Unit 4

VECTOR CALCULUS

Definitions

Gradient of a scalar

Let f(x, y, z) be a function which is differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f is noted by ∇f and written by grad f and is defined as

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}.$$

Divergence of a vector function

Let $\vec{f}$ be a vector function that is differentiable then the divergence of $\vec{f}$ is noted by $\nabla.\vec{v}$ and written by div. $\vec{f}$ and is defined as,

$$\nabla.\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.$$

Curl of a vector function

Let $\vec{f}$ be a vector function that is differentiable. Then the curl of $\vec{f}$ is noted by $\nabla\times\vec{f}$ and written by curl $\vec{f}$ and is defined as

$$\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \quad \text{for } \vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}.$$

Directional derivative of a function

Let f be given function. Then the directional derivative of f at a point f in the direction $\vec{a}$ is denoted by $D_{\vec{a}}^{f} = \text{grad}(f).\hat{a}$.

Here, $\hat{a}$ be unit vector of $\vec{a}$, is defined as $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

Exact Integral

Let $\iint_C (f_1dx + f_2dy + f_3dz)$ be an integral with the functions f_1, f_2 and f_3 are continuous and have continuous first order partial derivatives. Then the value under the integral sign, is exact if the conditions

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x},\quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x},\quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}$$

is satisfied.

Surface integral

Any integral which is to be evaluated over a surface, is called a surface integral.

Volume integral

Any integral which is to be evaluated over a volume, is called a volume integral.

Note:

(i) If curl $\vec{f} = 0$ i.e. $\nabla\times\vec{f} = 0$ then $\vec{f}$ is irretational.

(ii) If div $\vec{f} = 0$ i.e. $\nabla.\vec{f} = 0$ then $\vec{f}$ is solenoidal.

(iii) Let $\vec{f}$ be a function. Let P be a point on a surfaces and $\vec{n}$ be unit vector at P having direction of outward drawn normal to S at P. Then $\vec{f}.\vec{n}$ called normal component of $\vec{f}$ at P.

(iv) The integral of a normal component $\vec{f}.\vec{n}$ over s is called a flux of $\vec{f}$ over s.

That is, $\iint_S \vec{f}.\vec{n}\, ds$ is a flux.

Theorem: The necessary and sufficient condition for the vector function $\vec{a}$ of the scalar variable t to have constant magnitude is $\vec{a}.\dfrac{d\vec{a}}{dt} = 0$.

[2003 Fall Q.No. 3(a)]

Proof: (Necessary condition): Let $\vec{a}$ has a constant magnitude. Then we have to show

$$\vec{a}.\frac{d\vec{a}}{dt} = 0.$$

We know,

$$\vec{a}.\vec{a} = (|\vec{a}|)^2$$

Differentiating with respect to t,

$$\Rightarrow \frac{d}{dt}(\vec{a}.\vec{a}) = \frac{d}{dt}[(|\vec{a}|)^2]$$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0 \qquad \text{since } |\vec{a}| \text{ is constant.}$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 0$$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0.$$

Sufficient condition: Let $\vec{a}.\dfrac{d\vec{a}}{dt} = 0$, then we have to show that $\vec{a}$ has a constant magnitude.

We know, $\vec{a}.\vec{a} = (|\vec{a}|)^2$.

Differentiating with respect to t

$$\frac{d}{dt}(\vec{a}.\vec{a}) = (|\vec{a}|)^2$$

$$\Rightarrow \quad 2\left(\vec{a} \cdot \frac{d\vec{a}}{dt}\right) = 2|\vec{a}|\frac{d}{dt}(|\vec{a}|)$$

$$\Rightarrow \quad 0 = |\vec{a}|\frac{d}{dt}(|\vec{a}|).$$

Thus, we get $|\vec{a}|\frac{d}{dt}|\vec{a}| = 0 \Rightarrow \frac{d}{dt}|\vec{a}| = 0$, since $|\vec{a}| \neq 0$.

Thus we get $|\vec{a}|$ as a constant. That is $\vec{a}$ has a constant magnitude.

Theorem: The necessary and sufficient condition for the vector valued function $\vec{a}$ of the scalar variable t to have a constant direction is $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$.

[2013 Spring Q.No. 2(a)] [2009 Spring Q.No. 3(b)] [2002 Q.No. 3(a)]

Proof: Necessary condition: Let $\vec{a}$ has a constant direction, then we have to show that $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$.

We know, $\vec{a} = a\,\hat{a}$ and $\hat{a}$ is a constant vector in this case where a is a magnitude of $\vec{a}$ and $\hat{a}$ be as unit vector along $\vec{a}$. Then

$$\vec{a} \times \frac{d\vec{a}}{dt} = (a\,\hat{a}) \times \left(a\frac{d\hat{a}}{dt} + \frac{da}{dt}\hat{a}\right)$$

$$= a\,\hat{a} \times \left(\frac{da}{dt}\hat{a}\right) \qquad \text{Since } \frac{d\hat{a}}{dt} = \vec{0}$$

$$= \left(a\frac{da}{dt}\right)(\hat{a} \times \hat{a}) = \left(a\frac{da}{dt}\right)\vec{0} = \vec{0}.$$

Thus, $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$.

Sufficient Condition: Let $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$, then we have to show that $\vec{a}$ has a constant direction.

Here,

$$\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0} \quad \Rightarrow \quad (a\,\hat{a}) \times \frac{d}{dt}(a\,\hat{a}) = \vec{0}$$

$$\Rightarrow \quad a^2\left(\hat{a} \times \frac{d\hat{a}}{dt}\right) = \vec{0}$$

$$\Rightarrow \quad \hat{a} \times \frac{d\hat{a}}{dt} = \vec{0} \quad \text{...... (1)} \quad [\text{Since } a^2 \neq \vec{0}]$$

Also, we have

$$\hat{a} \cdot \frac{d\hat{a}}{dt} = 0 \quad \text{......... (2)} \quad [\text{Since } \hat{a} \text{ has constant magnitude}]$$

From (1) and (2), we get

either, $\hat{a} = \vec{0}$. or $\frac{d\hat{a}}{dt} = \vec{0}$.

Here, $\hat{a} \neq \vec{0}$, so $\frac{d\hat{a}}{dt} = \vec{0}$.

Thus by definition $\hat{a}$ is constant. So $\vec{a}$ has a constant direction.

Theorem: Let f be a continuous and differentiable scalar valued function, then curl (gradf) = $\vec{0}$. **[2001 Q.No. 3(b)]**

Proof: Let f be a scalar valued function. Then

$$\text{grad } f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)f$$

$$\Rightarrow \quad \text{grad } f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

So, $$\text{curl (gradf)} = \nabla \times \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{i} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) - \vec{j}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) + \vec{k}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right)$$

$$= \vec{i}\,0 + \vec{j}\,0 + \vec{k}\,0$$

$$= \vec{0}$$

Thus, we get curl grad(f) = $\vec{0}$.

Theorem: Let $\vec{v}$ be a vector valued function, which is continuous and differentiable, then div.(curl $\vec{v}$) = 0.

Proof: Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector valued function, which is continuous and differentiable. Then

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{i} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \vec{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

Again,

$$\text{div (curl } \vec{v}) = \nabla.\left[\vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \vec{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x\partial y} - \frac{\partial^2 v_2}{\partial x\partial z} - \frac{\partial^2 v_3}{\partial y\partial x} + \frac{\partial^2 v_1}{\partial y\partial z} + \frac{\partial^2 v_2}{\partial z\partial x} - \frac{\partial^2 v_1}{\partial x\partial y}$$

$$= 0.$$

Thus, div. (curl $\vec{v}$) = 0.

Green's Theorem in a Plane:

Theorem: Let R be a closed bound region in xy plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives everywhere in some domain containing R. Then

$$\oint_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

where integration along the entire boundary C of R is in anti clockwise direction.

Proof: Let us define a region R by

$$a \le x \le b, \quad u(x) \le y \le v(x) \quad \text{(fig. 1)}$$

$$\text{and } c \le y \le d, \quad p(y) \le x \le q(y) \quad \text{(fig. 2)}$$

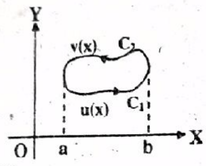


Fig. 1

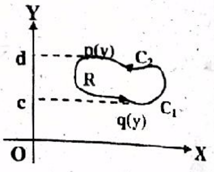


Fig. 2

Here,

$$\iint_R \frac{\partial F_1}{\partial y} dA = \int_a^b \int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy\, dx \quad \text{......... (1)}$$

For $\displaystyle\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy = [F_1(x, y)]_{u(x)}^{v(x)} = F_1[x, v(x)] - F_1[x, u(x)].$

Thus from equation (1), we get

$$\iint_R \frac{\partial F_1}{\partial y} dA = \int_a^b \{F_1[x, v(x)] - F_1[x, u(x)]\}\, dx$$

$$= \int_a^b F_1[x, v(x)]\, dx - \int_a^b F_1[x, u(x)]\, dx$$

$$= -\int_b^a F_1[x, v(x)]\, dx - \int_a^b F_1[x, u(x)]\, dx$$

Here $y = v(x)$ represents the curve C_2 and $y = u(x)$ represents C_1. Thus

$$\iint_R \frac{\partial F_1}{\partial y} dA = -\int_{C_2} F_1(x, y) dx - \int_{C_1} F_1(x, y) dx = -\oint_C F_1(x, y)\, dx$$

Similarly we get, $\displaystyle\iint_R \frac{\partial F_2}{\partial x} dA = \iint_R \frac{\partial F_2}{\partial x} dx\, dy$

$$= \int_c^d \int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x} dx\, dy$$

$$= \int_c^d F_2[q(y), y] dy + \int_d^c F_2(p(y), y) dy$$

$$= \oint_C F_2(x, y)\, dy$$

Thus we get, $\displaystyle\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C (F_1 dx + F_2 dy).$

***Note:** If $\vec{F} = F_1\vec{i} + F_2\vec{j}$. Then curl $\vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$*

and $\vec{F}.d\vec{r} = F_1 dx + F_2 dy$

Therefore Green Theorem can be written as,

$$\iint_R (\text{cun}\,\vec{F}.\vec{k})dA = \oint_C \vec{F}.d\vec{r}, \text{ where } \vec{k} \text{ be unit vector along } z\text{-axis.}$$

Gauss Divergence Theorem

[2009 Spring Q.No. 4(b)]

Statement: Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a vector valued function that is continuous and has continuous first order partial derivatives in some domain containing T. Then

$$\iint_S \vec{F}.\vec{n}\, dA = \iiint_T \text{div}\,\vec{F}.dV$$

where $\vec{n}$ is the outer unit normal vector on S.

That is the flux of $\vec{F}$ over S equals to the triple integral of the divergence of $\vec{F}$ over T.

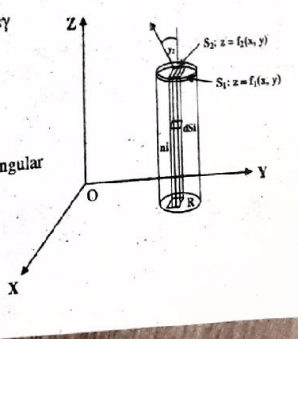
Proof: Let $\vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\theta\,\vec{k}$, where α, β, θ are angles which $\vec{n}$ makes the positive direction of x, y and z axes respectively. Then

$$\vec{F}.\vec{n} = (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).(\cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\theta\,\vec{k})$$

$$\vec{F}.\vec{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

Also,

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$



Thus divergence theorem in the rectangular form can be expressed as

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\theta)\, ds$$

To prove the theorem, consider a closed surface S which is such that any line parallel to the coordinate axes cuts S in two points only. Let the equations of the lower and upper portions S_1 and S_2 of S be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Let us projection of the surface S on the xy plane be denoted by R. Then,

$$\iiint_T \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\, dz \right\} dxdy$$

$$= \iint [F_3(x, y, z)]_{f_1}^{f_2}\, dxdy$$

$$= \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]dxdy \quad \text{........ (1)}$$

For upper portion S_2, we have

$$\delta x \delta y = \cos\theta_2\, \delta s_2 = \vec{k}.\vec{n}_2\, \delta s_2$$

and for the lower portion s_1, we have

$$\delta x \delta y = -\cos\theta_1 \delta s_1 = -\vec{k}.\vec{n}_1\, \delta s_1$$

Since θ_1 is the obtuse angle between the vector $\vec{n}_1$ and $\vec{k}$.

Thus, $\iint_R F_3(x, y, f_2)\, dx\, dy - \iint_R F_3(x, y, f_1)\, dx\, dy$ can be reduced to

$$\iint_{s2} F_3\vec{k}.\vec{n}_2 ds_2 + \iint_{s1} F_3\vec{k}\,\vec{n}_1 ds_1 = \iint_S F_3\vec{k}.\vec{n}\, ds$$

Therefore we get,

$$\iiint_T \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_S F_3\vec{k}.\vec{n}\, ds$$

Similarly by considering the projection of the surface S on other two coordinate planes, we have

$$\iiint_T \frac{\partial F_2}{\partial y}\, dx\, dy\, dz = \iint_S F_3\vec{j}.\vec{n}\, ds \quad \text{and} \quad \iiint_T \frac{\partial F_1}{\partial x}\, dx\, dy\, dz = \iint_S F_3\vec{i}.\vec{n}\, ds$$

Therefore, by adding them we get

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).\, \vec{n}\, ds$$

$$= \iint_S \vec{F}.\vec{n}\, ds$$

Thus we get, $\iiint_T \nabla \vec{F} dv = \iint_S \vec{F}.\vec{n}\, ds.$

Stoke's Theorem

Theorem: Let S be a piecewise smooth oriented surface in space and the boundary of S be a piecewise smooth simple closed curve C. Let $\vec{F}(x, y, z)$ be a continuous vector valued function that has continuous first partial derivatives in a domain in space containing S. Then

$$\iint_S (\text{curl}\,\vec{F}.\vec{n})ds = \oint_C \vec{F}d\vec{r},$$

where $\vec{n}$ is a unit normal vector of S and the integration around C is taken in anti-clockwise direction with respect to $\vec{n}$.

Proof: Let

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k} \quad \text{and} \quad \vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\theta\,\vec{k}$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\oint_C (F_1dx + F_2dy + F_3dz)$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\theta\right] ds$$

Let the equation of the surface S be $z = f(x, y)$ and let the projection of S on xy plane be the region R.

Also let the projection of the curve C on the xy plane be the curve denoted by C_1 bounding the region R.

Then,

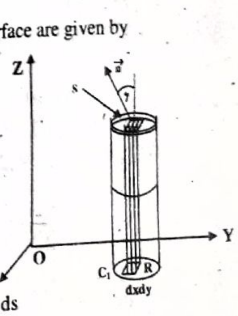
$$\oint_C F_1(x, y, z)\, dx = \iint_{c1} F_1\{x, y, f(x, y)\}\, dx$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x, y, f)\, dxdy, \text{ by Greens theorem}$$

$$= \iint \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y}\right) dxdy \quad \text{.........(1)}$$

We have the direction cosines of the normal to the surface are given by

$$\frac{\cos\alpha}{\partial f/\partial x} = \frac{\cos\beta}{\partial f/\partial y} = \frac{\cos\theta}{-1}$$



This gives

$$\frac{\partial f}{\partial y} = -\frac{\cos\beta}{\cos\theta}$$

Also we know, $dxdy = \cos\theta\, ds$ and then equation (1) reduces to

$$\oint_C F_1(x, y, z)dx = -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\cdot\frac{\cos\beta}{\cos\theta}\right)\cos\theta\, ds$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial y}\cos\theta - \frac{\partial F_1}{\partial z}\cdot\cos\beta\right) ds$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\theta - \frac{\partial F_1}{\partial y}\cdot\cos\theta\right) ds.$$

Similarly we can get

$$\oint_C F_2(x, y, z)dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\theta - \frac{\partial F_2}{\partial z}\cdot\cos\alpha\right) ds$$

and $$\oint_C F_3(x, y, z)dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cdot\cos\beta\right) ds$$

Adding these we get,

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\vec{n}\ ds.$$

This is the required form.