= P(X_0 = 2 \cap X_2 = 1)$

$= P(X_0 = 2) * P(X_2 = 1 / X_0 = 2)$

$= \frac{1}{3} \times 0.36 = 0.12$

- Revised prob

यदि MC 2 state = 2 ko awastha ma cha bhane, MC 0 step ma 2 ma Tani

(iii) $P(X_0 = 2 / X_2 = 1)$

$= \frac{P(X_0 = 2, X_2 = 1)}{P(X_2 = 1)}$ — (1) → MC takes the state after 2 step

Here

$P(X_2 = 1)$ = initial prob after 2 steps

$P_2 = P_0\, P(2)$

$= [1/3\ \ 1/3\ \ 1/3] \begin{bmatrix} 0.28 & 0.36 & 0.36 \\ 0.42 & 0.37 & 0.21 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$

$= [0.273 \quad 0.36 \quad 0.31]$

$P(X_0 = 1) = 0.36$

$P(X_0 = 2, X_2 = 1)$

$= \frac{P(X_0 = 2 \cap X_2 = 1)}{P(X_0 = 1)}$

Note $P(X_0 = 2 / X_2 = 1)$, it is the prob that MC take the state 2 initially when MC takes the state 1 after 2 steps.

Queue

M → arrival pattern
M → service pattern
1 : No. of servers
∞ → size of waiting room.

Traffic intensity –

$\rho = \frac{\lambda}{\mu} = \frac{\text{mean arrival time}}{\text{inter-service time}}$

I M/M/1 : (∞, FIFO)

(i) n customer in a system

$P_n = (1-\rho)\rho^n$

(ii) zero customer in a system

$P_0 = (1-\rho)$

(iii) avg no. of customer in the system

$L_s = \frac{\rho}{1-\rho}$ $L = \frac{\rho}{1-\rho}$

(iv) avg queue length

$L_q = L_s \rho.$ $L_q = L\rho$

(v) avg time customer spends in system

$W_s = \frac{L_s}{\lambda}$ $W = \frac{L}{\lambda}$

(vi) ~~avg no. of customer in queue waiting for service~~ is –

(vi) avg time a customer spends in queue is

$W_q = \frac{L_s}{\mu}$ $W_q = \frac{L}{\mu}$ $L/\mu.$

(vii) arriving unit has to wait for service $= P_w = \rho$

② $M/M/1 : (N, FIFO)$

i) $$P_n = \begin{cases} \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n ; & 0 \le n \le N, \ \rho \ne 1 \\ \frac{1}{n+1} ; & \rho = 1 \end{cases}$$

ii) $$P_0 = \begin{cases} \left(\frac{1-\rho}{1-\rho^{N+1}}\right) ; & \rho < 1 \\ \frac{\rho - 1}{\rho^{N+1} - 1} & \text{if } \rho > 1 \end{cases}$$

iii) Expected no. of customers in system

$$L = \begin{cases} \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} \\ N/2 ; \text{ if } \rho = 1 \end{cases}$$

iv) Expected no. of customer in queue waiting for service is

$$L_q = L - \rho ; \ \rho \ne 1$$

v) Expected (avg) waiting time of a customer in the system -

$$W = \frac{L}{\lambda(1-P_n)}$$

vi) waiting time of a customer in the queue

$$W_q = W - \frac{1}{\mu}$$

③ $M/M/S : (N, FIFO)$

known as birth-death model

$$\lambda_K = \lambda ; \ 0 \le K \le N$$

$$\mu_K = \begin{cases} K\mu ; & 0 \le K < S & \text{if } K < S = K\mu \\ S\mu ; & S \le K \le N & \text{if } S < K = S\mu \end{cases}$$

i) prob of K customers in the system

$$P_K = \begin{cases} \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K P_0 \ ; \ 0 \le K < S \\ \frac{1}{S!\,S^{K-S}}\left(\frac{\lambda}{\mu}\right)^K P_0 \ ; \ S \le K \le N \end{cases}$$

where

$$P_0 = \left[\sum_{K=0}^{S-1} \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K + \sum_{K=S}^{N} \frac{1}{S!\,S^{K-S}}\left(\frac{\lambda}{\mu}\right)^K\right]^{-1}$$

and $\rho = \frac{\lambda}{S\mu} < 1$

(ii) Avg no. of customer in queue is

$$L_q = \frac{(S\rho)^S \rho}{S!(1-\rho)^2}\left[1-\rho^{N-S+1} - (1-\rho)(N-S+1)\rho^{N-S}\right] P_0$$

(iii) Avg no. of customer in the system

$$L_s = L_q + \left(\frac{\lambda}{\mu}\right)(1-P_N)$$

(iv) Avg time customer spends in system

$$W_s = \frac{L_s}{\lambda(1-P_N)}$$

(v) Avg time customer spends in queue

$$W_q = W_s - \frac{1}{\mu}$$

or $W_q = \frac{L_q}{\lambda(1-P_N)}$

Imp

(4) $M/M/S : (M, GD)$ — server single; limited customers.

$\lambda_n = \lambda(M-n) \ ; \ n = 0,1,2,\dots,M$

~~$\mu_n = \mu \ ; \ n = 0,1,2,\dots,M$~~

(i) $P_0 = \left[\sum_{n=0}^{M} \frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$

(ii) $P_n = \frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 \ ; \ n = 0,1,2,\dots M$

(iii) $L_q = M - \left(\frac{\lambda+\mu}{\lambda}\right)(1-P_0)$

(iv) $L_s = M - \frac{\mu}{\lambda}(1-P_0)$

(v) $W_q = \frac{1}{\mu}\left[\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda}\right]$

(vi) $W_s = \frac{1}{\mu}\left[\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda} - 1\right]$

imp

(5) $M / E_K / 1 : (\infty, FIFO)$

where

E_K = erlang distn

= Gamma Distn

$k\lambda_n = \lambda$ (phases arrive per unit time)

$\mu_n = k\mu$ (phases service per unit time)

(i) Expected no. of phases (not customer) in system is

$$L_s(K) = \frac{K+1}{2} \frac{\lambda}{(\mu-\lambda)\mu}$$

(ii) Expected no. of phases in queue

$$L_q(K) = \frac{L_s(K)}{\mu}$$

(iii) Expected no. of customers in the queue

$$L_q = \left(\frac{K+1}{2K}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$$

(iv) Expected waiting time of a customer in queue,

$$W_q = \frac{L_q}{\lambda} \quad \underline{\underline{OR}} \quad W_q = \frac{K+1}{2K} \frac{\lambda}{\mu(\mu-\lambda)}$$

(v) Expected waiting time of a customer in system,

$$W_s = W_q + \frac{1}{\mu}$$

(vi) Expected no. of customer in system –

$$L_s = L_q + \frac{\lambda}{\mu}$$

imp

(6) $M | M | S : \infty, FIFO$

(i) $$L_q = \frac{\lambda\mu(\lambda/\mu)^s}{(s-1)!} \cdot \frac{P_0}{(s\mu-\lambda)^2}$$

where

$$P_0 = \left[\sum_{K=0}^{s-1} \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{s\mu}{s\mu-\lambda}\right]^{-1}$$

(ii) $$L_s = L_q + \frac{\lambda}{\mu}$$

(iii) $$W_q = \frac{L_q}{\lambda}$$

(iv) $$W_s = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

(v) $$\rho = \frac{\lambda}{s\mu}.$$

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