Stochastic Process:

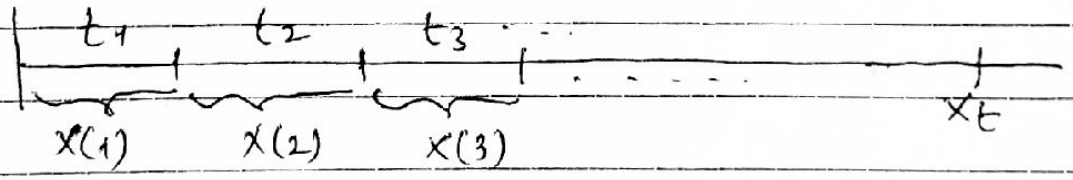
Def^n:

A family of random variables that is indexed by a parameter such as time is known as stochastic process (or chance or random process).

Mathmatically: A stochastic process is a family of random variables $\{X(t) \mid t \in T\}$, defined on a given prob^y space, indexed by the parameter t, where t varies over the indexed set $T\}$.

The values assumed the random variable $X(t)$ are called states and the set of all possible values from the state space of the process. The state space will be denoted by I.

A bachelor level student who appeared in all exams upto 8th semester and the marks or GPA secured by the student in 8 different semester gives the family of random variables.



The values assumed by $X(t)$ are called states and the set of all possible state is called state space. Every instant of time is one random variable in various time period there will be family of random variables.

The state space may be discrete or continuous and parameter T may also be discrete or continuous. Thus we have four different combinations.

i) Discrete time discrete state space (DT, DS)

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3 Queuing theory:

3.1 – Essential features of queuing ~~theory~~ system

3.2 – Specification and Measure of queuing System

3.3 – Probability distribution in queuing System

a) Distribution of arrival, pure birth process

b) Distribution of inter arrival times, distribution of departure, pure death process

3.4 Distribution of service time

3.5 The classical system: operating characteristics (Transit and steady state behavior, line length, Queue length Relationship among System characteristics)

3.6. Solution of queuing models,

The (M/M/1 : ∞/FCFS) model, M/M/1 : ∞/SIRO) model: The M/M/1 : N/FCFS model, The M/M/S : ∞/FCFS Model, The M/M/S : N/FCFS) model, The M/M/S : M/G/1) model, The M/Ek/1 : ∞ FCFS model. Application of Queuing theory in Computer science (Examples & numerical Problems)

ii) Discrete time continuous state space (DT, CS)
iii) Continuous time continuous state space (CT, CS)
iv) Continuous time discrete state space (CT, DS)

Classification of stochastic process:

i) Stationary process:

For a fixed time $t = t_1$, the term $X(t_1)$ is a simple random variable, that describe the state of processes at time t_1. For a fixed number x_1, take prob^y of the event $\{X(t_1) \le x_1\}$ gives the cumulative distribution function (CDF) of the random variable $X(t_1)$ denoted by

$$F(x_1, t_1) = P\{X(t_1) \le x_1\}$$

$F(x_1, t_1)$ is known as first order dist^n of the processes $\{X(t) / t \ge 0\}$.

For two time instant t_1 & t_2, $X(t_1)$ & $X(t_2)$ are two random variables on the same prob^y space. Then their joint distribution function is known as second order distribution of the process & is given by

$$F(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

In general,

The nth order joint distribution of the process $X(t), t \in T$ is

$$F(x, t) = P\{X(t_1) \le x_1, \cdots, X(t_n) \le x_n\} \quad (1)$$

for all $x = (x_1, x_2 \cdots x_n) \in R^n$, $t = (t_1, t_2 \cdots t_n) \in T^n$ such that $t_1 < t_2 < \cdots < t_n$.

Def^n: A stochastic process $\{X(t) / t \in T\}$ is said to be stationary if for $n \ge 1$, if its nth order joint CDF

Satisfies the condition,

$$F(x, t) = F(x; t+\tau,$$

for all vectors $x \in R^n$ & $t \in T^n$. The notation $t+\tau$ implies that scalar τ is added to all components of vector t.

ii) <u>Independent Process:</u>

A stochastic process $\{X(t) / t \in T\}$ is said to be independent process, provided its nth order joint probability distribution satisfies the condition.

$$F(x, t) = \prod_{i=1}^{n} P\{X(t_i) \leq x_i\}$$

Independent process have following special cases.

a) <u>Renewal Process:</u>

A renewal process is defined as discrete time independent process $\{X_n / n = 1, 2 \cdots\}$ where X_1, X_2 are independent and identically distributed non negative random variables.

For example consider a system in which a repair or replacement after failure is performed requiring negletble time. Now the times between successive failure might be independent, identically distributed random variables $\{X_n / n = 1, 2 \cdots\}$ of a renewal process.

b) <u>Markov Process:</u>

A stochastic process $\{X(t) / t \in T\}$ is called Markov process if for any $t_0 < t_1 < t_2 \cdots < t_n < t$, conditional distribution of $X(t)$ for given values $X(t_0)$, $X(t_1)$ ....., $X(t_n)$ depends only one $X(t_n)$:

$$P\{X(t) \leq x / X(t_n) = x_n, X(t_{n-1}) = x_{n-1} \cdots, X(t_0) = x_0\}$$

$$= P\{X(t) \leq x / X(t_n) = x_n\} \quad \text{--- (11)}$$

This definition applies to Markov process with continuous state space. Markov process or Markov chain will study both discrete time and continuous time Markov chain.

The conditional distribution function of eqn (II) has the property of invariance with respect to time origin t_n.

$$P\{X(t) \le x \mid X(t_n) = x_n\} = P\{X(t-t_n) \le X(0) = x_n\}$$

In this case M.C. is said to be (time) homogeneous. The stationarity of the ~~joint~~ conditional distribution function given by eqn (II) does not imply the stationarity of the joint distn function given by eqn (I). Thus homogeneous Markov process need not to be stationary stochastic process.

For homogeneous M.C. the past history of the processes is summarized in the current state, therefore the distribution of time Y, the process spends in a given state must be memoryless.

$$P\{Y \le \tau + t \mid Y \ge t\} = P\{Y \le \tau\} \quad \cdots (III)$$

But this implies that the time that a homogeneous time continuous Markov chain spends in a given state has an exponential distn. From (III) we have,

$$P\{Y \le \tau\} = \frac{P\{t \le Y \le t+\tau\}}{P(Y \ge t)}$$

i.e. $$F_Y(\tau) = \frac{F_Y(t+\tau) - F_Y(t)}{1 - F_Y(t)} \quad \text{— (IV)}$$

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Similarly, the time that a homogeneous discrete time Markov chain spends in a given state has a geometric distribution.

Transition Probability:

Movement of Probability from one state to another or remaining in the state during a single time period, i.e

$$P_{x_{n-1},\, x_n} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

$P\{X(t_n) = x_n\}$ identical to $P\{X(t_{n-1}) = x_{n-1}\}$, future behavior of the event can be calculated by the present probability, which is called the transition probability where n is the number of system.

From transition probability we can make transition probability Matrix.

Transition Probability Matrix

We can represent the Markov chain by a matrix containing the probabilities of the event for one state to another state which is called transition Matrix. Let the transition probability matrix is denoted by P and

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1m} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2m} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & P_{m3} & \cdots & P_{mm} \end{bmatrix}$$

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For Transition Probability Matrix,

- The sum of elements in each row of matrix is unity.

$$\sum_{j=1}^{m} P_{ij} = 1, \text{ for all } j. \quad \text{--- (I)}$$

- Each and every element in the matrix is non negative i.e $P_{ij} \geq 0$; for all i, j. ... (II)

Square matrix whose ~~te~~ elements satisfy the condition (I) & (II) is called a Markov matrix or stochastic matrix.

Markov Chain:

A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only in the state of the previous event.

A stochastic process containing random variable transitioning from one state to another depending on certain assumption and definite probabilistic rule. Let the random process $\{X_n, n \geq 0, \text{ie } n = 0, 1, 2 \cdots\}$, the process is the Markov chain only if,

$$P\{X_{n+1} = j / X_n = i\} \quad \forall \; i, j \geq 0.$$

Example:

Three boys A, B and C, playing a game, they throw a ball to ~~each~~ each other. A always throw the ball to B, B always throws a ball to C, but C is twice as likely to throw the ball to A as to B. Write the transition probability matrix and classify the states.

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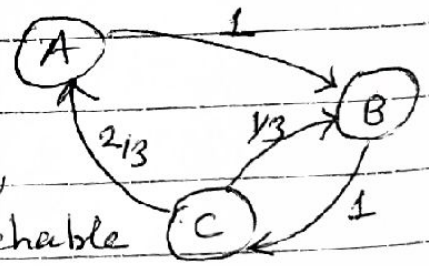
Sol^n: There are 3 states $\{A, B, C\}$ and P is a 3×3 matrix,
The transition probability matrix P is given as:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

A throws to B, so prob^y 1, but A do not throw to A and to C, so the prob^y is 0.
B throws to C prob^y 1

C throws either B or A, i.e C throws to B be x, so for A, it will be $2x$. i.e $x+2x=1$.
For C to B prob^y will be $1/3$ and for C to A probability will be $2/3$.

The probabilities can be represented by each state by a diagram which is called transition probability diagram or simply transition diagram



In diagram A can communicate or reachable B, B is reachable to C, and C can communicate or reachable to B as well as to A.

Here all the states are communicable i.e one state is linked by other with communication so the given chain is irreducible Markov chain and has only 3 states and it is finite

Fig shows we can reach from A to A in two ways i.e in 3 steps and also in 5 steps.

Period of $A = GCD\{3, 5 \cdots\} = 1$

Greatest common divison (GCD) of each step is 1,

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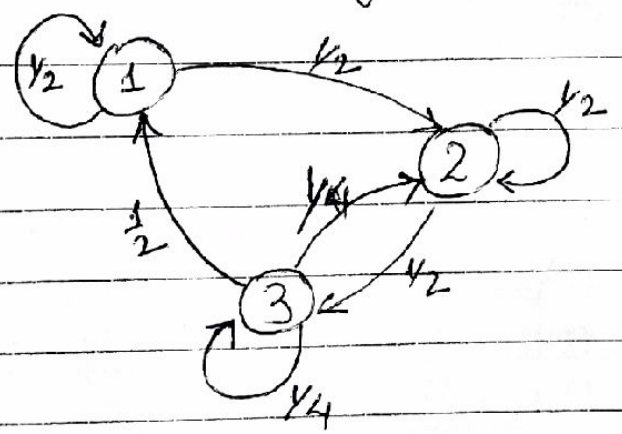
All the ways from A to A is with period 1, (i.e GCD = 1) and communicable, so A is non null presistent and by def^n A is ergodic.

For period of B = GCD$\{2, 3, 4 \cdots\} = 1$

if GCD is 1 i.e aperiodic and non null presistent, then B is also ergodic.

Period of C = GCD$\{2, 3, 4\} = 1$, State C is also a periodic and non null presistent, C is also ergodic.

Ex: A transition diagram of three states 1, 2 & 3 as shown in diagram.



We have 3 states, so we have 3×3 transition Prob^y Matrix.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \end{array}$$

A zero element in the transition matrix indicates the transition is not possible.

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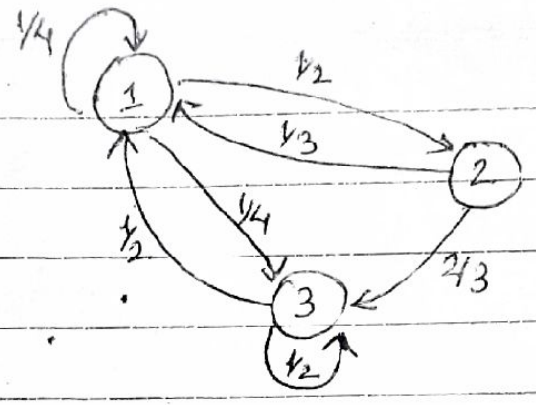
Ex: Consider the transition diagram and find.



From transition diagram we can form a transition probability matrix as:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/4 & 1/2 & 1/4 \\ 2 & 1/3 & 0 & 2/3 \\ 3 & 1/2 & 0 & 1/2 \end{array}$$

i) $P(X_4 = \overset{\text{column}}{3} / X_3 = \overset{\text{row}}{2})$ $\left[\because P_{ij} = P(X_n = j / X_{n-1} = i)\right.$ i - row, j column

$= P_{23} = 2/3$ (2nd row & 3rd column)

ii) $P(X_3 = 1 / X_2 = 1) = 1/4$ (1st row 1st column)

i.e $P_{11} = 1/4$

iii) If, $P(X_0 = 1) = 1/3$, then $P(X_0 = 1, X_1 = 2, X_2 = 3) = ?$

$P(X_0 = 1, X_1 = 2, X_2 = 3)$

$= P(X_0 = 1) \cdot P(X_1 = 2 / X_0 = 1) \cdot P(X_2 = 3 / X_1 = 2)$

$= \frac{1}{3} \times P_{12} \times P_{23}$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{12}$

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Regular Markov chain:

A transition matrix (Stochastic matrix) is said to be regular if some power of P has all positive entries (i.e strictly greater than zero), otherwise the chain is irregular.

Ex: Which of the following matrix is regular?

(a) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then,

$$P^2 = P.P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^5 = P^3 \cdot P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In any power of P i.e 2, 3, 4, 5 etc, all the elements of P-matrices are not greater than 0, so the given matrix P is not regular.

$$P = \begin{bmatrix} .2 & .8 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} .2 & .8 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} .2 & .8 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (.04+.8) & (.16+0) \\ (.2+0) & (.8+0) \end{bmatrix}$$

$$= \begin{bmatrix} .84 & .16 \\ .2 & .8 \end{bmatrix}$$ All entries are > 0.

So the given matrix is regular

(c) $\begin{bmatrix} .3 & .7 \\ .1 & .9 \end{bmatrix}$ All entries are > 0,

so given matrix is regular.

Properties of Regular Markov chain:

Let P be the transition matrix for a regular Markov Chain.

i) There is a unique stationary matrix S, that can be found by solving the equation $S.P = S$.

ii) Given any initial-state matrix S_0, the state matrix S_k approaches the stationary matrix S.

iii) The matrices P^k approaches a limiting matrix $\bar{P}$ is equ to the stationary matrix S.

Ex: Let the transition matrix for a Markov chain is:

$$P = \begin{bmatrix} .6 & .4 \\ .2 & .8 \end{bmatrix}$$

Find the stationary matrix S.

Sol^n: We have,

$$S.P = S$$

$$\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} .6 & .4 \\ .2 & .8 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$$

or $.6S_1 + 0.2S_2 = S_1$ — (I)

$.4S_1 + .8S_2 = S_2$ — (II)

$S_1 + S_2 = 1$ — (III) [∵ For stationary M.C. all elements of row sum is 1]

From (I)

$.6(1-S_2) + 0.2S_2 = 1 - S_2$ [∵ $S_1 = 1 - S_2$]

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or $0.6 - 0.6S_2 + 0.2S_2 + S_2 = 1$

or $0.6S_2 = 0.4$

$$S_2 = \frac{.4}{.6} = \frac{2}{3} \simeq 0.67$$

$$S_1 = 0.33$$

$\therefore$ $S = [0.33 \;\; 0.67]$, which is stationary Matrix.

2 A new rapid transit system has just started operating, it is found that 25% of the customers are using the system, while 75% still travel by automobiles. The following transition matrix was determined from records of the other rapid transit systems.

Next Month:

		Rapid Transit	Automobiles
Current Month:	Rapid transit	0.8	0.2
	Automobile	0.3	0.7

i) What is the initial state matrix ii) What percentage of the customers will be using the new system after 1 month? after 2 month? iii) Find the percentage of customer using each type of transportation after it has been service for a long time?

Sol^n:

i) The initial state matrix is

$$S_0 = [0.25 \quad 0.75]$$

ii) After 1 month percentage of customers using new system will be

$$S_1 = S_0 \cdot P$$

$$= \begin{bmatrix} .25 & .75 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$$

$$= \begin{bmatrix} (0.2 + 0.225) & (0.05 + 0.525) \end{bmatrix}$$

$\therefore S_1 = \begin{bmatrix} .425 & .575 \end{bmatrix}$ (Sum of entries equals 1)

1 month later 42.5% uses rapid transit and 57.5 uses automobiles.

<u>For after 2 month:</u>

$$S_2 = S_1 \cdot P$$

$$= \begin{bmatrix} .425 & .575 \end{bmatrix} \times \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$$

$$= \begin{bmatrix} (.34 + .1725) & (.085 + .4025) \end{bmatrix}$$

$S_2 = \begin{bmatrix} .5152 & .4875 \end{bmatrix}$ Sum of entries 1.

Here after 2 month from start 51.52% uses rapid transit and 48.75 uses automobiles.

iii) <u>For a long time:</u>

We have, $SP = S$

or $\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$

The system of equations are:

$.8S_1 + .3S_2 = S_1 \cdots$ (A)

$.2S_1 + .7S_2 = S_2$ (B)

$S_1 + S_2 = 1$ (C)

From (A) & (C)

$.8(1 - S_2) + .3S_2 = 1 - S_2$

or $.8 - .8S_2 + .3S_2 + S_2 = 1$

or $.5S_2 = 0.2$

or $S_2 = \frac{.2}{.5} = 0.4$

$S_1 = 0.6$

$\therefore S = [.6 \quad .4]$

So for a long run 60% uses rapid transit and 40% uses automobiles.

Absorbing Markov chain:

- A state in a Markov chain is called an absorbing state if once the state is entered, it is impossible to to leave.

Identify any absorbing states for the following transition matrices.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ .3 & .7 & 0 \\ 0 & .2 & .8 \end{bmatrix} \end{array}$$

State A to A there is 100% chance to communicate so, A is absorbing state.

From state B to state A there is 30% chance to communicate and state B to B there is 70 chance to communicate, so B is not absorbing state.

From state C to A 0% prob^y, from C to B 20% prob^y and from state C to C there 80% prob^y to communicate so, C is not absorbing state.

Next:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Here B is absorbing state & A & C is not absorbing state.

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Stationary Matrices & steady State Markov Chain

A stationary distribution of Markov chain is a prob distribution that remains unchanged in the Markov Chain as time progresses. Typically it is represented as a row vector whose entries are probabilities summing to 1, and given transition matrix P, it satisfy

Ex:

Consider an example where a company initially has 10% market share, using advertising campaign the trans matrix is given by.

$$P = \begin{array}{c} \\ A \\ A' \end{array} \begin{array}{c} \begin{array}{cc} A & A' \end{array} \\ \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix} \end{array}$$

A = uses brand A
A' = uses different brand

If the probabilities in P remain valid over a long period of time what happens to Companies market share?

Sol^n:

$$S_0 = \begin{bmatrix} .1 & .9 \end{bmatrix} \quad \left\{ \begin{array}{l} \because 10\% \text{ uses brand } A \\ 90\% \text{ uses brand } A' \end{array} \right.$$

Then for 1 month later,

$$P_1 = S_0 \cdot P = \begin{bmatrix} .1 & .9 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

$$= \begin{bmatrix} .62 & .38 \end{bmatrix}$$

One month later 62% uses brand A & 38% uses A'

2 month later,

$$P_2 = P_1 \cdot P = \begin{bmatrix} .62 & .38 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

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$P_2 = [.724 \quad .276]$

two month later 72.4% uses brand A & 27.6% uses A'.

Similarly,

$$P_3 = P_2 \cdot P$$

$$= [.724 \quad .276] \times \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

$$= [.7448 \quad .2552]$$

$$P_4 = P_3 \cdot P = [.7448 \quad .2552] \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

$$P_4 = [.74896 \quad .25104]$$

$$P_5 = [.749792 \quad .250208] \quad \text{using } P_4 \cdot P$$

$$P_6 = P_5 \cdot P = [.7499584 \quad .2500416]$$

It appears that the state matrices are getting close to

$$S = [.75 \quad .25]$$

In fact notice that

$$S \cdot P = [.75 \quad .25] \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

$$= [.75 \quad .25]$$

$$= S \quad \text{i.e no change occurs.}$$

The matrix $S = [.75 \quad .25]$ is called stationary matrix and the system is said to be steady-state. But every M.C. do not have unique stationary matrices, however for regular M.C. have unique stationary matrix.

After a long time 75% uses brand A & 25% uses brand A'.

n-Step transition Probability Matrix:

Consider a two state Markov Chain with transition Probability matrix.

$$P = \begin{bmatrix} 1-\alpha & \beta \\ \beta & 1-\beta \end{bmatrix}, \text{ where } 0<\alpha<1, \quad 0<\beta<1.$$

Then,

n-step transition probability matrix is given by

$$P^n = \frac{1}{\alpha+\beta}\left\{\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \alpha & -\alpha \\ \beta & -\beta \end{bmatrix}\right\}$$

$$= \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

We conclude that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \pi \end{bmatrix} = Q$$

Where Q is a matrix whose rows are identical and equal to the stationary distribution π.

Ex 1 One Step transition probability matrix P is given as:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

Then find n-step transition probability matrix P^n

Sol^n:

If $P = \begin{bmatrix} 1-\alpha & \beta \\ \beta & 1-\beta \end{bmatrix}$, $0<\alpha<1$, $0<\beta<1$,

Then

$$P^n = \frac{1}{\alpha+\beta}\left\{\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}\right\}$$

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Then $\alpha = 1-0.4 = 0.3$

$\beta = 1-0.4 = 0.6$

By using (1)

$$P^n = \frac{1}{0.9}\left\{\begin{bmatrix} 0.6 & 0.3 \\ 0.6 & 0.3 \end{bmatrix} + (0.1)^n \begin{bmatrix} 0.3 & -0.3 \\ -0.6 & 0.6 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 1/3 \end{bmatrix} + (0.1)^n \begin{bmatrix} \frac{1}{3} & -1/3 \\ -2/3 & 2/3 \end{bmatrix}$$

As $n \to \infty$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix} = Q$$

where, $\pi = [\pi_1 \ \pi_2]$

$= \left[\frac{2}{3}, \ 1/3\right]$

$\therefore \ Q = \left[\frac{2}{3}, \ 1/3\right] = [0.667 \quad 0.333]$

Given a transition probability matrix of a Markov chain below, obtain the matrix of n-step transition probabilities and then find two step and three step transition probabilities.

$$\begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Soln:

Given Matrix can be written as:

$$P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$$

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If P has the form,

$$P = \begin{bmatrix} 1-\alpha & \beta \\ \beta & 1-\beta \end{bmatrix}, \quad 0<\alpha<1, \quad 0<\beta<1,$$

Then,

$$P^n = \frac{1}{\alpha+\beta}\left\{\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}\right\}$$

Here, $\alpha = 1-0.25 = 0.75$

$\beta = 1-0.5 = 0.5$

$$P^n = \frac{1}{1.25}\left\{\begin{bmatrix} 0.5 & 0.75 \\ 0.5 & 0.75 \end{bmatrix} + (-0.25)^n \begin{bmatrix} 0.75 & -0.75 \\ -0.5 & 0.5 \end{bmatrix}\right.$$

$$= \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} + (-0.25)^n \begin{bmatrix} 0.6 & -0.6 \\ -0.4 & 0.4 \end{bmatrix}$$

As $n \to \infty$

$$Q = \lim_{n\to\infty} P^n = \begin{bmatrix} .4 & .6 \\ .4 & .6 \end{bmatrix}\begin{bmatrix} \pi \\ \pi \end{bmatrix}$$

$$\pi = [\pi_1 \quad \pi_2][.4 \quad .6]$$

$$\therefore Q = [.4 \quad .6]$$

Second step transition Prob^y Matrix

$$P^2 = P \cdot P = \begin{bmatrix} 0.5 & 0.75 \\ 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$$

$$=$$

Third step transition Matrix

$$P^3 = P^2 \cdot P =$$

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3 From the following transition Matrix, find n-step, 2nd step and 3rd-step transition Matrix.

$$P = \begin{bmatrix} 1/6 & 5/6 \\ 1/12 & 11/12 \end{bmatrix} = \begin{bmatrix} 0.17 & 0.83 \\ 0.08 & 0.92 \end{bmatrix}$$

Then n-step transition Proby Matrix

$$P^n = \frac{1}{(\alpha+\beta)} \left\{ \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} \right\}$$

$\alpha = 1 - 0.17 = 0.83$

$\beta = 1 - 0.92 = 0.08$

$$P^n = \frac{1}{0.91} \left\{ \begin{bmatrix} 0.08 & 0.83 \\ 0.08 & 0.83 \end{bmatrix} + (0.09)^n \begin{bmatrix} 0.83 & -0.83 \\ -0.08 & 0.08 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0.087 & 0.913 \\ 0.087 & 0.913 \end{bmatrix} + (0.09)^n \begin{bmatrix} 0.913 & -0.913 \\ -0.087 & 0.087 \end{bmatrix}$$

As $n \to \infty$.

$$Q = \lim_{n \to \infty} P^n = \begin{bmatrix} 0.087 & 0.913 \\ 0.087 & 0.913 \end{bmatrix} \begin{bmatrix} \pi \\ \pi \end{bmatrix}$$

$$\therefore Q = \pi = [\pi_1 \ \pi_2] \begin{bmatrix} 0.087 & 0.913 \end{bmatrix}$$

$$= [0.087 \ \ 0.913]$$

<u>2nd step</u> $P^2 = P \cdot P$

3rd step $P^3 = P^2 \cdot P$

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Ex: Calculate the 2nd step and 3rd step transitions Probability Matrices from the following Matrix.

$$P = \begin{bmatrix} .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix}$$

Soln:

Second Step transition probability matrix is

$$P^2 = P.P = \begin{bmatrix} .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix} \times \begin{bmatrix} .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix}$$

$$= \begin{bmatrix} .36 & .24 & .24 & .16 \\ .36 & .24 & .12 & .28 \\ .36 & .24 & .24 & .16 \\ .18 & .12 & .21 & .49 \end{bmatrix}$$

The third step transition Matrix is

$$P^3 = P^2 . P = \begin{bmatrix} .36 & .24 & .24 & .16 \\ .36 & .24 & .12 & .28 \\ .36 & .24 & .24 & .16 \\ .18 & .12 & .21 & .49 \end{bmatrix} \times \begin{bmatrix} .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix}$$

$$=$$

Irreducible Markov chain:

If all the states in the Markov chain belong to one closed communicating class, then the chain is called irreducible Markov chain. Irreducibility is a property of the chain, in an irreducible Markov chain the process can go from any state to any state whatever be the number of steps it requires.

Ex 1: Determine the given matrix is irreducible or not.

$$P = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Soln:

$$P^2 = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 & 0.49 & 0.35 \\ 0.07 & 0.32 & 0.60 \\ 0.02 & 0.24 & 0.74 \end{bmatrix}$$

Here all $P_{ij}^{(2)} > 0$,

All $P_{ij}^{(2)}$ elements are non negative and greater than 0, hence given matrix is irreducible.

Ex: A homogeneous Markov chain with transition prob'y matrix is given by

$$P = \begin{bmatrix} 0.8 & 0.2 & 0.3 \\ 0.1 & 0.6 & 0.2 \\ 0.5 & 0 & 0 \end{bmatrix}$$

Show that Markov chain is irreducible.

Sol^n:

$$P^2 = P.P = \begin{bmatrix} 0.8 & .2 & .3 \\ .1 & .6 & .2 \\ .5 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} .8 & .2 & .3 \\ .1 & .6 & .2 \\ .5 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.81 & 0.28 & .28 \\ 0.24 & 0.38 & 0.15 \\ 0.40 & 0.10 & 0.15 \end{bmatrix}$$

Here all $P_{ij}^{(2)} > 0$,

All $P_{ij}^{(2)}$ elements are non negative and greater than 0, hence given matrix is irreducible.

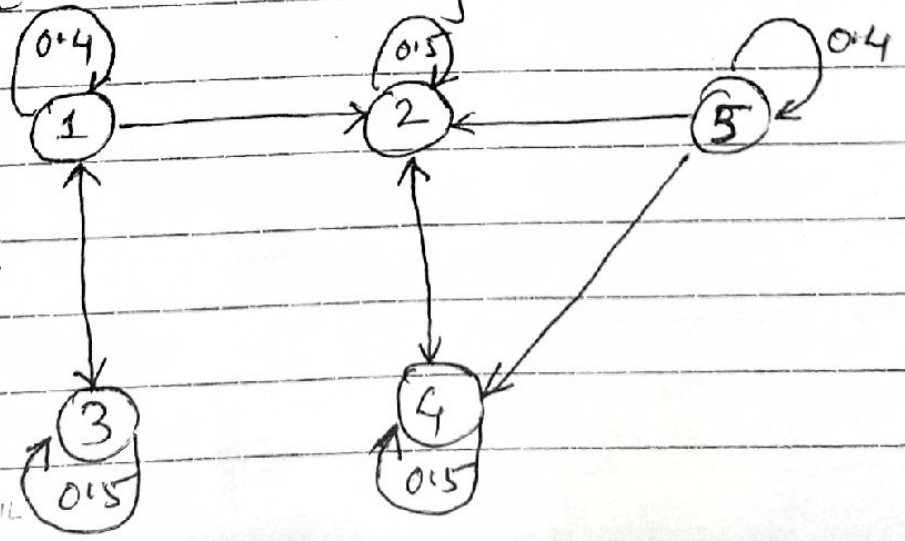
Transient and recurrent states in the Markov chains

- A set C is called irreducible if whenever $i, j \in C$, i communicate with j
- A set C is closed if it is impossible to get out.
- If C is finite closed and irreducible set, then all states in C are recurrent.
- If state A goes to state B, but state B does not go to state A then state A is transient.

Ex: Consider the following transition matrices, identify the transient, recurrent states and the irreducible closed sets in the Markov chain.

$$
P = \begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & .4 & .3 & .3 & 0 & 0 \\
2 & 0 & .5 & 0 & .5 & 0 \\
3 & .5 & 0 & .5 & 0 & 0 \\
4 & 0 & .5 & 0 & .5 & 0 \\
5 & 0 & .3 & 0 & .3 & .4
\end{array}
$$

The transition diagram will be



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$\longrightarrow$ indicates can communicate

$\not\longrightarrow$ indicates does not communicate.

In our Markov Chain

$1 \longrightarrow 1$ (i.e state 1 communicate to state 1)

$1 \longrightarrow 2$, $2 \not\longrightarrow 1 \Rightarrow 1$ is transient state

$3 \longrightarrow 2$, $2 \not\longrightarrow 3 \Rightarrow 3$ is transient state

$\left.\begin{matrix} 5 \longrightarrow 2 \\ 5 \longrightarrow 4 \end{matrix}\right\}$; $\left.\begin{matrix} 2 \\ 4 \end{matrix}\right\} \not\longrightarrow 5$, $\Rightarrow 5$ is transient.

$C = \{2, 4\}$, C is closed, $2 \longrightarrow 4$ and $4 \longrightarrow 2$ but does not communicate with other

So; $\{2, 4\}$ is closed, recurrent and irreducible.

Birth and Death process:

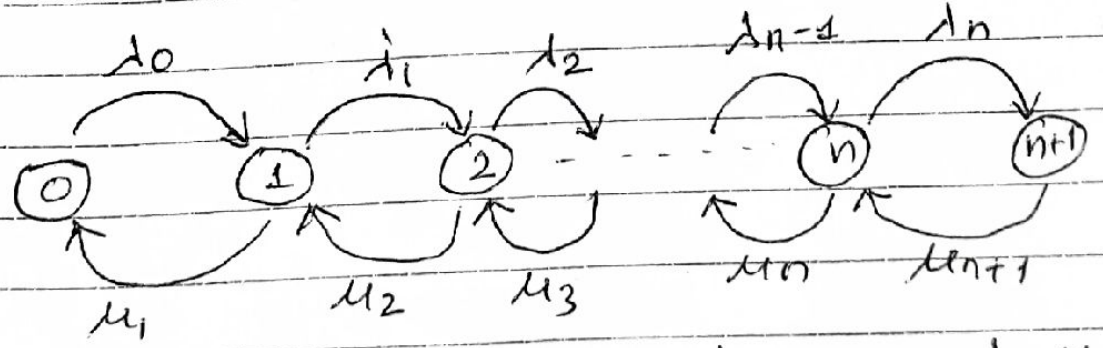
An important and useful class of Markov processes for analyzing queuing systems are birth and death process. The birth and death processes are designed after population models, which gain new members through birth and lose members through death. The arrival and departure of flight in an airport can be modelled as birth and death process. The arrival of flight can be taken as birth and departure of flight can be taken as death.

Mathematically,

A continuous Markov chain $X(t)$, with state space $S = \{0, 1, 2 \ldots\}$ is called a birth and death process if the following condition is satisfied.

$$P[X(t+h) - X(t) = k \mid X(t) = n] = \begin{cases} \lambda_n h + o(h), & \text{if } k = 1, n \geq 0 \\ \mu_n h + o(h), & \text{if } k = -1, n \geq 0 \\ 1 - (\lambda_n + \mu_n) h + o(h), & \text{if } k = 0, n \geq 0 \end{cases}$$

where, $\lambda_0, \lambda_1, \lambda_2 \ldots$ are positive constants called birth rates, $\mu_1, \mu_2, \mu_3 \ldots$ are positive constants called death rates.



State diagram for birth & death process.

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