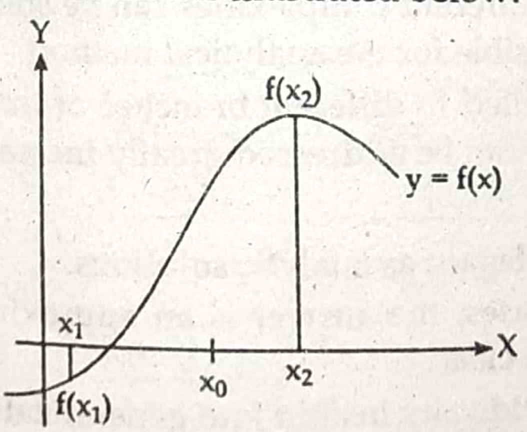
Chapter 2

SOLUTIONS OF NON–LINEAR EQUATIONS

2.1 Bisection Method

Bisection method is also known of half interval method or binary chopping method. This method is based on the theory that if $f(x_1)$ and $f(x_2)$ are two outputs of a function $f(x)$, then there exists at least one root in between x_1 and x_2 provided that $f(x_1)f(x_2) < 0$. The exact value of root is approached by bisecting the interval of x_1 and x_2 successively until the desired accuracy is achieved. The bisection method is illustrated below:



Let above figure is the plot of $y = f(x)$ where, $f(x_1)$ and $f(x_2)$ are outputs at x_1 and x_2 respectively such that $f(x_1)f(x_2) < 0$. Then from the assumption of bisection method, the root lies at any point x_0 in between x_1 and x_2. The initial value of x_0 is given by;

$$x_0 = \frac{x_1 + x_2}{2}$$

If $f(x_0) = 0$, the root lies at x_0

Otherwise, if $f(x_1)f(x_0) < 0$, then root lies between x_1 and x_0.

Thus, taking $x_2 = x_0$, we again make another assumption for x_0 as $\frac{x_1 + x_2}{2}$ and proceed.

Similarly, if $f(x_2)f(x_0) < 0$, then the root lies between x_2 and x_0.

Thus, taking $x_1 = x_0$, we again make another assumption for x_0 as $\frac{x_1 + x_2}{2}$ and proceed.

Example 2.1

Find the root of the equation $x^2 - 4x - 10 = 0$ correct up to three decimal places.

Solution:

For the given equation:

$$f(x) = x^2 - 4x - 10 = 0$$

Let's choose two points x_1 and x_2 such that;

$$f(x_1)f(x_0) < 0$$

Note

The two points can be chosen randomly by trial and error method. Otherwise the following method can be used for initial guess, which assumes that the desired interval is a set of interval $\pm|x_{max}|$.

Here,

$$|x_{max}| = 1 + \frac{1}{a_n} \times \text{Maximum}[|a_{n-1}|, |a_{n-2}|, \ldots\ldots, |a_0|]$$

where, a_n is the coefficient of the term with highest degree of x.

$$\text{Maximum}[|a_{n-1}|, |a_{n-2}|, \ldots\ldots, |a_0|] = \text{Maximum coefficient in equation}$$

Thus,

$$|x_{max}| = 1 + \frac{1}{1} \times 10 = 1 + 10 = 11$$

The desired interval lies in ± 11, so tabulate the values of function in the range ± 11.

x	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
f(x)	155	130	107	86	67	50	35	22	11	2	-5			

3	4	5	6	7	8	9	10	11

No need to make further calculation beyond - 1

Since, f(x) changes its sign form points - 2 to - 1

Here,

$$f(x_1) = f(-2) = 2$$
$$f(x_2) = f(-1) = -5$$

and, $f(-2)f(-1) < 0$; so, the solution lies between points - 2 and - 1.

For initial guess of x_0; we have,

$$x_0 = \frac{x_1 + x_2}{2} = \frac{-2-1}{2} = -1.5$$

and, $f(x_0) = f(-1.5) = -1.75$

Since, $f(x_1)f(x_0) < 0$, So root lies between x_1 and x_0. Thus, replacing x_2 with x_0, *i.e.,*, $x_2 = x_0$, we make another assumption for x_0 as;

$$x_0 = \frac{x_1 + x_2}{2} = \frac{-2-1.5}{2} = -1.75$$

This iteration is shown in the table below:

Iteration no.	(A)	(B)	(C)	(D)	(E)	(F)	
	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_0)$	Replace
1	- 2	- 1	- 1.5	2	- 5	- 1.75	$x_2 = x_0$
2	- 2	- 1.5	- 1.75	2	- 1.75	0.0625	$x_1 = x_0$
3	- 1.75	- 1.5	- 1.625	0.0625	- 1.75	- 0.8593	$x_2 = x_0$
4	- 1.75	- 1.625	- 1.6875	0.00625	- 0.8593	- 0.4023	$x_2 = x_0$
5	- 1.75	- 1.6875	- 1.734375	0.0625	- 0.4023	- 0.17089	$x_2 = x_0$
6	- 1.75	- 1.71875	- 1.734375	0.0625	- 0.17089	- 0.05444	$x_2 = x_0$
7	- 1.75	- 1.734375	- 1.74218	0.0625	- 0.05444	0.003967	$x_1 = x_0$
8	- 1.74218	- 1.734375	- 1.73827	- 0.0625	- 0.05444	- 0.02528	$x_2 = x_0$
9	- 1.74218	- 1.73827	- 1.740225	0.0039111	- 0.025337	- 0.010716	$x_2 = x_0$
10	- 1.74218	- 1.741205	- 1.741205	0.0039111	- 0.010716	- 0.0034	$x_2 = x_0$
11	- 1.74218	- 1.741205	- 1.7416925	0.0039111	- 0.0033	0.00026	$x_1 = x_0$
12	- 1.7416925	- 1.541205	- 1.741448				

Since, at iteration 12; x_1, x_2 and x_0 are identical up to 3 decimal places, we stop the iteration here.

Thus, the root of the equation $x^2 - 4x - 10 = 0$ is (-1.741).

Note

The above iteration table of calculated generally consumes lots of time. So, the above table can be programmed in programmable calculators (like fx-991 ES PLUS) as;

$A : B : C = \frac{A+B}{2} : D = A^2 - 4A - 10 : E = B^2 - 4B - 10 : F = C^2 - 4C - 10$ $\boxed{\text{CALC}}$

When the calculator is coded like above and $\boxed{\text{CALC}}$ button is pressed, following procedure should be followed:

i) When A? is displayed, give value of A and press (=)
ii) When B? is displayed, give value of B and press (=)
iii) Continue pressing (=) button, and the result for each column is displayed.
iv) Once the series is completed, provide replaced value of x_1 and x_2 for A? and B? for next iteration and proceed the calculation so on.

Example 2.2

Find a root of $x^3 - 2x - 5 = 0$ using bisection method with accuracy 0.08% and with error less than 0.00001.

Solution:

For the given equation;

$$f(x) = x^3 - 2x - 5 = 0$$

Let's choose two points x_1 and x_2 such that;

$$f(x_1)(x_2) < 0$$

Here,

$$|x_{max}| = 1 + \frac{1}{a_n} \times \text{Maximum}[|a_{n-1}|, |a_{n-2}|, \ldots\ldots, |a_0|]$$

$$= 1 + \frac{1}{1} \times 5$$

$$= 6$$

Thus, the desired interval lies in ±6, so tabulate the values of function in the range ±6.

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
f(x)	-209	-120	-61	-26	-9	-4	-5	-6	-1	16	51	110	199

Since, f(x) changed its sign form point 2 to 3, the root lies between 2 to 3.

Here,

$$f(x_1) = f(2) = -1$$

$$f(x_2) = f(3) = 16$$

and, $f(2)f(3) < 0$; so, the bisection is carried out in the interval of 2 and 3.

For initial guess of x_0; we have,

$$x_0 = \frac{x_1 + x_2}{2}$$

$$= \frac{2+3}{2} = 2.5$$

$$f(x_0) = 5.62$$

Since, $f(x_1)f(x_0) < 0$, So root lies between x_1 and x_0. Thus, replacing x_2 with x_0 *i.e.,,* $x_2 = x_0$, we make another assumption for x_0 as;

$$x_0 = \frac{x_1 + x_2}{2} = \frac{2 + 2.5}{2}$$

$$= 2.25$$

This iteration is shown in the table below:

Iteration no.	(A) x_1	(B) x_2	(C) x_0	(D) $f(x_1)$	(E) $f(x_2)$	(F) $f(x_0)$	(X) Error $\lvert x_1 + x_2 \rvert$	(Y) % Error $\frac{x_1 - x_2}{2} \times 100\%$	Replace
1	2	3	2.5	-1	16	5.62	1	40	$x_2 = x_0$
2	2	2.5	2.25	-1	5.62	1.89	0.5	2.22	$x_2 = x_0$
3	2	2.25	2.125	-1	1.89	0.34	0.25	11.76	$x_2 = x_0$
4	2	2.125	2.0625	-1	0.34	-0.35	0.125	6.06	$x_1 = x_0$
5	2.0625	2.125	2.09375	-0.35	0.34	-0.0089	0.0625	2.985	$x_1 = x_0$
6	2.09375	2.125	2.10937	-0.0089	0.34	0.166	0.0312	1.481	$x_2 = x_0$
7	2.09375	2.10937	2.10156	-0.0089	0.166	0.078	0.0156	0.74	$x_2 = x_0$
8	2.09375	2.10156	2.09765	-0.0089	0.078	0.034	0.0078	0.372	$x_2 = x_0$
9	2.09375	2.09765	2.0957	-0.0089	0.034	0.012	0.0039	0.186	$x_2 = x_0$
10	2.09375	2.0957	2.09472	-0.0089	0.012	0.0019	0.0019	0.09	$x_2 = x_0$
11	2.09375	2.09472	2.09423	-0.0089	0.0018	-0.0035	0.00097	0.046	$x_1 = x_0$
12	2.09423	2.09472	2.094475	-0.0035	0.0018	-0.0008	0.00049	0.023	$x_1 = x_0$
13	2.094475	2.09472	2.094597	-0.0008	0.0018	0.0005	0.00024	0.011	$x_2 = x_0$
14	2.094475	2.094597	2.094536	-0.0008	0.0005	-0.00017	0.00012	0.005	$x_1 = x_0$
15	2.094536	2.094597	2.094566	-0.00017	0.0005	0.00016	0.00006	0.0029	$x_2 = x_0$
16	2.094536	2.094566	2.094554	-0.00017	0.00016	-0.000005	0.00003	0.0014	$x_1 = x_0$
17	2.094551	2.094566	2.094558	-0.000005	0.00016	0.00007	0.000015	0.0007	$x_2 = x_0$
18	2.094551	2.094558	2.094554	-0.000005	0.00007	0.00003	0.000001	0.0003	

Since, at iteration 18, error is less than 0.00001 and accuracy more precise than 0.08%, we stop the iteration here.

Thus, the root of the equation $x^3 - 2x - 5$ is 2.094558.

Note

The code for calculator is;

$$A : B : C = \frac{A+B}{2} : D = A^3 - 2A - 5 : E = B^3 - 2B - 5 : F = C^3 - 2C - 5 :$$

$$X = A - B : Y = X \times \frac{100}{C} \boxed{\text{CALC}}$$

Example 2.3

Find the root of the equation $\sin x - 3x + 2 = 0$ Correct up to 3 decimal places.

Solution:

For the given equation:

$$f(x) = \sin x - 3x + 2 = 0$$

Let's choose two points x_1 and x_2 by inspection such that;

$$f(x_1)f(x_2) < 0$$

Let the two points be 0 and 1.

Here,

$$f(x_1) = f(0) = 2$$
$$f(x_2) = f(1) = -0.9825$$

and, $f(0)f(1) < 0$; so the bisection is carried out in the interval of 0 and 1.

For initial guess of x_0, we have

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$

and, $f(x_0) = 0.508$

Since $f(x_2)(x_0) < 0$, So root lies between x_0 and x_2. Thus, replacing x_1 with x_0, *i.e.,*, $x_1 = x_0$, we make another assumption for x_0 as;

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0.5+1}{2} = 0.75$$

This iteration is shown in the table below:

Iteration no.	(A) x_1	(B) x_2	(C) x_0	(D) $f(x_1)$	(E) $f(x_2)$	(F) $f(x_0)$	Replace
1	0	1	0.5	2	-0.98	0.50	$x_1 = x_0$
2	0.5	1	0.75	0.50	-0.98	-0.23	$x_2 = x_0$
3	0.5	0.75	0.625	0.50	-0.23	0.13	$x_1 = x_0$
4	0.625	0.75	0.6875	0.13	-0.23	-0.05	$x_2 = x_0$
5	0.625	0.6875	0.65625	0.13	-0.05	0.04	$x_1 = x_0$
6	0.65625	0.6875	0.67187	0.04	-0.05	-0.003	$x_2 = x_0$
7	0.65625	0.67187	0.66406	0.04	-0.003	0.01	$x_1 = x_0$
8	0.66406	0.67187	0.66796	0.01	-0.003	0.007	$x_1 = x_0$
9	0.66796	0.67187	0.66991	0.007	-0.003	0.001	$x_1 = x_0$
10	0.66991	0.67187	0.67089	0.001	-0.003	-0.0009	$x_2 = x_0$
11	0.66991	0.67089	0.6704	0.001	-0.0009	0.0005	$x_1 = x_0$
12	0.6704	0.67089	0.67064	0.0005	-0.0009	-0.0002	

Since, at iteration 12; x_1, x_2 and x_0 are identical up to 3 decimal places, we stop the iteration here.

Thus, the root of the equation $\sin x - 3x + 2$ is 0.67064.

Note

The code for calculator is:

$A : B : C = \frac{A+B}{2} : D = \sin(A) - 3A + 2 : E = \sin(B) - 3B + 2 : F = \sin(C) - 3C + 2$ $\boxed{\text{CALC}}$

Algorithm (Pseudo-Code) for Bisection Method

1. Start
2. Input initial guess x_1 and x_2 and initialize error E
3. Compute $f(x_1)$ and $f(x_2)$
4. If $f(x_1)(x_2) > 0$, go to step 2.
5. Compute $x_0 = \frac{x_1 + x_2}{2}$ and $f(x_0)$
6. If $f(x_1) \times f(x_0) < 0$

   Set $x_2 = x_0$

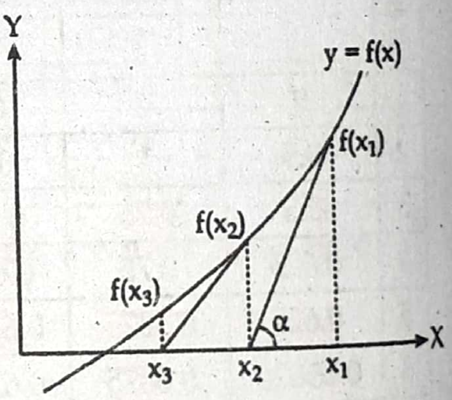
   Else

   Set $x_1 = x_0$
7. If $\left|\frac{x_1 + x_2}{x_0}\right| < E$, root $= x_0$

   Else

   go to step 5
8. Stop

2.2 Newton Raphson method (Two equation solution)



Newton Raphson method is a powerful technique to solve algebraic equations using differential calculus. It is based on the theory of linear approximation starting form a good initial guess. The Newton Raphson method is illustrated aside:

Let above figure is the plot of $y = f(x)$ where $f(x_1)$, $f(x_2)$ and $f(x_3)$ are outputs at x_1, x_2 and x_3 respectively. We draw a tangent line at point $y = f(x_1)$ which makes an angle α with positive x-axis. From the geometry of the above figure; we have,

$$\tan \alpha = \frac{f(x_1)}{x_1 - x_2}$$

or, $$f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

or, $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The above equation gives the approximating of x_2 form x_1. Similarly, the approximation of x_3 can be obtained from x_2. This process is continued until the solution converges to desired accuracy.

The general formula of Newton Raphson's iteration is given by;

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

where, $i = 1, 2, 3, \ldots\ldots$ and so on.

Example 2.4

Find the root of the equation $2x^2 + 4x - 5 = 0$ using Newton Raphson method correct up to 4 decimal places.

Solution:

The given equation is;

$$f(x) = 2x^2 + 4x - 5$$

The first derivative of given equation is given by:

$$f'(x) = 4x + 4$$

Let, the initial guess be $x_1 = 5$.

From Newton Raphson formula; we have,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The iteration table for Newton Raphson formula is given below:

Number of Iteration	(A) x_1	(B) $f(x_1)$	(C) $f'(x_1)$	(D) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	5	65	24	2.29166
2	2.29166	14.67005	13.16664	1.17747
3	1.17747	2.48275	8.70988	0.89242
4	0.89242	0.16251	7.56968	0.87095
5	0.87095	9.07×10^{-4}	7.4838	0.87082
6	0.87082	-6.51×10^{-5}	7.48328	0.87082
7	0.87082			

Since, at iteration 6 and 7, value of x_1 is identical up to 4 decimal places, we stop the iteration here.

Thus, the root of the equation $2x^2 + 4x - 5 = 0$ is 0.87082.

Note

Programming code calculator is;

$$A : B = 2A^2 + 4A - 5 : C = 4A + 4 : D = A - \frac{B}{C} \ \boxed{\text{CALC}}$$

Put the value of initial guess for A?

Example 2.5

Find the root of the equation $3x - e^{-x} - 2 = 0$. Correct up to 4 decimal places.

Solution:

The given equation is;

$$f(x) = 3x - e^{-x} - 2$$

The first derivative of given equation is given by;

$$f'(x) = 3 + e^{-x}$$

Let the initial guess be 4 *i.e.,,* $x_1 = 4$

From Newton Raphson formula; we have,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The iteration table for Newton Raphson formula is given below:

Number of Iteration	(A) x_1	(B) $f(x_1)$	(C) $f'(x_1)$	(D) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	4	9.98168	3.01831	0.69296
2	0.69296	- 0.42121	3.50009	0.81330
3	0.81330	-3.49×10^{-3}	3.44339	0.81431
4	0.81431	-1.48×10^{-5}	3.44294	0.81431
5	0.81431			

Since at iteration 4 and 5, value of x_1 is identical up to 4 decimal places, we stop the iteration here.

Thus, the root of the equation $3x - e^{-x} - 2 = 0$ is 0.81431.

Note

The programming code for the calculator is;

$$A : B = 3A - e^{-A} - 2 : C = 3 + e^{-A} : D = A - \frac{B}{C} \boxed{\text{CALC}}$$

Put the value of initial guess for A?

Example 2.6

Find the root of the equation $\sin x - 3x + 2 = 0$. Correct up to 4 decimal places.

Solution:

The given equation is;

$$f(x) = \sin x - 3x + 2 = 0$$

The first derivative of the given equation is given by;

$$f'(x) = \cos x - 3$$

Let the initial guess be 4 *i.e.,,* $x_1 = 4$

From Newton Raphson formula; we have,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The iteration table for Newton Raphson formula is given below:

Number of Iteration	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	4	- 9.93024	- 2.00243	- 0.95908

2	- 0.95908	4.86050	- 2.00014	1.47100
3	1.47100	- 2.38733	- 2.00033	0.27753
4	0.27753	1.17225	- 2.00001	0.86365
5	0.86365	- 0.57587	- 2.00011	0.57573
6	0.57573	0.28286	- 2.00005	0.71715
7	0.71715	- 0.13893	- 2.00007	0.64769
8	0.64769	0.06823	- 2.00006	0.68181
9	0.68181	- 0.03353	- 2.00007	0.66505
10	0.66505	0.01646	- 2.00006	0.67328
11	0.67328	-8.09×10^{-3}	- 2.00007	0.66924
12	0.66924	3.96×10^{-3}	- 2.00007	0.67122
13	0.67122	-1.94×10^{-3}	- 2.00007	0.67025
14	0.67025	9.47×10^{-4}	- 2.00007	0.67072
15	0.67072	-4.54×10^{-4}	- 2.00007	0.67049
16	0.67049	2.31×10^{-4}	- 2.00007	0.67061
17	0.67061	-1.26×10^{-4}	- 2.00007	0.67055
18	0.67055	5.30×10^{-5}	- 2.00007	0.67057
19	0.67057			

Since at iteration 18 and 19, value of x_1 is identical up to 4 decimal places; we stop the iteration here.

Thus, the root of the equation $\sin x - 3x + 2 = 0$ is 0.67057.

Note

Programming code for calculator is:

$$A : B = \sin(A) - 3A + 2 : C = \cos(A) - 3 : D = A - \frac{B}{C} \ \boxed{\text{CALC}}$$

Put the value of initial guess for A?

Algorithm (Pseudo Code) for Newton Raphson Method

1. Start
2. Input initial guess x_1 and initialize error E
3. Compute $f_1 = f(x_1)$ and $g_1 = f'(x_1)$
4. Compute $x_2 = x_1 - \frac{f_1}{g_1}$
5. If $\left|\frac{x_1 + x_2}{x_0}\right| < E$, then the root is x_1

   else,

   $x_1 = x_2$ and go to step (3)

6) Stop

Limitation of Using Newton Raphson Method

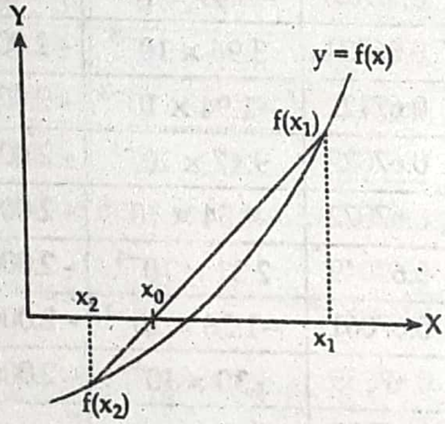
i) If $f'(x)$ is zero or near to zero, then division by zero occurs which cannot give the solution.

ii) If the initial guess is not a good guess, *i.e.,,* too far from exact root, then the process may converge to a garbage root.

2.3 Regula–Falsi Method, Secant Method

Regula-Falsi Method is a fast converging process or tool for solving algebraic equations. Similar to the bisection method, it uses two approximation's interval points that encloses the root and converges toward the root at a higher rate.

The Regula-Falsi method is illustrated below:



Let the above figure is the plot of $y = f(x)$

where, $f(x_1)$ and $f(x_2)$ and outputs at x_1 and x_2 respectively such that $f(x_1)f(x_2) < 0$.

From co-ordinate geometry, the equation of chord joining $f(x_1)$ and $f(x_2)$ is given as;

$$y - f(x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}(x - x_2)$$

This chord crosses the x-axis at point $(x_0, 0)$

Thus, the value of x_0 is given by;

$$-f(x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}(x_0 - x_2)$$

or, $$x_0 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

or, $$x_0 = \frac{f(x_1)x_2 - f(x_2)x_2 - f(x_2)x_1 + f(x_2)x_2}{f(x_1) - f(x_2)}$$

or, $$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

Getting the value of x_0, we find $f(x_0)$

If $f(x_0) = 0$, the root lies at x_0

Otherwise,

If $f(x_1)f(x_0) < 0$, then root lies between x_1 and x_0, so, we replace $x_2 = x_0$ and proceed the iteration.
Similarly,
If $f(x_1)f(x_0) < 0$, then, root lies between x_2 and x_0, so, we replace $x_1 = x_0$ and proceed the iteration.

Example 2.7

Find the root of the equation $x^2 - 4x + 3 = 0$ using regula-falsi method within the range $2 < x < 5$.

Solution:

For the given equation;

$$f(x) = x^2 - 4x + 3$$

Let's check for two boundary points $x_1 = 2$ and $x_2 = 5$ if $f(x_1)f(x_2) < 0$.

Here,

$$f(x_1) = f(2) = -1$$
$$f(x_2) = f(5) = 8$$

Since, $f(x_1)f(x_2) < 0$, so, the root lies between $x_1 = (2)$ and $x_2 = (5)$.

Now,

$$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)} = \frac{8 \times 2 - (-1) \times 5}{8 - (-1)} = 2.3333$$

and, $f(x_0) = -0.8889$

Since $f(x_2)f(x_0) < 0$, so, root lies between x_2 and x_0. Hence, replace $x_1 = x_0$ and precede the iteration.

The iteration table is shown below;

	(A)	(B)	(C)	(D)	(E)	(F)	
Iteration no.	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$	$f(x_0)$	Replace
1	2	5	- 1	8	2.3333	- 0.8889	$x_1 = x_0$
2	2.3333	5	- 0.89	8	2.5999	- 0.64	$x_1 = x_0$
3	2.5999	5	- 0.64	8	2.7777	- 0.39	$x_1 = x_0$
4	2.7777	5	- 0.39	8	2.8823	- 0.22	$x_1 = x_0$
5	2.8823	5	- 0.22	8	2.9394	- 0.11	$x_1 = x_0$
6	2.9394	5	- 0.11	8	2.9692	- 0.06	$x_1 = x_0$
7	2.9692	5	- 0.06	8	2.9845	- 0.03	$x_1 = x_0$

8	2.9845	5	- 0.03	8	2.9922	- 0.01	$x_1 = x_0$
9	2.9922	5	- 0.01	8	2.9961	-7.8×10^{-3}	$x_1 = x_0$
10	2.9961	5	-7.8×10^{-3}	8	2.9980	-3.9×10^{-3}	$x_1 = x_0$
11	2.9980	5	-3.9×10^{-3}	8	2.9989	-1.9×10^{-3}	$x_1 = x_0$

Since, at iteration 10 and 11 value of x_0 is identical up to 3 decimal places, so we stop the iteration here.

Thus, the root of the equation $x^2 - 4x + 3 = 0$ is 2.9989.

Note

The programming code for the calculator is;

$A : B : C = A^2 - 4A + 3 : D = B^2 - 4B + 3 : E = D \times A - C \times B / D - C : F = E^2 - 4E + 3$ $\boxed{\text{CALC}}$

Put x_1 and x_2 for A? and B? respectively.

Example 2.8

Find the root of the equation $3x - \cos x - 1 = 0$ correct up to 3 decimal places using Regula-Falsi method.

Solution:

For the given equation:

$$f(x) = 3x - \cos x - 1$$

Let's choose two points x_1 and x_2 such that;

$$f(x_1)f(x_2) < 0$$

Check for $x_1 = 1$ and $x_2 = 2$

$$f(1) = 1.0001$$
$$f(2) = 4.0006$$

Since, $f(x_1)f(x_2) > 0$, so, given interval do not bracket the root.

Again, check for $x_1 = 0$ and $x_2 = 1$

$$f(0) = -2$$
$$f(1) = 1.0001$$

Since $f(x_1)f(x_2) < 0$, so, given interval brackets the root.

Now,

$$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

$$= \frac{1.0001 \times 0 - (-2) \times 1}{1.0001 - (-2)} = 0.6666$$

and, $f(x_0) = -1.32 \times 10^{-4}$

Since $f(x_2)f(x_0) < 0$, so, root lies between x_2 and x_0. Hence, replace $x_1 = x_0$ and precede the iteration.

The iteration table is given below:

	(A)	(B)	(C)	(D)	(E)	(F)	
Iteration no.	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$	$f(x_0)$	Replace
1	0	1	– 2	1	0.6666	-3.38×10^{-5}	$x_1 = x_0$
2	0.6666	1	-1.32×10^{-4}	1	0.6666	-2.24×10^{-9}	

Since at iteration (1) and (2), value of x_0 is identical up to 3 decimal places, we stop the iteration there.

Thus, the root of the equation $3x - \cos x - 1 = 0$ is 0.6666.

Note

The programming code for calculator is

A : B : C = 3A – cos(A) – 1 : D = 3B – cos(B) – 1 : E = D × A – C × B/D – C : F = 3E – cos(E) – 1 $\boxed{\text{CALC}}$

Put x_1 and x_2 for A? and B? respectively.

Algorithm (Pseudo Code) for Regula-Falsi Method

1. Start
2. Input initial guess x_1 and x_2 and initialize error E.
3. Compute $f(x_1)$ and $f(x_2)$
4. If $f(x_1)f(x_2) > 0$, go to step 2
5. Compute x_0 and $f(x_0)$ where, $x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$
6. If $f(x_1)f(x_2) < 0$

   Set $x_2 = x_0$

   Else

   Set $x_1 = x_0$
7. If $\left|\frac{x_1 - x_2}{x_0}\right| < E$, root = x_0

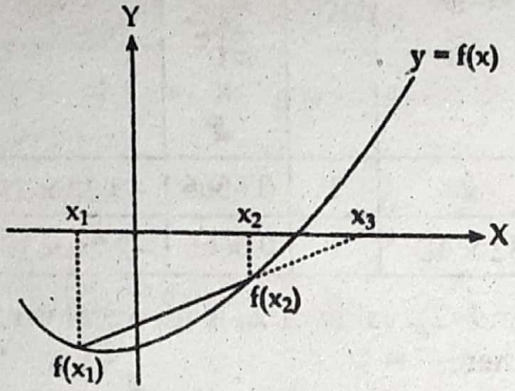
   Else

   go to step 5
8. Stop

Secant Method

The necessary condition for the Regula-Falsi method to converge to the root is that the initial guess interval of two points must include the root, *i.e.,,* $f(x_1)f(x_2) < 0$. The overcome this limitation, secant method was developed.

In secant method, an interval of two points is chosen which may or may not enclose the root. The root is approximated by the secant line passing through these two points. With each iteration, new approximations to the root are approximated until the desired accuracy of the root is obtained. The secant method is illustrated below:



Let above figure is the plot of $y = f(x)$ where, $f(x_1)$ and $f(x_2)$ are outputs at x_1 and x_2 respectively.
Form co-ordinate geometry, the equation of chord joining $f(x_1)$ and $f(x_2)$ is given as;

$$y - f(x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}(x - x_2)$$

This chord cuts x-axis at point $(x_3, 0)$
Thus, the value of x_3 is given by;

$$-f(x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}(x_3 - x_2)$$

or, $$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

or, $$x_3 = \frac{f(x_1)x_2 - f(x_2)x_2 - f(x_2)x_1 + f(x_2)x_2}{f(x_1) - f(x_2)}$$

or, $$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

Getting the value of x_3; we set,

$$x_1 = x_2$$
$$x_2 = x_3$$

and, find new x_3.
If the value of x_3 is identical up to required significant digits, the root is equal to x_3, otherwise, precede the iteration.

Example 2.9

Find the root of the equation $x^3 - 3x^2 - x + 3 = 0$ correct up to 3 decimal places using secant method.
Solution:
For the given equation:

$$f(x) = x^3 - 3x^2 - x + 3$$

Let's choose two boundary points $x_1 = 5$ and $x_2 = 8$

Here,

$$f(x_1) = f(5) = 48$$
$$f(x_2) = f(8) = 315$$

Now,

$$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)} = \frac{315 \times 5 - 48 \times 8}{315 - 48} = 4.4607$$

Set, $x_1 = x_2 = 8$

$x_2 = x_3 = 4.4607$

Calculate new x_3

The iteration table is given below;

	(A)	(B)	(C)	(D)	(E)
Iteration no.	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$
1	5	8	48	315	4.4607
2	8	4.4607	315	27.6033	4.1207
3	4.4607	4.1207	27.6033	17.9099	3.4927
4	4.1207	3.4927	17.9099	5.5170	3.2130
5	3.4927	3.2130	5.5170	1.9863	3.0557
6	3.2130	3.0557	1.9863	0.4647	3.0077
7	3.0557	3.0077	0.4647	0.0619	3.0003
8	3.0077	3.0003	0.0619	2.48×10^{-3}	3.0000

Since, at iteration (7) and (8), value of x_3 is identical up to 3 decimal places, we stop the iteration here.

Thus, the root of the equation $x^3 - 3x^2 - x + 3 = 0$ is 3.0000.

Note

The programming code for the calculator is;

$A : B : C = A^3 - 3A^2 - A + 3 : D = B^3 - 3B^2 - B + 3 : E = D \times A - C \times B / D - C : A = B : B = E$ $\boxed{\text{CALC}}$

Put x_1 and x_2 for A? and B? respectively.

Example 2.10

Calculate the root of the equation $f(x) = \cos x - 2x + 1$ using secant method correct up to 3 decimal places.

Solution:

For the given equation:

$$f(x) = \cos x - 2x + 1$$

Lets choose two boundary points $x_1 = 2$ and $x_2 = 3$

Here,

$$f(x_1) = f(2) = -2.0006$$
$$f(x_2) = f(3) = 4.0014$$

Now,

$$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)} = 1.0001$$

Set, $x_1 = x_2 = 3$

$x_2 = x_3 = 1.0001$

Calculate new x_3

The iteration table is given below:

	(A)	(B)	(C)	(D)	(E)
Iteration no.	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$
1	2	3	- 2.0006	- 4.0014	1.0001
2	3	1.0001	- 4.0014	-3.04×10^{-4}	0.9999
3	1.0001	0.9999	-3.04×10^{-4}	-4.6×10^{-8}	0.9999

Since, at iteration (2) and (3) value of x_3 is identical up to 3 decimal places, we stop the iteration here.

Thus, the root of the equation $\cos x - 2x + 1$ is 0.9999.

Note

The programming code for calculator is;
A : B : C = cos(A) – 2A + : D = cos(B) – 2B + 1 : E = D × A – C × B/D – C : A = B : B = E $\boxed{\text{CALC}}$
Put x_1 and x_2 for A? and B? respectively.

Algorithm for Secant Method

1. Start
2. Input initial guess x_1 and x_2 and initialize error E
3. Complete $f(x_1)$ and $f(x_2)$
4. Compute:
$$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$
5. Set
$$x_1 = x_2$$
$$x_2 = x_3$$
6. If $\left|\frac{x_1 - x_2}{x_3}\right| < E$, root $= x_3$
Else,
go to step (4)
7. Stop.

2.4 Fixed Point Iteration Method

As the name suggests, fixed point iteration method is the method of solving algebraic equation through successive iterations.

In this method, the given function is manipulated in a systematic order through algebraic operations in a way that a single variable of degree 1 and coefficient 1 is on left hand side while remaining other on the right hand side. Mathematically,

If $f(x) = 0$ is the given function then it is manipulated as:

$$x = g(x)$$

After such systematic manipulation, an initial guess of x is made to find the value of g(x) which is the value of root for the first iteration. The root of first iteration is the input for g(x) in second iteration and so on until in two successive iterations are identical up to desired accuracy.

Note

While making iteration, it should be noted that the solution is converging to a root or not. If the iteration is not converging to a root, the equations should be re-manipulated algebraically to make it a converging solution.

Example 2.11

Find the root of the equation $\sin x - 3x + 2 = 0$ corrects up to 3 decimal places using fixed point iteration method.

Solution:

The given equation is;

$$f(x) = \sin x - 3x + 2$$

Manipulating this equation in terms of x; we get,

$$\sin x - 3x + 2 = 0$$

or, $3x = \sin x + 2$

or, $x = \dfrac{\sin x + 2}{3}$

or, $x = g(x)$

where, $g(x) = \dfrac{\sin x + 2}{3}$

Let initial guess x_0 be 5.

Then,

$$x_1 = g(x_0) = 0.6957$$
$$x_2 = g(x_1) = 0.6707$$
$$x_3 = g(x_2) = 0.6706$$

Since at iteration (2) and (3), the value of x is identical up to 3 decimal places, we stop the iteration here.

Thus, the root of the equation $\sin x - 3x + 2 = 0$ is 0.6706.

Note

Programming code for calculator is;

$$A : B = \frac{\sin(A) + 2}{3} \boxed{\text{CALC}}$$

Put the value of initial guess for A? for first iteration and value of previous iteration for other iterations

Example 2.12

Find the square root of 8 using fixed point iteration method corrects up to 4 decimal places.

Solution:

The given equation is;

$$x = \sqrt{8}$$

or, $x^2 = 8$

or, $x^2 - 8 = 0$ (i)

Manipulating this equation terms of x; we get,

$$x^2 - 8 = 0$$

or, $x = \frac{8}{x}$

or, $x = g(x)$

where, $g(x) = \frac{8}{x}$

Let, initial guess x_0 be 2; then,

$$\left.\begin{aligned} x_1 &= g(x_0) = 4 \\ x_2 &= g(x_1) = 2 \\ x_3 &= g(x_2) = 4 \\ x_4 &= g(x_3) = 2 \end{aligned}\right\} \text{The root is not converging}$$

Since, the root is not converging, we re-manipulate equation (i) as;

$$x^2 - 8 = 0$$

or, $x = x^2 + x - 8$ [Adding x on both sides]

or, $x = g(x)$

where, $g(x) = x^2 + x - 8$.

Let initial guess x_0 be 2; then,

$$\left.\begin{aligned} x_1 &= g(x_0) = 4 \\ x_2 &= g(x_1) = 2 \\ x_3 &= g(x_2) = 4 \\ x_4 &= g(x_3) = 498 \\ x_5 &= g(x_4) = 248494 \end{aligned}\right\} \text{The root is not converging}$$

Since, the root is not converging, we re-manipulate equation (i) as;

$$x^2 - 8 = 0$$

or, $x = \frac{8}{x}$

or, $x + x = \frac{8}{x} + x$

or, $x = \frac{1}{2}\left(\frac{8}{x} + x\right)$

or, $x = g(x)$

where, $g(x) = \left(\frac{8}{x} + x\right) \div 2.$

Let initial guess x_0 be 2; then,

$$x_1 = g(x_0) = 3$$
$$x_2 = g(x_1) = 2.8333$$
$$x_3 = g(x_2) = 2.8284$$
$$x_4 = g(x_3) = 2.8284$$

Since, at iteration (3) and (4), the value of x is identical up to 3 decimal places, so we stop the iterative here.

Thus, the root of the equation $x = \sqrt{8}$ is 2.8284.

Note

Programming code for calculator is;

$$A : B = \left(\frac{8}{A} + A\right)/2\ \boxed{\text{CALC}}$$

Put value of initial guess for A? for first iteration and value of previous iteration for other iterations.

Example 2.13

Find the root of equation $x^3 - 3x^2 - x + 3 = 0$ using fixed point iteration method correct up to 3 decimal places.

Solution:

The given equation is;

$$f(x) = x^3 - 3x^2 - x + 3 \qquad \text{(i)}$$

Manipulating this equation in terms of x; we get,

$$x^3 - 3x^2 - x + 3 = 0$$

or, $x = x^3 - 3x^2 + 3$

or, $x = g(x)$

where, $g(x) = x^3 - 3x^2 + 3.$

Let initial guess x_0 be 5. Then,

$$\left.\begin{aligned} x_1 &= f(x_0) = 53 \\ x_2 &= f(x_1) = 140453 \\ x_3 &= f(x_2) = 2.77 \times 10^{15} \end{aligned}\right\} \text{Not converging}$$

Since, the root is not converging, we re–manipulate equation (i) as:

$$x^3 - 3x^2 - x + 3 = 0$$

or, $x = x^3 - 3x^2 + 3$

or, $x + x = x^3 - 3x^2 + x + 3$

or, $x = \frac{x^3 - 3x^2 + x + 3}{2}$

or, $x = g(x)$

where, $g(x) = \frac{x^3 - 3x^2 + x + 3}{2}$.

Let initial guess x_0 be 2; then,

$$\left.\begin{aligned} x_1 &= g(x_0) = 0.5 \\ x_2 &= g(x_1) = 1.4375 \\ x_3 &= g(x_2) = 0.6044 \\ x_4 &= g(x_3) = 1.3646 \\ x_5 &= g(x_4) = 0.6596 \\ x_6 &= g(x_5) = 1.3207 \\ x_7 &= g(x_6) = 0.6958 \end{aligned}\right\} \text{Not converging}$$

Since, the root is not converging, we re-manipulate equation (i) as;

$$x^3 - 3x^2 - x + 3 = 0$$

or, $3x^2 = x^3 - x + 3$

or, $x = \sqrt{\frac{x^3 - x + 3}{3}}$

or, $x = g(x)$

where, $g(x) = \sqrt{\frac{x^3 - x + 3}{3}}$.

Let the initial guess be $x_0 = 2$; then,

$$x_1 = g(x_0) = 1.7321$$
$$x_2 = g(x_1) = 1.4679$$
$$x_3 = g(x_2) = 1.2510$$
$$x_4 = g(x_3) = 1.1116$$
$$x_5 = g(x_4) = 1.0427$$
$$x_6 = g(x_5) = 1.0150$$
$$x_7 = g(x_6) = 1.0051$$
$$x_8 = g(x_7) = 1.0017$$
$$x_9 = g(x_8) = 1.0006$$
$$x_{10} = g(x_9) = 1.0002$$

Since, at iteration (9) and (10), the value of x is identical up to 3 decimal places.

Thus, the root of equation $x^3 - 3x^2 - x + 3 = 0$ is 1.0002.

Note

Programming code for calculator is;

$$A : B = ((x^3 - x + 3)/3)^{\frac{1}{2}} \boxed{\text{CALC}}$$

Put the value of initial guess for A? for first iteration and value previous iteration for other iterations.

2.5 Rate of Convergence and Comparisons of These Methods

Rate of convergence of a method is the rate at which the method yields or converges to the real root. It is the rate of convergence form an initial guess to exact root of the equation.

Comparisons of convergence of all these mentioned methods can be made through their calculation method and steps. Some of the methods always may or may not converge to the exact root. Similarly, the result processing speed of the different methods vary accordingly. The convergence rates are linear, quadratic or intermediate according to the steps involved during the solution of the equation.

2.6 EXAMINATION PROBLEMS

1. Calculate a real root $x^7 + \sin x - \cos x = 0$ accurate up to 3 decimal places using Bisection method. **[2071 Chaitra]**

Solution:

The given equation;

$$f(x) = x^7 + \sin x - \cos x$$

Let's choose two points x_1 + x_2 by inspection such that;

$$f(x_1) . f(x_2) < 0$$

Let, the two points be 0 and 1.

Here,

$$f(x_1) = f(0) = -1$$

$$f(x_2) = f(1) = 0.0176$$

and, f(0) . f(1) < 0, so the bisection is carried out in the interval of 0 and 1.

For initial guess of x_0, we have,

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$

and, $f(x_0) = -0.9834$

since, $f(x_2) . f(x_0) < 0$, so root lies between x_0 and x_2.

Thus, replacing x_1 with x_0, *i.e.*, $x_1 = x_0$, we make another assumption for x_0 as;

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0.5+1}{2} = 0.75$$

This iteration is shown in table below:

No. of iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_3)$	Replace
1	0	1	0.5	-1	0.0176	-0.9834	$x_1=x_0$
2	0.5	1	0.75	-0.9834	0.0176	-0.8533	$x_1=x_0$
3	0.75	1	0.875	-0.8533	0.0176	-0.5919	$x_1=x_0$
4	0.875	1	0.9375	-0.5919	0.0176	-0.3470	$x_1=x_0$
5	0.9375	1	0.9688	-0.3470	0.0176	-0.1827	$x_1=x_0$
6	0.9688	1	0.9844	-0.1822	0.0176	-0.0869	$x_1=x_0$
7	0.9844	1	0.9922	-0.0869	0.0176	-0.0359	$x_1=x_0$
8	0.9922	1	0.9961	-0.0359	0.0176	-0.0094	$x_1=x_0$
9	0.9961	1	0.9981	-0.0094	0.0176	+0.004	$x_2=x_0$
10	0.9961	0.9981	0.9971	-0.0094	0.0043	-0.003	$x_1=x_0$
11	0.9971	0.9981	0.9976	-0.0026	0.0043	+0.0009	$x_2=x_0$
12	0.9971	0.9976	0.9974	-0.0026	0.0009	-0.0008	END

Since, at iteration 12; x_1, x_2 and x_3 are identical up to 3 decimal places, we stop the iteration here.

Hence, the root of the equation $x^7 + \sin x - \cos x$ is 0.9974.

2. **Write Pseudo-code to find a real root of a given non-linear equation using false position method.** **[2071 Chaitra]**

Solution: See the definition part

3. **Discuss the limitations of Newton-Rapshon method in finding a real root of a non-linear equation.** **[2071 Chaitra]**

Ans: Newton Raphson method can be a very efficient method for finding the zero of a function. However, there are situations where the method will fail.

Consider what would have happened, if the initial estimate x_0 corresponds to a stationary point of the function. In this case, the tangent line is parallel to the axis, so, f(x) = 0, we are unable to compute x_1.

For, some functions, same starting points may enter an infinite cycle, preventing convergence.

In general, the behavior of the sequence can be very complex.

4. **Derive Newton Raphson iterative formula for solving non-linear equation, using Taylor Series.** **[2072 Ashwin]**

Solution: See the definition part

5. **Using the bisection method, find a real root of the equation $f(x) = 3x - \sqrt{1 + \sin x}$ correct up to three decimal points.** **[2072 Ashwin]**

Solution:

The given equation is;

$$f(x) = 3x - \sqrt{1 + \sin x}$$

Let's choose two points x_1 and x_2 by inspection such that,

$$f(x_1) \,.\, f(x_2) < 0$$

Let, two points be 0 and 1.

Here,

$$f(x_1) = f(0) = -1$$

$$f(x_2) = f(1) = 1.9913$$

and, f(0) . f(1) < 0, so the bisection is carried out in the interval of 0 and 1.

For initial guess of x_0; we have,

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0 + 1}{2} = 0.5$$

$$f(x_0) = f(0.5) = 0.4955$$

Since, $f(x_1) \,.\, f(x_0) < 0$, so root lies between x_0 and x_1.

Thus, replacing x_2 with x_0, *i.e.*, $x_2 = x_0$

The iteration table is as shown below:

No. of iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_3)$	Replace
1	0	1	0.5	-1	1.9913	0.4956	$x_2 = x_0$
2	0	0.5	0.25	-1	0.4956	-0.2522	$x_1 = x_0$
3	0.25	0.5	0.375	-0.2522	0.4956	0.1217	$x_2 = x_0$
4	0.25	0.375	0.3125	-0.2522	0.1217	-0.0652	$x_1 = x_0$
5	0.3125	0.375	0.3738	-0.0652	0.1217	0.0284	$x_2 = x_0$
6	0.3125	0.3458	0.3282	-0.2652	0.0284	-0.0184	$x_1 = x_0$
7	0.3282	0.3438	0.336	-0.0183	0.0284	0.0051	$x_2 = x_0$
8	0.3282	0.336	0.3321	-0.0183	0.0051	-0.0066	$x_1 = x_0$
9	0.3321	0.336	0.3341	-0.0066	0.0051	-0.0008	$x_1 = x_0$
10	0.3341	0.336	0.3351	-0.0006	0.0051	0.0022	$x_2 = x_0$
11	0.3341	0.3351	0.346	-0.0006	0.0024	0.0009	$x_2 = x_0$
12	0.3341	0.3346	0.3344	-0.0006	0.0009	0.0001	END

Since, at iteration 12, x_1, x_2, x_0 are identical up to 3 decimal places, use stop the iteration here.

Hence, the root of the equation $3x - \sqrt{1 + \sin x}$ is 0.3344.

6. Write an algorithm to solve a non-linear equation using secant method. [2072 Chaitra]

Solution: See the definition part

7. Find the positive root of equation cos x – 1.3x = 0, correct to six decimal places using Newton Raphson method. [2072 Chaitra]

Solution:

The given equation is;

$$f(x) = \cos x - 1.3x$$

The first derivative of the given equation is given by;

$$f'(x) = -\sin x - 1.3$$

Let, the initial guess be 4, *i.e.*, $x_1 = 4$

From Newton Raphson formula; we have,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The iteration table is given below:

No. of iteration	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	4	-4.202436	-1.369756	0.931983
2	0.931983	-0.211710	-1.316265	0.771141
3	0.771141	-0.002574	-1.313459	0.769181
4	0.769181	-0.00025	-1.313424	0.769162

5	0.769162	-7.06×10^{-7}	-1.313424	0.76191
6	0.769161	5.94×10^{-7}	-1.313424	0.769161
7	0.769161			

Since, at iterations 6 and 7, value of x_1 is identical up to 6 decimal places, we stop the iteration here.

Hence, the root of the equation cos x – 1.3x is 0.763161.

8. Discuss the limitations of fixed-point iteration methods graphically. **[2072 Chaitra]**

Solution: See the definition part

9. Find an approximation of the root of the equation $x^3 - x - 11 = 0$ by using bisection method correct to three decimal places. **[2073 Shrawan]**

Solution:

The given equation is;

$$f(x) = x^2 - x - 11$$

Let's choose two points x_1 and x_2 by inspection such that;

$$f(x_1) . f(x_2) < 0.$$

Let, two points be 2 and 2.5.

Here,

$$f(x_1) = f(2) = -5$$

$$f(x_2) = f(2.5) = 2.125$$

and, f(2) . f(2.5) < 0, so the bisection is carried out in the interval of 2 and 2.5

For initial guess of x_0; we have,

$$x_0 = \frac{x_1 + x_2}{2} = \frac{2 + 2.5}{2} = 2.25$$

$$f(x_0) = f(2.25) = -1.8594$$

Since, $f(x_2) . f(x_0) < 0$, so root lies between x_0 and x_2.

Thus, replacing x_1 with x_0 *i.e.*, $x_1 = x_0$

The iteration table is as shown below:

No. of iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_3)$	Replace
1	2	2.5	2.25	-5	2.125	-1.8594	$x_2 = x_0$
2	2.25	2.5	2.375	-1.8594	2.125	0.0215	$x_2 = x_0$
3	2.25	2.375	2.3125	-1.8594	0.0215	-0.9460	$x_1 = x_0$
4	2.3125	2.375	2.3438	-0.9460	0.0215	-0.4671	$x_1 = x_0$
5	2.3438	2.375	2.3594	-0.4684	0.0215	-0.2292	$x_1 = x_0$

6	2.3994	2.375	2.3672	-0.2252	0.0215	-0.1023	$x_1 = x_0$
7	2.3672	2.375	2.3711	-0.1623	0.0215	-0.0405	$x_1 = x_0$
8	2.3711	2.375	2.3731	-0.0405	0.0215	-0.0095	$x_1 = x_0$
9	2.3731	2.375	2.3741	-0.0084	0.0215	0.0064	$x_2 = x_0$
10	2.3731	2.3741	2.3736	-0.0087	0.0072	-0.0008	$x_1 = x_0$
11	2.3736	2.3741	2.3739	-0.0008	0.0072	0.0032	$x_2 = x_0$
12	2.3736	2.3739	2.3738	-0.0008	0.0040	0.0016	END

Since, at iteration 12, x_1, x_2 and x_0 are identical up to 3 decimal place; we stop the iteration here.

Hence, the root of the equation $x^3 - x - 11$ is 2.3738.

10. Write an algorithm for finding a real root of non-linear equation using Newton Raphson method. [2073 Shrawan]

Solution: See the definition part

11. Write an algorithm of secant method for finding a real root of a non-linear equation. [2073 Bhadra]

Solution: See the definition part

12. Find a real root of the equation sin x = e^{-x} correct up to four decimal places using N-R method. What are the limitations of this method? [2073 Bhadra]

Solution:

The given equation is;

$$f(x) = \sin x - e^{-x}$$

The first derivative of the given equation is given by;

$$f'(x) = \cos x + e^{-x}$$

Let, the initial guess be 3, i.e., $x_1 = 3$

From Newton Raphson formula; we have,

$$x_{i+1} = x_0 - \frac{f(x_i)}{f'(x_i)}$$

The iteration table is given below:

No. of iteration	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	3	0.00255	1.04842	2.99757
2	2.99757	0.002385	1.04854	2.995295
3	2.995295	0.002232	1.04866	2.99317
4	2.99317	0.00209	1.04876	2.99118
5	2.99118	0.00195	1.04887	2.989317
6	2.989317	0.001828	1.04895	2.98757

7	2.98757	0.00171	1.04905	2.98594
8	2.98957	0.001599	1.04913	2.98442
9	2.98442	0.001496	1.049213	2.98299
10	2.98299	0.001398	1.04929	2.98166
11	2.98166	0.00131	1.04935	2.98041
12	2.98041	0.00122	1.04972	2.97925
13	2.97925	0.00114	1.04948	2.97816
14	2.97816	0.00107	1.04954	2.97714
15	2.97714	0.000999	1.04959	2.97619
16	2.97619	0.00033	1.04964	2.97753
17	2.9753	0.00087	1.04968	2.97447
18	2.97447	0.00082	1.04973	2.9737
19	2.9737	0.00076	1.04977	2.97297
20	2.97297	0.00071	1.04981	2.97229
21	2.97229	0.00067	1.04984	2.9717
22	2.9717	0.00063	1.04987	2.9711
23	2.9711	0.00059	1.0499	2.9705
24	2.9705	0.00054	1.04993	2.96998
25	2.96998	0.00051	1.04996	2.5695
26	2.9695	0.00048	1.04999	2.9690
27	2.9690	0.00044	1.05001	2.9686
28	2.9686	0.00041	1.05003	2.9682
29	2.9682	0.00039	1.05005	2.9678
30	2.9678	0.00036	1.05008	2.36746
31	2.96746	0.00034	1.0501	2.9671
32	2.9671	0.00031	1.05011	2.9668
33	2.9668	0.00029	1.05013	2.9665
34	2.9695	0.00027	1.05014	2.9662
35	2.9662	0.00025	1.05016	2.9660
36	2.9660	0.00023	1.05017	2.9658
37	2.9658	0.00022	1.05018	2.9656
38	2.9656	0.00021	1.05019	2.9654
39	2.9654	0.00019	1.0502	2.9652
40	2.9652	0.00018	1.05021	2.9650
41	2.9650	0.00017	1.05022	2.9648
42	2.9648	0.00015	1.05023	2.9646
43	2.9646	0.00014	1.05024	2.9644
44	2.9644	0.00012	1.05025	2.9642

46	2.9642	0.00011	1.05026	2.9640
47	2.9640	0.00096	1.05027	2.9639
48	2.9639	0.00009	1.05028	2.9638
49	2.9638	0.00008	1.05028	2.9637
50	2.9637	0.00008	1.05029	2.9636
51	2.9636	0.00007	1.050295	2.9635
52	2.9635	0.00006	1.05030	2.9634
53	2.9634	0.00005	1.05031	2.9633
54	2.9633	0.00005	1.05031	2.9632
55	2.9632	0.00004	1.05032	2.9631
56	2.9631	0.00003	1.05032	2.9630
57	2.9630	0.00003	1.05033	2.9629
58	2.9629	0.00002	1.05033	2.9628
57	2.9628	0.00001	1.05034	2.9627
58	2.9627	0.00007	1.05034	2.9626
59	2.9626	-0.0000	1.05035	2.9626
60	2.9626			END

Here, the value of x_1 is same at iteration 59 and 60 up to 4 decimal place, use stop iteration here.

Hence, the root of given equation is 2.9626.

13. Find a real root of $\cos x + e^{-x} - 5 = 0$ accurate to 4 decimal places using Secant method. **[2073 Magh]**

Solution:

The given equation is;

$$\cos x + e^{-x} - 5 = 0$$

Let's choose two boundary points $x_1 = 1$ and $x_2 = 2$;

$$f(x_1) = f(1) = -1.28187$$

$$f(x_2) = f(2) = 3.38845$$

Now,

$$x_3 = \frac{f(x_2).\ x_1 - f(x_1)\ .\ x_2}{f(x_2) - f(x_1)} = 1.27447$$

Set, $x_1 = x_2 = 2$

$$x_2 = x_3 = 1.27447$$

Calculate new x_3

The iteration table is given below:

No. of iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_3 = \frac{(x_2).x_1 + f(x_1).x_2}{f(x_2) - f(x_1)}$
1	1	2	-1.28187	3.28845	1.27447
2	2	1.27447	3.38845	-0.42344	1.35507
3	1.27447	1.35507	-0.42344	-0.12325	1.38816

4	1.35507	1.38634	0.00718	-0.00011	1.38637
5	1.38816	1.38634	0.00718	-0.00011	1.38637
6	1.38634	1.38637	-0.0001	0.00001	1.38637

Since, at iteration (5) and (6) value of x_3 is identical up to 4 decimal places, we stop the iteration here.
Thus, the root of the equation $\cos x + e^{-x} - 5 = 0$ is 1.38637.

14. Write Pseudo-code to find a real root of a non-linear equation using the Bisection method. **[2073 Magh]**

Solution: See the definition part

15. Find a real root of $e^x - \cos x = 3$ correct to three places of decimal using the Bisection method. **[2073 Chaitra]**

Solution:

The given equation is;

$$e^x - \cos x - 3 = 0$$

i.e., $f(x) = e^x - \cos x - 3$

Let's choose two points x_1 and x_2 by inspection; such that,

$$f(x_1) . f(x_2) < 0$$

Let, two points be 1 and 2.

Here,

$$f(x_1) = f(1) = -1.28157$$

$$f(x_2) = f(2) = 3.38967$$

and, $f(1) . f(2) < 0$, so the bisection is carried out in the interval of 1 and 2.

For initial guess of x_0; we have,

$$x_0 = \frac{x_1 + x_2}{2} = \frac{1 + 2}{2} = 1.5$$

$$f(x_0) = f(1.5) = 0.48203$$

Since, $f(x_1) . f(x_0) < 0$, so root lies between x_0 and x_1.

Thus, replacing x_2 with x_0 *i.e.,* $x_2 = x_0$

The iteration table is as shown below:

No. of iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_3)$	Replace
1	1	2	1.5	-1.28157	3.38967	0.48203	$x_2 = x_0$
2	1	1.5	1.25	-1.28157	0.48203	-0.50942	$x_1 = x_0$
3	1.25	1.5	1.375	-0.50942	0.48203	-0.04464	$x_1 = x_0$
4	1.375	1.5	1.4375	-0.04464	0.48203	0.21047	$x_2 = x_0$
5	1.375	1.4375	1.40625	-0.04464	0.21047	0.08093	$x_2 = x_0$
6	1.375	1.40625	1.39063	-0.04464	0.08093	0.01767	$x_2 = x_0$
7	1.375	1.39063	1.38282	-0.04464	0.01767	-0.01360	$x_1 = x_0$

8	1.38282	1.39663	1.38673	-0.01360	0.017667	0.00202	$x_2 = x_0$
9	1.38282	1.38673	1.38478	-0.01358	0.00263	-0.00578	$x_1 = x_0$
10	1.38478	1.38673	1.38576	-0.00578	0.00203	-0.00186	$x_1 = x_0$
11	1.38576	1.38673	1.38625	-0.00186	0.00203	0.0001	$x_2 = x_0$
12	1.38576	1.38625	1.38601	-0.00186	0.00012	-0.00086	$x_1 = x_0$
13	1.38601	1.38625	1.38613	-0.00086	0.00012	-0.00036	END

Since at iteration 13, x_1, and x_0 are identical up to 3 decimal places, we stop the iteration here,

Hence, the root of the equation $e^x - \cos x - 3 = 0$ is 1.38613.

16. What are the drawbacks of Newton-Raphson method? Discuss. **[2073 Chaitra]**

Solution: See the definition part

17. Write Pseudo-code for finding a real root of a non-linear equation using the false position method. **[2074 Bhadra]**

Solution: See the definition part

18. Find a real root of the following equation, correct to six decimals, using fixed-point iteration method.
$\sin x + 3x - 2 = 0$

Solution: **[2074 Bhadra]**

The given equation is;

$f(x) = \sin x + 3x - 2$ (i)

Manipulating this equation is terms of x, we get;

$\sin x + 3x - 2 = 0$

or, $x = \frac{2 - \sin x}{3}$

or, $x = g(x)$

When, $g(x) = \frac{2 - \sin x}{3}$

Let, initial guess x_0 be 5. Then,

$x_1 = g(x_0) = g(5) = 0.6376148$

$x_2 = g(x_1) = 0.6629573$

$x_3 = g(x_2) = 0.6628098$

$x_4 = g(x_3) = 0.6628107$

$x_5 = g(x_4) = 0.6628107$

Since, at iteration (4) and (5), the value of x is identical up to 6 decimal places, we stop the iteration here.

Hence, the root of the equation $\sin x + 3x - 2 = 0$ is 0.6628167.

19. Write a pseudo-code to find a real root of a non-linear equation using false position method. **[2075 Ashwin]**

Solution: See the definition part

20. **Find a positive root of the equation $x^2 \sin x - e^x + 2 = 0$ correct to 3 decimals using bisection method.** **[2075 Ashwin]**

Solution:

For the given equation;

$$f(x) = x^2 \sin x - e^x + 2 = 0$$

Let's choose two points x_1 and x_2 by inspection such that;

$$f(x_1) \cdot f(x_2) < 0$$

Let the two points be 0 and 1.

Here,

$$f(x_1) = f(0) = 1$$

$$f(x_2) = f(1) = -0.7008$$

and, $f(0) \cdot f(1) < 0$; so, the bisection is carried out in the interval of 0 and 1.

For the initial guess of x_0; we have,

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$

and, $f(x_0) = f(0.5) = 3534$

Since, $f(x_2) \cdot f(x_0) < 0$; so, root lies between x_0 and x_2.

Thus, replacing x_1 with x_0, *i.e.*, $x_1 = x_0$; we make another assumption for x_0 as;

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0.5+1}{2} = 0.75$$

This iteration is shown in the table below:

No. of	A	B	C	D	E	F	
iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_0)$	Replace
1	0	1	0.5	1	– 0.7008	0.3534	$x_1 = x_0$
2	0.5	1	0.75	0.3534	– 0.7008	– 0.1096	$x_2 = x_0$
3	0.5	0.75	0.625	0.3534	– 0.1096	0.1360	$x_1 = x_0$
4	0.625	0.75	0.6875	0.1360	– 0.1096	0.0169	$x_1 = x_0$
5	0.6875	0.75	0.7187	0.0169	– 0.1096	– 0.0453	$x_2 = x_0$
6	0.6875	0.7187	0.7031	0.0169	– 0.0452	– 0.0139	$x_2 = x_0$
7	0.6875	0.7031	0.6953	0.0169	– 0.0139	0.0015	$x_1 = x_0$
8	0.6953	0.7031	0.6992	0.0015	– 0.0139	– 0.0061	$x_2 = x_0$
9	0.6953	0.6992	0.6972	0.0015	– 0.0061	– 0.0023	$x_2 = x_0$
10	0.6953	0.6972	0.6962	0.0015	– 0.0023	– 0.0003	$x_2 = x_0$
11	0.6953	0.6962	0.6957	0.0015	– 0.0003	– 0.006	$x_2 = x_0$
12	0.6953	0.6957	0.6955				

Since at iteration 12; x_1, x_2 and x_0 are identical up to 3 decimal places, we stop the iteration here.

Hence, the root of the equation is 0.6955.

21. Find a negative real root of the following equation correct to three decimal using bisection method. **[2075 Chaitra]**

$$\frac{1-(x+1)^4}{x}-1=0$$

Solution:

The given equation is;

$$\frac{1-(x+1)^4}{x}-1=0$$

or, $1-(x+1)^4-x=0$

$\therefore$ $f(x)=1-(x+1)^4-x$

Since, negative real root is desired, the range needs to be set below zero. Let the two points be (−1) and (−3).

Here,

$f(x_1)=f(-1)=2$

$f(x_2)=f(-3)=-12$

Since, $f(-1)f(-3)<0$; so, the bisection is carried out in the interval of (−1) and (−3).

For initial guess of x_0; we have,

$$x_0=\frac{x_1+x_2}{2}=\frac{-1-3}{2}=-2$$

and, $f(x_0)=2$

Since, $f(x_2)f(x_0)<0$; so, root lies between x_0 and x_2.

Thus, replacing x_1 with x_0; *i.e.*, $x_1=x_0$; we make another assumption for x_0 as;

$$x_0=\frac{x_1+x_2}{2}=\frac{-2+(-3)}{2}=-2.5$$

This iteration is shown in the table below:

No. of iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_0)$	Replace
1	−1	−3	−2	2	−12	2	$x_1=x_0$
2	−2	−3	−2.5	2	−12	−1.5625	$x_2=x_0$
3	−2	−2.5	−2.25	2	−1.5625	0.8085	$x_1=x_0$
4	−2.25	−2.5	−2.375	0.8085	−1.5625	−0.1994	$x_2=x_0$
5	−2.25	−2.375	−2.3125	0.8085	−0.1994	0.3449	$x_1=x_0$
6	−2.3125	−2.375	−2.3437	0.3449	−0.1994	0.0837	$x_1=x_0$
7	−2.3437	−2.375	−2.3593	0.0837	−0.1994	−0.0546	$x_2=x_0$
8	−2.3437	−2.3593	−2.3515	0.0837	−0.0546	0.0152	$x_1=x_0$
9	−2.3515	−2.3593	−2.3554	0.0152	−0.0546	−0.0195	$x_2=x_0$
10	−2.3515	−2.3554	−2.3534	0.0152	−0.0195	−0.0016	$x_2=x_0$
11	−2.3515	−2.3534	−2.3524	0.0152	−0.0016	0.0072	$x_1=x_0$

12	- 2.3529	- 2.3534	- 2.3529	0.0072	- 0.0016	0.0027	$x_1 = x_0$
13	- 2.3529	- 2.3534	- 2.3531	0.0072	- 0.0016	0.0009	$x_1 = x_0$
14	- 2.3531	- 2.3534	- 2.3532				

Since, at iteration 14; x_1, x_2 and x_0 are identical up to 3 decimal places, we stop the iteration here as the answer is correct up to 3 decimal places.

Thus, the negative real root of the equation $\frac{1-(x+1)^4}{x} - 1 = 0$ is (- 2.353).

22. **What are the limitations of Newton-Raphson method? Using Newton-Raphson method, find a root of the equation $x \sin x - \cos x = 0$ correct to four decimal places.** [2075 Chaitra]

Solution:

For limitations of Newton-Raphson method

See the solution of Q. no. 3

For the second part

The given equation is;

$$f(x) = x \sin x - \cos x = 0$$

The first derivative of the given equation is given by;

$$f'(x) = x \cos x + \sin x + \sin x$$
$$= x \cos x + 2 \sin x$$

Let the initial guess be 7.55 *i.e.*, $x_1 = 7.55$

From Newton-Raphson formula; we have,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The iteration table for Newton-Raphson formula is given below:

No. of iteration	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	7.55	0.00067	7.74732	7.54991
2	7.54991	0.00065	7.74723	7.54982
3	7.54982	0.00062	7.74714	7.54973
4	7.54973	0.0006	7.74705	7.54965
5	7.54965	0.00058	7.74697	7.54957
6	7.54957	0.00056	7.74689	7.54949
7	7.54949	0.00054	7.74681	7.54942
8	7.54942	0.00052		

Since, at iteration (7) and (8), value of x_1 is identical up to 4 decimal places, we stop the iteration here.

Thus, the root of the equation is 7.54942.

23. Write an algorithm/Pseudo-code to find a real root of a non-linear equation using bisection method. **[2076 Ashwin]**

Solution: See the definition part 2.1

24. Find a real root of the equation $3x + \sin x - e^x$ correct to 3 decimals using False position (Regular-Falsi) method. **[2076 Ashwin]**

Solution:

For the given equation

$$f(x) = 3x + \sin x - e^x$$

Lets choose two points x_1 and x_2 such that;

$$f(x_1) \cdot f(x_2) < 0$$

Check for $x_1 = 1$ and $x_2 = 2$

$$f(1) = 0.2991$$
$$f(2) = -1.3541$$

Since, $f(x_1) \cdot f(x_2) < 0$; so, given interval bracket the root.

Now,

$$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)} = \frac{-1.3541 \times 1 - 0.2991 \times 1}{-1.3541 - 0.2991} = 1.1809$$

and, $f(x_0) = 0.3060$

Since, $f(x_2) \cdot f(x_0) < 0$; so, root lies between x_2 and x_0. Hence, replace $x_1 = x_0$ and proceed the iteration.

The iteration table is given below:

Iteration number	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_0 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$	$f(x_0)$	Replace
1	1	2	0.2991	- 1.3541	1.1809	0.3060	$x_1 = x_0$
2	1.1809	2	0.3060	- 1.3541	1.3318	0.2307	$x_1 = x_0$
3	1.3318	2	0.2307	- 1.3541	1.4290	0.1373	$x_1 = x_0$
4	1.4290	2	0.1373	- 1.3541	1.4816	0.0706	$x_1 = x_0$
5	1.4816	2	0.706	- 1.3541	1.5073	0.0336	$x_1 = x_0$
6	1.5073	2	0.0336	- 1.3541	1.5192	0.0154	$x_1 = x_0$
7	1.5192	2	0.0154	- 1.3541	1.5246	0.007	$x_1 = x_0$
8	1.5246	2	0.007	- 1.3541	1.5270	0.0031	$x_1 = x_0$
9	1.5270	2	0.0031	- 1.3541	1.5281	0.0014	$x_1 = x_0$
10	1.5281	2	0.0031	- 1.3541	1.5281	0.0014	$x_1 = x_0$
11	1.5286	2					

Since, at iteration (9) and (10), value of x_0 is identical up to 3 decimal places, we stop the iteration there.

Thus, the root of the equation is 1.5286.