Software 3rd sem.
probability and Queuing Theory.

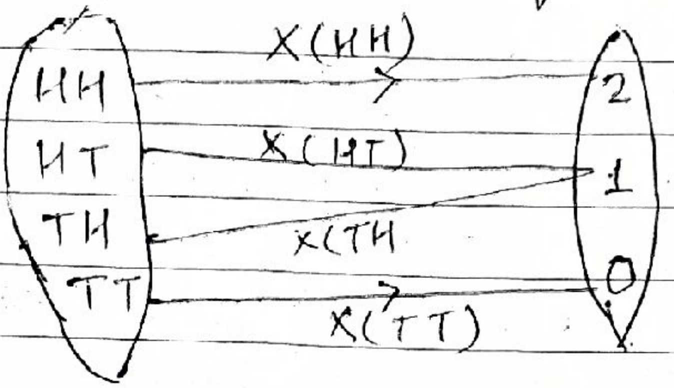
types of random variable.
- Probability distribution of discrete and Continuous random variables.
- Functions of random variables
- Mathematical expectation of continuous and discrete random variables
- Moments of continuous r.v; uses of Moments
- Binomial distribution; Poisson distribution
- Normal distribution, t-distribution, χ^2 distribution
F-distribution, β-distribution, Gamma distribution
Exponential distribution.
- Expectations and higher order moments
- Characteristics function
- Chebyshev inequality for continuous random variable
- Laws of large numbers, Weak laws and Strong laws of large numbers.
- Central limit theorem and its application.

Random Variable:

Consider a random experiment of tossing a coin 2 times, then the possible outcomes are:

Outcome(ω) = HH, HT, TH, TT

Let X be a real valued function of getting X,



Symbolically,

$$X = X(\omega) = \begin{cases} 2 \text{ if } \omega = HH \\ 1 \text{ if } \omega = HT, TH \\ 0 \text{ if } \omega = TT \end{cases}$$

Defn: A random variable (r.v.) is a real valued function $X(\omega)$ defined on sample space S (domain) and range $(-\infty, \infty)$ in random experiment. It is denoted by X, Y, Z.

If x is a real number, the outcome of random experiment is ω in S, then $X(\omega) = x$, is denoted briefly by writing $X = x$, So,

$$X = x \Rightarrow \{\omega : X(\omega) = x\}$$

Note:

If X_1 and X_2 are random variables and c is a constant then $cX_1 \pm cX_2$ and cX_1X_2 are also random variables.

Types of random Variables:

There are two types of random Variables, they are:

i) Discrete random Variable: A random variable is said to be discrete if it takes countable or whole number Values. Alternatively a real valued function defined on a discrete sample space is called discrete r.v.

The Examples of discrete r.v's are i) The no. of People in a queue in a hotel ii) The no. of Patients arrived in a certain hospital etc.

ii) Continuous random Variable:

A random Variable x is said to be continuous if it can take all possible values between certain limits. The continuous nature data such as height of People weight, temperature, area, Volume, acceleration due to gravity etc are continuous random Variables.

Discrete random Variables:

Probability Mass function:

Let x be a discrete r.v's which associates values $x_0, x_1, x_2 \ldots x_n$ with Probabilities $p(x_0), p(x_1), p(x_2) \ldots p(x_n)$ respectively. Then $p(x_i)$ $i = 0, 1, \ldots n$ is called Probability mass function (p.m.f) if it satisfies the following conditions.

i) $p(x_i) \geq 0, \ \forall \ i = 0, 1, 2 \ldots n$

ii) $\sum_{i=1}^{n} p(x_i) = 1$

or $p(x_i) \geq 0, \ \forall \ i = 0, 1, 2 \ldots \infty$ & $\sum_{i = all} p(x_i) = 1$

Probability Distribution function:

Let X be a discrete r.v. associates with values $x_0, x_1, x_2 \ldots$, with respective probabilities $P(x_0), P(x_1) \ldots P(x_\infty)$, then the cumulative probability of X, is called probability distribution function and denoted by $F(x)$.

Mathematically,

$$F(x) = P(X \le x_j)$$
$$= P(x_0) + P(x_1) + P(x_2) + \cdots P(x_j)$$
$$= \sum_{i=0}^{j} p(x_i)$$

Properties:

$F(x)$ lies between 0 to 1. i.e $0 \le F(x) \le 1$.

$$\left.\begin{aligned} F(-\infty) &= \lim_{x_i \to -\infty} F(x_i) = 0 \\ F(+\infty) &= \lim_{x_i \to \infty} F(x_i) = 1 \end{aligned}\right\} \begin{aligned} &P(a < X \le b) = F(b) - F(a) \\ &F(x_i) < F(x_j) \text{ if } x_i < x_j \end{aligned}$$

Ex: A fair coin is tossed 3 times, if X is the no. of heads find the probability distribution of X.

Soln:

When a coin is tossed 3 times, the possible out comes are:

$$S = \{HHH, HHT, HTT, TTT, THH, TTH, THT, HTH\}$$

Here x be the r.v. of getting head, ∴ X takes values 0, 1, 2, 3. Then

No. of head	0	1	2	3	Total
$P(x_i)$	1/8	3/8	3/8	1/8	1.
$F(x_j) = P(X \le x_j)$	1/8	4/8	7/8	8/8 = 1	

Here, $P(x_i)$ is Probability Mass function and $F(x_j) = P(X \leq x_j)$ is the Probability distribution function.

Ex: Let X be a discrete r.v. whose cumulative distribution function is,

$$F(x) = \begin{cases} 0, & \text{if } x < -3 \\ 1/6, & \text{if } -3 \leq x < 6 \\ 1/2, & \text{if } 6 \leq x < 10 \\ 1, & \text{if } x \geq 10 \end{cases}$$

Find i) $P(X \leq 4)$, $P(-5 < X \leq 4)$, $P(X = 3)$, $P(X = 10)$, $P(X = 6)$

ii) p.m.f. of X.

Solⁿ: $P(X \leq 4) = F(4) = 1/6$

$P(-5 < X \leq 4) = F(b) - F(a)$ [$\because P(a < x \leq b) = F(b) - F(a)$]

$= F(4) - F(-5)$

$= \frac{1}{6} - 0 = \frac{1}{6}$

Using the formula $P(X = x) = F(x) - F(x-0)$, then

$P(X = -3) = F(-3) - F(-3-0)$

$= \frac{1}{6} - 0 = \frac{1}{6}$

$P(X = 6) = F(6) - F(6-0)$

$= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$P(X = 10) = F(10) - F(10-0)$

$= 1 - \frac{1}{2} = 1/2$

ii) The Probability Mass function of X is,

x	$P(x)$
-3	$1/6$
6	$1/3$
10	$1/2$

Ex 3 A random variable X has the following probability distribution.

Value of X : x	0	1	2	3	4	5	6	7	8
p(x	a	3a	5a	7a	9a	11a	13a	15a	17a

i) Determine the value of a

ii) Find $P(X \le 3)$, $P(X \ge 3)$, $P(0 < X < 5)$, $F(4)$ and $F(7)$

iii) Find out the distribution function of r.v. X.

Soln:

i) We have, $\sum_{i=0}^{8} p(x_i) = 1$

or $a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

or $81a = 1$ or $a = 1/81$

ii) $P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 1/81 + 3/81 + 5/81 + 7/81$

$= 16/81$

$P(X \ge 3) = P(X=3) + P(X=4) + \cdots + P(X=8)$

$= 1 - P(X < 3)$

$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$

$= 1 - \left\{\frac{1}{81} + \frac{3}{81} + \frac{5}{81}\right\} = \frac{72}{81}$

$F(7) = P(X \le 7)$

$= 1 - P(X > 7)$

$= 1 - P(X=8) = 1 - 17/81 = 64/81$

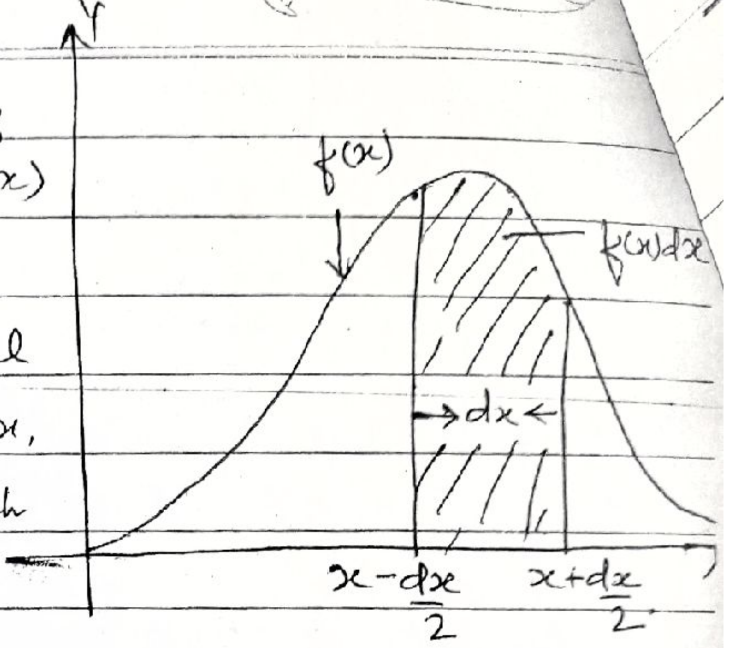
iii)

X	0	1	2	3	4	5	6	7	8
F(x)	1/81	4/81	9/81	16/81	25/81	36/81	49/81	64/81	81/81

Continuous random Variable:

Let x be a continuous r.v., which associates values x, $f(x)$ be any function of x.

Consider a small interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$ of length dx, along the curve $f(x)$, which is represented by $f(x)\,dx$.



$$\therefore P\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right] = f(x)\,dx$$

In figure $f(x)\,dx$ represents the area bounded by the curve $y = f(x)$ and at the points $x-\frac{dx}{2}$ and $x+\frac{dx}{2}$.

The function $f(x)$ is called probability density function (p.d.f) if it satisfies the following propert

i) $f(x) \geq 0, \ \forall \ x \in R$.

ii) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_R f(x)\,dx = 1,$$

where R is the collection of all points in the entire range of the variable.

iii) If x falls in the interval (α, β)

$$\int_{\alpha}^{\beta} f(x)\,dx = P(\alpha \leq X \leq \beta) = P(E)$$

Continuous Probability distribution function:

If X is a continuous r.v. with prob^y density function (p.d.f), $f(x)$, then the function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx, \quad -\infty < x < \infty$$

is called prob^y distribution function or Cumulative Probability distribution function of r.v. X.

Properties:

i) $0 \leq F(x) \leq 1$, if $-\infty < x < \infty$

ii) $dF(x) = f(x)\,dx$

i.e $F'(x) = f(x) \geq 0$. [$\because f(x)$ is p.d.f]

iii) $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 0$.

iv) $\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$.

v) $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b)$

$= P(a < X < b)$

$= F(b) - F(a)$

vi) $F(x) < F(y)$ if $x < y$.

Ex: 1. A r.v. X has the density function

$f(x) = a + bx^2, \quad 0 < x < 1$, find a & b so that $a+b=2$

We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

or $\int_0^1 (a + bx^2)\,dx = 1$

or $a\int_0^1 dx + b\int_0^1 x^2\,dx = 1$

or $a + \frac{b}{3} = 1$

or $3a + b = 3$ - (I)

Also,

Given $a + b = 2$ - (II)

Solving (I) & (II) we get

$a = \frac{1}{2}$

$b = 3/2$

Ex: 2 Let X be a r.v. with p.d.f.

$$f(x) \begin{cases} = ax, & 0 \le x \le 1 \\ = a, & 1 \le x \le 2 \\ = -ax + 3a, & 2 \le x \le 3 \\ = 0, & \text{otherwise} \end{cases}$$

i) Find a

ii) $P(X \le 1.5)$

Soln: We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$a\int_0^1 x\,dx + a\int_1^2 dx - a\int_2^3 x\,dx + 3a\int_2^3 dx = 1$$

or $a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 - a\left[\frac{x^2}{2}\right]_2^3 - 3a\left[x\right]_2^3 = 1$

or $\frac{a}{2} + a - a\left(\frac{9}{2} - \frac{4}{2}\right) + 3a \times 1 = 1$

$\therefore a = \frac{1}{2}$

ii) $P(X \leq 1.5) = \int_1^2 a\,dx$

$= a[x]_1^2$

$= \frac{1}{2} \times 1 = 0.5 \quad [\because a = \frac{1}{2}]$

Ex: 3 Given the p.d.f.

$$f(x) = \begin{cases} kx(1-x), & \text{for } 0 < x < 1 \\ 0, & \text{Otherwise} \end{cases}$$

Show that i) $k = 6$, ii) $P(0.3 < X < 0.5) = 0.2844$

iii) $P(X < 0.4) = 0.352$

Soln:

We have, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i) $\therefore k\int_0^1 x(1-x)\,dx = 1$

or $k\left[\frac{x^2}{2}\right]_0^1 - k\left[\frac{x^3}{3}\right]_0^1 = 1$

or $\frac{k}{2} - \frac{k}{3} = 1 \quad \therefore k = 6$

ii) $\int_{0.3}^{0.5} \left[\frac{kx^2}{2} - \frac{kx^3}{3}\right]_{0.3}^{0.5}$

$= k\left[\frac{x^2}{2}\right]_{0.3}^{0.5} - k\left[\frac{x^3}{3}\right]_{0.3}^{0.5}$

$= 6\left[\frac{(0.5)^2 - (0.3)^2}{2}\right] - 6\left[\frac{(0.5)^3 - (0.3)^3}{3}\right]$

$= 0.2844$

i.e. $P(X<0.4) = \int_0^{0.4} k x(1-x)\,dx$

$$= \left.\frac{kx^2}{2}\right]_0^{0.4} - \left.\frac{kx^3}{3}\right]_0^{0.4}$$

$$= 6\left[\frac{(0.4)^2}{2} - 0\right] - 6\left[\frac{(0.4)^3}{3} - 0\right]$$

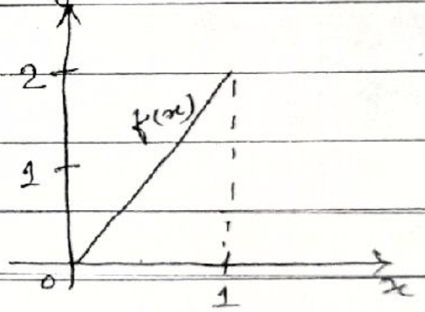
$$= 0.352$$

Ex: 4

A continuous r.v. X has the p.d.f.

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the cumulative distribution function (C.D.F.) for X, Also find $P\left[\frac{1}{4} < X < \frac{3}{4}\right]$.



Solⁿ:

$f(x) = 0$, if $x < 0$.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_0^1 2x\,dx$$

$$= 2\cdot\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\therefore F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Next $P\left(\frac{1}{4} < X < \frac{3}{4}\right) = 2\cdot\left[\frac{x^2}{2}\right]_{1/4}^{3/4} = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{2}$

Ex:5 A continuous distribution of a variable X, in the range (−3, 3) is defined by

$$f(x) = \frac{1}{16}(x+3)^2, \quad \text{for } -3 \le x < -1$$
$$= \frac{1}{16}(6-2x^2), \quad \text{for } -1 \le x \le 1$$
$$= \frac{1}{16}(3-x)^2, \quad \text{for } 1 \le x \le 3$$

i) Verify the area under the curve is unity,
ii) find $P(-0.5 < X < 1)$

Solⁿ:
i) Area under the curve $= \int_{-3}^{3} f(x)\,dx$

$$= \int_{-3}^{-1} \frac{(x+3)^2}{16}dx + \int_{-1}^{1} \frac{(6-2x^2)}{16}dx + \int_{1}^{3} \frac{(3-x)^2}{16}dx$$

$$= \frac{1}{16}\left\{\left[\frac{x^3}{3} + \frac{6x^2}{2} + 9x\right]_{-3}^{-1} + \left[6 - \frac{2x^3}{3}\right]_{-1}^{1} + \left[9x - \frac{6x^2}{2} + \frac{x^3}{3}\right]_{1}^{3}\right.$$

$$= \frac{1}{16}\left[\frac{-1+27}{3} + 3(1-9) + 9(-1+3) + \left(12 - \frac{4}{3}\right) + 9(3-1)\right.$$
$$\left. - 3(9-1) + \frac{26}{3}\right]$$

$$= \frac{1}{16}\left(\frac{52}{3} - 48 - 36 + 12 - \frac{4}{3}\right\} = \frac{48}{48} = 1$$

ii) $$P(-0.5 < x < 1) = \int_{-0.5}^{1} \frac{(6-2x^2)}{16}dx$$

$$= \frac{1}{16}\left\{6\int_{-0.5}^{1} dx - 2\int_{-0.5}^{1} x^2 dx\right.$$

$$= \frac{1}{16}\left\{6(1+0.5) - \frac{2}{3}(1-0.125)\right\}$$

$$= \frac{1}{16}\left\{9 - 0.5834\right\} = \frac{8.41}{16} = 0.52$$

Ex:6 A continuous r.v. X, has the probability density function $f(x) = k(1+x)$, $2 \le x \le 5$
Find $P(X \le 4)$.

Sol^n: We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

or $k\int_{2}^{5}(1+x)\,dx = 1$ or $k\left[x + \frac{x^2}{2}\right]_2^5 = 1$

or $k\left[3 + \frac{25-4}{2}\right] = 1$

or $k\left[\frac{6+21}{2}\right] = 1 \Rightarrow k = \frac{2}{27}$

$\therefore f(x) = \frac{2}{27}(1+x)$, $2 \le x \le 5$

Now,

$$P(X \le 4) = \int_{2}^{4} k(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{2}{27}\left[2 + \frac{16-4}{2}\right]$$

$$= \frac{2}{27}[8] = \frac{16}{27}$$

Ex:7 If $P(x) = \frac{x}{15}$, $x = 1,2,3,4,5$

$= 0$, Otherwise

Find $P\left\{\frac{1}{2} < X < \frac{5}{2} / X > 1\right\}$

Solⁿ:

$= P\{0.5 < X < 2.5 / X > 1)$

$= \frac{P(X=1) + P(X=2)}{P(X=2) + P(X=3) + P(X=4) + P(X=5)}$

$= \frac{1/15 + 2/15}{2/15 + 3/15 + 4/15 + 5/15} = 3/14$ ✓

Ex: 8 A lot of 10 items containing 3 defectives from which Sample of 4 items is drawn without replacement. Let X be the random variable being the no. of defective items in the sample. Find i) the Prob'y distribution of X, ii) $P(X < 1)$ iii) $P(0 < X < 2)$

Solⁿ:

$\boxed{\begin{matrix} 3D \\ 7ND \end{matrix}} \rangle$ 4 drawn

Let X be the no. of defective items

$P(X=0) = {}^7C_4 / {}^{10}C_4 = 35/210$

$P(X=1) = ({}^7C_3 \times {}^3C_1) / {}^{10}C_4 = 105/210$

$P(X=2) = ({}^7C_2 \times {}^3C_2) / {}^{10}C_4 = 63/210$

$P(X=3) = ({}^7C_1 \times {}^3C_3) / {}^{10}C_4 = 7/210$

$\left.\right\}$ $P(x)$

ii) $P(X \le 1) = P(X=0) + P(X=1) = 35/210 + \frac{105}{210} =$ Ans

iii) $P(0 < X < 2) = P(X=1) = \frac{105}{210}$

MATHEMATICAL EXPECTATIONS:

Mathematical expectation is a concept which is commonly used to refer to the value expected as the outcome of any game or strategy. Its origin refers to the theory of gambling and the games of chance.

Expectation of Discrete random Variable

Defn:

Let X be a discrete r.v. which associates values $x_1, x_2 \ldots x_n$, with probabilities $p(x_1), p(x_2), p(x_3) \ldots p(x_n)$ then mathematical expectation of X, denoted by E(X), is defined as:

$$E(X) = \sum_{i=1}^{n} x_i \, p(x_i), \quad i = 1, 2 \ldots n$$

If X can take any one of the values x_i, $i = 1, 2 \ldots \infty$ then,

$$E(X) = \sum_{i=1}^{\infty} x_i \, p(x_i), \quad \text{where } \sum_{i=1}^{\infty} p(x_i) = 1.$$

Theorems on expectation:

1. Addition Theorem:

Let $X, Y, Z \ldots T$ be the n discrete random variables. Then $E(X+Y+Z+\ldots+T) = E(X) + E(Y) + \ldots + E(T)$

Proof:

Let X be a discrete r.v. associates values $x_1, x_2 \ldots x_n$ with probabilities $p(x_1), p(x_2) \ldots p(x_n)$ and Y be a another r.v associates values $y_1, y_2 \ldots y_m$ with probabilities $p(y_1), p(y_2) \ldots p(y_m)$ respectively.

Then,

$$E(X) = \sum_{i=1}^{n} x_i\, p(x_i), \quad i = 1, 2 \cdots n,$$

and $$E(Y) = \sum_{j=1}^{m} y_j\, p(y_j), \quad j = 1, 2 \cdots m.$$

The Sum $(X+Y)$ is also a random Variable which takes $n \times m$ Values of $(x_i + y_j)$, $i = 1, 2 \cdots n$, $j = 1, 2 \cdots m$. Thus

$$E(X+Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(X = x_i, Y = y_j)\,(x_i + y_j)$$

$$= \sum_i \sum_j p(x_i, y_j)\,(x_i + y_j)$$

$$= \sum_i \sum_j p(x_i, y_j)\, x_i + \sum_i \sum_j p(x_i, y_j)\, y_j$$

$$= \sum_i x_i \sum_j p(x_i, y_j) + \sum_j y_j \sum_i p(x_i, y_j)$$

$$= \sum_i x_i\, p(x_i) + \sum_j y_j\, p(y_j) \quad \left[\because \sum_i p(x_i) = \sum_j p(y_j) = 1\right]$$

$$\therefore E(X+Y) = E(X) + E(Y)$$

Similarly for three Variables, X, Y, Z.

$$E(X+Y+Z) = E(M+Z) \quad [\because M = X+Y$$

$$= E(M) + E(Z)$$

$$= E(X+Y) + E(Z)$$

$$= E(X) + E(Y) + E(Z)$$

Similarly proceeding, the method of induction gives.

$$E(X+Y+Z+\cdots+T) = E(X) + E(Y) + E(Z) + \cdots + E(T)$$

Multiplication Theorem of Expectation:

The mathematical expectation, of the product of number of independent random variables is equal to the product of their expectations.

$$E(X \cdot Y \cdot Z \cdots T) = E(X) \cdot E(Y) \cdot E(Z) \cdots E(T)$$

Proof:

By Def^n we have,

$$E(X) = \sum_{i=1}^{n} x_i p(x_i), \quad E(Y) = \sum_{j=1}^{m} y_j p(y_j)$$

The joint Probability mass function

$$P(X = x_i, Y = y_j) = P(X = x_i)(Y = y_j) = P(x_i) \cdot p(y_j)$$

Then,

$$E(XY) = \sum_i \sum_j x_i, y_j \, p(x_i, y_j)$$

$$= \sum_i x_i p(x_i) \cdot \sum_j y_j \, p(y_j)$$

$$E(XY) = E(X) \cdot E(Y)$$

For 3 r.v.s $X, Y \& Z$

$$E(X \cdot Y \cdot Z) = E(MZ)$$

$$= E(M) \cdot E(Z) \quad [\because M = XY]$$

$$= E(X) \cdot E(Y) \cdot E(Z)$$

Hence by Method of induction we can get

$$E(X \cdot Y \cdot Z \cdots T) = E(X) \cdot E(Y) \cdot E(Z) \cdots E(T)$$

Note: Remarks

1) Expectation of Constant is constant itself.

$E(a) = a.$

ii) If a and b are constants then

$E(ax+b) = aE(x)+b.$

iii) If we take $\phi(x) = x^r$, then

$$E(x^r) = \sum_i p_i x_i^r$$

which is defined as r^{th} moment (about origin) of the probability distribution.

Thus

μ_r' (about origin) $= E(x^r)$ in particular

μ_1' (about origin) $= E(x) = \sum_i x_i \cdot p(x_i)$

μ_2' (about origin) $= E(x^2) = \sum_i x^2 p(x_i)$

Hence,

Mean $= \bar{x} = \mu_1'$ (about origin) $= E(x)$

$\mu_2 = \mu_2' - \mu_1'^2 = E(x^2) - [E(x)]^2$

$\mu_2 =$ Variance of x.

Particularly,

if $\phi(x) = (x-\bar{x})^r$, then

$$E(x-\bar{x})^r = \sum p_i (x_i - \bar{x})^r \quad \text{— (A)}$$

(A) is the r^{th} moment about mean.

If $r=2$, we get

$$\mu_2 = E(x-\bar{x})^2$$

$$= \sum_i p_i (x_i - \bar{x})^2$$

$$= E(x) - [E(x)]^2 \quad [\because \bar{x} = E(x)]$$

Variance of the random Variables:

Let X be a random variable with $E(X) = \mu$, then Variance of X is defined as:

$$V(X) = E\{X - E(X)\}^2$$
$$= E(X^2) - 2E(X)E(X) + [E(X)]^2$$
$$= E(X^2) - 2[E(X)]^2 + [E(X)]^2$$
$$= E(X^2) - [E(X)]^2$$
$$\therefore V(X) = E(X^2) - [E(X)]^2$$
$$= \sum_i x_i^2 p(x_i) - \left[\sum_i x_i p(x_i)\right]^2$$

Properties of Variance:

If a and b are two constants, then

i) Variance(a) = 0, i.e Variance of Constant is Zero.

ii) $V(ax+b) = a^2 V(x) + V(b)$
$$= a^2 V(x)$$

iii) $V(a_1 X \pm a_2 Y) = a_1^2 V(X) + a_2^2 V(Y) \pm 2a_1 a_2 Cov(X, Y)$

If X and Y are independent, then $Cov(X, Y) = 0$

and $V(a_1 X + a_2 Y) = a_1^2 V(X) + a_2^2 V(Y)$

Covariance:

The measure of Simultaneous Variation between the random Variables X & Y is called Covariance and denoted by $Cov(X,Y)$ or σ_{XY} & defined as:

$$Cov(X,Y) = E\{[X - E(X)][Y - E(Y)]\}$$
$$= E(XY) - E(X)E(Y) - E(X) \cdot E(Y) + E(X) \cdot E(Y)$$
$$= E(XY) - E(X) \cdot E(Y)$$

If X & Y are independent, then

$$Cov(XY) = E(X)E(Y) - E(X)E(Y) = 0$$

Expectation of Continuous random Variable

Let X be a continuous r.v. with probability density function $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

provided the integral is absolutely convergent.

Remark:

i) If $g(x)$ is a function of r.v. X, then

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Let $f_X(x)$ or $f(x)$ be probability density function of r.v. X and X is defined in $[a,b]$, then

Arithmetic Mean:

$$E(X) = \mu_1' = \int_a^b x f(x)\,dx$$

$$\mu_2' = E(x^2) = \int_a^b x^2 f(x)\,dx$$

$$V(X) = E(x^2) - [E(x)]^2$$

$$= \int_a^b x^2 f(x)\,dx - \left[\int_a^b x f(x)\,dx\right]^2$$

Covariance:

$$Cov(x,y) = E(XY) - E(X)\cdot E(Y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy - \int_{-\infty}^{\infty} x f(x)\,dx \int_{-\infty}^{\infty} y f(y)\,dy$$

Examples:

1. A random variable X has the prob distribution

X:	-1	0	1	2
$P(x)$:	$1/3$	$1/6$	$1/6$	$1/3$

Compute $E(X)$ & $V(X)$

Solⁿ:

$$E(X) = \sum_{x=1}^{2} x \cdot P(x)$$

$$= -1 \times \frac{1}{3} + 0 \times \frac{1}{6} + 1 \times \frac{1}{6} + 2 \times \frac{1}{3} = 1/2$$

$$E(X^2) = \sum x^2 \cdot P(x)$$

$$= (-1)^2 \times \frac{1}{3} + 0^2 \times \frac{1}{6} + 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{3} = \frac{11}{6}$$

$$\therefore V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{11}{6} - \left(\frac{1}{2}\right)^2 = \frac{19}{2}$$

2) An appliance dealer sells different models of freezer having capacity 13.5, 15.9, 19.1 Cu. ft of storage capacity. Let x be the amount of storage space purchased by the next customer to buy a freezer and $p(x)$ be the p.m.f given as

$X: x$	13.5	15.9	19.1
$P(x)$	0.2	0.5	0.3

Then a) Find $E(X)$ & $V(X)$

Solⁿ: $E(X) = \sum x \cdot P(x)$

$$= 13.5 \times 0.2 + 15.9 \times 0.5 + 19.1 \times 0.3$$

$$= 16.38$$

$E(x^2) = \sum x^2 \cdot P(x)$

$= 13.5^2 \times 0.2 + 15.9^2 \times 0.5 + 19.1 \times 0.3$

$= 272.29$

$V(x) = 272.29 - (16.38)^2 = 3.99.$

b) If the price of the freezer having capacity x cu.ft is $25x - 8.5$, what is the expected price paid by next Customer?

Solⁿ: Since $E(x) = 16.38$

E(Price Paid by next Customer)

$= 25 \times 16.38 - 8.5 = 401$

c) What is the Variance of the Price $25x - 8.5$ Paid by next customer?

Solⁿ:

Let $Y = 25x - 8.5$

$X: x$	$P(x)$	$Y: y$	$P(y)$
13.5	0.2	329	0.2
15.9	0.5	389	0.5
19.1	0.3	469	0.3

$E(Y) = \sum y \cdot P(y)$

$= 329 \times 0.2 + 389 \times 0.5 + 469 \times 0.3 = 401$

$E(Y^2) = 329^2 \times 0.2 + 389^2 \times 0.5 + 469^2 \times 0.3$

$= 163297$

$\therefore V(Y) = E(Y^2) - [E(Y)]^2$

$= 2496$

Hence Variance of the price Paid by next Customer

$= 2496$

Ex: 3 Find the expectation of the number on a die when thrown.
[or On average how many times must a die thrown until one get a 6 (or any 1, 2, 3, 4, 5)]

Solⁿ:

Let X be a r.v. representing the number on a die when thrown. X can take any one of the values $1, 2, 3 \ldots 6$, each with equal probability $\frac{1}{6}$, then

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5$$

$$E(X) = 3.5$$

or either 3 or 4 times on average to get 6.

Ex: 4 A coin is biased so that the head is twice as likely to appear as the tail, the coin is tossed twice, find the expected value of the number of heads.

Solⁿ: Let p & q denote the prob'y of getting head and tail, where $p + q = 1$.

Given that $p = 2q$.

$\therefore \quad p = \frac{2}{3}, \quad q = \frac{1}{3}$

Let X denote the no. of heads in two tosses

$(X = 0) = (TT) = q^2 = \frac{1}{9}$

$(X = 1) = (HT, TH) = 2pq = 2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$

$(X = 2) = (HH) = p^2 = \frac{4}{9}$

$E(x) = \Sigma x . p(x)$

$= 0 \times \frac{1}{9} + 1 \times \frac{4}{9} + 2 \times \frac{4}{9} = \frac{12}{9}$

$E(x) = 1\frac{1}{3}$ indicates that, average no. of heads that we are expected to get if the experiment performed repeatedly Since $1\frac{1}{3}$ is the actual values.

Ex 5: A gambler plays a game of rolling a die with the following rules. He will win 200 if he throws a 6, but will lose Rs 40 if throws 4 or 5 and lose Rs 20 if throws 1, 2 or 3, find the expected value that the gambler may gain.

Sol:

Let x denote the value that the gambler may win or lose,

$\{X = -40\} = \{4, 5\}$
$\{X = -20\} = \{1, 2, 3\}$ total 6 cases
$\{X = 200\} = \{6\}$

x	$p(x)$	$x . p(x)$
-40	$1/3$	$-40 \times 1/3$
-20	$1/2$	$-20 \times 1/3$
200	$1/6$	$200 \times 1/6$

$\therefore E(x) = \Sigma x . p(x)$

$= 10$

Thus the expected value that the gambler may gain is Rs 10.

(Refer Chebyshev's example)

Ex: 6 A pair of dice is rolled, if x is the sum of numbers find the probability distribution of X, & $E(X)$ & $V(X)$.

Soln: The prob'y distribution of x is

X:	2	3	4	5	6	7	8	9	10	11	12
$P(x)$:	$\frac{1}{36}$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Then $E(X) = \Sigma X . P(X)$

$= 2 \times 1/36 + 3 \times 2/36 + \cdots + 12 \times 1/36$

$= 7$

$E(x^2) = \Sigma x^2 . P(x)$

$= 2^2 \times 1/36 + 3^2 \times \frac{2}{36} + \cdots + 12^2 \times 1/36$

$= \frac{329}{6}$

$V(x) = E(x^2) - [E(x)]^2$

$= \frac{329}{6} - 49 = 5.833 = \frac{35}{6}$

Ex: 7 Six dice are thrown 729 times, how many times do you expected at least 3 dice show 5 or 6?

Soln:

Let X be a r.v. representing the no. on a die when thrown, X can take any one of the values 1, 2, 3 - - , 6 each with equal prob'y $\frac{1}{6}$.

$P(5) = \frac{1}{6}$, $P(6) = \frac{1}{6}$

$P(5 \text{ or } 6) = P(5) + P(6)$

$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

in 6 dice P(occurrence of at least 3) $= P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6}$

$\therefore$ When 6 dice thrown then Prob^y of occurrence of 5 or 6 $= \frac{1}{3} \times \frac{4}{6} = \frac{4}{18}$

When 6 dice thrown 729 times, then expected no. of times that at least 3 dice show 5 or 6

$= 729 \times \frac{4}{18} = 162$ #

Ex8: A coin is tossed until a head appears, what is the expected no. of tosses required?

Sol^n: Let p and q denote the probabilities of getting head and tail in a toss respectively. X ~~takes~~ denote the no. of tosses required until a head appears, then X takes values 1, 2, 3 ... with probabilities p, pq, pq^2, pq^3, pq^4 ... respectively. Here $p = q = \frac{1}{2}$.

Hence expected no. of tosses is given by

$$\mu = E(X) = \sum_x x.\,p(x)$$

$$= p + 2pq + 3pq^2 + 4pq^3 + \cdots$$

$$\text{or } E(X) = p(1 + 2q + 3q^2 + 4q^3 + 5q^4 + \cdots)$$

$$= p[1-q]^{-2}$$

$$= p(p)^{-2}$$

$$= (p)^{-1} = \left(\frac{1}{2}\right)^{-1}$$

$$\therefore E(X) = 2$$

(9) Five urn contains 5 green and 2 white balls, 7 Green and 6 white balls, 2G & 1W balls, 5G & 2W balls and 2G & 4W balls. One ball is drawn from each Urn find the expected number of white balls drawn out.

Soln:

I	II	III	IV	V
5G, 2W	2G, 1W	7G, 6W	5G, 2W	2G, 4W

Let X be the drawing a white ball in single draw.

For 1st Urn:

$$P(X=1) = \frac{^2C_1}{^7C_1} = 2/7$$

$$P(X=0) = 5/7$$

$$E(X_1) = \Sigma x.\, P(x)$$

$$= 1 \times \frac{2}{7} + 0 \times \frac{5}{7} = 2/7$$

Similarly for other 4 Urns

$$E(X_2) = 0 \times \frac{2}{3} + 1 \times 1/3 = 1/3$$

$$E(X_3) = 0 \times \frac{7}{13} + 1 \times \frac{6}{13} = 6/13$$

$$E(X_4) = 0 \times 5/7 + 1 \times 2/7 = 2/7$$

$$E(X_5) = 0 \times \frac{2}{6} + 1 \times 4/6 = 4/6$$

$$\therefore \quad E(X) = E(X_1) + E(X_2) + \cdots + E(X_5)$$

$$= \frac{2}{7} + \frac{1}{3} + \frac{6}{13} + \frac{2}{7} + \frac{4}{6} = 2.032$$

Ex1: The p.d.f of continuous r.v. is given as
$f(x) = Ax(1-x)^2, \quad 0 \le x \le 1$ elsewhere
find $E(x)$ and $V(x)$

Solⁿ:, We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1,$$

or $$A\int_0^1 (x^3 - 2x^2 + x)\,dx = 1$$

or $$A\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1$$

or $$A\left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right] = 1 \Rightarrow A = 12$$

$$E(x) = \int_0^1 x\,f(x)\,dx = 12\int_0^1 x.x(1-x)^2\,dx$$

$$= 12\int_0^1 (x^2 - 2x^3 + x^4)\,dx$$

$$= 12\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right)$$

$$= 12\left[\frac{20 - 30 + 12}{60}\right] = 2/5$$

$$E(x^2) = 12\cdot\int_0^1 x^3(1 - 2x + x^2)\,dx$$

$$= 12\left[\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right]_0^1 = \frac{1}{5}$$

$$\therefore V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{5} - \left(\frac{2}{5}\right)^2 = \frac{1}{25} \quad \text{Ans}$$

Ex 2: A random Variable X, has the density function $f(x) = a + bx^2$, $0 < x < 1$. Determine a and b So that mean is 2/3, also find the Variance of the distribution

Soln:

$E(X) = \mu = \frac{2}{3}$, $V(X) = ?$

We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

or $\int_0^1 (a + bx^2)\,dx = 1$

or $a\int_0^1 dx + b\int_0^1 x^2 dx = 1$

or $a[x]_0^1 + b\left[\frac{x^3}{3}\right]_0^1 = 1$

or $3a + b = 3$ — (I)

$E(X) = a\int_0^1 x\,dx + b\int_0^1 x^3 dx$

or $\frac{2}{3} = a\left[\frac{x^2}{2}\right]_0^1 + b\left[\frac{x^4}{4}\right]_0^1$

or $\frac{a}{2} + \frac{b}{4} = \frac{2}{3}$

or $6a + 3b = 8$ — (II)

Solving (I) & (II) we get $a = 1/3$, $b = 2$.

Again,

$$E(x^2) = \int_0^1 x^2(a + bx^2)\,dx$$

$$= a\left[\frac{x^3}{3}\right]_0^1 + b\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{a}{3} + \frac{b}{5} = \frac{1}{9} + \frac{2}{5} = \frac{23}{45}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{23}{45} - \left(\frac{2}{3}\right)^2 = \frac{3}{45} = \frac{1}{15}\ \text{Ans}$$

Ex 3: Let X be a continuous r.v. with p.d.f.

$$f(x) = ax, \quad 0 \le x \le 2$$
$$= a, \quad 2 \le x \le 4$$
$$= ax + 3a, \quad 4 \le x \le 6.$$
$$= 0, \quad \text{otherwise}$$

i) Determine a,

ii) Find $P(1.5 \le X \le 2.5)$

iii) Obtain $E(X)$

Soln: We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

or $$\int_0^2 ax\,dx + \int_2^4 a\,dx + \int_4^6 (ax+3a)\,dx = 1$$

or $$\left[\frac{ax^2}{2}\right]_0^2 + \left[ax\right]_2^4 + \left[\frac{ax^2}{2}\right]_4^6 + \left[3ax\right]_4^6 = 1$$

or $$2a + 2a + 10a + 6a = 1 \implies a = 1/20$$

ii) $$P(1.5 \le x \le 2.5) = \int_{1.5}^{2.5} f(x)\,dx$$

$$= \int_{1.5}^{2} ax\,dx + \int_2^{2.5} a\,dx$$

$$= \left[\frac{ax^2}{2}\right]_{1.5}^{2} + \left[ax\right]_2^{2.5}$$

$$= \frac{11}{40}$$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^2 ax^2 dx + \int_2^4 ax\, dx + \int_4^6 (ax^2 + 3ax) dx$$

$$= \left[\frac{ax^3}{3}\right]_0^2 + \left[\frac{ax^2}{2}\right]_2^4 + \left[\frac{ax^3}{3}\right]_4^6 + \left[\frac{3ax^2}{2}\right]_4^6$$

$$= \frac{8}{60} + \frac{12}{40} + \frac{152}{60} + \frac{80}{40} = 4.46$$ ✓

Ex:4 A r.v. x, has the p.d.f.

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find i) $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$ ii) $P\left(x > \frac{3}{4} \,/\, x > \frac{1}{2}\right)$

Sol^n: i) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} 2x\, dx = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

ii) $P\left(x > \frac{3}{4}\right) = \int_{3/4}^{1} 2x\, dx = \left[2 \cdot \frac{x^2}{2}\right]_{3/4}^{1} = \frac{7}{16}$

$P(x > 1/2 = \int_{1/2}^{1} 2x\, dx = \left[2 \cdot \frac{x^2}{2}\right]_{1/2}^{1} = 1 - \frac{1}{4} = 3/4$

$\therefore$ $P\left(x > 3/4 \,/\, x > 1/2\right)$

$$= \frac{7/16}{3/4} = \frac{7}{12}$$

Ex: 5 Let x be a continuous r.v. with p.d.f.

$$f(x) = Kx(2-x), \quad 0 < x < 2$$

find K, Mean, Variance and distribution function.

Solⁿ:

We have, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

or $\int_0^2 Kx(2-x)\,dx = 1 \Rightarrow K\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$

or $K\left[4/3\right] = 1 \quad \therefore K = 3/4$

$\therefore f(x) = \frac{3}{4}x(2-x), \quad 0 < x < 2$

Mean $E(X) = \int_0^2 x\, f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2-x)\,dx$

$= \frac{3}{4}\left[2.\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - 4\right] = 1.$

$\therefore E(X) = 1.$

Again,

$E(X^2) = \int_0^2 x^2 . \frac{3}{4}x(2-x)\,dx$

$= \frac{3}{4}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$

$= \frac{3}{4}\left[8 - \frac{32}{5}\right] = \frac{6}{5}$

$V(X) = E(X^2) - \left[E(X)\right]^2 = \frac{6}{5} - 1 = \frac{1}{5}$ #

To find Cumulative distribution function F.

if $x \le 0$, then $F(x) = P(X \le x) = 0$ [$\because 0 < x < 2$]

if $x \ge 2$, then $F(x) = P(X \le x) = 1$ [$\because$ Total prob$^y = 1$]

Then, by def^n,

$$F(x) = P(X < x), \text{ we let } 0 < x < 2$$

$$= \int_{t=0}^{x} f(t)\,dt$$

$$= \int_{0}^{x} \frac{3}{4} t(2-t)\,dt$$

$$= \frac{3}{4}\left[t^2 - \frac{t^3}{3}\right]_0^x$$

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]$$

Hence $F(x)$ is given by,

$$F(x) = \begin{cases} 0, & \text{for } x \le 0 \\ \frac{3}{4}\left(x^2 - \frac{x^3}{3}\right), & \text{for } 0 < x < 2 \\ 1, & \text{for } x \ge 2. \end{cases}$$

Ex:6 A Probability density function is given

$$f(x) = \begin{cases} \frac{x+2}{6}, & \text{if } 0 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find i) $P(X > 0)$

ii) $F(1.5)$

Solⁿ: $P(x > 0) = \int_0^2 f(x)\,dx + \int_2^\infty f(x)\,dx$

$= \int_0^2 \frac{(x+2)}{6}\,dx + 0$

$= \frac{1}{6}\left[\frac{x^2}{2} + 2x\right]_0^2 = 1.$

ii) $F(1.5) = P(X \le 1.5)$

$= \int_0^{1.5} \left(\frac{x+2}{6}\right) dx = \frac{1}{6}\left[\frac{x^2}{2} + 2x\right]_0^{1.5}$

$= \frac{1}{6}[1.125 + 3] = 0.6875$ #

Ex: 7 The diameter of an electric Cable say x, is assumed to be continuous r.v. with

$f(x) = 6x(1-x), \quad 0 \le x \le 1.$

i) Check whether f(x) is p.d.f.

ii) Find $P(-3 < x < 0.2)$

Solⁿ:

To be f(x) as pdf, we must have,

$\int_0^1 f(x)\,dx = 1.$

$\int_0^1 6x(1-x)\,dx = 1$

or $6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow 1 = 1$

Hence f(x) is p.d.f.

Ex:8 If x is a number of Points in a balance die, find the expected value of the random Variable $f(x) = 2x^2+1$.

Sol^n:

Here x denote the faces 1, 2, 3, 4, 5, 6. Then

$E(x) = \sum x \cdot p(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6}$

$= \frac{21}{6}$

$E(x^2) = 15.17$ using $\sum x^2 \cdot p(x)$

$E(2x^2+1) = 2E(x^2)+1$

$= 2 \times 15.17 + 1 = 31.34$

Ex 9: Let duration 'x' in minutes of long distance Calls from your home, follows exponential law with pdf $f(x) = \frac{1}{5} e^{-\frac{x}{5}}$ for $x>0$, and 0, otherwise. Then find $P(X>5)$, $P(3 \le X \le 6)$, Mean and Variance of X.

Sol^n:

$P(X>5) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{5}^{\infty} \frac{1}{5} e^{-\frac{x}{5}}\,dx$

or $P(X>5) = \left[-e^{-\frac{x}{5}}\right]_5^{\infty}$ $\left[\because \int e^{kx} = \frac{e^{kx}}{k}\right]$

$= e^{-1} = \frac{1}{e} = 0.3679$ #

Again,

$P(3 \le X \le 6) = \int_3^6 \frac{1}{5} e^{-\frac{x}{5}}\,dx$

$= \left[-e^{-\frac{x}{5}}\right]_3^6 = -e^{-\frac{6}{5}} + e^{-3/5} = 0.2476$ #

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} x \cdot \frac{1}{5} e^{-\frac{x}{5}} dx$$

Put $y = \frac{x}{5} \Rightarrow dx = 5dy$

or $E(X) = 5\int_{0}^{\infty} y e^{-y} dy$

Integrating we get

$E(X) = 5 \times \Gamma 2 \qquad \left[\because \int_{0}^{\infty} y e^{-y} = \Gamma 2 \right.$

$= 5$

Again,

$$E(X^2) = \int_{0}^{\infty} x^2 \cdot \frac{1}{5} e^{-\frac{x}{5}} dx$$

Put $y = \frac{x}{5}$

or $E(X^2) = 25\int_{0}^{\infty} y^2 \cdot e^{-\frac{x}{5}} dy$

Integrating,

$E(X^2) = 25 \times \Gamma 3 = 25 \times 2 = 50$

$\therefore V(X) = E(X^2) - [E(X)]^2$

$= 50 - 25$

$= 25$ #

Moments:

The r^{th} moment of a variable x, about any point $x=A$, usually denoted by μ_r' is defined as:

$$\mu_r' = \frac{1}{N} \sum f_i (x_i - A)^r, \quad \sum f_i = N$$

$$= \frac{1}{N} \sum f_i d_i^r, \quad \text{where } d_i = x_i - A$$

The r^{th} moment about variable mean $\bar{x}$, usually denoted by μ_r is given by

$$\mu_r = \frac{1}{N} \sum_i f_i (x_i - \bar{x})^r$$

$$= \frac{1}{N} \sum f_i z_i^r, \quad \text{where } z_i = (x_i - \bar{x})$$

In particular,

$$\mu_0 = \frac{1}{N} \sum_i f_i = 1, \quad \left[\because \frac{1}{N} \sum f_i (x_i - \bar{x})^0\right.$$

$$\mu_1 = \frac{1}{N} \sum_i f_i (x_i - \bar{x})^1$$

$$= 0 \quad \left[\because \sum (x_i - \bar{x}) = 0\right]$$

$$\mu_2 = \frac{1}{N} \sum f_i (x_i - \bar{x})^2 = \sigma^2$$

$$\therefore \left.\begin{aligned} \mu_0 &= 1, \\ \mu_1 &= 0 \\ \mu_2 &= \sigma^2 \end{aligned}\right\} \text{imp}$$

Relation between moments about mean interms of moments about any Point & vice versa:

we have,

$$\mu_r = \frac{1}{N}\sum_i f_i(x_i-\bar{x})^r$$

$$= \frac{1}{N}\sum_i f_i(x_i - A + A - \bar{x})^r$$

$$= \frac{1}{N}\sum_i f_i(d_i + A - \bar{x})^r \quad [\because d_i = x_i - A]$$

$$\text{or } \mu_r = \frac{1}{N}\sum_i f_i(d_i - \mu_1')^r \quad \left[\begin{array}{l} \because d_i = x_i - A \\ \bar{x} = A + \frac{1}{N}\sum f_i d_i = A + \mu_1' \end{array}\right]$$

$$\text{or } \mu_r = \frac{1}{N}\sum_i f_i\left[d_i^r - {}^rC_1 d_i^{r-1}\mu_1' + {}^rC_2 d_i^{r-2}\mu_1'^2 - {}^rC_3 d_i^{r-3}\mu_1'^3 + \cdots\right]$$

$$= \mu_r' - {}^rC_1 \mu_{r-1}'\mu_1' + {}^rC_2 \mu_{r-2}'\mu_1'^2 - \cdots + (-1)^r \mu_1'^r \quad \text{— (1)}$$

Putting $r = 2, 3, 4$ in (1) we get,

$$\mu_2 = \mu_2' - \mu_1'^2$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$\mu_4 = \mu_4' - 4\mu_2'\mu_1' + 6\mu_2'\mu_1' - 3\mu_1'^4$$

Conversely we can find,

$$\left.\begin{array}{l} \mu_2' = \mu_2 + \mu_1'^2 \\ \mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3 \\ \mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + \mu_1'^4 \end{array}\right\}$$

Moments of a random Variable:

Defn 1: Let X be a random Variable, k be any positive integers and c be a constant. Then the moment of order k or the k^{th} moment of X, about the point c is defined as: $E[(X-c)^k]$.

Defn 2: Let X be a random variable, the k^{th} moment of X, about the origin or simply the k^{th} moment of X, is denoted by μ_k' and defined as $E(X^k)$ i.e

$$\mu_k' = E(X^k), \quad k = 1, 2, 3, 4, \ldots$$

Here μ_k' are also known as raw moments. If X is a continuous random Variable, with pdf f, then

$$\mu_k' = \int_{-\infty}^{\infty} x^k f(x)\, dx.$$

If X is a discrete r.v, with pmf f, then

$$\mu_k' = \sum_i x_i^k f(x_i)$$

The moments of X, defined about $c = \mu$ are known as Central moments.

Note that $\mu_0' = 0$

$$\mu_1' = E(X) = \mu \text{ mean of } X.$$

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The moments of X, defined about $c = \mu$ are known as Central moments.

Note that $\mu_0' = 0$

$$\mu_1' = E(X) = \mu \text{ mean of } X.$$

Def^{n}3: Let X be a r.v., the k^{th} central moment of X is denoted by μ_k and is defined as:

$$\mu_k = E[(X-\mu)^k], \quad k = 1, 2, 3, 4.$$

If X is a continuous r.v. with pdf, f then

$$\mu_k = \int_{-\infty}^{\infty} (x-\mu)^k f(x)\, dx$$

If X is a discrete r.v. with p.m.f f, then

$$\mu_k = \sum_i (x-\mu)^k f(x_i)$$

Note: $\left.\begin{array}{l} \mu_0 = 1, \quad \mu_1 = 0 \\ \mu_2 = \mu_2' - \mu_1'^2 = \sigma^2 \end{array}\right\}$

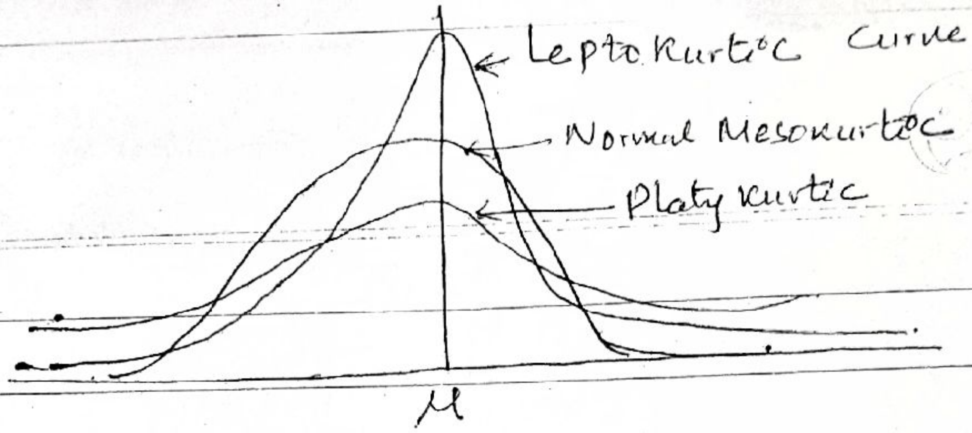
Def^{n}4: Karl Pearson's β & γ Coefficients:

Based on the first four central moments, Pearson's β and γ Coefficients can be defined as

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \quad \gamma_1 = +\sqrt{\beta_1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \quad \gamma_2 = \beta_2 - 3$$

The coefficients γ_1 and γ_2 are known as Coefficients of Skewness & Kurtosis, respectively.



If $\beta_2 = 3$, $\gamma_2 = 0$, the curve is mesokurtic i.e Normal.
$\beta_2 < 3$, $\gamma_2 < 0$, The curve is platykurtic
$\beta_2 > 3$, $\gamma_2 > 0$, the curve is leptokurtic.

For any distribution,

$\mu_0 = 1$

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2$

$\mu_3 = \mu_4' - 3\mu_3'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' + 3\mu_4'$

$\mu_0' = 0$

$\mu_1' = E(X) = \mu$

$\mu_2' = \mu_2 + \mu_1'^2$

$\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$

$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1' + \mu_1'^4$

Ex:1 Find the mean variance and coefficients β_1, β_2 of the random variable X, with p.d.f given by

$$f(x) = \begin{cases} k x^2 e^{-x} & , \text{ if } x \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

Soln:

We have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} k x^2 e^{-x}\,dx = 1$$

$$\Rightarrow k \int_0^{\infty} x^2 e^{-x}\,dx = 1$$

$$\Rightarrow k\,\Gamma 3 = 1 \Rightarrow k(2!) = 1$$

$\therefore\ k = 1/2$

Now the first four raw moments, μ_r' we have,

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \frac{1}{2}\int_0^{\infty} x^{r+2} e^{-x}\,dx = \frac{1}{2}\Gamma(r+3)$$

$$= \frac{(r+2)!}{2}$$

Hence, $\mu_1' = \frac{3!}{2} = \frac{3}{2}$, $\mu_2' = \frac{4!}{2} = 12$

$$\mu_3' = \frac{5!}{2} = 60, \quad \mu_4' = \frac{6!}{2} = 360$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 12 - 9 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 60 - 108 + 54 = 6$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 360 - 720 + 648 - 243$$

$$= 45$$

$\therefore$ Mean $\mu = \mu_1' = 3$

$\mu_2 = \sigma^2 = \mu_2' - \mu_1'^2 = 12 - 3^2 = 3$

$\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{36}{27} = \dfrac{4}{3}$ & $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{45}{9} = 5$

Ex 2:

The p.d.f. of X is given as:

$$f(x) = \begin{cases} 2(1-x) & , \text{ if } 0 < x < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

Find i) $E(X^r)$ ii) $E[(2x+1)^2]$

Soln:

By definition,

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^1 x^r \cdot 2(1-x)\,dx = 2\int_0^1 (x^r - x^{r+1})\,dx$$

$$= 2\left\{\int_0^1 x^r\,dx - \int_0^1 x^{r+1}\,dx\right\}$$

$$= 2\left\{\left[\frac{x^{r+1}}{r+1}\right]_0^1 - \left[\frac{x^{r+2}}{r+2}\right]_0^1\right\}$$

$$E(x^r) = \frac{2}{(r+1)(r+2)} \quad //$$

$$E\{(2x+1)^2\} = E(4x^2 + 4x + 1)$$

$$= 4E(x^2) + 4E(x) + 1$$

$$= 4\left(\frac{2}{3\times 4}\right) + 4\left(\frac{2}{2\times 3}\right) + 1 \quad \left[\because E(x^2) \Rightarrow r = 2\right]$$

$$= 3.$$

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Ex: 3 The density function of a random variable X is given by

$$f(x) = kx(2-x), \quad 0 \le x \le 2$$

Find k, Mean, Variance & r^{th} moment.

Solⁿ: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

or $k\int_0^2 x(2-x)\,dx = 1$

or $k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1 \Rightarrow k\left[4 - \frac{8}{3}\right] = 1$

$\therefore k = \frac{3}{4}$

Then the r^{th} raw moment of X, is

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^r \cdot x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\,dx$$

Integrating,

$$\mu_r' = \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[2\cdot\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$\therefore \mu_r' = \frac{3}{4}\left\{\frac{2 \cdot 2^r \cdot 2^2}{r+2} - \frac{2^r \cdot 2^3}{r+3}\right\}$$

$$= \frac{3}{4}\left\{\frac{2^{r+3}(r+3) - 2^{r+3}(r+2)}{(r+2)\cdot(r+3)}\right\}$$

$$= \frac{3}{4}\left\{\frac{2^{r+3}[\cancel{r}+3-\cancel{r}+2]}{(r+2)(r+3)}\right\}$$

$$= \frac{3 \cdot 2^{r+3} \cdot 2^{-2}}{(r+2)(r+3)} = \frac{3 \cdot 2^{r+1}}{(r+2)(r+3)} \#$$

Then, $\mu_1' = \frac{3 \cdot 2^{1+1}}{(1+2)(1+3)} = \frac{3 \times 4}{3 \times 4} = 1$

$\mu_2' = \frac{3 \cdot 2^3}{4 \times 5} = \frac{6}{5}$

Hence Mean $(x) = \mu_1' = 1$

$\text{Var}(x) = \mu_2' - \mu_1'^2 = \frac{6}{5} - 1 = \frac{1}{5} \#$

CHARACTERISTIC FUNCTION

Defⁿ: If X is any random variable, the complex valued function ϕ_X defined on R by,

$$\phi_X(t) = E(e^{itX}),\ t \in R$$

is called characteristic function (C.F). of the random variable X.

If X is a continuous r.v. with p.d.f f, then

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

If X is a discrete r.v X, with mass points x_i and probability mass function f, then

$$\phi_X(t) = \sum_i e^{itx_i} f(x_i)$$

Note:

i) The characteristic function is bounded by 1.

i.e.

$$|\phi_X(t)| \leq 1, \text{ for all } t.$$

ii) Let $\phi_X(t)$ be the characteristic function of the random variable X, then $\phi_X(0) = 1$.

$[\because\ \phi_X(0) = E(e^{i.0.x}) = E(1) = 1.$

Ex: 1) Find the Characteristic function of the discrete random variable X, whose probability function is given below.

X :	−1	1
f(X) :	1/2	1/2

Soln:

By definition Characteristic function of X is

$$\phi_x(t) = E(e^{itx}) = \sum_i e^{itx} f(x)$$

$$= e^{it(-1)} \cdot \frac{1}{2} + e^{it(1)} \cdot \frac{1}{2}$$

$$= \frac{1}{2}(e^{-it} + e^{it}) = \frac{1}{2}|2\cos t| = \cos t.$$

Ex2: Find the characteristic function of the Uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{Otherwise} \end{cases}$$

Also find Mean and Variance.

Soln:

By defn

$$\phi_x(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

or $$\phi_x(t) = \int_a^b e^{itx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{itx}}{it}\right]_a^b$$

or $\phi_x(t) = \frac{1}{b-a}\left[\frac{e^{itb}}{it} - \frac{e^{ita}}{it}\right]$

$$= \frac{e^{itb} - e^{ita}}{(b-a)\, it} \quad \#$$

Let $e^{itb} - e^{ita} = it(b-a) + \frac{i^2t^2(b^2-a^2)}{2!} + \frac{i^3t^3(b^3-a^3)}{3!} + \cdots$

$\therefore \phi_x(t) = \frac{e^{itb} - e^{ita}}{(b-a)\, it} = 1 + \frac{it(b+a)}{2} + \frac{i^2t^2(b^2+ab+a^2)}{3!} + \cdots$

Now,

$\mu_1' =$ coeff. of $\frac{it}{1!}$ in $\phi_x(t) = \frac{b+a}{2}$

$\mu_2' =$ coeff. $\frac{(it)^2}{2!}$ in $\phi_x(t) = \frac{b^2+ab+a^2}{3}$

$\therefore \mu = \mu_1' = \frac{a+b}{2}$

$Var(x) = \mu_2' - \mu_1'^2$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

Chebyshev's Inequality:

The role of standard deviation as a parameter to characterize variance is precisely interpreted by the means of Chebyshev's inequality. If σ is small there is high probability of getting values closer to the mean and if σ is large, there is corresponding high probability of getting values farther away from the mean.

The Chebyshev's inequality states that "If a probability distribution has mean μ and standard deviation σ, the probability of getting a value which deviates from μ, by at least $k\sigma$ is at most $1/k^2$.

Theorem on Chebyshev's inequality:

Let X be a random variable with mean μ and finite variance σ^2, then for any real number $k>0$, we have,

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\text{or} \quad P\{|X-\mu| < k\sigma\} \geq \left(1 - 1/k^2\right)$$

Ex: 1 A fair die is thrown and x denote the number then use chebyshev's inequality to show that
$$P\{|X-3.5|>2.5\} < 0.47$$

Sol: When a fair die is thrown, Then x denote the number 1, 2, 3, 4, 5, 6 with equal Prob'y $1/6$.

Then $E(x) = \Sigma x \cdot p(x) = \frac{1}{6}(1+2+\cdots+6) = 7/2 = 3.5$

$$E(x^2) = \Sigma x^2 p(x) = \frac{1}{6}(1^2+2^2+\cdots+6^2) = \frac{91}{6}$$

$$Var(x) = \sigma^2 = \frac{91}{6} - \frac{7}{2} = \frac{35}{12} = 2.9167$$

By using Chebyshev's inequality

$$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$$

Taking $c = 2.5$, from question we get
$$P\{|X-3.5|>2.5\} < \frac{2.9167}{6.25} = 0.4667$$

$$\therefore P\{|X-3.5|>2.5\} = 0.47 \ \#$$

Ex 2: Two unbiased dice are thrown, Let x be the random variable that represents the sum of the numbers showing up, prove by chebyshev's inequality
$$P\{|X-7| \geq 2\} \leq \frac{35}{24}$$
Compare this result with actual probability.

Sol^n:

The probability dist^n of the sum of numbers X is:

X	2	3	4	5	6	7	8	9	10	11	12
p(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

X:	favorable cases	probability
2	(1, 1)	1/36
3	(1,2) (2,1)	2/36
4	(1,3), (3,1), (2,2)	3/36
5	(1,4), (4,1), (2,3), (3,2)	4/36
6	(1,5), (5,1), (2,4), (4,2), (3,3)	5/36
7	(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)	6/36
8	(2,6), (6,2), (3,5), (5,3) (4,4)	5/36
9	(3,6), (6,3), (4,5), (5,4)	4/36
10	(4,6), (6,4), (5,5)	3/36
11	(5,6), (6,5)	2/36
12	(6,6)	1/36

$\therefore$ $E(x) = \Sigma x \cdot p(x) = 7$

$E(x^2)$ $\Sigma x^2 p(x) = 329/6$

$V(x) = E(x^2) - [E(x)]^2 = \frac{35}{6} = \sigma^2$

By using chebyshev's inequality

$$P\{|x-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

$$\text{or } P\{|x-7| \geq 2\} \leq \frac{35/6}{4} = \frac{35}{24} \quad [\because k=2]$$

Actual Probability

$$P\{|X-7| \geq 2\} = 1 - P\{|X-7| < 2\}$$
$$= 1 - P\{-2 < X-7 < 2\}$$
$$= 1 - P\{7-2 < X-7+7 < 2\}$$
$$= 1 - P\{5 < X < 9\}$$
$$= 1 - \{P(X=6) + P(X=7) + P(X=8)\}$$
$$= 1 - \left\{\frac{5}{36} + \frac{6}{36} + \frac{5}{36}\right\}$$
$$\therefore P\{|X-7| \geq 2\} = 1 - \frac{16}{36}$$
$$= \frac{20}{36} = 0.556$$

Thus actual probability is 0.556, while the upper bound given by Chebyshev's inequality is $\frac{35}{24} = 1.458$

Ex: 3 A random variable X has mean 10, variance 4 and unknown probability distribution, find the value of K such that $P\{|X-10| \geq K\} = 0.04$

Soln: Given that, $\mu = 10$, $\sigma^2 = 4$, then using Chebyshev's inequality

$$P\{|X-\mu| \geq K\} \leq \frac{\sigma^2}{K^2}$$

or $P\{|X-10| \geq K\} \leq \frac{4}{K^2}$

We find k, so that

$$\frac{4}{k^2} = 0.04 \Rightarrow k^2 = \frac{4}{0.04}$$

$$\Rightarrow k^2 = 100 \quad \therefore k = 10$$

The initial given inequality becomes.

$$P\{|X-10| \geq 10\} \doteq 0.04 \ \#$$

Ex: 4 A random variable X has a mean $\mu = 8$ and Variance $\sigma^2 = 9$, and an unknown probability distribution, then find i) $P(-4 < X < 20)$ & ii) $P(|X-8| \geq 6)$

Soln:

Using Chebyshev's inequality we have,

i) $$P\{|X-\mu| < k\} \geq 1 - \frac{\sigma^2}{k^2}$$

Thus,

i) $$P(-4 < X < 20) = P(|X-8| < 12) \geq 1 - \frac{9}{12^2}$$

$$= \frac{135}{144} = 0.9375$$

ii) Again,

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

$$P\{|X-8| \geq 6\} \leq \frac{9}{6^2} = \frac{1}{4}$$

$$\therefore P\{|X-8| \geq 6\} = 0.25$$

#

Chebyshev's Theorem with large no. of Bernoulli Trail

5. Show that for 40,000 flips of a balanced coin, the probability is at least 0.99, that the proportion of heads will fall between 0.475 and 0.525

Sol^n:

Let X be a random variable, that denote the no. of head, p be the proby of occurrence of head.

Then,

$$\text{Mean } \mu = np = 40{,}000 \times \frac{1}{2} = 20{,}000$$

$$\sigma = \sqrt{npq} = \sqrt{40{,}000 \times \frac{1}{2} \times \frac{1}{2}} = 100$$

and,

$$\{P|X-\mu| < \sigma k\} \geq 1 - \frac{1}{k^2}$$

Given that $1 - \frac{1}{k^2} = 0.99$

or $1 - 0.99 = \frac{1}{k^2}$ or $0.01k^2 = 1$

$\therefore k = 10.$

Chebyshev's theorem tells that the proby is at least 0.99 that we will get between $20{,}000 - 10(100) = 19{,}000$ & $20{,}000 + 10(100) = 21{,}000$. Hence the proby 0.99 that the proportion head falls between

$$\frac{19{,}000}{40{,}000} = 0.475 \quad \& \quad \frac{21{,}000}{40{,}000} = 0.525$$

Ex:6 Show that for 1 million flips of a balanced coin, the Prob^y is at least 0.99 that the proportion of heads will fall between 0.495 & 0.505.

Sol^n:

$n = 10,00,000$

p = Prob^y of getting head $= 1/2$, $q = 1/2$

Mean of the distribution

$\mu = np = 10,00,000 \times \frac{1}{2} = 5,00,000$

$\sigma = \sqrt{npq} = \sqrt{10,00,000 \times \frac{1}{2} \times \frac{1}{2}}$

$\therefore \sigma = 500$

According to Chebyshev's inequality

$P\{|X - \mu| < k\sigma\} \geq 1 - \frac{1}{k^2}$

where, $1 - \frac{1}{k^2} = 0.99 \Rightarrow 1 - 0.99 = 1/k^2$

or $0.01\,k^2 = 1 \quad \therefore k = 10$

Here,

$P\{|X - \mu| < k\sigma\} \geq 0.99$

$P\{|X - \mu| < k\sigma\} = P\{-k\sigma < (X - \mu) < k\sigma\}$

$= P\{\mu - k\sigma < X < \mu + k\sigma\}$

$= P\{500000 - 5000 < X < 500000 + 5000\}$

$= P\{495000 < X < 505000\}$

$\therefore$ Min no. of head $= 4,95,000$ & Max no of head $= 5,05,000$

The Min proportion of head $= \frac{495,000}{10,00,000} = 0.495$

Max proportion of head $= \frac{505,000}{10,00,000} = 0.505$

Q.7 Find the probability that the 10000 times tosses of a fair coin, the proportion of head will fall between 0.475 & 0.525.

Soln:

Prob of getting head $(P) = 1/2$, $q = 1/2$

Since the distribution is binomial,

Mean $E(X) = \mu = np = 10000 \times 1/2 = 5000$

$V(X) = npq = 10000 \times 1/2 \times 1/2 = 2500$

$\sigma = \sqrt{npq} = 50$

Now;

Lower limit $= 10{,}000 \times 0.475 = 4750$

Upper limit $= 10{,}000 \times 0.525 = 5250$

By chebyshev's inequality

$$P\{|X-\mu| < K\sigma\} \geq 1 - 1/K^2$$

or $P\{\mu - K\sigma < X < \mu + K\sigma\} \geq 1 - 1/K^2$

or $P\{4750 < X < 5250\} = ?$

Comparing lower limit

$$\mu - K\sigma = 4750$$

$$K = \frac{\mu - 4750}{\sigma} = \frac{5000-4750}{50}$$

$$K = \frac{\cancel{250}}{\cancel{50}} = 5$$

$\therefore$ $1 - 1/K^2 = 1 - 1/5^2 = \frac{24}{25} = 0.96$

$\therefore$ $P(4750 < X < 5250) = 0.96$ #