Queueing Theory:

Introduction:

Queuing theory is the mathematical study of waiting lines or queues. Queuing theory had its origin in 1909 when Agner Krarup Erlang (1878 – 1929), a Danish engineer who worked for the Copenhagen Telephone Exchange published the first paper (Erlang 1909).

Fundamental goal of Queueing theory is to desire an analytical or mathematical model of customers needing service and use the model to predict queue lengths and waiting time. A queue or waiting line is formed when units (customers or clients) requiring some kind of service arrive at a service counter or service chanel. A simple queueing model is:

Arrivials → $\boxed{\text{Queue}}$ → $\boxed{\text{Service}}$ → Departures

Basic characteristics of Queueing Model:

The basic characteristics of queueing model can be caterized into following 4 headings.

1. Arrivial Pattern or Input:

The input describes the manner in which customers (or units) arrive and join the Queueing system. We express the arrivial pattern of customers by the probability distribution of the number of arrivials per unit of time or inter-arrivial time.

We consider only those Queueing system

in which the number of arrivals per unit of time is a poisson random variable with mean λ. The number of customer may be finite or infinite sources. The input process should indicate the number of queues that are allowed to form, the maximum queue length, the maximum number of customers requiring service etc.

2. Service Mechanism:

In this system, we consider only those queuing syst in which the no. of customers serviced per unit of time has a poisson distribution with mean μ, or inter-service time has an exponential distribution with mean $1/\mu$.

3. Queue Descipline:

The various type of Queue descipline are:

i) FIFO or FCFS: first in first out or first come first served (It is most commonly used in servicing customers)

ii) LIFO or LCFS: Last in first out or Last come first served. (Commonly used in inventory systems)

iii) SIRO: Selection for service in random order.

iv) PIR: Priority in selection. (used in manual transmission messaging systems).

4. System capacity:

Number of customers in a Queueing system may be finite or infinite. In some models only limited customers or units are allowed in the system, when limiting value reached, no further customers are allowed to enter the Queueing system.

Transient and Steady States:

A queueing system is said to be transient state when the operating characteristics of the system depend on time, if the system is independent of time than it is called steady state.

Kendall's Notation of Queuing System:

Kendall 1951 provides very useful notation of queuing system. The notation has the form:

$$(a/b/c):(d/e)$$

where, a = inter arrival distribution
b = Service time distribution
c = No. of channels or servers
d = System capacity
e = Queue discipline

In Kendall's notation a and b take one of following symbols.
M: for Markovian or exponential distribution
G: for arbitrary or general distribution
D: for fixed or deterministic distribution.

Queue length:

For a queue, the line length or queue size is defined as the number of customers in the queuing system.

Queue length = Line length − No. of customers being served.

Notation / Terminology

n = No. of customers (units) in the system

$N(t)$ = No. of customers (units) in the system at time t.

$P_n(t)$ = The probability that there are exactly n customers at time t, i.e $P[N(t) = n]$

P_n = The Steady State probability, that exactly n customers in the system.

λ_n = Mean arrival rate, when there are n customers in the sy

μ_n = Mean service rate, when there are n customers in the system

λ = Mean arrival rate, when λ_n is constant for all n.

μ = Mean Service rate, when μ_n is constant for all n.

$\rho = \frac{\lambda}{\mu}$ = Traffic intensity or utilization factor.

$f_s(w)$ = The prob'y density function of waiting time in the system

$f_q(w)$ = The prob'y density function of waiting time in Queue.

L_s = the expected no of customers in the system or average line leng

L_q = The expected no. of customers in the queue or average queue leng

L_w = The expected no. of customers in the non empty queues.

W_s = The expected ~~no of customer~~ waiting time of a custom in the System.

W_q = The expected waiting time of a customer in the Queue

Transient ... Probabilities for Poisson Queue System

we desire the differential equations for transient state probabilities for the Poisson Queue system, which are known as pure birth process or immegration process.

Let $N(t)$ be the number of customers (or units) in the system at time t, and $P_n(t)$ be the probability that there are n customers in the system at time t, where $n \geq 1$, i.e

$$P_n(t) = P[N(t) = n]$$

First, we derive the differential equation satisfied $P_n(t)$ and then derive differential equation satisfied by P_n.

A Poisson queue model with mean arrival rate λ_n and mean service rate μ_n, we can make following assumptions.

when $N(t) = n$, the prob'y of arrival in $(t, t+\Delta t)$ is $\lambda_n \Delta t + O(\Delta t)$

when $N(t) = n$, the prob'y of departure is $(t, t+\Delta t)$ is $\mu_n \Delta t + O(\Delta t)$

when $N(t) = n$, the prob'y of more than one arrival and or more than one departure is $(t, t+\Delta t)$ is $O(\Delta t)$

Next we find differential equation satisfied by $P_n(t)$. For it we first find a formula for $P_n(t+\Delta t)$, i.e $P[N(t+\Delta t) = n]$.

The event $\{N(t+\Delta t) = n\}$ can happen in number of mutually exclusive ways. By assumption (iii) the event involving more than one arrival and/or more than one departure in $(t, t+\Delta t)$ is $O(\Delta t)$. There will remain four mutually exclusive events, which are described as:

A_{00}: $N(t) = n$ and no arrival or departure in $(t, t+\Delta t)$

A_{10} : $N(t) = n-1$ and one arrival and no departure in $(t, t+\Delta t)$

A_{01} : $N(t) = n+1$ and no arrival and one departure in $(t, t+\Delta t)$

A_{11}: $N(t) = n$ and One arrival and one departure in $(t, t+\Delta t)$

Then,

$$P(A_{00}) = P_n(t)\{1-(\lambda_n+\mu_n)\Delta t + O(\Delta t)\}$$

$$P(A_{10}) = P_{n-1}(t)\ \lambda_{n-1}\ \Delta t + O(\Delta t)$$

$$P(A_{01}) = P_{n+1}(t)\ \mu_{n+1}\ \Delta t + O(\Delta t)$$

$$P(A_{11}) = P_n(t)\{(\lambda_n \Delta t + O\Delta t)(\mu_n \Delta t + O(\Delta t)\}$$
$$= O(\Delta t).$$

Thus for $n \geq 1$, we have,

$$P_n(t+\Delta t) = P(A_{00}) + P(A_{10}) + P(A_{01}) + P(A_{11})$$
$$= P_n(t)[1-(\lambda_n+\mu_n)\Delta t] + \lambda_{n-1} P_{n-1}(t)\Delta t + \mu_{n+1} P_{n+1}(t)\Delta t + O\Delta t$$

Thus,

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda_n+\mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) + \frac{O(\Delta t)}{\Delta t}$$

When, $\Delta t \to 0$,

$$P_n'(t) = -(\lambda_n+\mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}$$
for $n \geq 1$, (I)

For $n=0$,

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad -(II)$$

The solution of eqn (I) & (II) yields the transient state proby $P_n(t)$ for $n \geq 0$.

Steady state prob'y for Poisson Queue System

The transient state probabilities for Poisson Queue system are given by the equation

$$P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \quad \text{--- (I)}$$

and,

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad \text{--- (II)}$$

The equation of steady state prob'y (P_n) can be obtained by putting $P_n'(t) = 0$ and replacing $P_n(t)$ by P_n in eq^n (I) & (II).

We can get,

$$0 = -(\lambda_n + \mu_n) P_n + \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} \quad \text{--- (III)}$$

and,

$$0 = -\lambda_0 P_0 + \mu_1 P_1 \quad \text{--- (IV)}$$

Equation (III) & (IV) are called balance or equilibrium equations of Poisson Queue system.

From (IV),

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \quad \text{--- (V)}$$

Putting $n = 1$ in (III) & using (V), we have,

$$\mu_2 P_2 = (\lambda_1 + \mu_1) P_1 - \lambda_0 P_0$$

$$= (\lambda_1 + \mu_1) \cdot \frac{\lambda_0}{\mu_1} P_0 - \lambda_0 P_0$$

$$\therefore P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 \quad \text{--- (VI)}$$

Next putting $n = 2$ in (III) & using (VI) we get

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$$

Iterating Similarly we get

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 \quad \text{--- (7)}$$

Since no. of customers in the system be 0, 1, 2 -- ∞, then

$$P_0 + P_1 + \cdots + P_n + \cdots = P_0 + \sum_{i=1}^{\infty} P_n = 1.$$

Thus,

$$P_0 + \sum_{i=1}^{\infty} \left[\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right] P_0 = 1.$$

$$\therefore \quad P_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \left[\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right]} \quad \text{--- (8)}$$

Eqn (7) & (8) play an important role characteristics of Poisson Queue model.

Model 1:

1. M/M/1 : ∞ FCFS Model

M/M/1 : ∞ FCFS model means arrival and departure distribution is Marcollian distribution with single server with infinite customer and service system may be first come first serve i.e FCFS or FIFO.

For this model we have the following assumptions:

- The mean arrival rate is constant i.e $\lambda_n = \lambda$ for all n.
- The mean service rate is constant $\mu_n = \mu$, for all n.
- The mean arrival rate is less than mean service rate

i.e $\lambda < \mu$

or $\rho = \frac{\lambda}{\mu} < 1$

i.e The traffic intensity $\rho < 1$.

The Required formulae for this model are;

1) Server utilisation or traffic intensity $\rho = \frac{\lambda}{\mu}$

2) Proby of n customer in the system

$$P_n = \rho^n (1-\rho) \quad (P_0 \text{ no customer in system})$$

3) Average or expected no. of customer in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

(4) Average or expected no. of customer in the queue i.e average length of queue

$$L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(5) Average or expected no. of customer in nonempty queue (L_w)

$$L_w = \frac{1}{1-\rho} = \frac{\mu}{\mu-\lambda}$$

(6) Expected waiting time of the customer in the system

$$W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda}$$

(7) Average waiting time of the customer in the ~~system~~ Queue

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

8) The probability that the waiting time of a customer in the queue exceeds t is

$$P(W_s > t) = P(T_q > t) = e^{-(\mu-\lambda)t}$$

<u>Example 1:</u>

Customer arrive at a one man barbor shop according to a Poisson process with a mean inter-arrival time of 20 minutes. Customer spend an average of 15 minute in the barber chair. The service time is exponentially distributed. If an hour is used as a unit of time, then i) what is the prob'y that a customer need not wait for hair cut? ii) what is the expected number of customer in the barber shop & in the queue? iii) How much time can a customer expect to spend in the shop? iv) Find the average time that a customer spends in the queue?

Sol^{n}:

$\lambda = 20\ min = \frac{1}{20}$ Per min.

$\mu = 15\ min = \frac{1}{15}$ Per min

$\rho = \frac{\lambda}{\mu} = \frac{1/20}{1/15} = \frac{15}{20} = \frac{3}{4}$ min

i) $P_0 = 1 - \rho = 1 - \frac{3}{4} = 1/4$

ii) $L_s = \frac{\rho}{1-\rho} = \frac{3/4}{1-3/4} = 3$ customers.

$L_q = \frac{\rho^2}{1-\rho} = \frac{(3/4)^2}{1-3/4} = \frac{9/16}{1/4} = \frac{9}{4} = 2.25$

iii) $W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{\frac{1}{15}\left(1-\frac{3}{4}\right)} = 60$ minutes

iv) $W_q = \frac{\rho}{\mu(1-\rho)} = \frac{3/4}{1/15\,(1/4)} = 45$ minutes.

Ex 2: Customers arrive at the express checkout lane in a Supermarket is a Poisson process with a rate of 15 Per hour. The ~~Customer~~ time to check out a Customer is an exponential random variable with mean of 12 minutes, find the average no. of customer present, what is the expected waiting time for a Customer in the system?

Sol^{n}:

$\lambda = 15$ per hour $= \frac{15}{60}$ per min $= \frac{1}{4}$ /min

$\mu = 2$ min $= 1/2$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/4}{1/2} = 1/2$$

i) $L_s = \dfrac{\rho}{1-\rho} = \dfrac{\frac{1}{2}}{1-1/2} = 1$ customer

$L_q = \dfrac{\rho^2}{1-\rho} = \dfrac{(1/2)^2}{1-1/2} = \dfrac{2}{4} = 1/2$ customers in the queue

iii) $W_s = \dfrac{1}{\mu(1-\rho)} = \dfrac{1}{\frac{1}{2}(1-1/2)} = 4$ minute waiting time

__Ex 3__ A supermarket has a single cashier. During peak hours, customers arrives at a rate of 20 per hour. The average no. of customers that can be processed by the cashier is 24 per hour. Calcute i) The cashier is idle ii) the average no. of customer in the Queueing system iii) The average time a customer spends in the system iv) The average no. of customer in the queue v) The average time a customer spends in the queue waiting for the service.

__Sol^n:__

$\lambda = 20$ per hour, $\mu = 24$ per hour, $\rho = \frac{\lambda}{\mu} = \frac{20}{24}$

i) The customer is idle (i.e no. customer in the system)

$$P^n = \rho^n(1-\rho)$$

$$P_0 = 1-\rho = 1-\frac{\lambda}{\mu} = 1-\frac{20}{24} = \frac{1}{6}$$

ii) The average no. of customer in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{20/24}{1-20/24} = \frac{20}{24-20} = 5$$

iii) The average time a customer spends in system

$$W_s = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda} = \frac{1}{24-20} = \frac{1}{4} \text{ hour}$$

$= 15$ minutes

iv) $Lq = \frac{\rho^2}{1-\rho} = \frac{(20/24)^2}{1-\frac{20}{24}} = \frac{400}{24 \times 4} = 4.166$

v) The average time a customer spends in a queue

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{24(24-20)}$$

$$= \frac{5}{24} \text{ hour} = 12.5 \text{ minutes}$$

*4: A xerox machine is maintained in an office and operated by a secretary who does other jobs also. The service rate is Poisson process with a mean service rate of 10 jobs per hour. Generally the requirements ~~for~~ for use are random over the entire 8-hours, working day, but arrive at a rate of 5 jobs per hour. Several people are noted that waiting line develops occasionally and have questioned the office policy of maintaining only one Xerox machine. If the time of secretary is valued at Rs 10 per hour, find i) Utilization of xerox machine ii) The proby that an arrival has ~~to sit~~ to wait iii) The mean no. of jobs of the system iv) The average cost per day due to waiting and operating the machine.

Soln:

Mean arrival rate $\lambda = 5$ jobs/hour

Mean service rate $\mu = 10$ jobs/hour

i) The utilization of xerox machine

$$\rho = \frac{\lambda}{\mu} = \frac{5}{10} = 0.5$$

Thus machine is uses 50% of the time.

ii) The Prob'y that an arrival has to wait

$P(N \geq 1) = \rho = 0.5$

iii) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{5}{10-5} = 1$

iv) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{10-5} = \frac{1}{5}$ hour or 12 minute.

v) Average cost perday = Average cost per job × no. of job processed perday

$= 8 \times 10 = 80$ RS.

Ex: 5 The arrival rates of telephone calls at telephone booth are according to Poisson process with average time of 12 minutes between arrival of two Consecutive calls. The le of telephone call is assumed to be exponentially distribu with mean 4 minutes. i) Determine the prob'y that the p son arriving at the booth will have to wait ii) Find the average queue length that formed from time to time iii) The telephone Company will install the second booth when Convinced that an arrival would expect to wait at leas 5 minutes for the Phone. Find the increase in flows of arrivals which will justify the 2nd booth. iv) What is the prob'y that an arrival will have to wait for more tha 15 minutes before the phone is free?

Sol^n:

Mean arrival time $= 1/\lambda = 12$ minutes

So mean arrival rate $= (\lambda) = 1/12$ per minute

$\mu = 1/4$ per minute

Traffic intensity $\rho = \dfrac{\lambda}{\mu} = \dfrac{\;}{\;} = \dfrac{1}{2}$

(ii) ... person arriving at booth will have to wait $P(n>0) = \rho = \dfrac{1}{2}$ ...

(iii) $L_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{\;}{\;(\;-\;)} = \dfrac{1}{2}$

(iv) The telephone company will install a second booth when $L_q > 2$ i.e.

$L_q = \dfrac{\lambda'^2}{\mu(\mu-\lambda')} > 2$

where λ' = the required arrival rate

or $\dfrac{\lambda'^2}{\frac{1}{4}\left(\frac{1}{4}-\lambda'\right)} > 2$

or $\lambda'^2 > \frac{1}{2}\left(\frac{1}{4}-\lambda'\right)$

or $\left(\lambda' + \frac{1}{4}\right)^2 > \frac{3}{16}$ or $\lambda' > \dfrac{\;}{\;}$

$\therefore \lambda' > \dfrac{1}{\;}$

(v) The average rate ... $\dfrac{\;}{\;} - \dfrac{1}{\;}$ ...

by putting the ...

$P(W > 10) = e^{-(\mu-\lambda)t}$

$= e^{\;}$

$= 0.0354$

Exercise:

1) In a public telephone booth having just one phone, the arrivals are considered to be Poisson with the average of 15 per hour. The length of call is assumed to be exponentially distributed with mean 3 minutes. i) Find the average no. of customers waiting in the system ii) Average no. of customer waiting in the queue iii) Proby that a person arriving at the booth will have to wait in the queue iv) Expected waiting time of a customer in the system and in the queue v) The percentage (i.e proby) of time that the telephone booth will be idle.

2) In a city airport, flights arrive at a rate of 24 flights per day. It is known that the inter-arrival time follows an exponential distn & the service time distn is also exponential with an average of 30 minutes. Find i) The proby that the system is idle ii) The mean queue size iii) The average no. of flights in the queue iv) The proby that the queue size exceeds 7.

Hint: $\lambda = \frac{24}{24} = 1$ flight/hour

$\mu = \frac{60}{30} = 2$ flight/hour

Model II:

M/M/1 : (N/FCFS) Model:

It means arrival and departure distribution is Markovian distribution with single server having first come first served service technique. N means finite customer capacity.

Then,

$$\text{Probability } P_0 = \frac{1-\rho}{1-\rho^{N+1}},$$

$$P_n = \frac{1-\rho}{1-\rho^{N+1}} \cdot \rho^n \quad \text{for } n = 0, 1, 2 \ldots N.$$

The measures of Model are:

i) $$L_s = \sum_{n=0}^{N} n P_n = \sum_{n=0}^{N} n \frac{(1-\rho)}{(1-\rho^{N+1})} \cdot \rho^n$$

$$= \frac{(1-\rho)}{(1-\rho^{N+1})} \sum_{n=0}^{N} n \cdot \rho^n = P_0 \sum_{n=0}^{N} n\rho^n$$

ii) $$L_q = L_s - \frac{\lambda}{\mu}$$

iii) $$W_s = L_s/\lambda$$

iv) $$W_q = W_s - 1/\mu = \frac{L_s}{\lambda} - \frac{1}{\mu}$$

$$= \frac{1}{\lambda}\left(L_s - \frac{\lambda}{\mu}\right) = L_q/\lambda$$

Ex: 1 Trains are arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity is limited to 4 trains find i) the prob.y that the yard is empty ii) The average no. of trains in the system

Sol^n: Arrival rate $(\lambda) = \frac{1}{15}$ trains/min

Service rate $(\mu) = \frac{1}{33}$ train/minute.

$N = 4.$

$\therefore$ Traffic intensity $\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/33} = 2.2$

i) $P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-2.2}{1-2.2^{4+1}} = \frac{-1.2}{-50.53} = 0.0237$

ii) $L_s = P_0 \sum_{n=0}^{N=4} n\rho^n$

$= P_0[0.\rho^0 + 1.\rho^1 + 2.\rho^2 + 3.\rho^3 + 4.\rho^4]$

$= 0.0237[2.2 + 2(2.2)^2 + 3(2.2)^3 + 4(2.2)^4]$

$= 3.259$

$= 3.26$

Ex: 2

Patients arrive at a clinic according to Poisson distribution, at the rate of 30 patients per hour. The accomodate capacity does not more than 14 patients. Examination time per patient is exponential with a mean rate of 20 per hour. i) Find the effective arrival rate at the clinic ii) What is the prob'y that an arriving patients will not wait? iii) What is the expected waiting time until the patient is discharged from the clinic?

Sol^n:

$\lambda = 30$ per hour $= \frac{30}{60}$ per minute $= \frac{1}{2}$ /min

$\mu = 20$ per hour $= \frac{20}{60} = \frac{1}{3}$ per minute

$k = 14 + 1 = 15$ (one patient is in service so $k = 14+1$)

$\rho = \frac{\lambda}{\mu} = \frac{1/2}{1/3} = 1.5$

$$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{1-1.5}{1-(1.5)^{16}} = 0.00076 \simeq 0.001.$$

i) $\lambda' = \mu(1-P_0) = \frac{1}{3}(1-0.001)$

$= 0.333$ per min.

ii) P(will not wait for Service)

$= P_0 = 0.001.$

iii) $W_s = \frac{L_s}{\lambda'}$

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$$

$$= \frac{1.5}{1-1.5} - \frac{16(1.5)^{16}}{1-(1.5)^{16}}$$

$= -3 + 16.02$

$= 13.02$

$\therefore W_s = \frac{13.02}{0.333} = 39$ minutes

Ex:3 The arrival of customers at a one window drive in a bank follows a Poisson distribution with mean 18 per hour. Service time per customer is exponential with mean ~~10 minutes~~ 6 per hour. The space infront of the window including that for serviced car can accommodate a maximum of 3 cars (including the car being served) and others can wait outside this space. Find i) Proby that an arriving customer will not wait ii)

Effective arrival rate iii) Expected no. of customer in the drive in bank. iv) Expected no. of customer in the queue. (v) Expected time a customer spends in the drive in bank vi) Expected waiting time of a customer in the queue.

Sol:

$\lambda = 18/hr, \ \mu = 6/hr \ \& \ K = 3.$ [K = N]

$\rho = \frac{\lambda}{\mu} = \frac{18}{6} = 3.$

i) The prob that arriving customer will not wait

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{\rho-1}{\rho^4-1} = \frac{2}{80} = 0.025$$

ii) The effective arrival rate

$$\lambda' = \mu(1-P_0) = 6 \times \left(1-\frac{1}{40}\right)$$

$$= 6 \times \frac{39}{40} = 5.85$$

iii) expected no. of customer in the drive in the bank

$$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \frac{\rho}{1-\rho} - \frac{4\rho^4}{1-\rho^4}$$

or $L_s = \frac{3}{1-3} - \frac{4 \times 81}{(-80)} = -1.5 + 4.05 = 2.55$

iv) Expected no. of customers in the queue is

$$L_q = L_s - \frac{\lambda'}{\mu} = 2.55 - \frac{5.85}{6} = 1.575$$

v) Expected time time a customer spends in the drive in bank

$$W_s = \frac{1}{\lambda'} L_s = \frac{1}{5.85} \times 2.55 = 0.4359 \ hr = 26.15 \ minute$$

vi) Expected waiting time of a customer in the queue is

$$W_q = \frac{1}{\lambda'} L_q = \frac{1}{5.85} \times 1.575 = 0.2692 \ hr$$

$= 16.1538$ minutes

(M/M/1) : (K/FCFS) Model:

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}}, \quad \rho = \frac{\lambda}{\mu}$$

Average expected no. of customer in the system

$$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} \quad \text{if } \rho \neq 1$$

Average expected no. of customer in the queue

$$L_q = L_s - \frac{\lambda}{\mu} \quad \text{--- (1)}$$

Effective arrival rate

$$\lambda' = \mu(1-P_0)$$

If $1 - P_0 \neq \frac{\lambda}{\mu}$, because the mean arrival rate is λ as long as there is vacancy in the queue and is zero when the system is full. This motivates us to define effective arrival rate λ'.

$$\therefore \lambda' = \mu(1-P_0)$$

$\therefore$ (1) becomes $L_q = L_s - \frac{\lambda'}{\mu}$

Average waiting time of a customer in the system

$$W_s = \frac{1}{\lambda'} L_s$$

Average waiting time of a customer in a queue

$$W_q = \frac{1}{\lambda'} L_q$$

M/M/S: (∞/FCFS) Model:

(Multiple server with infinite capacity)

Formula:

1) Server utilization $\rho = \frac{\lambda}{\mu s}$

2) $$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right]^{-1}$$

or

$$= \frac{1}{\left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{n}\right)^n + \frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s \right]}$$

3) Average or Expected no. of customers in the queue length

$$L_q = \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$$

or

$$L_q = \frac{1}{s \cdot s!} \frac{(s\rho)^{s+1}}{(1-\rho)^2} P_0$$

4) Average or expected no. of customer in the system:

$$L_s = L_q + \frac{\lambda}{\mu}$$

or

$$L_s = L_q + s\rho$$

5) Average waiting time of the Customer in the System (W_s)

$$W_s = \frac{L_s}{\lambda} \Rightarrow W_s = \frac{L_s}{\lambda}$$

6) Average waiting time of a Customer in a queue

$$W_q = \frac{L_q}{\lambda}$$

-> The probability that an arrival has to wait for Service:

$$P(N \geq s) = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$$

or

$$P(N \geq s) = \frac{(s\rho)^s}{s!(1-\rho)} P_0$$

8) The probability that an arrival enters the service without waiting

$= 1 - P(\text{an arrival has to wait})$

$= 1 - P(N \geq s)$

$$= 1 - \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$$

9) The mean waiting time in the queue for those who need to wait

$$E(T_q / T_s > 0) = \frac{1}{\mu s - \lambda}$$

10) The average or expected no. of customers in non empty queues: (L_n)

$$L_n = \left(\frac{\lambda}{\mu s}\right) \Big/ \left(1 - \frac{\lambda}{\mu s}\right)$$

11) The Probability that there will be some one waiting

$$P(N \geq s+1) = \frac{\left(\frac{\lambda}{\mu}\right)^s \left(\frac{\lambda}{\mu s}\right) P_0}{s! \left(1 - \frac{\lambda}{\mu s}\right)}$$

Example 1: There are 3 typist in an office, each typist can type 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour. i) What fraction of time all the typist will be busy ii) what is the average no. of letters waiting to be typed? iii) What is the average time a letter has to spend for waiting for being typed.

Soln:

$$\lambda = 15 \text{ Per hr} = \frac{15}{60} = \frac{1}{4} \text{ Per minute}$$

$$\mu = 6 \text{ Per hr}$$

$$= \frac{6}{60} = \frac{1}{10} \text{ Per minute}$$

$$s = 3.$$

$$\rho = \frac{\lambda}{\mu s} = \frac{\frac{1}{4}}{\left(\frac{1}{10}\right)3} = \frac{1}{4} \times \frac{10}{3} = \frac{5}{6} = 0.83$$

$$s\rho = 3 \times \frac{5}{6} = 2.5$$

$$P_0 = \left\{ \sum_{n=0}^{2} \frac{(2.5)^n}{n!} + \frac{(2.5)^3}{3!(1-0.83)} \right\}^{-1}$$

$$\therefore P_0 = \left\{1 + \frac{2.5}{1!} + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!(0.17)}\right\}^{-1}$$

$$= \left\{1 + 2.5 + 3.125 + 15.319\right\}^{-1}$$

$$= (21.94)^{-1}$$

$$\therefore P_0 = 0.045$$

i) $P(N \geq s) = \frac{(s\rho)^s}{s!(1-\rho)} P_0$

$$= \frac{(2.5)^3}{3!(1-0.83)} \times 0.045$$

$$= 0.69 \text{ (Typist busy proby)}$$

ii) $L_q = \frac{1}{s.s!} \cdot \frac{(s\rho)^{s+1}}{(1-\rho)^2} P_0$

$$= \frac{(2.5)^4}{3.3!(1-0.83)^2} \times 0.45$$

$$= \frac{1}{18} \times 60.824 = 3.38$$

iii) $W_s = \frac{L_s}{\lambda}$

$L_s = L_q + s\rho = 3.38 + 2.5 = 5.88$

$\therefore W_s = \frac{5.88}{1/4} = 4 \times 5.88 = 23.52$ minutes

Ex:2 A Supermarket has 2 girls running up sales at the customer. If the service time for each customer is exponential with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 per hour, find

i) What is the probability of having to wait for service?
ii) What is the expected percentage of idle time for each girl?
iii) What is the expected length of customer's waiting?

Soln:

$$\lambda = 10 \text{ per hr} = \frac{10}{60} = 1/6 \text{ Per minute}$$

$$\mu = 4 \text{ minute} = 1/4 \text{ Per minutes}$$

$$s = 2.$$

$$\rho = \frac{\lambda}{\mu s} = \frac{\frac{1}{6}}{\frac{1}{4} \times 2} = \frac{2}{6} = 0.33.$$

$$s\rho = 2 \times \frac{1}{3} = 0.67$$

$$P_0 = \left\{ \sum_{n=0}^{1} \frac{(0.67)^n}{n!} + \frac{(0.67)^2}{2!(1-0.33)} \right\}^{-1}$$

$$= \left\{ 1 + \frac{0.67}{1!} + \frac{(0.67)^2}{2!(1-0.33)} \right\}^{-1}$$

$$= \left\{ 1 + 0.67 + 0.335 \right\}^{-1} = 0.5$$

$$\therefore P_0 = 0.5$$

i) $$P(N \geq s) = \frac{(s\rho)^s P_0}{s!(1-\rho)} = \frac{(0.67)^2}{2!(1-0.33)} \times 0.5$$

$$= 0.168$$

The fraction of time when girls are idle $= 1-\rho$

$= 1-0.33 = 0.67$

Expected % value $= 67\%$ idle

$$W_q = \frac{L_q}{\lambda}$$

$$L_q = \frac{1}{s \cdot s!} \frac{(s\rho)^{s+1}}{(1-\rho)^2} P_0$$

$$= \frac{1}{2 \cdot 2!} \frac{(0.67)^3}{(1-0.33)^2} \times 0.5$$

$L_q = 0.084$

$$W_q = \frac{0.084}{1/6} = 0.504 \text{ minute}$$

A telephone exchange has two long distance operators. It has been found that long telephone calls arrive according to Poisson distribution at an average rate of 15 per hour. The length of service on these calls has been shown to be exponentially distributed with mean length of 2 minutes, find the
i) probability that a customer will have to wait for his long distance call, ii) Expected no. of Customer in the system iii) Expected no. of customer in the queue iv) Expected time a customer spends in the system v) Expected waiting time for a customer in the queue.

Sol^n: $\lambda = 15/hr = \frac{15}{60} = \frac{1}{4}$ Per minute

$\mu = \frac{1}{2}$ Per minute ; $S = 2$

$$\rho = \frac{\lambda}{\mu S} = \frac{\frac{1}{4}}{\frac{1}{2} \times 2} = \frac{1}{4} = 0.25$$

$$P_0 = \left\{ \sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)} \right\}^{-1}$$

$$= \left\{ \frac{(0.25)^0}{0!} + \frac{(0.25)^1}{1!} + \frac{(0.25)^2}{2!(1-0.25)} \right\}^{-1}$$

$$= \left\{ 1 + 0.25 + \frac{0.0625}{2 \times 0.75} \right\}^{-1}$$

$$\therefore \quad P_0 = \frac{3}{5} = 0.6$$

i) Prob^y that a Customer will have to wait

$$P(N \geq S) = \frac{(S\rho)^S}{S!(1-\rho)} P_0$$

$$= \frac{(2 \times 0.25)^2}{2!(1-0.25)} \times 0.6 = \frac{1}{10} = 0.1$$

ii) The expected no. of customer in the system is

$L_s = L_q + S\rho$

$$L_q = \frac{1}{S.S!} \frac{(S\rho)^{S+1}}{(1-\rho)^2} P_0$$

$$= \frac{1}{2.2!} \frac{(2 \times 0.25)^3}{(1-0.25)^2} \times 0.6$$

$$= \frac{0.125 \times 0.6}{4 \times 0.5625} = \frac{0.075}{2.25} = 0.0334$$

$L_s = L_q + S\rho$

$= 0.0334 + 2 \times 0.25 = 0.5334$

iii) The expected no. of customer in the queue

$L_q = 0.0334$

iv) The expected time a customer spends in the system

$$W_s = \frac{L_s}{\lambda}$$

$$= \frac{0.5334}{1/4} = 0.5334 \times 4$$

$$= 2.1336 \text{ minutes}$$

v) The expected waiting time for customer in a queue.

$$W_q = \frac{L_q}{\lambda} = \frac{0.0334}{1/4} = 0.0334 \times 4$$

$$= 0.1336 \text{ minutes}$$

2) A Petrol pump station has 4 pumps, the service time follows an exponential distribution with a mean of 6 minutes & car arrive for service in a Poisson process at the rate of 30 cars per hour.

i) What is the prob'y that an arrival will have to wait in the line? ii) Find the average waiting time in the queue, average time spent in the system and average no. of cars in the system. iii) For what percentage of time would the pumps be idle on an average?

Sol:

$\lambda = 30/hr$, $\mu = 6$ minutes $= \frac{1}{\frac{6}{60}} = \frac{10}{1}/hr$ $\left[\because \mu = \frac{1}{\lambda} \text{ exponential}\right]$

$S = 4$,

Then find i) $P(N \geq S)$ ii) W_q, W_s, L_s iii) $1 - P(N \geq 4) = ?$

Ex:5 A road transport company has two reservation clerks, serving the customers. The customer arrive in a Poisson fashion at the rate of 8 per hour. The service time for each customer is exponentially distributed with mean 10 minutes. Find the i) prob'y that a customer has to wait for service ii) Average no. of customer in a queue iii) Average no. of customer in a system iv) Expected waiting time of a customer in the queue v) Expected time customer spends in the system.

Soln: $\lambda = 8/\text{hrs}$, $\mu = \frac{1}{10/60}/\text{hr} = 6/\text{hr}$, $s = 2$.

Then,

$$P_0 = \left\{ \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s \right\}^{-1}$$

$$= \left\{ \sum_{n=0}^{1} \frac{1}{n!}\left(\frac{8}{6}\right)^n + \frac{1}{2!\left(1-\frac{8}{16}\right)}\left(\frac{8}{6}\right)^2 \right\}^{-1}$$

$$= \left\{ 1 + \frac{4}{3} + \left(\frac{3}{2} \times \frac{16}{9}\right) \right\}^{-1}$$

$$= \left\{ 1 + \frac{4}{3} + \frac{8}{3} \right\}^{-1} = (5)^{-1} = \frac{1}{5}$$

i) $$P(N \geq s) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{\left(1-\frac{\lambda}{s\mu}\right)} P_0$$

$$= \frac{1}{2!}\left(\frac{8}{6}\right)^2 \frac{1}{\left(1-\frac{8}{12}\right)} \times \frac{1}{5}$$

$$= \frac{1}{2} \times 16 \times 9 \times \frac{3}{1} \times \frac{1}{5}$$

$$= 0.5333$$

The average no. of Customer in queue

$$L_q = \frac{1}{S \cdot S!} \frac{(\lambda/\mu)^{S+1}}{\left(1-\frac{\lambda}{\mu S}\right)^2} P_0$$

$$= \frac{1}{2 \cdot 2!} \frac{\left(\frac{8}{6}\right)^3}{\left(1-\frac{8}{12}\right)^2} \times \frac{1}{5}$$

$$= \frac{1}{4} \times \frac{64/27}{1/9} \times 1/5 = \frac{16}{15} = 1.0667$$

(iii) The average no. of Customer in the System

$$L_s = L_q + \frac{\lambda}{\mu}$$

$$= \frac{16}{15} + \frac{8}{6} = \frac{12}{5} = 2.4$$

(iv) $W_q = \frac{1}{\lambda} \cdot L_q = \frac{1}{8} \cdot \frac{16}{15} = \frac{2}{15}$ hr $= 8$ minutes

$W_s = \frac{1}{\lambda} \cdot L_s = \frac{1}{8} \times \frac{12}{5} = \frac{3}{10}$ hour $= 18$ minutes

6) A Petrol Pump station has 3 Pumps, the Service time follow an exponential distribution with mean of 5 minutes and the car arrive for service in a Poisson Process at the rate of 8 cars per hour. Find the probability that a customer has to wait, and find

a) Average no. of customers waiting in the queue.
b) Average no. of cars in the system
c) Average waiting time a customer spends in the queue.

(M|M|S): (K/FCFS) Multiple server with finite capacity Model:

Formulae:

$$P_0 = \left\{ \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{\mu s}\right)^{n-s} \right\}^{-1}$$

The average or expected no. of customer in queue

$$L_q = P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left[1-\rho^{k-s}-(k-s)(1-\rho)\rho^{k-s}\right]$$

$$\rho = \frac{\lambda}{\mu s}$$

3) The average or expected no. of customer in system

$$L_s = L_q + \left\{ s - \sum_{n=0}^{s-1} (s-n) P_n \right\}$$

$$\frac{\lambda'}{\mu} = s - \sum_{n=0}^{s-1} (s-n) P_n$$

$$\therefore \lambda' = \mu\left\{ s - \sum_{n=0}^{s-1} (s-n) P_n \right\}$$

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 & \text{for } n \le s \\ \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 & \text{for } s \le n \le k \\ 0 & \text{for } n > k. \end{cases}$$

$$\therefore L_s = L_q + \frac{\lambda'}{\mu}$$

4) The average waiting time of a customer in a queue

$$W_q = \frac{1}{\lambda'} \times L_q = \frac{L_q}{\lambda'}$$

5) The average waiting time of a customer in System

$$W_s = \frac{L_s}{\lambda'}$$

Ex 1: A ~~doctor~~ dispensary has two doctors and four chairs in the waiting room, the patients who arrive at the dispensary leave when all four chairs in waiting room of the dispensary are occupied. It is known that patients arrive at the average rate of 8 per hour & spend an average of 10 minutes for their check up and medical consultation. The arrival process is Poisson and the service time is an exponential random variable, find i) Prob'y that an arriving patient will not wait ii) Effective arrival rate at the dispensary iii) Expected no of patients at the queue iv) Expected waiting time of a patient at the queue v) Expected no. of patients at the dispensary vi) Expected time a patients spends at the dispensary.

Sol^{n}:

$\lambda = 8/hr$, $\mu = 10 \text{ minutes} = 6/hr$

$S = 2$, $K = 2+4 = 6$

$$\rho = \frac{\lambda}{\mu s} = \frac{8}{6 \times 2} = \frac{\cancel{8}}{\cancel{12}} = \frac{2}{3} = 0.667 < 1$$

$$\left\{ \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu s}\right)^{n-s} \right\}^{-1}$$

$$= \left\{ \sum_{n=0}^{1} \frac{1}{n!}\left(\frac{8}{6}\right)^n + \frac{1}{2!}\left(\frac{8}{6}\right)^2 \sum_{n=2}^{6} \rho^{n-2} \right\}^{-1}$$

$$= \left\{ 1 + \frac{4}{3} + \frac{8}{9}\left(1 + \rho + \rho^2 + \rho^3 + \rho^4\right) \right\}^{-1}$$

$$= \left\{ 1 + \frac{4}{3} + \frac{8}{9}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4\right] \right\}^{-1}$$

$$= (5.0451)^{-1}$$

$$P_0 = 0.1982$$

The effective arrival rate at the dispensary

$$\lambda' = \mu\left\{ s - \sum_{n=0}^{s-1} (s-n) P_n \right\}$$

$$= 6\left\{ 2 - \sum_{n=0}^{1} (2-n) P_n \right\}$$

$$= 6\left[2 - 2P_0 - P_1\right]$$

$$P_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu}\right)^1 P_0 = \frac{8}{6} \times P_0 = \frac{4}{3} \times 0.1982$$
$$= 0.2643$$

$$\therefore \lambda' = 6\left\{ 2 - 2 \times 0.1982 - 0.2643 \right\}$$

$$= 8.0358$$

$$L_q = P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left[1 - \rho^{k-s} - (k-s)(1-\rho)\rho^{k-s}\right]$$

iii) $L_q = 0.1982 \times \left(\frac{4}{3}\right)^2 \times \frac{0.667}{2!(1-0.667)^2} \times [1-(0.67)^4-(0.33)\times 4 \times (0.6$

$= 0.7763.$

iv) $W_q = \frac{L_q}{\lambda'} = \frac{0.7763}{8.0358} = 0.0966 \text{ hr} = 5.79$ minutes

v) $L_s = L_q + \frac{\lambda'}{\mu}$

$= 0.7763 + \frac{8.0358}{6} = 2.1156$

vi) $W_s = \frac{L_s}{\lambda'} = \frac{2.1156}{8.0358} = 0.2633 \text{ hr} = 15.79$ minutes

Ex2: A beauty salon has 2 barbers and 6 chairs to accommodate waiting customers. Potential customer who arrive when all the 6 chairs are full, leave the salon immediately. Customer arrive at the salon at the average rate of 10 per hour and spend an average of 10 minutes in the barber's chair. The arrival process is Poisson & service time is exponential random variable, find

i) Prob of no customer in the beauty salon ii) Effective arrival rate at the beauty salon iii) Expected no. of customer in the queue iv) Expected waiting time of a customer at the queue v) Expected no. of customer at the salon vi) Expected time customer spends at the beauty salon.

$\lambda = 10/hr, \quad \mu = 10$ minute $= 6/hr$

$s = 2, \quad k = 2 + 6 = 8$

$$\rho = \frac{\lambda}{\mu s} = \frac{10}{6 \times 2} = \frac{10}{12} = \frac{5}{6} < 1$$

Prob'y no. Customer in the Salon

$$P_0 = \left\{ \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu s}\right)^{n-s} \right\}^{-1}$$

$$= \left\{ \sum_{n=0}^{1} \frac{1}{n!}\left(\frac{10}{6}\right)^n + \frac{1}{2!}\left(\frac{10}{6}\right)^2 \sum_{n=2}^{8} \rho^{n-2} \right\}^{-1}$$

$$= \left\{ \sum_{n=0}^{1} \frac{1}{n!}\left(\frac{5}{3}\right)^n + \frac{1}{2!}\left(\frac{5}{3}\right)^2 \sum_{m=0}^{6} \rho^{m} \right\}^{-1}$$

$$= \left\{ 1 + \frac{5}{3} * \left(\frac{1}{2} \times \frac{25}{9} \times \frac{1-\rho^7}{1-\rho} \right) \right\}^{-1}$$

$$= \left\{ 1 + \frac{5}{3} + \frac{18.023}{3} \right\}^{-1} = (8.6744)^{-1}$$
$$= 0.11153$$

$P_0 = 0.11153$

$$\lambda' = \mu \left\{ s - \sum_{n=0}^{s-1} (s-n) P_n \right\}$$

$$= 6 \left\{ 2 - \sum_{n=0}^{1} (2-n) P_n \right\}$$

$$= 6 [2 - 2P_0 - P_1]$$

$$P_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu}\right)^1 P_0$$

$$= \frac{8}{6} \times P_0 = \frac{4}{3} \times \overset{0.1153}{\cancel{0.1982}} = \cancel{0.2643}$$

$$= 0.1537$$

ii) $\lambda' = 6[2 - 2 \times 0.1153 - 0.1537)$

$\lambda' = 9.69$

iii) $$L_q = P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left[1 - \rho^{k-s} - (k-s)(1-\rho)\rho^{k-s}\right]$$

$$= 0.1153\left(\frac{10}{6}\right)^2 \times \frac{5/6}{2!(1-5/6)^2} \times \left[1 - \rho^6 - 6(1-\rho)\rho^6\right]$$

$$= 0.1153 \times 2.7778 \times 14.9876 \times 0.3302$$

$$= 1.5850$$

iv) $$W_q = \frac{L_q}{\lambda'} = \frac{1.5850}{9.69} = 0.1675 \text{ hr}$$

$$= 10.0488 \text{ min.}$$

v) $$L_s = L_q + \frac{\lambda'}{\mu} = 1.5850 + \frac{9.69}{6}$$

$$= 3.1623.$$

vi) $$W_s = \frac{L_s}{\lambda'} = \frac{3.1623}{9.69} = 0.3341 \text{ hr}$$

$$= 20.0488 \text{ min}$$

$(M/E_k/1):(\infty/FCFS)$ Model:

The arrival distribution is Markovian and departure distribution is Erlangian with Parameter k, with Single Server with infinite Capacity.

The Various measures of the model are:

1) Average number of units in the System.

$$L_s = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda} + \frac{\lambda}{\mu}$$

2) Average no. of Units in the queue.

$$L_q = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda}$$

3) Average time spent by a unit in the System

$$W_s = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

4) Average Waiting time of a unit in the queue.

$$W_q = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$$

Ex: Repairing a certain type of machine which breaks down in a given factory consists of 5 basic steps that must be performed sequentially. The time taken to Perform each of the 5 steps is found to have an exponential distn with mean 5 minutes and is independent of other steps. If these machines break down in a poisson fashion at an average rate of two per hour and if there is only one repairman, what is the average idle time for machine that has broken down?

No of Phases $k = 5$

Service time Per phase $= 5$ minutes

Service time Per unit $= 5 \times 5 = 25$ minutes

Service rate $(\mu) = \frac{1}{25}$ units/minute

$= \frac{1}{25} \times 60 = \frac{12}{5}$ units/hr

& arrival rate $\lambda = 2$ units/hrs

Now,

Average idle time for each machine

$=$ Average time spend by the machine in the system

$$W_s = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{5+1}{2 \times 5} \cdot \frac{2}{\frac{12}{5}\left(\frac{12}{5} - 2\right)} + \frac{1}{12/5}$$

$$= \frac{6}{10} \times \frac{2 \times 25}{24} + \frac{5}{12}$$

$$= \frac{5}{4} + \frac{5}{12} = \frac{20}{12} = \frac{5}{3} \text{ hr} = 100 \text{ minute}$$

$W_s = 100$ minutes.