Basic terms

(1) Random experiment
(2) Trial
(3) Event
(4) Exhaustive events
(5) Favourable events
(6) Equally like events
(7) Mutually exclusive events
(8) Independent events
(9) Dependent events
(10) Sample space.

(1) Random experiment: An experiment which has various possible outcomes ~~is called ran~~ but of whice one can happen is called random experiment.

(2) Trial: performing of a random experiment is called trial Eg: tossing a coin.

(3) Event: outcome or combination of outcomes of a random experiments is called event.

(4) Exhaustive events Total possible outcomes of a random experiment.

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$$= \lambda \left[1 - \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 \right]$$

i) $$P_0 = \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \sum_{n=s}^{N} \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$= \left[1 + \frac{15}{12} + \frac{1}{2!\, 2^0} \left(\frac{1/12}{1/15}\right)^2 + \frac{1}{2!\, 2} \left[\frac{1/12}{1/15}\right]^3 \right]^{-1}$$

$$= \left[1 + \frac{15}{12} + \frac{25}{32} + \frac{125}{256} \right]^{-1}$$

$$= \left[\frac{901}{256} \right]^{-1}$$

$$= \frac{256}{901}$$

$$\therefore \lambda_e = \frac{1}{12} \left[1 - \frac{1}{2!\, 2^{3-2}} \left(\frac{1/12}{1/15}\right)^3 \cdot \frac{256}{901} \right]$$

$$= \frac{1}{12} \left[1 - 0.[illegible] \right]$$

$$= [illegible]$$

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ii) $LS = ?$

Here,

$$L_q = \frac{(s\rho)^2}{s!\,(1-\rho)^2} \left[1 - \rho^{N-s+1} - (1-\rho)(N-s+1)\rho^{N-s} \right]$$

$$= \frac{(2 \times 5/8)^2}{2!\,(1 - 5/8)^2} \left[1 - \left(\frac{5}{8}\right)^{3-2+1} - \left(1 - \frac{5}{8}\right)(3-2+1)\left(\frac{5}{8}\right)^{3-2} \right]$$

$$= \frac{50}{9} \times \frac{9}{64}$$

$$= \frac{25}{32}$$

$$L_s = L_q + s - P_0 \sum_{n=0}^{s-1} \frac{(s-n)}{n!} \left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{25}{32} + 2 - \frac{256}{901} \sum_{n=0}^{1} \left[\frac{2-n}{n!} \cdot \left(\frac{15}{12}\right)^n \right]$$

$$= \frac{25}{32} + 2 - \frac{256}{901} \left[2 + 1 \cdot \frac{15}{12} \right]$$

$$= 1.857$$

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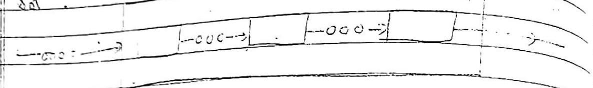
28\. Sol^n :



No. of servers (s) = 1

No. of phases (K) = 3

3 × 1.5 = 4.5 min = 1 customer served

$1 \text{ min} = \frac{1}{4.5} = \frac{2}{9}$ customer served

∴ Service rate $(\mu) = \frac{2}{9}$ customer per minute.

Arrival rate (λ) = 6 customer per hour

$= \frac{6}{60} = \frac{1}{10}$ customer per minute

This problem is related to $M/E_k/1 : (\infty, FIFO)$ queue

$W_s = ?$

Here,

$$W_s = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

$$= \frac{3+1}{2 \times 3} \times \frac{1/10}{2/9 \times (2/9 - 1/10)} + \frac{1}{2/9}$$

$$= \frac{163}{22}$$

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Also,

Most probable time in getting the service $= \frac{k-1}{k\mu}$

$$= \frac{3-1}{3 \times 2/9}$$

$$= 3 \text{ min}$$

26\. Sol^n :

limited no. of machine (M) = 4

no. of server (mechanic) (s) = 1

5 hours = 1 machine arrive to mechanic

1 hour = $\frac{1}{5}$ machine arrive to mechanic

Arrival time $(\lambda) = \frac{1}{5}$ machine per hour

1 hour = 1 machine served

Service rate (μ) = 1 machine per ~~pe~~ hour

This problem is related to $(M/M/1 : M/GD)$ queue

i) $P_0 = ?$

$P_3 = ?$

$L_s = ?$

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Then

$$\therefore P_0 = \left[\sum_{n=0}^{4} \frac{4!}{(4-n)!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$= \left[\frac{4!}{(4-0)!} \times 1 + \frac{4!}{3!} \cdot \left(\frac{1/5}{1}\right)^1 + \frac{4!}{2!} \left(\frac{1}{5}\right)^2 + \right.$$

$$\left. \frac{4!}{1!} \left(\frac{1}{5}\right)^3 \right]^{-1} + \frac{4!}{0!} \left(\frac{1}{5}\right)^4 \Big]$$

$$= \left[1 + 4 \times \frac{1}{5} + 12 \times \frac{1}{25} + \frac{4!}{0!}\left(\frac{1}{5}\right)^4 + 24 \frac{1}{125} \right]^{-1}$$

$$= \left(\frac{944}{625}\right)^{-1}$$

$$= \frac{625}{944}$$

6. Solⁿ:

Transmission line
[illegible] characters per second
[illegible] per min

1 message = 176 character
[illegible] per min.

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$$\mu = \frac{800}{176} \times 60 = 272.72 \simeq 273 \text{ msg/min}$$

$c = 1$

$N = \infty$

FIFO.

Questions Patterns

PQT Solution

1). Simple Probability
   (a) Rules of Prob
   (b) conditional prob

1 (a) (b)

2) Baye's Theorem

3) Random Variable, expectation, variance (1 LQ) 2 or 3

4) Joint Prob Distribution (1 LQ). 2 or 3

5) Theoretical Probability distribution
   (a) Binomial Distribution
   (b) Poisson Distribution
   Imp (c) Normal Distribution
   (d) Exponential "
   (e) Gamma "
   (f) Beta "
   (g) Uniform "

(2 LQ) Done

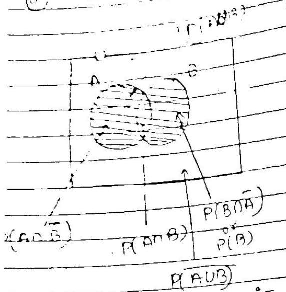
6) Central Limit Theorem, BLLN, Chebyshev's Inequality.
(1 LQ)

7) Markov's Chain (M.C) (1 LQ / 2 LQ.) Done.

8) Theory of Queue (3 LQ.)

(1) Simple Probability

(2) Rules of Probability



(1) Additional Rule :-

$P(A\cup B)$

↳ Prob of either A or B.

↳ Atleast one of them can happen.

$$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B).$$

(2) Multiplication Rule.

↳ Prob of $P(A\cap B)$

↳ Prob of A and B.

→ Both of them can happen

→ also known as joint prob of A and B

→ formula for independent events

$$P(A\cap B) = P(A) * P(B) \text{ or } P(B) * P(A)$$

Selection with replacement is independent.

↳ For dependent events - selection without replacement is dependent.

$$P(A\cap B) = P(A) \times P(B/A)$$

$$P(B\cap A) = P(B) \times P(A/B)$$

Note :

i) $P(A\cap\bar{B}) = P(A) - P(A\cap B)$.

ii) $P(B\cap\bar{A}) = P(B) - P(A\cap B)$

iii) $P(\overline{A\cup B}) = 1 - P(A\cup B)$.

Numericals.

(1) Mr. A and Mr. B are interested to attend a seminar. The chance of attending seminar by Mr. A is 0.6 and Mr. B is 0.3. They both can also attend the seminar. Attending the seminar by Mr. A and Mr. B are independent to eachother. What is the prob that :

a) Both of them can attend the seminar

b) atleast one of them can attend the seminar.

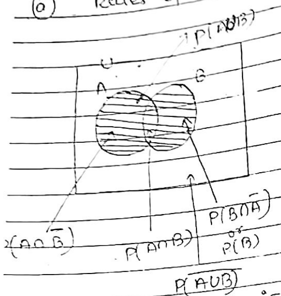
c) only Mr. A will attend the seminar

d) only Mr. B will attend the seminar

e) they will not have attend ~~the semin~~ in seminar.

① Simple Probability

ⓐ Rules of Probability



$P(U) = 1$

① Additional Rule :-

$P(A \cup B)$

↳ Prob of either A or B.

↳ Atleast one of them can happen.

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

② Multiplication Rule.

↳ Prob of $P(A \cap B)$

↳ Prob of A and B.

→ Both of them can happen

→ also known as joint prob of A and B

→ ~~EQ~~ formula for independent events

$P(A \cap B) = P(A) * P(B)$ or, $P(B) * P(A)$

Selection with replacement is independent.

↳ For dependent events - selection without replacement is dependent.

$$P(A \cap B) = P(A) \times P(B/A)$$

$$P(B \cap A) = P(B) \times P(A/B)$$

Note :

i) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

ii) $P(B \cap \bar{A}) = P(B) - P(A \cap B)$

iii) $P(\overline{A \cup B}) = 1 - P(A \cup B)$.

Numericals.

① Mr. A and Mr. B are interested to attend a seminar. The chance of attending seminar by Mr. A is 0.6 and Mr. B is 0.3. They both can also attend the seminar. Attending the seminar by Mr. A and Mr. B are independent to eachother. What is the prob that :

a) Both of them can attend the seminar

b) atleast one of them can attend the seminar

c) only Mr. A will attend the seminar

d) only Mr. B will attend the seminar

e) they will not have attend ~~the semin~~ in seminar

Soln

$P(A) = 0.6$

$P(B) = 0.3$

a) $P(A \cap B) = P(A) \times P(B)$

$= 0.18$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.3 - 0.18$

$= 0.72$

c) P(only Mr. A will attend)

$= P(A \cap \bar{B})$

$P(A) - P(A \cap B)$

$= 0.6 - 0.18$

$= 0.42$

d) P(only Mr. B will attend)

$= P(\bar{A} \cap B)$

$= P(B) - P(A \cap B)$

$= 0.3 - 0.18$

$= 0.12$

e) P(they will not have to attend seminars)

$= P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - 0.72$

$= 0.28$

2) A committee of 4 members has to be formed ~~from~~ from 5 IT engineer, 4 civil engineer, 3 software engineer and 2 elx. engineers. Find the prob that committee consists of :-

a) 2 IT and 2 civil engineer

b) No IT engineer

c) atleast one IT engineer.

d) atmost one IT engineer.

e) one IT, one civil, one software and one elx engineer.

Soln

Total possible no. of committee $= {}^nC_r$

$= {}^{14}C_4$

$= 1001$

(5) Favourable events: Desired outcomes of a random experiment which results in happening of an event.

(6) Equally like events: Each event has an equal chance of happenning.

(7) Mutually exclusive events: Events can't occur simultaneously in a single trial.

(8) Independent events: Happening of one event does not affect happening of other event in different trials.

(9) Dependent events: Happening of one event affects the happening of other event in different trial.

(10) Sample space: set of total possible outcomes of a random experiment.

For eg.

$S = \{H, T\}$ for a coin toss

$S = \{1, 2, 3, 4, 5, 6\}$ for a dice roll.

Defn of probability:

(1) Classical approach:

$$P(E) = \frac{\text{Favourable no. of events } (M)}{\text{Total no. of events } (N)}$$

This defn of probability holes when -

(i) events are equally like.

(ii) N is known.

Note:-

(a) $0 \leq P(E) \leq 1$

(b) $P(E) + P(\bar{E}) = 1$

(2) Statistical approach

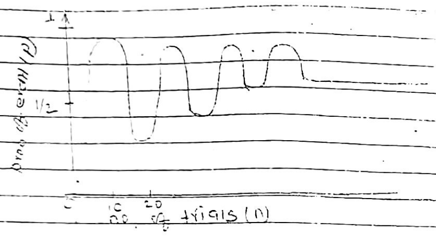
$$P(E) = \lim_{n \to \infty} \frac{X}{n}$$

Where X = No. of ~~represe~~ ~~repeatation of event~~ 't'.

n = total no. of trials.

This defination holds when all trials are performed under the similar conditions.

physical interpretation:



As $n \to \infty$, $\frac{X}{n}$ tends to attain single values (i.e $1/2$).

(3) Axiomatic approach:

It is based on set theory $E_1, E_2 \ldots E_n$ be mutually exclusive events having sample spaces. Then probability function $P(E_i)$ has to be satisfied following conditions.

(i) $0 \leq P(E_i) \leq 1$ for all i

(ii) $P(S) = 1$

(iii) $P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$.

(4) Subjective approach

It is based on personal believe or judgement of expertise.

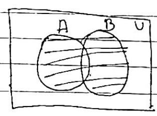
Rules of probability

(1) Additional rule

Statement:

let 'A' and 'B' be two events then probability that atleast one of them can occur is given by.

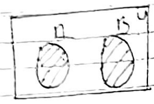
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Notes:

(i) $P(A \cup B) + P(\overline{A \cup B}) = 1$

(ii) If A and B mutually exclusive events then we can write $P(A \cup B) = P(A) + P(B)$ $[\because P(A \cap B) = 0]$



(iii) If A and B are independent events then we can write $P(A\cup B) = 1 - P(\bar{A}) \times P(\bar{B})$

where,

$P(\bar{A}) = 1 - P(A)$

$P(\bar{B}) = 1 - P(B)$

Proof:

We have,

$P(A\cup B) + P(\overline{A\cup B}) = 1$

or $P(A\cup B) = 1 - P(\overline{A\cup B})$

or $P(A\cup B) = 1 - P(\bar{A}\cap \bar{B})$ [By Demorgan's law]

or $\boxed{P(A\cup B) = 1 - P(\bar{A}) \times P(\bar{B})}$ [since A & B are independent]

(2) Multiplication Rule

Statement: Let A and B are two events then probability that both of them can occurs is given by

(i) $P(A\cap B) = P(A) \times P(B)$ for independent events

(ii) $P(A\cap B) = P(A) \times P(B/A)$ where

$P(B/A)$ = conditional prob. of B given that A has already occurred.

OR

$P(B\cap A) = P(B) \times P(A/B)$

Numerical

(1) If 3 cards are drawn from deck of 52 cards what is the probability of getting 3 kings.

Soln

Let K_1 = event of getting king in 1st drawn

K_2 = " " " " 2nd drawn

K_3 = " " " " 3rd drawn

$P(K_1\cap K_2\cap K_3) = P(K_1) \times P(K_2/K_1) \times P\left(\frac{K_3}{K_1\cap K_2}\right)$

$= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}$

$=$

OR

$P(3 \text{ kings}) = \frac{^{4}C_3}{^{52}C_3}$

$=$

(2) If 3 cards are drawn from deck of 52 cards, what is the prob. of getting 2 kings and 1 queen.

Soln

$$P(\text{2 king and 1 queen}) = \frac{4C_2 \times 4C_1}{52C_3}$$

$$= \frac{6 \times 4}{22100}$$

$$= 1.08 \times 10^{-3}$$

5. A committee of 4 members has to be formed from five engineer, four accountant, three politication, 2 doctors. (i) What is the prob. of that committs consists of each professionals

(ii) two engineers and two doctors

(iii) no engineers

(iv) at least one engineer

(v) at least two accountant

(vi) at most one doctor.

(vii) at most two politician.

Soln

5 Eng
4 acc
3 poli
2 doc

(i) prob. of (1 eng, 1 acc, 1 poli, 1 doct)

$$= \frac{5C_1 \times 4C_1 \times 3C_1 \times 2C_1}{14C_4}$$

$$= \frac{120}{1001}$$

$$= 0.12$$

(ii) prob. of (2 eng and 2 doc)

$$= \frac{5C_2 \times 2C_2}{14C_4}$$

$$= 0.11$$

(iii) prob. (No eng)

$= P(0 \text{ eng and } 4 \text{ others})$

$$= \frac{5C_0 \times 9C_4}{14C_4}$$

$=$

(iv) prob (at least one Eng)

$= P(\text{1 Eng and 3 others}) + P(\text{2 eng and 2 others}) + P(\text{3 eng and 1 others}) + P(\text{4 eng \& 0 others})$

$$= \frac{5C_1 \times 9C_3}{14C_4} + \frac{5C_2 \times 9C_2}{14C_4} + \frac{5C_3 \times 9C_1}{14C_4} + \frac{5C_4 \times 9C_0}{14C_4}$$

$=$

OR

P(at least one Eng)

$= 1 - P(\text{no-eng})$

$= 1 - \frac{5C_0 \times 9C_4}{14C_4}$

$= 1 - \frac{1 \times 126}{1001} = 1 - 0.125$

$= 0.8741$

(v) P(at least 2 Acc)

= P(2 Acc & 2 others) + P(3 acc & 1 other) + P(4 acc and 0 other)

$= \frac{4C_2 \times 10C_2}{14C_4} + \frac{4C_3 \times 10C_1}{14C_4} + \frac{4C_4 \times 10C_0}{14C_4}$

$= \frac{6 \times 45}{1001} + \frac{4 \times 10}{1001} + \frac{1 \times 1}{1001} = \frac{311}{1001}$

OR. $= 0.3106$

prob (at least 2 acc)

= 1 − prob (1 acc and 3 others) − prob (0 acc and 4 others)

$= 1 - \frac{4C_1 \times 10C_3}{14C_4} - \frac{4C_0 \times 10C_4}{14C_4}$

$= 1 - \frac{4 \times 120}{1001} - \frac{1 \times 210}{1001}$

$= 1 - 0.479 - 0.209 = 0.312$

(vi) prob (at most 1 doc)

= P(1 doc and 3 others) + P(0 doc and 4 others)

$= \frac{2C_1 \times 12C_3}{14C_4} + \frac{2C_0 \times 12C_4}{14C_4}$

=

=

ii) prob (at most 2 politicians)

= P(2 pol and 2 others) + P(1 pol and 3 others) + P(0 pol and 4 others)

$= \frac{3C_2 \times {}^{11}C_2}{14C_4} + \frac{3C_1 \times 11C_3}{14C_4} + \frac{3C_0 \times 11C_4}{14C_4}$

=

=

Conditional Probability

Let A and B be two dependent events. Then

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Numerical

Given a binary communication channel, where A is the input and B is the output, let $P(A) = 0.40$, $P(B/A) = 0.9$ and $P(\overline{B}/\overline{A}) = 0.6$. calculate $P(A/B)$, $P(A\cap B)$, $P(B)$, $P(A/B)$, $P(A/\overline{B})$.

Soln

$$P(\overline{B}/\overline{A}) = \frac{P(\overline{A}\cap\overline{B})}{P(\overline{A})}$$

$$\text{or } 0.60 = \frac{P(\overline{A\cup B})}{P(\overline{A})}$$

~~or 0.60 =~~

$$\text{or } 0.60 = \frac{1 - P(A\cup B)}{1 - P(A)}$$

$$\text{or } 0.60 = \frac{1 - P(A\cup B)}{1 - 0.40}$$

$$\text{or } 0.60 = \frac{1 - P(A\cup B)}{0.60}$$

$$\text{or } 1 - P(A\cup B) = 0.36$$

$$P(A\cup B) = 1 - 0.36$$

$$P(A\cup B) = 0.64$$

and

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

$$\text{or } 0.90 = \frac{P(A\cap B)}{0.40}$$

$$\text{or } P(A\cap B) = 0.36$$

Now

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\text{or, } 0.64 = 0.40 + P(B) - 0.36$$

$$\text{or, } 0.64 = 0.04 + P(B)$$

$$\therefore P(B) = 0.60$$

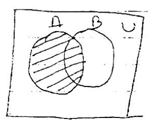
and,

$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{0.36}{0.60} = 0.6$$

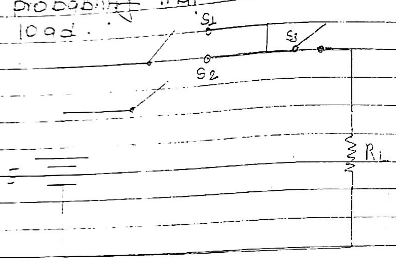
and

$$P(A/\overline{B}) = \frac{P(A\cap\overline{B})}{P(\overline{B})} = \frac{P(A) - P(A\cap B)}{1 - P(B)} = \frac{0.40 - 0.36}{1 - 0.60} = \frac{0.04}{0.40} = 0.1$$

#



Q. A battery is connected to a load through a series - parallel combinations of relay switches, each of which may fail to close with probability 0.20. Assume that separate switch actions are statistically independent. Find the probability that current will flow in this load.



prob(current flows through load)

$P[(S_1 \cup S_2) \cap S_3]$

$= P[(S_1 \cup S_2)] \cdot P(S_3)$

$= [P(S_1) + P(S_2) - P(S_1 \cap S_2)] \cdot P(S_3)$

$= [P(S_1) + P(S_2) - P(S_1) \cdot P(S_2)] P(S_3)$

$= [0.8 + 0.8 - (0.2)^2] (0.8)$

$= (0.4 - \cancel{0.04}) 0.2 \quad = (1.6 - 0.64) 0.8$

$= \cancel{0.36 \times 0.2} \quad = 0.768$

$= \cancel{0.072}$ #

OR.

$P[(S_1 \cup S_2) \cap S_3]$

$= P(S_1 \cup S_2) \cdot P(S_3)$

$= [1 - P(\bar{S_1}) \cdot P(\bar{S_2})] \; P(S_3)$

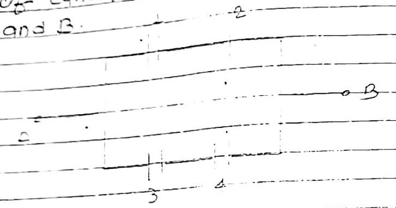
$= [(1 - 0.2 \times 0.2)] \; 0.8$

$= (1 - 0.04) \times 0.8$

$= 0.96 \times 0.8$

$= 0.768$ #

Q1. For a given circuit, the probability of closing of each switch is known to 0.3. Assume all switches act independently but not mutually exclusive. Find the probability of current exist between terminals A and B.



let E_1 = Event that switch 1 close
E_2 = " " " 2 "
E_3 = " " " 3 "
E_4 = " " " 4 "

No:01
prob (current exists betn A and B)

$= P[(E_1 \cap E_2)$ or $(E_3 \cap E_4)]$

$= P[(E_1 \cap E_2) \cup (E_3 \cap E_4)]$

$= P(E_1 \cap E_2) + P(E_3 \cap E_4) - P(E_1 \cap E_2 \cap E_3 \cap E_4)$

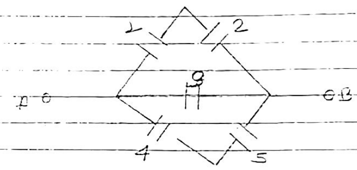
$= P(E_1) \cdot P(E_2) + P(E_3) \cdot P(E_4) - P(E_1) \cdot P(E_2) \cdot P(E_3) \cdot P(E_4)$

$= (0.3)^2 + (0.3)^2 - (0.3)^4$ [Given $P(E_1) = P(E_2) = P(E_3) = P(E_4) = 0.3$]

=

=

Q2.



prob (current exists betn A and B)

$= P[(E_1 \cap E_2) \cup (E_4 \cap E_5) \cup E_3]$

$= P(E_1 \cap E_2) + P(E_4 \cap E_5) + P(E_3) - P(E_1 \cap E_2 \cap E_4 \cap E_5) - P(E_4 \cap E_5 \cap E_3) - P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4 \cap E_5 \cap E_3)$

$= P(E_1) \cdot P(E_2) + P(E_4) \cdot P(E_5) + P(E_3) - P(E_1) \cdot P(E_2) \cdot P(E_4) \cdot P(E_5) - P(E_4) \cdot P(E_5) \cdot P(E_3) - P(E_1) \cdot P(E_2) \cdot P(E_3) + P(E_1) \cdot P(E_2) \cdot P(E_4) \cdot P(E_5) \cdot P(E_3)$

$= (0.3)^2 + (0.3)^2 + 0.3 - (0.3)^4 - (0.3)^3 - (0.3)^3 + (0.3)^5$

$$0.09 + 0.09 + 0.3 - 8.1\times10^{-3} - 0.027 - 0.027$$

$$0.03125$$

$$= 0.4203$$

OR

$$\boxed{P[(E_1 \cap E_2) \cup (E_4 \cap E_5) \cup E_3]}$$

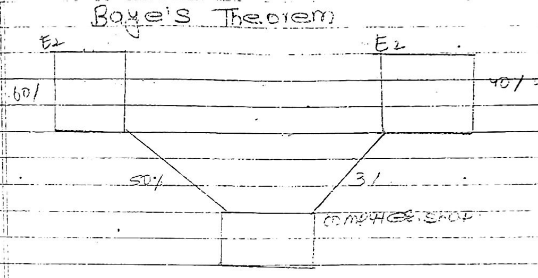
$$= 1 - P(\overline{E_1 \cap E_2}) \cdot P(\overline{E_4 \cap E_5}) \cdot P(\overline{E_3})$$

$$= 1 - [1 - P(E_1 \cap E_2)] \cdot [1 - P(E_4 \cap E_5)] [1 - P(E_3)]$$

$$= 1 - [1 - (0.3)^2] \cdot [1 - (0.3)^2] [1 - 0.3]$$

$$= 0.4203$$

Baye's Theorem



priori probabilities (sum = 1)

$$P(E_1) = 0.60$$
$$P(E_2) = 0.40$$

likelihood probabilities

$$P(D/E_1) = 0.05, \quad P(D/E_2) = 0.03$$

Revsed probabilities / posteriorz probabilities (sum = 1)

$$P(E_1/D)$$
$$P(E_2/D)$$

Case = defective computer)

Event	P(Event)	P(Case/event)	P(Event ∩ case)	P(Event/case)
E_1	$P(E_1) = 0.60$	$P(D/E_1) = 0.05$	$P(E_1 \cap D) = 0.6 \times 0.05 = 0.03$	$P(E_1/D) = \frac{P(E_1 \cap D)}{P(D)} = \frac{0.03}{0.042} = 0.714$
E_2	$P(E_2) = 0.40$	$P(D/E_2) = 0.03$	$P(E_2 \cap D) = 0.4 \times 0.03 = 0.012$	$P(E_2/D) = \frac{P(E_2 \cap D)}{P(D)} = \frac{0.012}{0.042} = 0.286$
	(prior probabilities)	likelihood probabilities	joint probabilities	
			$P(D) = 0.03 + 0.012 = 0.042$	

Formula

$$P(E_1/D) = \frac{P(E_1 \cap D)}{P(D)}$$

$$\boxed{P(E_1/D) = \frac{P(E_1) \times P(D/E_1)}{P(E_1) \times P(D/E_1) + P(E_2) \times P(D/E_2)}}$$

Similarly,

$$\boxed{P(E_2/D) = \frac{P(E_2) \times P(D/E_2)}{P(E_1) \times P(D/E_1) + P(E_2) \times P(D/E_2)}}$$

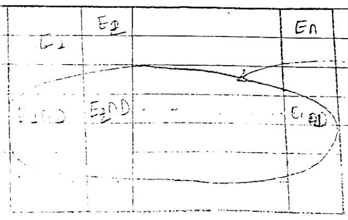
State and prove Baye's Theorem

Let $E_1, E_2, \ldots, E_n$ be the n mutually exclusive events. Event D can occurs conjunction with $E_1, E_2, \ldots, E_n$. Then probabilities of ~~every~~ event E_i given that ~~area~~ D has already occured is obtained as -

$$P(E_i/D) = \frac{P(E_i) \times P(D/E_i)}{\sum_{i=1} P(E_i) \times P(D/E_i)}$$

where, $i = 1, 2, 3, \ldots n$

Proof:



CDTO

From figure, when

$D = (E_1 \cap D) \cup (E_2 \cap D) \ldots \cup (E_n \cap D)$

Taking prob on both sides

$P(D) = P[(E_1 \cap D) \cup (E_2 \cap D) \ldots \cup (E_n \cap D)]$

$= P(E_1 \cap D) + P(E_2 \cap D) \ldots + P(E_n \cap D)$

$= P(E_1) \times P(D/E_1) + P(E_2) \times P(D/E_2) + \ldots + P(E_n) \times P(D/E_n)$

$$P(D) = \sum_{i=1}^{n} P(E_i) \times P(D/E_i)$$

Now

$P(E_1/D)$

$= \frac{P(E_1 \cap D)}{P(D)}$

$$\therefore \boxed{P(E_1/D) = \frac{P(E_1) \times P(D/E_1)}{\sum_{i=1}^{n} P(E_i) \times P(D/E_i)}}$$

and

$$P(E_2/D) = \frac{P(E_2 \cap D)}{P(D)}$$

$$= \frac{P(E_2) \times P(D/E_2)}{\sum_{i=1}^{n} P(E_i) \times P(D/E_i)}$$

In general,

$$\boxed{P(E_i/D) = \frac{P(E_i) \times P(D/E_i)}{\sum_{i=1}^{n} P(E_i) \times P(D/E_i)}}$$

Three urns of same appearance have the following ~~prop~~ proportional of balls.

2B	3B	4B
1R	2R	1R
1W	2W	3W
urn I	urn II	urn III

One urn is selected and two balls are drawn at random. If they turn out to be white and red, what is the probability that they are drawn from –

(i) urn I (ii) urn II.
(iii) urn III

Sol^n

Let

E_1 = Event that urn I is selected

E_2 = Event that urn II is "

E_3 = " " urn III " "

A = Event that selected balls are white and Red.

Given,

$P(E_1) = 1/3$

$P(E_2) = 1/3$

$P(E_3) = 1/3$

$$P(A/E_1) = \frac{{}^1C_1 \times {}^1C_1}{{}^4C_2} = 1/6$$

$$P(A/E_2) = \frac{{}^2C_1 \times {}^2C_1}{{}^7C_2} = 4/21$$

$$P(A/E_3) = \frac{{}^3C_1 \times {}^1C_1}{{}^8C_2} = \frac{3}{28}$$

(i) $P(E_1/A)$

$$= \frac{P(E_1) \times P(A/E_1)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2) + P(E_3) \times P(A/E_3)}$$

$$= \frac{1/3 \times 1/6}{1/3 \times 1/6 + 1/3 \times \frac{4}{21} + 1/3 \times \frac{3}{28}}$$

$$= \frac{1/18}{\frac{1}{18} + \frac{4}{63} + \frac{1}{28}} = \frac{1/18}{\frac{13}{84}} = \frac{1}{18} \times \frac{84}{13}$$

$$= \frac{84}{234}.$$

(ii) $P(E_2/A)$

$$= \frac{P(E_2) \times P(A/E_2)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2) + P(E_3) \times P(A/E_3)}$$

$$= \frac{1/3 \times 4/21}{1/3 \times 1/6 + 1/3 \times \frac{4}{21} + 1/3 \times 3/28}$$

$$= \frac{}{234} = \frac{4}{63} \times \frac{84}{13}$$

$$= \frac{336}{819}$$

(iii) $P(E_3/A)$

$$= \frac{P(E_3) \times P(A/E_3)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2) + P(E_3) \times P(A/E_3)}$$

$$= \frac{1/3 \times 3/28}{1/3 \times 1/6 + 1/3 \times 4/21 + 1/3 \times 3/28}$$

$= \frac{1}{28} \times \frac{84}{13}$

$= \frac{84}{364}$

2015 Fall

1(b) Sol^n

E_1 = Event that ELT is made by A
E_2 = event that ELT is " " B
E_3 = " " " " " " C
A = Event that ELT is defective.

$P(E_1) = 0.8$
$P(E_2) = 0.15$
$P(E_3) = 0.05$

$P(A/E_1) = 0.04$
$P(A/E_2) = 0.06$
$P(A/E_3) = 0.09$

(i) $P(E_1/A) = \frac{P(E_1) \times P(A/E_1)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2) + P(E_3) \times P(A/E_3)}$

$= 0.8 \times 0.04$

(ii) $P(E_2/A) =$

(iii) $P(E_2/\bar{A}) = 1 - P(E_2/A)$

$= 1 -$

2016 spring

1(b) E_1 = Event that product is manufactured by A
E_2 = " " " " " B
E_3 = " " " " " C

D = Event that product is defective

$P(E_1) = 0.40$
$P(E_2) = 0.35$
$P(E_3) = 0.25$

$P(D/E_1) = 0.08$

$P(D/E_2) = 0.1$

$P(D/E_3) = 0.05$

$$P(E_1/D) = \frac{P(E_1) \times P(D/E_1)}{P(E_1) \times P(D/E_1) + P(E_2) \times P(D/E_2) + P(E_3) \times P(D/E_3)}$$

$$= \frac{0.40 \times 0.08}{0.40 \times 0.08 + 0.35 \times 0.1 + 0.25 \times 0.05}$$

$$= \frac{0.032}{0.032 + 0.035 + 0.0125}$$

$$= 0.402$$

$\cancel{P(E_2/D) = E/E}$

$P(E_1/\bar{D}) = 1 - P(E_1/D)$

$= 1 - 0.402$

$= 0.598$

probability that the product is good.

$P(G/E_1) = 1 - 0.08 = 0.92$
$P(G/E_2) = 1 - 0.1 = 0.90$
$P(G/E_3) = 1 - 0.05 = 0.95$

$P(G) = P(E_1 \cap G) + P(E_2 \cap G) + P(E_3 \cap G)$
$= P(E_1) \times P(G/E_1) + P(E_2) \times P(G/E_2) + P(E_3) \times P(G/E_3)$
$= 0.40 \times 0.92 + 0.35 \times 0.90 + 0.25 \times 0.95$
$= 0.9205$

(i) $P(E_1/G)$

$$= \frac{P(E_1) \times P(G/E_1)}{P(E_1) \times P(G/E_1) + P(E_2) \times P(G/E_2) + P(E_3) \times P(G/E_3)}$$

$$= \frac{0.40 \times 0.92}{0.9205}$$

$=$

(ii) $P(E_2/G)$

$$= \frac{P(E_2) \times P(G/E_2)}{0.9205}$$

$$= \frac{0.35 \times 0.90}{0.9205} =$$

(iii) $P(E_3/G)$

$= \frac{P(E_3) \times P(G/E_3)}{0.9205}$

$= \frac{0.25 \times 0.95}{0.9205}$

$=$

2014 Fall

19) E_1 = Event that the plant receives voltage regulation from A.

E_2 = Event that the plant receives voltage regulation from B

E_3 = Event that plant receives voltage regulation from C

D = Event that perform according to the specification.

$P(E_1) = 0.6$
$P(E_2) = 0.3$
$P(E_3) = 0.1$

$P(D/E_1) = 0.95$
$P(D/E_2) = 0.65$
$P(D/E_3) = 0.80$

(i) $P(D) = P(E_1 \cap D) + P(E_2 \cap D) + P(E_3 \cap D)$

$= P(E_1) \times P(D/E_1) + P(E_2) \times P(D/E_2) + P(E_3) \times P(D/E_3)$

$= 0.6 \times 0.95 + 0.3 \times 0.65 + 0.1 \times 0.80$
$= 0.57 + 0.195 + 0.08$
$= 0.845$

(ii) $P(\bar{D}/E_1) = 0.05$

$P(\bar{D}/E_2) = 0.20$

$P(\bar{D}/E_3) = 1 - 0.65 = 0.35$

$P(E_3/\bar{D}) = \frac{P(E_3) \times P(\bar{D}/E_3)}{P(E_1) \times P(\bar{D}/E_1) + P(E_2) \times P(\bar{D}/E_2) + P(E_3) \times P(\bar{D}/E_3)}$

$= \frac{0.1 \times 0.35}{0.6 \times 0.05 + 0.3 \times 0.20 + 0.1 \times 0.35}$

$= \frac{0.035}{0.03 + 0.06 + 0.035}$

$= \frac{0.035}{0.125}$

$= 0.28$

2012 Fall

let,

1(a) E_1 = actually defective chips

E_2 = Non defective chips (good chips)

G = testers says good

D = tester say defective.

Given,

$P(E_1) = 0.02$

$P(E_2) = 0.98$

$P(G/E_2) = 0.95$

$P(D/E_1) = 0.94$

$P(D/E_2) = 1 - 0.95 = 0.05$

$P(G/E_1) = 1 - 0.94 = 0.06$

$P(E_1/D) = \frac{P(E_1) \times P(D/E_1)}{P(E_1) \times P(D/E_1) + P(E_2) \times P(D/E_2)}$

$= \frac{0.02 \times 0.94}{0.02 \times 0.94 + 0.98 \times 0.05}$

$= \frac{0.0188}{0.0188 + 0.049}$

$= \frac{0.0188}{0.0678}$

$= 0.2772$

2016 Fall

1(a) soln

T_0 = event that signals is transmitted as 0

T_1 = " " " " " as 1

R_0 = " " " " received as 0

R_1 = " " " " " as 1

$P(T_0) = 0.45$

$P(T_1) = 0.55$

$P(R_0/T_0) = 0.94 \Rightarrow P(R_1/T_0) = 0.06$

$P(R_1/T_1) = 0.91 \Rightarrow P(R_0/T_1) = 0.09$

(i) $P(R_1) = P(T_0 \cap R_1) + P(T_1 \cap R_1)$

$= P(T_0) \times P(R_1/T_0) + P(T_1) \times P(R_1/T_1)$

$=$

$=$

(ii) $P(R_0) = P(T_0 \cap R_0) + P(T_1 \cap R_0)$

$= P(T_0) \times P(R_0/T_0) + P(T_1) \times P(R_0/T_1)$

$=$

(iii) $P(T_1/R_1) = \frac{P(T_1 \cap R_1)}{P(R_1)}$

$= \frac{P(T_1) \times P(R_1/T_1)}{P(R_1)}$

$=$

Random variable

A variable whose values are determined by out comes of a random experiment is called random variable. It is denoted by X, Y ....

Types of random variable.

i) Discrete random variable

A random variable whose takes integer value is called discrete random variable. for eg: number of family member in certain village, no. of student in class.

ii) Continuous random variable;

A variable which takes a value within certain interval is called continuous random variable. eg: height, weight, temperature etc

Probability mass function;

A function $p(x)$ or $P(X=x)$ is said to be probability mass function. If it satisfies following condition.

(i) $0 \leq p(x) \leq 1$ for all x

(ii) $\sum_{x} p(x) = 1$

Q. Check ~~weather~~ ~~whea~~ whether following function is pmf or not.

$P(x) = x/6, \ x = 1, 2, 3, 4.$

Here,

X	P(x)
1	1/6
2	2/6
3	3/6
4	4/6

$\sum_{x} P(x)$

$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6}$

$= \frac{10}{6} = 1.6667$ where is not equal to 1

Here, the given function is not pmf.

* Probability Density Function.

A function $f(x)$ ~~or~~ of continuous random variables is said to be pdf if it satisfies following condition.

(i) $0 \leq f(x) \leq 1$ for all x

(ii) $\int_{x} f(x)\,dx = 1$

(iii) $P(a < x < b) = P(a < x \leq b)$

$= P(a \leq x < b) = P(a \leq x \leq b)$

$= \int_{a}^{b} f(x)\,dx$

2. Check whether the given function is pdf.

$f(x) = \frac{x}{6}; \ 0 < x < 1$

Sol^n

$P(0 < x < 1/2) = \int_{0}^{1/2} f(x)\,dx$

$= \int_{0}^{1/2} \frac{x}{6}\,dx$

$= \frac{1}{6}\left[\frac{x^2}{2}\right]_{0}^{1/2}$

$= \frac{1}{6\times} \times (1/2)^2$

$= \frac{1}{12} \times 1/4$

$= \frac{1}{48}$

Now,

$\int_x f(x)\,dx$

$= \int_0^1 \frac{x}{6}\,dx$

$= \frac{1}{6}\left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{6\times 2}\left[1\right]^2$

$= \frac{1}{12} \neq 1$

So, the given function is not pdf.

Q. Consider a coin is tossed three times. (i) Find the probability distribution of no. of tails (ii) Find the probability of (a) no tails (b) three tails (c) Atleast one tails (d) At most 2 tails (e) at least 2 heads (f) at most one head.

Soln

The possible outcomes are leasted below:

(i)

outcome		x - no. of tails
HHH	→	0
HTT	→	2
HTH	→	1
HHT	→	1
TTT	→	3
THH	→	1
THT	→	2
TTH	→	2

Probability distn of random variable is given by,

x	0	1	2	3
p(x)	1/8	3/8	3/8	1/8

(ii)

(a) p(no tails)

$= p(x=0)$

$= 1/8$

(b) p(3 tails)

$= p(x=3)$

$= 1/8$

(c) p(at least 1 tail)

$= p(X \geq 1)$

$= p(x=1) + p(x=2) + p(x=3)$

$= 3/8 + 3/8 + 1/8 = 7/8$

OR

$P(x \geq 1)$
$= 1 - P(x<1)$
$= 1 - P(x=0)$
$= 1 - 1/8$
$= 7/8$

(d) P(at most 2 tail)

$= P(x \leq 2)$
$= 1 - P(x>2)$
$= 1 - P(x=3)$
$= 1 - 1/8$
$= 7/8$

OR $P(x \leq 2)$
$= P(x=0) + P(x=1) + P(x=2)$
$= 1/8 + 3/8 + 3/8$
$= \frac{7}{8}$

(e) P(at least 2 head)
$= P$(at most 1 tail)
$= P(x \leq 1)$
$= P(x=0) + P(x=1)$
$= 1/8 + 3/8$
$= 4/8 = 1/2$

(f) P(at most 1 head)

$= P$(at least 2 tails)
$= P(x \geq 2)$
$= P(x=2) + P(x=3)$
$= 3/8 + 1/8$
$= 4/8 = 1/2$

Numerical:

(1) A rv x has following probability distn

X	0	1	2	3	4
P(x)	0.2	0.3	K	0.1	0.15

Find
(i) the value of K
(ii) $P(X<1)$, $P(X \geq 2)$, $P(X \geq 1)$ and $P(X=2)$
(iii) Expected value and variable of rv x.
(iv) $E[5X]$, $E[3X+2]$, $V[2X]$, $V[3X+6]$

mathematical expectation.
OR
Expected value of rv
OR
mean of rv
OR
Average of rv.

⇒ Let x be rv having pmf/pdf $p(x)$ | $f(x)$
Then expected value of rv X, denoted by $E(X)$, is given by -
$E(x) = \sum_x x p(x)$ if x is discrete rv

$\int_x x f(x)\,dx$ if X is continuous rv.

Properties of Expectation.

① Expected value of constant is always constant.

$E(\text{constant}) = \text{constant}$

For,

$E(a) = a$

$E(5) = 5$

Proof:

Here,

$E(a)$.

$= \sum_x a \cdot p(x)$

$= a \sum_a p(x)$

$= a \times 1$

$= a$

② ~~Expected value of a~~

$E[ax] = aE[x]$

For eg:

$E[3x] = 3E[x]$

③ $E[x+y] = E[x] + E[y]$

4) $E[ax+b]$

$= E[ax] + E[b]$

$= aE[x] + b$

⑤ If X and Y are independent rvs, then

$\boxed{E[XY] = E[X] \cdot E[Y]}$

Variance of rv

Let x be a rv having pmf/pdf $p(x)$/$f(x)$. Then variance of rv X, denoted by $V(x)$, is given by

$V(x) = E[X - E(x)]^2$

$\boxed{V(x) = E(X^2) - [E(x)]^2}$

where,

$E(x^2) = \begin{cases} \sum x^2 p(x) \\ \int x^2 f(x)\,dx \end{cases}$

properties of variance:

① $V(\text{constant}) = 0$

For,

$V(0) = 0$

$V(5) = 0$

(2) $V[ax] = a^2 V[X]$

For eg:

$V(3X) = 9 \cdot V(x)$

(3) $V[X+Y] = V[X] + V[Y] - 2COV(x,y)$

where,

$cov(x,y)$ = covariance betwn x and Y.

(4) $V[ax+b]$

$= a^2 V(x)$.

(i) sol^n

we have

$$\sum_{x=0}^{4} P(x) = 1.$$

or, $P(0) + P(1) + P(2) + P(3) + P(4) = 1$

or $0.2 + 0.3 + K + 0.1 + 0.15 = 1$

or $0.75 + K = 1$

$\therefore K = 0.25$

① $P(x<1)$

$= P(0)$

$= 0.2$

(ii) $P(x>2)$

$= P(3) + P(4)$

$= 0.1 + 0.15$

$= 0.25$

(iii) $P(x \geq 1)$

$= 1 - P(x<1)$

$= 1 - P(0)$

$= 1 - 0.2$

$= 0.8$

(iv) $P(x=2) = K = 0.25$

(iii) Expected value of r.v X is given by,

$$E(X) = \sum_{x} xP(x)$$

$$= \sum_{x=0}^{4} xP(x)$$

$= 0 \times P(0) + 1 \times P(1) + 2 \times P(2) + 3 \times P(3) + 4 \times P(4)$

$= 0 \times 0.2 + 1 \times 0.3 + 2 \times 0.25 + 3 \times 0.1 + 4 \times 0.15$

$= 1.65$

and

$E(x^2) = \sum_x x^2 p(x)$

$= 0^2 \times p(0) + 1^2 \times p(1) + 2^2 \times p(2) + 3^2 \times p(3) + 4^2 \times p(4)$

$= 0 \times 0.2 + 1 \times 0.3 + 4 \times 0.25 + 9 \times 0.1 + 16 \times 0.15$

$= 0 + 0.3 + 1 + 0.9 + 2.50$

$= 4.7$

we have,

$V(x) = E(x^2) - [E(x)]^2$

$= 0.7 - (1.65)^2$

$= 1.97$

(iv) $E[5X]$

$= 5E(X)$

$= 5 \times 1.65$

$= 8.25$

$E[3X + 2]$

$= 3E(X) + 2$

$= 3 \times 1.65 + 2$

$= 6.95$

$V(2X)$

$= 2^2 V(X)$

$= 4 \times 1.97$

$= 7.88$

$V(3X + 6)$

$= 3^2 V[X] + 0$

$= 9V(X)$

$= 9 \times 1.97$

$= 17.73$

A rv X has following probability density function $f(x) = Ax$; $0 < x < 2$

Find

(i) value of A

(ii) $P(X < 1)$

(iii) $P(X > 1.5)$

$P(1 < X < 1.75)$

(iii) mean and variance of rv X

(iv) $E(3X)$ $V(3X)$

$E(3X + 2)$ $V(3X + 2)$

Soln

(i) we have

$\int_x f(x)\,dx = 1$

or $\int_0^2 Ax\,dx = 1$

or, $A\left[\frac{x^2}{2}\right]_0^2 = 1$

or, $\frac{A}{2}[4-0] = 1$

$A = 1/2$

$\therefore f(x) = x/2; \quad 0<x<2$

(ii) $P(X<1)$

$P(0<X<1)$

$= \int_0^1 f(x)\,dx$

$= \int_0^1 Ax\,dx$

$= A\left[\frac{x^2}{2}\right]_0^1$

$= \frac{A}{2}[1-0]$

$= A/2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\times$ $P(X>1.5)$

$= P(1.5<X<2)$

$= \int_{1.5}^{2} f(x)\,dx$

$= \int_{1.5}^{2} Ax\,dx$

$= A\left[\frac{x^2}{2}\right]_{1.5}^{2}$

$= \frac{A}{2}(4-2.25)$

$= \frac{1}{4}(1.75)$

$= 0.4375.$

$\times$ $P(1 \le X \le 1.75)$

$= \int_1^{1.75} f(x)\,dx$

$= A\left[\frac{x^2}{2}\right]_1^{1.75}$

$= \frac{A}{2}(3.0625-1)$

$= \frac{1}{4} \times 2.0625$

$= 0.5156$

(ii) mean of rv X is given by

$E(X) = \int_x x f(x)\,dx$

$= \int_0^2 x \cdot \frac{x}{2} dx$

$= \frac{1}{2}\int_0^2 x^2 dx$

$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$

$= \frac{1}{2\times 3}\left[2^3 - 0\right]$

$= \frac{8}{6} = \frac{4}{3}$ #

(iv) ~~E(3X)~~

And

$E(X^2) = \int_x x^2 f(x) dx$

$= \int_0^2 x^2 \cdot \frac{x}{2} dx$

$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2$

$= \frac{1}{8}\left[16 - 0\right]$

$= 2$ #

we have,

$V(X) = E(X^2) - [E(X)]^2$

$= 2 - \left(\frac{8}{6}\right)^2$

$= 2 - \frac{64}{36}$

$= 0.77$

(v) $E(3X) = 3E[X]$

$= 3 \times \frac{4}{2} = 4$ #

$E[3X+2] = 3E(X)+2 = 3\times\frac{8}{6}+2$

$= 4+2 = 6$ #

$V[3X] = 3^2 V(X) = 9\times 0.22$

$= 1.98$

$V[3X+2] = 3^2 V(X) + 0 = 9 * 0.22$

$= 1.98$

(i) $\int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$

(ii) $\Gamma n = (n-1)!$

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2 (b)

A random variable has the probability density function $f(x) = \frac{1}{2} e^{-x/2}$, $x > 0$. Find the mean and varience.

Sol^n

mean is given by

$$E(x) = \int_x x f(x) dx$$

$$= \int_0^{\infty} x \frac{1}{2} e^{-x/2} dx.$$

$$= \frac{1}{2} \int_0^{\infty} x e^{-x/2} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-x/2} x^{2-1} dx$$

$$= \frac{1}{2} \frac{\Gamma 2}{(1/2)^2}$$

$$= \frac{1}{2} \times (2-1)! \times 4$$

$$= 1/2 \times 1 \times 4$$

$$= 2$$

#

$$E(x^2) = \int_x x^2 f(x) dx.$$

$$= \int_0^{\infty} x^2 \, 1/2 \, e^{-1/2 x} dx.$$

$$= \frac{1}{2} \int_0^{\infty} x^2 e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-\frac{1}{2}x} x^{3-1} dx$$

$$= \frac{1}{2} \cdot \frac{\Gamma 3}{(1/2)^3}$$

$$= \frac{1}{2} \times (3-1)! \times 8$$

$$= \frac{1}{\cancel{2}} \times 2! \times \cancel{8}\,4$$

$$= 8$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= 8 - (2)^2$$

$$= 8 - 4 = 4$$

#

Bivariate probability distribution

Discrete bivariate probability distribution

(i) Joint probability mass function (JPMF)

Let X and Y be two discrete random variables. Then a function $p(x,y)$ is said to be JPMF if it satisfies following condition.

(a) $0 \le p(x,y) \le 1$ for all x and y

(b) $\sum_x \sum_y p(x,y) = 1$

(ii) Marginal PMF

Let x and y be 2 discrete random variable with JPMF $p(x,y)$ then marginal pmf of X is given by -

$$p(x) = \sum_y p(x,y)$$

and marginal pmf of Y is given by

$$p(y) = \sum_x p(x,y)$$

(iii) Conditional pmf

Let X and Y be two discrete random variable with JPMF $p(x,y)$ then conditional pmf of x given y is given by

$$p(x/y) = \frac{p(x,y)}{p(y)} \;;\quad p(y) \neq 0$$

and similarly

$$p(y/x) = \frac{p(x,y)}{p(x)} \;;\quad p(x) \neq 0$$

Note: If X and Y are independent discrete random variable then

$$\boxed{p(x) \cdot p(y) = p(x,y)}$$

Continuous bivariate probability distribution

(i) Joint probability density function (JPdf)

A function $f(x,y)$ is said to be JPdf if it satisfies following condition

(a) $0 \le f(x,y) \le 1$ for all x and y

(b) $\int_x \int_y f(x,y)\, dy\, dx = 1$

(ii) Marginal pdf

Let X and Y be two continuous rvs with JPdf $f(x,y)$. Then marginal pdf of X is given by

$$f(x) = \int_y f(x,y)\, dy$$

and similarly

$$f(y) = \int_x f(x,y)\, dx$$

(iii) Conditional pdf

let X and Y be two continuous rvs with JPdf $f(x,y)$. Then conditional pdf of x given y is given by -

$$f(x/y) = \frac{f(x,y)}{f(y)} ; \quad f(y) \neq 0$$

and similarly

$$f(y/x) = \frac{f(x,y)}{f(x)} ; \quad f \neq 0$$

Note: If X and Y are two independent condition continuous random variable then

$$\boxed{f(x) \cdot f(y) = f(x,y)}$$

Joint prob. distn function OR

Joint cummulative distn function

let X and Y be two rvs. Then joint prob distribution function of X and Y is given by

$$F(x,y) = P(X \leq x, Y \leq y)$$

$$= \begin{cases} \sum_{-\infty}^{x} \sum_{-\infty}^{y} P(x,y), & \text{if } x \text{ and } y \text{ are discrete rvs} \\ \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dy\, dx, & \text{if X and Y are continuous rvs} \end{cases}$$

where, $f(x,y)$ = Joint prob. density function

$F(x,y)$ = Joint prob. distn function

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2(b) The joint prob. density function of X and Y is given by

$$f(x,y) = \begin{cases} kxy & \text{if } 0<x<2,\ 0<y<4 \\ 0 & \text{else where,} \end{cases}$$

Find

(i) the value of k

(ii) marginal pdf of X and that of Y

(iii) conditional pdf of X given Y

(iv) conditional pdf of Y given X

(v) Are they independent rvs?

(vi) Joint prob. distribution function of X and Y.

(vii) $E(X), E(Y), V(X), V(Y)$.

Sol^n

(i) By defn of JPdf

$$\int_x \int_y f(x,y)\,dy\,dx = 1$$

$$\int_x \int_y kxy\,dy\,dx = 1$$

or $k\int_0^2 \int_0^4 xy\,dy\,dx = 1$

or $k\int_0^2 x\left[\frac{y^2}{2}\right]_0^4 dx = 1$

or, $\frac{k}{2}(16-0)\left[\frac{x^2}{2}\right]_0^2 = 1$

or, $\frac{k}{\cancel{2}} \times 16 \times \cancel{2} = 1$

$\therefore k = \frac{1}{16}$

(ii) marginal pdf of x is

$$f(x) = \int_y f(x,y)\,dy$$

$$= \int_0^4 \frac{1}{16}xy\,dy$$

$$= \frac{x}{16}\left[\frac{y^2}{2}\right]_0^4$$

$$= \frac{x}{\cancel{16}} \times \frac{\cancel{16}}{2} = x/2$$

$$\boxed{f(x) = \frac{x}{2}}\ ;\quad 0<x<2$$

and marginal pdf of y is.

$$f(y) = \int_x f(x,y)\,dx$$

$$= \int_0^2 \frac{1}{16} xy\,dx$$

$$= \frac{y}{16}\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{y}{8}$$

$$\therefore \boxed{f(y) = \frac{y}{8}} \quad 0 < y < 4$$

(iii) Conditional pdf of X given y is

$$f(x/y) = \frac{F(x,y)}{f(y)}$$

$$= \frac{\frac{1}{16}xy}{y/8}$$

$$= \frac{xy}{\cancel{16}_2} \times \frac{\cancel{8}}{y}$$

$$= x/2$$

$$\therefore f(x/y) = x/2$$

(iv). Conditional pdf of Y given X is

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{xy}{16}}{x/2}$$

$$= \frac{xy}{16} \times \frac{2}{x}$$

$$= \frac{y}{8}$$

(v) Here,

$$f(x) \cdot f(y)$$

$$= \frac{x}{2} \times y/8$$

$$= \frac{xy}{16} = f(x,y)$$

Hence, they are independent.

(vi) joint prob. distn function is given by.

$$F(x,y) = P(X \le x, Y \le y)$$

$$= \int_0^x \int_0^y f(x,z)\,dz\,dx$$

$$= \int_0^x \int_0^y \frac{1}{16} xy \, dy \, dx$$

$$= \frac{1}{16} \int_0^x x \left[\frac{y^2}{2}\right]_0^y dx$$

$$= \frac{y^2}{32} \int_0^x x \, dx$$

$$= \frac{y^2}{32} \left[\frac{x^2}{2}\right]_0^x$$

$$\boxed{F(x,y) = \frac{x^2 y^2}{64}} \; ; \quad 0 < x < 2, \; 0 < y < 4$$

(vii) We have,

$$E(x) = \int_x x f(x) \, dx$$

$$= \int_0^2 x \cdot \frac{x}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2} \left[\frac{2^3}{3}\right]$$

$$= 4/3$$

and

$$E(X^2) = \int_x x^2 f(x) \, dx$$

$$= \int_0 x^2 \cdot \frac{x}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4}\right]_0^2 = 2$$

Then,

$$V(x) = E(x^2) - [E(x)]^2$$

$$= 2 - (4/3)^2$$

$$= 2 - \frac{16}{9}$$

$$= 2/9$$

$$E(y) = \int_y y f(y) \, dy$$

$$= \int_0^4 y \cdot y/8 \, dy$$

$$= \frac{1}{8} \left[\frac{y^3}{3}\right]_0^4$$

$$= \frac{1}{8} \times \frac{64}{3}$$

$$= 8/3$$

and $E(y^2) = \int_y y^2 f(y)\,dy$

$= \int_0^4 y^2 \cdot \frac{y}{8}\,dy$

$= \frac{1}{8}\left[\frac{y^4}{4}\right]_0^4$

$= \frac{1}{8}\left[\frac{256}{4}\right]$

$= 8$

Then,

$V(Y) = E(Y^2) - [E(Y)]^2$

$= 8 - 8/3$

$= 16/3$

Numerical 2

Consider two rvs (X, Y) have following prob distribution.

Y \ X	-1	0	1
0	0.20	K	0
1	0.10	0.15	0.25

Find
(i) value of K
(ii) marginal prob. distn of rv X
(iii) marginal prob. distn of rv Y
(iv) Conditional prob. distn of X given $Y=0$
(v) " " " Y given $X=1$
(vi) $F(0,1)$
(vii) Are they independent rvs
(viii) $E(X)$, $E(Y)$, $V(X)$, $V(Y)$

Soln

(i) $0.20 + 0.10 + 0.15 + K + 0 + 0.25 = 1$
$\therefore K = 0.3$

(ii) Marginal prob. distribution of rvs is given by

~~X -1 0 1~~
~~P(X) 0.20+0~~

X	-1	0	1
P(X)	0.20+0.10 = 0.3	0.3+0.15 = 0.45	0+0.25 = 0.25

(iv) Conditional prob. distn of X given $Y=0$

X	-1	0	1
$P(X/Y=0)$	2/5	3/5	0

We have,

$P(X/Y) = \frac{P(X,Y)}{P(Y)}$

Now,

$$P(-1/y=0) = \frac{P(-1,0)}{P(y=0)} = \frac{0.20}{0.20+0.30+0}$$

$$= \frac{0.20}{0.50}$$

$$= 2/5$$

(vi) $F(0,1)$

$= P(X \le 0, Y \le 1)$

$= 0.20 + 0.30 + 0.10 + 0.15$

$=$

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2 b

Given,

$$f(x,y) = \begin{cases} 2/3\,(x+2y); & 0<x<1,\ 0<y<1 \\ 0; & \text{else where} \end{cases}$$

(i) Marginal pdf of X is given by

$$f(x) = \int_y f(x,y)\,dy$$

$$= \int_0^1 \frac{2}{3}(x+2y)\,dy$$

$$= \left\{ \frac{2}{3} x [y]_0^1 + 2\left[\frac{y^2}{2}\right]_0^1 \right\}$$

$$\boxed{f(x) = \left[\frac{2}{3}(x+1)\right]; \quad 0<x<1}$$

and marginal pdf of y is given by

$$f(y) = \int_x f(x,y)\,dx$$

$$= \int_0^1 \frac{2}{3}(x+2y)\,dx$$

$$= \frac{2}{3}\left\{ \left[\frac{x^2}{2}\right]_0^1 + 2y[x]_0^1 \right\}$$

$$\boxed{f(y) = \frac{2}{3}\left(\frac{1}{2}+2y\right); \quad 0<y<1}$$

(ii) Conditional pdf of X given Y = y is given by

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{2/3\,(x+2y)}{2/3\,(1/2+2y)}$$

$$\boxed{f(x/y) = \frac{(x+2y)}{(1/2+2y)}} ; \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

(iii) Here,

$f(x) \cdot f(y)$

$= \frac{2}{3}(x+1) \cdot \frac{2}{3}(1/2 + 2y)$

$= \frac{4}{9}(x+1)\left(\frac{1}{2} + 2y\right) \neq f(x,y)$

Hence, they are not independent rvs

(iv) Find

$P(X < 1/2)$

$P(X+Y < 1)$

Soln

$\# P(X < 1/2)$

$= \int_0^{1/2} f(x)\,dx$

$= \int_0^{1} \frac{2}{3}(x+1)\,dx \qquad \left\{ \because f(x) = \frac{2}{3}(x+1) \right\}$

$= \frac{2}{3}\left\{ \left[\frac{x^2}{2}\right]_0^{1/2} + [x]_0^{1/2} \right\}$

$= \frac{2}{3}\left[\frac{1}{8} + \frac{1}{2}\right]$

$= \frac{1}{3}\left[\frac{1}{4} + 1\right]$

$= 5/12$

$\# P(x+y) < 1)$

$= \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$

$= \int_0^1 \left\{ \int_0^{1-x} \frac{2}{3}(x+2y)\,dy \right\} dx$

$= \frac{2}{3}\int_0^1 \left\{ x[y]_0^{1-x} + [y^2]_0^{1-x} \right\} dx$

$= \frac{2}{3}\int_0^1 \left\{ x(1-x) + (1-x)^2 \right\} dx$

$= \frac{2}{3}\int_0^1 (x - x^2 + 1 - 2x + x^2)\,dx$

$= \frac{2}{3}\int_0^1 (1-x)\,dx$

$= \frac{2}{3}\left\{ [x]_0^1 - \left[\frac{x^2}{2}\right]_0^1 \right\}$

$= \frac{2}{3}(1 - 1/2) = \frac{2}{3} \times 1/2 = 1/3$

$\#$

(v) Find the joint distn function of X and Y.

We have,

$F(x,y) = P(X \le x, Y \le y)$

$= \int_0^x \int_0^y f(x,y)\, dy\, dx$

$= \int_0^x \int_0^y \frac{2}{3}(x+2y)\, dy\, dx$

$= \frac{2}{3}\int_0^x 2\left[\frac{y^2}{2}\right]_0^y dx$

$= \frac{2}{6}(x^2y + 2y^2x)$

(vi) Find covariance between X and Y.

We have,

$\text{cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$

Here,

$E(X,Y) = \int_x \int_y xy\, f(x,y)\, dy\, dx$

$= \int_0^1 \int_0^1 xy\; 2/3\,(x+2y)\, dy\, dx$

$= \frac{2}{3}\int_0^1 x \left\{ x\left[\frac{y^2}{2}\right]_0^1 + 2\left[\frac{y^3}{3}\right]_0^1 \right\} dx$

$= \frac{2}{3}\int_0^1 x\left[\frac{x}{2} + \frac{2x}{3}\right] dx$

$= \frac{2}{3}\left\{ \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{2}{3}\left[\frac{x^2}{2}\right]_0^1 \right\}$

$= \frac{2}{3}\left[\frac{1}{6} + 1/3\right]$

$= \frac{2}{3}\left[\frac{2+1}{6}\right]$

$= 1/3$

$E(X) = \int_x x\, f(x)\, dx$

$= \int_0^1 x\, \frac{2}{3}(x+1)\, dx$

$= \frac{2}{3}\int_0^1 (x^2+x)\, dx$

$= \frac{2}{3}\left\{ \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 \right\}$

$= \frac{2}{3}\left[1/3 + 1/2\right]$

$= \frac{2}{3} \times \frac{5}{6} = \frac{5}{9}$

$E(Y) = \int_y y\, f(y)\, dy$

$= \int_0^1 y \cdot \frac{2}{3}\left(1/2 + 2y\right) dy$

$= \frac{2}{3}\int_0^1 \left(\frac{y}{2} + 2y^2\right) dy$

$= \frac{2}{3}\left\{ \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1 + 2\left[\frac{y^3}{3}\right]_0^1 \right\}$

$= \frac{2}{3}\left(1/4 + \frac{2}{3}\right)$

$= \frac{2}{3} \times \frac{11}{12}$

$= \frac{11}{18}$

Then,

$cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

$= 1/3 - \frac{5}{9} \times \frac{11}{18}$

$= 1/3 - \frac{55}{162}$

$=$

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5(a) $p(x,y) = \frac{x+y}{21}$ $X = 1,2,3$

$Y = 1,2.$

y \ x	1	2	3
1	2/21	3/21	4/21
2	3/21	4/21	5/21

(i) marginal probability dist^n of X is:

X	1	2	3
P(x)	$\frac{2}{21} + \frac{3}{21} = \frac{5}{21}$	$\frac{3}{21} + \frac{4}{21} = \frac{7}{21}$	$\frac{4}{21} + \frac{5}{21} = \frac{9}{21}$

We have,

$P(x) = \sum_y p(x,y)$

and marginal probability dist^n of Y is given by

$P(y) = \sum_x p(x,y)$

Y	1	2
P(y)	$\frac{2}{21} + \frac{3}{21} + \frac{4}{21} = \frac{9}{21}$	$\frac{3}{21} + \frac{4}{21} + \frac{5}{21} = \frac{12}{12}$

(ii) Conditional prob. dist^n of X given Y = 2 is obtained by.

$$p(X = x/y = 2) = \frac{p(X = x, Y = 2)}{p(Y = 2)}$$

X	1	2	3
$p(X = x/y = 2)$	$\frac{3/21}{12/21} = \frac{1}{4}$	$\frac{4/21}{12/21} = \frac{1}{3}$	$\frac{5/21}{12/21} = \frac{5}{12}$

(iii) $$V\left(\frac{X = x}{y = 2}\right) = E\left[\frac{x^2}{y = 2}\right] - \left[E\left(\frac{Y}{Y = 2}\right)\right]^2$$

Now,

$$E[x^2/y = 2] = \Sigma x^2 \cdot p\left(\frac{x}{y} = 2\right)$$

$$= (1)^2 \times \frac{1}{4} + (2)^2 \times 1/3 + (3)^2 \times \frac{5}{22}$$

$$= \frac{16}{3}$$

and

$$E\left(\frac{X}{y=2}\right) = \sum_x x \cdot p(x/y = 2)$$

$$= 1 \times \frac{1}{4} + 2 \times 1/3 + 3 \times \frac{5}{12}$$

$$= \frac{3 + 8 + 15}{12} = \frac{26}{12}$$

Theoritical probability distribution:

Discrete prob. distⁿ	Continous prob. distⁿ
(i) Binomial distⁿ	(i) Normal distⁿ.
(ii) poisson distⁿ	(ii) Gamma distⁿ.
	(iii) Beta distⁿ.
	(iv) exponential distⁿ.

Binomial distribution

conditions to apply binomial distribution.

(i) No. of trials (n) is finite and independent
ii) Each trial has only two possible outcomes.
iii) probability of success in each trial (p) remains constant. Consequently prob. of failure in each trial (q) remains constant i.e.

$$p + q = 1$$

Defⁿ of Binomial Distribution.

A discrete random variable is said to follow binomial distribution with parameters (n, p) if its pmf is given by

$$\boxed{p(X = x) = nC_x\, p^x q^{n-x}}$$ where, $x = 0, 1, 2, \ldots n$

n = no. of trials.
p = prob. of sucess in each trial.
q = " " failure " " "
x = no. of success (es)

Notation:
$X \sim B(n, p)$

Mean and variance:

let $X \sim B(n, p)$. Then

$\boxed{mean = E(X) = np}$

$\boxed{Variance = V(X) = npq}$

Q. If a coin is tossed 10 times, what is the prob. of getting.

(i) NO heads
(ii) one heads
(iii) two heads
(iv) at least one head
(v) at must two heads
(vi) two tails
(vii) one tail
(viii) at least one tail
(ix) ~~at least one tail~~

(ix) at most two tails.

sol^n

no. of trials (n) = 10

let X = no. of heads

prob of head in each trial (p) = $\frac{1}{2}$
" " tail " " " (q) = $\frac{1}{2}$

We have,
$p(X = x) = {}^nC_x p^x q^{n-x}$

or $p(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$

$\boxed{\therefore p(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10}}$, $x = 0, 1, 2 \cdots 10$

(i) p(no head)

i.e $p(X=0) = {}^{10}C_0 \left(\frac{1}{2}\right)^{10}$
$= \frac{1}{1024}$

(ii) p(one head)

i.e $p(X=1) = {}^{10}C_1 \left(\frac{1}{2}\right)^{10}$
$= 10 \times \frac{1}{1024} = \frac{10}{1024}$

iii) p (two head)

i.e.

$P(X=2) = {}^{10}C_2 \left(\frac{1}{2}\right)^{10}$

$= \frac{45}{1024}$

(iv) $p(X \geq 1)$

$= 1 - P(X<1)$

$= 1 - \frac{1}{1024}$

$= \frac{1023}{1024}$

(v) $p(X \leq 2)$

$= P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{1024} + \frac{10}{1024} + \frac{45}{1024}$

$= \frac{56}{1024}$

#

(vi) prob (two tails)

= prob (eigth heads)

$= P(X=8)$

$= {}^{10}C_8 \left(1/2\right)^{10}$

$= \frac{45}{1024}$

(vii) p (one tail)

= p (9 head)

$= p(X=9)$

$= {}^{10}C_9 \left(\frac{1}{2}\right)^{10}$

$= \frac{10}{1024}$

viii) p (at least one tail)

= p (at most 9 heads)

$= p(X \leq 9)$

$= 1 - P(X>9)$

$= 1 - P(X=10)$

$= 1 - \frac{1}{1024}$

$= \frac{1023}{1024}$

(ix) p (at most 2 tails)

= p (at least 8 heads)

$= p(X \geq 8)$

$= p(X=8) + p(X=9) + p(X=10)$

$= \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$

$= \frac{56}{1024}$

2015 Fall

3(c) An information source emits a ten digit message into a channel in binary code. Each digit is chosen independently and is one with prob. 0.3. Find the prob. that message contains.

(i) Three ones

(ii) Between 2 and 4 ones (inclusive)

(iii) No less than 2 zeroes

Soln

let X = no. of 1 digit in message.

Given

no. of digits = 10

prob. of one (p) = 0.30

prob. of zero (q) = 1 − p = 0.70

we have,

$P(X=x) = {}^{10}C_x (0.30)^x (0.70)^{10-x}$

(i) prob (three one's)

$= P(X=3)$

$= {}^{10}C_3 (0.30)^3 (0.70)^7$

$= 0.266$

(ii) prob $(2 \le X \le 4)$

$= P(X=2) + P(X=3) + P(X=4)$

$= 0.699$

iii) prob (no less than 2)

= prob (at least 2 zero)

= prob (at most 9 1's)

$= P(X \le 8)$

$= 1 - P(X > 8)$

$= 1 - [P(X=9) + P(X=10)]$

$= 1 - [{}^{10}C_9 (0.3)^9 (0.7) + {}^{10}C_{10} (0.3)^{10}]$

$= 0.99868$

2011 Fall

4(a) During one stage in the manufacture of integrated chips, a coating must be applied. If 70% of chips receives a thick coating. Find the probability that among 15 chips.

(i) At least 12 will have thick enough coatings

(ii) At most 6 will have thick enough coatings

(iii) Exactly 10 will have thick enough coatings

Soln

X = no. of chips that received a thick coating.

Given,

no. of chips (n) = 15

prob. of thick coating (p) = 0.70

prob. of not thick coating (q) = 0.30

we have,

$P(X=x) = {}^{15}C_x (0.70)^x (0.30)^{15-x}$

$x = 0, 1, 2 \cdots 15$

(i) $P(X \ge 12)$

$= P(X=12) + P(X=13) + P(X=14) + P(X=15)$

$= [{}^{15}C_{12} (0.7)^{12} (0.3)^3] + [{}^{15}C_{13} (0.7)^{13} (0.3)^2]$

$+ [{}^{15}C_{14} (0.7)^{14} (0.3)] + [{}^{15}C_{15} (0.7)^{15}]$

$= 0.2968$

(ii) prob $(X \leq 6)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$
$+ P(X=5) + P(X=6)$

$=$

(iii) $P(X=10)$

$= {}^{10}C_{10}\ (0.70)^{10}\ (0.30)^{15}$

$=$

Poisson distribution

Assumptions / Conditions to apply poisson distn.

(i) $n \to \infty$
(ii) $p \to 0$
iii) λ is finite

→ No of trials (n) is indefinitely large i.e $(n \to \infty)$
→ prob. of success is very small i.e $(p \to 0)$
→ λ is finite (Avg occurrance of event per unit time $(\lambda) = np$ is fine)

Def^n:

A discrete random variable X is said to follow poisson distn, if its pmf is given by

$$\boxed{P(X = x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}}, \quad x = 0, 1, 2, 3 \ldots$$

Notation: $X \sim P(\lambda)$

Mean and variance

$$\boxed{\begin{array}{l} \text{Mean} = E(X) = \lambda \\ \text{variance} = V(X) = \lambda \end{array}}$$

Note: It is noted that mean and variance of poisson distn are equal.

Q) Average no. of customers arriving at certain bank per hour is 2. Find the prob that there is -

(i) No customers in 1 hour
(ii) 1 " " 1 hour
(iii) 2 " " " "

(iv) at least 2 customers in 1 hour.
(v) at most 1 customers in 1 "
(vi) 2 customers in 2 hour.
(vii) at least one customer in 2 hour.

Solⁿ
let X = no. of customers arrives per hour
given,
$\lambda = 2$ customers per hour.
we have,

$$\boxed{P(X=x) = \frac{e^{-2}\, 2^x}{x!}} \quad x = 0,1,2 \ldots$$

(i) p(no customer per hour)

$= P(X=0)$

$= \frac{e^{-2}\, 2^0}{0!}$

$= \frac{e^{-2} \times 1}{1}$

$= e^{-2}$

(ii) prob(1 customer per hour)

$= P(X=1)$

$= \frac{e^{-2}\, 2^1}{1!}$

$= 2e^{-2}$

(iii) prob (at least 2 customers per hour)

$= P(X \geq 2)$

$= 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [e^{-2} + 2e^{-2}]$

$= 1 - 3e^{-2}$

(v) prob. (at most 1 customer per hour)

$= P(X \leq 1)$

$= P(X=0) + P(X=1)$

$= e^{-2} + 2e^{-2}$

$= 3e^{-2}$

Again, $X = 2 \times 2 = 4$ customers in two hours

we have,

$$\boxed{P(X=x) = \frac{e^{-4} \cdot 4^x}{x!}}, \quad x = 0,1,2,3 \ldots$$

(vi) prob. (2 customers in two hrs)

$= P(X=2)$

$= \frac{e^{-4}\, 4^2}{2!}$

$= \frac{e^{-4}\, 4^2}{2}$

vii) p (at least 1 customer in 2 hours)

$= p(X \geq 1)$

$= 1 - p(X=0)$

$= 1 - \frac{e^{-4}\, 4^0}{0!}$

$= 1 - e^{-4}$

viii) prob (1 customer per hour)

$= p(X=1)$

$= \frac{e^{-2} \cdot 2^1}{1!}$

$= 2e^{-2}$

2015 Fall

3(b) An office switch board receives an average of 4 emergency calls in a 10 minute interval. What is the prob. that

(i) there is no call in a 10 min interval

(ii) there is at most 2 calls in 5 minutes intervals?

Soln

let X = no. of emergency calls in 10 min.

Given,

$\lambda = 4$ emergency calls in 10 min.

we have,

$p(X=x) = \frac{e^{-4}\, 4^x}{x!}$ $x = 0, 1, 2, 3 \ldots$

(i) p (no call)

$= p(X=0)$

$= \frac{e^{-4}\, 4^0}{0!}$

$= \frac{e^{-4} \times 1}{1} = e^{-4}$

Again, $\lambda = 2$ emergency calls in 5 min

$p(X=x) = \frac{e^{-2}\, 2^x}{x!}$

(ii) p (at most 2)

$= p(X \leq 2)$

$= p(X=0) + p(X=1) + p(X=2)$

Q2 It is known that 5% of items manufactured by a company are defective. If a sample of 10 items is selected at random, find the prob. that sample contain.

(i) exactly 2 defective items.
(ii) at least 2 " "
(iii) at most 1 defective item.

soln.

Given,

no. of items $(n) = 20$

prob. of defective in a sample $(p) = 5\%$
$= 0.05$

Here,

mean defective per sample $(\lambda) = np =$
$= 20 \times 0.05$
$= 1$

we have,

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$\therefore \boxed{P(X=x) = \frac{e^{-1} \cdot 1^x}{x!}} \quad x = 0, 1, 2, 3 \ldots$$

(i) $P(X=2)$

$= \frac{e^{-1} \cdot 1^2}{2!}$

$= \frac{e^{-1}}{2}$

(ii) $P(X \geq 2)$

$= 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[\frac{e^{-1} \cdot 1^0}{0!} + \frac{e^{-1} \cdot 1^1}{1!}\right]$

$= 1 - [e^{-1} + e^{-1}]$

$= 1 - 2e^{-1}$

(iii) $P(X \leq 2)$

$= P(X=0) + P(X=1)$

$= \frac{e^{-1} \cdot 1^0}{0!} + \frac{e^{-1} \cdot 1^1}{1!}$

Data is said to be follow normal distribution if
(i) mean, median and mode are equal
(ii) curve of ~~norma~~ normal distribution is bell shaped
(iii) curve of normal distribution is symmetric.

Defn:
A continuous random variable X is said to following normals distn if its probability density function is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$

where, μ = mean
σ = S.D

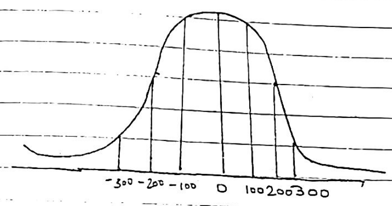
Notation:
$X \sim N(\mu, \sigma^2)$

If $X \sim N(\mu, \sigma^2)$, then SNV is defined as

$$Z = \frac{X-\mu}{\sigma}$$

Here
$E(Z) = 0$
$V(Z) = 1$

$\therefore Z \sim N(0,1)$



$\therefore P(-1 < z < 1) = 0.6826$
$P(-2 < z < 2) = 0.9544$
$P(-3 < z < 3) = 0.9974$

Mean and S.D of marks obtained by students are 60 and 10 respectively. Find the probability a student will get marks.

(i) between 60 to 78.
(ii) between 43 to 60.
(iii) greater then 73.
(iv) less then 53.
(v) greater than 42
vi) less than 67
vii) between 67 to 77
viii) bet^n 55 to 58.

Assume that marks follows Normal distribution.

Sol^n

mean $(\mu) = 60$

S.D $(\sigma) = 10$

Let X = mark of students

(i) $\text{prob}(60 < X < 78)$

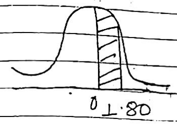
$$= \text{prob}\left(\frac{60-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{78-\mu}{\sigma}\right)$$

$$= \text{prob}\left(\frac{60-60}{10} < \frac{X-60}{10} < \frac{78-60}{10}\right)$$

$$= \text{prob}\left(0 < z < \frac{78-60}{10}\right)$$

$$= P(0 < z < 1.80)$$

$$= 0.4641$$



(ii) $\text{prob}(43 < X < 60)$

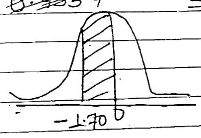
$$= \text{prob}\left(\frac{43-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{60-\mu}{\sigma}\right)$$

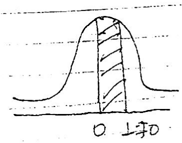
$$= \text{prob}\left(\frac{43-60}{10} < z < \frac{60-60}{10}\right)$$

$$= \text{prob}(-1.70 < z < 0)$$

$$= \text{prob}(0 < z < 1.70)$$

$= \cancel{0.4554}$ $= 0.4554$ (By symmetric property)





(iii) prob $(X > 73)$

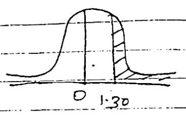
$= \text{prob}\left(\frac{X-\mu}{\sigma} > \frac{73-\mu}{\sigma}\right)$

$= \text{prob}\left(Z > \frac{73-60}{10}\right)$

$= \text{prob}(Z > 1.30)$

$= 0.50 - P(0 < Z < 1.30)$

$= 0.50 - 0.4032$



(iv) prob (less than 53)

$= \text{prob}(X < 53)$

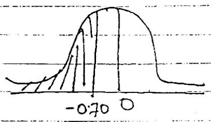
$= \text{prob}\left(\frac{X-\mu}{\sigma} < \frac{53-\mu}{\sigma}\right)$

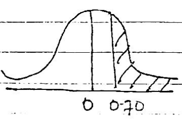
$= \text{prob}\left(Z < \frac{53-60}{\sigma}\right)$

$= \text{prob}(Z < -0.70)$

$= \text{prob}(Z > 0.70)$

By symmetrical property.





(v) prob (greater than 42)

$= \text{prob}(X > 42)$

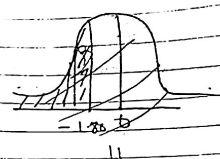
$= \text{prob}\left(\frac{X-\mu}{\sigma} > \frac{42-\mu}{\sigma}\right)$

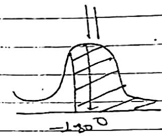
$= \text{prob}\left(Z > \frac{42-60}{10}\right)$

$= P(Z > -1.80)$

$= 0.50 + P(0 < Z < 1.80)$

$= 0.50 + 0.4641$





(vi) less than 67

$= \text{prob}(X < 67)$

$= \text{prob}\left(\frac{X-\mu}{\sigma} < \frac{67-\mu}{\sigma}\right)$

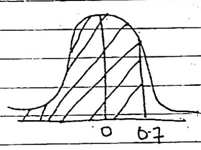
$= \text{prob}\left(Z < \frac{67-60}{10}\right)$

$= \text{prob}(Z < 0.7)$

$= 0.50 + P(0 < Z < 0.7)$

$= (0.50 + 0.2580)$

$= 0.7580$



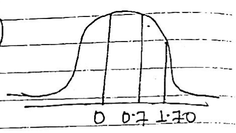
(VII) betⁿ 67 to 77

prob $(67 < X < 77)$

$= \text{prob}\left(\frac{67-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{77-\mu}{\sigma}\right)$

$= \text{prob}\left(\frac{67-60}{10} < Z < \frac{77-60}{10}\right)$

$= \text{prob}(0.7 < Z < 1.70)$



$= \text{prob}(0 < Z < 1.70) - p(0 < Z < 0.70)$

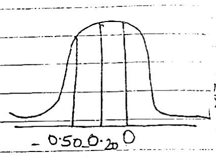
$= 0.4554 - 0.2580$

$= 0.1974$

VIII) betⁿ 55 to 58

$= \text{prob}(55 < X < 58)$

$= \text{prob}\left(\frac{55-60}{10} < Z < \frac{58-60}{10}\right)$

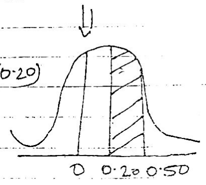


$= \text{prob}(-0.50 < Z < -0.20)$

$= \text{prob}(0.20 < Z < 0.50)$

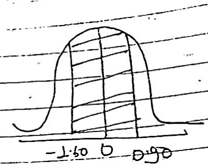
$= \text{prob}(0 < Z < 0.50) - p(0 < Z < 0.20)$

$=$



(IX) betⁿ 44 to 69.

$= p(44 < X < 69)$



$= p\left(\frac{44-60}{10} < Z < \frac{69-60}{10}\right)$

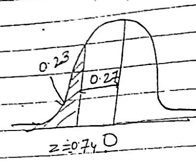
$= p(-1.60 < Z < 0.90)$

$= p(-1.60 < Z < 0) + p(0 < Z < 0.90)$

$= p(0 < Z < 1.60) + p(0 < Z < 0.90)$

$=$

Find the minimum pass mark if 23% of student had failed in exam.



we have,

$Z = \frac{X-\mu}{\sigma}$

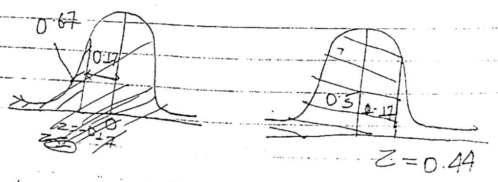
or, $-0.74 = \frac{X-60}{10}$

or, $X = 60 - 0.74 \times 10$

$x = 60 - 7.4$

$x = 52.6$ #

Find the minimum pass mark of 67% of student had failed in exam.



we have,

$Z = \frac{X-\mu}{\sigma}$

or, $\cancel{0.17} =$

or, $0.44 = \frac{X-60}{10}$

$X = \frac{0.44 \times 10}{60}$

$0.44 = \frac{X-60}{10}$

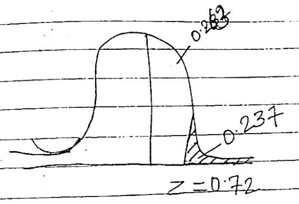
or $X - 60 = 0.44 \times 10$

or $X = 4.4 + 60$

$\therefore X = 64.4$

#

If 23.7% of student got distinction. Find the distinction mark.



we have,

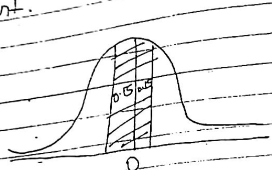
$Z = \frac{X-\mu}{\sigma}$

or $0.72 = \frac{X-60}{10}$

$\therefore X = 67.2$

#

* Find the limits of middle 30% of student.



$z = 0.39$, $z = -0.39$

Now,

$z = \frac{X-\mu}{\sigma}$

or, $0.39 = \frac{X-60}{10}$

or, $X - 60 = 0.39 \times 10$

or, $X - 60 = 3.9$

or, $X = 3.9 + 60$

or, $X = 63.9$

$z = \frac{X-\mu}{\sigma}$

or, $-0.39 = \frac{X-60}{10}$

or, $X - 60 = -3.9$

or, $X = -3.9 + 60$

$X = \cancel{20}\ 56.1$

* Find the minimum mark of top 69% of student.

$z = 0.50$

$z = \frac{X-\mu}{\sigma}$

or, $-0.50 = \frac{X-60}{10}$

or, $-5 = X - 60$

or, $X = 55$

2015 Fall

4(a) Given,

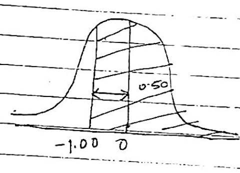
Number of students (N) = 1000

average (μ) = 60

S.D (σ) = 10

Let, X = score

) prob $(X > 50)$



$= P\left(\frac{X-\mu}{\sigma} > \frac{50-\mu}{\sigma}\right)$

$= P\left(Z > \frac{50-60}{10}\right)$

$= P(Z > -1.00)$

$= 0.50 + 0.3413$

$= 0.8413$

$\therefore$ Number of student exceeding 50 mark $= 100 \times 0.8413$
$= 841.3 \approx 841$

(ii) $prob(44 < X < 74)$

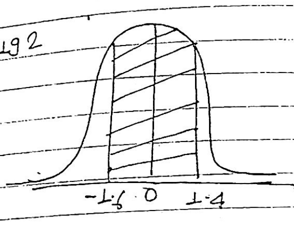
$= prob\left(\frac{44-\mu}{\sigma} < Z < \frac{74-\mu}{\sigma}\right)$

$= prob\left(\frac{44-60}{10} < Z < \frac{74-60}{10}\right)$

$= prob(-1.6 < Z < 1.4)$

$= 0.4452 + 0.4192$

$= 0.8644$



No. of ~~student of~~ students lying between 44 and 74 $= 1000 \times 0.8644$

$= 864.4$

≈ 864 #

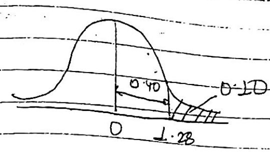
(iii) soln

% of top 100 student $= \frac{100}{1000} \times 100$

$= 10\%$

0.08
0.6
~~0.68~~

From question



we have,

$Z = \frac{X-\mu}{\sigma}$

or, $1.29 = \frac{X-60}{10}$

$X = 60 + 12.9$

$X = 72.9$

#

2013 Spring

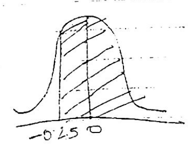
3(a)

$N = 500$

$\mu = 45$

$\sigma = 20$

(i) $P(X > 40)$

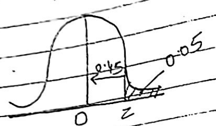


$P\left(Z > \frac{40-45}{20}\right)$

$P(Z > -0.25)$

$= 0.68 + 0.5$

(11) From ~~quest~~ question.



~~Here~~,
$z = 1.65$

we have,

$z = \frac{X-\mu}{20}$

or $1.65 = \frac{X-45}{20}$.

or $X = 21.65 + 45$

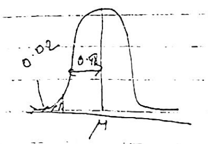
$X = 66.65$

2014 Fall

3(b) $\mu = ?$

$\sigma = 0.04$ ounce

$P(X<4) = 0.02$



$z = -2.00$

we have,

$z = \frac{X-\mu}{\sigma}$

or, $-2.06 = \frac{4-\mu}{0.04}$.

$\therefore \mu = 4.0828$ #

Central limit theorem (CLT)

Whatever be the distribution of population sampling distribution of sample mean is normal as simple size $(n) \to \infty$.

i.e.

If $X \sim N(\mu, \sigma^2)$ then

$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$

Application of CLT

(I) To sum

If $X \sim N(\mu, \sigma^2)$ then

$S_n \sim N(n\mu, n\sigma^2)$ as $n \to \infty$.

and

$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$

(II) To binomial distn

If $X \sim B(n, p)$ then

$S_n \sim N(np, npq)$ as $n \to \infty$

and

$\bar{X}_n \sim N(p, \frac{pq}{n})$ as $n \to \infty$

(III) To poisson distn

If $X \sim P(\lambda)$ then

$S_n \sim N(n\lambda, n\lambda)$ as $n \to \infty$

and

$\bar{X}_n \sim N(\lambda, \frac{\lambda}{n})$ as $n \to \infty$.

Q. A sky lift is designed with 10000 lbs limit. If mean and SD of weight of all person using this left is 190 lbs and 25 lbs respectively. Find the probability that a group of 50 persons exceed load time limit.

Given,

load limit = 10,000

mean (μ) = 190 lbs

S.D (σ) = 25 lbs

variance $(\sigma^2) = (25)^2 = 625$ lbs^2

sample size $(n) = 50$

Let S_n = sum of weights of 50 person.

∴ From CLT

$S_n \sim N(n\mu, n\sigma^2)$ as $n \to \infty$

so,

$E(S_n) = n\mu = 50 \times 190 = 9500$ lbs

$V(S_n) = n\sigma^2 = 50 \times 625 = 31250$ lbs

SD of $S_n = \sqrt{31250} = 176.77$.

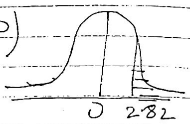
Now,

prob $(S_n > 101000)$

$= \text{Prob}\left[\frac{S_n - E(S_n)}{SD} > \frac{101000 - E(S_n)}{S.D}\right]$

$= \text{Prob}\left(Z > \frac{101{,}000 - 9500}{176.77}\right)$

$= \text{prob}(Z > 2.82)$



$= 0.50 - 0.4976$

$=$

Q In a survey of a company mean salary of employes is 29321 \$ with s.D of 2120\$. consider the ~~sim~~ sample of hundred employeses and find the probability their mean salary will be less ~~ze~~ then 29000\$

Soln

mean $(\mu) = 29321\$$

S.D $(\sigma) = 2120\$$

variance $(\sigma^2) = (2120\$)^2 = 4494400\$$

sample size $(n) = 100$

$\bar{X}_n$ = mean salary of 100 employees.

From CLT.

$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ as $n \to \infty$.

so,

$E(\bar{X}_n) = \mu = 29321$

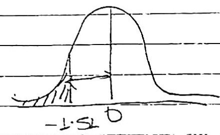
$V(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{(2120)^2}{100} = 44944$

$\Rightarrow$ S.D of $\bar{X}_n = \sqrt{44944} = 212$.

prob $(\bar{X}_n < 29000)$

$= P\left(Z < \frac{29000 - 29321}{212}\right)$

$= P(Z < -1.51)$



$= 0.50 -$

(3) The average score of a subject is 2.89 for the whole class, with a standard deviation of 0.630. If a sample of 25 student is being taken, then find the probability of getting the average of this sample to be more than 3.

Sol^n

mean $(\mu) = 2.89$

$S.D(\sigma) = 0.630$

variance $(\sigma^2) = (0.630)^2 = 0.3969$

Sample size $(n) = 25$

Let $\bar{X}_n$ = average sample more than 3.

From CLT

$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$

So,

$E(\bar{X}_n) = \mu = 2.89$

$V(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{0.3969}{25} = \cancel{0.01478} = 0.015870$

$\Rightarrow$ SD of $\bar{X}_n = \sqrt{0.015876} = 0.126$

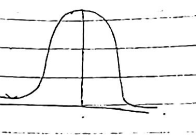
prob $(\bar{X}_n > 3)$

$= P\left(Z > \frac{3 - 2.89}{0.126}\right)$

$= \cancel{P(Z > 0.905)}$

$= 0.50 -$

$= P\left(Z > \frac{3 - 2.89}{0.126}\right)$



$= P(Z > 0.873)$

$= 0.5 - 0.3078$

$= 0.1922.$

(4) An insurance company wants to audit health insurance ~~clam~~ claims in its very large database of transactions. In a quick attempt to assess the level of overstatement of this database, the insurance company selects at random 400 items from the database (each item represents a dollar amount). Suppose that the population mean overstatement of the entire database is \$8, with population

Standard deviation \$20. (c) Find the

Soln

size of sample (n) = 400

$\mu = 8\$$

$\sigma = 20\$$

$\sigma^2 = 400\$^2$

Now

$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Here

$\mu = 8$

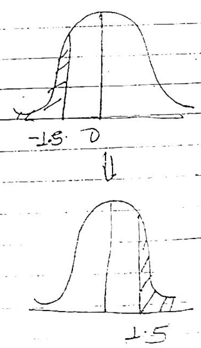
$S.D\left(\frac{\sigma^2}{n}\right) = 1$

Now

Prob $[\bar{X}_n < 6.5]$

$= \text{Prob}\left[Z_n < \frac{6.5-8}{1}\right]$

Prob $[Z_n < -1.5]$



$\therefore$ Prob $(0 > Z_n > 1.5)$

$= 0.5 - 0.4332$

$= 0.0668$

Exponential distribution

Defn:

A continuous rv 'X' is said to follow exponential distribution if its pdf is given by,

$f(x) = \lambda e^{-\lambda x}$

$-\infty < x < \infty$ and $0 < \lambda < \infty$

Mean and variance of exponential distn

Mean $= E(X) = 1/\lambda$ OR $E(X) = \int_0^\infty x f(x)\, dx$

Variance $= V(X) = 1/\lambda^2$ $E(X^2) = \int_0^\infty x^2 f(x)\, dx$

$V(X) = E[X^2] - [E(X)]^2$

Numerical

Suppose that the time (hrs) of a battery is a random variable having exponential distn with $\lambda = \frac{1}{25}$. Find the probability that such battery

(i) will last least 20 hrs

(ii) will last at least 30 hrs

(iii) will last at most 35 hrs

(iv) will operating at least 30 hrs

given that it has already operated at least 20hrs.

Solⁿ

$\lambda = \frac{1}{25}$

$$\boxed{\therefore f(x) = \frac{1}{25} e^{-\frac{1}{25}x}} , 0 < x < \infty$$

let,

X = lifetime (hrs) of a battery.

(i) Prob (X ≥ 20hrs)

$$= \int_{20}^{\infty} f(x)\, dx$$

$$= \int_{20}^{\infty} \frac{1}{25} e^{-\frac{1}{25}x} dx$$

$$= \frac{1}{25} \left[\frac{e^{-\frac{1}{25}x}}{-\frac{1}{25}} \right]_{20}^{\infty}$$

$$= -1 \left[e^{-\infty} - e^{-20/25} \right]$$

$$= -1 \left[0 - e^{-\frac{20}{25}} \right]$$

$$= e^{-\frac{20}{25}}$$

(ii) prob (X ≥ 30hrs)

$$= \int_{30}^{\infty} f(x)\, dx$$

$$= \int_{30}^{\infty} \frac{1}{25} e^{-\frac{1}{25}x} dx$$

$$= \frac{1}{25} \left[\frac{e^{-\frac{1}{25}x}}{-\frac{1}{25}} \right]_{30}^{\infty}$$

$$= -1 \left[e^{-\infty} - e^{-\frac{30}{25}} \right]$$

$$= -1 \left[0 - e^{-30/25} \right]$$

$$= e^{-30/25}$$

(iii) Prob (X ≤ 35hrs)

$$= \int_{0}^{35} f(x)\, dx$$

$= \int_0^{35} \frac{1}{25} e^{-1/25 x} dx$

$= \int_0^{35} \frac{1}{25} \left[\frac{e^{-\frac{1}{25}x}}{-1/25} \right]_0^{35}$

$= \frac{-1}{25} \left[e^{-\frac{35}{25}} - e^0 \right]$

$= 1 - e^{-\frac{35}{25}}$

v) $P(X \geq 30 / X \geq 20)$

$= \frac{P(X \geq 30 \cap X \geq 20)}{P(X \geq 20)}$

$= \frac{P(X \geq 30)}{P(X \geq 20)}$

$= \frac{e^{-30/25}}{e^{-20/25}}$

$= e^{-10/25}$

OR

By memoryless property

~~$P(X \geq 20)$~~

$P(X \geq 30 / X \geq 20)$

$= P(X \geq 10)$

$= \int_{10}^{\infty} f(x) dx$

$= \int_{10}^{\infty} \frac{1}{25} e^{-1/25 x} dx$

$= e^{-10/25}$

Memoryless property of exponential distⁿ:-

For any two intergers 's' and 't'

$P[X \geq (s+t) / X \geq s] = P(X \geq t)$

Let X be a exponential rv with pdf $f(x) = \alpha e^{-\alpha x}$; $0 < x < \infty$

LHS

$P[X \geq (s+t) / X \geq s]$

$$= \frac{P[X \geq (s+t) \cap X \geq s]}{P[X \geq s]}$$

$$= \frac{P[X \geq (s+t)]}{P[X \geq s]}$$

$$= \frac{\int_{(s+t)}^{\infty} f(x)\,dx}{\int_{s}^{\infty} f(x)\,dx}$$

$$= \frac{\int_{(s+t)}^{\infty} \alpha e^{-\alpha x}\,dx}{\int_{s}^{\infty} \alpha e^{-\alpha x}\,dx}$$

$$= \frac{\alpha\left[\frac{e^{-\alpha x}}{-\alpha}\right]_{s+t}^{\infty}}{\alpha\left[\frac{e^{-\alpha x}}{-\alpha}\right]_{s}^{\infty}}$$

$$= \frac{-[e^{-\infty} - e^{-\alpha(s+t)}]}{-[e^{-\infty} - e^{-\alpha s}]}$$

$$= \frac{e^{-\alpha(s+t)}}{e^{-\alpha s}}$$

$$= e^{-\alpha t}$$

R.H.S.

$P(X \geq t)$

$$= \int_{t}^{\infty} f(x)\,dx$$

$$= \int_{t}^{\infty} \alpha e^{-\alpha x}\,dx$$

$$= \alpha\left[\frac{e^{-\alpha x}}{-\alpha}\right]_{t}^{\infty}$$

$$= e^{-\alpha t}$$

$\therefore$ L.H.S = R.H.S; Hence, proved #

Gamma distⁿ

A continuous random variable 'X' is said to follow gamma distⁿ with ~~param~~ parameters 'α' and 'β' if its pdf is given by.

$$F(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} e^{-x/\beta} x^{\alpha-1}$$

, $0 < x < \infty$
$0 < \alpha < \infty$
$0 < \beta < \infty$

Mean and variance

mean = $E(X) = \alpha\beta$
variance = $V(X) = \alpha\beta^2$

In a certain city, the daily consumption of electric power (in million of kw-hours) can be treated as rv having gamma distn with $\alpha = 3$ and $\beta = 2$. If the power plant of this city has a daily capacity of 12 million kw-hours. What is the prob that this power supply will be inadequate on any given day?

Soln let,
X = daily consumption of electricity

Given,
$\alpha = 3$
$\beta = 2$

$\Gamma\alpha = \Gamma 3 = (3-1)! = 2! = 2$

$\beta^\alpha = 2^3 = 8$.

we have,
$$f(x) = \frac{1}{\beta^\alpha \Gamma\alpha} e^{-x/\beta} x^{\alpha-1}$$

So $\boxed{f(x) = \frac{1}{16} e^{-x/2} x^2}$; $0 < x < \infty$

Now,
$P(X > 12)$

$= 1 - P(X \leq 12)$

$= 1 - \int_0^{12} \frac{1}{16} e^{-x/2} x^2 dx$

$= 1 - \frac{1}{16} \int_0^{12} e^{-x/2} x^2 dx$

$= 1 - \frac{1}{16}\left[x^2 \frac{e^{-x/2}}{-1/2} - 2x \cdot \frac{e^{-x/2}}{(-1/2)^2} + \frac{2e^{-x/2}}{(-1/2)^3} \right]_0^{12}$

$= 1 - \frac{1}{16}\left[-2x^2 e^{-x/2} - 8x e^{-x/2} - 16 e^{-x/2} \right]_0^{12}$

$= 1 + \frac{2}{16}\left[x^2 e^{-x/2} + 4x e^{-x/2} + 8 e^{-x/2} \right]_0^{12}$

$= 1 + \frac{1}{8}\left\{ \left[(12)^2 e^{-12/2} + 4 \times 12\, e^{-12/2} + 8 e^{-12/2} \right] - \left[0 + 0 + 8 \times e^0 \right] \right\}$

$= 1 + \frac{1}{8}\left\{ \left[144 e^{-6} + 48 e^{-6} + 8e^{0} \right] - \left[8 \right] \right\}$

$= 1 + 1/8 \left\{ \left[200e^{-6} \right] - 8 \right\}$

$= 1 + \frac{1}{8} \times 200e^{-6} - 8 \times 1/8$

$= \cancel{1 + 1/8} \times$

$= \cancel{1} + 25e^{-6} - \cancel{1}$

$= 25e^{-6}$

$= 0.0619$

3) Light bulbs are installed successively into a socket. Assume that each has a mean life of 2 months with a standard deviation of 1/4 month. Find the prob. that 40 bulbs last at least 7 years. Also find how many bulbs should be bought so that one can 95% sure that supply will last 5 years.

Soln

Given

mean $(\bar{X}) = 2$

SD $(\sigma) = 1/4$

variance $(\sigma^2) = 1/16$.

Let X be a r.v.

$n = 40$

$S_n \sim N(n\mu, n\sigma^2)$

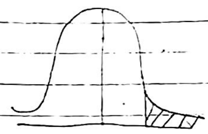
mean $E(S_n) = 40 \times 2 = 80$

$V(S_n) = 40 \times 1/16 = \frac{40}{16}$

Now,

$P(X_n \geq 7 \times 12$

$= P\left(Z \geq \frac{87 - 80}{40/16}\right)$



$= 0.5 - P(z > 1.6)$

$= 0.5 - 0.4452$

$= 0.0548$

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room 2 = $\frac{3}{5}$

Last chapter

Queue Theory

Queue Theory deals with problems that involve waiting (or queuing). Example of queue:

1. Waiting for service in a bank.
2. Waiting for bus in station.
3. Waiting in doctor's clinic.
4. Waiting a barber saloon.

Symbols and Notations

n = no. of customers in system (queue + server).

μ = average no. of customer being served per unit time (i.e., service rate).

λ = average no. of customers arrive to ~~sy~~ per unit of time (i.e., arrival rate).

s = no. of server

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Elements of queuing system

1. Arrival pattern

Arrival can be measured as arrival rate or inter-arrival time (time between two arrivals).

$$\text{Inter-arrival time} = \frac{1}{\text{arrival rate}}$$

These quantity (arrival rate and inter-arrival time) can be measured deterministic or stochastic. Generally, it is assumed that customers joining a queue follows Poission distribution / Poission process. Inter-arrival time follows exponential distribution with parameter λ (Markov n-distribution).

2. Service pattern

It may ~~de~~ be deterministic or stochastic. Generally, it is assumed that customer being served (departure) follows Poission distribution and service time follows Exponential distribution with parameter μ.

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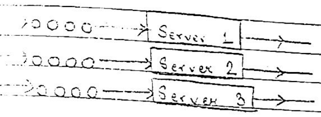
3. Queue discipline (QD)

This is the manner by which customers are selected by service. Some of QD are as follows:

i) First in first out (FIFO) / FCFS
ii) Last in first out (LIFO)
iii) Service in random order (SIRO)

4. Number of servers

i)


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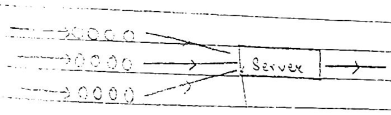

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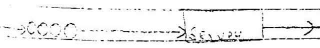
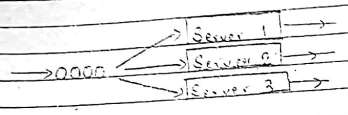


d) 

e) → OOOO → Server 1 → OOO → Server 2 →

5 Size of waiting room (N)

It gives capacity of queuing system to store customers. Generally it is assumed to be infinite.

Kendall's Notation

It is defined as

A | B | S : (N, QD)

where,

A = inter-arrival time distribution

B = service time distribution

S = no. of servers

N = size of waiting room

QD = Queue discipline

Commonly used distribution curve inter-arrival time (A) and service time (B) are

i) Markovian (M) [Exponential distribution].

ii) Deterministic distribution (D)

iii) Erlang distribution (E_k) [Gamma distribution]

Queue Model

1. M/M/1 : (∞, FIFO) Model → OOOOOO → [ ] → OOO

- Arrival pattern (M)
- Service pattern (M)
- No. of servers (S)
  → Single server
- Size of waiting (N)
  → N = ∞
- QD
  → FIFO

2. M/M/1 : (∞, SIRO) queue Model

3. M/M/1 : (N, FIFO) queue Model

4. M/M/S : (∞, FIFO) queue Model

5. M/M/S : (N, FIFO) queue Model

6. M/M/1 : (M/GD)

7. M/M/S : (M/GD)

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8 $M / E_k / 1 : (\infty, FIFO)$

Operating characteristics (OS) / performance measures of Queue Model.

1. P_0 = prob that no customer in system / prob. that server is idle.
2. P_n = prob. of n customers in system. $[n \geq 1]$
3. L_s = average no of customers in system.
4. W_s = average time a customer spends in system.
5. L_q = average no. of customer in queue.
6. W_q = average time for customer in queue
7. P_w = prob that a customer has to wait for service.

5 Sol^n:

No. of server (s) = 1

Arrival rate (λ) = 10 cars per hour

3 mins = 1 car served

1 min = 1/3 car served

60 min = $\frac{1}{3} \times 60 = 20$ cars served

Service rate (μ) = 20 cars per hour

Size of waiting room (N) = ∞

QD = F.I.F.O

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$M/M/1 : (\infty, FIFO)$ queue

Here,

traffic intensity $(\rho) = \frac{\lambda}{\mu} = \frac{10}{20} = \frac{1}{2}$

i) L_s = ?

ii) W_q = ?

iii) = L_q = ?

iv) $P_n = P_2$ = ?

i) $L_s = \frac{\rho}{1-\rho}$

$= \frac{1/2}{1 - 1/2}$

$= 1$

ii) ~~W_q~~ $\Rightarrow W_q = \frac{L_q}{\lambda} = \frac{\rho L_s}{\lambda}$

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Soln :

$s = 1$

9 minutes = 1 call arrive

1 minute = 1/9 call arrive

arrival rate $(\lambda) = \frac{1}{9}$ call per minute

And,

3 min = 1 call served

1 min = 1/3 call served

service rate $(\mu) = 1/3$ call per min

size of waiting $(N) = \infty$

QD = FIFO

Then

i) $P_w = ?$

ii) $L_q = ?$

We have,

i) $P_w = \rho = \frac{\lambda}{\mu} = \frac{1/9}{1/3} = \frac{1}{3}$

ii) $L_q = \rho L_s = \rho \times \frac{\rho}{1-\rho}$

$= 1/3 \times \frac{1/3}{1 - 1/3}$

$= \frac{1}{6}$

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Soln :

$s = 1$

Arrival rate $(\lambda) = 30$ trains per day

$N = 9$

36 mins = 1 train served

1 min = 1/36 train served

1 hrs = 60 min = $\frac{1}{36} \times 60$

24 hrs = 1 day = $\frac{1}{36} \times 60 \times 24 = 40$ trains per day

QD = FIFO.

Then

$\rho = \frac{\lambda}{\mu} = \frac{30}{40} = \frac{3}{4}$

$\lambda_e = \lambda(1 - P_N)$

or, $\lambda_e = 30(1 - P_9)$

or, $\lambda_e = 30(1 - P_0 \rho^9)$ = 0.2649

or, $\lambda_e = 30\left[1 - 0.2649 \times \left(\frac{3}{4}\right)^9\right]$

or, $\lambda_e = 29.40$

$L_s = 0.2649$

$W_q = \frac{L_q}{\lambda_e} = \frac{L_s - \lambda/\mu}{\lambda_e}$

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$$= \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$= \frac{3/4}{1-3/4} - \frac{(9+1)(3/4)^{9+1}}{1-(3/4)^{9+1}}$$

$$= \frac{29.40}{40}$$

= day

customer served

1/4 customers

customers per min

40 ... 40 customers

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$\lambda = \frac{1}{6}$ customers per min

GD = FIFO , $N = \infty$.

$M/M/S : (\infty\ FIFO)$

Then

$$\rho = \frac{\lambda}{\mu c}$$

$$= \frac{1}{3}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{1}{s!} \frac{(s\rho)^s}{1-\rho} \right]^{-1}$$

$$= \left[1 + \frac{\left(\frac{2}{3}\right)^1}{1!} + \frac{1}{2} \times \left(\frac{2}{3}\right)^2 \times \frac{3}{2} \right]^{-1}$$

$$= \left(1 + \frac{2}{3} + \frac{1}{3}\right)^{-1}$$

$$= \frac{1}{2}$$

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$$P_n = \begin{cases} \frac{\rho^n}{n!} P_0 = \frac{(1/3)^n}{n!} \cdot \frac{1}{2} & \text{if } n \le s = 2 \\ \frac{\rho^n}{s!\, s^{n-s}} P_0 = \frac{(1/3)^n}{2!\cdot 2^{n-2}} \times \frac{1}{2} = \frac{(1/3)^n}{2^n} & \text{if } n > s = 2 \end{cases}$$

$$L_q = \left[\frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{(1-\rho)^2} \right] P_0$$

$$= \left[\frac{1}{2!} \left(\frac{1/6}{1/4}\right)^2 \left(\frac{1/3}{(1-1/3)^2}\right) \right] \times \frac{1}{2}$$

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Sol^n:

Given,

Number of server (S) = ~~1~~ 2

Size of waiting room (N) = 3

Arrival rate (λ) = 5 customer per hour

$= \frac{5}{60} = \frac{1}{12}$ customer per ~~hour~~ minute.

15 minutes = 1 customer served

or, 1 min = $\frac{1}{15}$ customer served.

$\therefore$ Service rate (μ) = $\frac{1}{15}$ customer per minute

QD = FIFO.

This question is related to M/M/S : (N, FIFO) queue model

Here,

traffic intensity (ρ) = $\frac{\lambda}{s\mu}$

$= \frac{1/12}{2 \times 1/15}$

$= \frac{5}{8}$

Effective arrival rate (λ_e) = $\lambda(1 - P_N)$

$= \lambda(1 - P_3)$