Chapter 5

NUMERICAL DIFFERENTIATION AND INTEGRATION

Introduction

When the function, $y = f(x)$ is exactly defined, such as $y = x^n$, $y = e^x$, etc. we find $\frac{dy}{dx}$ by calculus methods. But sometimes, the function $y = f(x)$ is defined in the tabular form, *viz;* by means of the values $y_0, y_1, \ldots\ldots, y_n$ of y corresponding to the values $x_0, x_1, \ldots\ldots, x_n$ of x. in such situations, the actual value of f(x) cannot be found out, but f(x) is approximated by any convenient interpolating polynomial P(x). Then $\frac{dy}{dx}$ is found out as $P'(x)$. The process of computing $\left(\frac{dy}{dx}\right)_{x=x_k}$ by this method is called numerical differentiation where, $x_0 \le x_k \le x_n$ or x_k is close to x_0 or x_n, if it lies outside the range (x_0, x_n).

It the values of 'x' are equally spaced, any one of the interpolating polynomials discussed in chapter 4 is used. If the values of x are not equally space, either of x are not equally spaced, either Newton's divided difference formula or Lagrange's interpolation formula discussed in chapter 4 is used for numerical differentiation.

Note

Though $f(x_k)$ is approximated by $P(x_k)$, $f'(x_k)$ need not be approximately equal to $P'(x_k)$ in all situation. $P'(x_k)$ can be assumed to be assumed to be a good approximation of $f'(x_k)$ only it the difference $f(x)$ of some order are equal.

5.1 Numerical Differentiation Formulae

Values of the Derivatives of 'y' Based on Newton's Forward Interpolation Formula

When $(n + 1)$ pairs of values of values (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) are given, Newton's forward interpolation formula takes the form:

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots \quad (1)$$

where, $u = \dfrac{x - x_0}{h}$

Differentiating both sides of (1) with respect to x; we have,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or,

$$\frac{dy}{dx} = \frac{1}{h}\Big[\Delta y_0 + \frac{1}{2!}(2u-1)\Delta^2 y_0 + \frac{1}{3!}(3u^2 - 6u + 2)\Delta^3 y_0 + \frac{1}{4!}(4u^3 - 18u^2 + 224 - 6)\Delta^4 y_0 + \dots\dots\dots\Big] \quad (2)$$

Differentiating (2) further with respect to x; we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\Big[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{1}{12}(6u^2 - 18u + 11)\Delta^4 y_0 + \dots\dots\dots\Big] \quad (3)$$

Differentiating (3) further with respect to x; we get,

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\Big[\Delta^3 y_0 + \frac{1}{2}(2u-3)\Delta^4 y_0 + \dots\dots\dots\Big] \quad (4)$$

Formulas (2), (3) and (4) may be used to find the first three derivatives of y at any point $x = x_k$, where, $x_0 \le x_k \le x_n$ or x_k is close to x_0 or x_n, if it lies outside the range (x_0, x_n).

The above formulas take the following simpler forms when x_k concedes with x_0 *i.e.*, when $u = 0$.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\Big[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \dots\dots\dots\Big] \quad (5)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\ldots\right] \quad (6)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 - \ldots\ldots\right] \quad (7)$$

Values of Derivatives of y, Based On Newton's Backward Interpolating Formula

When $(n+1)$ pairs of values $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_n, y_n)$ are given, Newton's backward interpolation formula takes the form:

$$y = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_0 + \ldots\ldots \quad (1)$$

where, $u = \frac{x - x_n}{h}$

Differentiating both sides of (1) successively with respect to x; we have,

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2!}(2u+1)\nabla^2 y_n + \frac{1}{3!}(3u^2 + 6u + 2)\nabla^3 y_n + \frac{1}{4!}(4u^3 + 18u^2 + 224 + 6)\Delta^4 y_n + \ldots\ldots\right] \quad (2)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{1}{12}(6u^2 + 18u + 11)\nabla^4 y_n + \ldots\ldots\right] \quad (3)$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_0 + \frac{1}{2}(2u - 3)\nabla^4 y_0 + \ldots\ldots\right] \quad (4)$$

Formulas (2), (3) and (4) may be used to find the first three derivatives of y at any point $x = x_k$ where, $x_0 \leq x_k \leq x_n$ or x_k is close to x_0 or x_n, if it lies outside the range (x_0, x_n).

The above formulas take the following simpler forms when x_k coincides with x_n *i.e.*, when, $u = 0$.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2!}\nabla^2 y_n + \frac{1}{3!}\nabla^3 y_n + \frac{1}{4!}\nabla^4 y_n + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\ldots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \frac{7}{4}\nabla^5 y_n + \ldots\ldots\right]$$

Values of Derivatives of 'y', Based on Starlings Formulas

When the pairs of values $(x_{-3}, y_{-3}), (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1)$, etc. are given, sterling's interpolating formula takes the form:

$$y = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2 - 1^2)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}$$

$$+\frac{u_2(u^2-1^2)}{4!}\Delta^4 y_{-2}+\ldots\ldots\ldots$$

where, $u=\frac{x-x_0}{h}$.

Differentiating both sides of (1) successively with respect to ...; we have,

$$\frac{dy}{dx}=\frac{1}{h}\left[\frac{(\Delta y_0+\Delta y_{-1})}{2}+4\Delta^2 y_{-1}+\frac{(3u^2-1)}{6}\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2}+\frac{(2u^3-4)}{12}\Delta^4 y_{-2}+\ldots\ldots\ldots\right] \quad (2)$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[\Delta^2 y_{-1}+4\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2}+\frac{(6u^2-1)}{12}\Delta^4 y_{-2}+\ldots\ldots\right] \quad (3)$$

$$\frac{d^3y}{dx^3}=\frac{1}{h^3}\left[\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2}+\Delta^4 y_{-2}+\ldots\ldots\ldots\right] \quad (4)$$

Putting $x=x_0$ or $u=0$, formulas (2), (3) and (4) take the following simpler forms:

$$\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left[\frac{(\Delta y_0+\Delta y_{-1})}{2}-\frac{1}{6}\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2}+\frac{1}{30}\frac{(\Delta^5 y_{-2}+\Delta^5 y_{-3})}{2}+\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0}=\frac{1}{h^2}\left[\Delta^2 y_{-1}-\frac{1}{12}\Delta^4 y_{-2}+\ldots\ldots\ldots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0}=\frac{1}{h^3}\left[\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2}+\ldots\ldots\ldots\right]$$

In a similar manner, we can derive the formulas for the derivatives using other central difference interpolating formulas also.

Note

If the values of x are not equally spaced, we may use Newton's divided difference formula or Lagrange's interpolation formula to get the values of $\frac{dy}{dx}$ at any point 'x' and at any specified point $x = x_k$.

Example 5.1

Find the first and second derivatives of $y = f(x)$ at $x = 1.5$ from the data. Also find $f'(x)$ at $x = 3.5$ in two ways.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.0	13.625	24.0	38.875	59.0

Solution:

From the given;

x:	1.5	2.0	2.5	3.0	3.5	4.0
y:	3.375	7.0	13.625	24.0	38.875	59.0

Since we have to find $f'(x)$ and $f''(x)$ at the initial point $x = x_0$, we will use the simpler forms of the formulas for $\left(\frac{dy}{dx}\right)_{x=x_0}$ and $\left(\frac{d^2y}{dx^2}\right)_{x=x_0}$ derived from Newton's forward interpolation formula; namely,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\dots\right] \quad (1)$$

and, $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\dots\right] \quad (2)$$

Generating Newton's forward and backward interpolation table; we have,

x	y	Δy or ∇y	$\Delta^2 y$ or $\nabla^2 y$	$\Delta^3 y$ or $\nabla^3 y$	$\Delta^4 y$ or $\nabla^4 y$	$\Delta^5 y$ or $\nabla^5 y$
1.5	3.375					
		$\boxed{3.625}$				
2.0	7.0		$\boxed{3.0}$			
		6.625		$\boxed{0.75}$		
2.5	13.625		3.75		$\boxed{0}$	
		10.375		0.75		$\boxed{0}$
3.0	24.0		4.5		$\boxed{0}$	
		14.875		$\boxed{0.75}$		
3.5	38.875		$\boxed{5.25}$			
		$\boxed{20.125}$				
4.0	59.0					

Now, using the values form table above in (1); we get,

We have,

$h =$ Interval of $x = 2.0 - 1.5 = 0.5$

$\Delta y_0 = 3.625$

$\Delta^2 y_0 = 3.0$

$\Delta^3 y_0 = 0.75$

$\Delta^4 y_0 = 0$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}\times 3 + \frac{1}{3}\times 0.75 - \frac{1}{4}\times 0\right] = 4.75$$

Using (2); we get,

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.5} = \frac{1}{(0.5)^2}\left[3 - 0.75 + \frac{11}{12}\times 0\right] = 9$$

The formula for the derivative of f(x) at any point x, derived from Newton's forward interpolation formula is;

$$\frac{dy}{dx}=\frac{1}{h}\Big[\Delta y_0+\frac{1}{2!}(2u-1)\Delta^2 y_0+\frac{1}{3!}(3u^2-6u+2)\Delta^3 y_0+\frac{1}{4!}(4u^3-18u^2$$
$$+224-6)\Delta^4 y_0+\frac{1}{5!}(5u^4-40u^3+105u^2-100u+24)\Delta^5 y_0+\dots\Big] \quad (3)$$

where, $u=\frac{x-x_0}{h}$.

Here,

h = Interval of $x = 0.5$

At $x = 3.5$;

$$u=\frac{3.5-1.5}{0.5}=4$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=3.5}=\left(\frac{dy}{dx}\right)_{x=4}$$

$$\text{or,} \quad \left(\frac{dy}{dx}\right)_{x=3.5}=\frac{1}{0.5}\Big[3.625+\frac{1}{2}(2\times 4-1)\times 3+\frac{1}{2}\{3\times(4)^2-6\times 4+2\}\times 0.75+0\Big]$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=3.5}=34.75$$

Now, let us obtain the value of $\left(\frac{dy}{dx}\right)_{x=3.5}$ using Newton's backward interpolation formula.

The formula for $f'(x)$ is;

$$\frac{dy}{dx}=\frac{1}{h}\Big[\nabla y_n+\frac{1}{2!}(2u+1)\nabla^2 y_n+\frac{1}{3!}(3u^2+6u+2)\nabla^3 y_n+\frac{1}{4!}(4u^3+18u^2$$
$$+224+6)\nabla^4 y_n+\frac{1}{5!}(5u^4+40u^3+105u^2+100u+24)\nabla^5 y_n+\dots\dots\Big] \quad (4)$$

where, $u=\frac{x-x_n}{h}$

where, $h = 0.5$

and, $x = 3.5$

and, $x_n = 4$

$\therefore \quad u=\frac{x-x_n}{h}$

i.e., $\quad u=\frac{3.5-4}{0.5}=-1$

Also, from above table,

$\nabla y_n = 20.125$

$\nabla^2 y_n = 5.250$

$\nabla^3 y_n = 0.75$

$$\nabla^4 y_n = 0$$

$$\nabla^5 y_n = 0$$

Now, putting these values in (4); we have,

$$\left(\frac{dy}{dx}\right)_{x=3.5} = \frac{1}{0.5}\left[20.125 + \frac{1}{2}(-2+1)5.25 + \frac{1}{6}(3-6+2)0.75 + 0\right]$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=3.5} = 34.75$$

5.2 Maxima and Minima

It is known that the maximum and minimum values of a function can be found by equating the first derivative to zero and solving for the variable. Consider Newton's forward difference formula:

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots$$

Differentiating this with respect to 'u'; we obtain,

$$\frac{dy}{du} = \Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \dots\dots\dots \qquad (1)$$

For maxima or minima $\frac{dy}{du} = 0$,

Hence, terminating the right-hand side, for simplicity, after the third difference and equating it to zero, we obtain the quadratic for 'u'

$$\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 = 0$$

i.e., $\left(\frac{1}{2}\Delta^3 y_0\right)u^2 + (\Delta^2 y_0 - \Delta^3 y_0)u + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$

Substituting the valid of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ from the difference table, we solve this quadratic for 'u' then the corresponding values of x are given by $x = x_0 + uh$ at which 'y' is maximum or minimum.

Example 5.2

From the following table, find x, correct to two decimal places, for which y is maximum and find this value of y:

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

Solution:

From the given;

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

At first, generating the difference table,

x	y	Δy or ∇y	$\Delta^2 y$ or $\nabla^2 y$
1.2	0.9320		
		0.0316	
			- 0.0097
1.3	0.9636		
		0.0219	
			- 0.0099
1.4	0.9855		
		0.0120	
			- 0.0099
1.5	0.9975		
		0.0021	
1.6	0.9996		

Taking,

$$x_0 = 1.2,\ y_0 = 0.9320,\ \Delta y_0 = 0.0316,\ \Delta^2 y_0 = -0.0097$$

Therefore, Newton's forward difference formula gives;

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$$

or, $$y = 0.9320 + \frac{u}{1} \times 0.0316 + \frac{u(u-1)}{2!}(-0.0097)$$

Differentiating it with respect to u; we have,

$$\frac{dy}{du} = 0.0316 + \frac{(2u-1)}{2}(-0.0097)$$

For y to be maximum;

$$\frac{dy}{du} = 0$$

or, $$0.0316 + \frac{(2u-1)}{2}(-0.0097) = 0$$

or, $$u = 3.755$$

Hence,

$$x = x_0 + uh$$

where, h = Interval of x = 0.1.

$$x_0 = 1.2$$
$$u = 3.755$$

$\therefore$ $$x = x_0 + uh = 1.2 + 3.755 \times 0.1 = 1.58$$

Again, for this value of x; we have, to use Newton's backward formula at $x_n = 1.6$;

$$y = y_n + \frac{u}{1!}\nabla y_0 + \frac{u(u+1)}{2!}\nabla^2 y_n$$

where, $u = \frac{x - x_n}{h} = \frac{1.58 - 1.6}{0.1} = -0.2$

Also,

$$y_n = 0.9996$$

$$\nabla y_0 = 0.0021$$
$$\nabla^2 y_n = -0.0099$$

Now, putting these values; we get,

$$y(1.58) = 0.9996 + (-0.2)(0.0021) + \frac{(-0.2)(-0.2+1)}{2}(-0.0099)$$

$\therefore$ $y(1.58) = 0.9999$

Numerical Integration

Introduction

$\int_a^b f(x)$ can be evaluated exactly, if f(x) is explicitly defined as a mathematical function and it is integrable. But sometimes, the function $y = f(x)$ defined by means of $(n+1)$ paired values (x_i, y_i); $i = 0, 1, 2, \ldots, n$. In such situations, we resort to numerical integration, in which we replace f(x) by any convenient interpolating polynomial P(x) and compute $\int_a^b P(x)\,dx$ that is approximately equal $\int_a^b f(x)\,dx$. The process of numerical integration of a function of a single variable is sometimes called mechanical quadrature or simply quadrature and that of a function of two variables (*viz;* the computation of a double integral) is called mechanical cubature or simply cubature.

5.3 Newton's–Cote General Quadrature Formula

Let, $y_0, y_1, y_2, \ldots\ldots\ldots, y_n$ be the values of $y = f(x)$ corresponding to $x = x_0, x_1, x_2, \ldots\ldots\ldots, x_n$ which are equally spaced with step-size h.

Then, by Newton's forward interpolation formula.

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots \quad (1)$$

where, $u = \dfrac{x - x_0}{h}$

Now,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx$$

$$= \int_0^n \left[y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right] h\,du$$

On replacing y by interpolation polynomial in (1) and putting $dx = h\,du$;

or, $$\int_{x_0}^{x_n} y\,dx = h\left[y_0 + \Delta y_0 + \frac{(u^2-u)}{2}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{6}\Delta^3 y_0 + \ldots\right] du$$

or, $$\int_{x_0}^{x_n} y\,dx = h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0\right.$$

$$+\frac{1}{6}\left(\frac{u^4}{4}-u^3+u^2\right)\Delta^3 y_0+\ldots\ldots\Big]_0^n$$

or, $$\int_{x_0}^{x_n} y\,dx = h\Big[ny_0+\frac{n^2}{2}\Delta y_0+\frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0+\frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0$$
$$+\frac{1}{24}\left(\frac{n^5}{5}-\frac{3n^4}{2}+\frac{11n^3}{3}-3n^2\right)\Delta^4 y_0+\frac{1}{120}\left(\frac{n^6}{6}-2n^5+\frac{35n^4}{4}-\frac{50n^3}{3}\right.$$
$$\left.+12n^2\right)\Delta^5 y_0+\frac{1}{720}\left(\frac{n^7}{7}-\frac{15n^6}{6}+17n^5-\frac{225n^4}{4}+\frac{274n^3}{3}\right.$$
$$\left.-60n^2\right)\Delta^6 y_0+\ldots\ldots \quad (2)$$

Result (2) is called Newton-cote's quadrature formula.

From the general formula (2), we can obtain a variety of special quadrature formulas by putting $n = 1, 2, 3, \ldots\ldots$ as discussed below:

5.4 Trapezoidal, Simpson's $\frac{3}{8}$ Rule

Trapezoidal Rule

By putting $n = 1$ in (2) above *i.e.*, in Newton's-cote's quadrature formula and assuming that there are only two paired values of x and y or that the interpolating polynomial is linear; we get,

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0+\frac{\Delta y_0}{2}\right], \text{ as higher order differences do not exist.}$$

or, $$\int_{x_0}^{x_1} y\,dx = h\left[y_0+\frac{1}{2}(y_1-y_0)\right]$$

or, $$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0+y_1) \quad (1)$$

Composite Trapezoidal Rule

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx$$

or, $$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx+\int_{x_1}^{x_2} y\,dx+\ldots\ldots+\int_{x_{n-1}}^{x_n} y\,dx$$

where, $x_i - x_{i-1} = h;\ i = 0, 1, 2, \ldots\ldots, n$

or, $$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}(y_0+y_1)+\frac{h}{2}(y_1+y_2)+\ldots\ldots+\frac{h}{2}(y_{n-1}+y_2) \quad \text{[From (i)]}$$

or, $$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\ldots\ldots+y_{n-1})] \quad (2)$$

Formula (2) is known as composite trapezoidal or simply trapezoidal rule.

The geometric significance of these rules is that we replace the graph of the given function $y = f(x)$ by 'n' linear segments and hence the figure formed by the linear segment in (x_{i-1}, x_i) the two ordinates at $x = x_{i-1}$ and $x = x_i$ and the x-axis is a trapezium.

Simpson's One-Third Rule

By putting n = 2 in Newton-cote's quadrature formula *i.e.*, assuming that there are only three paired values of x and y or that the interpolating polynomial is a second degree polynomial; we get,

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$

as $\Delta^3 y_0$ and higher order differences do not exist.

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

Since, $\Delta^2 y_0 = y_2 - 2y_1 + y_0$

$$\int_{x_0}^{x_2} y\,dx = \frac{1}{3}(y_0 - 4y_1 + y_2) \qquad (1)$$

Composite Simpson's One-Third Rule

$\int_{x_0}^{x_{2n}} y\,dx = \int_{x_0}^{x_0+2nh} y\,dx$, assuming that (x_0, x_{2n}) is divided into 2n equal sub-intervals.

$$\int_{x_0}^{x_{2n}} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \ldots\ldots\ldots + \int_{x_{2n-2}}^{x_2} y\,dx$$

or, $$\int_{x_0}^{x_{2n}} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \ldots\ldots\ldots + \frac{h}{3}(y_{2n-2} + 4y_{2n-1} + y_{2n})$$

or, $$\int_{x_0}^{x_{2n}} y\,dx = \frac{h}{3}[(y_0 + y_{2n}) + 4(y_1 + y_3 + \ldots\ldots\ldots + y_{2n-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{2n-2})] \qquad (2)$$

Formula (2) is known as composite Simpson's one-third rule or simply Simpson's one-third rule.

The geometric significance of this rule is that we replace the graph of the given function of y = f(x) in (x_0, x_{2n}) by 'n' arcs of second degree parabolas with vertical axes.

Note

To apply composite Simpson's Rule, the interval of integration must be divided into an even number of sub-intervals each of width 'h'.

Simpson's Three-Eights Rule (3/8 Rule)

By putting n = 3 in Newton's cote's quadrature formula and omitting terms with $\Delta^4 y_0$ and higher order differences in the interpolating polynomial; we get,

$$\int_{x_0}^{x_3} y\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(9 - \frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

or, $$\int_{x_0}^{x_3} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 2y_1 - y_0)\right]$$

or, $\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$ (1)

Composite Simpson's Three-Eight's rule

$\int_{x_0}^{x_{3n}} y\,dx = \int_{x_0}^{x_0+3nh} y\,dx$ assuming that (x_0, x_{3n}) is divided into 3n equal sub-intervals.

$$\int_{x_0}^{x_{3n}} y\,dx = \int_{x_0}^{x_3} y\,dx + \int_{x_0}^{x_6} y\,dx + \ldots\ldots\ldots + \int_{x_{3n-3}}^{x_{3n}} y\,dx$$

or, $\int_{x_0}^{x_{3n}} y\,dx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \ldots\ldots + (y_{3n-3} + 3y_{3n-2} + 3y_{3n-1} + y_{3n})]$

or, $\int_{x_0}^{x_{3n}} y\,dx = \frac{3h}{8}[(y_0 + 3y_{3n}) + 3(y_1 + y_4 + y_7 + \ldots\ldots\ldots) + 3(y_2 + y_5 + y_8 + \ldots\ldots\ldots) + 2(y_3 + y_6 + y_9 + \ldots\ldots\ldots + y_{3n})]$ (2)

Formula (2) is known as composite Simpson's three-eight rule.

To apply this rule, the interval of integration must be divided into a 3-multiple number of sub-intervals each of width h.

Example 5.3

Using the data of the following table, compute the integrals $\int_{0.5}^{1.1} x^2y\,dx$ and $\int_{0.5}^{1.1} xy^2\,dx$ by trapezoidal rule.

x	0.5	0.6	0.7	0.8	0.9	1.0	1.1
y	0.4804	0.5669	0.6490	0.7262	0.7985	0.8658	0.9281

Solution:

From the given table;

x	0.5	0.6	0.7	0.8	0.9	1.0	1.1
y	0.4804	0.5669	0.6490	0.7262	0.7985	0.8658	0.9281

Here, we have to find $\int_{0.5}^{1.1} x^2y\,dx$ and $\int_{0.5}^{1.1} xy^2\,dx$ by using trapezoidal rule.

i) Let, $Z = x^2y$; then,

$$\int_{0.5}^{1.1} x^2y\,dx = \int_{0.5}^{1.1} Z\,dx$$

Then, by trapezoidal rule,

$$\int_{0.5}^{1.1} Z\,dx = \frac{h}{2}[(Z_0 + Z_6) + 2(Z_1 + Z_2 + Z_3 + Z_4 + Z_5)] \quad (1)$$

Since, interval is from 0.5 to 1.1 *i.e.*, interval is from Z_0 to Z_6.

Now, finding Z_0 to Z_6 by using the values of x_0 to x_6 and y_0 to y_6 from given table;

x	y	$Z = x^2y$
0.5	0.4804	$Z_0 = 0.1201$
0.6	0.5669	$Z_1 = 0.204084$
0.7	0.6490	$Z_2 = 0.31801$
0.8	0.7262	$Z_3 = 0.464768$
0.9	0.7985	$Z_4 = 0.646785$
1.0	0.8658	$Z_5 = 0.8658$
1.1	0.9281	$Z_6 = 1.123001$

Hence,

$$h = \text{Interval of } x = 0.6 - 0.5 = 0.1$$

Now, putting these values of Z_0 to Z_6 in (1); we get,

$$\int_{0.5}^{1.1} Z\,dx = \frac{0.1}{2}[(0.1201 + 1.123001) + 2(0.204084 + 0.31801 + 0.464768 + 0.646785 + 0.8658)]$$

or, $\int_{0.5}^{1.1} Z\,dx = 0.31209975$

$\therefore \quad \int_{0.5}^{1.1} x^2y\,dx = 0.3121$

ii) Let $u = xy^2$; then,

$$\int_{0.5}^{1.1} xy^2\,dx = \int_{0.5}^{1.1} u\,dx$$

Since interval is from 0.5 to 1.1 *i.e.*, from u_0 to u_6; so, by trapezoidal rule; we have,

$$\int_{0.5}^{1.1} u\,dx = \frac{h}{2}[(u_0 + u_6) + 2(u_1 + u_2 + u_3 + u_4 + u_5)] \quad (2)$$

Now, finding the values of u_0 to u_6;

x	y	$u = xy^2$
0.5	0.4804	$u_0 = 0.115392$
0.6	0.5669	$u_1 = 0.192825$
0.7	0.6490	$u_2 = 0.294840$
0.8	0.7262	$u_3 = 0.421893$
0.9	0.7985	$u_4 = 0.573842$
1.0	0.8658	$u_5 = 0.749609$
1.1	0.9281	$u_6 = 0.947506$

Here,

$$h = \text{Interval of } x = 0.6 - 0.5 = 0.1$$

Now, putting these values from u_0 to u_6 in (2); we get,

$$\int_{0.5}^{1.1} u\,dx = \frac{0.1}{2}[(0.115392 + 0.947506) + 2(+0.192825 + 0.294840$$
$$+0.421893 + 0.573842 + 0.749609)]$$

$$\therefore \quad \int_{0.5}^{1.1} u\,dx = 0.2764$$

Example 5.4

Compute the value of π from the formula $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$ using trapezoidal rule with 10 sub-intervals. When the interval (0, 1) is divided into 10 Sub-interval, then h = 0.1

The values of $y = \frac{1}{1+x^2}$ required for the use of trapezoidal rule are given in the following data:

x	$y = \frac{1}{1+x^2}$
0	1.0000
0.1	0.9901
0.2	0.9615
0.3	0.9174
0.4	0.8621
0.5	0.8000
0.6	0.7353
0.7	0.6711
0.8	0.6098
0.9	0.5525
1.0	0.5000

Solution:
From given table; we have,

x	$y = \frac{1}{1+x^2}$
0	$y_0 = 1.0000$
0.1	$y_1 = 0.9901$
0.2	$y_2 = 0.9615$
0.3	$y_3 = 0.9174$
0.4	$y_4 = 0.8621$
0.5	$y_5 = 0.8000$
0.6	$y_6 = 0.7353$
0.7	$y_7 = 0.6711$

0.8	$y_8 = 0.6098$
0.9	$y_9 = 0.5525$
1.0	$y_{10} = 0.5000$

Here, at first we have to find the value of $\int_0^1 \frac{dx}{1+x^2}$ *i.e.,* $\int_0^1 y\,dx$

Since, there is 11 terms *i.e.,* from y_0 to y_{10}; so, by using trapezoidal rule;

$$\int_0^1 y\,dx = \frac{h}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)] \quad (1)$$

Here,

$$h = \text{Interval of } x = 0.2 - 0.1 = 0.1$$

Now, putting the value of y_0 to y_{10} in (1), we get:

$$\int_0^1 y\,dx = \frac{h}{2}[(1.000 + 0.5000) + 2(0.9901 + 0.9615 + 0.9174 + 0.8621 + 0.8000 + 0.7353 + 0.6711 + 0.6098 + 0.5525)]$$

$$\therefore \quad \int_0^1 y\,dx = \int_0^1 \frac{dx}{1+x^2} = 0.78498$$

Now, from the given above;

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

or, $\frac{\pi}{4} = 0.78498$

$\therefore \quad \pi = 3.13992$

But, the actual value of $\pi = \frac{22}{7} = 3.14285$

Then,

Error committed $= 3.13992 - 3.14285 = -0.00293$

Example 5.5

The velocity 'V' of a particle at distance 'S' form a point on its linear path is given in the following data:

S (m)	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
V (m/sec.)	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 meters, using Simpson's one-third rule.

Solution:

From the given table; we have,

S (m)	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
V (m/sec.)	16	19	21	22	20	17	13	11	9

We have when a particle moves on a straight line, the velocity 'V' at a distance 'S' from a point on the line is given by;

$$V = \frac{dS}{dt}$$

or, $$\frac{1}{V} = \frac{dS}{dt}$$

or, $$dt = \frac{1}{V}dS$$

Integrating on both the sides; we have,

$$t = \int \frac{1}{V}dS$$

$\therefore$ $$\text{Time required} = \int_0^{20} \frac{1}{V}dS$$

Let, $$t = \frac{1}{V}$$

Now, finding 'y' corresponding to 'S' given in the table,

S	V	$y = \frac{1}{V}$
0	16	$y_0 = 0.0625$
0.1	19	$y_1 = 0.0625$
0.2	21	$y_2 = 0.0476$
0.3	22	$y_3 = 0.0454$
0.4	20	$y_4 = 0.050$
0.5	17	$y_5 = 0.0588$
0.6	13	$y_6 = 0.0769$
0.7	11	$y_7 = 0.0909$
0.8	9	$y_8 = 0.111$

Since, there is table nine-terms, *i.e.*, from y_0 to y_8 for S = 0 to 20 m.

Now, by Simpson's one-third rule; we have,

$$\int_{S_0}^{S_8} y\,dS = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_2 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)] \quad (1)$$

where, h = Interval of S = 2.5 − 0 = 2.5

Now, putting the values of y_0 to y_8 and 'h' in (1); we get,

$\therefore$ $$\int_0^{20} \frac{1}{V}dS = \frac{2.5}{3}[(0.0625 + 0.1111) + 4(0.0526 + 0.0454 + 0.0588 + 0.0909) + 2(0.0476 + 0.050 + 0.0769)]$$

$\therefore$ $$\int_0^{20} \frac{1}{V}dS = 1.2611$$

Hence, the taken by the particle to traverse the distance of 20 meters is 1.2611 seconds.

Example 5.6

A penter vase has circular cross-section and the areas of its section at various distances from the base are given in the following data:

Distance 'h' form base in cm	Area 'A' of cross-section in cm^2
0	44.375
1.25	86.250
2.50	110.625
3.75	75.625
5.00	25.000
7.50	25.000
10.00	28.125
12.50	37.500
15.00	60.00

Find the volume of the penter vase approximately, using Simpson's one-third rule of integration.

The volume of the vase is given by $V = \int_0^{15} A\, dh$

Solution:

Here, for the application of Simpson's one-third rule, the interval (0, 15) must have been divided into sub-interval of equal width.

But here, the interval (0, 5) is divided into equal Sub-intervals of width 1.25 and the interval (5, 15) is divided into equal sub-interval of width 2.5.

Hence, the required volume 'V' is takes as;

$$V = V_1 + V_2 \qquad (1)$$

where, $V_1 = \int_0^5 A\, dh$

and, $V_2 = \int_5^{15} A\, dh$

Now, from the given table for V_1

Distance 'h' form base in cm	Area 'A' of cross-section in cm^2
0	$A_0 = 44.375$
1.25	$A_1 = 86.250$
2.50	$A_2 = 110.625$
3.75	$A_3 = 75.625$
5.00	$A_4 = 25.000$

Here,

Interval of h = 1.25

Now, by using Simpson's one-third rule; we have,

$$V_1 = \int_0^5 A\,dh$$

or, $$V_1 = \frac{\text{Interval of h}}{3}[(A_0 + A_4) + 4(A_1 + A_3) + 2(A_2)]$$

or, $$V_1 = \frac{1.25}{3}[(44.375 + 25) + 4(86.250 + 75.625) + 2 \times 110.625]$$

or, $$V_1 = 390.8854 \text{ cm}^3$$

Again, from the given table for V_2

Distance 'h' form base in cm	Area 'A' of cross-section in cm^2
5.00	$A_0 = 25.000$
7.50	$A_1 = 25.000$
10.00	$A_2 = 28.125$
12.50	$A_3 = 37.500$
15.00	$A_4 = 60.000$

Here,

Interval of h = 2.50

Now, by using Simpson's one-third rule; we have,

$$V_2 = \int_5^{15} A\,dh$$

or, $$V_2 = \frac{\text{Interval of h}}{3}[(A_0 + A_4) + 4(A_1 + A_3) + 2(A_2)]$$

or, $$V_2 = \frac{2.5}{3}[(25.000 + 60.000) + 4(25.000 + 37.500) + 2 \times 28.125]$$

or, $$V_2 = 326.04167 \text{ cm}^3$$

Hence, the required total volume is;

$$V = V_1 + V_2 = 390.8854 + 326.04167 = 716.9270 \text{ cm}^3$$

Example 5.7

Compute $\int_0^{\frac{\pi}{2}} \sin x\,dx$, using Simpson's three-eighths rule of numerical integration.

Solution:

Here, to apply Simpson's three-eight's rule, the interval of integration $\left(0, \frac{\pi}{2}\right)$ must be divided into a (3-multiple) number of sub-interval of equal width.

Hence, let us divide $\left(0, \frac{\pi}{2}\right)$ into 9 sub-intervals each of width $\frac{\pi}{18}$ where, $\pi = 180°$

x	y = sin x
0	$y_0 = 0$
$\frac{\pi}{18}$	$y_1 = 0.1736$
$\frac{2\pi}{18}$	$y_2 = 0.3420$
$\frac{3\pi}{18}$	$y_3 = 0.5000$
$\frac{4\pi}{18}$	$y_4 = 0.6428$
$\frac{5\pi}{18}$	$y_5 = 0.7660$
$\frac{6\pi}{18}$	$y_6 = 0.8660$
$\frac{7\pi}{18}$	$y_7 = 0.9397$
$\frac{8\pi}{18}$	$y_8 = 0.9848$
$\frac{9\pi}{18}$	$y_9 = 1.000$

Here,

$$\text{Interval of } x = \frac{\pi}{18} = h$$

Now, by using Simpson's three-eight's rule; we have,

$$\int_{x_0}^{x_9} y\,dx = \frac{3h}{8}[(y_0 + y_9) + 3(y_1 + y_4 + y_7) + 3(y_2 + y_5 + y_8) + 2(y_3 + y_6)]$$

or, $$\int_{x_0}^{x_9} y\,dx = \frac{3}{8} \times \frac{\pi}{18}[(0 + 1.000) + 3(0.1736 + 0.6428 + 0.9397) + 3(0.3420 + 0.7660 + 0.9848) + 2(0.5000 + 0.8660)]$$

∴ $$\int_{x_0}^{x_9} y\,dx = 0.31830625\pi$$

Here, we use the values of π in radian *i.e.*, $\pi = 3.14285$

$$\int_{x_0}^{x_9} y\,dx = 0.31830625 \times 3.14285 = 1.000$$

Hence,

$$\int_{x_0}^{x_9} y\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx = 1.000$$

Example 5.8

S is the Specific heat of a body at temperature θ°C. Find the total heat required to raise the temperature of the body of weight gram from 0°C to 12°C, using following data of values and Sampson's three-eight's rule:

θ	0	2	4	6	8	10	12
S	1.00664	1.00543	1.00435	1.00331	1.00233	1.00149	1.00078

Solution:

From the given table; we have,

θ	0	2	4	6	8	10	12
S	1.00664	1.00543	1.00435	1.00331	1.00233	1.00149	1.00078

From Simpson's three eight's rule; we have,
The total heat required is given by;

$$H = \int_{\theta_0}^{\theta_6} S\, d\theta$$

or, $$H = \frac{3h}{8}[(S_0 + S_6) + 3(S_1 + S_4) + 3(S_2 + S_5) + 2 \times S_3] \quad (1)$$

where, $\theta_0 = 0$.

and, $\theta_6 = 12$

$h = \text{Interval of } \theta = 2 - 0 = 2$

Now, putting these values in (1); we get,

$$H = \int_0^{12} S\, d\theta = \frac{3 \times 2}{8}[(1.00664 + 1.00078) + 3(1.00543 + 1.00233) + 3(1.00435 + 1.00149) + 2 \times 1.00331]$$

$$\therefore \quad \int_0^{12} S\, d\theta = 12.04113$$

Hence, the total heat required to raise to raise the temperature of body of weight 1 gram from 0°C to 12°C is 12.04113.

5.5 Romberg Integration

As above, in 5.3 and 5.4; we have, derived approximate quadrature formulae with the help of finite difference method. Romberg's method provides a simple modification to these quadrature formulas for finding their better approximations. As an illustration, let us improve upon the value of the integral.

$I = \int_a^b f(x)\, dx$; by trapezoidal rule

If I_1, I_2 be the values of with Sub-intervals of width h_1, h_2 and E_1, E_2 be their corresponding errors respectively; then,

$$E_1 = -\frac{(b-a)h_1^2}{12} y''(x)$$

$$E_2 = -\frac{(b-a)h_2^2}{12} y''(\overline{x})$$

Since, $y''(\overline{x})$ is also the largest value of $y''(x)$, we can reasonably assume that $y''(x)$ and $y''(\overline{x})$ are very nearly equal.

$$\frac{E_1}{E_2} = \frac{h_1^2}{h_2^2}$$

or, $$\frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2} \quad (1)$$

Since, $I = I_1 + E_1 = I_2 + E_2 \quad (2)$

$$\therefore \quad E_2 - E_1 = I_1 - I_2$$

From (1) and (2); we have,

$$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(E_2 - E_1)$$

or, $$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

Hence,

$$I = I_1 + E_1 = I_1 + \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

or, $$I = \frac{I_1 h_2^2 - I_1 h_1^2 + I_1 h_1^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

i.e., $$I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}; \qquad (3)$$

which is a better approximation of I.

To evaluate I systematically; we take,

$$h_1 = h$$

and, $$h_2 = \frac{h}{2}$$

so, from (3); we get,

$$I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2 h^2}{\left(\frac{h}{2}\right)^2 - h^2} = \frac{\frac{I_1 h^2 - 4I_2 h^2}{4}}{\frac{h^2 - 4h^2}{4}} = \frac{(I_1 - 4I_2)h^2}{-3h^2} = \frac{4I_2 - I_1}{3}$$

i.e., $$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] \qquad (4)$$

Now, we use the trapezoidal rule several times successively having 'h' and apply (4) to each pair of values as per the following scheme.

$I(h)$			
	$I\left(h, \frac{h}{2}\right)$		
$I\left(\frac{h}{2}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}\right)$	
	$I\left(\frac{h}{2}, \frac{h}{4}\right)$		$I\left(h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$
$I\left(\frac{h}{4}\right)$		$I\left(\frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right)$	
	$I\left(\frac{h}{4}, \frac{h}{8}\right)$		
$I\left(\frac{h}{8}\right)$			

The computation is continued till successive values are close to each other. This method is called Richardson's deferred approach to the limit and its systematic refinement is called Romberg's method.

Example 5.9

Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places using Romberg's method hence, find the value of $\log_e 2$.

Solution:

Let, $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

so, taking $h = 0.5$, 0.25 and 0.125 successively, let us evaluate the given integral by trapezoidal rule.

i) When, $h = 0.5$, the values of $y = (1+x)^{-1}$ are;

x:	0	0.5	1
y:	1	0.6667	0.5

Now, from trapezoidal rule; we have,

$$I(h) = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{0.5}{2}[1 + 0.5 + 2 \times 0.6667] = 0.7083$$

ii) When $\frac{h}{2} = \frac{0.5}{2} = 0.25$, the values of $y = (1+x)^{-1}$ are;

x:	0	0.25	0.5	0.75	1
y:	1	0.8	0.6667	0.5714	0.5

Now, from trapezoidal rule; we have,

$$I\left(\frac{h}{2}\right) = \frac{\left(\frac{h}{2}\right)}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[(1 + 0.5) + 2(0.8 + 0.6667 + 0.5714)]$$

$$\therefore \quad I\left(\frac{h}{2}\right) = 0.6970$$

iii) When $\frac{h}{4} = \frac{0.5}{4} = 0.125$, the values of $y = (1+x)^{-1}$ are;

x	0	0.125	0.25	0.375	0.50	0.625	0.75	0.875	1
y	1	0.8889	0.8	0.7272	0.6667	0.6153	0.5714	0.5333	0.5

Now, from trapezoidal rule; we have,

$$I\left(\frac{h}{4}\right) = \frac{\left(\frac{h}{4}\right)}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[(1 + 0.5) + 2(0.8889 + 0.8 + 0.7272 + 0.6667 + 0.6153 + 0.5714 + 0.5333)]$$

$$\therefore \quad I\left(\frac{h}{4}\right) = 0.6941$$

Now, using Romberg's formulae; we have,

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = \frac{1}{3}(4 \times 0.6970 - 0.7083) = 0.6923$$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = \frac{1}{3}(4 \times 0.6941 - 0.6970) = 0.6931$$

Again,

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right]$$

$$= \frac{1}{3}(4 \times 0.6931 - 0.6932) = 0.6931$$

Hence, the value of the integral corrected to three decimal places is

$$\int_0^1 \frac{dx}{1+x} = I\left(h, \frac{h}{2}, \frac{h}{4}\right) = 0.693 \qquad (1)$$

Also; we have,

$$\int_0^1 \frac{dx}{1+x} = |\log_e(1+x)|_0^1 = \log_e 2 \qquad (2)$$

From (1) and (2); we have,

$$\log_e 2 = 0.693$$

Example 5.10

Use Romberg integration method to evaluate $\int_0^2 \frac{e^x + e^{-x}}{2}dx$, correct up to three decimal place up to three decimal place.

Solution:

Let, $h = \frac{b-a}{2} = \frac{2-0}{2} = 1$

so, taking h = 1, 0.5 and 0.25 successively, let us evaluate the given integral by trapezoidal rule;

so, assume $y_i = \frac{e^x + e^{-x}}{2}$

i) When h = 1, then the values of $\frac{e^x + e^{-x}}{2}$ *i.e.,* y_i are;

x	0	1	2
y	1	1.5430	3.7621

Now, using trapezoidal rule; we have,

$$I(h) = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[1 + 3.7621 + 2 \times 1.5430] = 3.924$$

ii) When $\frac{h}{2} = \frac{1}{2} = 0.5$, then the values of $y_i = \frac{e^x + e^{-x}}{2}$ are;

x	0	0.5	1.0	1.5	2.0
y	1	1.1276	1.5430	2.3524	3.7621

Now, using trapezoidal rule; we have,

$$I\left(\frac{h}{2}\right) = \frac{\left(\frac{h}{2}\right)}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[(1 + 3.7621) + 2(1.1276 + 1.5430 + 2.3524)]$$

$$\therefore \quad I\left(\frac{h}{2}\right) = 3.7020$$

iii) When $\frac{h}{4} = \frac{1.0}{4} = 0.25$, then the values of $y_i = \frac{e^x + e^{-x}}{2}$ are;

x	0	0.25	0.50	0.75	1	1.25	1.50	1.75	2.0
y	1	1.0314	1.1276	1.2946	1.5430	1.8884	2.3524	2.9641	3.7621

Using trapezoidal rule; we have,

$$I\left(\frac{h}{4}\right) = \frac{\left(\frac{h}{4}\right)}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[(1 + 3.7621) + 2(1.0314 + 1.1276 + 1.2946 + 1.5430 + 1.8884 + 2.3524 + 2.9641)]$$

$$\therefore \quad I\left(\frac{h}{4}\right) = 3.6456$$

Now, using Romberg's formula; we get,

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = \frac{1}{3}(4 \times 3.7020 - 3.924) = 3.6280$$

Again,

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = \frac{1}{3}(4 \times 3.6456 - 3.7020) = 3.6268$$

Again,

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right]$$

$$= \frac{1}{3}(4 \times 3.6268 - 3.6280) = 3.6264$$

Hence, the value of the integral $I = \int_0^2 \frac{e^x + e^{-x}}{2}\, dx$ corrected to three decimal place is;

$$I = \int_0^2 \frac{e^x + e^{-x}}{2}\, dx = 3.6264$$

5.6 Gaussian Integration (Gaussian–Legendre Formula 2 Point And 3 Point)

As the formulae derived for evaluation of $\int_a^b f(x)\, dx$ required the values of the function at equally spaced points of the interval. Gauss derived a formula which uses the same number of functional values but with different spacing and yields better accuracy.

i) Gauss formula is expressed as;

$$\int_{-1}^{1} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2) + \ldots\ldots\ldots + w_n f(x_n)$$

or, $$\int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i) \tag{1}$$

where, w_i and x_i are called the weights and abscissa respectively. The abscissa and weights are symmetrical with respect to the middle point of the interval. There being 2n unknowns in (1), 2n relations between them are necessary so that the formula is exact for all polynomials of degree not exceeding $2n - 1$.

Thus; we consider,

$$f(x) = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots + C_{2n-1} x^{2n-1} \tag{2}$$

Then (1) gives,

$$\int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} (C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots + C_{2n-1} x^{2n-1})\, dx$$

or, $$\int_{-1}^{1} f(x)\, dx = 2C_0 + \frac{2}{3} C_2 + \frac{2}{5} C_4 + \ldots\ldots\ldots \tag{3}$$

Putting $x = x_i$ in (2); we get,

$$f(x_i) = C_0 + C_1 x_i + C_2 x_i^2 + \ldots\ldots\ldots + C_{2n-1} x_i^{2n-1}$$

Substituting these values on the right hand side of (1); we obtain,

$$\begin{aligned}
\int_{-1}^{1} f(x)\, dx = {} & w_1 (C_0 + C_1 x_1 + C_2 x_1^2 + C_3 x_1^2 + \ldots\ldots + C_{2n-1} x_1^{2n-1}) \\
& + w_2 (C_0 + C_1 x_2 + C_2 x_2^2 + C_3 x_2^2 + \ldots\ldots + C_{2n-1} x_2^{2n-1}) \\
& + w_3 (C_0 + C_1 x_3 + C_2 x_3^2 + C_3 x_3^2 + \ldots\ldots + C_{2n-1} x_3^{2n-1}) \\
& + \ldots\ldots + w_n (C_0 + C_1 x_n + C_2 x_n^2 + C_3 x_n^2 + \ldots\ldots + C_{2n-1} x_n^{2n-1})
\end{aligned}$$

or, $$\begin{aligned}
\int_{-1}^{1} f(x)\, dx = {} & C_0 (w_1 + w_2 + w_3 + \ldots\ldots + w_n) \\
& + C_1 (w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots\ldots + w_n x_n) \\
& + C_2 (w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + \ldots\ldots + w_n x_n^2) + \ldots\ldots\ldots \\
& + C_{2n-1} (w_1 x_1^{2n-1} + w_2 x_2^{2n-1} + w_3 x_3^{2n-1} + \ldots\ldots + w_n x_n^{2n-1})
\end{aligned} \tag{4}$$

But the equations (3) and (4) are identical for all values of C_i, hence comparing coefficients of C_i, we obtain 2n equations in 2n unknowns w_i and $x_i (i = 1, 2, 3, \ldots\ldots\ldots, n)$

$$\left.\begin{aligned} & w_1 + w_2 + w_3 + \dots\dots + w_n = 2 \\ & w_1x_1 + w_2x_2 + w_3x_3 + \dots\dots + w_nx_n = 0 \\ & w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \dots\dots + w_nx_n^2 = \frac{2}{3} \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_3^{2n-1} + \dots\dots + w_nx_n^{2n-1} = 0 \end{aligned}\right\} \quad (5)$$

The solutions of the above equations are extremely complicated. It can however, be shown that x_i are the zeros of the $(n+1)^{th}$ Legendre polynomial.

ii) Gauss formula for $n = 2$ is;

$$\int_{-1}^{1} f(x)\,dx = w_1f(x_1) + w_2f(x_2)$$

Then, the equations (5) becomes;

$$w_1 + w_2 = 2$$
$$w_1x_1 + w_2x_2 = 0$$
$$w_1x_1^2 + w_2x_2^2 = \frac{2}{3}$$
$$w_1x_1^3 + w_2x_2^3 = 0$$

Solving these equations, we obtain.

$$w_1 = w_2 = 1$$
$$x_1 = -\frac{1}{\sqrt{3}}$$

and, $x_2 = \frac{1}{\sqrt{3}}$

Thus, Gauss formula for $n = 2$ is;

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right); \quad (6)$$

which gives the correct value of the integral of $f(x)$ in the range $(-1, 1)$ for any function up to third order.

Equation (6) is also known as "Gauss-Legendre formula".

iii) Gauss formula for $n = 3$ is;

$$\int_{-1}^{1} f(x)\,dx = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]; \quad (7)$$

which is exact for polynomials up to degree 5.

Example 5.11

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for $n = 2$ and $n = 3$.

Solution:

Here, we have to evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$

By using Gauss formula for n = 2 and n = 3

i) Gauss formula for n = 2

We have for n = 2, Gauss formula is given by;

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

where, $f(x) = \frac{1}{1+x^2}$

$$\therefore \quad f\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\left(-\frac{1}{\sqrt{3}}\right)^2} = \frac{3}{4}$$

and, $f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{3}{4}$

$$\therefore \quad I = \int_{-1}^{1} \frac{dx}{1+x^2} = \frac{3}{4} + \frac{3}{4} = 1.5$$

Hence, for n = 2;

$$\int_{-1}^{1} \frac{dx}{1+x^2} = 1.5$$

ii) Gauss formula for n = 3

We have for n = 3, Gauss formula is given by;

$$I = \int_{-1}^{1} f(x)\,dx = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

where, $f(x) = \frac{1}{1+x^2}$

$$\therefore \quad f(0) = \frac{1}{1+(0)^2} = 1$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{1}{1+\left(-\sqrt{\frac{3}{5}}\right)^2} = \frac{5}{8}$$

and, $f\left(\sqrt{\frac{3}{5}}\right) = \frac{1}{1+\left(\sqrt{\frac{3}{5}}\right)^2} = \frac{5}{8}$

$$\therefore \quad I = \int_{-1}^{1} \frac{dx}{1+x^2} = \frac{8}{9} \times 1 + \frac{5}{9}\left[\frac{5}{8} + \frac{5}{8}\right] = 1.5833$$

Hence, for n = 3;

$$\int_{-1}^{1} \frac{dx}{1+x^2} = 1.5833$$

Note

Gauss formula imposes a restriction on the limits of integration to be from −1 to 1.

In general, the limits of the integral $\int_a^b f(x)\,dx$ are changed to −1 to 1 by means of the transformation.

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) \qquad \text{(a)}$$

Example 5.12

Using three point Gaussian quadrature formula, evaluate $\int_0^1 \frac{dx}{1+x}$

Solution:

We have to find the value of $\int_0^1 \frac{dx}{1+x}$ by using Gauss formula for n = 3.

Here, at first, we change the limits (0, 1) to −1 to 1 by using (a), so that;

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

where, b = 1 and a = 0.

or, $x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) = \frac{u}{2} + \frac{1}{2}$

∴ $x = \frac{1}{2}(u+1)$

and, $dx = \frac{1}{2}du$

∴ $$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{\frac{1}{2}du}{1+\frac{1}{2}(u+1)} = \int_{-1}^{1} \frac{du}{u+3}$$

Now, gauss formula for n = 3 is;

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

where, $f(u) = \frac{1}{u+3}$

$$f(0) = \frac{1}{0+3} = \frac{1}{3}$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{1}{-\sqrt{\frac{3}{5}}+3} = 0.44935$$

and, $f\left(\sqrt{\frac{3}{5}}\right) = \dfrac{1}{\sqrt{\frac{3}{5}} + 3} = 0.26492$

$$\therefore \quad I = \frac{8}{9} \times \frac{1}{3} + \frac{5}{9}[0.44935 + 0.26492] = 0.69311$$

Thus, for n = 3;

$$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^1 \frac{du}{u+3} = 0.69311$$

5.7 EXAMINATION PROBLEMS

1. **Find approximate values of y'(3) and y''(3) from the following function.**

x	2	2.5	3	3.5	4
y	5.53	5.14	4.62	2.96	2.89

[2071 Chaitra]

Solution:

The difference table can be presented as;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	5.53				
		– 0.39			
2.5	5.14		– 0.13		
		– 0.52		– 1.01	
3	4.62		– 1.14		3.74
		– 1.66		2.73	
3.5	2.96		1.59		
		– 0.07			
4	2.89				

Using, Newton's forward interpolation function:

Here, we have;

$$h = 2.5 - 2 = 0.5$$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]$$

or, $$y'(3) = \frac{1}{0.5}\left[-1.66 - \frac{1}{2}\times 1.59\right] = -4.91$$

∴ $$y'(3) = -4.91$$

and, $$y''(3) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

or, $$y''(3) = \frac{1}{0.5^2}[1.59 - 0] = 6.36$$

or, $$y''(3) = 6.36$$

2. **Write an algorithm to calculate the definite integral $\int_a^b f(x)\,dx$ using composite Simpson's $\frac{1}{3}$ rule.** [2072 Ashwin]

Solution: See the definition part

3. **The distance traveled by a vertical at intervals of 2 minutes are given as follows:**

Time (min):	2	4	6	8	10	12
Distance (km):	0.25	1	2.2	4	6.5	8.5

Evaluate the velocity and acceleration of the vertical at t = 3 minutes. **[2072 Ashwin]**

Solution:

Now, presenting the above data is difference table as below:

x (min)	y (km)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	0.25					
		0.75				
4	1		0.45			
		1.2		0.15		
6	2.2		0.6		– 0.05	
		1.8		0.10		– 1.25
8	4		0.7		– 1.30	
		2.5		– 1.2		
10	6.5		– 0.5			
		2.0				
12	8.5					

Using Newton's forward interpolation formula;

Here, we have;

$h = 2$ min

For velocity,

$$\left(\frac{dy}{dx}\right) = y'(t_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + \ldots\ldots\right]$$

Note

Precede same as the solution of Q. no. 1

4. **Derive Newton-codes quadrature formula for integration and use it to obtain the Trapezoidal rule of integration.** **[2072 Chaitra]**

Solution: See the definition part

5. **Find the value of cos (1.74) from the following table.**

x	1.7	1.74	1.78	1.82	1.86
sin x	0.9916	0.9857	0.9781	0.9691	0.9587

[2073 Shrawan, 2073 Bhadra]

Solution:

The difference table is presented below:

x	y = sin x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	0.9916				
		-0.0059			
1.74	0.9857		-0.0017		
		-0.0076		0.0003	
1.78	0.9781		-0.0014		- 0.0006
		-0.0090		-0.0003	
1.82	0.9691		-0.0017		
		-0.0107			
1.86	0.9587				

Using, Newton's forward interpolation formula

Here, we have;

$$h = 1.74 - 1.70 = 0.04$$

$$y'(x) = \cos(x) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]$$

$$\cos(1.74) = \frac{1}{0.04}\left[-0.0076 + \left(\frac{1}{2}\times 0.0014\right) - \frac{1}{3}\times(0.0003) + 0\right]$$

$$= -0.175$$

$\therefore$ $\cos(1.74) = -0.175$

6. Derive composite Simpson's three-eight formula for the integration. **[2073 Shrawan]**

Solution: See the definition part

7. Evaluate $\int_{0.2}^{1.5} e^{-(x2)}\,dx$ using the 3 point Gaussian quadrature formula. **[2073 Bhadra]**

Solution:

We have to find the value of $\int_{0.2}^{1.5} e^{-(x^2)}\,dx$ by using Gauss, formula for n = 3

Here, at first, we change the limits from (0.2, 1.5) to (–1, 1)

so, $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(1.5-0.2)u + \frac{1}{2}(1.5+0.)$

$= (0.65u + 0.85)$

so, $dx = 0.65\,du$

$\therefore$ $I = \int_{0.2}^{1.5} e^{-(x^2)}\,dx = \int_{-1}^{1} e^{-(0.054+0.85)^2}\times 0.65\,du$

Here,

$f(u)\ e^{-(0.654+0.85)^2}\times 0.65$

so, $f(0) = e^{-(0.65\times 0+0.85)^2+0.65)} = 0.4855 \times 0.65 = 0.3156$

$$f\left(-\sqrt{\frac{3}{5}}\right) = e^{0.65\times -\sqrt{\frac{3}{5}}+0.85^2} \times 0.65 = 0.8869 \times 0.65 = 0.5765$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = e^{0.65\times -\sqrt{\frac{3}{5}}+0.85^2} \times 0.65 = 0.1601 \times 0.65 = 0.1041$$

Here, we have the Gauss 3 point formula;

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9} + 0.3156 + \frac{5}{9}[0.5765 + 0.1041]$$

$$= 0.2805 + 0.3781$$

$$= 0.6586$$

8. **Derive the formula to evaluate y'(a) y''(x) from Newton's forward Interpolation formula.** **[2073 Magh]**

Solution: See the definition part

9. **Evaluate $\int_0^{1.4}(\sin x^3+\cos x^2).\,dx$ using Gaussian-3 point formula.**

[2073 Magh]

Solution:

Here, we have to evaluate $\int_0^{1.4}(\sin x^3 + \cos x^2)\,dx$ by using Gauss formula for n = 3.

Here, at first, we have to change the limit force (0, 1.4) to (–1 , 1).

so, $x = \frac{1}{2}(b - a)\,u + \frac{1}{2}(b + a)$

$$= \frac{1}{2}(1.4 - 0)\,u + \frac{1}{2}(1.4 + 0)$$

$$= 0.74 + 0.7$$

$$= 0.7\,(4 + 1)$$

$\therefore$ $dx = 0.7\,du$

$\therefore$ $I = \int_0^{1.4}(\sin x^3 + \cos x^2)\,dx$

$$= \int_{-1}^{1}[\sin\{0.7\,(4 + 1)\}^3 + \cos\{0.7\,(4 + 1)\}^2] + 0.7\,du$$

$$= \int_{-1}^{1}[\sin\{0.373\,(4 + 1)^3\} + \cos\{0.49(4 + 1)^2\}] + 0.7\,du$$

$\therefore$ $f(u) = 0.7\,[\sin\{0.343\,(4 + 1)^3\} + \cos\{0.19\,(4 + 1)^2\}]$

and, $f(0) = 0.7042$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.70$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.7232$$

Now, using 3-point Gauss formula;

$$I = \frac{8}{9}(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9} + 0.7042 + \frac{5}{9}[0.70 + 0.7232]$$

$$= 0.626 + 0.7907$$

$$= 1.4167$$

10. Derive Newton-cotes quadrature formula for integration.

[2073 Chaitra]

Solution: See the definition part

11. Evaluate $\int_{-1}^{2} e^{-x^2}\,dx$ using 3-point Gaussian quadrature formula.

[2073 Chaitra]

Solution:

Here, we have to evaluate $\int_{-1}^{2} e^{-x^2}\,dx$ using Gauss formula for n = 3

Here, at first we have to change the limit from (–1, 2) to (–1, 1)

Putting, b = 2 and a = – 1

so, $x = \frac{1}{2}(2+1)\,u + \frac{1}{2}(2-1) = \frac{3}{2}u + \frac{1}{2}8 = \frac{1}{2}(3u+1)$

$\therefore$ $2x = 1.5\,du$

$\therefore$ $I = \int_{-1}^{2} e^{-x^2}\,dx = \int_{-1}^{1} e^{-(1.54+0.5)^2} \times 1.5\,du = \int_{-1}^{1} 1.5\,e^{-(1.54+0.5)^2}\,du$

so, $f(u) = 1.5 \times e^{-(1.54+0.5)^2}$

$f(0) = 1.1682$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.0948$$

Now, using Gauss 3-point formula;

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9} \times 1.1682 + \frac{5}{9}[0.9676 + 0.0948]$$

$$= 1.0384 + 0.5904$$

$$= 1.6288$$

12. Evaluate $\int_0^2 (\sin x + \cos x)dx$ using Gaussian 3-point formula. **[2074 Ashwin]**

Solution:

First we change the limit of integration from $(0, 2)$ to $(-1, 1)$ using transformation.

$$x = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right) = \left(\frac{2-0}{2}\right)z + \left(\frac{2+0}{2}\right) = z + 1$$

$\therefore \quad dx = dz$

$\therefore \quad g(z) = \sin(z+1) + \cos(z+1)$

Now,

$$\int_{-1}^{1} g(z)\,dz = \left(\frac{b-a}{2}\right) \sum_{i=1}^{n} w_i g(z_i)$$

$$= 1 \times [w_1 g(z_1) + w_2 g(z_2) + w_3 g(z_3)]$$

$$= \frac{5}{9} \times \left[\sin\left(\frac{-\sqrt{3}}{5} + 1\right) + \cos\left(\frac{-\sqrt{3}}{5} + 1\right)\right]$$

$$+ \frac{8}{9}[\sin(0+1) + \cos(0+1)]$$

$$+ \frac{5}{9}\left[\sin\left(\frac{\sqrt{3}}{5} + 1\right) + \cos\left(\frac{\sqrt{3}}{5} + 1\right)\right]$$

$$= 0.5618 + 0.904 + 0.5684$$

$$= 2.0344$$

13. Derive the formula for computing first and second derivative using Newton's forward difference interpolation formula. **[2074 Ashwin]**

Solution: See the definition part 5.1 and derive equations 1, 2, 3, 5 and 6

14. Derive Newton's cote general quadrature formula and hence use it to obtain Simpson's $\left(\frac{1}{3}\right)$ rule of integration. **[2074 Chaitra]**

Solution: See the definition part 5.3 and 5.4

15. Evaluate $\int_0^1 \frac{\tan^{-1}x}{x} dx$ using Gaussian 3-point formula.

[2074 Chaitra]

Solution:

First we change the limit of integration from (0, 1) to (-1, 1) using transformation.

$$x = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right)$$

$$= \left(\frac{1-0}{2}\right)z + \left(\frac{1+0}{2}\right)$$

$$= 0.5z + 0.5$$

$\therefore \quad dx = \frac{1}{2} dz$

$\therefore \quad g(z) = \frac{\tan^{-1}(0.5z + 0.5)}{0.5z + 0.5}$

Now,

$$\int_{-1}^{1} g(z)\, dz = \left(\frac{b-a}{2}\right) \sum_{i=1}^{n} w_i g(z_i)$$

$$= \frac{1-0}{2} \times [w_1 g(z_1) + w_2 g(z_2) + w_3 g(z_3)]$$

$$= \frac{1}{2} \times \left[\frac{5}{9} \times \frac{\tan^{-1}\left(0.5 \times \frac{-\sqrt{3}}{5} + 0.5\right)}{0.5 \times \frac{-\sqrt{3}}{5} + 0.5} + \frac{8}{9} \times \frac{\tan^{-1}(0.5)}{0.5} + \frac{5}{9} \times \frac{\tan^{-1}\left(0.5 \times \frac{\sqrt{3}}{5} + 0.5\right)}{0.5 \times \frac{\sqrt{3}}{5} + 0.5}\right]$$

$$= \frac{1}{2} \times (30.765 + 47.22 + 28.0157)$$
$$= 53$$

16. **Derive an expression to evaluate first derivative from Newton's backward interpolation formula and evaluate $\frac{dy}{dx}$ at $x = 8$ for the following table.** **[2075 Ashwin]**

x	2	4	6	8
y	- 0.7553	- 11.2151	34.2867	- 8.3226

Solution:

For the first part

See the definition part 5.1 [derive equations (1) and (2)]

Also, write

At $x = x_n$;

$$u = \frac{x - x_n}{h} = \frac{0}{h} = 0$$

so, equation (2) becomes;

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2!}\nabla^2 y_n + \frac{1}{3!}\nabla^3 y_n + \frac{1}{4!}\nabla^4 y_n + \ldots\ldots\right]$$

For the second part

The difference table is;

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
2	– 0.7553			
		– 10.4598		
4	– 11.2151		55.9616	
		45.5018		– 144.0727
6	34.2867		– 88.1111	
		– 42.6093		
8	– 8.3226			

At x = 8;

$$u = \frac{x - x_n}{h} = \frac{8-8}{2} = 0$$

so, we use the equation;

$$\left(\frac{dy}{dx}\right)_{x=8} = \frac{1}{h}\left[\nabla y_8 + \frac{1}{2!}\nabla^2 y_8 + \frac{1}{3!}\nabla^3 y_8 + \frac{1}{4!}\nabla^4 y_8 + \ldots\ldots\right]$$

$$= \frac{1}{2}\left[-42.6093 + \frac{1}{2!}(-88.1111) + \frac{1}{3!}(-144.0727) + \ldots\ldots\right]$$

$$= -55.3384$$

17. Derive the general Newton-Cotes quadrature formula and hence use it to obtain Simpson's $\left(\frac{3}{8}\right)$ formula. [2075 Ashwin]

Solution: See the definition part 5.3 and 5.4

18. Derive composite Simpson's $\left(\frac{3}{8}\right)$ formula for integration. [2075 Chaitra]

Solution: See the definition part 5.4

19. Use Romberg's method of compute $\int_0^1 \frac{1}{1+x^2}\,dx$ correct to three decimal places. [2075 Chaitra]

Solution: See the definition part 5.4

Let, h = 0.5; then,

$$\frac{h}{2} = 0.25$$

$$\frac{h}{4} = 0.125$$

Let's evaluate the integral using trapezoidal rule.

i) When h = 0.5, the value of $y = \frac{1}{1+x^2}$ is shown in the table below:

x	0	0.5	1
y	1	0.8	0.5
	y_0	y_1	y_2

$$\therefore \quad I(h) = I(0.5) = \frac{h}{2}[y_0 + 2y_1 + y_2]$$
$$= \frac{0.5}{2}[1 + 2 \times 0.8 + 0.5]$$
$$= 0.775$$

ii) When h = 0.25;

x	0	0.25	0.5	0.75	1
y	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

$$\therefore \quad I\left(\frac{h}{2}\right) = I(0.25) = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$$
$$= \frac{0.25}{2}[1 + 2 \times (0.9411 + 0.8 + 0.64) + 0.5]$$
$$= 0.782775$$

iii) When h = 0.125;

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64	0.5663	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\therefore \quad I\left(\frac{h}{4}\right) = I(0.25) = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \dots\dots\dots + y_7) + y_8]$$
$$= \frac{0.125}{2}[1 + 2 \times (0.9846 + 0.9411 + \dots + 0.5663)$$
$$+ 0.5]$$
$$= 0.784725$$

$$\therefore \quad I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right]$$
$$= \frac{1}{3}[4 \times 0.782775 - 0.775] = 0.78537$$

$$\therefore \quad I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right]$$
$$= \frac{1}{3}[4 \times 0.784725 - 0.782775] = 0.78537$$

$$\therefore \quad I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right]$$
$$= \frac{1}{3}[4 \times 0.785375 - 0.78537] = 0.785376$$

By calculator

$$I = \int_0^1 \frac{1}{1+x^2}\, dx = 0.78539$$

20. Using Gauss-Legendre 3-point formula, evaluate $\int_1^3 (x \sin x + \log_e x)\, dx$ [2076 Ashwin]

Solution:

Given that;

$$\int_1^3 (x \sin x + \log_e x)\, dx$$

Changing the limit of integration from 1 to 3 to – 1 to 1 by;

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(3-1)u + \frac{1}{2}(3+1)$$

$$x = u + 2$$

$$dx = du$$

Now,

$$I = \int_1^3 (x \sin x + \log_e x)\, dx$$

$$= \int_{-1}^1 \{u + \sin(u+2) + \log_e(u+2)\}\, du$$

$$= \int_{-1}^1 f(u)\, du$$

so that;

$$f(u) = (u+2)\sin(u+2) + \log_e(u+2)$$

Now,

$$f(0) = (0+2)\sin(0+2) + \log_e(0+2) = 2.5117$$

Also,

$$f\left(-\sqrt{\frac{3}{5}}\right) = \left(-\sqrt{\frac{3}{5}}+2\right)\sin\left(-\sqrt{\frac{3}{5}}+2\right) + \log_e\left(-\sqrt{\frac{3}{5}}+2\right)$$

or, $$f\left(-\sqrt{\frac{3}{5}}\right) = 1.3563$$

Again,

$$f\left(\sqrt{\frac{3}{5}}\right) = 2.0161$$

Using Gauss 3-point formula; we have,

$$I = \int_{-1}^1 f(u)\, du$$

or, $$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

or, $$I = \frac{8}{9} \times 2.5117 + \frac{5}{9}[1.3563 + 2.0161]$$

∴ $$I = 4.1061$$