EXERCISE – 1.1

1. **If** $A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & 5 & -3 \end{bmatrix}$. **Find 3A – 4B.**

Solution: Given matrices are

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & 5 & -3 \end{bmatrix}$$

Now,

$$3A - 4B = 3\begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 0 \\ 12 & 3 & -3 \\ 3 & 6 & 6 \end{bmatrix} - \begin{bmatrix} 20 & 4 & 12 \\ 8 & 4 & 4 \\ -4 & 20 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 9-20 & 6-4 & 0-12 \\ 12-8 & 3-4 & -3-4 \\ 3+4 & 6-20 & 6+12 \end{bmatrix} = \begin{bmatrix} -11 & 2 & -12 \\ 4 & -1 & -7 \\ 7 & -14 & 18 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$. **Show that AB ≠ BA.**

Solution: Given that,

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+3\times1+0\times(-1) & 1\times3+3\times2+0\times1 & 1\times4+3\times3+0\times2 \\ (-1)\times2+2\times1+1\times(-1) & (-1)\times3+2\times2+1\times1 & (-1)\times4+2\times3+1\times2 \\ 0\times2+0\times1+2\times(-1) & 0\times3+0\times2+2\times1 & 0\times4+0\times3+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}.$$

And,

$$BA = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3+0 & 6+6+0 & 0+3+8 \\ 1-2+0 & 3+4+0 & 0+2+6 \\ -1-1+0 & -3+2+0 & 0+1+4 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}.$$

Thus, $BA = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix} = BA.$

$\Rightarrow AB \neq BA.$

3. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$. **Find AB and explain why BA is not defined.**

Solution: Given that,

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0-1+4 & 0+0-2 \\ 1-2+6 & -2+0-3 \\ 2-3+8 & -4+0-4 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

And,

$$BA = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$$

Here the first row of first matrix has 2 values but the first column of second matrix has 3 values. So, we cannot operate the system of multiplication between the matrices. Thus, BA is undefined.

Note: Remember that the multiplication between 2–matrices is possible only if number of column(s) in first matrix is same as the number of row(s) in second matrix.

4. If $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$. Verify AB = BA = I.

Solution: Given that,

$$A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+6-3 & -6+6+0 & 2-3+1 \\ -1+4-3 & -3+4+0 & 1-2+1 \\ -6+18-12 & -18+18+0 & 6-9+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Also,

$$BA = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+6 & 3+6-9 & -1-3+4 \\ -4-2+6 & 6+4-9 & -2-2+4 \\ -6+0+6 & 9+0-9 & -3+0+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus, AB = BA = I.

5. **Find AB – BA, where** $A = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix}$.

Solution: Given that,

$$A = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2+63 & 10+18 \\ 4+21 & 20+6 \end{bmatrix} = \begin{bmatrix} 65 & 28 \\ 25 & 26 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2+20 & 9+15 \\ 14+8 & 63+6 \end{bmatrix} = \begin{bmatrix} 22 & 24 \\ 22 & 69 \end{bmatrix}$$

Now,

$$AB - BA = \begin{bmatrix} 65 & 28 \\ 25 & 26 \end{bmatrix} - \begin{bmatrix} 22 & 24 \\ 22 & 69 \end{bmatrix} = \begin{bmatrix} 43 & 4 \\ 3 & -43 \end{bmatrix}.$$

6. **If** $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. **Find AB or BA which ever exists.**

Solution: Given that,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Here, matrix A has 4–columns but matrix B has only 3–rows. So, AB does not exist being (number of column of A) = 4 ≠ 3 = (number of rows of B).

And, B has 3–columns and A has 3–rows. Thus, (number of columns of B) = (number of rows of A). So, BA is defined.

Now,

$$BA = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+0 & 4+0+0 & 8+2+0 \\ 3+4+3 & 6+0+1 & 12+4+5 \\ 1+0+3 & 2+0+1 & 4+0+5 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}.$$

7. **If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$. Show that (AB)C = A(BC) and A(B + C) = AB + AC.**

Solution: Given that,

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2+4 & 1+6 \\ -4+6 & -2+9 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -6+2 & 2+0 \\ -6+6 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -3+4 & 1+0 \\ 6+6 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

So that,

$$(AB)C = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -18+14 & 6+0 \\ -6+14 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4+0 & 2+4 \\ 8+0 & -4+6 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

Thus,

$$(AB)C = A(BC)$$

Also,

$$A(B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -1+8 & 2+6 \\ 2+12 & -4+9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

And,

$$AB + AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 12 & -12 \end{bmatrix} = \begin{bmatrix} 6+1 & 7+1 \\ 2+12 & 7-12 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

Thus,

$$A(B + C) = AB + AC$$

8. **If $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that (2I – A) (10I – A) = I.**

Solution: Given that,

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$2I - A = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix}$$

and,

$$10I - A = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$$

Now,

$$(2I - A)(10I - A) = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9 & 0+0 \\ -24+24 & 9+0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 9I$$

$$\Rightarrow (2I - A)(10I - A) = 9I$$

9. **If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. Find AB.**

Solution: Given that.

$$A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Then,

$$(2A) = [(A + B) + (A - B)] = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

And,

$$A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} - A = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2+0 & -2+0 \\ -2+1 & -2-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}.$$

10. **If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 5A + 7I = 0$, where I is a unit matrix of size 2×2.**

Solution: Let, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then,

$$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Now,

$$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Thus, $A^2 - 5A + 7I = 0$.

11. **Let $A = \begin{bmatrix} -3 & 2 \\ 1 & -3 \\ -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$. Show that AB and AC does not exist.**

Solution: Given that,

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -3 \\ -3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$$

Since the matrix A has 2–columns but both B and C have 3–rows. Thus, (number of column of A) ≠ (number of row of B or C). So, AB and AC are not defined. Therefore, AB and AC does not exist.

12. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ **where $i^2 = -1$, then show that $(A + B)^2 = A^2 + B^2$.**

Solution: Let,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \text{where } i^2 = -1$$

Then,

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$$

So,

$$\begin{aligned}(A+B)^2 &= (A+B).(A+B) \\ &= \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 + (1-i)(1+i) & 0+0 \\ 0+0 & (1-i)(1+i)+0 \end{bmatrix} \\ &= \begin{bmatrix} 1-i^2 & 0 \\ 0 & 1-i^2 \end{bmatrix} = \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\end{aligned}$$

And, $A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Also, $B^2 = B.B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0-i^2 & 0+0 \\ 0+0 & -i^2+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So that,

$$A^2 + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = (A + B)^2$$

Thus, $(A + B)^2 = A^2 + B^2$

Note: *In general, $(A + B)^2$ may not be equal to $A^2 + B^2$. For this, check the condition with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.*

13. **If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$. Show that $A^3 - A^2 - 18A - 30I = 0$.**

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\begin{aligned}A^2 = A.A &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8+3 \\ 2-2+12 & 4+1+4 & 6-4+4 \\ 3+2+3 & 6-1+1 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}\end{aligned}$$

And,

$$\begin{aligned}A^3 = A.A^2 &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 14+24+24 & 3+18+18 & 14+12+42 \\ 28-12+32 & 6-9+24 & 28-6+56 \\ 42+12+8 & 9+9+6 & 42+6+14 \end{bmatrix} = \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix}\end{aligned}$$

Now,

$$\begin{aligned}&A^3 - A^2 - 18A - 30I \\ &= \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - 30 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 62-14-18-30 & 39-3-36-0 & 68-14-54-0 \\ 48-12-36-0 & 21-9+18-30 & 78-6-72-0 \\ 62-8-54-0 & 24-6-18-0 & 6214-18-30 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0\end{aligned}$$

Thus, $A^3 - A^2 - 18A - 30I = 0$.

14. **If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$. Show that $A^3 - 4A^2 - 3A + 11I = 0$, where I is the unit matrix of order 3.**

Solution: Process as 13.

EXERCISE – 1.2

1. **If $A = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}, D = [4 \;\; 3 \;\; 0]$.**

   Calculate the following products or give reasons why they are not defined.

   (i) BA, A^TB, AB (iii) A^TD, A^TD^T, DA, AD

   (ii) C^2, C^TC, CC^T (iv) BB^T, B^TB, BB^T, B

Solution: Let,

$$A = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}, D = [4 \;\; 3 \;\; 0].$$

Then, the transpose of the matrices be

$$A^T = [1 \;\; 4 \;\; 3], B^T = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}, C^T = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}, D^T = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

(i) (a) BA– Since (number of column in B) ≠ (number of rows in A). So, BA is undefined.

(b) A^TB – Here, A^T has 3–columns and B has 3–rows. So, A^TB is defined. And,

$$\begin{aligned}A^TB &= [1 \;\; 4 \;\; 3] \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \\ &= [2+0+0 \quad -3+8+3] = [2 \quad 8].\end{aligned}$$

(c) AB – Here, A has 1–column and B has 3–rows. So, (number of column in A) ≠ (number of rows in B). Therefore, AB is not defined.

(ii) (a) $C^2 = C.C$ – Since both C are 3×3 square matrices. So, C^2 is defined.

Here,

$$C^2 = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+36+4 & 24+0+6 & 8+18-2 \\ 24+0+6 & 36+0+9 & 12+0-3 \\ 8+18-2 & 12+0-3 & 4+9+1 \end{bmatrix} = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}.$$

(b) C^TC, CC^T – Since $C^T = C$. So, $C^TC = CC^T = CC = C^2$.

Therefore, see (a) for $C^TC = CC^T$.

(iii) (a) A^TD – Here, A^T has 3–columns and D has 1–rows.

So, (number of columns in A^T) $\neq$ (number of rows in D).

Therefore, A^TD is undefined.

(b) A^TD^T – Here, (number of columns in A) = (number of rows in D^T).

So, A^TD^T is defined. And,

$$A^TD^T = [1 \quad 4 \quad 3]\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = (4 + 12 + 0) = (16).$$

(c) DA – Here D has 3–columns and A has 3–rows. So, DA is defined. And,

$$DA = [4 \quad 3 \quad 0]\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = (4 + 12 + 0) = (16)$$

(d) AD – Here A has 1–column and D has 1–row. So, AD is defined. And,

$$AD = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}[4 \quad 3 \quad 0] = \begin{bmatrix} 4 & 3 & 0 \\ 16 & 12 & 0 \\ 12 & 9 & 0 \end{bmatrix}$$

(iv) (a) BB^T – Here B has 2–columns and B^T has 2–rows. So, BB^T is defined. And,

$$BB^T = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9 & 0-6 & 0-3 \\ 0-6 & 0+4 & 0+2 \\ 0-3 & 0+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 13 & -6 & -3 \\ -6 & 4 & 2 \\ -3 & 2 & 1 \end{bmatrix}.$$

(b) B^TB – Here B^T has 3–columns and B has 3–rows. So, B^TB is defined. And,

$$B^TB = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+0 & -6+0+0 \\ -6+0+0 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$$

(c) BB^TB – Since we know $BB^TB = (BB^T)B = B(B^TB)$.

Since, B has 2–columns and B^TS has 2–rows.

So, $B(B^TB) = (BB^TB)$ is defined.

Now,

$$BB^TB = B(B^TB) = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 8+18 & -12-42 \\ 0-12 & 0+2 \\ 0-6 & 0+14 \end{bmatrix} = \begin{bmatrix} 24 & -54 \\ -12 & 28 \\ -6 & 14 \end{bmatrix}.$$

2. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Verify that $(AB)^T = B^TA^T$.

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}$$

And,

$$B^TA^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+4+5 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 9 \\ -3 & 6 & 0 \end{bmatrix}$$

Thus,

$$(AB)^T = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix} = B^TA^T$$

This verifies $(AB)^T = B^TA^T$.

3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$, show that $(A^2)^T = (A^T)^2$.

Solution: Let, $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix}$

Now,

$$A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 4+15 & 6-21 \\ 10-35 & 15+49 \end{bmatrix} = \begin{bmatrix} 19 & 15 \\ -25 & 64 \end{bmatrix}.$$

And

$$(A^T)^2 = A^T.A^T = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 4+15 & 10-35 \\ 6-21 & 15+49 \end{bmatrix} = \begin{bmatrix} 19 & -25 \\ 15 & 64 \end{bmatrix}.$$

Thus,

$$(A^2)^T = \begin{bmatrix} 19 & 15 \\ -25 & 64 \end{bmatrix}^T = \begin{bmatrix} 19 & -25 \\ 15 & 64 \end{bmatrix} = (A^T)^2$$

4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Verify $(AB)^T = B^TA^T$.

Solution: Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

So,

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Here, A has 2-columns and B has 2-rows. So, AB is defined. And, B^T has 2-columns and A^T has 2-rows. So, B^TA^T is defined.

Now,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6 & 1+4 & 3+2 \\ 6+12 & 3+8 & 9+4 \\ 10+18 & 5+12 & 15+6 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 18 & 11 & 13 \\ 28 & 17 & 21 \end{bmatrix}.$$

So,

$$(AB)^T = \begin{bmatrix} 8 & 18 & 28 \\ 5 & 11 & 17 \\ 5 & 13 & 21 \end{bmatrix}$$

And,

$$B^TA^T = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6 & 6+12 & 10+18 \\ 1+4 & 3+8 & 5+12 \\ 3+2 & 9+4 & 15+6 \end{bmatrix} = \begin{bmatrix} 8 & 18 & 28 \\ 5 & 11 & 17 \\ 5 & 13 & 21 \end{bmatrix}.$$

Thus, $(AB)^T = B^TA^T$.

5. **Find A^*, where $A = \begin{bmatrix} 1+i & 2 & 3i \\ 2+i & i & 3 \end{bmatrix}$.**

Solution: Given that,

$$A = \begin{bmatrix} 1+i & 2 & 3i \\ 2+i & i & 3 \end{bmatrix}$$

So, $\bar{A} = \begin{bmatrix} 1-i & 2 & -3i \\ 2-i & -i & 3 \end{bmatrix}$ being $\bar{A}$ is a complex conjugate of A.

Now,

$$A^* = (\bar{A})^T = \begin{bmatrix} 1-i & 2 & -3i \\ 2-i & -i & 3 \end{bmatrix}^T = \begin{bmatrix} 1-i & 2-i \\ 2 & -i \\ -3i & 3 \end{bmatrix}.$$

6. **Show that the matrix $\begin{bmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{bmatrix}$ is Hermitian.**

Solution: Let, the given matrix is

$$A = \begin{bmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{bmatrix}$$

Then the complex conjugate $\bar{A}$ of A is,

$$\bar{A} = \begin{bmatrix} 7 & 7+4i & 1-i \\ 7-4i & 2 & 2+i \\ 1+i & 2-i & 1 \end{bmatrix}$$

So,

$$A^* = (\bar{A})^T = \begin{bmatrix} 7 & 7+4i & 1-i \\ 7-4i & 2 & 2+i \\ 1+i & 2-i & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{bmatrix} = A$$

Thus,

$$A^* = (\bar{A})^T = A$$

So, the given matrix A is Hermitian.

7. **Show that the square matrix $A = \begin{bmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{bmatrix}$ is skew Hermitian matrix.**

Solution: Let

$$A = \begin{bmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{bmatrix}$$

Then the complex conjugate $\bar{A}$ of A is,

$$\bar{A} = \begin{bmatrix} -i & 2-i & 3+i \\ -2-i & -2i & 2 \\ -3+i & -2 & i \end{bmatrix}$$

Then,

$$(\bar{A})^T = \begin{bmatrix} -i & -2-i & -3+i \\ 2-i & -2i & -2 \\ 3+i & 2 & i \end{bmatrix}^T = -\begin{bmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{bmatrix} = -A$$

Thus,

$$A^* = (\bar{A})^T = -A$$

Therefore, A is skew-Hermitian matrix.

8. **If $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ then show that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric matrix.**

Solution: Let, $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Now,

$$A + A^T = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}.$$

Here $a_{ij} = a_{ji}$ for $a_{ij} \in (A + A^T)$. So $(A + A^T)$ is symmetric.

And,

$$A - A^T = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

Here, $a_{ij} = -a_{ji}$ for $a_{ij} \in (A - A^T)$. So, $A - A^T$ is skew-symmetric.

9. **Prove that the following.**

(i) The sum of two symmetric matrix is symmetric.

Solution: Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} \text{ and } B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{pmatrix}$$

are two symmetric matrices.

Thus, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$.

Now,

$$A+B=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{nn}+b_{nn}\end{pmatrix}$$

Since, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. So, $a_{12} + b_{12} = a_{21} + b_{21}$, $a_{m1} + b_{m1} = a_{1n} + b_{1n}$ and so on.
This shows that A + B is a symmetric matrix.

(ii) The sum of two skew-symmetric matrix is skew-symmetric.

Solution: Let,

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} \text{ and } B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{pmatrix}$$

are two skew-symmetric matrices.

Therefore, $a_{ij} = -a_{ji}$ $\forall\ i \neq j$ and $a_{ij} = a_{ji}$ $\forall\ i = j$.

Also, $b_{ij} = -b_{ji}$ $\forall\ i \neq j$ and $b_{ij} = b_{ji}$ $\forall\ i = j$.

Now,

$$A+B=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn}\end{pmatrix}$$

Here, $a_{ij} + b_{ij} = (-a_{ji}) + (-b_{ji}) = -(a_{ji} + b_{ji})$ $\forall\ i \neq j$.

Also, $a_{ij} + b_{ij} = a_{ji} + b_{ji}$ $\forall\ i = j$.

This proves that A + B is skew-symmetric matrix.

(iii) If A is a square matrix then A + A^T is symmetric and A –A^T skew symmetric.

Solution: Let,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$$

be a square matrix. Then, its transpose matrix be

$$A^T=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$$

Now,

$$A+A^T=\begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \cdots & a_{1n}+a_{n1}\\ a_{21}+a_{12} & a_{22}+a_{22} & \cdots & a_{2n}+a_{n2}\\ \cdots & \cdots & \cdots & \cdots\\ a_{1n}+a_{1n} & a_{n2}+a_{2n} & \cdots & a_{nn}+a_{nn}\end{bmatrix}$$

This matrix has element, $a_{ij} + b_{ji}$ on upper triangular form and $a_{ji} + a_{ij}$ on the lower triangular form.

Since, $a_{ij} + a_{ji} = a_{ji} + a_{ij}$ $\forall\ i, j$

Thus, the element on the upper triangular is equal to the term on respective lower triangular position.

So, A+A^T is symmetric matrix.

Next,

$$A-A^T=\begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & \cdots & a_{1n}-a_{n1}\\ a_{21}-a_{12} & a_{22}-a_{22} & \cdots & a_{2n}-a_{n2}\\ \cdots & \cdots & \cdots & \cdots\\ a_{1n}-a_{1n} & a_{n2}-a_{2n} & \cdots & a_{nn}-a_{nn}\end{bmatrix}$$

Here, the element is the upper triangular position has the form $a_{ij} - a_{ji}$ and in the lower triangular position has the form $a_{ji} - a_{ij}$, $\forall$ i and $\forall$ j.

Since, $a_{ij} - a_{ji} = -(a_{ji} - a_{ij})$

This shows that every element in upper triangular has same value but in negative form, in the respective lower triangular position. So, A – A^T is skew-symmetric.

(iv) Every square matrix can be expressed as the sum of a symmetric and skew-symmetric matrix.

Solution: Let A be a square matrix. Then A + A^T is symmetric and A – A^T is skew-symmetric matrices.

Moreover, $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is a skew-symmetric matrix.

Here, $A = \frac{1}{2}[(A + A^T) + (A - A^T)]$.

This shows that A can be expressed as the sum of its symmetric and skew-symmetric form.

EXERCISE 1.3

1. **Without expanding, show that each of the following determinants vanishes.**

(i) $\begin{vmatrix} 43 & 1 & 6\\ 35 & 7 & 4\\ 17 & 3 & 2\end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 43 & 1 & 6\\ 35 & 7 & 4\\ 17 & 3 & 2\end{vmatrix}$$

Applyng $C_1 = C_1 - C_2 - C_3$ then,

$$=\begin{vmatrix} 36 & 1 & 6\\ 24 & 7 & 4\\ 12 & 3 & 2\end{vmatrix}$$

Taking the common value 6 from first column. Then,

$$=6\begin{vmatrix} 6 & 1 & 6\\ 4 & 7 & 4\\ 2 & 3 & 2\end{vmatrix}$$

Here, $C_1 = C_3$. So, the property of determinant tells us that the value of the determinant is 0. So,

$$\begin{vmatrix} 43 & 1 & 6\\ 35 & 7 & 4\\ 17 & 3 & 2\end{vmatrix} = 6\begin{bmatrix} 6 & 1 & 6\\ 4 & 7 & 4\\ 2 & 3 & 2\end{bmatrix} = 6 \times 0 = 0$$

(ii) $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$ and $C_3 = C_3 - C_1$. Then,

$$= \begin{vmatrix} 265 & 21 & 46 \\ 240 & 27 & 42 \\ 219 & 17 & 38 \end{vmatrix}$$

(iii) $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$, $R_2 = R_2 - R_3$, $R_3 = R_3 - R_4$. Then

$$= \begin{vmatrix} -3 & -5 & 1 & 0 \\ 18 & 14 & 4 & 7 \\ 1 & 1 & 1 & 1 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Again applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$, $C_3 = C_3 - C_4$. Then,

$$= \begin{vmatrix} 2 & -6 & 1 & 0 \\ 4 & 10 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -1 & 2 \end{vmatrix}$$

Again applying $R_1 = R_1 = R_4$ then,

$$= \begin{vmatrix} 0 & 0 & 0 & 2 \\ 4 & 10 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -1 & 2 \end{vmatrix}$$

Again applying $R_1 = R_1 - 2R_3$ then

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 4 & 10 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -1 & 2 \end{vmatrix} = 0.$$

being the first row has zero value in each position.

(iv) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Applying $C_3 = C_3 + C_2$ then,

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = (a+b+c)\times 0 = 0$$

because if two columns have same value then the value of determinant, is zero.

(v) $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$, $R_2 = R_2 - R_3$ then,

$$= \begin{vmatrix} 0 & a-b & a^2 - b^2 + ac - bc \\ 0 & b-c & b^2 - c^2 + ab - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b+c) \\ 0 & b-c & (b-c)(a+b+c) \\ 1 & c & c^2 - ab \end{vmatrix}$$

Taking–out the common value $a - b$ from R_1 and $(b - c)$ from R_2 then,

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & c & c^2 - ab \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ then

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & a+b+c \\ 1 & c & c^2 - ab \end{vmatrix} = 0, \text{ being each value in } R_1 \text{ is zero.}$$

(vi) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Applying $C_3 = C_3 + C_2$ then,

$$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Taking–out the common value $ab + bc + ca$ from C_3 then

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} = (ab+bc+ca)\times 0 = 0$$

because if two columns of a determinant has same value then the determinant determines zero value.

(vii) $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$$

Applying $R_1 = aR_1$, $R_2 = bR_2$ and $R_3 = cR_3$ then

$$= \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

Taking-out the common value abc from C_3. Then,

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} = 1\times 0 = 0$$

because if the determinant has same value in two columns the value of determinant zero.

(viii) $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+acd \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$ **[2004 Fall Q.No. 1(a)]**

Solution: Here,

$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+acd \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & acd \\ 1 & c & c^2 & abd \\ 1 & d & d^2 & abc \end{vmatrix}$$

Applying for second matrix as $R_1 = aR_1$, $R_2 = bR_2$, $R_3 = cR_3$ and $R_4 = dR_4$. Then,

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$$

Taking-out the common value abcd from second matrix. Then,

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \frac{abcd}{abcd}\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + (-1)^3\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = 0 + 0$$

being C_4 cross three parallel columns in second matrix.

$= 0$, adding 1st and 2nd matrices.

(ix) $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ac & a+c \\ a^2b^2 & ab & a+b \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ac & a+c \\ a^2b^2 & ab & a+b \end{vmatrix}$$

Multiply R_1 by a^2, R_2 by b^2 and R_3 by c^2 and then taking out the column value $a^2b^2c^2$ from c_1 then,

$$= \frac{a^2b^2c^2}{a^2b^2c^2}\begin{vmatrix} 1 & a^2bc & a^2(b+c) \\ 1 & ab^2c & b^2(a+c) \\ 1 & abc^2 & c^2(a+b) \end{vmatrix}$$

Taking out the common value abc from c_2 then,

$$= abc\begin{vmatrix} 1 & a & (a^2b+a^2c) \\ 0 & b-a & (ab^2+b^2c) \\ 0 & c-a & (ac^2+bc^2) \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ then,

$$= abc\begin{vmatrix} 1 & a & (a^2b+a^2c) \\ 0 & b-a & (ab^2+b^2c-a^2b+a^2c) \\ 0 & c-a & (ac^2+bc^2-a^2b-a^2c) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2b+a^2c \\ 0 & b-a & ab^2-a^2b+b^2c-abc+abc-ca^2 \\ 0 & c-a & ac^2-a^2c+bc^2-abc+abc-a^2b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2b+a^2c \\ 0 & b-a & (b-a)(ab+bc+ca) \\ 1 & c-a & (c-a)(ab+bc+ca) \end{vmatrix}$$

Taking the common factor $(b-a)$ from R_2 and $(c-a)$ from R_3 then,

$$= abc\,(b-a)(c-a)\begin{vmatrix} 1 & a & a^2b+a^2c \\ 0 & 1 & ab+bc+ca \\ 0 & 1 & ab+bc+ca \end{vmatrix}$$

Again, applying $R_3 \to R_3 - R_2$ then,

$$= abc\,(b-a)(c-a)\begin{vmatrix} 1 & a & a^2b+a^2c \\ 0 & 1 & ab+bc+ca \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0$, being all values in R_3 are zero.

(x) $\begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{vmatrix}$$

Applying $R_3 = R_3 - R_2$ and $R_2 = R_2 - R_1$ then,

$$= \begin{vmatrix} x+a & x+2a & x+3a \\ a & a & a \\ 2a & 2a & 2a \end{vmatrix}$$

Taking-out the common value a from R_2 and 2a from R_3 then

$$= 2a^2\begin{vmatrix} x+a & x+2a & x+3a \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2a^2\times 0 = 0, \text{ being } R_2 = R_3.$$

2. Show that:

(i) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-z)(x-y)(z-x)$

Solution: Here,

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ and $R_2 = R_2 - R_3$ then

$$= \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Taking–out the common value $x - y$ from R_1 and $y - z$ from R_2 then,

$$= (x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$

Now, expanding from C_1 then,

$$= (x-y)(y-z)\{(y+z)-(x+y)\}$$

$$= (x-y)(y-z)(z-x)$$

(ii) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution: Here,

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3$ then

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking–out the common value $a + b + c$ from R_1 then

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$ then

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Taking the common value $(a + b + c)$ from C_1 and C_2 then,

$$= (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expanding the determinant from R_1 then

$$= (a+b+c)^3\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = (a+b+c)^3$$

(iii) $\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$

Solution: Here,

$$\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix}$$

Applying $R_1 = R_1 - R_2 - R_3$ then

$$= \begin{vmatrix} 0 & -2c & -2c \\ a & a+c & c \\ b & c & b+c \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$ then

$$= \begin{vmatrix} 0 & 0 & -2c \\ a & a & c \\ b & -b & b+c \end{vmatrix}$$

Now, expanding from R_1 then

$$= -2c\begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2c(-ab-ab) = 4abc.$$

(iv) $\begin{vmatrix} b+c & a & b \\ a+c & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2$

Solution: Here,

$$\begin{vmatrix} b+c & a & b \\ a+c & c & a \\ a+b & b & c \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3$ then,

$$= \begin{vmatrix} 2(a+b+c) & a+b+c & a+b+c \\ a+c & c & a \\ a+b & b & c \end{vmatrix}$$

Taking–out the common value $(a + b + c)$ from R_1 then

$$= (a+b+c)\begin{vmatrix} 2 & 1 & 1 \\ a+c & c & a \\ a+b & b & c \end{vmatrix}$$

Applying $C_1 = C_1 - C_2 - C_3$ and $C_2 = C_2 - C_3$ then

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & c-a & a \\ a-c & b-c & c \end{vmatrix}$$

Now, expanding from R_1 then

$$= (a+b+c)(a-c)^2$$

(v) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = x^3(4+x)$

Solution: Here,

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3 + R_4$ then,

$$= \begin{vmatrix} 4+x & 4+x & 4+x & 4+x \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

Taking-out the common value 4+x from R_1 then,

$$= (4+x) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

Again applying $C_2 = C_2 - C_1$, $C_3 = C_3 - C_1$, $C_4 = C_4 - C_1$ then

$$= (4+x) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x & 0 & 0 \\ 1 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix}$$

Expanding the determinant from C_1 then

$$= (4+x)(x^3)$$
$$= (4+x)\,x^3$$

(vi) $$\begin{vmatrix} a & x & x & x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix} = (a+3x)(a-x)^3$$

Solution: Here,

$$\begin{vmatrix} a & x & x & x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3 + R_4$ then

$$= \begin{vmatrix} a+3x & a+3x & a+3x & a+3x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix}$$

Taking-out the common value $(a+3x)$ from R_1 then

$$= (a+3x) \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix}$$

Applying $C_2 = C_2 - C_1$, $C_3 = C_3 - C_1$, $C_4 = C_4 - C_1$, then

$$= (a+3x) \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & a-x & 0 & 0 \\ x & 0 & a-x & 0 \\ x & 0 & 0 & a-x \end{vmatrix}$$

Now, expanding the determinant from R_1 then

$$= (a+3x)(a-x)^3$$

(vii) $$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} = 1+a+b+c+d$$

Solution: Here,

$$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

Applying $C_1 = C_1 + C_2 + C_3 + C_4$ then

$$= \begin{vmatrix} 1+a+b+c+d & b & c & d \\ 1+a+b+c+d & 1+b & c & d \\ 1+a+b+c+d & b & 1+c & d \\ 1+a+b+c+d & b & c & 1+d \end{vmatrix}$$

Taking-out the common value $1+a+b+c+d$ then,

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & 1+b & c & d \\ 1 & b & 1+c & d \\ 1 & b & c & 1+d \end{vmatrix}$$

Applying $R_2 = R_2 - R_1$, $R_3 = R_3 - R_1$, $R_4 = R_4 - R_1$ then

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Expanding the determinant from C_1 then

$$= (1+a+b+c+d)$$

(viii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Solution: Here,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$ and $C_2 = C_2 - C_3$ then

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

Taking-out the common value $(a-b)$ from C_1 and $(b-c)$ from C_2 then

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Now, expanding from R_1 then

$$= (a-b)(b-c) \begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$
$$= (a-b)(b-c)\,[(b^2+bc+c^2) - (a^2+ab+b^2)]$$
$$= (a-b)(b-c)(c-a)(a+b+c)$$

(ix) $$\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = (x-2y+z)^2$$

Solution: Here,

$$\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$$

Applying $R_1 = R_1 - 2R_2 + R_3$ then

$$= \begin{vmatrix} 0 & 0 & 0 & x-2y+z \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$$

Expanding from R_1 we get,

$$= -(x-2y+z)\begin{vmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ x & y & z \end{vmatrix}$$

Applying $C_1 = C_1 - 2C_2 + C_3$ then

$$= -(x-2y+z)\begin{vmatrix} 0 & 6 & 7 \\ 0 & 7 & 8 \\ x-2y+z & y & z \end{vmatrix}$$

Expanding from C_1 then

$$= -(x-2y+z)(x-2y+z)(48-49)$$
$$= (x-2y+z)^2$$

(x) $\begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix} = -(a-b)^4$

Solution: Here,

$$\begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$. Then

$$= \begin{vmatrix} a-b & b & a & a \\ a-b & b & b & b \\ 0 & b & b & a \\ 0 & a & b & a \end{vmatrix}$$

Taking the common value $(a-b)$ from C_1 then

$$= (a-b)\begin{vmatrix} 1 & b & a & a \\ 1 & b & b & b \\ 0 & b & b & a \\ 0 & a & b & a \end{vmatrix}$$

Again applying $R_2 = R_2 - R_1$. Then

$$= (a-b)\begin{vmatrix} 1 & b & a & a \\ 0 & 0 & b-a & b-a \\ 0 & b & b & a \\ 0 & a & b & a \end{vmatrix}$$

Expanding from C_1 then

$$= (a-b)\begin{vmatrix} 0 & b-a & b-a \\ b & b & a \\ a & b & a \end{vmatrix}$$

Taking the common value $(b-a)$ from R_1 then

$$= (a-b)(b-a)\begin{vmatrix} 0 & 1 & 1 \\ b & b & a \\ a & b & a \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$. Then

$$= -(a-b)(a-b)\begin{vmatrix} 0 & 0 & 1 \\ b & b-a & a \\ a & b-a & a \end{vmatrix}$$

Expanding from R_1 then

$$= -(a-b)^2[b(b-a) - a(b-a)]$$
$$= -(a-b)^2(b-a)^2$$
$$= -(a-b)^4$$

(xi) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (b-c)(c-a)(a-b)$

Solution: Here,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ and $R_2 = R_2 - R_3$ then

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ bc-ca & ca-ab & ab \end{vmatrix}$$

Taking common value $(a-b)$ from C_1 and $(b-c)$ from C_2 then

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ c & a & ab \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 1 & 1 \\ c & a \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

(xii) $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

[2006 Spring Q.No. 1(a) OR]

Solution: Here,

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$, $R_2 = R_2 - R_3$ then

$$= \begin{vmatrix} x-y & x^2-y^2 & yz-zx \\ y-z & y^2-z^2 & zx-xy \\ z & z^2 & xy \end{vmatrix}$$

Taking-out the common value $x - y$ from R_1 and $y - z$ from R_2 then

$$= (x-y)(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ then,

$$= (x-y)(y-z)\begin{vmatrix} 0 & x-z & x-z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Taking-out the common value $x - z$ from R_1 then

$$= (x-y)(y-z)(x-z)\begin{vmatrix} 0 & 1 & 1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$ then

$$= (x-y)(y-z)(x-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & x+y+z & -x \\ z & z^2-xy & xy \end{vmatrix}$$

Expanding from R_1 then

$$= (x-y)(y-z)(x-z)\begin{vmatrix} 1 & x+y+z \\ z & z^2-xy \end{vmatrix}$$

$$= (x-y)(y-z)(x-z)(z^2 - xy - xz - yz - z^2)$$

$$= (x-y)(y-z)(x-z)(-1)(xy + yz + zx)$$

$$= (x-y)(y-z)(z-x)(xy + yz + zx)$$

(xiii) $\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$ **[2007 Fall Q.No. 1(a) OR]**

Solution: Here,

$$\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix}$$

Multiplying R_1 by c, R_2 by a and R_3 by b then

$$= \frac{1}{abc}\begin{vmatrix} c(a+b)^2 & c^2a & bc^2 \\ ca^2 & a(b+c)^2 & a^2b \\ b^2c & ab^2 & b(c+a)^2 \end{vmatrix}$$

Taking-out the common value c from C_1, a from C_2 and b from C_3 then

$$= \frac{cab}{abc}\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

Applying $C_1 = C_1 - C_3$ and $C_2 = C_2 - C_3$ then

$$= \begin{vmatrix} (a+b)^2 - c^2 & 0 & c^2 \\ 0 & (b+c)^2 & a^2 \\ b^2 - (c+a)^2 & b^2 - (c+a)^2 & (c+a)^2 \end{vmatrix}$$

Taking the common value $(a + b + c)$ from C_1 and C_2 then

$$= (a+b+c)^2\begin{vmatrix} a+b-c & 0 & c^2 \\ 0 & b+c-a & a^2 \\ b-c-a & b-c-a & (c+a)^2 \end{vmatrix}$$

Applying $R_3 = R_3 - R_1 - R_2$ then

$$= (a+b+c)^2\begin{vmatrix} a+b-c & 0 & c^2 \\ 0 & b+c-a & a^2 \\ -2a & -2c & 2ca \end{vmatrix}$$

Now, expanding we get,

$$= (a+b+c)^2\left\{(a+b-c)\begin{vmatrix} b+c-a & a^2 \\ -2c & 2ca \end{vmatrix} + c^2\begin{vmatrix} 0 & b+c-a \\ -2a & -2ca \end{vmatrix}\right\}$$

$$= (a+b+c)^2\{(a+b+c)[2abc + 2c^2a - 2ca^2 + 2ca^2] + c^2(2ab+2ac-2a^2)\}$$

$$= (a+b+c)^2(2a^2bc + 2a^2c^2 + 2ab^2c + 2abc^2 - 2abc^2 - 2ac^3 + 2abc^2 + 2ac^3 - 2a^2c^2)$$

$$= 2abc(a+b+c)^2(a+b+c)$$

$$= 2abc(a+b+c)^3$$

(xiv) $\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2 + d^2$

[2014 Spring Q. No. 1(a) OR]

[2011 Spring Q.No. 1(a)] [2010 Fall Q.No. 1(a) OR] [2005 Fall Q.No. 1(a)]

Solution: Here,

$$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b, R_3 by c and R_4 by d then,

$$= \frac{1}{abcd}\begin{vmatrix} a(a^2+1) & a^2b & a^2c & a^2d \\ ab^2 & b(b^2+1) & b^2c & b^2d \\ ac^2 & bc^2 & c(c^2+1) & c^2d \\ ad^2 & bd^2 & d^2c & d(d^2+1) \end{vmatrix}$$

Taking-out the common value a from C_1, b from C_2, c from C_3, d from C_4 then

$$= \frac{abcd}{abcd}\begin{vmatrix} a^2+1 & a^2 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 & b^2 \\ c^2 & c^2 & c^2+1 & c^2 \\ d^2 & d^2 & d^2 & d^2+1 \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3 + R_4$ and the taking-out the common value $1 + a^2 + b^2 + c^2 + d^2$ from R_1. So that,

$$= (1 + a^2 + b^2 + c^2 + d^2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 & b^2 \\ c^2 & c^2 & c^2+1 & c^2 \\ d^2 & d^2 & d^2 & d^2+1 \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$, $C_3 = C_3 - C_4$. Then,

$$= (1 + a^2 + b^2 + c^2 + d^2)\begin{vmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & b^2 \\ 0 & -1 & 1 & c^2 \\ 0 & 0 & -1 & d^2+1 \end{vmatrix}$$

Now, expanding the determinant from R_1 then

$$= (-1)(1 + a^2 + b^2 + c^2 + d^2)\begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{vmatrix}$$

Again applying $C_2 = C_2 + C_1$ then

$$= (-1)(1 + a^2 + b^2 + c^2 + d^2)\begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{vmatrix}$$

Expanding the determinant from R_1 we get,

$$= (-1)^2 (1 + a^2 + b^2 + c^2 + d^2)(1 - 0)$$
$$= (1 + a^2 + b^2 + c^2 + d^2)$$

(xv) $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3(a^2 + b^2 + c^2 + d^2 + \lambda)$

[2008 Spring Q.No. 1(b) OR]

Solution: Here,

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b, R_3 by c, R_4 by d then

$$= \frac{1}{abcd} \begin{vmatrix} a(a^2+\lambda) & a^2b & a^2c & a^2d \\ ab^2 & b(b^2+\lambda) & b^2c & b^2d \\ ac^2 & bc^2 & c(c^2+\lambda) & c^2d \\ ad^2 & bd^2 & cd^2 & d(d^2+\lambda) \end{vmatrix}$$

Then taking–out the common value a from C_1, b from C_2, c from C_3 and d from C_4 then

$$= \frac{abcd}{abcd} \begin{vmatrix} a^2+\lambda & a^2 & a^2 & a^2 \\ b^2 & b^2+\lambda & b^2 & b^2 \\ c^2 & c^2 & c^2+\lambda & c^2 \\ d^2 & d^2 & d^2 & d^2+\lambda \end{vmatrix}$$

Now, applying $R_1 = R_1 + R_2 + R_3 + R_4$ and the taking–out the common value $a^2 + b^2 + c^2 + d^2 + \lambda$ from R_1. So that,

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^2 & b^2+\lambda & b^2 & b^2 \\ c^2 & c^2 & c^2+\lambda & c^2 \\ d^2 & d^2 & d^2 & d^2+\lambda \end{vmatrix}$$

Applying $C_2 = C_2 - C_1$, $C_3 = C_3 - C_1$, $C_4 = C_4 - C_1$ then

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \begin{vmatrix} 1 & 0 & 0 & 0 \\ b^2 & \lambda & 0 & 0 \\ c^2 & 0 & \lambda & 0 \\ d^2 & 0 & 0 & \lambda \end{vmatrix}$$

Expanding the determinant from R_1 then

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

Expanding from C_1 then

$$= (a^2 + b^2 + c^2 + d^2 + \lambda)\lambda^3$$

(xvi) $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$

Solution: Here,

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

Taking–out the common value a from C_1, b from C_2, c from C_3 and d from C_4 then

$$= abcd \begin{vmatrix} (1/a)+1 & 1/b & 1/c & 1/d \\ 1/a & (1/b)+1 & 1/c & 1/d \\ 1/a & 1/b & (1/c)+1 & 1/d \\ 1/a & 1/b & 1/c & 1+(1/d) \end{vmatrix}$$

Then applying $C_1 = C_1 + C_2 + C_3 + C_4$ and the taking–out the common value $\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$ from C_1 then

$$= (abcd)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \begin{vmatrix} 1 & 1/b & 1/c & 1/d \\ 1 & 1+(1/b) & 1/c & 1/d \\ 1 & 1/b & 1+(1/c) & 1/d \\ 1 & 1/b & 1/c & 1+(1/d) \end{vmatrix}$$

Again applying $R_2 = R_2 - R_1$, $R_3 = R_3 - R_1$, $R_4 = R_4 - R_1$ then

$$= (abcd)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \begin{vmatrix} 1 & 1/b & 1/c & 1/d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Now, expanding the determinant from C_1 then

$$= (abcd)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Again expanding from R_1 then

$$= abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)(1)$$
$$= abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

(xvii) $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a - b)(b - c)(c - a)(a + b + c)$

Solution: Here,

$$\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying $R_1 = R_1 + R_2$ then

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Taking–out the common value $(a + b + c)$ from R_1 then

$$= (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying $C_2 = C_2 - C_1$ and $C_3 = C_3 - C_1$ then

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b+c & b-a & b-c \\ a^2 & b^2-a^2 & c^2-b^2 \end{vmatrix}$$

Taking–out the common value (b – a) from C_2 and (b – c) from C_3 then

$$= (a+b+c)(b-a)(b-c)\begin{vmatrix} 1 & 0 & 0 \\ b+c & 1 & 1 \\ a^2 & b+a & c+b \end{vmatrix}$$

Applying $C_3 = C_3 - C_2$ then

$$= (a+b+c)(b-a)(b-c)\begin{vmatrix} 1 & 0 & 0 \\ b+c & 1 & 0 \\ a^2 & b+a & c-a \end{vmatrix}$$

Expanding from R_1 then

$$= (a+b+c)(b-a)(b-c)(c-a)$$
$$= -(a-b)(b-c)(c-a)(a+b+c)$$

(xviii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

Taking–out the common value

$$\Rightarrow abc\begin{vmatrix} a & bc & 1 \\ b & ca & 1 \\ c & ab & 1 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b and R_3 by c on the left determinants then

$$\Rightarrow \begin{vmatrix} a^2 & abc & a \\ b^2 & abc & b \\ c^2 & abc & c \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Taking–out the common value on the left determinant then,

$$\Rightarrow abc\begin{vmatrix} a^2 & 1 & a \\ b^2 & 1 & b \\ c^2 & 1 & c \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow (-1)\,abc\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow (-1)^2\,abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

This completes the requirement.

(xix) $\begin{vmatrix} bcd & 1 & a & a^2 \\ acd & 1 & b & b^2 \\ abd & 1 & c & c^2 \\ abc & 1 & d & d^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

Solution: Here,

$$\begin{vmatrix} bcd & 1 & a & a^2 \\ acd & 1 & b & b^2 \\ abd & 1 & c & c^2 \\ abc & 1 & d & d^2 \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b, R_3 by c and R_4 by d and then taking out the common value abcd from C_1. So that,

$$= \frac{abcd}{abcd}\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

(xx) $\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$ **[2012 Fall Q.No. 1(a)]**

Solution: Here,

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b, R_3 by c and then applying $R_1 \to R_1 - R_2 - R_3$ then,

$$= \frac{1}{abc}\begin{vmatrix} 0 & -2bc^2 & -2b^2c \\ ab^2 & bc^2 + a^2b & b^2c \\ ac^2 & bc^2 & a^2c + b^2c \end{vmatrix}$$

Thing out common value –2bc from R_1 then,

$$= \frac{-2bc}{abc}\begin{vmatrix} 0 & c & b \\ ab^2 & bc^2 + a^2b & b^2c \\ ac^2 & bc^2 & a^2c + b^2c \end{vmatrix}$$

Taking out the common value 'a' from c_1, 'b' from R_2 and 'c' from R_3 then,

$$= -2bc\left\{0 - b\begin{vmatrix} c & b \\ bc & a^2+b^2 \end{vmatrix} + c\begin{vmatrix} c & b \\ c^2+a^2 & bc \end{vmatrix}\right\}$$
$$= -2bc\{0 - b(a^2c + b^2c - b^2c) + c(bc^2 - bc^2 - a^2b)\}$$
$$= -2bc\{-b(a^2c) + c(-a^2b)\}$$
$$= 2bc(a^2bc + a^2bc)$$
$$= 2bc \times 2a^2bc$$
$$= 4a^2b^2c^2$$

(xxi) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution: Here,

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Taking–out the common value a from C_1, b from C_2, c from C_3 then,

$$= abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Applying $C_1 = C_1 + C_3$ and $C_2 = C_2 - C_3$ then,

$$= abc\begin{vmatrix} 0 & 0 & a \\ 0 & -2b & b \\ 2c & 2c & -c \end{vmatrix}$$

Now, expanding from R_1 then

$$= abc \cdot a\begin{vmatrix} 0 & -2b \\ 2c & 2c \end{vmatrix} = a^2bc\,(0 + 4bc) = 4a^2b^2c^2$$

(xxii) $\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix} = x^2(6 + x)$

Solution: Here,

$$\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix}$$

Applying $C_1 = C_1 + C_2 + C_3$ then

$$= \begin{vmatrix} 6+x & 2 & 3 \\ 6+x & 2+x & 3 \\ 6+x & 2 & 3+x \end{vmatrix}.$$

Taking–out the common value 6+x from C_1 then

$$= (6 + x)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix}$$

Appling $R_1 = R_1 - R_2$ and $R_2 = R_2 - R_3$ then

$$= (6 + x)\begin{vmatrix} 0 & -x & 0 \\ 0 & x & -x \\ 1 & 2 & 3-x \end{vmatrix}$$

Now, expanding the determinant from C_1 then,

$$= (6 + x).1\begin{vmatrix} -x & 0 \\ x & -x \end{vmatrix} = (6 + x)(x^2 - 0) = x^2(6 + x)$$

(xxiii) $\begin{vmatrix} a & b & ax + by \\ b & c & bx + cy \\ ax + by & bx + cy & 0 \end{vmatrix} = (b^2 - ac)(ax^2 + 2bxy + cy^2)$

Solution: Here,

$$\begin{vmatrix} a & b & ax + by \\ b & c & bx + cy \\ ax + by & bx + cy & 0 \end{vmatrix}$$

Multiplying R_1 by x and R_2 by y then

$$= \frac{1}{xy}\begin{vmatrix} ax & bx & ax^2 + bxy \\ by & cy & bxy + cy^2 \\ ax + by & bx + cy & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 - R_3$ then,

$$= \frac{1}{xy}\begin{vmatrix} 0 & 0 & ax^2 + 2bxy + cy^2 \\ by & cy & bxy + cy^2 \\ ax + by & bx + cy & 0 \end{vmatrix}$$

$$= \frac{1}{xy}(ax^2 + 2bxy + cy^2)\begin{vmatrix} by & cy \\ ax + by & bx + cy \end{vmatrix}$$

$$= \frac{1}{xy}(ax^2 + 2bxy + cy^2)\, y\,[b^2x + bcy - acx - bcy]$$

$$= \frac{1}{x}(ax^2 + 2bxy + cy^2)\,[b^2x - acx]$$

$$= (b^2 - ac)(ax^2 + 2bxy + cy^2).$$

(xxiv) $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a - b)(b - c)(c - a).$ **[2009 Spring Q.No. 1(b)]**

Solution: Here,

$$\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ and $R_2 = R_2 - R_3$ then

$$= \begin{vmatrix} 0 & b-a & b^2-a^2 \\ 0 & c-b & c^2-b^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Taking–out the common value (b – a) from R_1 and (c – b) from R_2 then

$$= (b - a)(c - b)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Again applying $R_1 = R_1 - R_2$ then

$$= (b - a)(c - b)\begin{vmatrix} 0 & 0 & a-c \\ 0 & 1 & b+c \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Now, expanding the determinant from C_1 then

$$= (b - a)(c - b)\{-(a - c)\}$$

$$= (a - b)(b - c)(c - a).$$

(xxv) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca)$

Solution: Here,

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$ then

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2 - a^2 & b^3 - a^3 \\ 0 & c^2 - a^2 & c^3 - a^3 \end{vmatrix}$$

Taking–out the common value (b – a) from R_2 and (c – a) from R_3 then

$$= (b - a)(c - a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b + a & b^2 + ab + a^2 \\ 0 & c + a & c^2 + ac + a^2 \end{vmatrix}$$

Now, expanding the determinant from C_1 then

$$= (b - a)(c - a)\begin{vmatrix} b+a & b^2 + ab + a^2 \\ c+a & c^2 + ac + a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\,(bc^2+abc + a^2b + ac^2 + a^2c + a^3 - b^2c - abc - a^2c - ab^2 - a^2b - a^3)$$

$= (b-a)(c-a)(bc^2 + ac^2 - b^2c - ab^2)$

$= (b-a)(c-a)[(c-b)bc + a(c^2 - b^2)]$

$= (b-a)(c-a)(c-b)(bc + a(b+c))$

$= (a-b)(b-c)(c-a)(bc + ab + ca)$

$= (a-b)(b-c)(c-a)(ab + bc + ca).$

3. Solve the following equations

(i) $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$

Solution: Given that,

$$\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$$

Applying $C_1 = C_1 + C_3 - 2C_2$ then,

$$\Rightarrow \begin{vmatrix} 0 & 2x+1 & 3x+1 \\ -3 & 4x+3 & 6x+3 \\ -3 & 6x+4 & 8x+4 \end{vmatrix} = 0$$

Again, applying $R_3 = R_3 - R_1$ then

$$\Rightarrow \begin{vmatrix} 0 & 2x+1 & 3x+1 \\ -3 & 4x+3 & 6x+3 \\ -3 & 4x+3 & 5x+3 \end{vmatrix} = 0$$

Again, applying $R_2 = R_2 - R_3$ then

$$\Rightarrow \begin{vmatrix} 0 & 2x+1 & 3x+1 \\ 0 & 0 & x \\ -3 & 4x+3 & 5x+3 \end{vmatrix} = 0$$

Now, expanding the determinant from C_1 then,

$$\Rightarrow (-3)\begin{vmatrix} 2x+1 & 3x+1 \\ 0 & x \end{vmatrix} = 0$$

$\Rightarrow (-3)(-x)(2x+1) = 0$

$\Rightarrow x(2x+1) = 0 \qquad [\because 3 \neq 0]$

$\Rightarrow x = 0, \frac{1}{2}$

(ii) $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$

Solution: Here,

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$$

Solution: Here,

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 2R_1$ then,

$$\Rightarrow \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 2x+2 & 3x+1 & 4x-1 \\ 1 & -4 & 0 \end{vmatrix} = 0$$

Again, applying $C_2 = C_2 + 4C_1$ then

$$\Rightarrow \begin{vmatrix} 4x & 22x+2 & 8x+1 \\ 2x+2 & 11x+9 & 4x-1 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Again, applying $R_1 = R_1 - 2R_2$ then

$$\Rightarrow \begin{vmatrix} -4 & -16 & 3 \\ 2x+2 & 11x+9 & 4x-1 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Now, expanding the determinant from R_3 then,

$$\Rightarrow \begin{vmatrix} -16 & 3 \\ 11x+9 & 4x-1 \end{vmatrix} = 0$$

$\Rightarrow -64x + 16 - 33x - 27 = 0$

$\Rightarrow 97x + 11 = 0$

$\Rightarrow x = \frac{-11}{97}$

4. If $a + b + c = 0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$

[2011 Fall Q.No. 1(a) OR] [2010 Spring Q.No. 1(a) OR]

Solution: Let, $a + b + c = 0$

Here,

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Applying $c_1 \to c_1 + c_2 + c_3$ and then taking out the common value $a + b + c - x$ from c_1. Then,

$$\Rightarrow (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$$\Rightarrow (-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0 \qquad [\because a+b+c=0]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ then,

$$\Rightarrow (-x)\begin{vmatrix} 1 & c & b \\ 0 & b-c-x & a-b \\ 0 & a-x & c-b-x \end{vmatrix} = 0$$

$\Rightarrow (-x)[(b-c-x)(c-b-x) - (a-b)(a-c)] = 0$

$\Rightarrow (-x)[-(b-c-x)(b-c+x) + (a-b)(c-a)] = 0$

$\Rightarrow (-x)[-(b-c)^2 + x^2 + (a-b)(c-a)] = 0$

$\Rightarrow x[x^2 + (a-b)(c-a) - (b-c)^2] = 0$

$\Rightarrow x[x^2 + ac - bc - a^2 + ab - b^2 + 2bc - c^2] = 0$

$\Rightarrow x[x^2 + ac + bc + ab - a^2 - b^2 - c^2] = 0$

$\Rightarrow x[x^2 - (a+b+c)^2 + 3(ac + bc + ab)] = 0$

$$\Rightarrow x[x^2 + 3(ac + bc + ab)] = 0 \qquad [\because a + b + c = 0]$$

This gives,

$$x = 0, \text{ or, } x = \pm\sqrt{3ab + bc + ac}.$$

5. **Show that:**

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$$

Solution: Here,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$

[$\because$ interchanging C_2 and C_3]

$$= \begin{vmatrix} a & b & c \\ -b & c & a \\ -c & a & b \end{vmatrix}\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -bc+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

Next, we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = [a(bc-a^2) - b(b^2 - ca) + c(ab - c^2)]^2$$

$$= [abc - a^3 - b^3 + abc + abc - c^3]^2$$

$$= (-1)^2 [a^3 + b^3 + c^3 - 3abc]^2$$

$$= (a^3 + b^3 + c^3 - 3abc)^2$$

6. **Show that:**

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$ **as the product of two determinants and equals to $4a^2b^2c^2$.**

Solution: Here, $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking out the common value 'a' from C_1, 'b' from C_2 and 'c' from C_3. Then,

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 - C_3$ then,

$$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$

Again applying $R_2 \to R_2 - R_3$ then,

$$= abc \begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 2b & b+c & c \end{vmatrix}$$

$$= 2ab^2c\,(ac - c^2 + ac + bc^2)$$

$$= 4a^2b^2c^2$$

$$= \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2 = \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}\begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

7. **Show that the determinant**

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix}$$ **is a perfect square and the value is equals to $4a^2b^2c^2$**

Solution: Here,

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+0+ca \\ 0+0+ab & c^2+0+a^2 & 0+0+bc \\ 0+0+ca & 0+0+bc & b^2+a^2+0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

This shows that the given determinant is a perfect square a determinant.

Moreover,

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = [0 - c(0 - ab) + b(ac - 0)]^2$$

$$= (abc + abc)^2$$

$$= 4a^2b^2c^2$$

EXERCISE 1.4

1. **If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$. Show that its adjoint is itself.**

Solution: Let

$$A = \begin{vmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{vmatrix}$$

Here,

$$\text{Cofactor of } -4 = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0 - 4 = -4$$

$$\text{Cofactor of } -3 = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(3 - 4) = 1$$

$$\text{Cofactor of } -3 = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = (4 - 0) = 4$$

$$\text{Cofactor of } 1 = (-1)^{2+1}\begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -(-9+12) = -3$$

$$\text{Cofactor of } 0 = (-1)^{2+2}\begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = (-12+12) = 0$$

$$\text{Cofactor of } 1 = (-1)^{2+3}\begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = -(-16+12) = 4$$

$$\text{Cofactor of } 4 = (-1)^{3+1}\begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = (-3-0) = -3$$

$$\text{Cofactor of } 4 = (-1)^{3+2}\begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = -(-4+3) = 1$$

$$\text{Cofactor of } 3 = (-1)^{3+3}\begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = (0+3) = 3$$

Thus, the adjoint matrix of A be

Adj.(A) = Transpose of cofactor of A

$$= \begin{vmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{vmatrix}^T = \begin{vmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{vmatrix} = A$$

Hence, Adj. (A) = A

2. **If** $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$. **Find adj. A and show that A(adj. A) = adj(A)A = |A|I**

Solution: Let, $A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}$

Here,

$$\text{Cofactor of } 1 = (a_{11}) = (-1)^{1+1}\begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = (36-25) = 11$$

$$\text{Cofactor of } 2 = (a_{12}) = (-1)^{1+2}\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -(12-5) = -7$$

$$\text{Cofactor of } 3 = (a_{13}) = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = (5-3) = 2$$

$$\text{Cofactor of } 1 = (a_{21}) = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -(4-15) = -9$$

$$\text{Cofactor of } 3 = (a_{22}) = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = (12-3) = 9$$

$$\text{Cofactor of } 3 = (a_{23}) = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -(5-2) = -3$$

$$\text{Cofactor of } 1 = (a_{31}) = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = (10-9) = 1$$

$$\text{Cofactor of } 5 = (a_{32}) = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(5-3) = -2$$

$$\text{Cofactor of } 12 = (a_{33}) = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (3-2) = 1$$

Thus, the adjoint matrix of A be

Adj. (A) = Transpose of cofactor matrix of A

$$= \begin{vmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{vmatrix}^T = \begin{vmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

Now,

$$\text{Adj. (A).A} = \begin{vmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{vmatrix}\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}$$

$$= \begin{vmatrix} 11-9+1 & 22-27+5 & 33-45+12 \\ -7+9-2 & -14+27-10 & -21+45-24 \\ 2-3+1 & 4-9+5 & 6-15+12 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

Also,

$$\text{A.Adj.(A)} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}\begin{vmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 11-14+6 & -9+18-9 & 1-4+3 \\ 11-21+10 & -9+27-15 & 1-6+5 \\ 11-35+24 & -9+45-36 & 1-10+12 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

And,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix} = 1(36-25) - 1(24-15) + 1(10-9)$$

$$= 11 - 9 + 1$$

$$= 3$$

So, $|A|I = 3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

Thus,

Adj. (A).A = A.Adj.(A) = |A|I

3. **Find the inverse of each of the following matrices if exists:**

(i) $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$

Solution: Let,

$$A = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix}$$

Here,

$$\text{Cofactor of } 1 = (a_{11}) = (-1)^{1+1}\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = (2-4) = -2$$

$$\text{Cofactor of } 2 = (a_{12}) = (-1)^{1+2}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -(6-4) = -2$$

$$\text{Cofactor of } 5 = (a_{13}) = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = (3-1) = 2$$

$$\text{Cofactor of } 3 = (a_{21}) = (-1)^{2+1}\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = -(4-5) = 1$$

$$\text{Cofactor of } 1 = (a_{22}) = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} = (2-5) = -3$$

$$\text{Cofactor of } 4 = (a_{23}) = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1-2) = 1$$

$$\text{Cofactor of } 1 = (a_{31}) = (-1)^{3+1}\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = (8-5) = 3$$

$$\text{Cofactor of } 1 = (a_{32}) = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = -(4-15) = 11$$

$$\text{Cofactor of } 2 = (a_{33}) = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = (1-6) = -5$$

Thus, the adjoint matrix of A is

$$\text{Adj.(A)} = \text{Transpose of cofactor matrix of A}$$

$$= \begin{bmatrix} -2 & -2 & 2 \\ 1 & -3 & 1 \\ 3 & 11 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

And,

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-4) - 2(6-4) + 5(3-1)$$

$$= -2 - 4 + 10 = 4.$$

Here, $|A| = 4 \neq 0$. So, A^{-1} exists.

Now,

$$A^{-1} = \frac{\text{Adj. (A)}}{|A|} = \frac{1}{4}\begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

Note: If $|A| = 0$ then A^{-1} is not exist.

(ii) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -1 & 5 \\ 2 & 6 & 4 \\ 5 & 5 & 9 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 1 & 4 & 9 \end{bmatrix}$ **(vii)** $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: Process as (i).

4. **If** $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. **Verify that $(AB)^{-1} = B^{-1}A^{-1}$.**

Solution: Let,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Here,

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{vmatrix} = 1(18-12) - 1(2-8) + 1(3-18)$$

$$= 6 + 6 - 15 = -3$$

Thus, $|A| = -3 \neq 0$. So A^{-1} exists.

Also,

$$|B| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 1(9-1) - 2(6+1) + 0$$

$$= 8 - 14 = -6$$

Thus, $|B| = -6 \neq 0$. So B^{-1} exists.

Here, for cofactor matrix of A,

Cofactor of $a_{11} = (18-12) = 6$

Cofactor of $a_{12} = -(2-3) = 1$

Cofactor of $a_{13} = (4-9) = -5$

Cofactor of $a_{21} = -(2-8) = 6$

Cofactor of $a_{22} = (2-2) = 0$

Cofactor of $a_{23} = -(4-1) = -3$

Cofactor of $a_{31} = (3-18) = -15$

Cofactor of $a_{32} = -(3-2) = -1$

Cofactor of $a_{33} = (9-1) = 8$

For cofactor matrix of B

Cofactor of $a_{11} = (9-1) = 8$

Cofactor of $a_{12} = -(6+1) = -7$

Cofactor of $a_{13} = (-2-3) = -5$

Cofactor of $a_{21} = -(6-0) = -6$

Cofactor of $a_{22} = (3-0) = 3$

Cofactor of $a_{23} = -(-1-2) = 3$

Cofactor of $a_{31} = (-2-0) = -2$

Cofactor of $a_{32} = -(-1-0) = 1$

Cofactor of $a_{33} = (3-4) = -1$

Thus, the adjoint matrix of A is

$$\text{Adj.(A)} = \text{Transpose of cofactor matrix of A}$$

$$= \begin{bmatrix} 6 & 1 & -5 \\ 6 & 0 & -3 \\ -15 & -1 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

And, the adjoint matrix of B is

$$\text{Adj.(B)} = \text{Transpose of cofactor matrix of B}$$

$$= \begin{bmatrix} 8 & -7 & -5 \\ -6 & 3 & 3 \\ -2 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 8 & 6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

Hence, the inverse matrix of A and B are,

$$A^{-1} = \frac{\text{Adj. (A)}}{|A|} = -\frac{1}{3}\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

and

$$B^{-1} = \frac{\text{Adj. (B)}}{|B|} = -\frac{1}{6}\begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

Now,

$$B^{-1}A^{-1} = \frac{1}{18}\begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 8 & -6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 48-6+10 & 48+0+6 & -120+6-16 \\ -42+3-5 & -42+0-3 & 105-3+8 \\ -30+3+5 & -30+0+3 & 75-3-8 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 641 \end{bmatrix} \quad \text{......... (1)}$$

Next,

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+1 & 2+3-2 & 0-1+6 \\ 1+18+3 & 2+27-3 & 0-9+9 \\ 1+8+2 & 2+12-2 & 0-4+6 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 5 \\ 22 & 26 & 0 \\ 11 & 12 & 2 \end{bmatrix}$$

Then,

$$|AB| = \begin{vmatrix} 5 & 3 & 5 \\ 22 & 26 & 0 \\ 11 & 12 & 2 \end{vmatrix} = 5(22-0) - 3(44-0) + 5(264-286)$$

$$= 260 - 132 - 110 = 18$$

Thus, $|AB| = 18 \neq 0$. So, $(AB)^{-1}$ exists.

Here for cofactor matrix of AB

Cofactor of 4 = (52 − 0) = 52

Cofactor of 3 = − (44 − 0) = − 44

Cofactor of 5 = (264 − 286) = − 22

Cofactor of 22 = − (6 − 60) = 54

Cofactor of 26 = (10 − 55) = − 45

Cofactor of 0 = − (60 − 33) = −27

Cofactor of 11 = (0 − 130) = − 130

Cofactor of 12 = − (0 − 110) = 110

Cofactor of 2 = (130 − 66) = 64

Thus, the adjoint matrix of AB is

Adj.(AB) = Transpose of cofactor matrix of AB

$$= \begin{bmatrix} 52 & -44 & -22 \\ 54 & -45 & -27 \\ -130 & 110 & 64 \end{bmatrix}^T = \begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix}$$

Now,

$$(AB)^{-1} = \frac{\text{Adj. (AB)}}{|AB|} = \frac{1}{18}\begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix} \quad ....(2)$$

Hence, by (1) and (2), we conclude that,

$(AB)^{-1} = B^{-1}A^{-1}$

5. **If** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. **Verify that** $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: Process as Q. 4

6. **Find A, where** $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solution: Since we know that $(A^{-1})^{-1} = A$.

Given that, $A^{-1} = \begin{vmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{vmatrix}$

So, we wish to find the inverse of A^{-1}

Here,

$$|A^{-1}| = \begin{vmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{vmatrix} = 1(0-1) + 3(0+2) + 2(3-6)$$

$$= -1 + 6 - 6 = -1 \neq 0$$

So, inverse of A^{-1} exists

For cofactor of A^{-1},

Cofactor of $a_{11} = 0 - 1 = -1$

Cofactor of $a_{12} = -(0 + 2) = -2$

Cofactor of $a_{13} = 3 - 6 = -3$

Cofactor of $a_{21} = -(0 + 2) = -2$

Cofactor of $a_{22} = 0 - 4 = -4$

Cofactor of $a_{23} = -(-1 + 6) = -5$

Cofactor of $a_{31} = 3 - 6 = -3$

Cofactor of $a_{32} = -(-1 + 6) = -5$

Cofactor of $a_{33} = (3 - 9) = -6$

Thus, the adjoint matrix of A is,

Adj. (A^{-1}) = Transpose of Cofactor matrix of A^{-1}

$$= \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix}^T = (-1)\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Hence the inverse of A^{-1} is,

$$A = (A^{-1})^{-1} = \frac{\text{Adj }(A^{-1})}{|A^{-1}|} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

EXERCISE – 1.5

Solve the following system of linear equations by using Gauss elimination method:

1. $\mathbf{6x + 4y = 2}$

$\mathbf{3x - 5y = -34}$

Solution: Given system of linear equations be

$6x + 4y = 2$

$3x - 5y = -34$

The augment matrix of given equations be

$$\begin{bmatrix} 6 & 4 & : & 2 \\ 3 & -5 & : & -34 \end{bmatrix}$$

Applying $R_2 = 2R_2 - R_1$ then

$$\begin{bmatrix} 6 & 4 & : & 2 \\ 0 & -14 & : & -70 \end{bmatrix}$$

Thus, from R_2, we have

$-14y = -70 \Rightarrow y = 5$

and from R_2, we have

$6x + 4y = 2 \Rightarrow 6x = 2 - 4y \Rightarrow 6x = 2 - 20$ [$\because y = 5$]

$\Rightarrow 6x = -18$

$\Rightarrow x = -3$

Hence $x = -3$, $y = 5$ be required solution.

2. $\mathbf{3x - 0.5y = 0.6}$

$\mathbf{1.5x + 4.5y = 6}$

Solution: Given system of linear equations be

$3x - 0.5y = 0.6$

$1.5x + 4.5y = 6$

The augment matrix of given equations be

$$\begin{bmatrix} 3 & -0.5 & : & 0.6 \\ 1.5 & 4.5 & : & 6 \end{bmatrix}$$

Applying $R_2 = 2R_2 - R_1$ then,

$$\begin{bmatrix} 3 & -0.5 & : & 0.6 \\ 0 & 9.5 & : & 11.4 \end{bmatrix}$$

Thus, from R_2 we have,

$9.5y = 11.4 \Rightarrow y = 1.2$

and from R_1, we have

$3x - 0.5y = 0.6 \Rightarrow 3x - 0.6 = 0.6 \Rightarrow 3x = 1.2$

$\Rightarrow x = 0.4$

Hence, $x = 0.4$, $y = 1.2$ are solution of given equations.

3. $\mathbf{x + y - z = 9}$

$\mathbf{8y + 6z = -6}$

$\mathbf{-2x + 4y - 6z = 40}$

Solution: Given that,

$x + y - z = 9$

$8y + 6z = -6$

$-2x + 4y - 6z = 40$

The augment matrix of given system of equations be

$$\begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ -2 & 4 & -6 & : & 40 \end{bmatrix}$$

Applying $R_3 = R_3 + 2R_1$ then

$$\begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ 0 & 6 & -8 & : & 58 \end{bmatrix}$$

Again applying $R_3 = 4R_3 - 3R_2$ then

$$\begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ 0 & 0 & -50 & : & 2.50 \end{bmatrix}$$

Thus, from R_3, we have,

$-50z = 2.50 \Rightarrow z = -5$

and from R_2,

$8y + 6z = -6 \Rightarrow 8y = -6 - 6z = -6 + 30 = 24$

$\Rightarrow y = 3$

Also, from R_1,

$x + y - z = 9 \Rightarrow x = 9 - y + z = 9 - 3 - 5 = 1$

Hence, $x = 1$, $y = 3$, $z = -5$ are solutions of given equations.

4. $\mathbf{13x + 12y = -6}$

$\mathbf{-4x + 7y = -73}$

$\mathbf{11x - 13y = 157}$

Solution: Given equations are

$13x + 12y = -6$

$-4x + 7y = -73$

$11x - 13y = 157$

The augmented matrix of given equation be

$$\begin{bmatrix} 13 & 12 & 0 & : & -6 \\ -4 & 7 & 0 & : & -73 \\ 11 & -13 & 0 & : & 157 \end{bmatrix}$$

Applying $R_2 = 13R_2 + 4R_1$ and $R_3 = 13R_3 - 11R_1$ then

$$\begin{bmatrix} 13 & 12 & 0 & : & -6 \\ 0 & 139 & 0 & : & -973 \\ 0 & -301 & 0 & : & 2107 \end{bmatrix}$$

Applying $R_2 = \frac{R_2}{139}$ and $R_3 = \frac{R_3}{-301}$ then

$$\begin{bmatrix} 13 & 12 & 0 & : & -6 \\ 0 & 1 & 0 & : & -7 \\ 0 & 1 & 0 & : & -7 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$\begin{bmatrix} 13 & 12 & 0 & : & -6 \\ 0 & 1 & 0 & : & -7 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

This shows the system is valid for only values (as required). From R_2,

$y = -7$

and from R_1,

$13x + 12y = -6 \Rightarrow 13x = -6 - 12y = -6 + 84 = 78$

$\Rightarrow x = 6$

Hence, $x = 6$, $y = -7$ are required solution for the given equations.

5. $\mathbf{4y + 3z = 8}$

$\mathbf{2x - z = 2}$

$\mathbf{3x + 2y = 5}$

Solution: Given system of equations be

$4y + 3z = 8$

$2x - z = 2$

$3x + 2y = 5$

The augmented matrix of given system of equations be

$$\begin{bmatrix} 0 & 4 & 3 & : & 8 \\ 2 & 0 & -1 & : & 2 \\ 3 & 2 & 0 & : & 5 \end{bmatrix}$$

Here the first value in R_1 is zero which we need non-zero. So, interchanging the R_1 and R_2 in the augmented matrix that has no significant change in the system. Therefore,

$$\begin{bmatrix} 2 & 0 & -1 & : & 2 \\ 0 & 4 & 3 & : & 8 \\ 3 & 2 & 0 & : & 5 \end{bmatrix}$$

Applying $R_3 = 2R_3 - 3R_1$ then

$$\begin{bmatrix} 2 & 0 & -1 & : & 2 \\ 0 & 4 & 3 & : & 8 \\ 0 & 4 & 3 & : & 1 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$\begin{bmatrix} 2 & 0 & -1 & : & 2 \\ 0 & 4 & 3 & : & 8 \\ 0 & 0 & 0 & : & -7 \end{bmatrix}$$

In R_3, we see that $0 + 0 + 0 = -7$ which is not possible. So, the given system of equation has no solution.

6. $\mathbf{7x - 4y - 2z = -6}$

$\mathbf{16x + 2y + z = 3}$

Solution: Given system of equation be

$$7x - 4y - 2z = -6$$
$$16x + 2y + z = 3$$

The augmented matrix of given equation be

$$\begin{bmatrix} 7 & -4 & -2 & : & -6 \\ 16 & 2 & 1 & : & 3 \end{bmatrix}$$

Applying $R_2 = 7R_2 - 16R_1$ then

$$\begin{bmatrix} 7 & -4 & -2 & : & -6 \\ 0 & 78 & 39 & : & 117 \end{bmatrix}$$

From R_2, we have

$$78y + 39z = 117 \Rightarrow 2y + z = 3$$
$$\Rightarrow z = 3 - 2y$$

And from R_1, we have

$$7x - 4y - 2z = -6 \Rightarrow 7x - 4y - 6 + 4y = -6$$
$$\Rightarrow 7x = 0 \Rightarrow x = 0$$

Thus, $x = 0$, $z = 3 - 2y$ are solution of given system. This implies that, the system has no unique solution.

7. $\mathbf{x - 2y + 3z = 11}$

$\mathbf{3x + y - z = 2}$

$\mathbf{5x + 3y + 2z = 3}$

Solution: Given system of linear equations be,

$$x - 2y + 3z = 11$$
$$3x + y - z = 2$$
$$5x + 3y + 2z = 3$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & -2 & 3 : 11 \\ 3 & 1 & -1 : 2 \\ 5 & 3 & 2 : 3 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 5R_1$, then

$$\begin{bmatrix} 1 & -2 & 3 : 11 \\ 0 & 7 & -10 : -31 \\ 0 & 13 & -13 : -52 \end{bmatrix}$$

Applying $R_3 \to \frac{1}{13} R_3$, then

$$\begin{bmatrix} 1 & -2 & 3 : 11 \\ 0 & 7 & -10 : -31 \\ 0 & 1 & -1 : -4 \end{bmatrix}$$

Again applying $R_3 \to 7R_3 - R_2$, then

$$\begin{bmatrix} 1 & -2 & 3 : 11 \\ 0 & 7 & -10 : -31 \\ 0 & 0 & 3 : 3 \end{bmatrix}$$

Now, from R_3, $3z = 3 \Rightarrow z = 1$.

And from R_2, $7y - 10z = -31 \Rightarrow 7y = -31 + 10 = -21 \Rightarrow y = -3$.

Also, from R_1, $x = 11 + 2y - 3z = 11 - 6 - 3 = 2$.

Thus, $x = 2$, $y = -3$ and $z = 1$ be the solution of gives system.

8. $\mathbf{2x + 3y + 4z = 20}$

$\mathbf{3x + 4y + 5z = 26}$

$\mathbf{3x + 5y + 6z = 31}$

Solution: Given system of equations be

$$2x + 3y + 4z = 20$$
$$3x + 4y + 5z = 26$$
$$3x + 5y + 6z = 31$$

The augment matrix of given system of equations be

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 3 & 4 & 5 & : & 26 \\ 3 & 5 & 6 & : & 31 \end{bmatrix}$$

Applying $R_2 = 2R_2 - 3R_1$ and $R_3 = 2R_3 - 3R_1$ then

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 0 & -1 & -2 & : & -8 \\ 0 & 1 & 0 & : & 2 \end{bmatrix}$$

Again, applying $R_3 = R_3 + R_2$ then,

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 0 & -1 & -2 & : & -8 \\ 0 & 0 & -2 & : & -6 \end{bmatrix}$$

Here, from R_3, we have

$$-2z = -6 \qquad \Rightarrow z = 3$$

and from R_2, we have

$$-y - 2z = -8 \Rightarrow y = 8 - 2z = 8 - 6 = 2$$

Also, from R_1, we have

$$2x + 3y + 4z = 20 \Rightarrow 2x + 6 + 12 = 20$$
$$\Rightarrow 2x = 18 \Rightarrow x = 1$$

Thus, $x = 1$, $y = 2$, $z = 3$ be solution of given system of equations.

9. $\mathbf{3x - y + z = -2}$

$\mathbf{x + 5y + 2z = 6}$

$\mathbf{2x + 3y + z = 0}$

Solution: Given system of equations be

$$3x - y + z = -2$$
$$x + 5y + 2z = 6$$
$$2x + 3y + z = 0$$

The augmented matrix of given equations be

$$\begin{bmatrix} 3 & -1 & 1 & : & -2 \\ 1 & 5 & 2 & : & 6 \\ 2 & 3 & 1 & : & 0 \end{bmatrix}$$

Applying $R_2 = 3R_2 - R_1$ and $R_3 = 3R_3 - 2R_1$ then

$$\begin{bmatrix} 3 & -1 & 1 & : & -2 \\ 0 & 16 & 5 & : & 20 \\ 0 & 11 & 1 & : & 4 \end{bmatrix}$$

Again applying $R_3 = 16R_3 - 11R_2$ then

$$\begin{bmatrix} 3 & -1 & 1 & : & -2 \\ 0 & 16 & 5 & : & 20 \\ 0 & 0 & -39 & : & -156 \end{bmatrix}$$

Thus, from R_3, we have

$$-39z = -156 \Rightarrow z = 4$$

And, from R_2, we have

$$16y + 5z = 20 \quad \Rightarrow 16y + 20 = 20 \quad \Rightarrow y = 0$$

Also, from R_1, we have

$$3x - y + z = -2 \quad \Rightarrow \quad 3x - 0 + 4 = -2$$
$$\Rightarrow \quad 3x = -6 \quad \Rightarrow \quad x = -2$$

Hence, $x = -2$, $y = 0$, $z = 4$ be solution of given system of equations.

10. $\mathbf{5x + 5y - 10z = 0}$

$\mathbf{2w - 3x - 3y + 6z = 2}$

$\mathbf{4w + x + y - 2z = 4}$

Solution: Given system of linear equations be

$$5x + 5y - 10z = 0$$
$$2w - 3x - 3y + 6z = 2$$
$$4w + x + y - 2z = 4$$

The system has w with 0 as its coefficient is the first equation which is not valid the process of solving equations by Gauss–elimination method.

So, interchanging the first and second equations then the above equation can written as,

$$2w - 3x - 3y + 6z = 2$$
$$5x + 5y - 10z = 0$$
$$4w + x + y - 2z = 4$$

The augmented matrix of the system of equation be

$$\begin{bmatrix} 2 & -3 & -3 & 6 & : & 2 \\ 0 & 5 & 5 & -10 & : & 0 \\ 4 & 1 & 1 & -2 & : & 4 \end{bmatrix}$$

Applying $R_3 = R_3 - 2R_1$ then

$$\begin{bmatrix} 2 & -3 & -3 & 6 & : & 2 \\ 0 & 5 & 5 & -10 & : & 0 \\ 0 & 7 & 7 & -14 & : & 0 \end{bmatrix}$$

Again applying $R_3 = 5R_3 - 7R_2$ then

$$\begin{bmatrix} 2 & -3 & -3 & 6 & : & 2 \\ 0 & 5 & 5 & -10 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Thus, R_3 is free. So, from R_2, we have

$$5x + 5y - 10z = 0$$
$$\Rightarrow \quad x + y - 2z = 0$$

And from R_1 we have,

$$2w - 3x - 3y + 6z = 2 \quad \Rightarrow \quad 2w - 3(x + y + 2z) = 2$$
$$\Rightarrow \quad 2w - 3.0 = 2$$
$$\Rightarrow \quad 2w = 2 \quad \Rightarrow \quad w = 1.$$

Thus, $w = 1$, $x + y = 2z$ be solution of given system. This shows that the system has no unique solution.

11. $\mathbf{10x + 4y - 2z = -4}$

$\mathbf{-3w - 17x + y + 2z = 2}$

$\mathbf{w + x + y = 6}$

$\mathbf{8w - 34x + 16y - 10z = 4}$

Solution: Given system of equations be

$$10x + 4y - 2z = -4$$
$$-3w - 17x + y + 2z = 2$$
$$w + x + y = 6$$
$$8w - 34x + 16y - 10z = 4$$

Here the first equation is free from first variable w in the system which is not acceptable. So, rearranging the equations as,

$$w + x + y = 6$$
$$10x + 4y - 2z = -4$$
$$-3w - 17x + y + 2z = 2$$
$$8w - 34x + 16y - 10z = 4$$

The augment matrix of the system be

$$\begin{bmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ -3 & -17 & 1 & 2 & : & 2 \\ 8 & -34 & 16 & -10 & : & 4 \end{bmatrix}$$

Applying $R_3 = R_3 + 3R_1$ and $R_4 = R_4 - 8R_1$ then

$$\begin{bmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & -14 & 4 & 2 & : & 20 \\ 0 & -42 & 8 & -10 & : & -44 \end{bmatrix}$$

Again applying $R_3 = 5R_3 + 7R_2$, $R_4 = 5R_4 + 21R_2$ then,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & 0 & 48 & -4 & : & 72 \\ 0 & 0 & 124 & -92 & : & -304 \end{bmatrix}$$

Applying $R_3 = \frac{R_3}{4}$ and $R_4 = \frac{R_4}{4}$ then

$$\begin{bmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & 0 & 12 & -1 & : & 18 \\ 0 & 0 & 31 & -23 & : & -76 \end{bmatrix}$$

Again applying $R_4 = 14R_4 - 31R_3$ then

$$\begin{bmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & 0 & 12 & -1 & : & 18 \\ 0 & 0 & 0 & -245 & : & -1470 \end{bmatrix}$$

Thus, from R_4,

$$-245z = -1470 \Rightarrow z = 6$$

And from R_3,

$$12y - z = 18 \Rightarrow y = \frac{1}{12}[18 + z] = \frac{1}{12}[18 + 6] = 2$$

And from R_2,

$$10x + 4y - 2z = -4 \Rightarrow 10x = -4 - 4y + 2z = -4 - 8 + 12 = 0$$

$$\Rightarrow x = 0.$$

Also, from R_1,

$$w + x + y = 6 \Rightarrow w = 6 - x - y = 6 - 0 - 2 = 4.$$

Thus, $w = 4$, $x = 0$, $y = 2$, $z = 6$ are required solution of given system.

EXERCISE – 1.6

Solve by using Cramer's rule of the following system of linear equations:

1. $\mathbf{5x - 3y = 37}$

$\mathbf{-2x + 7y = -38}$

Solution: Given system of linear equations be,

$$5x - 3y = 37$$

$$-2x + 7y = -38$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 5 & -3 \\ -2 & 7 \end{vmatrix} = 35 - 6 = 29 \neq 0$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 37 & -3 \\ -38 & 7 \end{vmatrix} = 259 - 114 = 145$$

And,

$$D_2 = \begin{vmatrix} 5 & 37 \\ -2 & -38 \end{vmatrix} = -190 + 74 = -116$$

Now, by Cramer's rule,

$$x = \frac{D_1}{D} = \frac{145}{29} = 5 \quad \text{and} \quad y = \frac{D_2}{D} = \frac{-116}{29} = -4.$$

Thus, $x = 5$, $y = -4$ are solution of given system of equations.

2. $\mathbf{3x + 7y + 8z = -13}$

$\mathbf{2x + 9z = -5}$

$\mathbf{-4x + y - 26z = 2}$

Solution: Given system of liner equations be

$$3x + 7y + 8z = -13$$

$$2x + 9z = -5$$

$$-4x + y - 26z = 2$$

Here the determinant of the coefficients be

$$D = \begin{vmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{vmatrix} = 3(-9) - 7(36 - 52) + 8(2)$$

$$= -27 + 112 + 16 = 101 \neq 0.$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} -13 & 7 & 8 \\ -5 & 0 & 9 \\ 2 & 1 & -26 \end{vmatrix} = -13(1 - 9) - 7(130 - 18) + 8(-5 - 0)$$

$$= -707$$

And,

$$D_2 = \begin{vmatrix} 3 & -13 & 8 \\ 2 & -5 & 9 \\ -4 & 2 & -26 \end{vmatrix} = 3(130 - 18) + 13(-52 + 36) + 8(4 - 20)$$

$$= 336 - 208 - 128 = 0$$

Also,

$$D_3 = \begin{vmatrix} 3 & 7 & -13 \\ 2 & 0 & -5 \\ -4 & 1 & 2 \end{vmatrix} = 3(0 + 5) - 7(4 - 20) - 13(2 - 0)$$

$$= 15 + 112 - 26 = 101$$

Now, by Cramer's rule,

$$x = \frac{D_1}{D} = \frac{-707}{101} = -7, \quad y = \frac{D_2}{D} = \frac{0}{101} = 0 \quad \text{and,} \quad z = \frac{D_3}{D} = \frac{101}{101} = 1$$

Thus, $x = -7$, $y = 0$ and $z = 1$ be the solution of the given system of equations.

3. $\mathbf{x + y + z = 0}$

$\mathbf{2x + 5y + 3z = 1}$

$\mathbf{-x + 2y + z = 2}$

Solution: Given system of linear equations

$$x + y + z = 0$$

$$2x + 5y + 3z = 1$$

$$-x + 2y + z = 2$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 3 \\ -1 & 2 & 1 \end{vmatrix} = 1(5-6) - 1(2+3) + 1(4+5)$$

$$= -1 - 5 + 9 = 3 \neq 0$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 5 & 3 \\ 2 & 2 & 1 \end{vmatrix} = 0 - 1(1-6) + 1(2-10) = 0 + 5 - 8 = -3$$

and,

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = 1(1-6) - 0 + 1(4+1) = -5 - 0 + 5 = 0.$$

also,

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1(10-2) - 1(4+1) + 0 = 8 - 5 + 0 = 3$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-3}{3} = -1, \quad y = \frac{D_2}{D} = \frac{0}{3} = 0 \quad \text{and,} \quad z = \frac{D_3}{D} = \frac{3}{3} = 1$$

Thus, $x = -1$, $y = 0$ and $z = 1$ be the solution of the given system of equations.

4. $\mathbf{3x + y + 2z = 3}$

$\mathbf{2x - 3y - z = -3}$

$\mathbf{x + 2y + z = 4}$

Solution: Given system of equations be,

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(2+1) + 2(4+3)$$

$$= -3 - 3 + 14 = 8 \neq 0.$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(-3+4) + 2(-6+12)$$

$$= -3 - 1 + 12 = 8.$$

and,

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 3(-3+4) - 3(2+1) + 2(8+3)$$

$$= 3 - 9 + 22 = 16.$$

also,

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = 3(-12+6) - 1(8+3) + 3(4+3)$$

$$= -18 - 11 + 21 = -8$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{8}{8} = 1, \quad y = \frac{D_2}{D} = \frac{16}{8} = 2 \quad \text{and,} \quad z = \frac{D_3}{D} = \frac{-8}{8} = -1.$$

Thus, $x = 1$, $y = 2$ and $z = -1$ be the solution of given equations.

5. $\mathbf{x + y + z = 1}$

$\mathbf{2x + 3y + 2z = 2}$

$\mathbf{3x + 3y + 4z = 1}$

Solution: Given system of equations is,

$$x + y + z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} = 1(12-6) - 1(8-6) + 1(6-9)$$

$$= 6 - 2 - 3 = 1 \neq 0$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 3 & 4 \end{vmatrix} = 1(12-6) - 1(8-2) + 1(6-3)$$

$$= 6 - 6 + 4 = 3.$$

and,

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 4 \end{vmatrix} = 1(8-2) - 1(8-6) + 1(2-6)$$

$$= 6 - 2 - 4 = 0.$$

also,

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 1 \end{vmatrix} = 1(3-6) - 1(2-6) + 1(6-9)$$

$$= -3 + 4 - 3 = -2.$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{4}{1} = 4, \quad y = \frac{D_2}{D} = \frac{0}{1} = 0 \quad \text{and,} \quad z = \frac{D_3}{D} = \frac{-2}{1} = -2.$$

Thus, $x = 4$, $y = 0$ and $z = -2$ is the solution set of given equations.

6. $\mathbf{x + 2y + 3z = 6}$

$\mathbf{2x + 4y + z = 7}$

$\mathbf{3x + 3y + 9z = 14}$

Solution: Given system of equations be,

$x + 2y + 3z = 6$

$2x + 4y + z = 7$

$3x + 3y + 9z = 14$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 3 & 9 \end{vmatrix} = 1(36-2) - 2(18-3) + 3(4-12)$$

$$= 34 - 30 - 24 = -20 \neq 0.$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 3 & 9 \end{vmatrix} = 6(36-2) - 2(63-14) + 3(14-56)$$

$$= 204 - 98 - 126 = -20$$

and,

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = 1(63-14) - 6(18-3) + 3(28-21)$$

$$= 49 - 90 + 21 = -20$$

also,

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = 1(56-14) - 2(28-21) + 6(4-12)$$

$$= 42 - 14 - 48 = -20$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-20}{-20} = 1, \quad y = \frac{D_2}{D} = \frac{-20}{-20} = 1 \quad \text{and,} \quad z = \frac{D_3}{D} = \frac{-20}{-20} = 1$$

Thus, $x = 1$, $y = 1$ and $z = 1$ is the solution set of given equations.

7. $\mathbf{x + 3y + 6z = 2}$

$\mathbf{3x - y + 4z = 9}$

$\mathbf{x - 4y + 2z = 7}$

Solution: Given system of equations is,

$x + 3y + 6z = 2$

$3x - y + 4z = 9$

$x - 4y + 2z = 7$

Then the determinant of coefficients of the equations be

$$D = \begin{vmatrix} 1 & 3 & 6 \\ 3 & -1 & 4 \\ 1 & -4 & 2 \end{vmatrix} = 1(-2+16) - 3(6-4) + 6(-12+1)$$

$$= 14 - 6 - 66 = -58 \neq 0$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 2 & 3 & 6 \\ 3 & -1 & 4 \\ 7 & -4 & 2 \end{vmatrix} = 2(-2+16) - 3(18-28) + 6(-36+7)$$

$$= 28 + 30 - 174 = -116$$

and,

$$D_2 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & 9 & 4 \\ 1 & 7 & 2 \end{vmatrix} = 1(18-28) - 2(6-4) + 6(21-9) = 58$$

also,

$$D_3 = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 9 \\ 1 & -4 & 7 \end{vmatrix} = 1(-7+36) - 3(21-9) + 2(-12+1)$$

$$= 29 - 36 - 22 = -29.$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-116}{-58} = 2, \quad y = \frac{D_2}{D} = \frac{58}{-58} = -1 \text{ and,} \quad z = \frac{D_3}{D} = \frac{-29}{-58} = \frac{1}{2}$$

Thus, $x = 2$, $y = -1$ and $z = ½$ is the solution set of given equations.

EXAMPLE 1.7

A. Are the following sets of vectors linearly independent or dependent?

1. (1, 0, 0), (1, 1, 0), (1, 1, 1)

Solution: Given vectors are $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$ and $v_3 = (1, 1, 1)$.

And let a, b, c are any scalars.

Now, $av_1 + bv_2 + cv_3 = 0$

$\Rightarrow a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1) = 0 = (0, 0, 0)$

$\Rightarrow (a + b + c, b + c, c) = (0, 0, 0)$

This gives,

$a + b + c = 0, \quad b + c = 0, \quad c = 0.$

Solving we get, $c = 0$, $b = 0$, $a = 0$.

This shows that the given vectors are linearly independent.

2. (–1, 5, 0), (16, 8, –3), (– 64, 56, 9)

Solution: Given vectors are $v_1 = (-1, 5, 0)$, $v_2 = (16, 8, -3)$ and $v_3 = (-64, 56, 9)$.

Let a, b, c are any three scalars.

Now, $av_1 + bv_2 + cv_3 = 0$

$\Rightarrow a(-1, 5, 0) + b(16, 8, -3) + c(-64, 56, 9) = 0 = (0, 0, 0)$

$\Rightarrow (-a + 16b - 54c, 5a + 8b + 56c, -3b + 9c) = (0, 0, 0)$

This gives,

$-a + 16b - 54c = 0, \quad 5a + 8b + 56c = 0, \quad -3b + 9c = 0.$

Solving the equations, we get, $a = 0$, $b = 0$, $c = 0$.

This shows that the given vectors are linearly independent.

3. (2, – 4), (1, 9), (3, 5)

Solution: Given vectors are $v_1 = (2, -4)$, $v_2 = (1, 9)$, $v_3 = (3, 5)$.

Let a, b, c are any three scalars.

Now, $av_1 + bv_2 + cv_3 = 0$

$\Rightarrow a(2, -4) + b(1, 9) + c(3, 5) = 0$

$\Rightarrow (2a + b + 3c, -4a + 9b + 5c) = 0 = (0, 0)$

This gives

$2a + b + 3c = 0, \quad -4a + 9b + 5c = 0$

Solving we get, $a = -c$ and $b = -c$

This shows that the given vectors are linearly dependent.

4. (1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 8)

Solution: Given vectors are $v_1 = (1, 9, 9, 8)$, $v_2 = (2, 0, 0, 3)$ and $v_3 = (2, 0, 8)$.

Let a, b, c are any scalars.

Now, $av_1 + bv_2 + cv_3 = 0$

$\Rightarrow a(1, 9, 9, 8) + b(2, 0, 0, 3) + c(2, 0, 0, 8) = 0$

$\Rightarrow (a + 2b + 2c, 9a, 9a, 8a + 3b + 8c) = 0 = (0, 0, 0)$

This gives,

$a + 2b + 2c = 0, 9a = 0, 9a = 0, 8a + 3b + 8c = 0$

Solving we get, $a = 0, b = 0, c = 0$.

This shows that the vectors are linearly independent.

B. Show that the following sets of vectors are linearly independent:

1. (1, 2), (1, 3)
2. (1, 1, 1), (1, –1, 0), (0, 1, 1)
3. (2, 3, 5), (4, 9, 11)
4. (1, 1, 2), (3, 1, 2), (0, 1, 4)
5. (1, 0, 0), (0, 1, 0), (0, 0, 1)

C. Check dependent or independent of the following:

1. (0, 1, 0), (0, 0, 1), (1, 1, 1)
2. (1, 0, 1), (1, 1, 0), (–1, 0, –1)
3. (1, 2, –1), (2, 3, 0), (0, 0, 0)
4. (2, 1, 1), (3, –2, 2), (–1, 2, –1)

Solution: Process as in A.

EXERCISE 1.8

A. Find rank of the following matrices.

1. $A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$

Applying $C_2 = 2C_2 + C_1$ then

$A \sim \begin{bmatrix} 8 & 0 \\ -2 & 0 \\ 6 & 0 \end{bmatrix}$

Here, $\{C_1\}$ is linearly independent. So, rank of A = 1.

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Clearly, the matrix is in echelon form. Here, $\{R_1, R_2\}$ be linearly independent. So rank of A = 2.

3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Solution: Let, $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Clearly A has $\{R_1\}$ as linearly independent. So, rank of A = 1.

4. $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

Solution: Let, $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

Applying $R_2 = 3R_2 - 2R_1$ then

$A = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

5. $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{bmatrix}$

Applying $R_2 = R_2 - 2R_1$ then

$A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -8 & -7 \\ 0 & 8 & 7 \end{bmatrix}$

Again applying $R_3 = R_3 + R_2$ then

$A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -8 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

6. $A = \begin{bmatrix} 8 & -3 & 7 \\ -20 & -17 & -15 \\ 11 & 2 & 9 \end{bmatrix}$
7. $\begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 10 & 0 & 14 \end{bmatrix}$

Solution: Process as Q. 5

8. $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Solution: Let,

$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Applying $R_2 = R_2 - 2R_1$ then

$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Again applying $R_3 = R_3 - R_2$ then

$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This shows that A has $\{R_1, R_2\}$ as independent rows. So, rank of A = 2.

9. $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

Solution: Let

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

Applying $R_3 = R_3 + R_1$ and $R_4 = 3R_4 - R_1$ then

$$A \sim \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \end{bmatrix}$$

Applying, $R_3 = R_3 - R_2$, $R_4 = R_4 - R_2$ then

$$A \sim \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, applying $R_1 = 2R_1 - R_2$ then

$$A \sim \begin{bmatrix} 3 & -3 & 0 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

10. $A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$

Solution: Let,

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Applying $R_2 = 3R_2 - R_1$, $R_3 = 9R_3 - R_1$ then,

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 6 & 8 & 9 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Again applying $R_3 = R_3 + 2R_2$, $R_4 = R_4 - 2R_2$ then

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & -3 & 45 \end{bmatrix}$$

Again applying $R_4 = 4R_4 + R_3$ then

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & 0 & 153 \end{bmatrix}$$

This shows that A has $\{R_1, R_2, R_3, R_4\}$ as linearly independent rows. So, rank of A = 4.

11. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

Solution: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$$

Here A is 2×4 matrix. So, maximum rank of A = 2.

Applying $R_2 = R_2 + 2R_1$ then

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 11 & 15 \end{bmatrix}$$

This shows that A has 2–independent rows R_1 and R_2. So, rank of A = 2.

2. $A = \begin{bmatrix} 1 & 2 & 9 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & 9 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Clearly A is of 3×3 matrix. So, maximum rank of A = 3.

Here, applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 2R_1$ then

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Applying $R_3 = R_3 - R_2$ then,

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that A has 2–independent rows. So, rank of A = 2.

B. Find rank of AB, where

1. $A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $B = [0 \quad 3 \quad 4]$

Solution: Let

$$A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } B = [0 \quad 3 \quad 4]$$

Here A is 3×1 and B is 1×3 matrix. So, AB is defined. And,

$$AB = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} [0 \quad 3 \quad 4] = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ then

$$AB \sim \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that AB has only R_1 as independent row. So, rank of (AB) = 1.

2. $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Here A is 2x3 and B is 3x2 matrix. So, AB is defined. Then,

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{pmatrix} 1+4+0 & 4-2+0 \\ 3-4+10 & 12-4+10 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 9 & 18 \end{pmatrix}$$

Applying $C_2 = C_2 - 2C_1$ then

$$AB \sim \begin{bmatrix} 5 & -8 \\ 9 & 0 \end{bmatrix}$$

This shows that A has 2–independent rows. So, rank of AB = 2.

EXERCISE – 1.9

Check following system of linear equations is consistence or not, if consistence then solve it.

1. $x + y + z = -3$
   $3x + y - 2z = -2$
   $2x + 4y + 7z = 7$

Solution: Given system of linear equations is

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = 7$$

The augmented matrix of given equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 3 & 1 & -2 & : & -2 \\ 2 & 4 & 7 & : & 7 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - 2R_1$ then

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 2 & 5 & : & 11 \end{bmatrix}$$

Again applying $R_3 = R_3 + R_2$ then

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 0 & 0 & : & 18 \end{bmatrix}$$

Here, R_3 shows that $0.x + 0.y + 0.z = 19$ which is impossible. So, the given system is inconsistence.

2. $3x - 4y = 2$
   $-x + 3y = 1$

Solution: Given system of linear equation is

$$3x - 4y = 2$$
$$-x + 3y = 1$$

The augmented matrix of given equation is

$$[A : B] = \begin{bmatrix} 3 & -4 & : & 2 \\ -1 & 3 & : & 1 \end{bmatrix}$$

Applying $R_2 = 3R_2 + R_1$ then

$$[A : B] \sim \begin{bmatrix} 3 & -4 & : & 2 \\ 0 & 5 & : & 5 \end{bmatrix}$$

This shows that the rank of coefficient matrix A is equal to the rank of augment matrix [A : B]. So, the given system is linear equations are consistence.

And, from last matrix we have,

$3x - 4y = 30$ and $0.x + 5y = 5.$

$\Rightarrow 15x = 30$ and $\Rightarrow 5y = 5$

$\Rightarrow x = 2, y = 1$

Thus, x = 2, y = 1 be solution of given equations.

3. $x + y + z = 6$
   $x + 7y + 3z = 10$
   $x + 2y + 4z = 1$

Solution: Given system of linear equations be

$$x + y + z = 6$$
$$x + 7y + 3z = 10$$
$$x + 2y + 4z = 1$$

The augmented matrix of given equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 7 & 3 & : & 10 \\ 1 & 2 & 4 & : & 1 \end{bmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$ then

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 6 & 2 & : & 4 \\ 0 & 1 & 3 & : & -5 \end{bmatrix}$$

Again applying $R_3 = 6R_3 - R_2$ then

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 6 & 2 & : & 4 \\ 0 & 0 & 16 & : & -34 \end{bmatrix}$$

This shows that the rank of coefficient matrix (A) is equal to the rank of the augmented matrix [A:B]. So, the given system of linear equations is consistence.

And the last matrix gives us,

$x + y + z = 6$ ....(i)

$6y + 2z = 4$ ....(ii)

$16z = -34$ ....(iii)

From (iii) we get,

$$z = -\frac{34}{16} \Rightarrow z = -\frac{17}{8}$$

From (ii), we get

$$6y + 2\left(-\frac{17}{8}\right) = 4 \Rightarrow 6y = 4 + \frac{17}{4} = \frac{33}{4} \Rightarrow y = \frac{11}{8}$$

And from (i) we get,

$$x = 6 - y - z = 6 - \frac{11}{8} + \frac{17}{8} \Rightarrow x = \frac{54}{8} = \frac{27}{4}$$

Thus, $x = \frac{27}{4}$, $y = \frac{11}{8}$, $z = -\frac{17}{8}$ be solution of given equations.

4. $\mathbf{x + y + z = 3}$
$\mathbf{x + 2y + 3z = 4}$
$\mathbf{2x + 3y + 4z = 9}$

Solution: Given system of linear equations is

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$2x + 3y + 4z = 9$$

The augmented matrix of given system of equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 2 & 3 & 4 & : & 9 \end{bmatrix}$$

Applying $R_3 = R_3 - R_1 - R_2$ then

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 0 & 0 & 0 & : & 2 \end{bmatrix}$$

Here, R_3 shows that $0.x + 0.y + 0z = 2$ is impossible. So, the system of linear equations is inconsistent.

5. $\mathbf{x - y + 2z = 4}$
$\mathbf{3x + y + 4z = 6}$
$\mathbf{x + y + z = 1}$

Solution: Given system of linear equations is

$$x - y + 2z = 4$$
$$3x + y + 4z = 6$$
$$x + y + z = 1$$

The augmented matrix of given system of equation is

$$[A : B] = \begin{bmatrix} 1 & -1 & 2 & : & 4 \\ 3 & 1 & 4 & : & 6 \\ 1 & 1 & 1 & : & 1 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - R_1$ then

$$[A : B] \sim \begin{bmatrix} 1 & -1 & 0 & : & -2 \\ 0 & 4 & 0 & : & 0 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

Also, applying $R_1 = 4R_1 + R_2$ then

$$[A : B] \sim \begin{bmatrix} 1 & 0 & 0 & : & -8 \\ 0 & 4 & 0 & : & 0 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent. And, from last matrix we have,

$$x = -8,$$
$$4y = 0 \Rightarrow y = 0$$
$$\text{and } -z = -3 \Rightarrow z = 3$$

Thus, $x = -8$, $y = 0$ and $z = 3$ is the solution set of given system of linear equations.

6. $\mathbf{2x + 5y + 6z = 13}$
$\mathbf{3x + y - 4z = 0}$
$\mathbf{x - 3y - 8z = -10}$

Solution: Given system of linear equations is

$$2x + 5y + 6z = 13$$
$$3x + y - 4z = 0$$
$$x - 3y - 8z = -10$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 2 & 5 & 6 & : & 13 \\ 3 & 1 & -4 & : & 0 \\ 1 & -3 & -8 & : & -10 \end{bmatrix}$$

Applying $R_2 = 2R_2 - 3R_1$ and $R_3 = 2R_3 - R_1$ then

$$[A : B] = \begin{bmatrix} 2 & 5 & 6 & : & 13 \\ 0 & -13 & -26 & : & -39 \\ 0 & -11 & -22 & : & -33 \end{bmatrix}$$

Taking out the common value -13 from R_2 and -11 from R_3 then,

$$[A : B] = 143 \begin{bmatrix} 2 & 5 & 6 & : & 13 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & 2 & : & 3 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ and $R_1 = R_1 - 3R_2$ then

$$[A : B] = 143 \begin{bmatrix} 2 & 2 & 0 & : & 4 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = 2 = (rank of augmented matrix [A : B]).

So, the given system is consistent. Then by the last matrix, we see

$$2x + 2y = 4 \quad \text{and} \quad y + 2z = 3$$
$$\Rightarrow x + y = 2, \quad \Rightarrow y + 2z = 3$$

That is, $x = 2 - y = 2 - 3 + 2z = 2z - 1$ and $y = 3 - 2z$.

Thus, $x = 2z - 1$, $y = 3 - 2z$, $z = z$ is the solution set of given system of linear equations.

7. $\mathbf{x + y + z = 8}$
$\mathbf{x - y + z = 6}$
$\mathbf{3x + 5y - 7z = 14}$

Solution: Given system of linear equations is

$$x + y + z = 8$$
$$x - y + z = 6$$
$$3x + 5y - 7z = 14$$

The augmented matrix of given system of linear equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & -1 & 1 & : & 6 \\ 3 & 5 & -7 & ; & 14 \end{bmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 3_1$ then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 0 & : & -2 \\ 0 & 2 & -10 & : & -10 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the given system is consistent.

Here from last matrix, we have

$$x + zy = 8, \qquad -2y = -2 \qquad \text{and} \qquad -10z = -12$$

$$\Rightarrow x = 8 - y - z, \qquad y = 1 \qquad \text{and} \qquad z = \frac{6}{5}$$

$$\Rightarrow x = 8 - 1 - \frac{6}{5}, \qquad y = 1 \qquad \text{and} \qquad z = \frac{6}{5}$$

$$\Rightarrow x = \frac{29}{5}, \qquad y = 1, \qquad z = \frac{6}{5}$$

Thus, $x = \frac{29}{5}$, $y = 1$, $z = \frac{6}{5}$ is the solution set of given system of equations.

8. $\mathbf{x + y + z = 6}$

$\mathbf{x - y + z = 5}$

$\mathbf{3x + y + z = 8}$

Solution: Given system of linear equations is

$$x + y + z = 6$$
$$x - y + z = 5$$
$$3x + y + z = 8$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 1 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 3R_1$ then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 0 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 0 & : & -1 \\ 0 & 0 & -2 & : & -9 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent.

Here, from last matrix, we have

$$x + y + z = 6, \qquad -2y = -1 \qquad \text{and} \qquad -2z = -9$$

$$\Rightarrow x = 6 - y - z, \qquad y = \frac{1}{2} \qquad \text{and} \qquad z = \frac{9}{2}$$

$$\Rightarrow x = 6 - \frac{1}{2} - \frac{9}{2}, \qquad y = \frac{1}{2} \qquad \text{and} \qquad z = \frac{9}{2}$$

$$\Rightarrow x = 1, \qquad y = \frac{1}{2} \qquad \text{and} \qquad z = \frac{9}{2}$$

Thus, $x = 1$, $y = \frac{1}{2}$ and $z = \frac{9}{2}$ be solution of given system of linear equations.

9. $\mathbf{x + 2y + 3z = 1}$

$\mathbf{2x + 3y + 2z = 2}$

$\mathbf{2x + 3y + 4z = 1}$

Solution: Given system of linear equations is

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$2x + 3y + 4z = 1$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 2 & 3 & 2 & : & 2 \\ 2 & 3 & 4 & : & 1 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ and $R_3 = R_3 - 2R_1$ then

$$[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & -1 & -4 & : & 0 \\ 0 & -1 & -2 & : & -1 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & -1 & -4 & : & 0 \\ 0 & 0 & 2 & : & -1 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, system is consistent.

Now, from last matrix, we have

$$2z = -1 \Rightarrow z = -\frac{1}{2}$$

And, $-y - 4z = 0 \Rightarrow y = -4z \Rightarrow y = -2$

Also, $x + 2y + 3z = 1 \Rightarrow x = 1 - 2y - 3z,$

$$\Rightarrow x = 1 - 2y - 3z,$$

$$\Rightarrow x = 1 - 4 + \frac{3}{2} \Rightarrow x = -\frac{3}{2}$$

Thus, $x = -\frac{3}{2}$, $y = 2$, $z = -\frac{1}{2}$ is solution set of given system of linear equations.

10. $x + 2y - z = 3$
$3x - y + z = 1$
$2x - 2y + 3z = 2$ [2011 Fall Q.No. 1(b)]

Solution: Given system of linear equation is

$$x + 2y - z = 3$$
$$3x - y + z = 1$$
$$2x - 2y + 3z = 2$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 1 & : & 1 \\ 2 & -2 & 3 & : & 2 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - 2R_1$ then

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 4 & : & -8 \\ 0 & -6 & 5 & : & -4 \end{bmatrix}$$

Again applying $R_3 = 7R_3 - 6R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 4 & : & -8 \\ 0 & 0 & 11 & : & 20 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent.

Now, from above matrix, we have,

$$11z = 20 \Rightarrow z = \frac{20}{11}$$

And, $-7y + 4z = -8 \Rightarrow -7y = -8 - 4z \Rightarrow y = \frac{24}{11}$

Also, $x + 2y - z = 3 \Rightarrow x = 3 - 2y + z \Rightarrow x = \frac{5}{11}$

Thus, $x = \frac{5}{11}, y = \frac{24}{11}$ and $z = \frac{20}{11}$ is solution set of given system of linear equations.

11. $\mathbf{2x - 3y + 7z = 5}$
$\mathbf{3x + y - 3z = 13}$
$\mathbf{2x + 19y - 47z = 32}$

Solution: Given system of linear equations is

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

Applying $R_2 = 2R_2 - 3R_1$ and $R_3 = R_3 - R_1$ then

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 22 & -54 & : & 27 \end{bmatrix}$$

Again applying $R_3 = R_3 - 2R_2$ then

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 0 & 0 & : & 5 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) ≠ (rank of augmented matrix [A : B]).

So, the given system is inconsistent.

12. $\mathbf{4x - 2y + 6z = 8}$
$\mathbf{x + y - 3z = -1}$

Solution: Given system of linear equations is,

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 1 & 1 & -3 & : & -1 \end{bmatrix}$$

Applying $R_2 = 4R_2 - R_1$ then

$$[A : B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 0 & 6 & -18 & : & -12 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent. Now, from the last matrix we have,

$4x - 2y + 6z = 8$ and $6y - 18z = -12$

$\Rightarrow 2x - y + 3z = 4$ and $y - 3z = -2$

$\Rightarrow 2x = 4 + y - 3z$ and $y = 3z - 2$

$\Rightarrow 2x = 4 + 3z - 2 - 3z$ and $y = 3z - 2$

$\Rightarrow x = 1$ and $y = 3z - 2$

Thus, $x = 1$, $y = 3z - 2$, $z = z$ is solution set of given system of linear equations.

EXERCISE – 1.10

1. **Let $V = R^2$ be a vector space. Show that $W = \{(x, y): x + 2y = 0\}$ is a vector subspace of V.**

Solution: Here V is a vector space over a field F of real numbers.

Here, $W = \{(x, y): x + 2y = 0\}$. Then W is a subset of V.

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ are in W and a, b, ∈ F.

Now,

$$au + bv = a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2) \quad ....(1)$$

Here,

$$(ax_1 + bx_2, ay_1 + by_2) = (ax_1 + bx_2) + 2(ay_1 + by_2)$$
$$= a(x_1 + 2y_1) + b(x_2 + 2y_2) \quad = a.\ 0 + b.0 = 0$$

So, $au + bv \in W$.

This means W is a vector subspace of V.

2. Let $V = R^3$ be a vector space. Show that **[2014 Fall Q. No. 7(d)]**

(i) $W = \{(x, y, z): x + 2y + z = 0\}$

(ii) $W = \{(x, y, z) : 2x + y + 2z = 0\}$

are vector subspace of V.

Solution: (i) Given $V = R^3 = \{(x, y, z): x, y, z \in R\}$ is a vector space over a field F of real numbers.

Also given $W = \{(x, y, z): x + 2y + z = 0\}$. Then W is a subset of V.

And, let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are in W.

Then $x_1 + 2y_1 + z_1 = 0$ and $x_2 + 2y_2 + z_2 = 0$.

Now, for a, b, ∈ F,

$$au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \quad ....(1)$$

Here,

$$(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) = ax_1 + bx_2 + 2(ay_1 + by_2) + az_1 + bz_2$$
$$= a(x_1 + 2y_1 + z_1) + b(x_2 + 2y_2 + z_2)$$
$$= a.0 + b.0 = 0$$

Then by (1), we have $au + bv \in W$.

This means W is a vector subspace of V.

(ii) Given $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be a vector space over a field F.

Also given $W = \{(x, y, z) : 2x + y + 2z = 0\}$ then W is a subset of V.

And, let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are in W.

Then, $2x_1 + y_1 + 2z_1 = 0$ and $2x_2 + y_2 + 2z_2 = 0$

Now, for a, b ∈ F;

$$au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \quad .... (1)$$

Here,

$$(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$
$$= 2(ax_1 + bx_2) + ay_1 + by_2 + 2(az_1 + bz_2)$$
$$= a(2x_1 + y_1 + 2z_1) + b(2x_2 + y_2 + 2z_2) \quad = a.0 + b.0 = 0$$

Thus, from (1), we have $au + bv \in W$.

This means W is a vector subspace of V.

3. Let $V = R^3$ be a vector space. Let

(i) $W_1 = \{(x, 0, z): x, z \in R\}$ **(ii) $W_2 = \{(0, y, z): y, z \in R\}$**

Show that W_1, W_2 and $W_1 \cap W_2$ are vector subspace of V.

Solution: Given, $V = R^3 = \{(x, y, z): x, y, z \in R\}$ be a vector space over a field F of real numbers.

Also given $W_1 = \{(x, 0, z) : x, z \in R\}$ and $W_2 = \{(0, y, z): y, z \in R\}$.

Then W_1 and W_2 are subset of V.

Let $u_1 = (x_1, 0, z_1)$, $v_1 = (x_2, 0, z_2)$ are in W_1. And $u_2 = (0, y_3, z_3)$, $v_2 = (0, y_4, z_4)$ are in W_2.

Now, for a, b ∈ F.

$$au_1 + bv_1 = a(x_1, 0, z_1) + b(x_2, 0, z_2)$$
$$= (ax_1 + bx_2, 0, az_1 + bz_2) \in W_1$$

So W_1 is a vector subspace of V.

And.

$$au_2 + bv_2 = a(0, y_3, z_3) + b(0, y_4, z_4)$$
$$= (0, ay_3 + by_4, az_3 + bz_4) \in W_2$$

Thus, W_2 is a vector subspace of v.

Also, Let $u = (0, 0, z_1)$ and $v = (0, 0, z_2)$ are in $W_1 \cap W_2$.

Then, $W_1 \cap W_2$ be a subset of V.

Now, for a, b ∈ F.

$$au + bv = a(0, 0, z_1) + b(0, 0, z_2)$$
$$= (0, 0, az_1 + bz_2)$$

Since, $z_1, z_2 \in R$ then $az_1 + bz_2 \in R$. So, $au + bv \in W_1 \cap W_2$.

This means $W_1 \cap W_2$ be a vector subspace of V.

4. Let V = set of all 2×2 matrices, be a vector space. Let

(i) $W_1 = \left\{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in R\right\}$ **(ii) $W_2 = \left\{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} : c, d \in R\right\}$**

Show that W_1, W_2 are vector subspace of V.

Solution: Given V = {set of all 2×2 matrices}, be a vector space over a field F.

(i) Given $W_1 = \left\{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in R\right\}$. Then W_1 is a subset of V.

Let $u_1 = \begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then $u_1, u_2 \in W_1$.

Now, for a, b, ∈ F,

$$au_1 + bu_2 = a\begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix} + b\begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & ab_1 + bb_2 \\ ac_1 + bc_2 & ad_1 + bd_2 \end{pmatrix}$$

Since $b_1, b_2, c_1, c_2, d_1, d_2 \in R$.

So, $ab_1 + bb_2$, $ac_1 + bc_2$, $ad_1 + bd_2 \in R$ for a, b, ∈ R. So, $au_1 + bu_2 \in W_1$.

This means W_1 is a vector subspace of V.

(ii) Given $W_2 = \left\{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} : c, d \in R\right\}$. Then W_2 is a subset of V.

Let $v_1 = \begin{pmatrix} 0 & 0 \\ c_3 & d_3 \end{pmatrix}$ for $c_3, d_3 \in R$; and $v_2 = \begin{pmatrix} 0 & 0 \\ c_4 & d_4 \end{pmatrix}$ for $c_4, d_4 \in R$.

Then $v_1, v_2 \in W_2$.

Now, for a, b, ∈ F,

$$av_1 + bv_2 = a\begin{pmatrix} 0 & 0 \\ c_3 & d_3 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ c_4 & d_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ ac_3 & ad_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ bc_4 & bd_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ac_3+bc_4 & ad_3+bd_4 \end{pmatrix}$$

Since, $c_3, d_3, c_4, d_4 \in R$. So, $ac_3 + bc_4$, $ac_4 + bd_4 \in R$.

So that, $av_1 + bv_2 \in W_2$.

This means W_2 is a vector subspace of V.

5. Check which of the following forms a basis of R^3.

(i) (1, 1, 1), (1, 3, 2), (–1, 0, 1) (ii) (1, 2, 1), (2, 1, 0), (1, –1, 2)

(iii) (1, 1, 0), (1, 0, 1) (iv) (1, 1, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4)

Solution: (i) Let $V = R^3 = \{(x, y, z): x, y, z \in R\}$ be a vector space over a field F.

Then we wish to show (1, 1, 1), (1, 3, 2), (–1, 0, 1) form a basis for v.

Now, for a, b, c, $\in$ F,

$$a(1, 1, 1) + b(1, 3, 2) + c(-1, 0, 1) = 0$$

$$\Rightarrow (a + b - c, a + 3b, a + 2b + c) = 0 = (0, 0, 0)$$

This gives $a + b - c = 0$, $a + 3b = 0$, $a + 2b + c = 0$

Solving these equations we get $a = 0$, $b = 0$, $c = 0$.

Thus, (1, 1, 1), (1, 3, 2) and (–1, 0, 1) are linearly independent.

Next, let $(x, y, z) \in v$ then we wish to show that (x, y, z) can be written as linear combination of (1, 1, 1), (1, 3, 2) and (–1, 0, 1).

For this, let

$$(x, y, z) = \alpha(1, 1, 1) + \beta(1, 3, 2) + r(-1, 0, 1) \quad \text{.......(i)}$$
$$= (\alpha + \beta - \gamma, \alpha + 3\beta, \alpha + 2\beta + \gamma)$$

This gives,

$$x = \alpha + \beta - \gamma,\ y = \alpha + 3\beta,\ z = \alpha + 2\beta + \gamma$$

Solving we get,

$$\alpha = x - y + z,\ \beta = \frac{1}{3}(2y - x - z) \text{ and } \gamma = -\frac{1}{3}(x + y - 2z)$$

Thus (1) becomes,

$$(x, y, z) = (x - y + z)(1, 1, 1) + \frac{1}{3}(2y - x - z)(1, 3, 2) - \frac{1}{3}(x + y - 2z)(-1, 0, 1)$$

This shows that the form (1, 1, 1), (1, 3, 2), (–1, 0, 1) are in the linear combination of (x, y, z). So, the form generates R^3. Hence, the form a basis of $V = R^3$.

Solution: (ii) Given vectors are (1, 2, 1), (2, 1, 0), (1, –1, 2)

Let $V = R^3 = \{(x, y, z): x, y, z \in R\}$ be a vector space over a field F.

Then we wish to show the forms a basis for V.

For this first we will show the forms are linearly independent.

Now, for a, b, c $\in$ F,

$$a(1, 2, 1) + b(2, 1, 0) + c(1, -1, 2) = 0 = (0, 0, 0)$$

$$\Rightarrow (a + 2b + c, 2a + b - c, a + 2c) = (0, 0, 0)$$

This gives,

$$a + 2b + c = 0,\quad 2a + b - c = 0,\quad a + 2c = 0$$

Solving we get, $a = 0$, $b = 0$, $c = 0$

This shows that the forms are linearly independent.

Next, let $(x, y, z) \in V$. Then we will show that (x, y, z) can be written as linear combination of the form (1, 2, 1), (2, 1, 0), (1, –1, 2).

For this, let

$$(x, y, z) = \alpha(1, 2, 1) + \beta(2, 1, 0) + \gamma(1, -1, 2)$$
$$= (\alpha + 2\beta + \gamma, 2\alpha + \beta - \gamma, \alpha + 2\gamma)$$

This gives,

$$x = \alpha + 2\beta + \gamma,\ y = 2\alpha + \beta - \gamma,\ z = \alpha + 2$$

Solving we get,

$$\alpha = \frac{1}{9}(4y - 2x + 3z),\ \beta = \frac{1}{9}(5x - y - 3z),\ \gamma = \frac{1}{9}(x - 2y + 3z)$$

This shows that (x, y, z) can be written as a linear combination of the forms (1, 2, 1), (2, 1, 0), (1, –1, 2).

Therefore, the forms generates R^3. Hence, the forms is a basis of $V = R^3$.

Solution: (iii) Given vectors are (1, 1, 0), (1, 0, 1).

Let $V = R^3 = \{(x, y, z): x, y, z \in R\}$ be a vector space over a field F.

Then we wish to show the vectors forms a basis for R^3. For this first we will show the forms are linearly independent.

Now, for a, b, $\in$ F,

$$a(1, 1, 0) + b(1, 0, 1) = 0 = (0, 0, 0)$$

$$\Rightarrow (a + b, a, b) = (0, 0, 0)$$

This gives, $a + b = 0$, $a = 0$, $b = 0$.

This shows that the forms are linearly independent.

Next, let $(x, y, z) \in V = R^3$. Then we will show (x, y, z) can be written as a linear combination of given forms.

Let

$$(x, y, z) = \alpha(1, 1, 0) + \beta(1, 0, 1)$$
$$= (\alpha + \beta, \alpha, \beta)$$

This gives

$$x = \alpha + \beta,\ y = \alpha,\ z = \beta$$

$$\Rightarrow x = y + z,\ y = \alpha,\ z = \beta$$

This shows that (x, y, z) can not be written as linear combination of given forms.

So, the forms (1, 1, 0), (1, 0, 1) is not a basis of R^3.

Solution: (iv) Given vectors are (1, 1, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4).

Then we wish to examine the vector forms a basis of R^3 or not.

For this, first we examine linearly independence of the forms.

Let, a, b, c, d $\in$ F. Then,

$$a(1, 1, 0) + b(0, 1, 0) + c(0, 0, 1) + d(2, 3, 4) = 0 = (0, 0, 0)$$

$$\Rightarrow (a + 2d, a + b + 3d, c + 4d) = (0, 0, 0)$$

This gives, $a + 2d = 0$, $a + b + 3d = 0$, $c + 4d = 0$

Solving we get, $a = -2d$, $b = -d$, $c = -4d$.

This shows that the forms are not linearly independent. So, the forms is not a basis R^3.

6. **Check the following transformations are linear or not.**

(i) **T: $R^2 \to R$ be defined by $T(x, y) = x + y$.** **[2013 Spring Q. No. 7(a)]**

Solution: Given that a map T: $R^2 \to R$ is defined by $T(x, y) = x + y$.

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ then u, v ∈ R. And let a, b, ∈ F. Now,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= ax_1 + bx_2 + ay_1 + by_2 \\ &= a(x_1 + y_1) + b(x_2 + y_2) \\ &= aT(x_1, y_1) + bT(x_2, y_2) = aT(u) + bT(v) \end{aligned}$$

This shows that T is a linear transformation.

(ii) **T : $R^2 \to R^3$ be defined by $T(x, y) = (x, y, xy)$**

Solution: Given that a transformation T: $R^2 \to R^3$ is defined by

$$T(x, y) = (x, y, xy)$$

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ then u, v ∈ R^2.

Then T is linear only if $T(au + bv) = aT(u) + bT(v)$.

Now, for a, b ∈ F,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, ax_1 + bx_2)(ay_1 + by_2)) \quad \text{......(1)} \end{aligned}$$

Next,

$$\begin{aligned} aT(u) + bT(v) &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= a(x_1, y_1, x_1, y_1) + b(x_2, y_2, x_2y_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, ax_1y_1 + bx_2y_2) \quad \text{.......(2)} \end{aligned}$$

Thus from (1) and (2), we observe that

$$T(au + bv) \neq aT(u) + bT(v)$$

So, T is not a linear.

(iii) **T : $R^2 \to R^2$ be defined by $T(x, y) = (x + 3, y)$** **[2014 Spring Q. No. 7(c)]**

Solution: Given that a transformation T: $R^2 \to R^2$ is defined by $T(x, y) = (x + 3, y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then u, v ∈ R^2.

Also, let a, b ∈ F

Then T is linear only if $T(au + bv) = aT(u) + bT(v)$.

Now,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2 + 3, dy_1 + by_2) \quad \text{.......(1)} \end{aligned}$$

Next,

$$\begin{aligned} aT(u) + bT(v) &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= a(x_1 + 3, y_1) + b(x_2 + 3, y_2) \\ &= (ax_1 + bx_2 + 3(a + b), ay_1 + by_2) \quad \text{......(2)} \end{aligned}$$

From (1) and (2), we observe that $T(au + bv) \neq aT(u) + bT(v)$. So, T is not linear transformation.

(iv) **T : R → R be defined by $T(x) = x + 4$.**

Solution: Given transformation T: R → R is defined by $T(x) = x + 4$.

Let, x, y ∈ R and a, b ∈ F.

Then T is linear only if $T(ax + by) = aT(x) + bT(y)$.

Here,

$$T(ax + by) = ax + by + 4 \quad \text{........(1)}$$

Next,

$$\begin{aligned} aT(x) + bT(y) &= a(x + 4) + b(y + 4) \\ &= ax + by + 4(a + b) \neq ax + by + 4 = T(ax + by) \end{aligned}$$

This shows that T is not a linear transformation.

(v) **T : $R^2 \to R^2$ be defined by $T(x, y) = (x, -2y)$.**

Solution: Given transformation T: $R^2 \to R^2$ is defined by $T(x, y) = (x, -2y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then u, v ∈ R^2.

The transformation T is linear only if $T(au + bv) = aT(u) + bT(v)$.

Here, for a, b ∈ F,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2, -2(ay_1 + by_2)) \\ &= (ax_1 + bx_2, -2ay_1 - 2by_2) \quad \text{........(1)} \end{aligned}$$

Next,

$$\begin{aligned} aT(u) + bT(v) &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= a(x_1, -2y_1) + b(x_2, -2y_1) \\ &= (ax_1 + bx_2, -2ay_1 - 2by_2) \\ &= T(au + bv) \quad [\because \text{ using } (1)] \end{aligned}$$

This shows that T is linear.

EXERCISE – 1.11

1. **Find eigen value and eigen vector of the following matrices:**

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let λ and X are corresponding eigen value and eigen vector of A. The characteristics equation of A is

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1 - \lambda)(-1 - \lambda) = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Thus, the eigen value of A are $\lambda = 1, \lambda = -1$.

And for the corresponding eigen vector is

$$AX = \lambda X \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{at } \lambda = 1$$

$$\Rightarrow \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This gives,

$x = x$ and $-y = y$

$\Rightarrow$ $x = x$ (i.e. x is free) and $y = 0$

Thus, the required eigen vector with corresponding to $\lambda = 1$ is (x, 0).

And, at $\lambda = -1$,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

This gives,

$x = -x$ and $-y = -y$

$\Rightarrow$ $x = 0$ and $y = y$ (i.e. y is free).

Thus the eigen vector with corresponding to $\lambda = -1$ is (0, y).

Hence, the required eigen vector of A be (x, 0) associated with $\lambda = 1$ and (0, y) associated with $\lambda = -1$.

(ii) $\begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$ (iii) $\begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

Solution (ii) — (vi): process as (i).

(vii) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Solution: Let,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Let λ and X are corresponding eigen value and eigen vector of A.

The characteristic equation of A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \quad \Rightarrow \lambda = 3, 1$$

Thus, the eigen value of A are $\lambda = 1, \lambda = 3$.

And, the corresponding eigen vector is

$$AX = \lambda X \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$

This gives $2x + y = \lambda x$ and $x + 2y = \lambda y$

At $\lambda = 1$,

$2x + y = x$ and $x + 2y = y$

$\Rightarrow x + y = 0$ and $x + y = 0$

Thus, at $\lambda = 1$, the corresponding eigen vector be (x, –x)

And at $\lambda = 3$,

$2x + y = 3x$ and $x + 2y = 3y$

$\Rightarrow x - y = 0$ and $x - y = 0$

This shows that at $\lambda = 3$, the corresponding eigen vector is (x, y).

Hence, the required eigen vector of A is (x, –x) associated with the eigen value $\lambda = 1$ and (x, x) associated with the eigen value $\lambda = 3$.

(viii) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Let λ and X are corresponding eigen value and eigen vector of A. The characteristic equation of A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 6$$

Thus, the eigen value of A are $\lambda = 1, \lambda = 6$.

And, the corresponding eigen vector is,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5x + 4y \\ x + 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This gives, $5x + 4y = \lambda x$ and $x + 2y = \lambda y$

At $\lambda = 1$, $5x + 4y = x$ and $x + 2y = y$

$\Rightarrow 4x + 4y = 0$ and $\Rightarrow x + y = 0$

$\Rightarrow x + y = 0$ and $\Rightarrow x + y = 0$

Thus, at $\lambda = 1$, the corresponding Eigen vector is (x, –x).

At $\lambda = 6$, $5x + 4y = 6x$ and $x + 2y = 6y$

$\Rightarrow x - 4y = 0$ and $\Rightarrow x - 4y = 0$

Thus, at $\lambda = 6$, the corresponding Eigen vector is, (4x, x).

2. **Find eigen value as well as vector of the following:**

(i) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Solution: Let,

$$A = \begin{vmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

Let, λ and x are eigen value and eigen vector of A respectively. The characteristic equation of A is,

$|Ax - \lambda xI| = 0.$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -8-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-8-\lambda)(4-\lambda) = 0$$

$$\Rightarrow (3-\lambda)(8+\lambda)(4-\lambda) = 0$$

$$\Rightarrow \lambda = 3, 4, -8$$

Thus, the eigen value of A are $\lambda = 3$, $\lambda = 4$ and $\lambda = -8$

And the corresponding eigen vector of A is,

$Ax = \lambda x$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} 3x \\ -8y \\ 4z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

$$\Rightarrow 3x = \lambda x,\ -8y = \lambda y \text{ and } 4z = \lambda z$$

At $\lambda = 3$,

$3x = 3x, \quad -8y = 3y \quad$ and $\quad 4z = 3z$

These gives, $x = x$, $y = 0$ and $z = 0$.

Thus, the corresponding eigen vector to eigen value $\lambda = 3$ is $(x, 0, 0)$

And at $\lambda = 4$,

$3x = 4x, \quad -8y = 4y \quad$ and $\quad 4z = -8z$

These gives $x = 0$, $y = y$ and $z = 0$.

Thus, the corresponding eigen vector to eigen value $\lambda = 3, 4, -8$ and the corresponding Eigen vectors are $(x, 0, 0)$, $(0, 0, z)$ and $(0, y, 0)$.

(ii) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: process as (i)

(v) $\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ **[2014 Fall Q. No. 1(b)]**

Solution: Let,

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Let λ and x are eigen value and eigen vector of A, respectively.

The characteristic equation of A is,

$|Ax - \lambda xI| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(-1-\lambda)(1-\lambda) - 3] = 0$$

$$\Rightarrow (2-\lambda)[(1+\lambda)(1-\lambda) + 3] = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda^2+3) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-4) = 0$$

$$\Rightarrow \lambda = 2, -2$$

These are the eigen value of A.

And the corresponding eigen vector of A is,

$Ax = \lambda x$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+y+2z \\ -y+3z \\ y+z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

$$\Rightarrow 2x+y+2z = \lambda x, \quad -y+3z = \lambda y, \quad y+z = \lambda z$$

At $\lambda = 2$,

$2x + y + 2z = 2x, \quad -y + 3z = 2y, \quad y + z = 2z$

$\Rightarrow 2x + y + 2z = 2x, \quad 3y - 3z = 0, \quad y - z = 0$

$\Rightarrow 2x + y + 2z = 2x$ ... (i) $\quad y - z = 0$ ... (ii) $\quad y - z = 0$ ... (iii)

From (ii) and (iii) we have, $y = z$

And from (i), it is possible only if $y = z = 0$ then $2x = 2x$.

Thus, the corresponding vector is $(x, 0, 0)$.

Next, at $\lambda = -2$,

$2x + y + 2z = -2x, \quad -y + 3z = -2y, \quad y + z = -2z$

$\Rightarrow 4x + y + 2z = 0, \quad y + 3z = 0, \quad y + 3z = 0$

$\Rightarrow 4x + y + 2z = 0$ ... (iv) $\quad y + 3z = 0$ ... (v) $\quad y + 3z = 0$ ... (vi)

From (v) $y = -3z$ then (iv) gives us,

$4x - 3z + 2z = 0 \Rightarrow 4x = z$

Thus, for $x = 1$, $z = 4$ and $y = -12$

Therefore, the corresponding eigen vector is, $(1, -12, 4)$.

Hence, the eigen value of A are $\lambda = 2$, and $\lambda = -2$ and their corresponding eigen vector are $(x, 0, 0)$ and $(x, -12y, 4z)$.

(vi) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Let λ and x are eigen value and eigen vector respectively to corresponding matrix A

The characteristic equation A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)^2 (2-\lambda) - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)\{(3-\lambda)^2 - 1\} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda-1)(3-\lambda+1) = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda)(4-\lambda) = 0$$

$$\Rightarrow \lambda = 2, 2, 4$$

Thus, the eigen value are $\lambda = 2$ and $\lambda = 4$.

And, the corresponding eigen vector is,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$3x + y = \lambda x; \quad x + 3y = \lambda y; \quad 2z = \lambda z$$

At, $\lambda = 2$,

$x + y = 0$

$x + y = 0$

$2z = 2z$

This shows $x = -y$, $z = z$

At $\lambda = 4$

$-x + y = 0$

$x - y = 0$

$2z = 0$

This shows $x = y$, $z = 0$

Thus, the eigen vector be $(x, -x, z)$ at $\lambda = 2$ and $(x, x, 0)$ at $\lambda = 4$.

(vii) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Solution: Process as (vi)

(xi) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Let λ and x are eigen value and eigen vector respectively to the corresponding matrix A.

The characteristics equation of A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Applying $R_2 \to R_2 + R_3$ then,

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 + C_3$ then,

$$\Rightarrow \begin{vmatrix} 6-\lambda & 0 & 2 \\ 0 & 4-2\lambda & 2-\lambda \\ 2 & 2-\lambda & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\{(4-2\lambda)(3-\lambda) - (2-\lambda)^2\} = 0$$

$$\Rightarrow 2(6-\lambda)(2-\lambda)(3-\lambda-2+\lambda) = 0$$

$$\Rightarrow (6-\lambda)(2-\lambda) = 0 \quad [\because 2 \neq 0]$$

$$\Rightarrow \lambda = 2, 6$$

Thus, the eigen value of A is 2 and 6.

And the corresponding eigen vector be,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$6x - 2y + 2z = \lambda x; \quad -2x + 3y - z = \lambda y; \quad 2x - y + 3z = \lambda z$$

At $\lambda = 2$,

$$6x - 2y + 2z = 0; \quad 2x - y + z = 0; \quad 2x - y + z = 0$$

This implies,

$$2x - y + z = 0 \; [\because 2 \neq 0]; \quad 2x - y + z = 0; \quad 2x - y + z = 0.$$

(xii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (xiii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **[2013 Fall Q. No. 1(b)]**

Solution: Process as (vi)

(xiv) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Let λ and x are eigen value and eigen vector respectively to the matrix A.

The characteristic equation to A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\Rightarrow \lambda = 2, 3, 5$$

Thus, the eigen value to A are $\lambda = 2, \lambda = 3, \lambda = 5$.

The corresponding eigen vector be,

$$AX = \lambda X \quad \Rightarrow \quad \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$3x + y + 4z = \lambda x; \quad 2y = \lambda\, y; \qquad 5z = \lambda\, z$$

At, $\lambda = 2$,

$$x + y + 4z = 0$$
$$2y = 2y$$
$$3z = 0.$$

This shows that the corresponding vector is $(x, -x, 0)$.

At $\lambda = 3$,

$$y + 4z = 0$$
$$y = 0$$
$$2z = 0.$$

Here, x is free and $y = 0$, $z = 0$. so, the corresponding vector is $(x, 0, 0)$.

At $\lambda = 5$,

$$-2x + y + 4z = 0$$
$$3y = 0$$
$$5z = 5z.$$

Here, $y = 0$, z is free and $x = 2z$. Therefore the corresponding vector is, $(2x, 0, x)$.

Thus, the corresponding eigen vector are $(x, -x, 0)$ at $\lambda = 2$, $(x, 0, 0)$ at $\lambda = 3$ and $(2x, 0, x)$ at $\lambda = 5$.

3. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

Solution: Given matrix be,

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Let λ and x are eigen value and eigen vector of A, respectively.

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)[(2-\lambda)(1-\lambda) - 6] - 4[3(1-\lambda) - 14] + 1[9 - 7(2-\lambda)] = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2 - 3\lambda + 2 - 6) - 4(3 - 3\lambda - 14) + (9 - 14 - 7\lambda) = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda + 12\lambda + 44 - 7\lambda - 5 = 0$$

$$\Rightarrow \quad \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0 \qquad \ldots\ldots (1)$$

This is required characteristic equation of A.

To find the inverse of A, first we satisfy the Cayley–Hamiltom condition. For this we should show the corresponding condition of (1),

$$A^3 - 4A^2 - 20A - 35I = 0 \qquad \ldots\ldots\ldots (2)$$

Here, $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

So, $A^2 = A.A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

And $\quad A^3 = A.A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$

$$= \begin{bmatrix} 20+45+70 & 23+66+63 & 23+111+98 \\ 80+30+30 & 92+44+27 & 92+74+42 \\ 20+30+10 & 23+44+9 & 23+74+14 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Now, $\quad A^3 - 4A^2 - 20A - 35I$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60-0 & 232-92-140-0 \\ 140-60-80-0 & 163-88-40-35 & 208-148-68-0 \\ 60-40-20-0 & 76-36-40-0 & 111-56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$$\Rightarrow \quad A^3 - 4A^2 - 20A - 35I = 0 \qquad \ldots\ldots\ldots (3)$$

Now, multiplying (3) by A^{-1} then

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{35}(A^2 - 4A - 20I)$$

$$= \frac{1}{35}\left(\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{35}\begin{bmatrix} 20-4-20 & 23-12-0 & 23-28-0 \\ 15-16-0 & 22-8-20 & 37-12-0 \\ 10-4-0 & 9-8-0 & 14-4-20 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}.$$

4. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A.

Solution: Given matrix is, $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

Let, λ and x are eigen value and eigen vector of A, respectively.

Now, the characteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)[(-1-\lambda)^2 - 4] - 2[2(-1-\lambda) - 3] + 3[8 - 3(-1-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2+2\lambda-3)-2(2\lambda-5)+3(3\lambda+11)=0$$

$$\Rightarrow \lambda^2+2\lambda-3-\lambda^3-2\lambda^2+3\lambda+4\lambda+10+9\lambda+33=0$$

$$\Rightarrow \lambda^3+\lambda^2-18\lambda-40=0 \quad \ldots\ldots\ldots \text{(i)}$$

The characteristics equation (i) will satisfy by A if we able to show

$$A^3+A^2-18A-40I=0 \ldots\ldots\ldots \text{(ii)}$$

Here,

$$A^2=A.A=\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9&2-2+3&3+8-3\\2-2+3&4+1+4&6-4-4\\3+8-3&6-1-1&9+4+1\end{bmatrix}=\begin{bmatrix}14&3&8\\12&9&-2\\2&4&14\end{bmatrix}.$$

And $A^3=A^2.A=\begin{bmatrix}14&3&8\\12&9&-2\\2&4&14\end{bmatrix}\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}$

$$=\begin{bmatrix}14+24+6&3+18+12&8-4+42\\28-12+8&6-9+16&16+2+56\\42+12-2&9+9-4&24-2-14\end{bmatrix}=\begin{bmatrix}44&33&46\\24&13&74\\52&14&8\end{bmatrix}$$

Then,

$$A^3+A^2-18A-40I$$

$$=\begin{bmatrix}44&33&46\\24&13&74\\52&14&8\end{bmatrix}+\begin{bmatrix}14&3&8\\12&9&-2\\2&4&14\end{bmatrix}-18\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}-40\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}44+14-18-40&33+3-36-0&46+8-54-0\\24+12-36-0&13+9+18-40&74-2-72-0\\52+2-54-0&14+4-18-0&8+14+18-40\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0.$$

Thus, the equation (ii) is satisfied.

5. **Using Cayley Hamiltan theorem find the inverse of**

(i) $\begin{bmatrix}2&3\\3&5\end{bmatrix}$ (ii) $\begin{bmatrix}7&-1&3\\6&1&4\\2&4&8\end{bmatrix}$

Solution: (i) Let, $A=\begin{bmatrix}2&3\\3&5\end{bmatrix}$

The characteristics equation of A be,

$$|A-\lambda I|=0$$

where, λ is eigen value of A

$$\Rightarrow \begin{bmatrix}2-\lambda&3\\3&5-\lambda\end{bmatrix}=0$$

$$\Rightarrow 10-2\lambda-5\lambda+\lambda^2-9=0$$

$$\Rightarrow \lambda^2-7\lambda+1=0 \quad \ldots\ldots\ldots\ldots (1)$$

This is required characteristics equation of A.

The corresponding equation of A to (1) is,

$$A^2-7A+I=0 \quad \ldots\ldots\ldots\ldots (2)$$

This condition is the Cayley-Hamilton theorem.

Here,

$$A^2=A.A=\begin{bmatrix}2&3\\3&5\end{bmatrix}\begin{bmatrix}2&3\\3&5\end{bmatrix}=\begin{bmatrix}4+9&6+15\\6+15&9+25\end{bmatrix}=\begin{bmatrix}13&21\\21&34\end{bmatrix}$$

Then,

$$A^2-7A+I=\begin{bmatrix}13&21\\21&34\end{bmatrix}-7\begin{bmatrix}2&3\\3&5\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}13-14+1&21-21+0\\21-21+0&34-35+1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}=0$$

Thus, (2) is verified. So, Cayley–Hamilton theorem is satisfied.

Now, multiplying the equation $A^2-7A+I=0$ by A^{-1} then

$$A-7I+A^{-1}=0$$

$$\Rightarrow A^{-1}=-A+7I=-\begin{bmatrix}2&3\\3&5\end{bmatrix}+7\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}-2+7&-3\\-3&-5+7\end{bmatrix}=\begin{bmatrix}5&-3\\-3&2\end{bmatrix}$$

(ii) Given matrix is, $A=\begin{bmatrix}7&-1&3\\6&1&4\\2&4&8\end{bmatrix}$

Let λ is eigen value of A. The characteristics equation of A is

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix}7-\lambda&-1&3\\6&1-\lambda&4\\2&4&8-\lambda\end{vmatrix}=0$$

$$\Rightarrow (7-\lambda)[(1-\lambda)(8-\lambda)-16]+1[6(8-\lambda)-8]+3[24-2(1-\lambda)]=0$$

$$\Rightarrow (7-\lambda)(8-9\lambda+\lambda^2-16)+(48-6\lambda-8)+3(24-2+2\lambda)=0$$

$$\Rightarrow -56-63\lambda+7\lambda^2+8\lambda+9\lambda^2-\lambda^3-6\lambda+40+6\lambda+66=0$$

$$\Rightarrow \lambda^3-16\lambda^2+55\lambda+50=0 \quad \ldots\ldots\ldots (1)$$

The corresponding value of A is,

$$A^3-16A^2+55A-50I=0 \quad \ldots\ldots\ldots\ldots (2)$$

The condition (2) is Cayley–Hemilton condition for A.

Here,

$$A^2=A.A=\begin{bmatrix}7&-1&3\\6&1&4\\2&4&8\end{bmatrix}\begin{bmatrix}7&-1&3\\6&1&4\\2&4&8\end{bmatrix}$$

$$=\begin{bmatrix}49-6+6&-7-1+12&21-4+24\\42+6+8&-6+1+16&18+4+32\\14+24+16&-21+4+32&6+16+64\end{bmatrix}=\begin{bmatrix}49&4&41\\56&11&54\\54&34&86\end{bmatrix}$$

And,

$$A^3=A.A^2=\begin{bmatrix}7&-1&3\\6&1&4\\2&4&8\end{bmatrix}\begin{bmatrix}49&4&41\\56&11&54\\54&34&86\end{bmatrix}$$

$$=\begin{bmatrix}343-56+162 & 28-11+102 & 287-54+258\\ 294+56+216 & 24+11+136 & 246+54+344\\ 98+224+432 & 8+44+272 & 82+216+688\end{bmatrix}$$

$$=\begin{bmatrix}449 & 99 & 491\\ 296 & 171 & 644\\ 754 & 324 & 986\end{bmatrix}$$

Then,

$A^3-16A^2-55A+521$

$$=\begin{bmatrix}449 & 99 & 491\\ 296 & 171 & 644\\ 754 & 324 & 986\end{bmatrix}-16\begin{bmatrix}49 & 4 & 41\\ 56 & 11 & 54\\ 54 & 34 & 86\end{bmatrix}+55\begin{bmatrix}7 & -1 & 3\\ 6 & 1 & 4\\ 2 & 4 & 8\end{bmatrix}-50\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}449-784+385-50 & 99-64-55-0 & 491-656+165-0\\ 296-896+330-0 & 171-176+55-50 & 644-544+220-0\\ 754-864+110-0 & 324-544+200-0 & 986-1376+440-50\end{bmatrix}$$

6. Diogonalise the following matrices and obtain the modal matrix in each case:

(i) $\begin{bmatrix}-1 & 1 & 2\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}$ (ii) $\begin{bmatrix}9 & -1 & -9\\ 3 & -1 & 3\\ -7 & 1 & -7\end{bmatrix}$

Solution: (i) Let, $A=\begin{bmatrix}-1 & 1 & 2\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}$

Let λ be eigen value of A. Then the characteristic equation of A is,

$$|A-\lambda I|=0 \Rightarrow \begin{bmatrix}-1-\lambda & 1 & 2\\ 0 & -2-\lambda & 1\\ 0 & 0 & -3-\lambda\end{bmatrix}=0$$

$$\Rightarrow (-1-\lambda)[(-2-\lambda)(-3-\lambda)-0]=0$$

$$\Rightarrow (1+\lambda)(2+\lambda)(3+\lambda)=0$$

$$\Rightarrow \lambda=-1,-2,-3$$

Thus, $\lambda=-1, \lambda=-2, \lambda=-3$ are eigen value of A.

And, the corresponding eigen vector of A is,

$$Ax=\lambda x \Rightarrow \begin{bmatrix}-1 & 1 & 2\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\lambda\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-x+y+2z\\ -2y+z\\ -3z\end{bmatrix}=\begin{bmatrix}\lambda x\\ \lambda y\\ \lambda z\end{bmatrix}$$

This gives,

$-x+y+2z=\lambda x$, $-2y+z=\lambda y$ and $-3z=\lambda z$

At $z=-1$,

$-x+y+2z=-x$, $-2y+z=-y$ and $-3z=-z$

$\Rightarrow -x+y+2z=-x$, $y-z=0$ and $2z=0$

This implies $z=0$, $y=0$ and $x=x$.

Thus, the eigen vector $(x, 0, 0)$ is required vector corresponding to $\lambda=-1$.

At, $\lambda=-2$,

$-x+y+2z=-2x$, $-2y+z=-2y$ and $-3z=-2y$

$\Rightarrow x+y+2z=0$, $2y-z=2y$ and $z=0$

This implies $z=0$, $y=y$ and $x=-y$.

Thus, the corresponding Eigen vector to $\lambda=-2$ is $(x,-x,0)$.

Also, at $\lambda=-3$,

$-x+y+2z=-3x$; $-2y+z=-3y$ and $-3z=-3z$

$\Rightarrow 2x=y+2z=0$; $y+z=0$ and $z=z$

This implies $z=z$, $y=-z$ and $x=\frac{1}{2}(-y-2z)=\frac{1}{2}(z-2z) \Rightarrow z=-2x$.

Thus, the corresponding eigen vector to $\lambda=-3$ is $(-2x, 2x, -2x)$.

Hence, the modal matrix of A is,

$$B=\begin{bmatrix}1 & 1 & -2\\ 0 & -1 & 2\\ 0 & 0 & -2\end{bmatrix}$$

Next we wish to find out B^{-1} matrix.

For this,

Cofacter of a_{11} (1) = 2; Cofacter of a_{12} (1) = 0; Cofacter of a_{13} (–2) = 0

Cofacter of a_{21} (0) = –2; Cofacter of a_{22} (–1) = –2; Cofacter of a_{23} (–2) = 0

Cofacter of a_{31} (0) = 0; Cofacter of a_{32} (0) = –2; Cofacter of a_{33} (–2) = –1

And,

$$|B|=\begin{vmatrix}1 & 1 & -2\\ 0 & -1 & 2\\ 0 & 0 & -2\end{vmatrix}=1(2-0)=2\neq 0.$$

So, B^{-1} exists.

And,

$$B^{-1}=\frac{1}{|B|}\begin{bmatrix}2 & 0 & 0\\ -2 & -2 & -2\\ 0 & -2 & -1\end{bmatrix}^T=\frac{1}{2}\begin{bmatrix}2 & 0 & 0\\ -2 & -2 & -2\\ 0 & -2 & -1\end{bmatrix}$$

We know the diagonal matrix of A is $B^{-1}AB$. That is,

$$B^{-1}AB=\frac{1}{2}\begin{bmatrix}2 & -2 & 0\\ 0 & -2 & -2\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & -2\\ 0 & -1 & 2\\ 0 & 0 & -2\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2 & -2 & 0\\ 0 & -2 & -2\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}-1+0+0 & -1-1+0 & 2+2+4\\ 0+0+0 & 0+2+0 & 0-4+2\\ 0+0+0 & 0+0+0 & 0+0-6\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2 & -2 & 0\\ 0 & -2 & -2\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}-1 & -2 & 8\\ 0 & 2 & -2\\ 0 & 0 & -6\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-2+0+0 & -4-4+0 & 16+4+0\\ 0+0+0 & 0-4+0 & 0+4+12\\ 0+0+0 & 0+0+0 & 0+0+6\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-2 & -8 & 20\\ 0 & -4 & 16\\ 0 & 0 & 6\end{bmatrix}=\begin{bmatrix}-1 & -4 & 10\\ 0 & -2 & 8\\ 0 & 0 & 3\end{bmatrix}$$

Thus, the model matrix of A is $\begin{bmatrix} -1 & -4 & 10 \\ 0 & -2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$.

(ii) **Similar to (i).**

OTHER IMPORTANT QUESTION FROM FINAL EXAM

2012 Fall Q.No. 1(a)

What is the Hermitian matrix? Show that the square matric A is skew

$$\textbf{Hermitain } A = \begin{pmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{pmatrix}.$$

Solution: See definition of Hermitian matrix.

For problem part, see solution of Exercise 1.2 – Q.7.

2010 Fall Q.No. 1(a); 2009 Fall Q.No. 1(a); 2007 Fall Q.No. 1(a)

Define transpose of the matrix. Show that $(AB)^t = B^t A^t$, where A and B are matrices of specific order.

OR – Define transpose of a matrix. If A and B are two matrices of order m × p and p × n prove that $(AB)^T = B^T A^T$. [2009 Spring Q.No. 1(a)]

Solution: See definition of transpose of a matrix.

For problem part see the theoretical solution of $(AB)^T = B^T A^T$.

2008 Spring Q.No. 1(a)

Define inverse of a matrix. Let A and B be two matrices of specified order, then show that $(AB)^{-1} = B^{-1}A^{-1}$ where A and B are non–singular matrix.

Solution: See definition of inverse of a matrix.

For problem part see the theoretical solution of $(AB)^{-1} = B^{-1}A^{-1}$.

2003 Spring Q.No. 1(a)

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix}$, show that: A(AdjA) = (AdjA)A.

Solution: Let, $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix}$

Then the cofactor elements of A are,

Cofactor of $a_{11} = 27 - 20 = 7$ Cofactor of $a_{12} = -(18 - 16) = -2$

Cofactor of $a_{13} = 10 - 12 = -2$ Cofactor of $a_{21} = -(27 - 25) = -2$

Cofactor of $a_{22} = 9 - 20 = -12$ Cofactor of $a_{23} = -(5 - 12) = 7$

Cofactor of $a_{31} = 12 - 15 = -3$ Cofactor of $a_{32} = -(4 - 10) = 6$

Cofactor of $a_{33} = 3 - 6 = -3$

Then the adjoint matrix of A is,

Adj. (A) = Transpose of cofactor matrix of A

$$= \begin{bmatrix} 7 & -2 & -2 \\ -2 & -12 & 7 \\ -3 & 6 & -3 \end{bmatrix}^T = \begin{bmatrix} 7 & -2 & -3 \\ -2 & -12 & 6 \\ -2 & 7 & -3 \end{bmatrix}$$

Now,

$$A\,[\text{Adj. }(A)] = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix}\begin{bmatrix} 7 & -2 & -3 \\ -2 & -12 & 6 \\ -2 & 7 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 7-6-10 & -2-36+35 & -3+18-15 \\ 14-6-8 & -4-36+28 & -6+18-12 \\ 28-10-18 & -8-60+63 & -12+30-27 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -3 & 0 \\ 0 & -12 & 0 \\ 0 & -5 & -9 \end{bmatrix}$$

Also,

$$(\text{Adj. }A)\,A = \begin{bmatrix} 7 & -2 & -3 \\ -2 & -12 & 6 \\ -2 & 7 & -3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 7-4-12 & -21-6+15 & 35-8-27 \\ -2-24+24 & -6-36+30 & -10-48+54 \\ -2+14-12 & -6+21-15 & -10+28-27 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & & 0 \\ & -12 & \\ 0 & & -9 \end{bmatrix}$$

2001 Q.No. 1(a)

Define transpose of a matrix. Show that $(A + B)^T = A^T + B^T$.

Solution: See the definition of transpose of a matrix and see remark for problem.

USE OF PROPERTIES OF DETERMINANT

2009 Fall Q.No. 1(a) OR

Using the properties of determinant, show that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution: See exercise 1.3 Q.2 (xii).

2002 Q.No. 1(a) OR

Show that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

Solution: We have,

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Apply, $C_2: C_2 - C_1$, $C_3: C_3 - C_1$ we get,

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & b^2-b^2 \\ c^2 & c^2-c^2 & (a+b)^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a-b-c)(a+b+c) \\ b^2 & (c+a-b)(a+b+c) & 0 \\ c^2 & 0 & (a+b-c)(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Apply $R_1 : R_1 - R_2 - R_3$

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Expanding from R_1

$$= (a+b+c)^2 [2bc (c+a-b)(a+b-c) + 2cb^2(a+b-c) - 2b\{-c^2(c+a-b)\}]$$

$$= 2(a+b+c)^2 [bc(ac+bc-c^2+a^2+ab-ac-ab-b^2+bc) + b^2ac + b^3c - b^2c^2 + bc^3 + abc^2 - b^2c^2]$$

$$= 2(a+b+c)^2 (ab^2 + b^2c^2 - bc^3 + a^2bc - abc^2 - b^3c + b^2c^2 + b^2ac + b^3c - b^2c^2 + bc^3 + abc^2 - b^2c^2)$$

$$= 2(a+b+c)^2 (abc^2 + a^2bc + b^2ac)$$

$$= 2abc\,(a+b+c)^3.$$

CONSISTENCY OF SYSTEM OF LINEAR EQUATIONS

2012 Fall Q.No. 1(b) OR

Check the consistency of the given system of equations and solve it:

$5x + 3y + 7z = 4$; $3x + 26y + 2z = 9$; $7x + 2y + 10z = 5$.

Solution: Given system of equations be,

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

The augmented matrix of the given system is,

$$[A : C] = \begin{bmatrix} 5 & 3 & 7:4 \\ 3 & 26 & 2:9 \\ 7 & 2 & 10:5 \end{bmatrix}$$

Applying $R_2 \rightarrow 5R_2 - 3R_1$ and $R_3 \rightarrow 5R_3 - 7R_1$ then,

$$[A : C] = \begin{bmatrix} 5 & 3 & 7:4 \\ 0 & 121 & -11:33 \\ 0 & -11 & 1:-3 \end{bmatrix}$$

Again applying $R_2 \rightarrow \frac{R_2}{11}$ then,

$$[A : C] = \begin{bmatrix} 5 & 3 & 7:4 \\ 0 & 11 & -1:3 \\ 0 & -11 & 1:-3 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$, then

$$[A : C] = \begin{bmatrix} 5 & 3 & 7:4 \\ 0 & 11 & -1:3 \\ 0 & 0 & 0:0 \end{bmatrix} \qquad \text{......... (i)}$$

Here, rank of the coefficient matrix (A) is 2 and rank of the augmented matrix ([A : C]) is also 2.

Thus, rank of A = rank of [A : C].

This, shows that the given system is consistence.

Now, from R_2 in (i), $11y - z = 3$

and from R_1 in (i), $5x + 3y + 7z = 4$

Thus, $5x + 80y = 25$, $11y - z = 3$, $z = z$ be solution of given system.

2011 Spring Q.No. 1(b)

Check whether the following system of linear equation is consistent or not. If consistent solve it by using Gauss elimination method.

$2x - y + 3z = 9$; $x + y + z = 6$; $x - y + z = 2$.

Solution: Given system of linear equations are

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2$$

Rewrite the equations as,

$$x - y + z = 2$$
$$2x - y + 3z = 9$$
$$x + y + z = 6$$

The argumented matrix of the system is,

$$[A : C] = \begin{bmatrix} 1 & -1 & 1:2 \\ 2 & -1 & 3:9 \\ 1 & 1 & 1:6 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_1$ and $R_2 \rightarrow R_2 - 2R_1$ then,

$$[A : C] = \begin{bmatrix} 1 & -1 & 1:2 \\ 0 & 1 & 1:5 \\ 0 & 2 & 0:4 \end{bmatrix}$$

Again applying $R_3 \rightarrow R_3 - 2R_2$ then

$$[A : C] = \begin{bmatrix} 1 & -1 & 1:2 \\ 0 & 1 & 1:5 \\ 0 & 0 & -2:-6 \end{bmatrix}$$

Also, applying $R_3 \rightarrow \frac{-R_3}{3}$ then,

$$[A : C] = \begin{bmatrix} 1 & -1 & 1:2 \\ 0 & 1 & 1:5 \\ 0 & 0 & 1:3 \end{bmatrix} \qquad \text{............ (i)}$$

This shows that the rank of coefficient matrix (A) = 3 and rank of augmented matrix ([A : C]) = 3

That is rank of A = rank of [A : C]

Therefore, the given system is consistence

From R_3 of (i), $z = 3$.

R_2 of (i), $y + z = 5 \Rightarrow y = 5 - 3 = 2$.

R_1 of (i), $x - y + z = 2 \Rightarrow x = 2 + 2 - 3 = 1$.

Thus, $x = 1$, $y = 2$, $z = 3$ be solution of given system.

2010 Fall Q.No. 2(a)

Define consistence and inconsistence of a system of linear equations. Check consistence of given system of linear equation.

$$x + 2y + 3z = 1;\ 2x + 3y + 2z = 2;\ 2x + 3y + 4z = 1.$$

Solution: See the definition of consistence and inconsistence of a system of linear equations.

For the problem see exercise 1.9 Q. 9.

2010 Spring Q.No. 1(b)

Check whether the system of linear equations is consistent or not, if consistent solve it by using Gauss elimination method.

$$x + 6y + 2z = 0;\ 7x + 3y + z = 13;\ x + 2y + 3z + 20.$$

Solution: Given system of linear equations are,

$$x + 6y + 2z = 0$$
$$7x + 3y + z = 13$$
$$x + 2y + 3z = 20$$

The argumented matrix of given system be,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 : 0 \\ 7 & 3 & 1 : 13 \\ 1 & 2 & 3 : 20 \end{bmatrix}$$

Applying $R_2 \to R_2 - 7R_1$ and $R_3 - R_1$ then,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 : 0 \\ 0 & -39 & -13 : 13 \\ 0 & -4 & 1 : 20 \end{bmatrix}$$

Applying $R_2 \to \frac{-R_2}{39}$ and $R_3 \to \frac{-R_3}{4}$ then,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 : 0 \\ 0 & 1 & 1/3 : -1/3 \\ 0 & 1 & -1/4 : -5 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ then,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 : 0 \\ 0 & 1 & 1/3 : -1/3 \\ 0 & 0 & -7/12 : -14/3 \end{bmatrix}$$

Applying $R_3 \to \frac{-12\, R_3}{7}$ then

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 : 0 \\ 0 & 1 & 1/3 : -1/3 \\ 0 & 0 & 1 : 8 \end{bmatrix} \quad \text{.............. (i)}$$

This shows that rank of $(A) = 3 =$ rank of $[A : C]$

That means the given system is consistence.

Now,

From R_3 of (i), $z = 8$

From R_2 of (i), $y + \frac{z}{3} = \frac{-1}{3} \Rightarrow y = \frac{-1}{3} - \frac{8}{3} = -3$

From R_1 of (i), $x + 6y + 2z = 0 \Rightarrow x = 18 - 16 = 2$

Thus, $x = 2$, $y = -3$, $z = 8$ be solution of given system.