SEMESTER IV

MA1252 – PROBABILITY AND QUEUEING THEORY

(Common to CSE and IT)

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UNIT I RANDOM VARIABLES 9

Discrete and continuous random variables – Moments – Moment generating functions and their properties – Binomial – Poisson – Geometric – Negative binomial – Uniform – Exponential – Gamma and Weibull distribution .

UNIT II TWO DIMENSIONAL RANDOM VARIABLES 9

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and regression – Transformation of random variables – Central limit theorem.

UNIT III MARKOV PROCESSES AND MARKOV CHAINS 9

Classification – Stationary process – Markov process – Markov chains – Transition probabilities – Limiting distributions – Poisson process.

UNIT IV QUEUEING THEORY 9

Markovian models – Birth and death queuing models – Steady state results – Single and multiple server queuing models – Queues with finite waiting rooms – Finite source models – Little's formula.

UNIT V NON-MARKOVIAN QUEUES AND QUEUE NETWORKS 9

M/G/1 queue – Pollaczek – Khintchine formula – Series queues – Open and closed networks.

L: 45 T: 15 Total: 60

TEXT BOOKS

1. Ibe, O.C., "Fundamentals of Applied Probability and Random Processes", Elsevier, First Indian Reprint, 2007.

2. Gross, D. and Harris, C.M., "Fundamentals of Queuing Theory", Wiley Student Edition, 2004.

REFERENCES

1. Allen, A.O., "Probability, Statistics and Queueing Theory with Computer Applications", 2nd Edition, Elsevier, 2005.

2. Taha, H.A., "Operations Research", 8th Edition, Pearson Education, 2007

3. Trivedi, K.S., "Probability and Statistics with Reliability, Queueing andComputer Science Applications", 2nd Edition, John Wiley and Sons, 2002.

UNIT I RANDOM VARIABLES 9

Discrete and continuous random variables – Moments – Moment generating functions and their properties – Binomial – Poisson – Geometric – Negative binomial – Uniform – Exponential – Gamma and Weibull distribution .

UNIT-I

PROBABILITY AND RANDOM VARIABLES

- → Probability concepts
- → Random variables
- → Moment Generating fn/.
- → Standard distributions:
  - * Binomial
  - * Poisson
  - * Rectangular or Uniform
  - * Normal
  - * Exponential distributions
- → Fns/. of Random variables
- → Two dimensional Random variables.

PROBABILITY CONCEPTS

Probability theory had its origin in the analysis of certain games of chance that were popular in the seventeenth century. It has since found applications in many branches of science and Engineering and this extensive application makes it an important branch of study. Probability theory, as a matter of fact, is a study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

DEFINITIONS

EXPERIMENT :

Any performance which results in an outcome is called as an experiment.

Experiments may be classified as

(i) Deterministic experiment

(ii) Random experiment (or) Non-deterministic experiment.

(i) DETERMINISTIC EXPERIMENT :

An experiment whose outcome or result can be predicted with certainty is called as deterministic experiment

EXAMPLE : If the potential difference E between the two ends of a conductor and the resistance R are known, the current I flowing in the conductor is uniquely determined by Ohm's law $I = \frac{E}{R}$

(ii) RANDOM EXPERIMENT :

An experiment whose outcome cannot be predicted with certainty is called as Random experiment.

EXAMPLE : Whenever a fair 6-faced dice is rolled, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be, when the dice is rolled at a point of time.

(iv) SURE EVENT :

If ~~the~~ an event corresponds to all outcomes of the ~~ex~~ random experiment, it is called as a sure event. A sure event is also called as a certain event.

EXAMPLE :

Let B be the event of getting a nos. which is less than 7. when a die is thrown.

$B = \{1, 2, 3, 4, 5, 6\}$.

MUTUALLY EXCLUSIVE EVENTS :

Two or more events are said to be mutually exclusive if the happening of one event completely excludes the happening of the other events.

EXAMPLE :

Consider an experiment of throwing a die.

$\therefore \quad S = \{1, 2, 3, 4, 5, 6\}$

Let A be the event of getting nos. less than or equal to 2

$\therefore \quad A = \{1, 2\}$

Let B be the event of getting nos. greater than 4

$B = \{5, 6\}$.

Here the occurrence of A excludes the occurrence of B and vice versa.

→ $\therefore$ If A and B are mutually exclusive events then $P(A \cap B) = 0 \quad (\because A \cap B = \emptyset)$

NOTE :

- All Equally likely events need not be M.e.e.
- ~~M.e.e~~ Equally likely events when we select. M.e.e occur when it occurs

(3)

INDEPENDENT AND DEPENDENT EVENTS:

An event whose occurrence is not influenced by the occurrence or non-occurrence of any other event is called as an independent event.

EXAMPLE:

Consider an experiment of throwing 'n' coins successfully.

Here the outcomes of any trial are not influenced by the outcomes of the previous trials.

Let H_1 : Getting a head in first trial

H_2 : Getting a head in second trial

T_1 : Getting a tail in first trial

T_2 : Getting a tail in second trial.

→ Here T_2 or H_2 do not depend on T_1 or H_1. Hence T_2 and H_2 are independent of T_1 and H_1.

→ Here T_1 occurs only if H_1 fails to occur and vice versa. ie., the occurrence of T_1 depends on non-occurrence of H_1.

Hence T_1 and H_1 are dependent events.

NOTE: Mutually exclusive events are dependent events.

EQUALLY LIKELY EVENTS:

Two or more events are said to be equally likely if we do not have any reason to prefer one event to the other event.

EXAMPLE: Consider an experiment of tossing a coin.

Let A be the event of getting Head.

Let B be the event of getting tail.

When we toss a coin, we cannot predict or prefer that surely event A will occur.

EXHAUSTIVE EVENTS:

The totality of all outcomes of the random experiment is called as exhaustive events.

EXAMPLE:

Consider an experiment of throwing a die.

$\therefore S = \{1, 2, 3, 4, 5, 6\}$

Here the nos. of exhaustive events are 6.

FAVOURABLE EVENTS

The nos. of cases which favour the happening of an event is called as favourable nos. of cases or favourable events.

EXAMPLE:

In throwing of a die, the nos. of favourable events to the appearance of multiple of 3 are two (i.e., 3 and 6).

MATHEMATICAL (OR) APRIORI DEFN. OF PROBABILITY:

Let S be the sample space and A be an event associated with a random experiment. Let $n(S)$ and $n(A)$ be the no. of elements of S and A. Then the proby. of event A occurring, denoted as $P(A)$ is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{No. of cases favourable to } A}{\text{Exhaustive no. of cases in } S}$$

EXAMPLE:

The proby. of getting an even no. in die throwing experiment is $\frac{1}{2}$ as

$$S = \{1, 2, 3, 4, 5, 6\}, \quad n(S) = 6$$

$$A = \{2, 4, 6\} \qquad n(A) = 3$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$

STATISTICAL OR APOSTERIORI PROBY.:

Let a random experiment be repeated 'n' times and let an event A occur n_A times out of the 'n' trials. The ratio $\frac{n_A}{n}$ is called the relative frequency of the event A.

As n increases, $\frac{n_A}{n}$ shows a tendency to stabilise and to approach a constt. value. This value, denoted by $P(A)$ is called the proby. of the event A.

i.e., $P(A) = \lim_{n \to \infty} \frac{n_A}{n}$.

ADDITION THEOREM OF PROBABILITY

STATEMENT : If A and B are any two events which are not disjoint

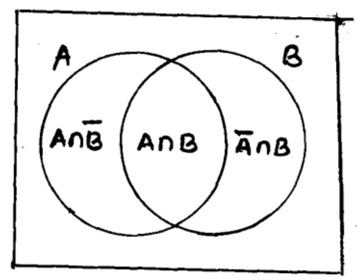
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

PROOF:

Given : A and B are not disjoint

consider

$$A \cup B = A \cup (\bar{A} \cap B)$$



$$\Rightarrow \quad P(A \cup B) = P[A \cup [\bar{A} \cap B]]$$

$$= P(A) + P(\bar{A} \cap B) \quad (\because A \text{ and } \bar{A} \cap B \text{ are m.e.e})$$

$$\Rightarrow \quad P(\bar{A} \cap B) = P(A \cup B) - P(A) \quad \text{---} (1)$$

consider $B = (A \cap B) \cup (\bar{A} \cap B)$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B) \quad (\because A \cap B \text{ \& } \bar{A} \cap B \text{ are m.e})$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) \quad \text{---} (2)$$

$\therefore$ From (1) & (2) we have

$$P(A \cup B) - P(A) = P(B) - P(A \cap B)$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

NOTE : If A and B are disjoint, then $A \cap B = \phi$
$\therefore P(A \cap B) = 0$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - 0$$

$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

ADDITION THEOREM FOR THREE EVENTS

STATEMENT: If A, B and C are any three events which are not disjoint, then

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

PROOF:

$$P(A\cup B\cup C) = P[(A\cup B)\cup C]$$

$$= P[A\cup B]+P(C)-P[(A\cup B)\cap C]$$

$$= P(A\cup B)+P(C)-P[(A\cap C)\cup(B\cap C)] \quad (\because \text{Distributive law})$$

$$= \underline{P(A\cup B)}+P(C)-\{P(A\cap C)+P(B\cap C)-P(A\cap B\cap C)\}$$

$$= \{P(A)+P(B)-P(A\cap B)\}+P(C)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

$$= P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

NOTE: GENERALISATION OF ADDITION THEOREM OR EXTENSION OF ADDITION THEOREM

$$P(A_1\cup A_2) = \sum_{i=1}^{2} P(A_i) - P(A_1\cap A_2)$$

$$P(A_1\cup A_2\cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i\neq j=1}^{3} P(A_i\cap A_j) + P(A_1\cap A_2\cap A_3)$$

$$P(A_1\cup A_2\cup\cdots\cup A_r) = \sum_{i=1}^{r} P(A_i) - \sum_{i\neq j=1}^{r} P(A_i\cap A_j) + \sum_{i\neq j\neq k=1}^{r} P(A_i\cap A_j\cap A_k)+\cdots+(-1)^{r-1}P(A_1\cap A_2\cap\cdots\cap A_r)$$

BASIC PROBLEMS

(6)

SS1) From a pack of nos. numbered from 1 to 10, one no. is selected at random. What is the chance that it is a no. greater than 6.

Soln:-

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Let A : Event of getting no. greater than 6.

$A = \{7, 8, 9, 10\}$

$P(A) = \dfrac{4C_1}{10C_1} = \dfrac{4}{10} = \dfrac{2}{5}$

SS2) From a pack of cards containing 4 white cards & 7 yellow cards, two cards are drawn at random. What is the probability that

(i) both are white.

(ii) One is white and one is yellow.

Soln:-

$S = \{4W, 7Y\}$

(i) $P(\text{Both are white}) = \dfrac{4C_2}{11C_2} = \dfrac{6}{55}$

(ii) $P(\text{One is white and one is yellow}) = \dfrac{4C_1 \times 7C_1}{11C_2} = \dfrac{4 \times 7}{55} = \dfrac{28}{55}$

SSQ) Three coins are thrown simultaneously. what is the proby. of getting 2 tails.

Soln

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$P(2 \text{ tails}) = \frac{3C_1}{8C_1} = \frac{3}{8}$.

SSQ) Two dice are rolled simultaneously. what is the chan. that the sum of outcomes is 7?

Soln

$$S = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{array} \right\}$$

$p(\text{sum is } 7) = \frac{6C_1}{36C_1} = \frac{6}{36} = \frac{1}{6}$.

SSQ) A committee of 5 members is to be formed from a group containing 6 advocates, 7 auditors and 8 managers. what is the chance that the committee contains atleast one manager?

Soln

$S = \{6 \text{ advocates}, 7 \text{ auditors}, 8 \text{ managers}\}$

$$\begin{aligned} p(\text{atleast one manager}) &= 1 - P(\text{none is a manager}) \\ &= 1 - \frac{13C_5}{21C_5} \\ &= 0.94. \end{aligned}$$

BINOMIAL DISTRIBUTION

ASSUMPTIONS:

(1) The random experiment corresponds to a two possible outcomes

(2) The no. of trials is finite.

(3) The trials are independent

(4) The proby. of success is a const. from to trial

NOTATIONS:

$n \rightarrow$ no. of trials

$p \rightarrow$ proby. of success

$q \rightarrow$ proby. of failure

$X \rightarrow$ A r.v which represents the no. of suc

PROBABILITY MASS FN.:

A discrete r.v X is said to follow Binomial distn. if its probability mass fn. is

$$p(x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2 \ldots n$$

MOMENT GENERATING FUNCTION:

The M.G.F of a Binomial variate is

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$M_x(t) = \sum_{x=0}^{n} e^{tx} \, nC_x p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= \{nC_0 \, q^n + nC_1 (pe^t) q^{n-1} + \cdots + nC_n (pe^t)^n\}$$

$$= \{q^n + nC_1 (pe^t) q^{n-1} + \cdots + (pe^t)^n\}$$

$$= (q + pe^t)^n$$

$$M_x(t) = (pe^t + q)^n$$

RAW MOMENTS (OR) TO FIND MEAN AND VARIANCE FROM M.G.F

The relation between M.G.F & raw moments is given by

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0} \quad \text{——} (1)$$

TO FIND μ_1' (MEAN)

Put $r=1$ in (1)

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$\mu_1' = \left[\frac{d}{dt}(q+pe^t)^n\right]_{t=0}$$

$$=\left[n(q+pe^t)^{n-1}(pe^t)\right]_{t=0}$$

$$= n(q+pe^0)^{n-1}(pe^0)$$

$$= n(q+p)^{n-1}p$$

$$= n(1)^{n-1}p \qquad (\because q+p=1)$$

$$\mu_1' = np$$

i.e., $\text{Mean} = \mu_1' = np$

TO FIND μ_2':

put $r=2$ in (1)

$$\mu_2' = \left\{\frac{d^2}{dt^2}(q+pe^t)\right\}_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{d}{dt}(q+pe^t)\right\}\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[n(q+pe^t)^{n-1}pe^t\right]\right\}_{t=0}$$

$$= np\left\{\frac{d}{dt}((q+pe^t)^{n-1}e^t)\right\}_{t=0}$$

$$= np\left\{(n-1)(q+pe^t)^{n-2}(pe^t)e^t + (q+pe^t)^{n-1}e^t\right\}_{t=0}$$

$$= np\left\{(n-1)(q+p)^{n-2}p + (q+p)^{n-1}\right\}$$

$$= np\left\{(n-1)p+1\right\}$$

$$= np\{(n-1)p+1\}$$

$$\mu_2' = n(n-1)p^2 + np$$

VARIANCE OF BINOMIAL DISTN.

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= (n(n-1)p^2 + np) - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\mu_2 = npq$$

$$\text{VARIANCE} = \mu_2 = npq$$

NOTE:

$$\sum_{x=0}^{n} nC_x\, p^x q^{n-x} = (p+q)^n$$

$$\sum_{x=1}^{n} (n-1)C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)} = (p+q)^{n-1}$$

EXPECTATION OF BINOMIAL DISTRIBUTION
(MEAN OF BINOMIAL DISTRIBUTION)

$$E(X) = \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \frac{n(n-1)!}{\cancel{x}(x-1)!(n-x)!} p^x q^{n-x}$$

$$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} (n-1)\, C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(p+q)^{n-1}$$

$$E(X) = np$$

VARIANCE OF BINOMIAL DISTN.

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} x^2 \, nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1) + x] \, nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \, nC_x \, p^x q^{n-x} + \sum_{x=0}^{n} x \, nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^x q^{n-x} + \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x q$$

$$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)} \; n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!\,[(n-2)-(x-2)]!} p^2 . p^{x-2} q^{(n-2)-(x}$$

$$+ \quad E(X)$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} p^{(x-2)} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$E(X^2) = n(n-1)p^2 + np$$

(44)

$$V(x) = E(x^2) - (E(x))^2$$

$$= (n(n-1)p^2 + np) - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$V(x) = npq$$

ADDITIVE PROPERTY OF BINOMIAL DISTN.

Let x_1 be a binomial r.variate with parameters n_1 and p. Let x_2 be a binomial r.variate with parameters n_2 and p.

Then $x_1 + x_2$ is a binomial r.variate with parameters n_1+n_2 & p

PROOF

$$X_1 \sim B(n_1, p)$$

$$X_2 \sim B(n_2, p)$$

If X is a binomial random variate with parameters n & p, then W.K.T $M_X(t) = (pe^t+q)^n$

$$\therefore M_{X_1}(t) = (pe^t+q)^{n_1}$$

$$M_{X_2}(t) = (pe^t+q)^{n_2}$$

$$\therefore M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$$

$$= (pe^t+q)^{n_1}(pe^t+q)^{n_2}$$

$$M_{X_1+X_2}(t) = (pe^t+q)^{n_1+n_2}$$

$\Rightarrow X_1+X_2$ is also a binomial variate with parameters (n_1+n_2, p)

RECURRENCE RELATION FOR CENTRAL MOMENTS OF BINOMIAL DISTRIBUTION

The recurrence relation for the central mome[nts] of binomial distn. is gn. by

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

PROOF:

W.k.T $\mu_r = E(X - E(X))^r$

$\mu_r = E(X - np)^r \quad (\because E(X) = np)$

$$= \sum_{x=0}^{n} (x - np)^r p(x)$$

$$\mu_r = \sum_{x=0}^{n} (x-np)^r \, nC_x p^x q^{n-x}$$

Diff' w.r.t 'p'

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} r(x-np)^{r-1}(-n)\, nC_x p^x q^{n-x} + \sum_{x=0}^{n} (x-np)^r nC_x x p^{x-1} q^{n-x}$$

$$+ \sum_{x=0}^{n} (x-np)^r nC_x p^x (n-x)(1-p)^{n-x-1}(-1)$$

$$= -nr\sum_{x=0}^{n} (x-np)^{r-1} nC_x p^x q^{n-x} + \sum_{x=0}^{n} (x-np)^r nC_x p^x q^{n-x}\left\{\frac{x}{p} - \frac{(n-x)}{q}\right\}$$

$$= -nr\sum_{x=0}^{n} (x-np)^{r-1} p(x) + \sum_{x=0}^{n} (x-np)^r nC_x p^x q^{n-x}\left\{\frac{xq - np + xp}{pq}\right\}$$

$$= -nr\mu_{r-1} + \sum_{x=0}^{n} (x-np)^r nC_x p^x q^{n-x}\left\{\frac{x(p+q) - np}{pq}\right\}$$

$$= -nr\mu_{r-1} + \sum_{x=0}^{n} (x-np)^r nC_x p^x q^{n-x}\left\{\frac{x - np}{pq}\right\}$$

R

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$$\frac{d\mu_r}{dp} = -nr\mu_{r-1} + \frac{1}{pq}\sum_{x=0}^{n} (x-np)^{r+1}\, nC_x\, p^x q^{n-x}$$

$$\frac{d\mu_r}{dp} = -nr\mu_{r-1} + \frac{\mu_{r+1}}{pq}$$

$$\Rightarrow \quad \frac{\mu_{r+1}}{pq} = \left[\frac{d\mu_r}{dp} + nr\mu_{r-1}\right]$$

$$\Rightarrow \quad \mu_{r+1} = pq\left[\frac{d\mu_r}{dp} + nr\mu_{r-1}\right]$$

POISSON DISTRIBUTION

The binomial distn. tends to poisson distn. when

(i) The no. of trials is indefinitely large ie., $n \to \infty$

(ii) The proby. of success is very small (ie, $p \to 0$)

(iii) np is a const. ie., $np = \lambda$, where λ is a +ve real no.

PROBABILITY MASS FN:

If X is a discrete R.V that can assume the values $0, 1, 2, \ldots$ such that its probability mass fn. is gn. by

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2 \ldots, \lambda > 0.$$

then X is said to follow a poisson distn. with parameter λ

POISSON DISTN. AS LIMITING FORM OF BINOMIAL DISTN.

Poisson distn. is a limiting case of binomial distn. under the following conditions:

(i) n, the no. of trials is indefinite large ie, $n \to \infty$

(ii) p, the proby. of success is very small ie, $p \to 0$

(iii) $np (= \lambda)$ is finite or $p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$ where λ is a +ve real no.

(iv) P(children of both sexes)

$= 1 - P$ (children of the same sex)

$= 1 - \{P$ (all are boys) $+ P$(all are girls)$\}$

$= 1 - \{4C_4 (1/2)^4 + 4C_0 (1/2)^4\}$

$= 1 - \{(1/2)^4 + (1/2)^4\}$

$= 1 - 2(1/2)^4$

$= 1 - \frac{\cancel{2}^1}{\cancel{16}_8}$

$= 7/8$.

No/. of families having children of both sexes

$= 800 \times \frac{7}{8}$

$= 700$

TVA) An irregular 6-faced dice is such that the proby/. that it gives 3 even nos/. in 5 throws is twice the proby/. that it gives 2 even nos/. in 5 throws. How many sets of exactly 5 trials can be expected to give no even no/. out of 250 sets?

Soln Let the proby/. of getting an even no/. with the unfair dice is p.

Let X denote the no/. of even nos/. obtained in 5 trials (throws)

PROBLEMS (UNIFORM DISTR.)

55Q) If x is uniformly distd. with mean 1 & variance $4/3$ find $P(x<0)$

Soln

$$\text{mean} = \frac{a+b}{2} \qquad V(x) = \frac{4}{3}$$

mean = 1

$\Rightarrow \frac{a+b}{2} = 1$

$\Rightarrow a+b = 2$ — (1)

$V(x) = 4/3$

$\Rightarrow \frac{(a-b)^2}{12} = 4/3$

$(a-b)^2 = 16$

$a-b = \pm 4$ — (2)

(1)+(2) $\Rightarrow$ $2a = 6$ or $2a = -2$

$a = 3$ $a = -1$

$\therefore a+b = 2$

when $a=3$ $\Rightarrow b = -1$ when $a = -1$, $b = 3$

$\therefore a=3, b=-1$ or $a=-1, b=3$

The first combination is not possible because $a<b$.

$\therefore a=-1, b=3$

$$\therefore f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{4} & -1<x<3 \\ 0 & \text{otherwise} \end{cases}$$

$$P(x<0) = \int_{-1}^{0} f(x)\,dx$$

$$= \int_{-1}^{0} 1/4\,dx$$

$$= 1/4\,(x)_{-1}^{0}$$

$$= 1/4\,(1)$$

$$P(x<0) = 1/4$$

55) If X has uniform distn. in [0,1]. find the pdf of $-2\log X$

Soln

$$X \sim U(0,1)$$

$$f(x) = \begin{cases} \frac{1}{1-0} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y = 2\log X$$

Let $G(y)$ be the distn. fn. of Y

$$G(y) = P(Y \le y)$$

$$= P(-2\log X \le y)$$

$$= P(\log X \geq -y/2)$$

$$= P(X \geq e^{-y/2})$$

$$= 1 - P(X < e^{-y/2})$$

$$= 1 - \int_0^{e^{-y/2}} f(x)\,dx$$

$$= 1 - \int_0^{e^{-y/2}} 1\,dx$$

$$= 1 - (x)_0^{e^{-y/2}}$$

$$= 1 - (e^{-y/2} - 0)$$

$$= 1 - e^{-y/2}$$

$\therefore$ p.d.f of $Y = 2\log X$

$$= \quad g(y) = \frac{d}{dy} G(y)$$

$$= \frac{d}{dy}(1 - e^{-y/2})$$

$$= -e^{-y/2}(-1/2)$$

$$g(y) = \frac{1}{2} e^{-y/2}, \quad 0 < y < \infty$$

880) A R.V X has a uniform distn over (-4, 4). compute $P(|X| > 2)$

Soln $X \sim U(-4,4) \Rightarrow f(x) = \begin{cases} \frac{1}{8} & -4 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

$$P(|X| > 2) = 1 - P(|X| \le 2)$$

$$= 1 - P(-2 \le X \le 2)$$

$$= 1 - \int_{-2}^{2} f(x)\,dx$$

$$= 1 - \int_{-2}^{2} \frac{1}{8}\,dx = 1 - \frac{1}{8}(x)_{-2}^{2} = 1 - 4/8 = 1/2.$$

PROBLEMS (EXPONENTIAL DISTN)

TVQ) The mileage which car owners get with a certain kind of radial tyre is a R.V having an exponential distn, with mean 40,000 km. Find the prob. that one of these tires will last

(i) atleast 20,000 km

(ii) atmost 30,000 km

Soln

$$\text{mean} = \frac{1}{\lambda}$$

$$\Rightarrow \quad \frac{1}{\lambda} = 40{,}000 \ (\text{Given})$$

$$\therefore \quad \lambda = \frac{1}{40{,}000}$$

$$\therefore \quad f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{40000} e^{-x/40000} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) $P(X \geq 20{,}000) = \int_{20000}^{\infty} f(x)\,dx \longrightarrow \int_{20000}^{\infty} \frac{1}{40000} e^{\frac{-x}{40000}} \, dx$

$$= \frac{1}{\cancel{40000}} \left[\frac{e^{-\frac{x}{40000}}}{-\frac{1}{\cancel{40000}}} \right]_{20000}^{\infty}$$

$$= 0 - \left(-e^{-\frac{20000}{40000}}\right)$$

$$= e^{-0.5}$$

$$= 0.6065$$

(ii) $P(X \le 30{,}000) = \int_0^{30{,}000} f(x)\,dx$

$$= \int_0^{30{,}000} \frac{1}{40000} e^{-\frac{x}{40000}} dx$$

$$= \frac{1}{40000}\left\{ \frac{e^{-\frac{x}{40000}}}{-\frac{1}{40000}} \right\}\Bigg|_0^{30{,}000}$$

$$= -e^{-3/4} + 1$$

$$= 1 - e^{-0.75}$$

$$= 0.5270$$

IV(a) If the time T to failure of a component is exponentially distd.. with parameter λ and if n such components are installed, what is the proby, that one half or more of these components are still fn..ing at the end of 't' hours?

Soln: The density fn. of T is gn. by

$$f(t) = \lambda e^{-\lambda t}, \quad t \ge 0$$

P(a component fn.. at the end or after t hours)

$$P(T \ge t) = \int_t^{\infty} \lambda e^{-\lambda t} dt = \lambda\left(\frac{e^{-\lambda t}}{-\lambda}\right)_t^{\infty} = e^{-\lambda t}$$

If we consider a component fn..ing at the end or after 't' hours as a success in a single trial, we have

$$p = e^{-\lambda t}, \quad q = 1 - e^{-\lambda t}$$

Then the no. X of successes in n independent trials follows a binomial distn. with parameters n & p

$$P(X=r) = nC_r p^r q^{n-r} \quad , r = 0, 1, 2 \ldots n$$

If n is even the required proby. is gn. by

$$\sum_{r=n/2}^{n} P(X=r) = \sum_{r=n/2}^{n} nC_r e^{-\lambda t} (1-e^{-\lambda t})^{n-r}$$

If n is odd, the required proby. is gn. by

$$\sum_{r=\frac{n+1}{2}}^{n} P(X=r) = \sum_{r=\frac{n+1}{2}}^{n} nC_r e^{-\lambda t} (1-e^{-\lambda t})^{n-r}$$

IV 30) The time (in hours) required to repair a machine is exponentially distd. with parameter $\lambda = \frac{1}{2}$

a) what is the proby. that the repair time exceeds 2h?

b) what is the conditional proby. that a repair takes atleast 10 h gn. that its duration exceeds 9 h?

Soln

$X \to$ the time to repair the machine.

$$\therefore f(x) = \lambda e^{-\lambda x}$$
$$= \frac{1}{2} e^{-x/2} \quad , x > 0$$

a) $P(X>2) = \int_2^{\infty} \frac{1}{2} e^{-x/2} dx = -\left(e^{-x/2}\right)_2^{\infty} = e^{-1} = 0.3679$

b) $P(X \geq 10 / X > 9) = P(X > 1)$ (by memoryless property)

$$= \int_1^{\infty} \frac{1}{2} e^{-x/2} dx = -\left(e^{-x/2}\right)_0^{\infty} = e^{-0.5} = 0.6065$$

UNIT-I
RANDOM VARIABLES
PART-A(2 Marks)

1. If X is a discrete random variable with probability distribution P(X=x)=kx,x=1,2,3,4 find P(2<X<4).
2. Find K,if the p.d.f of X is

X	-1	0	1	2	3
P(X=x)	2k	3k	4k	$6k^2$	$4k^2$

3.The p.d.f of a continuous random variable X is f(x)= $\begin{cases} k(3+2x), \text{ for } 2 \le x \le 4 \\ 0, \text{ otherwise} \end{cases}$ find the value of K.

4.The continuous random variable x has a probability density function f(x)=k(1+x), 2 ≤x≤5.Find P(X<4).

5. A random variable X has the p.d.f f(x) given by f(x)= $\begin{cases} cxe^{-x}, \text{ if } x > 0 \\ 0, \text{ if } x \le 0. \end{cases}$ find the value of and C.D.F of X.
6. A continuous random variable X has the p.d.f f(x) given by f(x)= $Ce^{-|x|}$, $-\infty < x < \infty$. Find the value of C.
7. Find the mean and variance of the distribution whose moment generating function is $(0.4e^t+0.6)^2$.
8. What is the moment generating function of a random variable?
9. Find the moment generating function of continuous probability distribution whose density is $2e^{-2x}$, x≥0
10. The first four moments of a distribution about X = 4 are 1,4,10 and 45 respectively. Show that the mean is 5 , variance is 3 , $\mu_3=0$ and $\mu_4=26$.
11. The mean and variance of a binomial variate X are 4 and 4/3 respectively. Find $P(X \ge 1)$
12. Find the MGF of binomial distribution.
13. If the independent random variables X , Y are binomially distributed respectively with n=3 , p= $\frac{1}{3}$ and n=5 , p= $\frac{1}{3}$, find P(X+Y≥1).
14. If 3% of the electric bulbs manufactured by a company are defective , find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective. (e^{-3}=0.0498)

(iv) atmost 4 defectives **(8)**

b. Find the moment generating function of the random variable with the probability law $P(X=x)=q^{x-1}p$;x=1,2,…..Find the mean and variance. **(8)**

8.a. Define Geometric distribution.If the probability that an applicant for a driver's licence will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test;
(i) on the fourth trial
(ii) in fewer than 4 trials. **(8)**

b. Find the mean, variance and m.g.f. of a r.v uniformly distributed in the interval (a,b) **(8)**

9.a. A random variable X has an uniform distribution over the interval (-3,3). Compute
(i) P[X=2]
(ii) P[X<2]
(iii) P [$|X|<2$]
(iv) P [$|X-2|<2$]
(v)Find k such that $P(X>k)=\frac{1}{3}$ **(8)**

b. Obtain the M.G.F of exponential distribution. From the M.G.F.,find the mean and variance and r^th^ moment about origin. **(8)**

10.a. The time in minutes that girl speaks over phone is a r.v. X with pdf $f(x)=Ae^{-\frac{x}{5}}$, x>0. Find the probability that she uses phone
(i) for atleast 5 minutes
(ii) for atmost 10 minutes
(iii) between 5 and 10 minutes. **(8)**

b. State and prove memoryless property of exponential distribution. **(8)**

11.a. Derive the MGF of Gamma distribution and find its mean and variance. **(8)**

b. If the life (in yrs) of a certain type of car has a weibull distribution with the parameter β = 2 ,find the value of te parameter α,given that the probability that parameter $\alpha = \frac{1}{2}$, what is the probability the required time (a) exceeds 2 hours and (b) exceeds 5 hours. **(8)**

UNIT II TWO DIMENSIONAL RANDOM VARIABLES 9

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and regression – Transformation of random variables – Central limit theorem.

TWO DIMENSIONAL RANDOM VARIABLE

DEFN.: Let S be the sample space associated with a random experiment E. Let $X=X(s)$ and $Y=Y(s)$ be two fns. each assigning a real no. to each outcomes $s \in S$. Then (X,Y) is called a two-dimensional r.v.

TWO DIMENSIONAL DISCRETE RANDOM VARIABLE:

If the possible values of (X,Y) are finite or countably infinite, (X,Y) is called a 2-Dal discrete r.v. The possible values of (X,Y) may be represented as (x_i, y_j), $i=1,2,\ldots m,\ldots$, $j=1,2,\ldots n,\ldots$

TWO DIMENSIONAL CONTINUOUS R.V:

If (X,Y) can assume all values in a specified region R in the xy-plane, then (X,Y) is called a 2-Dal continuous R.V.

DEFNS. FOR 2-DAL R.V

PROBABILITY MASS FN. OR PROBABILITY FN. OF (X,Y) (DISCRETE R.V)

If (X,Y) is a two-dimensional discrete R.V such that $P(x=x_i, y=y_j)=p_{ij}$ then p_{ij} is called probability mass fn. (p.m.f) of (X,Y) provided the follg. conditions are satisfied

(i) $p_{ij} \geq 0 \quad \forall i,j$

(ii) $\sum_j \sum_i p_{ij} = 1$

~~If~~ the set of triples $\{x_i, y_j, p_{ij}\}$, $i=1,2\ldots m,\ldots$ $j=1,2\ldots n,\ldots$ is called the joint proby. distn. of (X,Y).

~~CUMULATIVE DISTN. FN.~~

~~If (X,Y) is a two dimensional R.V (discrete or continuous), then~~

JOINT PROBABILITY DENSITY FN. (CONTINUOUS R.V)

If (X,Y) is a two-dimensional continuous R.V such that $P\left\{x-\frac{dx}{2} \le X \le x+\frac{dx}{2} \text{ and } y-\frac{dy}{2} \le Y \le y+\frac{dy}{2}\right\} = f(x,y)\,dx\,dy$

then $f(x,y)$ is called the joint pdf of (X,Y), provided $f(x,y)$ satisfies the following conditions:

(i) $f(x,y) \ge 0 \quad \forall \ (x,y) \in R$, where R is the range space

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

NOTE: $P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f(x,y)\,dx\,dy$.

CUMULATIVE DISTN. FN.

If (X,Y) is a 2-Dal R.V (discrete or continuous), then $F(x,y) = P(X \le x \ \& \ Y \le y)$ is called the cdf of (X,Y)

If (X,Y) is a Discrete R.V:

$$F(x,y) = \sum_j \sum_i p_{ij}$$

If (X,Y) is a continuous R.V:

$$F(x,y) = \int^{y} \int^{x} f(x,y)\,dx\,dy$$

PROBLEMS (TWO DIMENSIONAL R.V)

550) X & Y are two r.v's having the joint density fn. $f(x,y) = \frac{1}{27}(x+2y)$ where x & y can assume only the integer values 0, 1 & 2. Find the marginal distns. Also find the conditional distn of Y given X=x.

Soln

Y \ X	0	1	2	P(Y)
0	0/27	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
P(X)	6/27	9/27	12/27	1

MARGINAL DISTN. OF X:

X:	0	1	2
P(X):	6/27	9/27	12/27

MARGINAL DISTN. OF Y:

Y:	0	1	2
P(Y):	3/27	9/27	15/27

CONDITIONAL DISTN. OF Y GN. X=0:

$$P(Y=y/X=0) = \frac{P(Y=y, X=0)}{P(X=0)}$$

(i) y=0: $P(Y=0/X=0) = \frac{P(Y=0, X=0)}{P(X=0)} = \frac{0/27}{6/27} = 0$

(ii) $P(Y=1/X=0) = \frac{P(Y=1, X=0)}{P(X=0)} = \frac{2/\cancel{27}}{6/\cancel{27}} = \frac{1}{3}$

(iii) $P(Y=2/X=0) = \frac{P(Y=2, X=0)}{P(X=0)} = \frac{4/\cancel{27}}{6/\cancel{27}} = \frac{2}{3}$

∴ conditional distn. of Y given X=0 is

Y	$P(Y=y/X=0)$
0	0
1	$\frac{1}{3}$
2	$\frac{2}{3}$

conditional distn. of Y given X=1 : is

$$P(Y=y/X=1) = \frac{P(Y=y, X=1)}{P(X=1)}$$

Y	$P(Y=y/X=1)$
0	$\frac{P(Y=0, X=1)}{P(X=1)} = \frac{1/27}{9/27} = \frac{1}{9}$
1	$\frac{P(Y=1, X=1)}{P(X=1)} = \frac{3/27}{9/27} = \frac{1}{3}$
2	$\frac{P(Y=2, X=1)}{P(X=1)} = \frac{5/27}{9/27} = \frac{5}{9}$

conditional distn. of Y given X=2 :

$$P(Y=y/X=2) = \frac{P(Y=y, X=2)}{P(X=2)}$$

Y	$P(Y=y/X=2)$
0	$\frac{P(Y=0, X=2)}{P(X=2)} = \frac{2/27}{12/27} = \frac{1}{6}$
1	$\frac{P(Y=1, X=2)}{P(X=2)} = \frac{4/27}{12/27} = \frac{1}{3}$
2	$\frac{P(Y=2, X=2)}{P(X=2)} = \frac{6/27}{12/27} = \frac{1}{2}$.

80) Let X & Y be two r.v's with joint pdf

$$f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the marginal proby/. density fn/. of X & Y

(ii) Find $P(X \le 1/4 \mid 1/2 \le Y \le 3/4)$

(iii) Verify whether X & Y are independent

(iv) Find $E(X)$ & $E(Y)$

Soln (i) Marginal proby density fn/. of X

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{1} 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_x^1 = \overset{4}{\cancel{8}}x\left(\tfrac{1}{2}-\tfrac{x^2}{2}\right) = 4x(1-x^2)$$

$$f(x) = \begin{cases} 4x(1-x^2) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Marginal proby/. density fn/. of Y

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{y} 8xy\,dx = 8y\left(\frac{x^2}{2}\right)_0^y = 8y\left(\frac{y^2}{2} - \frac{0}{2}\right) = 4y(y^2) = 4y^3$$

$$f(y) = \begin{cases} 4y^3 & 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $P\left(X \le \frac{1}{4} \,/\, \frac{1}{2} \le Y \le \frac{3}{4}\right) = \dfrac{P\left(X \le \frac{1}{4} \cap \frac{1}{2} \le Y \le \frac{3}{4}\right)}{P\left(\frac{1}{2} \le Y \le \frac{3}{4}\right)}$.

$$P\left(X \le \tfrac{1}{4}\right) = \int_{-\infty}^{1/4} f(x)\,dx$$

$$P\left(X \le \tfrac{1}{4} \cap \tfrac{1}{2} \le Y \le \tfrac{3}{4}\right) = \int_{0}^{1/4}\int_{1/2}^{3/4} f(x,y)\,dy\,dx$$

$$= \int_{0}^{1/4}\int_{1/2}^{3/4} 8xy\,dy\,dx$$

$$= \int_{0}^{1/4} \cancel{8}^{4} x \left(\frac{y^2}{2}\right)_{1/2}^{3/4} dx$$

$$= \int_{0}^{1/4} 4x\left(\frac{9}{16} - \frac{1}{4}\right) dx$$

$$= \int_{0}^{1/4} 4x\left(\frac{5}{\cancel{16}_{4}}\right) dx$$

$$= \frac{5}{4}\left(\frac{x^2}{2}\right)_{0}^{1/4}$$

$$= \frac{5}{8}\left(\frac{1}{16}\right)$$

$$= \frac{5}{128}$$

$$P\left(\tfrac{1}{2} \le Y \le \tfrac{3}{4}\right) = \int_{1/2}^{3/4} f(y)\,dy = \int_{1/2}^{3/4} 4y^3\,dy = \cancel{4}\frac{y^4}{\cancel{4}}\Big|_{1/2}^{3/4}$$

$$= \left(\frac{3}{4}\right)^4 - \left(\frac{1}{2}\right)^4$$

$$= \frac{49}{256}$$

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$$P(X \le 1/4 / 1/2 \le Y \le 3/4) = \frac{P(X \le 1/4 \cap 1/2 \le Y \le 3/4)}{P(1/2 \le Y \le 3/4)}$$

$$= \frac{5/128}{49/256}$$

$$= \frac{10}{49}$$

(ii) To check whether X & Y are independent

$$f(x,y) = f(x)f(y)$$

$$8xy \neq 4x(1-x^2)\, 4y^3.$$

(iii) $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_0^1 x\, 4x(1-x^2) dx$$

$$= 4\int_0^1 (x^2 - x^4) dx$$

$$= 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= 4\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 4\left(\frac{2}{15}\right)$$

$$E(X) = \frac{8}{15}$$

$E(Y) = \int_{-\infty}^{\infty} y f(y) dy$

$$= \int_0^1 y \cdot 4y^3 dy$$

$$= 4 \left.\frac{y^5}{5}\right|_0^1$$

$$E(Y) = \frac{4}{5}$$

TVQ) The joint pdf of the R.V (X,Y) is gn by

$$f(x,y) = kxye^{-(x^2+y^2)} \quad x>0,\ y>0$$

Find the value of k & prove also X & Y are independent

Soln W.k.T $f(x,y)$ is a pdf

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} kxye^{-(+x^2+y^2)}\,dx\,dy = 1$$

$$k\int_{0}^{\infty} y\, e^{-y^2}\left(\int_{0}^{\infty} x e^{-x^2} dx\right) dy = 1$$

put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$k\int_{0}^{\infty} ye^{-y^2} dy \left(\int_{0}^{\infty} e^{-t}\frac{dt}{2}\right) = 1$$

$$k\int_{0}^{\infty} ye^{-y^2} dy \left(\frac{e^{-t}}{2}\right)_0^{\infty} = 1$$

$$k\int_{0}^{\infty} ye^{-y^2} dy \left(\frac{1}{2}\right) = 1$$

$$k\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = 1$$

$$\boxed{k = 4}$$

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$\therefore f(x,y) = 4xy\, e^{-(x^2+y^2)}$

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_{0}^{\infty} 4xy\, e^{-(x^2+y^2)}\,dy$

$= 4xe^{-x^2} \int_{0}^{\infty} y e^{-y^2}\,dy$

$= 4xe^{-x^2} \left(\frac{1}{2}\right)$

$f(x) = 2xe^{-x^2} \quad , x > 0$

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_{0}^{\infty} 4xy\, e^{-(x^2+y^2)}\,dx$

$= 4ye^{-y^2} \int_{0}^{\infty} x e^{-x^2}\,dx$

$= 4ye^{-y^2} \left(\frac{1}{2}\right)$

$f(y) = 2ye^{-y^2} \quad , y > 0$

$f(x,y) = 4xy\, e^{-x^2} \cdot e^{-y^2}$

$= 2xe^{-x^2}\, 2ye^{-y^2}$

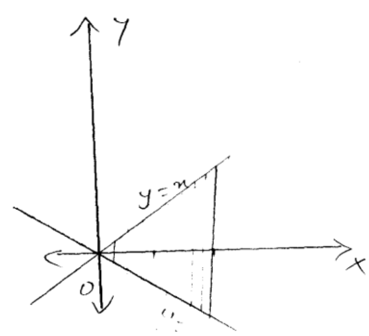
$f(x,y) = f(x)\, f(y)$

$\therefore$ X & Y are independent .

TVQ Cmp. $f(x,y) = cx(x-y)$, $0 < x < 2$, $-x < y < x$ & 0 elsewhere a) evaluate c b) $f_X(x)$ c) $f_{Y/X}(y/x)$ d) $f_Y(y)$

Soln W.K.T $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$\int_{0}^{2}\int_{-x}^{x} cx(x-y)\,dy\,dx = 1$



$$c\int_0^2\int_{-x}^{x}(x^2-xy)\,dy\,dx=1$$

$$c\int_0^2\left(x^2y-\frac{xy^2}{2}\right)_{-x}^{x}dx=1$$

$$c\int_0^2\left(\left(x^3-\frac{x^3}{2}\right)-\left(-x^3-\frac{x^3}{2}\right)\right)dx=1$$

$$c\int_0^2\left(\frac{x^3}{2}+\frac{3x^3}{2}\right)dx=1$$

$$\frac{c}{2}\left(\frac{x^4}{4}+\frac{3x^4}{4}\right)_0^2=1$$

$$\frac{c}{2}\left(\frac{16}{4}+\frac{3(16)}{4}\right)=1$$

$$\frac{c}{2}(16)=1$$

$$\boxed{c=\frac{1}{8}}$$

(i) $f(x)=\int_{-\infty}^{\infty}f(x,y)\,dy$

$$=\int_{-x}^{x}\frac{1}{8}x(x-y)\,dy=\frac{1}{8}\int_{-x}^{x}(x^2-xy)\,dy=\frac{1}{8}\left(x^2y-\frac{xy^2}{2}\right)_{-x}^{x}$$

$$=\frac{1}{8}\left[\left(x^3-\frac{x^3}{2}\right)-\left(-x^3-\frac{x^3}{2}\right)\right]$$

$$=\frac{1}{8}\left[\frac{x^3}{2}+\frac{3x^3}{2}\right]$$

$$=\frac{1}{16}4x^3=\frac{x^3}{4}\qquad 0<x<2$$

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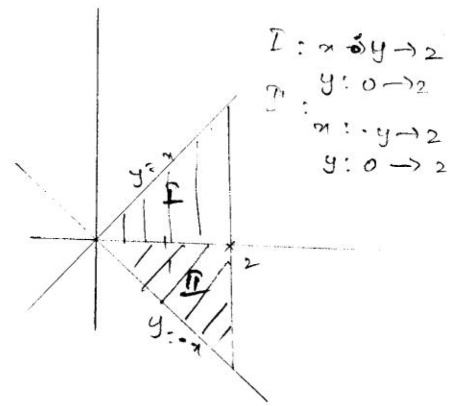
c) $f(y/x) = \dfrac{f(x,y)}{f(x)} = \dfrac{\frac{1}{8}x(x-y)}{x^3/4} = \dfrac{(x-y)}{2x^2}, \quad -x < y < x.$

d) $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$



In I:

$f(y) = \int_{y}^{2} \frac{1}{8} x(x-y)\,dx$

$= \frac{1}{8} \int_{y}^{2} (x^2 - xy)\,dx$

$= \frac{1}{8} \left(\frac{x^3}{3} - \frac{x^2}{2} y \right)_y^2$

$= \frac{1}{8} \left[\left(\frac{8}{3} - 2y\right) - \left(\frac{y^3}{3} - \frac{y^3}{2} \right) \right]$

$= \frac{1}{8} \left[\frac{8}{3} - 2y + \frac{y^3}{6} \right]$

$= \frac{1}{3} - \frac{y}{4} + \frac{y^3}{48} \qquad -2 \le y \le 0.$

In II:

$f(y) = \int_{-y}^{2} f(x,y)\,dx$

$= \frac{1}{8} \int_{-y}^{2} (x^2 - xy)\,dx$

$= \frac{1}{8} \left(\frac{x^3}{3} - \frac{x^2}{2} y \right)_{-y}^{+2}$

$= \frac{1}{8} \left[\left(8/3 - 2y \right) - \left(-\frac{y^3}{3} - \frac{y^3}{2} \right) \right]$

$= \frac{1}{8} \left[\frac{8}{3} - 2y + \frac{5y^3}{6} \right]$

$= \frac{1}{3} - \frac{y}{4} + \frac{5}{48} y^3, \quad 0 \le y \le 2$

$\therefore f(y) = \begin{cases} \frac{1}{3} - y/4 + \frac{5}{48} y^3 & 0 \le y \le 2 \\ \frac{1}{3} - y/4 + \frac{y^3}{48} & -2 \le y \le 0. \end{cases}$

NOTE :

(i) $F(x)$ is a non-decreasing fn. of x ie, $x_1 < x_2$, then $F(x_1) \le F(x_2)$

(ii) $F(-\infty) = 0$ & $F(\infty) = 1$

(iii) If X is a discrete R.V taking values $x_1, x_2, \ldots$ where $x_1 < x_2 < x_3 < \ldots < x_{i-1} < x_i < \ldots$ then $P(X = x_i) = F(x_i) - F(x_{i-1})$

(iv) If x is a continuous R.V then

$$\frac{d}{dx} F(x) = f(x)$$

at all pts. where $F(x)$ is differentiable.

(100) NV12

TRANSFORMATION OF TWO DIMENSIONAL R.V

consider a two dimensional r.v (X,Y).
consider the r.v's U & V where

$U = u(x,y)$
$V = v(x,y)$

Here u & v are continuously diff'e fns/. for which the Jacobian of transformation is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The joint pdf $g(u,v)$ of the transformed r.v's U & V is gn/. by

$$g(u,v) = f(x,y)|J|$$

where $|J|$ is the modulus value of the Jacobian & $f(x,y)$ is expressed in terms of u & v.

NOTE: If X & Y are independent continuous r.v's then the p.d.f of $U = X+Y$ is gn by

$$h(u) = \int_{-\infty}^{\infty} f_x(v) f_y(u-v)\,dv.$$

PROBLEMS (FNS of R.V's – 2D)

580) Let (X, Y) be a 2-Dal R.V having the joint density

$$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)}, & x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

P.T the density fn. of $u = \sqrt{x^2+y^2}$ is

$$h(u) = \begin{cases} 2u^3e^{-u^2} & 0 \leq u \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

Soln

G.M/. $u = \sqrt{x^2+y^2}$ Let $v = x$.

$u = \sqrt{v^2+y^2}$

$u^2 = v^2 + y^2$

$\Rightarrow u^2 \geq v^2$ $y \geq 0$.

$\Rightarrow u \geq v$.

$x \geq 0, y \geq 0$

$\Rightarrow u \geq 0, v \geq 0, u \geq v$

$$\frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = \sqrt{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}(2x)$

$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$v = x$

$\frac{\partial v}{\partial x} = 1$

$\frac{\partial v}{\partial y} = 0$.

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & 1 \\ \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$\frac{1}{J} = \frac{-y}{\sqrt{x^2+y^2}}$$

$$\therefore |J| = \left|\frac{-y}{\sqrt{x^2+y^2}}\right| = \frac{y}{\sqrt{x^2+y^2}}$$

$\therefore$ P.d.f of u & v is

$$g(u,v) = f(x,y)\,|J|$$

$$= 4xy e^{-(x^2+y^2)} \cdot \frac{\sqrt{x^2+y^2}}{y}$$

$$= 4v e^{-u^2} \cdot u$$

$$g(u,v) = 4uve^{-u^2}$$

$$\therefore g(u,v) = \begin{cases} 4uve^{-u^2} & , u \geq 0, \; 0 \leq v \leq u \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ Density fn. of $u = \sqrt{x^2+y^2}$

$$g(u) = \int_{-\infty}^{\infty} g(u,v)\,dv$$

$$= \int_0^u 4uve^{-u^2}\,dv$$

$$= 4ue^{-u^2}\int_0^u v\,dv$$

$$= 4ue^{-u^2}\left(\frac{v^2}{2}\right)_0^u$$

$$= \overset{2}{\cancel{4}}ue^{-u^2}\frac{u^2}{\cancel{2}}$$

$$g(u) = \begin{cases} 2u^3e^{-u^2} & , u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

UNIT - II
TWO DIMENSIONAL RANDOM VARIABLES
PART-A(2 Marks)

1.The joint probability density function of the random variable (X,Y) is given by
f(x,y)=Kxy $e^{-(x^2+y^2)}$ x>0, y>0. Find the value of K.

2.The joint p.d.f. of (X,Y) is f(x,y)=4xy, 0<x,y<1 and f(x,y)=0, otherwise, Find E(XY).

3. The joint p.d.f. of (X,Y) is f(x,y)= $\begin{cases} 4xy, 0 < x, y < 1 \\ 0, elsewhere \end{cases}$ Examine X and Y are independent.

4. Define joint distributions of two random variables X and Y and state its properties.

5. If the probability density function of X is $f_x(x)$=2x, 0<x<1, find the probability density function of Y=3X+1.

Find the correlation coefficient between X and Y. **(8)**

8.a. Obtain the two regression equations to the following data:

X:	65	66	67	67	68	69	70	72
Y:	67	68	65	68	72	72	69	71

(8)

b. If the joint p.d.f. of (X,Y) is given by f(x,y)=x+y,0≤x,y≤1,find the p.d.f. of U=XY. **(8)**

9.a. Let (X,Y) be a two-dimensional non-negative continuous random variable having the joint density f(x,y) = $\begin{cases} 4xye^{-(x^2+y^2)}, x \geq 0, y \geq 0 \\ 0, elsewhere \end{cases}$

Find the density function of U=$\sqrt{(X^2+Y^2)}$ **(8)**

b. The joint p.d.f of X and Y is given by f(x,y) = $e^{-(x+y)}$ x>0,y>0,find the probability density function of U = (X+Y)/2. **(8)**

10.a. State and prove central limit theorem.

b. A distribution with unknown mean µ has variance equal to 1.5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be atleast 0.95 that the sample mean will be within 0.5 of the population mean. **(8)**

11.a. If $X_1,X_2,X_3,\ldots\ldots X_n$ are Poisson variates with mean 2,use central limit theorem to evaluate $P(120<s_n<160)$ where $S_n=X_1+X_2+X_3.\ \ldots.X_n$ and n=75. **(8)**

b. A sample of size 100 is taken from a population of mean 60 and variance 400. Using CLT find the probability that the sample mean will not differ from the population mean by more than four. **(8)**

UNIT III MARKOV PROCESSES AND MARKOV CHAINS 9

Classification – Stationary process – Markov process – Markov chains – Transition probabilities – Limiting distributions – Poisson process.

RANDOM PROCESSES.

A random Variable is a rule (or function) that assigns a real number to every outcome of a random experiment.

For example, Consider the random experiment of tossing a dice at $t=0$ and observing the number on the top face. The sample space of this experiment consists of the outcomes $\{1,2,3,\ldots,6\}$. For each outcome of the experiment, let us arbitrarily assign a function of time t $(0 \le t < \infty)$ in the following manner.

outcome: 1 2 3 4 5 6

function of time: $x_1(t)$ $x_2(t)$ $x_3(t)$ $x_4(t)$ $x_5(t)$ $x_6(t)$
$=-4$ $=-2$ $=2$ $=4$ $=-t/2$ $=t/2$

The set of functions $\{x_1(t), x_2(t), \ldots, x_6(t)\}$ represents a random process.

Definition:

A random process is a collection (or ensemble) of RV's $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set).

The set of possible values of any individual member of the random process is called state space. Any individual member itself is called a sample function or a realisation of the process.

Note: (i) If s and t are fixed, $\{X(s,t)\}$ is a number.
(ii) If t is fixed $\{X(s,t)\}$ is a RV.
(iii) If s is fixed, $\{X(s,t)\}$ is a single time function.
(iv) If s and t are variables, $\{X(s,t)\}$ is a collection of RV's that are time functions.

Notation:

As the dependence of a random process on s is obvious, s will be omitted hereafter in the notation of a random process. If the parameter set T is discrete, the random process will be noted by $\{X(n)\}$ or $\{X_n\}$.

If the parameter T is continuous, the process will be denoted by $\{X(t)\}$.

Classification of Random Processes:

Depending on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into four types:

(i) If both T and S are discrete, the random process is called a discrete random sequence. For example, if X_n represents the outcome of the n^{th} toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, \ldots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$.

(ii) If T is discrete and S is continuous, the random process is called a continuous random sequence.

For example, if X_n represents the temperature at the end of the n^{th} hour of a day, then $\{X_n, 1 \leq n \leq 24\}$ is

Sub $t = 0$ and $n=0$ in ④, we have

$$P_0(0) = \frac{\lambda^0 t^0}{0!} f(0)$$

(ie) $f(0) = P_0(0)$

$= P[X(0) = 0]$

$= P[\text{no event occurs in } (0,0)]$

$= 1. \longrightarrow ⑦$

when $t = 0$, Equation ⑥ becomes

$$f(0) = k\, e^{-\lambda(0)}$$

$$f(0) = k \longrightarrow ⑧$$

Substituting ⑦ in ⑧, we have

$$1 = k.$$

(ie) $k = 1.$

Now equation ⑥ becomes

$$f(t) = e^{-\lambda t} \longrightarrow ⑨$$

Substituting ⑨ in ④, we have

$$P_n(t) = \frac{\lambda^n t^n}{n!} e^{-\lambda t},$$

$$P_n(t) = P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots \infty$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt.

Note:

We have assumed that the rate of occurrence of the event λ is a constant, but it can be function of t also. When λ is a constant, the process is called a homogeneous-poisson process. Unless specified otherwise, the poisson process will be assumed homogeneous.

Second-order probability function of a Homogeneous Poisson Process:

$P[X(t_1)=n_1, X(t_2)=n_2]$

$= P[X(t_1)=n_1]\,P[X(t_2)=n_2 / X(t_1)=n_1],\ t_2>t_1$

$= P[X(t_1)=n_1]\,P[\text{the event occurs } (n_2-n_1) \text{ times in the interval } (t_2-t_1)]$

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!}\ \frac{e^{-\lambda(t_2-t_1)}[\lambda(t_2-t_1)]^{n_2-n_1}}{(n_2-n_1)!}\ \text{if } n_2 \geq n_1$$

$$= \begin{cases} \dfrac{e^{-\lambda t_1}\lambda^{n_1} t_1^{n_1} e^{-\lambda t_2} e^{\lambda t_1} \lambda^{n_2-n_1} (t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}, & n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}, & n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases}$$

Proceeding similarly, we can get the third-order probability function as

$P[X(t_1)=n_1, X(t_2)=n_2, X(t_3)=n_3]$

$$= \begin{cases} \dfrac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{(n_1)!(n_2-n_1)!(n_3-n_2)!}, & n_3 \geq n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases}$$

Mean and Autocorrelation of the Poisson Process:

The probability law of the Poisson process $\{X(t)\}$ is the same as that of a Poisson distribution with parameter λt.

Therefore, $E[X(t)] = Var[X(t)] = \lambda t$

$E[X^2(t)] = Var[X(t)] + (E[X(t)])^2$

$= \lambda t + \lambda^2 t^2$

$= \lambda t + (\lambda t)^2$

$$= e^{-\lambda t} \lambda^2 t^2 \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right]$$

$$+ e^{-\lambda t} \lambda t \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right].$$

$$= e^{-\lambda t} \lambda^2 t^2 \cdot e^{\lambda t} + e^{-\lambda t} \lambda t e^{\lambda t}$$

$$= \lambda^2 t^2 + \lambda t \longrightarrow (2)$$

Substituting (2) in (1), we have

$$\text{Variance} = \lambda^2 t^2 + \lambda t - (\lambda t)^2$$

$$= \lambda^2 t^2 + \lambda t - \lambda^2 t^2$$

$$= \lambda t$$

Hence, mean and Variance of the poisson process coincides and the Value is λt.

2. <u>Additive Property:</u> <u>U.Q</u>

Sum of two independent Poisson processes is a poisson process.

<u>Proof:</u>

Let $X(t) = X_1(t) + X_2(t)$.

$$P[X(t) = n] = \sum_{r=0}^{n} P[X_1(t) = r]\, P[X_2(t) = n-r]$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \sum_{r=0}^{n} \frac{(\lambda_1)^r (t)^r (\lambda_2)^{n-r} (t)^{n-r}}{r!\,(n-r)!}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{t^n (\lambda_1)^r (\lambda_2)^{n-r}}{r!\,(n-r)!}$$

$$\left[\text{We know that } nC_r = \frac{n!}{r!(n-r)!} \quad (n-r)!\, r! = \frac{nC_r}{n!} \right]$$

Variance of Poisson Process

$$\text{Variance} = E[x^2(t)] - [E[x(t)]]^2 \rightarrow (1)$$

$$E[x^2(t)] = \sum_{x=0}^{\infty} x^2 \left[P[X = x(t)]\right]$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \left[\frac{e^{-\lambda t}(\lambda t)^x}{x!}\right]$$

$$= e^{-\lambda t} \sum_{x=0}^{\infty} \frac{x(x-1)}{x!} (\lambda t)^{x} + e^{-\lambda t} \sum_{x=0}^{\infty} \frac{x(\lambda t)^{x}}{x!}$$

$$= e^{-\lambda t} \sum_{x=0}^{\infty} \frac{\cancel{x}(\cancel{x-1})(\lambda t)^{x}}{\cancel{x}(\cancel{x-1})(x-2)!} + e^{-\lambda t} \sum_{x=0}^{\infty} \frac{\cancel{x}(\lambda t)^{x}}{\cancel{x}(x-1)!}$$

$$= e^{-\lambda t} \sum_{x=2}^{\infty} \frac{(\lambda t)^{x}}{(x-2)!} + e^{-\lambda t} \sum_{x=1}^{\infty} \frac{(\lambda t)^{x}}{(x-1)!}$$

$$= e^{-\lambda t}\left[\frac{(\lambda t)^{2}}{0!} + \frac{(\lambda t)^{3}}{1!} + \frac{(\lambda t)^{4}}{2!} + \cdots\right] + e^{-\lambda t}\left[\frac{\lambda t}{0!} + \frac{(\lambda t)^{2}}{1!} + \frac{(\lambda t)^{3}}{2!} + \cdots\right]$$

$$= e^{-\lambda t} \lambda^2 t^2 \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right]$$

$$+ e^{-\lambda t} \lambda t \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right].$$

$$= e^{-\lambda t} \lambda^2 t^2 \cdot e^{\lambda t} + e^{-\lambda t} \lambda t e^{\lambda t}$$

$$= \lambda^2 t^2 + \lambda t \longrightarrow (2)$$

Substituting (2) in (1), we have

$$\text{Variance} = \lambda^2 t^2 + \lambda t - (\lambda t)^2$$

$$= \lambda^2 t^2 + \lambda t - \lambda^2 t^2$$

$$= \lambda t$$

Hence, mean and Variance of the poisson process coincides and the Value is λt.

$$= \frac{e^{-(\lambda_1+\lambda_2)t}\left[(\lambda_1+\lambda_2)t\right]^n}{n!}$$

Therefore, $X_1(t)+X_2(t)$ is a poisson process with parameter $(\lambda_1+\lambda_2)t$.

Note: The additive property holds good for any number of independent Poisson processes.

3. Difference of two independent Poisson processes is not a poisson process.

Proof:

Let $X(t) = X_1(t) - X_2(t)$

$E[X(t)] = E[X_1(t)] - E[X_2(t)]$

$= \lambda_1 t - \lambda_2 t$

$= (\lambda_1 - \lambda_2) t$

$E[x^2(t)] = E([x_1(t) - x_2(t)]^2)$

$= E[x_1^2(t)] + E[x_2^2(t)] - 2E[x_1(t)]\,E[x_2(t)]$

[By independence]

$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t)$
$- 2(\lambda_1 t)(\lambda_2 t)$

$= \lambda_1^2 t^2 + \lambda_2^2 t^2 - 2\lambda_1 \lambda_2 t^2 + \lambda_1 t + \lambda_2 t$

$= t(\lambda_1 + \lambda_2) + t^2(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2)$

$= (\lambda_1 + \lambda_2) t + (\lambda_1 - \lambda_2)^2 t^2$

$\neq (\lambda_1 - \lambda_2) t + (\lambda_1 - \lambda_2)^2 t^2.$

We know that $E[X^2(t)]$ for a Poisson process $\{X(t)\}$ with parameter λ is given by $E[X^2(t)] = \lambda t + \lambda^2 t^2$.

Therefore, $\{X_1(t) - X_2(t)\}$ is not a Poisson process.

4. The interarrival time of a Poisson process, (ie) the interval between two successive occurrences of a Poisson process with parameter λ has an exponential distribution with mean $1/\lambda$.

Proof:

Let two consecutive occurrences of the event be E_i and E_{i+1}.

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i and E_{i+1}. T is a Continuous RV.

$P[T > t] = P[E_{i+1}$ did not occur in $(t_i, t_i + t)]$

$= P$[No event occurs in an interval of length t].

$= P[X(t) = 0]$

$= e^{-\lambda t}$

Therefore, the cdf of T is given by

$F(t) = P[T \leq t]$

$= 1 - e^{-\lambda t}$

Therefore, the pdf of T is given by

$f(t) = 0 - (-\lambda) e^{-\lambda t}$

$f(t) = \lambda e^{-\lambda t} \; (t \geq 0)$.

Which is an exponential distribution with mean $1/\lambda$

$$= \frac{e^{-\lambda t}(\lambda p t)^n e^{\lambda t} e^{-\lambda p t}}{n!}$$

$$= \frac{e^{-\lambda p t}(\lambda p t)^n}{n!}$$

1. Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. find the probability that during a time interval of 2 min

(i) exactly 4 customers arrive and

(ii) more than 4 customers arrive.

Mean of the poisson process $= \lambda t$

Mean arrival rate = mean number of arrivals per minute (unit time) $= \lambda$

$$= \frac{e^{-\lambda t}(\lambda p t)^n e^{\lambda t} e^{-\lambda p t}}{n!}$$

$$= \frac{e^{-\lambda p t}(\lambda p t)^n}{n!}$$

1. Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. find the probability that during a time interval of 2 min
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1. Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. find the probability that during a time interval of 2 min

(i) exactly 4 customers arrive and

(ii) more than 4 customers arrive.

Mean of the poisson process $= \lambda t$

Mean arrival rate = mean number of arrivals per minute (unit time) $= \lambda$

5. If the number of occurrences of an event ~~E~~ E in an interval of length t is a Poisson process $\{X(t)\}$ with parameter λ and if each occurrence of E has a constant probability p of being recorded and the recordings and independent of each other, then the number $N(t)$ of the recorded occurrences in t is also a Poisson process with parameter λp.

<u>Proof:</u>

$$P[N(t)=n] = \sum_{r=0}^{\infty} P\left[\begin{array}{l}E \text{ occurs } (n+r) \text{ times} \\ \text{in } t \text{ and } n \text{ of them are} \\ \text{recorded}\end{array}\right]$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \, {}^{(n+r)}C_n \, p^n q^r, \quad q = 1-p.$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(\cancel{n+r})!} \cdot \frac{(\cancel{n+r})!}{n!(\cancel{n}+r-\cancel{n})!} p^n q^r$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n (\lambda t)^r p^n q^r}{n! \; r!}$$

$$= \frac{e^{-\lambda t}(\lambda p t)^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{r!}$$

$$= \frac{e^{-\lambda t}(\lambda p t)^n}{n!} \left[1 + \frac{\lambda q t}{1!} + \frac{(\lambda q t)^2}{2!} + \cdots \right]$$

$$= \frac{e^{-\lambda t}(\lambda p t)^n}{n!} e^{\lambda q t}$$

$$= \frac{e^{-\lambda t}(\lambda p t)^n e^{\lambda(1-p)t}}{n!}$$

(ii) $P[1 < T < 2] = \int_1^2 2e^{-2t}dt$

$= 2\left[\frac{e^{-2t}}{-2}\right]_1^2$

$= -\frac{2}{2}\left(e^{-4} - e^{-2}\right)$

$= e^{-2} - e^{-4}$

$= 0.1353 - 0.0183.$

$= 0.117.$

(iii) $P[T \leq 4] = \int_0^4 2e^{-2t}dt$

$= 2\left[\frac{e^{-2t}}{-2}\right]_0^4$

$= -\frac{2}{2}\left[e^{-8} - 1\right]$

$= 1 - e^{-8}$

$= 1 - 0.00033546$

$= 0.9996$ //

3. A radioactive source emits particles at a rate of 5 per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4-min period.

By property 5 of Poisson processes, the number of recorded particles $N(t)$ follows a Poisson process with parameter λp.

Here $\lambda = 5$ and $p = 0.6$

$\therefore P\{N(t) = k\} = \dfrac{e^{-3t}(3t)^k}{k!}$

$$= 1 - \left[\frac{e^{-6} \cdot (6)^0}{0!} + \frac{e^{-6} (6)^1}{1!} + \frac{e^{-6} (6)^2}{2!} + \frac{e^{-6} (6)^3}{3!} + \frac{e^{-6} (6)^4}{4!} \right].$$

$$= 1 - e^{-6} \left[1 + 6 + \frac{36}{2} + \frac{216}{6} + \frac{1296}{24} \right]$$

$$= 1 - 0.002478 \left[1 + 6 + 18 + 36 + 54 \right].$$

$$= 1 - 0.002478 (115).$$

$$= 1 - 0.28497$$

$$= 0.71503$$

2. If customers arrive at a Counter in accordance with a Poisson process with a mean rate of 2 per minute find the probability that the interval between 2 consecutive arrivals is (i) more than 1 min, (ii) between 1 min and 2 min and (iii) 4 min or less.

By property 4 of Poisson processes. The interval T between 2 consecutive arrivals follows an exponential distribution with parameter $\lambda = 2$. $f(t) = \lambda e^{-\lambda t}$,

(i) $P[T > 1] = \int_1^{\infty} 2 e^{-2t}\, dt$.

$$= 2 \left[\frac{e^{-2t}}{-2}\right]_1^{\infty}$$

$$= -\frac{2}{2}\left[0 - e^{-2}\right] = e^{-2}$$

$= 0.1353$.

$$P\{N(4) = 10\} = \frac{e^{-12}(12)^{10}}{10!}$$

$$= \frac{0.00000614 \times 0.0000000004}{}$$

$$= \frac{380172\dot{6}163}{3628800}$$

$$= 0.10476.$$

4. If $\{N_1(t)\}$ and $\{N_2(t)\}$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, show that

$$P[N_1(t) = k / \{N_1(t) + N_2(t) = n\}] = nC_k \, p^k q^{n-k}$$

where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$.

Soln:

$$P[N_1(t)=k / \{N_1(t)+N_2(t)=n\}]$$

$$= \frac{P[\{N_1(t)=k\} \cap \{N_1(t)+N_2(t)=n\}]}{P[N_1(t)+N_2(t)=n]}$$

$$= \frac{P[\{N_1(t)=k\} \cap \{N_2(t)=n-k\}]}{P\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \times \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{n!}}$$

$$= \frac{\cancel{e^{-\lambda_1 t}}(\lambda_1 \cancel{t})^k \cancel{e^{-\lambda_2 t}}(\lambda_2 \cancel{t})^{n-k} \cdot n!}{k!\,(n-k)!\,\cancel{e^{-\lambda_1 t}}\cancel{e^{-\lambda_2 t}}(\lambda_1 \cancel{t})^n (\lambda_2 t)^n}$$

$$= \frac{e^{-\cancel{\lambda_1 t}} \lambda_1^k t^k e^{-\cancel{\lambda_2 t}} \lambda_2^{n-k} t^{n-k} \times n!}{k!(n-k)!\,(\lambda_1+\lambda_2)^n t^n e^{-\cancel{\lambda_1 t}} e^{-\cancel{\lambda_2 t}}}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{\cancel{t^n}\,\lambda_1^k \lambda_2^{n-k}}{\cancel{t^n}\,(\lambda_1+\lambda_2)^k (\lambda_1+\lambda_2)^{n-k}}$$

$$= nC_k \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= nC_k\, p^k q^{n-k}$$

5. If $\{X(t)\}$ is a poisson process, prove that

$$P[X(s)=r / X(t)=n] = nC_r \left(\frac{s}{t}\right)^r \left(1-\frac{s}{t}\right)^{n-r}$$ where $s<t$.

$$P[x(s)=r / x(t)=n] = \frac{P[\{x(s)=r\} \cap \{x(t)=n\}]}{P[\{x(t)=n\}]}$$

$$= \frac{P[\{x(s)\}=r \cap \{x(t-s)\}=n-r]}{P[\{x(t)\}=n]}$$

$$= \frac{P[x(s)=r]\,P[x(t-s)=n-r]}{P[x(t)=n]}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^{r}}{r!} \cdot \dfrac{e^{-\lambda(t-s)}[\lambda(t-s)]^{n-r}}{(n-r)!}}{\dfrac{e^{-\lambda t}(\lambda t)^{n}}{n!}}$$

$$= \frac{n!}{r!(n-r)!} \cdot \frac{e^{-\lambda s}\lambda^{r} s^{r}\, \cancel{e^{-\lambda t}} \cdot e^{\lambda s}\lambda^{n-r}(t-s)^{n-r}}{\cancel{e^{-\lambda t}}\lambda^{n} t^{n}}$$

$$= nC_r \frac{\cancel{\lambda^{n}} s^{r}(t-s)^{n-r}}{\cancel{\lambda^{n}} t^{n}}$$

$$= nC_r \frac{s^r (t-s)^{n-r}}{t^n}$$

$$= nC_r \frac{s^r (t)^{n-r} \left[1 - \frac{s}{t}\right]^{n-r}}{t^n}$$

$$= nC_r \frac{s^r \left(1 - \frac{s}{t}\right)^{n-r}}{t^n \cdot t^{-(n-r)}}$$

$$= nC_r \frac{s^r \left(1 - \frac{s}{t}\right)^{n-r}}{t^r}$$

$$= nC_r \left(\frac{s}{t}\right)^r \left(1 - \frac{s}{t}\right)^{n-r}$$

6. If the probability that a person suffers from a disease is 0.001, find the probability that out of 3000 persons, (a) exactly 4 (b) more than 2 persons will suffer from the disease.

Here $p = 0.001$

$n = 3000$

$np = \lambda = 3000 \times .001 = 3.$

Poisson distribution is given by

$$P[X = x] = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad x = 0, 1, 2, \ldots$$

(a) $P[X = 4] = \dfrac{e^{-3} (3)^4}{4!}$

$= \dfrac{0.0497870684 \times 81}{24}$

$= 0.16803.$

UNIT – III
MARKOV PROCESSES AND MARKOV CHAIN
PART – A (2 Marks)

1. What do you understand by stationary process?
2. {X(s,t)} is a random process , what is the nature of X(s,t) when (a) s is fixed (b) t is fixed?
3. Define wide – sense stationary process.
4. Define strict sense stationary random process.
5. Consider the random process X(t) = cos (t+ϕ),where ϕ is uniformly distributed in $(-\pi/2, \pi/2)$. Check whether the process is stationary?
6. Give an example of Markov process.
7. Define a Markov process and a Markov chain.
8. State Chapman-Kolmogorov theorem.
9. Define irreducible Markov chain.
10. Prove that the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ is the tpm of an irreducible Markov chain.
11. Define one-step Transition probability.
12. Define Transition probability matrix.
13. If the transition probability matrix of a Markov chain is $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, find the limiting

5.a. The tpm of a Markov chain with 3 states 0,1,2 is P $\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ and the initial state distribution of the chain is $P(X_0=i) = \frac{1}{3}$, i = 0,1,2. Find (i) $P(X_2=2)$ (ii) $P(X_3=1;X_2=2;X_1=1;X_0=2)$ and (iii) $P(X_1=1/X_2=2)$. **(8)**

b. Consider a Markov chain with state space {0,1} and the tpm P=$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(i) Draw a transition diagram.
(ii) Show that state 1 is transient.
(iii) Is the chain irreducible. **(8)**

6.a. Find the nature of the states of the Markov chain with the transition probability matrix $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$. **(8)**

b. Suppose that the probability of a dry day following a rainy day is $\frac{1}{3}$ and that the probability of a rainy day following a dry day is $\frac{1}{2}$.Given that may 1 is a dry day. Find the probability that may 3 is a dry day and also may 5 is a dry day. **(8)**

7.a. Let $\{X_n : n=1,2,3,..\}$ be a Markov chain on the space S = {1,2,3} with one step transition probabilities P = $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$

(i) Sketch the transition diagram.
(ii) Is the chain irreducible?Explain.
(iii) Is the chain Ergodic?Explain. **(8)**

b. Three boys A,B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian and prove that the chain is irreducible. Find also the steady-state distribution of the chain. **(8)**

8.a. Prove that the sum of two independent Poisson processes is a Poisson process but the difference is not a Poisson process. **(8)**

b. If customers arrive at a counter in accordance with a poisson process with a mean rate of 2 per minute,find the probability that the interval between 2 consecutive arrivals is (i) more than 1 min (ii) between min.1 and min.2 and (iii) 4 min.or less. **(8)**

9.a. Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute ; find the probability that during a time interval of 2 min (i) exactly 4 customers arrive and (ii) more than 4 customers arrive. **(8)**

b. State the postulates of Poisson process and also prove that the inter-arrival time of Poisson process follows exponential distribution. **(8)**

10.a. Define the Poisson process and obtain its probability distribution. **(8)**

b. If $\{N_1(t)\}$ and $\{N_2(t)\}$ are two independent Poisson process with parameters λ_1 and λ_2 respectively, show that

$$P[N_1(t) = k / \{ N_1(t) + N_2(t) = n\}] = \binom{n}{k} p^k q^{n-k},$$

Where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ and $q = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$. **(8)**

UNIT IV QUEUEING THEORY

Markovian models – Birth and death queuing models – Steady state results – Single and multiple server queuing models – Queues with finite waiting rooms – Finite source models – Little's formula.

20/03/10.

QUEUING THEORY

Generalisation of Birth and Death process is called Queuing process.

Queuing theory describes the problem of interest is how to achieve a balance b/w the cost associated with long waiting (queue) and the cost associated with the prevention of waiting in order to maximize the profits.

Charac. of Queuing system:

1. I/p (the arrival) - no. of arrivals in one time period or the interval b/w successive arrivals is not treated as a constant, but a RV. So the model of arrivals of the customer is expressed by means of the prob. distribution of the no. of arrivals.

   The no. of arrivals per unit of time has a Poisson distribution with mean λ. The time b/w consecutive arrivals has an exponential distribution with mean $1/\lambda$.

2. Service rate: The mode of service is represented by means of the prob. dist. of the no. of customers per unit of time has a Poisson distribution with mean μ, 2 i.e. service time has an expon. dist. with mean $1/\mu$.

3. Queue discipline.

(i) FIFO [First In First Out]

(ii) LIFO [Last In First Out]

(iii) Selection in Random order

4. Customer behaviour:

(i) Balking (ii) Jockeying (iii) Priority

Kendal's Notation:

Symbolic representation of a Queuing model.

$(a/b/c) : (d/e)$

where a – no. of arrivals per unit time

b – type of dist. of the service time

c – no. of servers

d – capacity of the system

e – queue discipline.

Models:

1. Single server & infinite capacity
$(M/M/1) : (\infty/FIFO)$
2. Single server & finite capacity
$(M/M/1) : (S/FIFO)$
3. Multi server & infinite capacity
$(M/M/M) : (\infty/FIFO)$
4. Multi server & finite capacity
$(M/M/K) : (S, FIFO)$.

Values of P_0 and P_n for Poisson Queuing Systems:

In the steady state $P_n(t)$ and $P_0(t)$ are independent of time and hence $P_n'(t)$ and $P_0'(t)$ becomes zero. Hence we can derive the following difference eqn.

$$\lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0 \quad -(1)$$

$$-\lambda_0 P_0 + \mu_1 P_1 = 0 \quad -(2)$$

$$(1) \Rightarrow P_{n+1} = \frac{(\lambda_n + \mu_n) P_n - \lambda_{n-1} P_{n-1}}{\mu_{n+1}} \quad -(3)$$

$$(2) \Rightarrow P_1 = \frac{\lambda_0 P_0}{\mu_1} \quad -(4)$$

For $n=1$ in eqn. (3)

$$P_2 = \frac{(\lambda_1 + \mu_1) P_1 - \lambda_0 P_0}{\mu_2}$$

$$= \left(\frac{\lambda_1 + \mu_1}{\mu_2}\right)\left(\frac{\lambda_0 P_0}{\mu_1}\right) - \left(\frac{\lambda_0 P_0}{\mu_2}\right)$$

$$= \lambda_0 P_0 \left[\frac{\lambda_1 + \mu_1}{\mu_1 \mu_2} - \frac{1}{\mu_2}\right]$$

$$= \lambda_0 P_0 \left[\frac{\mu_1 + \lambda_1 - \mu_1}{\mu_1 \mu_2}\right] = \left[\frac{\lambda_0 \lambda_1}{\mu_1 * \mu_2}\right] P_0$$

$$P_2 = \left[\frac{\lambda_0 \lambda_1}{\mu_1 * \mu_2}\right] P_0$$

$$\Rightarrow P_3 = \left[\frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3}\right] P_0$$

In general,

$$P_n = \left(\frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right) P_0$$

$$P_0 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right) P_0 = 1$$

$$P_0 \left[1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right) \right] = 1$$

$$P_0 = 1 \Big/ 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right)$$

$$= 1 \Big/ 1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n$$

$$= 1 \Big/ 1 + \{ (\lambda/\mu)^1 + (\lambda/\mu)^2 + (\lambda/\mu)^3 + \cdots \}$$

$$= 1 \Big/ (1 - \lambda/\mu)^{-1} = 1 - \lambda/\mu$$

Characteristics of Infinite capacity, Single server Poisson Queuing model: [Model I]:

* L_s - Avg. no. of customers in the system
* N denote the no. of customers in the system

* N is the discrete RV which can take the values $0, 1, 2, \cdots \infty$ such that $P_n = (\lambda/\mu)^n P_0$ which is equal to $(\lambda/\mu)^n (1 - \lambda/\mu)$.

Now, $L_s = E(N) = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

$$= (1 - \lambda/\mu) \left[\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2 + 3\left(\frac{\lambda}{\mu}\right)^3 + \cdots \right]$$

$$= (1 - \lambda/\mu)(\lambda/\mu) \left[1 + 2\left(\frac{\lambda}{\mu}\right) + 3\left(\frac{\lambda}{\mu}\right)^2 + \cdots \right]$$

(6) Pdf of waiting time & waiting time distribution fn. of the queue.

Let T_q be the RV that represents the waiting time of the customer in the queue. Let the pdf of T_q be $f_q(w)$ and let $f_q(w/n)$ be the density fn. of T_q given that there are n customers in the system {n−1 in queue & 1 being serviced}.

$$\therefore f_q(w) = \sum_{n=1}^{\infty} f_q(w/n)\, P_n$$

where $f_q(w/n)$ = pdf of sum of n service times

= pdf of sum of n independent RV, each of which is exponentially distributed with parameter μ.

$$f_q(w/n) = \frac{\mu^n e^{-\mu w} w^{n-1}}{(n-1)!}, \quad w > 0$$

⇒ the pdf of gamma distribution,

$$f_q(w) = \sum_{n=1}^{\infty} \frac{\mu^n . e^{-\mu w} . w^{n-1}}{(n-1)!} P_n$$

$$= \sum_{n=1}^{\infty} \frac{\mu^n . e^{-\mu w} . w^{n-1}}{(n-1)!} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \lambda/\mu\right)$$

$$= \left(1 - \lambda/\mu\right) e^{-\mu w} . \lambda \sum_{n=1}^{\infty} \frac{\mu^n (\lambda w)^{n-1}}{(n-1)!}$$

$$= \frac{\lambda}{\mu}(\mu - \lambda) e^{-\mu w} e^{-w(\mu - \lambda)}$$

$= \left(\frac{\lambda}{\mu}\right)(1-\lambda/\mu)(1-\lambda/\mu)^{-2} = \left(\frac{\lambda}{\mu}\right)(1-\lambda/\mu)^{-1}$

$= \frac{\lambda/\mu}{\frac{\mu-\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$

(2) Traffic intensity $\rho = \lambda/\mu$

(3) To find the expected no. of customers in the queue (or) average length of the queue $= L_q$

queue length = queue size − no. of customers being served

If the no. of customers in the system is n, then the no. of customers in queue $= (n-1)$, (excluding 1 being serviced)

$L_q = \sum_{n=1}^{\infty} (n-1) P_n = E(n-1)$

$= \sum_{n=1}^{\infty} nP_n - P_n = \sum_{n=0}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_0\right]$

$= L_s - [1-P_0] = \frac{\lambda}{\mu-\lambda} - 1 + 1 - \frac{\lambda}{\mu}$

$= \lambda\left[\frac{\mu-(\mu-\lambda)}{\mu(\mu-\lambda)}\right] = \frac{\lambda^2}{\mu(\mu-\lambda)}$

(4) To find the expected no. of customers in non-empty queue.

$L_n = \frac{E[n-1]}{P(n-1>0)} = \frac{E(n-1)}{P(n>1)} = \frac{L_q}{\sum_{n=2}^{\infty} P_n}$

$= \frac{\lambda^2/\mu(\mu-\lambda)}{(1-\lambda/\mu)\sum_{n=2}^{\infty}(\lambda/\mu)^n} = \frac{\lambda^2/\mu(\mu-\lambda)}{(1-\lambda/\mu)\sum_{n=0}^{\infty}(\lambda/\mu)^{n+2}}$

(10) Pdf of the waiting time & waiting time distribution of the system:

Let T_s be the RV that represents the waiting time of the customer in the system. Let the pdf of T_s be $f_s(\omega)$ and $f_s(\omega/n)$ be the density fn. of T_s given that there are n customers in the system when the customers arrived.

$\therefore f_s(\omega) = \sum_{n=0}^{\infty} f_s(\omega/n) * P_n$

$f_s(\omega/n)$ = pdf of sum of $n+1$ service times ($n \Rightarrow$ complete service time + service time of customer in the service)

= pdf of sum of $n+1$ indep. RV each of which follows exponential distribution with parameter μ.

$f_s(\omega/n) = \frac{\mu^{n+1} \cdot e^{-\mu\omega} \cdot \omega^n}{n!}, \quad \omega > 0$

$f_s(\omega) = \sum_{n=0}^{\infty} \frac{\mu^{n+1} \cdot e^{-\mu\omega} \cdot \omega^n}{n!} \cdot \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$

$= (1-\lambda/\mu)\, e^{-\mu\omega} \cdot \mu \sum_{n=0}^{\infty} \frac{(\omega\lambda)^n}{n!}$

$= (1-\lambda/\mu)\, e^{-\mu\omega} \cdot \mu\, [e^{\lambda\omega}]$

$= (\mu-\lambda)\, e^{-\omega(\mu-\lambda)}$

Characteristics of a

Poisson queuing model: $(M/M/s):(\infty/FIFO)$

Values of P_0 & P_n:

For Poisson queue system, P_n is given by

$$P_n = \left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)P_0, \quad n \ge 1$$

where $$P_0 = \left[1+\sum_{n=1}^{\infty}\left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)\right]^{-1}$$

If there is a single server, $\mu_n = \mu$ for all μ. But there are s servers working independently of each server.

If no. of severs is less than s, i.e. $n<s$, only n servers will be busy and others will be idle.

Hence, the mean service rate will be $n\mu$.

If $n>s$, all s severs will be busy. then the mean service rate $= \mu_n = \begin{cases} n\mu, & 0 \le n < s \\ s\mu, & n \ge s \end{cases}$

Characteristics of a

Poisson queuing model: $(M/M/s):(\infty/FIFO)$

Values of P_0 & P_n:

For Poisson queue system, P_n is given by

$$P_n = \left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)P_0, \quad n \ge 1$$

where $$P_0 = \left[1+\sum_{n=1}^{\infty}\left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)\right]^{-1}$$

If there is a single server, $\mu_n = \mu$ for all μ. But there are s servers working independently of each server.

If no. of severs is less than s, i.e. $n<s$, only n servers will be busy and others will be idle.

Hence, the mean service rate will be $n\mu$.

If $n>s$, all s severs will be busy. then the mean service rate $= \mu_n = \begin{cases} n\mu, & 0 \le n < s \\ s\mu, & n \ge s \end{cases}$

$$= \frac{\lambda^n}{s!\,\mu^n\, s^{n-s}} P_0$$

$$P_n = \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 \quad , \; n \geq s$$

P_0 is given by $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{\infty} \left[\frac{1}{s!\, s^{n-s}} \left(\lambda/\mu\right)^n\right] \right] = 1$$

$$P_0 \left[\sum_{n=0}^{s} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^n \frac{1}{s!\, s^{s}} \right] = 1$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{s^s}{s!\, s^s} \left(\frac{\lambda}{\mu s}\right)^s \left[1 + \frac{\lambda}{\mu s} + \cdots\right] \right] = 1$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{s^s}{s!} \left(\frac{\lambda}{\mu s}\right)^s \left(1 - \frac{\lambda}{\mu s}\right)^{-1} \right] = 1$$

2. Avg. no of customers in the queue (or) avg. queue length:

$$L_q = E[N_q] \Rightarrow E[(N-s)] = \sum_{n=s}^{\infty} (n-s) P_n$$

$$= \sum_{n=0}^{\infty} n\, P_{(n+s)} \quad \{ \text{Replace } n \text{ by } n+s \}$$

$$= \sum_{n=0}^{\infty} n \left\{ \frac{1}{s!\, s^{n+s-s}} \left(\frac{\lambda}{\mu}\right)^{n+s} P_0 \right\}$$

$$= \sum_{n=0}^{\infty} n \left\{ \frac{1}{s!\, s^n} \left(\frac{\lambda}{\mu}\right)^{n+s} P_0 \right\}$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu s}\right)^n P_0$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \left[1 \cdot \frac{\lambda}{\mu s} + 2\left(\frac{\lambda}{\mu s}\right)^2 + \cdots \right] P_0$$

$$= \frac{\lambda^n}{s!\,\mu^n\, s^{n-s}} P_0$$

$$P_n = \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 \;, \quad n \geq s$$

P_0 is given by $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{\infty} \left[\frac{1}{s!\, s^{n-s}} \left(\lambda/\mu\right)^n\right]\right] =$$

$$P_0 \left[\sum_{n=0}^{s} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^n \frac{1}{s!\, s^s} \right] = 1$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{s^s}{s!\, s^n} \left(\frac{\lambda}{\mu s}\right)^s \left[1 + \frac{\lambda}{\mu s} + \cdots\right]\right] =$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{s^s}{s!} \left(\frac{\lambda}{\mu s}\right)^s \left(1 - \frac{\lambda}{\mu s}\right)^{-1}\right] =$$

2. Avg. no of customers in the queue (or) avg. queue length:

$$L_q = E[N_q] \Rightarrow E[(N-s)] = \sum_{n=s}^{\infty} (n-s) P_n$$

$$= \sum_{n=0}^{\infty} n\, P_{(n+s)} \quad \{\text{Replace } n \text{ by } n+s\}$$

$$= \sum_{n=0}^{\infty} n \left\{ \frac{1}{s!\, s^{n+s-s}} \left(\frac{\lambda}{\mu}\right)^{n+s} P_0 \right\}$$

$$= \sum_{n=0}^{\infty} n \left\{ \frac{1}{s!\, s^n} \left(\frac{\lambda}{\mu}\right)^{n+s} P_0 \right\}$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu s}\right)^n P_0$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \left[1 \cdot \frac{\lambda}{\mu s} + 2\left(\frac{\lambda}{\mu s}\right)^2 + \cdots\right] P_0$$

⑤ Avg. time a customer has to spend in the queue.

$$E[W_q] = \frac{E[N_q]}{\lambda} = \frac{\frac{1}{\mu} \cdot \frac{1}{s \cdot s!}}{\left(1 - \frac{\lambda}{\mu s}\right)^2}$$

⑥ Prob. that an arrival has to wait

Required prob = prob. that there are s more customers in the system

i.e. $P(W_s > 0) = P(N \geq s)$

$$= \sum_{n=s}^{\infty} P_n = \sum_{n=s}^{\infty} \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right.$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s P_0 \sum_{n=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^n$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s P_0 \left[1 - \frac{\lambda}{\mu s}\right]^{-1}$$

$$P(W_s > 0) = \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \left[1 - \lambda/\mu s\right]^{-1}$$

⑦ Prob. that an arrival enters the service without waiting = 1 − P(an arrival has to wait)

$$= 1 - \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \left[1 - \lambda/\mu s\right]^{-1}$$

⑧ Mean waiting time in the queue for those who actually wait:

$$= E(W_q / W_q > 0) = \frac{E(W_q)}{P(W_q > 0)}$$

$$= \frac{\frac{1}{\mu s \cdot s!} \dfrac{(\lambda/\mu)^s}{(1 - \lambda/\mu s)^2} P_0}{\frac{1}{s!} (\lambda/\mu)^s \; \frac{1}{(1 - \lambda/\mu s)}}$$

$$= \frac{1}{\mu s} \cdot \frac{1}{1 - \lambda/\mu s}$$

$$= \frac{1}{\mu s - \lambda}$$

(9) Prob. that there will be someone waiting

Required prob. $= P(N \geq s+1)$

$$= \sum_{n=s+1}^{\infty} P_n$$

$$= \sum_{n=s}^{\infty} P_n - P(N=s)$$

$$= \sum_{n=s}^{\infty} \frac{1}{s!} \frac{1}{s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 - \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!}$$

$$= \frac{\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s P_0}{(1 - \lambda/\mu s)} - \frac{(\lambda/\mu)^s P_0}{s!}$$

$$= \frac{P_0 (\lambda/\mu)^s \left[1 - (1 - \lambda/\mu s)\right]}{s! \, (1 - \lambda/\mu s)}$$

$$= \frac{P_0 (\lambda/\mu)^s (\lambda/\mu s)}{s \cdot s! \, (1 - \lambda/\mu s)}$$

$$= \frac{P_0 (\lambda/\mu)^{s+1}}{s \cdot s! \, (1 - \lambda/\mu s)}$$

UNIT-IV
QUEUEING THEORY
PART-A (2 Marks)

1. Define birth and death process.
2. What are the basic characteristics of a queueing system?
3. Define Kendall's notation.
4. Give the formulas for the waiting time of a customer in the queue and in the system for the (M/M/1):(∞/FIFO) model.
5. In the usual notation of an M/M/1 queueing system, if λ=3/hour and μ=4/hour, find $P(X \geq 5)$ where X is the number of customers in the system.
6. In the usual notation of an M/M/1 queueing system, if λ=12/hour and μ=24/hour, find the average number of customers in the system.
7. Derive the average number of customers in the system for (M/M/1):(∞/FIFO) model.
8. State Little's formula for an (M/M/1):(∞/FIFO) queueing model.
9. In a given (M/M/1):(∞/FCFS) queue, ρ=0.6, what is the probability that the queue contains 5 or more customers?
10. What is the probability that a customer has to wait more than 15 minutes to get his service completed in a (M/M/1):(∞/FIFO) queue system, if λ=6 per hour and μ=10 per hour?
11. Consider an M/M/1 queueing system. If λ=6 and μ=8, find the probability of atleast 10 customers in the system.
12. Consider an M/M/1 queueing system. Find the probability of finding atleast 'n' customers in the system.
13. What is the probability that an arrival to an infinite capacity 3 server Poisson queue with $\lambda/c\mu = 2/3$ and $P_0=1/9$ enters the service without waiting?
14. Consider an M/M/C queueing system. Find the probability that an arriving customer is forced to join the queue.
15. For (M/M/C):(N/FIFO)model, write down the formula for
    (i) Average number of customers in the queue
    (ii)Average waiting time in the system
16. Give the formulas for the average number of customers in the queue and in the system for the (M/M/s):(∞/FIFO) queueing model.
17. What is the effective arrival rate for (M/M/1):(4/FCFS) queueing model when λ=2 and μ=5.

18. Give the probability that there is no customer in an (M/M/1):(k/FIFO) queueing system
19. Write the formulas for the average number of customers in the (M/M/1):(k/FIFO) queueing system and also in the queue.
20. Define effective arrival rate with respect to an (M/M/1):(k/FIFO) and (M/M/s):(k/FIFO) queueing models.

PART-B(16 Marks)

1.a. Derive the balance equation of birth and death process. **(8)**

b. Discuss the pure birth process and hence obtain its probabilities, mean and variance.(

2.a. Arrivals at a telephone booth are considered to be poisson with an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.

(a) Find the average number of persons waiting in the system.
(b) What is the probability that it will take him more than 10min. altogether to wait for the phone and complete his call?
(c) Estimate the fraction of the day when the phone will be in use
(d) What is the average length of the queue that forms time to time? **(8)**

b. In a railway marshalling yard goods-trains arrive at an average rate of 30 per day according to Poisson distribution. If the mechanic services according exponential distribution with a mean of 36 minutes, find L_s,L_q and W_s. **(8)**

3.a. Customers arrive at a one-man barber shop according to a Poisson process with a mean interarrival time of 12 min. Customers spend an average of 10min in the barber'

(a) What is the expected number of customers in the barber shop and in the queue?
(b) How much time can a customer expect to spend in the barber's shop?
(c) What is the average time customer spends in the queue?
(d) What is the probability that more than 3 customers are in the system? **(8)**

b. A T.V. repairman finds that the time spent on his jobs follows exponential distribution with a mean of 30 minutes. If he repairs the sets in the order of their arrival according to Poisson distribution at an average rate of 10 per 8 hour-day, find his expected idle time on each day and also the total number of sets in his shop.(8

4.a. Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes, and the service time is an exponential random variable with mean 8 minutes. Find the average number of customers L_S, the average waiting time a customer spends in the shop W_S and the average time a customer spends in the waiting for service W_q. **(8)**

b. A duplicating machine maintained for office use is operated by an office assistant who earns Rs. 5 per hour. The time to complete each jobs varies according to an exponential distribution with mean 6 mins. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If an 8-hrs day is used as a base , determine

(1) The percentage idle time of the machine.
(2) The average time a job is in the system.
(3) The average earning per day of the assistant. **(8)**

5.a. There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour.

(a) what fraction of the time all the typists will be busy?
(b) What is the average number of letters waiting to be typed?
(c) What is the average time a letter has to spend for waiting and for being typed?

(d) What is the probability that a letter will take longer than 20 min. waiting to be typed and being typed? **(8)**

b. A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min and cars arrive for service in a poisson process at the rate of 30 cars per hour.

(a) what is the probability that an arrival would have to wait in line?

(b) find the average waiting time, average time spend in the system and the average number of cars in the system.

(c) For what percentage of time would a pump be idle on an average? **(8)**

6.a. A supermarket has 2 girls attending to sales at the counters. If the service times for each customer is exponential with mean 4min and if people arrive in Poisson fashion at the rate of 10per hour,

(a) what is the probability that a customer has to wait for service?

(b) What is the expected percentage of idle time for each girl?

(c) If the customer has to wait in the queue, what is the expected length of his waiting time? **(8)**

b. A bank has two tellers working on savings account. Service time for each customer is 3 minutes and customers arrive at an average rate of 30 per hour. Assuming Poisson arrivals and exponential services, find the probability for a customer has to wait for service and also the expected waiting time. **(8)**

7.a. A tax consulting firm has three counters in its office to receive people who have problems concerning their income and sales taxes. On the average 48 persons arrive in an 8-hour day. Each tax advisor spends 15 minutes on the average on an arrival. If the arrivals are Poisson distributed and service times are exponentially distributed, find:

(a) Average number of customers in the system

(b) Average number of customers waiting to be served.

(c) Average time a customer spends in the system

(d) Average waiting time for a customer. **(8)**

b. The local one person barber shop can accommodate a maximum of 5 people at a time(4waiting and 1 getting hair-cut) Customers arrive according to a Poisson distribution with mean 5 per hour. The barber cuts hair at an average rate of 4 per hour

(a) what percentage of time is the barber idle?

(b) What fraction of the potential customers are turned away?

(c) What is the expected number of customers waiting for a hair-cut?

(d) How much time can a customer expect to spend in the barber shop? **(8)**

8.a. Patients arrive at clinic according to poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(a) find the effective arrival rate at the clinic.

(b) What is the probability that an arriving patient will not wait?

(c) What is the expected waiting time until a patients is discharged from the clinic? **(8)**

b. Customers arrive at a one window drive-in bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space in front of the window including that for the serviced car can accommodate a maximum of 3 cars. Other can wait outside this space.

(1) What is the probability that an arriving customer can drive directly to the space in front of the window?
(2) What is the probability that an arriving customer will have to wait outside the indicated space?
(3) How long the arriving customer is expected to wait before starting service? **(8)**

9.a. A 2-person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full, leave without entering barber shop. Customers arrive at the average rate of 4 per hour and spend an average of 12 min in the barber's chair. Compute $P_0, P_1, P_7, E[N_q]$ and $E[W]$. **(8)**

b. A car servicing station has 2 bays where service can be offered simultaneously. Because of space limitation, only 4 cars are accepted for servicing. The arrival pattern is poisson with 12 cars per day. The service time in both the bays is exponentially distributed with μ=8 cars per day per boy. Find the average number of cars in the service station, the average number of cars waiting for service and the average time a car spends in the system. **(8)**

10.a. For an M/M/2 queueing system with a waiting room of capacity 5, find the average number of customers in the system, assuming that arrival rate as 4 per hour and mean service time 30 minutes. **(8)**

b. In a machine repair station, the machine mechanic repairs four machines. Meantime between service requirements is 5 hours for each machine with Exponential distribution and mean repair time is one hour with exponential distribution. Find
(a) Probability that the service will be idle.
(b) Average number of machine waiting to be repaired and being repaired.
(c) Expected time a machine will wait in queue to be repaired. **(8)**

UNIT V NON-MARKOVIAN QUEUES AND QUEUE NETWORKS 9

M/G/1 queue – Pollaczek – Khintchine formula – Series queues – Open and closed networks.

Charac. of finite capacity on single server Poisson queue $[(M/M/1):(K/FIFO)]$

1. Values of P_0 and P_n

For the poisson queue system, P_n

$P_n = P(N=n)$, in the steady state is given by the difference equations

$$\lambda_{n-1}P_{n-1} - (\lambda_n + \mu_n)P_n + \mu_{n+1}P_{n+1} = 0 \; ; n > 0$$

$$\& \; -\lambda_0 P_0 + \mu_1 P_1 = 0 \; ; \; n = 0$$

This model represents ~~estimation~~ situation in which the system can accomodate only a finite no. of arrivals (k).

If the customer arrives & the queue is full, the customer leaves without joining the queue.

$\therefore$ for this model,

$$\mu_n = \mu \; ; \; n = 1, 2, 3, \ldots$$

$$\& \; \lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, \ldots (k-1) \\ 0, & n = k, k+1, \ldots \end{cases}$$

Using these values ... in the eqns. given above, we have

$\mu P_1 = \lambda P_0$ — ①

$\mu P_{n+1} = (\lambda+\mu) P_n - \lambda P_{n-1}$, for $1 \le n \le$ — ②

$\mu P_k = \lambda P_{k-1}$, $n = k$ — ③

($\because P_{k+1}$ has no meaning)

From eqn ①

$$P_1 = \frac{\lambda}{\mu} P_0$$

From eqn. ②

for $n=1$, $\mu P_2 = (\lambda+\mu) P_1 - \lambda P_0$

$$\mu P_2 = (\lambda+\mu)\left(\frac{\lambda}{\mu} P_0\right) - \lambda P_0$$

$$P_2 = \frac{(\lambda+\mu)}{\mu}\left(\frac{\lambda}{\mu}\right) P_0 - \lambda P_0$$

$$= \left[\left(\frac{\lambda}{\mu}\right)^2 + \frac{\lambda}{\mu}\right] P_0 - \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

From eqn. ③

$$\mu P_k = \lambda P_{k-1}$$

$$P_k = \left(\frac{\lambda}{\mu}\right) P_{k-1} = \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{k-1} P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^k P_0$$

In general,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Now

$$\sum_{n=0}^{k} P_n = 1 \Rightarrow \sum_{n=0}^{k} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$P_0\left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots\left(\frac{\lambda}{\mu}\right)^k\right]=1$$

$$P_0\left[1-\lambda/\mu\right]^{-1}=1$$

$$P_0\left[\frac{\mu-\lambda}{\mu}\right]^{-1}=1 \Rightarrow \frac{\mu P_0}{\mu-\lambda}=1$$

$$P_0=\frac{\mu-\lambda}{\mu}$$

$$P_0\frac{\left\{1-(\lambda/\mu)^{k+1}\right\}}{1-\lambda/\mu}=1$$ which is valid for ~~even~~ $\lambda\neq\mu$

$$P_0=\begin{cases}\dfrac{1-\lambda/\mu}{1-(\lambda/\mu)^{k+1}}, & \text{if } \lambda\neq\mu\\[2ex] 1/k+1, & \text{if } \lambda=\mu\end{cases}$$

$$\because \lim_{\lambda/\mu\to 1}\frac{(1-\lambda/\mu)}{1-(\lambda/\mu)^{k+1}}=1/k+1$$

$$\therefore P_n=\begin{cases}\left(\dfrac{\lambda}{\mu}\right)^n\left\{\dfrac{1-\lambda/\mu}{1-(\lambda/\mu)^{k+1}}\right\}, & \lambda\neq\mu\\[2ex] \dfrac{1}{k+1}, & \lambda=\mu\end{cases}$$

2. Average no. of customers in the system

$$E(N)=\sum_{n=0}^{k}nP_n=\frac{1-\lambda/\mu}{\left(1-(\lambda/\mu)^{k+1}\right)}\sum_{n=0}^{k}n\left(\frac{\lambda}{\mu}\right)^n$$

let $x=\lambda/\mu$

$$=\frac{(1-x)\cdot x}{1-x^{k+1}}\sum_{n=0}^{k}\frac{d}{dx}(x^n)$$

$$=\frac{(1-x)x}{1-x^{k+1}}\cdot\frac{d}{dx}\left(\frac{1-x^{k+1}}{1-x}\right)$$

$$= \frac{(1-x)\,x}{1-x^{k+1}} \left[\frac{(1-x)\left[-(k+1)x^{k}\right] - (1-x^{k+1})(-1)}{(1-x)^2} \right]$$

$$= \frac{(x-1)\,x}{1-x^{k+1}} \left[\frac{(1-x)(k+1)x^{k} - (1-x^{k+1})}{(1-x)^2} \right]$$

$$= \frac{x}{1-x^{k+1}} \left[\frac{(k+1)x^{k} - (k+1)x^{k+1} - 1 + x^{k+1}}{x-1} \right]$$

$$= \frac{x\left[kx^{k} + x^{k} - kx^{k+1} - \cancel{x^{k+1}} - 1 + \cancel{x^{k+1}}\right]}{(1-x^{k+1})(x-1)}$$

$$= \frac{kx^{k+1} + x^{k+1} - kx^{k+2} - x}{(1-x^{k+1})(x-1)}$$

$$= \frac{(k+1)x^{k+1} - kx^{k+2}}{(1-x^{k+1})(x-1)}$$

$$= \frac{(k+1)x^{k+1}}{(1-x^{k+1})(x-1)} - \frac{kx^{k+2}}{(1-x^{k+1})(x-1)}$$

$$= \frac{x}{1-x} - \frac{(k+1)x^{k+1}}{1-x^{k+1}} \qquad \text{if } \lambda \neq \mu$$

$$= \left(\frac{\lambda/\mu}{1-\lambda/\mu}\right) - \frac{(k+1)(\lambda/\mu)^{k+1}}{1-(\lambda/\mu)^{k+1}}$$

if $\lambda = \mu \qquad E(N) = \sum_{n=0}^{k} \frac{n}{k+1} = \frac{k}{2}$

Charac. of finite capacity on single server Poisson queue $[(M/M/1):(K/FIFO)]$

1. Values of P_0 and P_n

For the poisson queue system, P_n

$P_n = P(N=n)$, in the steady state is given by the difference equations

$$\lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0 \; ; n > 0$$

$$\& \; -\lambda_0 P_0 + \mu_1 P_1 = 0 \; ; \; n = 0$$

This model represents ~~estimation~~ situation in which the system can accomodate only a finite no. of arrivals (k).

If the customer arrives & the queue is full, the customer leaves without joining the queue.

$\therefore$ For this model,

$\mu_n = \mu \; ; n = 1, 2, 3, \ldots$

$$\& \; \lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, \ldots (k-1) \\ 0, & n = k, k+1, \ldots \end{cases}$$

3. Average no. of customers in the queue.

$$E(N_q) = E(N-1) = \sum_{n=1}^{k} (n-1) P_n$$

$$= \sum_{n=0}^{k} n P_n - \sum_{n=1}^{k} P_n$$

$$= E(N) - (1 - P_0) \qquad \left\{ \because \sum_{n=0}^{k} P_n = 1 \right\}$$

As per Little's formula,

$$E(N_q) = E(N) - \lambda/\mu$$

which is true when the avg. arrival rate is λ throughout. From (A) $1 - P_0 \neq \frac{\lambda}{\mu}$, because the avg. arrival rate is λ as long as there is a vacancy in the queue and is zero when the system is full.

Hence we define the overall effective arrival rate, denoted by λ', by using (A) & Little's formula as,

$$\frac{\lambda'}{\mu} = 1 - P_0 \quad (\text{or}) \quad \boxed{\lambda' = \mu(1 - P_0)}$$

$$\therefore E(N_q) = E(N) - \frac{\lambda'}{\mu} \Rightarrow \text{modified Little's formula}$$

4. Avg. waiting time in the system and in the queue.

By modified Little's formula,

$$E(W_s) = 1/\lambda' \, E(N)$$

$$E(W_q) = 1/\lambda' \, E(N_q)$$

$$= \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\lambda^s}{s!\,\mu^s}\left[1 + \frac{\lambda}{\mu s} + \left(\frac{\lambda}{\mu s}\right)^2 + \cdots + \left(\frac{\lambda}{\mu s}\right)^{k-s}\right]$$

$$P_0^{-1} = \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{\mu s}\right)^{n-s}$$

$$P_n = \begin{cases} \frac{1}{n!}\left(\lambda/\mu\right)^n P_0, & \text{for } n \le s \\ \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{for } s < n \le k \\ 0, & \text{for } n > k \end{cases}$$

2. Avg. no. of customers in the queue:

$$E(N_q) = E(N-s) = \sum_{n=s}^{k} (n-s) P_n$$

$$= \frac{P_0}{s!} \sum_{n=s}^{k} \frac{(n-s)\left(\frac{\lambda}{\mu}\right)^n}{s^{n-s}}$$

$$= \frac{(\lambda/\mu)^s}{s!} P_0 \cdot \sum_{x=0}^{k-s} x \cdot \left(\frac{\lambda}{\mu s}\right)^x$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0 \rho}{s!} \sum_{x=0}^{k-s} x \cdot \rho^{x-1} \qquad \rho = \frac{\lambda}{\mu s}$$

$$= \left(\frac{\lambda}{\mu}\right)^s \frac{P_0 \rho}{s!} \sum_{x=0}^{k-s} \frac{d}{d\rho}(\rho^x)$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0 \rho}{s!} \cdot \frac{d}{d\rho}\left\{\frac{1-\rho^{k-s+1}}{1-\rho}\right\}$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0 \rho}{s!}\left[\frac{-(1-\rho)(k-s+1)\rho^{k-s} + (1-\rho^{k-s+1})}{(1-\rho)^2}\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0 \rho}{s!}\left[\frac{-(k-s)(1-\rho)\rho^{k-s} - (1-\rho)\rho^{k-s} + 1 - \rho^{k-s+1}}{(1-\rho)^2}\right]$$

$$= P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!}\left[\frac{-(k-s)(1-\rho)\rho^{k-s}+1-\rho^{k-s}(1-\rho+\rho)}{(1-\rho)^2}\right]$$

$$= P_0\cdot\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!\,(1-\rho)^2}\left[1-\rho^{k-s}-(k-s)(1-\rho)\rho^{k-s}\right]$$

where $\rho = \lambda/\mu s$

3. Average no. of customers in the system.

$$E(N) = \sum_{n=0}^{k} n P_n = \sum_{n=0}^{s-1} n P_n + \sum_{n=s}^{k} n P_n$$

$$= \sum_{n=0}^{s-1} n P_n + \sum_{n=s}^{k} (n-s) P_n + \sum_{n=s}^{k} s P_n$$

$$= \sum_{n=0}^{s-1} n P_n + E(N_q) + s\left\{\sum_{n=0}^{k} P_n - \sum_{n=0}^{s-1} P_n\right\}$$

$$= E(N_q) + s - \sum_{n=0}^{s-1} (s-n) P_n \quad \left\{\because \sum_{n=0}^{k} P_n = 1\right\} \quad -(1)$$

Obviously, $\left\{s - \sum_{n=0}^{s-1} (s-n) P_n\right\} \neq \frac{\lambda}{\mu}$,

so that step (1) represents Little's formula.

In order to make (1) to assume the form of Little's formula, we define the overall effective arrival rate λ' or λ_{eff} as follows:

$$\frac{\lambda'}{\mu} = s - \sum_{n=0}^{s-1} (s-n) P_n$$

$$\text{ie. } \lambda' = \mu\left[s - \sum_{n=0}^{s-1} (s-n) P_n\right]$$

With this defn. of λ', step (1) becomes

$$E(N) = E(N_q) + \frac{\lambda'}{\mu}$$

which is the modified Little's formula for this model.

4. Avg. waiting time in the system & in the queue

By the modified Little's formula,

$$E(W_s) = \frac{1}{\lambda'} E(N)$$

and $E(W_q) = \frac{1}{\lambda'} E(N_q)$

1. A petrol pump station has 4 pumps. The service time follows the exponential distribution with a mean of 6 min and cars arrive for service in a Poisson process at the rate of 30 cars per hour.

a) What is the prob. that an arrival would have to wait in line?

b) Find the avg. w. time, avg. time spent in the system & the avg. no. of cars in the system.

c) For what % of time would a pump be idle on an average?

$\frac{1}{\mu} = 6$ min $\quad \mu = \frac{1}{6}$ /min

$s = 4$. $\lambda = 30/$hr $\quad \mu = 10/$hr

$\Rightarrow (M/M/s):(\infty/FIFO)$

a) P(an arrival has to wait)

$= P(W > 0)$

$$= \frac{(\lambda/\mu)^s \cdot P_0}{s!\,(1-\lambda/\mu s)} = \frac{3^4 \cdot P_0}{4!\,(1-3/4)} = \frac{81\,P_0}{6}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{(\lambda/\mu)^s}{s!\,(1-\lambda/\mu s)}\right]$$

$$= \left[\sum_{n=0}^{3} \frac{1}{n!}(3)^n + \frac{3^4}{24\,(1-3/4)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{3} \frac{3^n}{n!} + 13.5\right]^{-1}$$

$$= \left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + 13.5\right]^{-1}$$

$$= \left[1 + 3 + \frac{9}{2} + \frac{27}{6} + 13.5\right]^{-1}$$

$$= [26.5]^{-1} = 0.0377$$

$\Rightarrow P(W > 0) = 13.5 \times 0.0377 = 0.509$

$$) \quad E(W_q) = \frac{1}{\mu} \cdot \frac{1}{s \times s!} \cdot \frac{(\lambda/\mu)^s}{(1-\lambda/\mu s)^2} \times P_0$$

$$= \frac{1}{10 \times 4 \times 24} \cdot \frac{3^4}{(1/4)^2} = \frac{81}{\cancel{40} \times \cancel{24}} \times \frac{1}{\cancel{4 \times 4}} \times 0.0377$$

$$= \frac{81}{60} \times 0.0377 = 0.0509 \text{ h}$$

$$E(W_s) = \frac{1}{\mu} + E(W_q) = \frac{1}{10} + 0.0509 = 0.1509 \text{ h}$$

$$E(N) = \frac{1}{s \times s!} \cdot \frac{(\lambda/\mu)^{s+1}}{(1-\lambda/\mu s)^2} \times P_0 + \frac{\lambda}{\mu}$$

$$= \frac{1}{4 \times 24} \times \frac{3^5}{(1-3/4)^2} \times 0.0377 + 3$$

$$= \frac{81 \times \cancel{8}}{\cancel{8}2} \times 0.0377 + 3$$

$$= 1.526 + 3 = 4.526 \text{ cars}$$

c) fraction of time when the pumps are busy

$$= \frac{\lambda}{\mu s} = \frac{3}{4}$$

$\therefore$ fraction of time when the pumps are idle $= 1/4$

% required $= 25\ \%$

Non - Markovian Queueing model :

$(M/G/1) : (\infty/GD)$

In Markovian queueing model, the

$M \rightarrow$ the no. of arrivals in time t follows poisson process.

$G \rightarrow$ the service time follows a general (arbitrary) distribution

$GD \rightarrow$ General queue disciple (FIFO, LIFO)

Pollaczek - Khinchine formula

Let N and N' be the no. of customers in the system at times t and $t+T$, when 2 consecutive customers have just left the system after getting serviced. Thus T is the random service time, which is a continuous RV. Let $f(t)$, $E(T)$, $Var(T)$ be the pdf, mean and variance of T.

Also let M be the no. of customers arriving in the system during the service time T.

$$N' = \begin{cases} M, & \text{if } N = 0 \\ N-1+M, & \text{if } N > 0 \end{cases}$$

where M is a discrete RV, taking the values $0, 1, 2, \ldots$

Equivalently, $N' = N - 1 + M + \delta$ — (1)

where $\delta = \begin{cases} 1, & \text{if } N = 0 \\ 0, & \text{if } N > 0 \end{cases}$

$\therefore E(N') = E(N) - 1 + E(M) + E(\delta)$ — (2)

When the system has reached steady state, the prob. of no. of customers in the system will be a constant. Hence $E(N) = E(N')$, $E(N^2) = E(N'^2)$.

Using this we get $E(\delta) = 1 - E(M)$ — (3)

Squaring on both sides on

$$N'^2 = N^2 + (M-1)^2 + \delta^2 + 2N(M-1) + 2(M-1)\delta + 2N\delta \quad — (4)$$

Now $\delta^2 = \delta$ $\quad (\because \delta = 0 \text{ or } 1)$

$$N\delta = \begin{cases} 0 \times 1, & \text{if } N = 0 \\ N \times 0, & \text{if } N > 0 \end{cases}$$

$$N\delta = 0$$

(4) $\Rightarrow$

$$\therefore N'^2 = N^2 + (M-1)^2 + \delta + 2N(M-1) + 2(M-1)$$

$$= N^2 + M^2 + 2N(M-1) + (2M-1)\delta - 2M + 1$$

$$2N(1-M) = N^2 - N'^2 + M^2 + (2M-1)\delta - 2M + 1$$

$$2E(N(1-M)) = E(N^2) - E(N'^2) + E(M^2) + E[(2M-1)\delta] - 2E(M) + 1$$

By independence,

$$2E(N)E(1-M) = E(M^2) + (2E(M) - 1)E(\delta) - 2E(M) + 1$$

$$\therefore E(N) = \frac{E(M^2) + \{2E(M) - 1\}\{1 - E(M)\} - 2E(M) + 1}{2\{1 - E(M)\}}$$

$$= \frac{E(M^2) - 2E^2(M) + E(M)}{2\{1 - E(M)\}} \quad — (5)$$

$\because$ The no. M of arrivals ... ...
poisson process with parameter λ, say,
then $E(M) = \lambda T$ and $Var(M) = \lambda T$, and

$E(M^2) = (\lambda T)^2 + \lambda T$

$\therefore E(M) = E\left(E(M/T)\right) = E(\lambda T) = \lambda E(T)$ — (6)

$$E(M^2) = E\left[E(M^2/T)\right]$$

$$= E\left[\lambda^2 T^2 + \lambda T\right]$$

$$= \lambda^2 \{Var(T) + E^2(T)\} + \lambda E(T) \quad \text{— (7)}$$

Using (7) & (6) in (5)

$$E(N) = L_s = \frac{\lambda^2 \{var(T) + E^2(T)\} + \lambda E(T) - 2\{\lambda E(T)\}^2 + \lambda E(T)}{2\{1 - \lambda E(T)\}}$$

$$= 2\lambda E(T) + \frac{\lambda^2 \{Var(T) + E^2(T)\}}{2\{1 - \lambda E(T)\}}$$

In this M/G/1 model, if G=M, the service time T follows an exponential distribution:

$$E(T) = 1/\mu \quad \& \quad Var(T) = 1/\mu^2$$

$$\therefore L_s = \lambda \cdot 1/\mu + \frac{\lambda^2 \{ 1/\mu^2 + (1/\mu)^2 \}}{2\{1 - \lambda \cdot 1/\mu\}}$$

$$= \lambda/\mu + \frac{\cancel{2}\lambda^2/\mu^{\cancel{2}}}{\cancel{2}\{\frac{\mu-\lambda}{\mu}\}} = \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Using this we get $E(\delta) = 1 - E(M)$ — (3)

Squaring on both sides on

$$N'^2 = N^2 + (M-1)^2 + \delta^2 + 2N(M-1) + 2(M-1)\delta + 2N\delta \quad \text{— (4)}$$

Now $\delta^2 = \delta$ $\quad (\because \delta = 0 \text{ or } 1)$

$$N\delta = \begin{cases} 0 \times 1, & \text{if } N=0 \\ N \times 0, & \text{if } N>0 \end{cases}$$

$$N\delta = 0$$

(4) ⇒ $\therefore\ N'^2 = N^2 + (M-1)^2 + \delta + 2N(M-1) + 2(M-1)\delta$

$= N^2 + M^2 + 2N(M-1) + (2M-1)\delta - 2M + 1$

$2N(1-M) = N^2 - N'^2 + M^2 + (2M-1)\delta - 2M + 1$

$2E(N(1-M)) = E(N^2) - E(N'^2) + E(M^2) + E[(2M-1)\delta] - 2E(M) + 1$

By independence,

$2E(N)\,E(1-M) = E(M^2) + (2E(M)-1)E(\delta) - 2E(M) + 1$

$\therefore\ E(N) = \dfrac{E(M^2) + \{2E(M)-1\}\{1-E(M)\} - 2E(M) + 1}{2\{1-E(M)\}}$

$= \dfrac{E(M^2) - 2E^2(M) + E(M)}{2\{1-E(M)\}}$ — (5)

$$= \frac{\lambda\mu - \lambda^2 + \lambda^2}{\mu(\mu-\lambda)} = \frac{\lambda\mu}{\mu(\mu-\lambda)} = \frac{\lambda}{\mu-\lambda}$$

1. Patients arrive at a clinic according to Poisson dist. at a rate of 30 patients per hr. The waiting room does not accomodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

a) Find the effective arrival rate at the clinic.

b) What is the prob. that an arriving patient will not wait?

c) What is the E(W) until a patient is discharged from the clinic?

$\lambda = 30$ per hour

$\mu = 20$ per hour   Model : $(M \backslash M \backslash 1):(k \backslash FIFO)$

$k = 14 + 1 = 15$

$$P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{k+1}} \quad \{\because \lambda \neq \mu\}$$

$$= \frac{1-30/20}{1-(30/20)^{16}} = \frac{1-1.5}{1-(1.5)^{16}} = \frac{-0.5}{-655.84}$$

$$= 0.00076$$

effective arrival rate $= \mu(1-P_0)$

$$\lambda' = 20(1-0.00076) = 19.9848$$

b) P.(a patient will not wait)

$= P_0 = 0.00076$

c) $E(N) = L_s$

$$= \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-(\lambda/\mu)^{k+1}}$$

$$= \frac{30}{20-30} - \frac{16\,(1.5)^{16}}{1-(1.5)^{16}}$$

$$= -3 - \frac{10509.45}{-655.84}$$

$= -3 + 16.02$

$= 13.02 \simeq 13$ patients

$E(W) = \frac{E(N)}{\lambda'} = \frac{13}{19.98} = 0.65$ h

UNIT-V
NON-MARKOVIAN QUEUES AND QUEUE NETWORKS
PART-A (2 Marks)

1. Write Pollaczek-Khintchine formula and explain the notations.
2. What you mean by M/G/1 queue?
3. In an M/G/1/FCFS with infinite capacity queue, the arrival rate $\lambda = 5$ and the mean service time E(S)=1/8 hour and Var(S) = 0. Compute the mean waiting time W_q in the queue.
4. Define series queues.
5. Define series queues with blocking.
6. Define Jackson networks.
7. Define open Jackson networks.
8. Write the traffic equations in open Jackson networks.
9. Define closed Jackson networks.
10. Write the flow balance equations in closed Jackson networks.

PART-B(16 Marks)

1.a. Derive Pollaczek-Khinchine formula for the average number of customers in the M/G/1 queueing system. **(10)**

b. A one-man barber shop takes exactly 25 minutes to complete one hair-cut. If customers arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer spends in the shop? Also find the average time a customer must wait for service. **(6)**

2.a. Automatic car wash facility operates with only one boy. Cars arrive according to a Poisson process, with mean of 4 cars per hour and may wait in the facility's parking lot if the boy is busy. If the service time for all cars is constant and equal to 10 min, determine
(1) Mean number of customers in the system.
(2) Mean number of customers in the queue.
(3) Mean waiting time in the system.
(4) Mean waiting time in the queue. **(8)**

b. A car wash facility operates with only one boy. Cars arrive according to a Poisson distribution with mean a of 4 cars per hour and may wait in the facility's parking lot if the boy is busy. The parking lot is large enough to accommodate any number of cars. If the time for washing and cleaning a car follows normal distribution with mean 12 minutes and S.D 3 minutes, find the average number of cars waiting in the parking lot. Also find the mean waiting time of cars in the parking lot. **(8)**

3.a. A patient who goes a single doctor clinic for a general check-up has to go through 4 phases.The doctor takes on the average 4 minutes for each for each phase of the check-up and the time taken for each phase is exponentially distributed.If the arrivals of the patients at the clinic are approximately Poisson at the average rate of 3 per hour,what is the average time spent by a patient (i) in the examination? (ii) waiting in the clinic? **(8)**

b. In a car manufacturing plant, a loading crane takes exactly 10 minutes to load a car into wagon and again come back to position to load another car. If the arrival of cars is a Poisson stream at an average of 1 every 20 minutes, calculate the following
(i) The average number of cars in the system
(ii) The average number of cars in the queue
(iii) The average waiting time of a car in the system
(iv) The average waiting time of a car in the queue. **(8)**

4. A car wash facility operates with only one boy. Cars arrive according to a Poisson distribution with mean a of 4 cars per hour and may wait in the facility's parking lot if the boy is busy. The parking lot is large enough to accommodate any number of cars. Find the average number of cars waiting in the parking lot, if the time for washing and cleaning a car follows
(a) Uniform distribution between 8 and 12 minutes.
(b) a normal distribution with mean 12 minutes and S.D 3 minutes.
(c) a discrete distribution with values equal to 4, 8 and 15 minutes and corresponding probabilities 0.2, 0.6 and 0.2. **(16)**

5.a. Prove that in a series queues inter-departure times follows an exponential distribution with parameter λ. **(8)**

b. Derive the steady state probabilities on series queues with blocking. **(8)**

6. The supermarket owner is experimenting with a new store design and has remodeled one of his stores as follows. Instead of the usual checkout-counter design, the store has been remodeled to include a check out "lounge". As customers complete their shopping, they enter the lounge with their carts. If all checkers are busy the customers receive a number. They then park their carts and take a seat. When a checker is free, the next number is called and the customer having that particular number enters the available check out counters. The store has been enlarged so that for practical purposes, there is no limit on either the number of shoppers that can be in the shopping section or the number that can wait in the lounge, even during peak periods. It has been estimated that during peak hours, customers arrive according to a Poissor process at a mean rate of 40/hr and it takes a customer on the average, $\frac{3}{4}$ hr to fill his shopping cart. The filling times are exponentially distributed. Furthermore, the checkout times are also approximately exponentially distributed with a mean of 4 minutes, irrespective of the particular checkout counter(during peak periods each counter has a cashier and bagger, hence the low mean checkout time). The management wishes to know the following:
1. Minimum number of checkout counters required in operation during peak periods.
2. If it is decided to add one more counter than the minimum number of counters required, then what is the average waiting time in the lounge?
3. How many people, on the average will be in the lounge?
4. How many people, on the average will be in the entire supermarket? **(16)**

7. a. There are 2 salesman in a supermarket. Out of the 2 salesman, one is incharge of billing and receiving payment while the other salesman is incharge of weighing and delivering the items. Due to lack of space, only if the billing clerk is free. The customer who has finished his billing job has to wait until the delivery section becomes free. If customers arrive according to a Poisson process at rate 1 per hr and the service times of 2 clerks are independent and have exponential rates of 3 per hr and 2 per hr. Find
(i) The proportion of customers who enter the supermarket.
(ii) The average number of customers in the supermarket.
(iii) The average amount of time a customer spends in the shop. **(8)**

b. There are 2 salesman in a supermarket. Out of the 2 salesman, one is incharge of billing and receiving payment while the other salesman is incharge of weighing and delivering the items. Due to lack of space, only if the billing clerk is free. The customer who has finished his billing job has to wait until the delivery section becomes free. If customers arrive according to a Poisson process at rate of 5 per hr and both the salesman take 6 per hr. Find
(i) The proportion of customers who enter the supermarket.
(ii) The average number of customers in the supermarket.
(iii) The average amount of time a customer spends in the shop. **(8)**

8.a. A TVS company in MADURAI containing a repair section shared by a large number of machines has 2 sequential stations with respective service rates of 3 per hour and 4 per hour. The cumulative failure rate of all the machines is 1 per hour. Assuming that the system behaviour can be approximated by a 2-stage tandem queue, find:
(i) The probability that both the service stations are idle
(ii) The average repair time including the waiting time and
(iii) The bottleneck of the repair facility. **(8)**

b. Define open Jackson network and discuss for the steady state solution. **(8)**

9.a. In a book shop, there are 2 sections, one for engineering books and the other section for mathematics books. There is only one salesman in each section. Customers from outside arrive at the engineering book section at a Poisson rate of 4 per hour and at the mathematics book section at a Poisson rate of 3 per hour.

15. (a) If $X(t) = Y\cos t + Z\sin t$ where Y and Z are independent random variables, each of which assumes the values -1 and 2 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively, prove that $\{X(t)\}$ is a WSS process. (8)

(b) The T.P.M. of a Markov chain with three states 0, 1, 2 is $p=\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$

and the initial state distribution of the chain is $P[X_0=i]=\frac{1}{3},\ i=0,1,2$.

Find (i) $P[X_2=2]$ (ii) $P[X_3=1, X_2=2, X_1=1, X_0=2]$. (8)

Or

16. (a) Define random process. Classify the random process with one example for each. (8)

(b) Suppose that customers arrive at a bank according to a poisson process with mean rate of 3 per minute. Find the probability that during a time interval of two minutes (i) more than 4 customers arrive (ii) atmost 4 customers arrive. (8)

17. (a) A supermarket has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Find (i) the average number of customers in the queue (ii) the average number of customers in the system (iii) the average time a customer spends in the system and in the queue. (8)

(b) Patients arrive at a clinic according to a Poisson distribution at a rate of 30 patients per hour. The waiting time does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(i) Find the effective arrival rate at the clinic.

(ii) What is the expected waiting time until a patient is discharged from the clinic? (8)

Or

Reg. No. :

G 0405

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2010

FOURTH SEMESTER

COMPUTER SCIENCE ENGINEERING

MA1252 PROBABILITY AND QUEUEING THEORY

(Common to IT)

(REGULATION 2008)

Time : Three hours | Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. A continuous random variable X can assume any value between $x=2$ and $x=5$ and has the p.d.f. $f(x)=k(1+x)$. Find $P(X<4)$.

2. State the p.d.f of Weibull distribution.

3. If the joint p.d.f. of (X,Y) is $f(x,y)=\frac{1}{4}, 0\le x, y<2$ find $P(X+Y\le 1)$.

4. X and Y are independent random variables with variances 2 and 3. Find the variance of $3X+4Y$.

5. Define stationary process.

6. What is meant by one-step transition probability?

7. What is meant by steady-state queueing system?

8. Write down the Little's formula for an $(M/M/1):(\infty/FIFO)$ queueing model.

9. When the service is constant, give the reduced form of P-K formula?

10. Define a two-stage series queue.

PART B — (5 × 16 = 80 marks)

11. (a) If the density function of a continuous R.V. X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find the value of a

(ii) Find the c.d.f. of X and

(iii) $P(X > 1.5)$ (8)

(b) Obtain the MGF of Geometric distribution. Hence find the mean and variance of the distribution. (8)

Or

12. (a) It is known that the probability of an item produced by a certain machine will be defective is 5%. If the produced items are sent to the market in packets of 20, find the number of packets containing

(i) Atleast two defective items

(ii) Atmost two defective items in a consignment of 1000 packets using Poisson distribution. (8)

(b) Define Gamma distribution. Find its mean and variance. (8)

13. (a) The joint p.d.f. of the random variables X and Y is given by $f(x, y) = xe^{-(y+1)x}$ where $0 \le x,\ y < \infty$.

(i) Find the marginal density functions of X and Y.

(ii) Are X and Y independent?

(iii) Find the regression curve of Y on X. (8)

(b) If X and Y are independent random variables with joint p.d.f. $f(x, y) = 3e^{-(x+3y)}$, $x \ge 0$, $y \ge 0$, find the p.d.f. of $Z = \dfrac{X}{Y}$. (8)

Or

14. (a) If the joint p.d.f. of (X, Y) is given by $f(x, y) = x + y$, $0 \le x$, $y \le 1$, find the correlation coefficient between X and Y. (8)

(b) If $X_1, X_2, \ldots\ldots X_n$ are Poisson variables with parameter $\lambda = 2$, use the central limit theorem to find $P(120 \le S_n \le 160)$ where $S_n = X_1 + X_2 + \ldots. X_n$ and $n = 75$. (8)

15. (a) If $X(t) = Y \cos t + Z \sin t$ where Y and Z are independent random variables, each of which assumes the values -1 and 2 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively, prove that $\{X(t)\}$ is a WSS process. (8)

(b) The T.P.M. of a Markov chain with three states 0, 1, 2 is $p=\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$

and the initial state distribution of the chain is $P[X_0=i]=\frac{1}{3},\ i=0,1,2$.

Find (i) $P[X_2=2]$ (ii) $P[X_3=1, X_2=2, X_1=1, X_0=2]$. (8)

Or

16. (a) Define random process. Classify the random process with one example for each. (8)

(b) Suppose that customers arrive at a bank according to a poisson process with mean rate of 3 per minute. Find the probability that during a time interval of two minutes (i) more than 4 customers arrive (ii) atmost 4 customers arrive. (8)

17. (a) A supermarket has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Find (i) the average number of customers in the queue (ii) the average number of customers in the system (iii) the average time a customer spends in the system and in the queue. (8)

(b) Patients arrive at a clinic according to a Poisson distribution at a rate of 30 patients per hour. The waiting time does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(i) Find the effective arrival rate at the clinic.

(ii) What is the expected waiting time until a patient is discharged from the clinic? (8)

Or

18. A petrol pump station has 4 pumps. The service time follow the exponential distribution with a mean of 6 min and cars arrive for service in a poisson process at the rate of 30 cars per hour.

(a) What is the probability that an arrival would have to wait in line? (b) Find

the average waiting time.

(c) Find the averag~ time spent in the system.

(d) Find the average number of cars in the system and in the queue.

(e) For what percentage of time would a pump be idle on an average? (16)

19. A car wash facility operates with only one bay. Cars arrive according to a poisson distribution with a mean of 4 cars per hour and may wait in the faCtory's parking lot if the bay is busy. The parking lot is large enough to accommodate any number of cars. Find the average number of cars waiting in the parking lot and the average hme a car spends in the facility if (a) the time for washing and cleaning a car follows uniform distribution between 8 and 12 minutes (b) the service time T is 10 minutes for a car. (16)

20. (a) In an ophthalmic clinic, there are sections-one section for assessing the power approximately and the other for final assessment and prescription of glasses. Patients arrive at the clinic in a poison fashion at the rate of 3 per hour. The assistant in the first section takes nearly 15 minutes per patient for approximate assessment of power and the doctor in the second section takes nearly 6 minutes per patient for final prescription. If the service times in the two sections are approximately exponential, find the probability that there are 3 patients in the first section and 2 patients in the second section. Find also the average number of patients in the clinic and the average waiting time of a patient in the clinic. Assume that enough space is available for the patients to wait in front of both sections.

(8)

(b) In a network of three service stations 81, 82, 83 customers arrive at 81, 82, 83 from outside in accordance with poisson process having rates 5, 10, 15 respectively. The service times at the 3 stations are exponential with respective rates 10, 50, 100. A customer completing service 81 is equally likely to (i) go to 82 (ii) go to 83 or (iii) leave the system. A customer departing from service at 82 always goes to 83. A departure from service at 83 is equally likely to go to 82 or leave the system.

(1) What is the average number of customers in the system, consisting of all the three stations?

(2) What is the average time a customer spends in the system? (8)

(G **0405)**