Chapter 3

SOLUTION OF SYSTEM OF LINEAR ALGEBRAIC EQUATION

3.1 Gauss Elimination Method with Pivoting Strategies

Gauss elimination method is the method of solving linear algebraic equation by converting the augmented matrix into upper triangular matrix and solving for the unknown through backward substitution or by converting the augmented matrix into lower triangular matrix and solving for the unknown thorough forward substitution.

Mathematically;

If a set of linear algebraic equations are given as;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then the equations can be written in matrix form as;

$$[A][X] = [B]$$

where, $[A] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[B] = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The augmented matrix for the equation $AX = B$ is given as;

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

i) The augmented matrix can be converted to upper triangular matrix through forward elimination and the unknowns can be found through backward substitution as;

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_2'' & d_3'' \end{array}\right]$$

ii) The augmented matrix can be converted to lower triangular matrix through backward elimination and the unknowns can be found through forward substitution as;

$$[A|B] = \left[\begin{array}{ccc|c} a_1'' & 0 & 0 & d_1'' \\ a_2' & b_2' & 0 & d_2' \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Example 3.1

Solve for the following set of equations by Gauss elimination method with forward elimination.

$2x + 2y + 4z = 18$

$x + 3y + 2z = 13$

$3x + y + 3z = 14$

Solution:

The above set of equations can be written in matrix form $[A][X] = [B]$.

where, $[A] = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{bmatrix}$, $[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[B] = \begin{bmatrix} 18 \\ 13 \\ 14 \end{bmatrix}$

The augmented matrix for the above set of equation can be written as;

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - \frac{1}{2}R_1$, $R_3 \leftarrow R_3 - \frac{3}{2}R_1$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -13 \end{array}\right]$$

Applying $R_3 \leftarrow R_3 + R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

Now, applying backward substitution; we have,

$-3z = -9$

or, $z = 3$

and, $2y = 4$

or, $y = 2$

and, $2x + 2y + 4z = 18$

or, $2x + 2 \times 2 + 4 \times 3 = 18$

or, $2x = 18 - 4 - 12$

or, $2x = 2$

or, $x = 1$

Thus, the required solution is $[X] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Example 3.2

Solve for the following set of equations by Gauss elimination method with backward elimination.

$2x + 2y + 4z = 18$

$x + 3y + 2z = 13$

$3x + y + 3z = 14$

Solution:

The above set of equations can be written in matrix form [A][X] = [B].

where, $[A] = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{bmatrix}$, $[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[B] = \begin{bmatrix} 18 \\ 13 \\ 14 \end{bmatrix}$

The Augmented matrix for the above set of equation can be written as;

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - \frac{4}{3}R_3$, $R_2 \leftarrow R_2 - \frac{2}{3}R_3$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} -2 & \frac{2}{3} & 0 & -\frac{2}{3} \\ -1 & \frac{7}{3} & 0 & \frac{11}{3} \\ 3 & 1 & 3 & 14 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - \frac{2}{7}R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc} -\frac{12}{7} & 0 & 0 \\ -7 & \frac{7}{3} & 0 \\ 3 & 1 & 3 \end{array}\right]$$

Now, applying forward substitution; we have,

$-\frac{12}{7}x = -\frac{12}{7}$

or, $x = 1$

and, $-x + \frac{7}{3}y = \frac{11}{3}$

or, $\frac{7}{3}y = \frac{11}{3} + 1$

or, $\frac{7}{3}y = \frac{14}{3}$

or, $y = 2$

and, $3x + y + 3z = 14$

or, $3 \times 1 + 2 + 3z = 14$

or, $3z = 14 - 3 - 2$

or, $z = 3$

Thus, the required solution is $[X] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Pivoting Strategies in Gauss Elimination Method

There are two types of pivoting strategies in Gauss elimination method and they are;

i) Partial pivoting

In partial pivoting strategy, interchanges of rows are done to place the pivot element (element with highest absolute value) in the columns at their respective place.

Note

For partial pivoting, columns are the pivoting zone. For column 1, Pivot element needs to be placed at a_{11} position, for column 2, pivot element needs to be placed at a_{22} position and so on. Pivoting is not done for a_{nn} position of last column.

Example 3.3

Solve for the following set of equation by Gauss elimination method with partial pivoting.

$13x + 5y + 3z = 10$

$20x + 10y + 5z = 15$

$10x - 40y - 16z = 9$

Solution:

The above set of equation can be written in matrix form $[A][X] = [B]$.

where, $[A] = \begin{bmatrix} 13 & 5 & 3 \\ 20 & 10 & 5 \\ 10 & -40 & -16 \end{bmatrix}$, $[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[B] = \begin{bmatrix} 10 \\ 15 \\ 9 \end{bmatrix}$

The augmented matrix for the above set of equations can be written as;

$$[A|B] = \left[\begin{array}{ccc|c} 13 & 5 & 3 & 10 \\ 20 & 10 & 5 & 15 \\ 10 & -40 & -16 & 9 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 13 & 5 & 3 & 10 \\ 10 & -40 & -16 & 9 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - \frac{13}{20}R_1$, $R_3 \leftarrow R_3 - \frac{1}{2}R_1$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 0 & -1.5 & -0.25 & 0.25 \\ 0 & -45 & -18.5 & 1.5 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 0 & -45 & -18.5 & 1.5 \\ 0 & -1.5 & -0.25 & 0.25 \end{array}\right]$$

Applying $R_3 \leftarrow R_3 - \frac{1.5}{45}R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 0 & -45 & -18.5 & 1.5 \\ 0 & 0 & 0.3667 & 0.2 \end{array}\right]$$

Now, applying backward substitution; we have,

$0.3667z = 0.2$

or, $z = 0.5454$

and, $-45y - 18.5z = 1.5$

or, $-45y = 1.5 + 18.5 \times 0.5454$

$\therefore$ $y = -0.2575$

and, $20x + 10y + 5z = 15$

or, $20x + 10 \times (-0.2575) + 5 \times 0.5454 = 15$

$\therefore$ $x = 0.7424$

Thus, the required solution is $[X] = \begin{bmatrix} 0.7424 \\ -0.2575 \\ 0.5454 \end{bmatrix}$.

ii) Complete pivoting

In complete pivoting strategy, interchange of both rows and columns are done to place the pivot element (element with highest absolute value) in the matrix system at their respective place.

For complete pivoting, the whole matrix system is pivoting zone.

Example 3.4

Solve for the following set of equations by Gauss elimination method with complete pivoting.

$13x + 5y + 3z = 10$

$20x + 10y + 5z = 15$

$10x - 40y - 16z = 9$

Solution:

The above set of equations can be written in matrix form $[A][X] = [B]$.

$$\text{where, } [A] = \begin{bmatrix} 13 & 5 & 3 \\ 20 & 10 & 5 \\ 10 & -40 & -16 \end{bmatrix}, [X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } [B] = \begin{bmatrix} 10 \\ 15 \\ 9 \end{bmatrix}.$$

The augmented matrix for the above set of equations can be written as;

$$[A|B] = \left[\begin{array}{ccc|c} 13 & 5 & 3 & 10 \\ 20 & 10 & 5 & 15 \\ 10 & -40 & -16 & 9 \end{array}\right]$$

Applying $C_1 \leftrightarrow C_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 13 & 3 & 10 \\ 10 & 20 & 5 & 15 \\ -40 & 10 & -16 & 9 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} -40 & 10 & -16 & 9 \\ 10 & 20 & 5 & 15 \\ 5 & 13 & 3 & 10 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 + \frac{10}{40}R_1$, $R_3 \leftarrow R_3 + \frac{5}{40}R_1$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} -40 & 10 & -16 & 9 \\ 0 & 22.5 & 1 & 17.25 \\ 0 & 14.25 & 1 & 11.125 \end{array}\right]$$

Since, pivot element is already on top left position of pivot zone, no need of interchanging rows and columns.

Applying $R_3 \leftarrow R_3 - \frac{14.25}{22.5}R_2$

$$[A|B] = \left[\begin{array}{ccc|c} -40 & 10 & -16 & 9 \\ 0 & 22.5 & 1 & 17.25 \\ 0 & 0 & 0.3667 & 0.2 \end{array}\right]$$

Now, applying backward substitution; we have,

$0.3667z = 0.2$

or, $z = 0.5454$

and, $22.5x + z = 14.25$

or, $22.5x = 14.25 - 0.5454$

or, $x = 0.7424$

and, $-40y + 10x - 16z = 9$

or, $-40y + 10 \times 0.7424 - 16 \times 0.5454 = 9$

or, $y = -0.2575$

Thus, the required solution is $[X] = \begin{bmatrix} 0.7424 \\ -0.2575 \\ 0.5454 \end{bmatrix}$.

3.2 Gauss–Jordan Method

Gauss–Jordan method is the method of solving linear algebraic equation by converting augmented matrix into unit matrix to obtain the direct solution.

Gauss–Jordan method implies two methods of solution of linear equation and they are;

i) Simple Gauss–Jordan method

In simple Gauss Jordan, the augmented matrix is converted into unit matrix obtain the direct solution.

Example 3.5

Solve for the following set of equations by simple Gauss-Jordan method.

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Solution:

The above set of equations can be written in matrix form [A][X] = [B].

where, $[A] = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, $[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[B] = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$.

The augmented matrix for the above f equations can be written as;

$$[A|B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_1 - 9R_3$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

Applying $R_2 \leftarrow 2R_3 - R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 1 & 58 & 59 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 + 8R_2$, $R_3 \leftarrow R_3 - 9R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{array}\right]$$

Applying $R_3 \leftarrow R_3/(-473)$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - 420R_3$, $R_2 \leftarrow R_2 - 58R_3$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, the required solution is $[X] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

ii) Inverse Gauss–Jordan method

In inverse Gauss-Jordan method, the augmented matrix and unit matrix are placed side by side and further calculation is made to convert the augmented matrix into unit matrix.

Example 3.6

Solve for the flowing set of equations by inverse Gauss–Jordan method.

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Solution:

The above set of equations can be written in matrix form [A][X] = [B].

where, $[A] = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, $[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[B] = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$.

The augmented matrix for the above set of equation can be written as;

$$[A|I] = \left[\begin{array}{ccc|ccc} 10 & 1 & 1 & 1 & 0 & 0 \\ 2 & 10 & 1 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - 9R_3$; we get,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -8 & -44 & 1 & 0 & -9 \\ 2 & 10 & 1 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$; we get,

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -8 & -44 & 1 & 0 & -9 \\ 0 & 26 & 89 & -2 & 1 & 18 \\ 0 & 9 & 49 & -1 & 0 & 10 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - 3R_3 - R_2$; we get,

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -8 & -44 & 1 & 0 & -9 \\ 0 & 1 & 58 & -1 & -1 & 12 \\ 0 & 9 & 49 & -1 & 0 & 10 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 + 8R_2$, $R_3 \leftarrow R_3 - 9R_2$; we get,

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 420 & -7 & -8 & 87 \\ 0 & 1 & 58 & -1 & -1 & 12 \\ 0 & 0 & -473 & 8 & 9 & -98 \end{array}\right]$$

Applying $R_3 \leftarrow R_3/(-473)$; we get,

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 420 & -7 & -8 & 87 \\ 0 & 1 & 58 & -1 & -1 & 12 \\ 0 & 0 & 1 & -0.0169 & -0.0190 & 0.2072 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - 420R_3$, $R_2 \leftarrow R_2 - 58R_3$; we get,

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.11 & -0.02 & -0.02 \\ 0 & 1 & 0 & -0.02 & 0.1 & -0.0176 \\ 0 & 0 & 1 & -0.01692 & -0.019 & 0.2072 \end{array}\right]$$

This is in the form $[I][A^{-1}]$.

where, $$[A^{-1}] = \begin{bmatrix} 0.11 & -0.02 & -0.02 \\ -0.02 & 0.1 & -0.0176 \\ -0.01692 & -0.019 & 0.2072 \end{bmatrix}.$$

We have,

$$[A][X] = [B]$$

or, $$[X] = [A^{-1}][B]$$

$$= \begin{bmatrix} 0.11 & -0.02 & -0.02 \\ -0.02 & 0.1 & -0.0176 \\ -0.01692 & -0.019 & 0.2072 \end{bmatrix} \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1.32 - 0.26 - 0.14 \\ -0.24 + 1.3 - 0.123 \\ -0.20304 - 0.247 + 1.45 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 0.94 \\ 0.999 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$; which is the required solution.

3.3 LU Factorization

LU Factorization method is the method of solving linear algebraic equation by factorizing the matrix of coefficients into upper and lower triangular matrix and solving for the unknown step by step.

Mathematically;

In the equation [A][X] = [B], matrix A can be spitted as;

$$[A] = [L][U]$$

$$\text{where, } [L] = \begin{bmatrix} L_{11} & 0 & 0 & \cdots\cdots & 0 \\ L_{21} & L_{22} & 0 & \cdots\cdots & 0 \\ L_{31} & L_{32} & L_{33} & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots \\ & & & \cdots\cdots & L_{nn} \end{bmatrix}$$

$$[U] = \begin{bmatrix} U_{11} & U_{21} & \cdots\cdots & U_{1n} \\ 0 & U_{22} & \cdots\cdots & U_{2n} \\ \vdots & 0 & \cdots\cdots & \vdots \\ \vdots & \vdots & \cdots\cdots & U_{nn} \end{bmatrix}$$

Now, equation [A][X] = [B] can be written as;

$$[L][U][X] = [B]$$

or, $[L][Z] = [B]$

where, Z is an unknown matrix given by [Z] = [U][X].

Thus, solving for the equations; we have,

$$[L][Z] = [B]$$

and, $[U][X] = [Z]$

We obtain the required solution.

Example 3.7

Solve the following set of equations by decomposition method.

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Solution:

The above set of equations can be written in matrix form as [A][X] = [B].

$$\text{where, } [A] = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix}, [B] = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix} \text{ and } [X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Factorizing matrix A into upper and lower matrix; we get,

$$[A] = [L][U]$$

$$\text{or, } \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

For factorization, the diagonal element of either lower triangular matrix or upper triangular matrix is assumed to be unity.

Let's assume diagonal elements of lower triangular matrix be unit (which is called Do-little LU decomposition and the reverse is called Group LU decomposition).

Then,

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = [A]$$

or, $$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = [A]$$

Comparing the corresponding elements of the two matrixes; we get,

$U_{11} = 10$

$U_{12} = 1$

$U_{13} = 1$

$L_{21}U_{11} = 2$

or, $L_{21} = 0.2$

$L_{21}U_{12} + U_{22} = 10$

or, $U_{22} = 9.8$

$L_{21}U_{13} + U_{23} = 1$

or, $U_{23} = 0.8$

$L_{31}U_{12} + L_{32}U_{22} = 1$

or, $L_{32} = 0.092$

$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 5$

or, $U_{33} = 4.826$

We have,

$[A][X] = [B]$

or, $[L][U][X] = [B]$

or, $[L][Z] = [B]$ (1)

where, $[Z] = [U][X]$. (2)

Solving for equation (1); we get,

$[L][Z] = [B]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.21 & 1 & 0 \\ 0.1 & 0.092 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

Applying forward substitution; we have,

$Z_1 = 12$

and, $0.2Z_1 + Z_2 = 13$

or, $Z_2 = 10.6$

and, $0.1Z_1 + 0.092Z_2 + Z_3 = 7$

or, $Z_3 = 7 - 1.2 - 0.092 \times 10.6$

or, $Z_3 = 4.825$

Thus,

$$Z = \begin{bmatrix} 12 \\ 10.6 \\ 4.825 \end{bmatrix}$$

Again, solving for equation (2); we get,

$$[U][X] = [Z]$$

or, $$\begin{bmatrix} 10 & 1 & 1 \\ 0 & 9.8 & 0.8 \\ 0 & 0 & 4.826 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 10.6 \\ 4.825 \end{bmatrix}$$

Applying backward substitution; we have,

$4.826Z = 4.825$

or, $Z = 1$

and, $9.8y + 0.8z = 10.6$

or, $9.8y = 10.6 - 0.8$

or, $y = 1$

and, $10x + y + z = 12$

or, $10x + 1 + 1 = 12$

or, $10x = 10$

or, $x = 1$

Thus, the required solution is;

$$[X] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.4 Iterative Method (Jacobi Method, Gauss–Seidel Method)

Iterative method of solution of linear algebraic equation is the method in which an initial guess of solution is made and through successive iterations, the guess is converged to exact solution correct up to desired significant digits.

The common iterative methods are:

i) Jacobi method

Jacobi method of solution of linear algebraic equations is the method in which an initial guess of solution is made initially and the outputs form initial guess is used for the next iteration and so on until the desired accuracy is achieved.

Example 3.8

Solve for the following set of equations using Jacobi method:

$20x + 3y - z = 23$

$4x + 23y + 2z = 56$

$2x - 3y + 25z = 71$

Solution:

Rearranging the above equations in terms of x, y and z; we get,

$$x = \frac{23 - 3y + z}{20}$$

$$y = \frac{56 - 4x - 2z}{23}$$

$$z = \frac{71 - 2x + 3y}{25}$$

Note

While re-arranging the equations, the equation of x is chosen from the equation having highest coefficient of x and so on.

Let the initial guess be $x = 0$, $y = 0$ and $z = 0$.

Then, for the first iteration;

$$x = \frac{23}{20} = 1.15$$

$$y = \frac{56}{23} = 2.4348$$

$$z = \frac{71}{25} = 2.84$$

These outputs are used for next iteration and so on.

The iteration table is given below:

No. of iteration	(A) Value of x	(B) Value of y	(C) Value of z
1	0	0	0
2	1.15	2.4348	2.84
3	0.9268	1.9878	3.0402
4	1.0038	2.0092	3.0044
5	0.9988	1.9989	3.0008
6	1.0002	2.0001	2.9999
7	0.9999	1.9999	3.0000
8	1.0000	2.0000	3.0000

Since, at iteration 7 and 8, the values are nearly same, so we stop the iteration here.

Thus, the required solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 2.0000 \\ 3.0000 \end{bmatrix}$.

Note

The programming code for calculator is:

A : B : C : D = (23 − 3B + C)/20 : E = (56 − 4A − 2C)/23 : F = (71 − 2A + 3B)/25 : A = D : B = E : C = F $\boxed{\text{CALC}}$

Put value of initial guess for A?, B? and C? and continue pressing $\boxed{=}$ button for values of successive iterations.

Example 3.9

Solve for the following set of equations Jacobi method.

$2x - 4y + 32z = 23$

$40x - 5y + 6z = 144.5$

$3x + 24y + 5z = 56.5$

Solution:

Re-arranging the above equations in terms of x, y and z; we get,

$$x = \frac{144.5 + 5y - 6z}{40}$$

$$y = \frac{56.5 - 3x - 5z}{24}$$

$$z = \frac{65 - 2x + 4y}{32}$$

Let initial guess be x = 0, y = 0 and z = 0.

Then, for the first iteration;

$$x = \frac{144.5}{40} = 3.6125$$

$$y = \frac{56.5}{24} = 2.3542$$

$$z = \frac{65}{32} = 2.0313$$

These outputs are used for next iteration and so on.

The iteration table is given below:

No. of iteration	(A) Value of x	(B) Value of y	(C) Value of z
1	0	0	0
2	3.6125	2.3542	2.0313
3	3.6021	1.4794	2.0997
4	3.4825	1.4665	1.90104
5	3.4972	1.5041	1.9969
6	3.5009	1.5010	2.0007

7	3.5000	1.4997	2.0001
8	3.4999	1.4999	1.9999
9	3.5000	1.5000	2.0000
10	3.5000	1.4999	2.0000

Since, at iteration (9) and (10) the values are nearly same, so we stop the iteration here.

Thus, the required solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3.5 \\ 1.5 \\ 2.0 \end{bmatrix}$.

Note

The programming code for calculator is;
A : B : C : D = (144.5 + 5B − 6C) ÷ 40 : E = (56.5 − 3A − 5C) ÷ 24 : F = (65 − 2A + 4B) ÷ 32 : A = D : B = E : C = F $\boxed{\text{CALC}}$
Put the value of initial guess for A?, B? and C? and continue pressing $\boxed{=}$ button for values of successive iterations.

ii) Gauss–Seidel method

Gauss–Seidel method is similar to the Jacobi method except that instead of using all values of previous iteration, latest updated values are used for further calculation.

Example 3.10

Solve for the following set of equations using Gauss–Seidel method.

$20x + 3y - z = 23$
$4x + 23y + 2z = 56$
$2x - 3y + 25z = 71$

Solution:

Re–arranging the above equations in terms of x, y and z; we get,

$$x = (23 - 3y + z) \div 20$$
$$y = (56 - 4x - 2z) \div 23$$
$$z = (71 - 2x + 3y) \div 25$$

Let initial guess be x = 0, y = 0 and z = 0.
Then for the first iteration;

$$x = \frac{23}{20} = 1.15$$
$$y = \frac{56 - 4x - 2z}{23} = \frac{56 - 4 \times 1.15 - 0}{23} = 2.2348$$
$$z = \frac{71 - 2x + 3y}{25} = \frac{71 - 2 \times 1.15 + 3 \times 2.2348}{25} = 3.0162$$

Similarly, further calculations are made for successive iterations.
The iteration table is given below:

No. of iteration	(A) Value of x	(B) Value of y	(C) Value of z
1	0	0	0
2	1.15	2.2348	3.0162
3	0.9656	2.0046	3.0033
4	0.9995	1.9998	3.0000
5	1.0000	1.9999	2.9999
6	1.0000	2.0000	2.9999

Since, at iteration (5) and (6), values are nearly same, so we stop the iteration here.

Thus, the required solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 2.0000 \\ 3.0000 \end{bmatrix}$.

Note

The programming code for calculator is;
A : B : C : D = (23 − 3B + C) ÷ 20 : A = D : E = (56 − 4A − 2C) ÷ 23 : B = E : F = (71 − 2A + 3B) ÷ 25 : C = F $\boxed{\text{CALC}}$
Put the value of initial guess for A?, B? and C? and continue pressing $\boxed{=}$ button for values of successive iterations.

Example 3.11

Solve for the following set of equations using Gauss-Seidel method.

$2x - 4y + 32z = 23$

$40x - 5y + 6z = 144.5$

$3x + 24y + 5z = 56.5$

Solution:

Re-arranging the above equations in terms of x, y and z; we get,

$$x = \frac{144.5 + 5y - 6z}{40}$$

$$y = \frac{56.5 - 3x - 5z}{24}$$

$$z = \frac{65 - 2x + 4y}{32}$$

Let initial guess be $x = 0$, $y = 0$ and $z = 0$.

Then, for the first iteration;

$$x = \frac{144.5}{40} = 3.6125$$

$$y = \frac{56.5 - 3x - 5z}{24} = \frac{56.5 - 3 \times 3.6125 - 0}{24} = 1.9026$$

$$z = \frac{65 - 2x + 4y}{32} = \frac{65 - 2 \times 3.6125 + 4 \times 1.9026}{32} = 2.0433$$

These outputs are used for next iteration and so on.
The iteration table is given below:

No. of iteration	(A) Value of x	(B) Value of y	(C) Value of z
1	0	0	0
2	3.6125	1.9026	2.0433
3	3.5438	1.4855	1.9954
4	3.4989	1.5011	2.0002
5	3.5001	1.4999	1.9999
6	3.4999	1.5000	2.0000
7	3.5000	1.4999	1.9999

Since, at iteration (6) and (7), the values are nearly same, so, we stop the iteration here.

Thus, the required solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3.5000 \\ 1.4999 \\ 1.9999 \end{bmatrix} \cong \begin{bmatrix} 3.5 \\ 1.5 \\ 2.0 \end{bmatrix}$.

Note

The programming code for calculator is;
A : B : C : D = (144.5 + 5B − 6C) ÷ 40 : A = D : E = (56.5 − 3A − 5C) ÷ 24 : B = E : F = (65 − 2A + 4B) ÷ 32 : C = F $\boxed{\text{CALC}}$
Put the value of initial guess for A?, B? and C? and continue pressing $\boxed{=}$ button for values of successive iterations.

3.5 Eigen Value and Eigen Vector Using Power Method

A non-zero vector V is called an Eigen vector of a square matrix A if AV is a scalar multiple of V. Mathematically;

$$AV = \lambda V$$

where, λ is a scalar known as the eigen value.

Example 3.12

Find the largest eigen value and corresponding eigen vector of the given matrix $[A] = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ using power method.

Solution:

Let, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Note

X can be chosen $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1\end{bmatrix}$

We have,

At Iteration 1,

$$Y = [A][X] = \begin{bmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\3\\1\end{bmatrix} = 3\begin{bmatrix}-0.67\\1\\0.33\end{bmatrix}$$

At iteration 2,

$$Y = [A][X] = \begin{bmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}-0.67\\1\\0.33\end{bmatrix} = \begin{bmatrix}-2.02\\2.32\\2\end{bmatrix} = 2.32\begin{bmatrix}-0.87\\1\\0.86\end{bmatrix}$$

At iteration 3,

$$Y = [A][X] = \begin{bmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}-0.87\\1\\0.86\end{bmatrix} = \begin{bmatrix}-0.3\\2.98\\2.73\end{bmatrix} = 2.98\begin{bmatrix}-0.1\\1\\0.92\end{bmatrix}$$

At iteration 4,

$$Y = [A][X] = \begin{bmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}-0.1\\1\\0.92\end{bmatrix} = \begin{bmatrix}1.48\\4.64\\2.02\end{bmatrix} = 4.64\begin{bmatrix}0.32\\1\\0.44\end{bmatrix}$$

At iteration 5,

$$Y = [A][X]$$

$$= \begin{bmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}0.32\\1\\0.44\end{bmatrix} = \begin{bmatrix}0.4\\4.52\\1.12\end{bmatrix} = 4.52\begin{bmatrix}0.09\\1\\0.25\end{bmatrix}$$

At iteration 6,

$$Y = [A][X]$$

$$= \begin{bmatrix}2 & -2 & 4\\2 & 3 & 2\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}0.09\\1\\0.25\end{bmatrix} = \begin{bmatrix}-0.82\\3.68\\1.16\end{bmatrix} = 3.68\begin{bmatrix}-0.22\\1\\0.32\end{bmatrix}$$

Since, two consecutive sealers are smaller in value than that of iteration (4), we stop the iterations here and take the values of iteration (4) as the solution.

Thus, the largest eigen value = 4.64

The corresponding eigen vector = $\begin{bmatrix}0.32\\1\\0.44\end{bmatrix}$

Example 3.13

Find the largest eigen value and corresponding eigen vector of the given matrix $[A] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **using power method.**

Solution:

Let, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

At iteration (1),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (2),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix} = 2.5\begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (3),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \\ 0 \end{bmatrix} = 2.8\begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (4),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix} = 2.93\begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (5),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 2.98 \\ 0 \end{bmatrix} = 2.98\begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (6),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.99 \\ 0 \end{bmatrix} = 2.99\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (7),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

At iteration (8),

$$Y = [A][X] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since, the further iterations give the same values, so, we stop the iteration here. The highest value of scalar is the value in iteration (8), so, we take its values as the solution.

Thus, the largest eigen value = 3

The corresponding eigen vector = $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3.6 EXAMINATION PROBLEMS

1. **Solve the given system of linear equation using partial pivoting.**

[2067 Ashadh]

$2x_1 + x_2 + x_3 - 2x_4 = 10$

$4x_1 + 2x_3 + x_4 = 8$

$3x_1 + 2x_3 + 2x_3 = 7$

$x_1 + 3x_2 + 2x_3 - 4x_4 = -5$

Solution:

The above set of equation can be written in matrix form [A] [X] = [B].

where, $[A] = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

and, $[B] = \begin{bmatrix} 10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$

The augmented matrix for the above set of equation can be written as;

$$[A \mid B] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -4 & -5 \end{array}\right]$$

In the $R_1 \leftrightarrow R_2$; we get,

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & -8 \\ 2 & 1 & 1 & 2 & -10 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -4 & -5 \end{array}\right] \text{pivot}$$

Applying $R_2 \leftrightarrow R_2 + \frac{1}{2}R_1$, $R_3 \leftrightarrow R_3 \frac{-3}{4} R_1$ and $R_4 \leftrightarrow \frac{1}{4} R_1$; we get,

$$\begin{bmatrix} 4 & 0 & 2 & 1 & -8 \\ 0 & 1 & 0 & -\frac{5}{2} & -14 \\ 0 & 2 & \frac{1}{2} & -\frac{3}{4} & 1 \\ 0 & 3 & \frac{3}{2} & -\frac{17}{4} & -7 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_4$; we get,

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & -8 \\ 0 & 1 & \frac{3}{2} & -\frac{17}{4} & 7 \\ 0 & 2 & \frac{1}{2} & -\frac{3}{4} & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & -14 \end{array}\right] \text{Pivot}$$

Applying $R_3 \leftrightarrow R_3 - \frac{2}{3}R_2$, $R_4 \leftrightarrow R_4 - \frac{1}{3}R_1$; we get,

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & -8 \\ 0 & 1 & \frac{3}{2} & -\frac{17}{4} & 7 \\ 0 & 2 & -\frac{1}{2} & \frac{25}{12} & -\frac{11}{3} \\ 0 & 0 & 0 & 1 & -20 \end{array}\right]$$

Using backward substitution (forward elimination) from R_4; we get,

$$x_4 = -20$$

From R_3; we get,

$$\frac{-1}{2}x_3 + \frac{25}{12}x_4 = -\frac{-11}{3}$$

$\therefore \quad x_3 = -76.$

From R_2; we get,

$$3x_2 + \frac{3}{2}x_3 - \frac{-17}{4}x_1 = 7$$

$\therefore \quad x_2 = 12$

From R_1; we get,

$$4x_1 + 0 + 2x_3 + x_4 = 8$$

$\therefore \quad x_1 = 43$

Hence,

$$[X] = \begin{bmatrix} 45 \\ 12 \\ -76 \\ -20 \end{bmatrix}$$

i.e., $x_1 = 45$

$x_2 = 12$

$x_3 = -76$

$x_4 = -20$

2. **Find the largest eigen value correct to three significant digit and corresponding eigen vector of the following matrix using power method.** **[2068 Baishakh]**

Solution:

Given that;

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

and, let $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

At iteration 1,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 0.428 \\ 0.1428 \\ 1 \end{bmatrix}$$

At iteration 2,

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.428 \\ 0.1428 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.7142 \\ 4.7142 \\ 6.8571 \end{bmatrix} = 6.8571 \begin{bmatrix} 0.5416 \\ 0.6875 \\ 1 \end{bmatrix}$$

At iteration 3,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.5416 \\ 0.6875 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.9166 \\ 4.3958 \\ 9.373 \end{bmatrix} = 9.373 \begin{bmatrix} 0.5244 \\ 0.4688 \\ 1 \end{bmatrix}$$

At iteration 4:

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.5244 \\ 0.4688 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.4622 \\ 4.6 \\ 8.4489 \end{bmatrix} = 8.4489 \begin{bmatrix} 0.528 \\ 0.642 \\ 1 \end{bmatrix}$$

At iteration 5,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.528 \\ 0.542 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.6123 \\ 4.5142 \\ 8.7528 \end{bmatrix} = 8.7528 \begin{bmatrix} 0.526 \\ 0.515 \\ 1 \end{bmatrix}$$

At iteration 6,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.526 \\ 0.515 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5584 \\ 4.638 \\ 8.6438 \end{bmatrix} = 8.6438 \begin{bmatrix} 0.527 \\ 0.525 \\ 1 \end{bmatrix}$$

At iteration 7,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.527 \\ 0.525 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5774 \\ 4.5297 \\ 8.6822 \end{bmatrix} = 8.6822 \begin{bmatrix} 0.527 \\ 0.521 \\ 1 \end{bmatrix}$$

At iteration 8,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.527 \\ 0.525 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5774 \\ 4.5297 \\ 8.6822 \end{bmatrix} = 8.6822 \begin{bmatrix} 0.527 \\ 0.521 \\ 1 \end{bmatrix}$$

At iteration 9,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.573 \\ 4.5316 \\ 8.6734 \end{bmatrix} = 8.6734 \begin{bmatrix} 0.5272 \\ 0.5224 \\ 1 \end{bmatrix}$$

At iteration 10,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.572 \\ 4.532 \\ 8.6717 \end{bmatrix} = 8.6717 \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix}$$

At iteration 11,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5725 \\ 4.5318 \\ 8.6722 \end{bmatrix} = 8.6722 \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix}$$

At iteration 12,

$$Y = [A]\,[X] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5724 \\ 4.5319 \\ 8.6721 \end{bmatrix} = 8.6721 \begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix}$$

Hence,

Largest eigen value = 8.672

and, Corresponding eigen vector = $\begin{bmatrix} 0.527 \\ 0.522 \\ 1 \end{bmatrix}$

3. **Find the largest eigen value and corresponding vector of the following matrix using power method.** **[2070 Magh]**

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}$$

Solution:

Here, all the row contains non-zero elements.

$\therefore$ Initial eigen vector; $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

At iteration (1),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix} = 14 \begin{bmatrix} 0.571 \\ 0.429 \\ 1 \end{bmatrix}$$

At iteration (2),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.571 \\ 0.429 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.287 \\ 4.997 \\ 11.858 \end{bmatrix} = 11.858 \begin{bmatrix} 0.62 \\ 0.421 \\ 1 \end{bmatrix}$$

At iteration (3),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.362 \\ 0.421 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.829 \\ 3.698 \\ 11.625 \end{bmatrix} = 11.625 \begin{bmatrix} 0.329 \\ 0.341 \\ 1 \end{bmatrix}$$

At iteration (4),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.329 \\ 0.341 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.363 \\ 3.963 \\ 10.337 \end{bmatrix} = 10.337 \begin{bmatrix} 0.325 \\ 0.383 \\ 1 \end{bmatrix}$$

At iteration (5),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.325 \\ 0.383 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.565 \\ 3.859 \\ 11.474 \end{bmatrix} = 11.474 \begin{bmatrix} 0.311 \\ 0.336 \\ 1 \end{bmatrix}$$

At iteration (6),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.311 \\ 0.336 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.302 \\ 3.883 \\ 11.319 \end{bmatrix} = 11.319 \begin{bmatrix} 0.292 \\ 0.343 \\ 1 \end{bmatrix}$$

At iteration (7),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.292 \\ 0.343 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.299 \\ 3.774 \\ 11.321 \end{bmatrix} = 11.321 \begin{bmatrix} 0.291 \\ 0.333 \\ 1 \end{bmatrix}$$

At iteration (8),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.291 \\ 0.333 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.247 \\ 3.789 \\ 11.290 \end{bmatrix} = 11.290 \begin{bmatrix} 0.288 \\ 0.336 \\ 1 \end{bmatrix}$$

At iteration (9),

$$Y = [A]\,[X] = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.288 \\ 0.336 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.256 \\ 3.768 \\ 11.296 \end{bmatrix} = 11.296 \begin{bmatrix} 0.288 \\ 0.334 \\ 1 \end{bmatrix}$$

$\therefore$ Largest eigen value = 11.296

and, Eigen vector = $\begin{bmatrix} 0.288 \\ 0.334 \\ 1 \end{bmatrix}$

4. Solve the following linear equation using Gauss eliminating or Gauss Gordon method using partial pivoting.

$2x + 3y + 2z = 2$

$10x + 3y + 4z = 16$

$3x + 6y + z = 6$ **[2071 Bhadra]**

Solution:

The above set of equation can be written in matrix form [A] [x] = [B]

where, $[A] = \begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$[B] = \begin{bmatrix} 2 \\ 16 \\ 6 \end{bmatrix}$$

The augmented matrix for the above set of equation can be written as;

$$[A + B] = \left[\begin{array}{ccc|c} 2 & 3 & 2 & 2 \\ 10 & 3 & 4 & 16 \\ 3 & 6 & 1 & 6 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$; we have,

$$[A|B] = \left[\begin{array}{ccc|c} 10 & 3 & 4 & 16 \\ 2 & 3 & 2 & 2 \\ 3 & 6 & 1 & 6 \end{array}\right] \text{pivot}$$

Applying $R_2 \leftrightarrow R_2 - \frac{R_1}{5}$, $R_3 \leftrightarrow R_3 - \frac{3}{10} R_1$; we have,

$$[A|B] = \left[\begin{array}{ccc|c} 10 & 3 & 4 & 16 \\ 0 & \frac{12}{5} & \frac{6}{5} & -\frac{6}{5} \\ 0 & \frac{51}{10} & -\frac{1}{5} & \frac{6}{5} \end{array}\right] \text{pivot}$$

Applying $R_3 \leftrightarrow R_3 - \frac{8}{17} R_2$; we have,

$$[A|B] = \left[\begin{array}{ccc|c} 10 & 3 & 4 & 16 \\ 0 & \frac{51}{10} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 0 & \frac{22}{7} & -\frac{30}{17} \end{array}\right]$$

Now, applying forward elimination (Backward substitution); we have,

$$\frac{22}{17} z = \frac{-30}{17}$$

$\therefore$ $$z = \frac{-30}{22} = \frac{-15}{11}$$

and, $$\frac{51}{10} y \frac{-1}{5} z = \frac{6}{5}$$

or, $$\frac{51}{10} y - \frac{1}{5} \times \left(\frac{-30}{22}\right) = \frac{6}{5}$$

$\therefore$ $$y = \frac{2}{11}$$

and, $$10x + 3y + 4z = 16$$

or, $$10x + 3 \times \frac{2}{11} = 4 \left(\frac{-30}{22}\right) = 16$$

$\therefore$ $$x = \frac{23}{11}$$

Thus, the required solution is $x = \begin{bmatrix} \frac{23}{11} \\ \frac{2}{11} \\ -\frac{15}{11} \end{bmatrix}$.

5. **Find the largest eigen-value and the corresponding eigen-vector of the following matrix.** **[2071 Bhadra]**

$$\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

Solution:

Let, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and, $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$

At iteration 1,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 0.3333 \\ 1 \\ 0 \end{bmatrix}$$

At iteration 2,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 0.3333 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9.3333 \\ 3.3333 \\ 0.6660 \end{bmatrix} = 9.333\begin{bmatrix} 1 \\ 0.3571 \\ 0.1714 \end{bmatrix}$$

At iteration 3,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.3571 \\ 0.1714 \end{bmatrix} = \begin{bmatrix} 25.500 \\ 2.0714 \\ 1.7142 \end{bmatrix} = 25.500\begin{bmatrix} 1 \\ 0.0812 \\ 0.0672 \end{bmatrix}$$

At iteration 4,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0812 \\ 0.0672 \end{bmatrix} = \begin{bmatrix} 25.215 \\ 1.243 \\ 1.731 \end{bmatrix} = 25.215\begin{bmatrix} 1 \\ 0.0493 \\ 0.0686 \end{bmatrix}$$

At iteration 5,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0493 \\ 0.0686 \end{bmatrix} = \begin{bmatrix} 25.187 \\ 1.1479 \\ 1.725 \end{bmatrix} = 25.187\begin{bmatrix} 1 \\ 0.0455 \\ 0.0685 \end{bmatrix}$$

At iteration 6,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0455 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.183 \\ 1.1367 \\ 1.726 \end{bmatrix} = 25.183\begin{bmatrix} 1 \\ 0.451 \\ 0.0685 \end{bmatrix}$$

At iteration 7,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.182 \\ 1.1354 \\ 1.7258 \end{bmatrix} = 25.182 \begin{bmatrix} 1 \\ 0.045 \\ 0.0685 \end{bmatrix}$$

At iteration 8,

$$Y = [A][X] = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.182 \\ 1.135 \\ 1.125 \end{bmatrix} = 25.182 \begin{bmatrix} 1 \\ 0.045 \\ 0.068 \end{bmatrix}$$

Since, further iteration gives the same value so, we stop the iteration here.

Largest eigen value = 25.182

and, Corresponding eigen vector = $\begin{bmatrix} 1 \\ 0.045 \\ 0.068 \end{bmatrix}$

6. Find the inverse of matrix A = $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$. [2071 Magh]

Solution:

Let, $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

and, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Writing the given matrix side by side; we have,

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - 2R_2$, $R_2 \leftarrow R_2 + 2R_3$; we get,

$$\left[\begin{array}{ccc|ccc} -2 & -8 & 0 & 1 & -2 & 0 \\ 0 & 5 & 4 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_3 \leftarrow R_3 - \frac{1}{2} R_1$; we get,

$$\left[\begin{array}{ccc|ccc} -2 & -8 & 0 & 1 & -2 & 0 \\ 0 & 5 & 4 & 0 & 1 & 2 \\ 0 & 5 & 1 & -1/2 & 1 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - 4R_3$; we get,

$$\left[\begin{array}{ccc|ccc} -2 & -8 & 0 & 1 & -2 & 0 \\ 0 & -15 & 0 & 2 & -3 & -2 \\ 0 & 5 & 1 & -1/2 & 1 & 1 \end{array}\right]$$

Applying $R_3 \leftarrow R_3 + \frac{1}{3} R_2$; we get,

$$\left[\begin{array}{ccc|ccc} -2 & -8 & 0 & 1 & -2 & 0 \\ 0 & -15 & 0 & 2 & -3 & -2 \\ 0 & 0 & 1 & -1/5 & 0 & 1/3 \end{array}\right]$$

Applying $R_1 \leftarrow -\frac{1}{2}R_1$ and $R_2 \leftarrow \frac{R_2}{-15}$; we get,

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1/2 & 1 & 0 \\ 0 & 1 & 0 & -2/-15 & 1/5 & 2/15 \\ 0 & 0 & 1 & 1/6 & 0 & 1/3 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - 4R_2$; we get,

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1/30 & 1/5 & -8/15 \\ 0 & 1 & 0 & -2/15 & 1/5 & 2/15 \\ 0 & 0 & 1 & 1/6 & 0 & 1/3 \end{array}\right]$$

Hence, inverse of matrix $A = \begin{bmatrix} 1/30 & 1/5 & -8/15 \\ -2/15 & 1/5 & 2/15 \\ 1/6 & 0 & 1/3 \end{bmatrix}$.

7. **Find the largest Eigen value and the corresponding Eigen vector of the following matrix using the power method with an accuracy of 2 decimal points.** **[2072 Ashwin]**

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

Solution:

Let, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

At iteration 1,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$$

At iteration 2,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4.5 \\ 1 \end{bmatrix} = 4.5\begin{bmatrix} 0.667 \\ 1 \\ 0.222 \end{bmatrix}$$

At iteration 3,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0.667 \\ 1 \\ 0.222 \end{bmatrix} = \begin{bmatrix} 2.889 \\ 2.777 \\ 2.444 \end{bmatrix} = 2.889\begin{bmatrix} 1 \\ 0.961 \\ 0.846 \end{bmatrix}$$

At iteration 4,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.961 \\ 0.846 \end{bmatrix} = \begin{bmatrix} 3.763 \\ 4.653 \\ 2.076 \end{bmatrix} = 4.653 \begin{bmatrix} 0.81 \\ 1 \\ 0.446 \end{bmatrix}$$

At iteration 5,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.81 \\ 1 \\ 0.446 \end{bmatrix} = \begin{bmatrix} 3.256 \\ 3.513 \\ 2.364 \end{bmatrix} = 3.513 \begin{bmatrix} 0.927 \\ 1 \\ 0.673 \end{bmatrix}$$

At iteration 6,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.927 \\ 1 \\ 0.673 \end{bmatrix} = \begin{bmatrix} 3.599 \\ 4.199 \\ 2.254 \end{bmatrix} = 4.199 \begin{bmatrix} 0.857 \\ 1 \\ 0.536 \end{bmatrix}$$

At iteration 7,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.857 \\ 1 \\ 0.536 \end{bmatrix} = \begin{bmatrix} 3.594 \\ 3.788 \\ 2.321 \end{bmatrix} = 3.788 \begin{bmatrix} 0.896 \\ 1 \\ 0.612 \end{bmatrix}$$

At iteration 8,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.896 \\ 1 \\ 0.612 \end{bmatrix} = \begin{bmatrix} 3.509 \\ 4.0178 \\ 2.2836 \end{bmatrix} = 4.0178 \begin{bmatrix} 0.8733 \\ 1 \\ 0.5683 \end{bmatrix}$$

At iteration 9,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.8733 \\ 1 \\ 0.5683 \end{bmatrix} = \begin{bmatrix} 3.4417 \\ 3.8835 \\ 2.3049 \end{bmatrix} = 3.8835 \begin{bmatrix} 0.8862 \\ 1 \\ 0.5933 \end{bmatrix}$$

At iteration 10,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.8862 \\ 1 \\ 0.5935 \end{bmatrix} = \begin{bmatrix} 3.4798 \\ 3.9595 \\ 2.2927 \end{bmatrix} = 3.9595 \begin{bmatrix} 0.8788 \\ 1 \\ 0.579 \end{bmatrix}$$

At iteration 11,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.8788 \\ 1 \\ 0.579 \end{bmatrix} = \begin{bmatrix} 3.4579 \\ 3.9158 \\ 2.2998 \end{bmatrix} = 3.9158 \begin{bmatrix} 0.883 \\ 1 \\ 0.5873 \end{bmatrix}$$

At iteration 12,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.883 \\ 1 \\ 0.5873 \end{bmatrix} = \begin{bmatrix} 3.4704 \\ 3.9408 \\ 2.2958 \end{bmatrix} = 3.9408 \begin{bmatrix} 0.8806 \\ 1 \\ 0.5825 \end{bmatrix}$$

At iteration 13,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.8806 \\ 1 \\ 0.5825 \end{bmatrix} = \begin{bmatrix} 3.4632 \\ 3.9264 \\ 2.298 \end{bmatrix} = 3.9264 \begin{bmatrix} 0.8820 \\ 1 \\ 0.5852 \end{bmatrix}$$

At iteration 14,

$$Y = [A][X] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0.8820 \\ 1 \\ 0.5852 \end{bmatrix} = \begin{bmatrix} 3.4673 \\ 3.9346 \\ 2.2967 \end{bmatrix} = 3.9346 \begin{bmatrix} 0.8812 \\ 1 \\ 0.5837 \end{bmatrix}$$

Since, further iteration gives the same value up to two decimal point. So, we stop the iteration here.

Largest eigen value = 3.93

and, Corresponding eigen vector = $\begin{bmatrix} 0.88 \\ 1 \\ 0.58 \end{bmatrix}$

8. **Find the largest eigen value and the corresponding eigen vector of the following matrix.** **[2072 Magh]**

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

Solution:

Let, $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

and, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

At iteration 1,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0.3333 \\ 1 \\ 0.3333 \end{bmatrix}$$

At iteration 2,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0.3333 \\ 1 \\ 0.3333 \end{bmatrix} = \begin{bmatrix} 2 \\ 3.3333 \\ 2 \end{bmatrix} = 3.3333 \begin{bmatrix} 0.6 \\ 1 \\ 0.6 \end{bmatrix}$$

At iteration 3,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0.6 \\ 1 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 3.6 \\ 2.8 \end{bmatrix} = 3.6 \begin{bmatrix} 0.7777 \\ 1 \\ 0.7777 \end{bmatrix}$$

At iteration 4,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0.7777 \\ 1 \\ 0.7777 \end{bmatrix} = \begin{bmatrix} 3.3333 \\ 3.7778 \\ 3.3333 \end{bmatrix}$$

$$= 3.7778 \begin{bmatrix} 0.8823 \\ 1 \\ 0.8823 \end{bmatrix}$$

At iteration 5,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.8823 \\ 1 \\ 0.8823 \end{bmatrix} = \begin{bmatrix} 3.647 \\ 3.8823 \\ 3.647 \end{bmatrix}$$

$$= 3.8823\begin{bmatrix} 0.9394 \\ 1 \\ 0.9394 \end{bmatrix}$$

At iteration 6,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.9394 \\ 1 \\ 0.9394 \end{bmatrix} = \begin{bmatrix} 3.8182 \\ 3.9394 \\ 3.8182 \end{bmatrix}$$

$$= 3.9394\begin{bmatrix} 0.9692 \\ 1 \\ 0.9692 \end{bmatrix}$$

At iteration 7,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.9692 \\ 1 \\ 0.9692 \end{bmatrix} = \begin{bmatrix} 3.9077 \\ 3.9692 \\ 3.9077 \end{bmatrix}$$

$$= 3.9692\begin{bmatrix} 0.9845 \\ 1 \\ 0.9845 \end{bmatrix}$$

At iteration 8,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.9845 \\ 1 \\ 0.9845 \end{bmatrix} = \begin{bmatrix} 3.9535 \\ 3.9845 \\ 3.9535 \end{bmatrix} = 3.9845\begin{bmatrix} 0.99 \\ 1 \\ 0.99 \end{bmatrix}$$

At iteration 9,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.99 \\ 1 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 3.9767 \\ 3.9923 \\ 3.9767 \end{bmatrix} = 3.9923\begin{bmatrix} 0.9961 \\ 1 \\ 0.9961 \end{bmatrix}$$

At iteration 10,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.9961 \\ 1 \\ 0.9961 \end{bmatrix} = \begin{bmatrix} 3.9883 \\ 3.9961 \\ 3.9883 \end{bmatrix}$$

$$= 3.9961\begin{bmatrix} 0.998 \\ 1 \\ 0.998 \end{bmatrix}$$

At iteration 11,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0.998 \\ 1 \\ 0.998 \end{bmatrix} = \begin{bmatrix} 3.9941 \\ 3.998 \\ 3.9941 \end{bmatrix} = 3.998\begin{bmatrix} 0.999 \\ 1 \\ 0.999 \end{bmatrix}$$

At iteration 12,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0.999 \\ 1 \\ 0.999 \end{bmatrix} = \begin{bmatrix} 3.997 \\ 3.999 \\ 3.997 \end{bmatrix}$$

$$= 3.999 \begin{bmatrix} 0.999 \\ 1 \\ 0.999 \end{bmatrix}$$

At iteration 13,

$$Y = [A][X] = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since, further iteration gives the same value. So, we stop the iteration here.

Largest eigen value = 4.

and, Corresponding eigen vector $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

9. Write the algorithm for computing the dominant Eigen value and corresponding vector of a square matrix using power method. [2073 Magh]

Solution:

Algorithm for computing the dominant Eigen value and corresponding vector of a square and corresponding square matrix using power method is as follows;

1. Start
2. Define matrix X
3. Calculate Y = AX
4. Find the largest element in magnitude of matrix Y and assign it to K.
5. Calculate fresh value

   $Y = \left(\frac{1}{K}\right) \times Y$
6. If [Kn - K (n - 1) > delta, go to step 3.
7. Stop.

10. Compute the inverse of following matrix using the Gauss Jordon method. [2073 Magh]

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Solution:

Let, $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

and, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Writing the given matrix side by side; we have,

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - R_3$; we get

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 + R_2$, $R_3 \leftarrow R_3 - 2R_2$; we get

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - \frac{1}{2} R_1$; we get,

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 + 3R_3$; we get

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & -\frac{1}{2} & -5.5 & 3.5 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow \frac{1}{2} R_1$ and $R_3 \leftarrow R_3 - R_2$; we get,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & -1 & 0 & -0.5 & -5.5 & 3.5 \\ 0 & 0 & -1 & 0.5 & 3.5 & -2.5 \end{array}\right]$$

Applying $R_1 \leftarrow (-1) R_2$, $R_3 \leftarrow (-1) R_3$; we get,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 5.5 & -3.5 \\ 0 & 0 & 1 & -0.5 & -3.5 & 2.5 \end{array}\right]$$

Hence, the inverse of given matrix is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{11}{2} & -\frac{7}{2} \\ -\frac{1}{2} & -\frac{7}{2} & \frac{5}{2} \end{bmatrix}$.

11. **Solve the following system of equation using LU factorization method.** **[2074 Bhadra]**

$5x_1 + 2x_2 + 3x_3 = 31$

$3x_1 + 3x_2 + 2x_3 = 25$

$x_1 + 2X_2 + 4X_3 = 25$

Solution:

The above set of equation can be written in matrix form as [A][X] = [B].

where, $[A] = \begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, $[B] = \begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}$ and $[X] = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

$$\begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Assuming diagonal element of lower triangle matrix to be unity;

$$\begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

or, $$\begin{bmatrix} 5 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} \end{bmatrix}$$

Comparing the corresponding element of the two matrix; we get,

$U_{11} = 5$

$U_{12} = 2$

$U_{13} = 3$

$L_{21}U_{11} = 3$

or, $L_{21} \times 5 = 3$

$\therefore$ $L_{21} = \frac{3}{5} = 0.6$

Let $L_{21}U_{12} + U_{22} = 3$

or, $\frac{3}{5} \times 2 + U_{22} = 3$

$\therefore$ $U_{22} = \frac{9}{5} = 1.8$

$L_{31}\ U_{11} = 1$

or, $L_{31} \times 5 = 1$

$\therefore$ $L_{31} = 0.2$

$L_{31}U_{12} + L_{32}\ U_{22} = 2$

or, $0.2 \times 2 + L_{32} \times 1.8 = 2$

$\therefore$ $L_{32} = 0.8889$

$L_{21} U_{13} + U_{23} = 2$

or, $0.6 \times 3 + U_{23} = 2$

$\therefore$ $U_{23} = 0.2$

$L_{31} U_{13} + L_{32} U_{23} + U_{33} = 4$

or, $0.5 \times 3 + 0.8889 \times 0.2 + U_{33} = 4$

$\therefore$ $U_{33} = 3.2222$

We have,

$[A][X] = [B]$

$[L] [U] [X] = [B]$

$[L] [Z] = [B]$ (i)

where, $[Z] = [U] [X]$ (ii)

Solving for equation (i); we get,

$[L][Z] = [B]$

or, $$\begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.2 & 0.8889 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 25 \\ 25 \end{bmatrix}$$

Applying forward substitution; we get,

$Z_1 = 31$

or, $0.6Z_1 + Z_2 = 25$

or, $0.6 \times 31 + Z_2 = 25$

$\therefore$ $Z_2 = 6.4$

$0.2 \times Z_1 + 0.8889Z_2 + Z_3 = 25$

or, $0.2 \times 31 + 0.8889 \times 6.4 + Z_3 = 25$

$\therefore$ $Z_3 = 13.111$

Thus,

$$Z = \begin{bmatrix} 31 \\ 6.4 \\ 13.111 \end{bmatrix}$$

Again solving for equation (ii); we have,

$[U] [X] = [Z]$

or, $$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 1.8 & 0.2 \\ 0 & 0 & 3.222 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 6.4 \\ 13.111 \end{bmatrix}$$

Applying backward substitution; we have,

$3.222\, X_3 = 13.111$

$\therefore$ $X_3 = 4.069$

$1.8 \times 1 + 0.2 \times 3 = 6.4$

$\therefore \quad X_2 = 3.103$

$5 \times 1 + 2X_2 + 3 \times 3 = 31$

or, $\quad X_1 = 2.517$

Thus, the required solution is;

$$X = \begin{bmatrix} 2.517 \\ 3.103 \\ 4.069 \end{bmatrix}$$

12. **Find the dominant Eigen value and eigen vector of the matrix.** **[2075 Baishakh]**

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution:

Let, $\quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

and, $\quad X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

At iteration 1,

$$Y = [A]\,[x] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 0.5 \\ 1 \\ -0.5 \end{bmatrix}$$

At iteration 2,

$$Y = [A]\,[X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 0.6667 \\ 1 \\ -0.6667 \end{bmatrix}$$

At iteration 3,

$$Y = [A][X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.6667 \\ 1 \\ -0.6667 \end{bmatrix} = \begin{bmatrix} -2.3333 \\ 3.3333 \\ 2.3333 \end{bmatrix} = 3.3333\begin{bmatrix} -0.7 \\ 1 \\ -0.7 \end{bmatrix}$$

At iteration 4,

$$Y = [A]\,[X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.7 \\ 1 \\ -0.7 \end{bmatrix} = \begin{bmatrix} -2.4 \\ 3.4 \\ -2.4 \end{bmatrix} = 3.4\begin{bmatrix} 0.7058 \\ 1 \\ -0.7058 \end{bmatrix}$$

At iteration 5,

$$Y = [A][X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.7058 \\ 1 \\ -0.7058 \end{bmatrix} = \begin{bmatrix} -2.4117 \\ 3.4117 \\ -2.4117 \end{bmatrix} = 3.4117\begin{bmatrix} 0.7069 \\ 1 \\ -0.7069 \end{bmatrix}$$

At iteration 6,

$$Y = [A][X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.7069 \\ 1 \\ -0.7069 \end{bmatrix} = \begin{bmatrix} -2.4138 \\ 3.4138 \\ -2.4138 \end{bmatrix} = 3.4138\begin{bmatrix} -0.707 \\ 1 \\ -0.707 \end{bmatrix}$$

At iteration 7,

$$Y = [A][X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.707 \\ 1 \\ -0.707 \end{bmatrix} = \begin{bmatrix} -2.4142 \\ 3.4142 \\ -2.4142 \end{bmatrix} = 3.4142\begin{bmatrix} -0.7071 \\ 1 \\ -0.7071 \end{bmatrix}$$

At iteration 8,

$$Y = [A][X] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.707 \\ 1 \\ -0.7071 \end{bmatrix} = \begin{bmatrix} -2.4142 \\ 3.4142 \\ -2.4142 \end{bmatrix} = 3.4142\begin{bmatrix} -0.7071 \\ 1 \\ -0.7071 \end{bmatrix}$$

Since, the further iteration gives the same value. So, we stop iteration here.

Dominant eigen value = 3.4142

and, Corresponding eigen vector = $\begin{bmatrix} -0.7071 \\ 1 \\ -0.7071 \end{bmatrix}$

13. Using L-U method, solve the following system of equations:

$2x + 3y + z = 1$

$6x - 3y + 4z = 17$

$5x + 7y + 6z = 10$ **[2075 Ashwin]**

Solution: Proceed same as Q. no. 11

14. Determine the dominant eigen value and corresponding vector of the following matrix using the power method.

$$\begin{bmatrix} 2 & 6 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$$ **[2075 Ashwin]**

Solution: Proceed same as example 3.12

15. Solve the following system of linear equation, using Gauss-elimination method with partial pivoting technique. [2075 Chaitra]

$x_1 - 3x_2 + 8x_3 = 3$

$5x_1 + x_2 + 2x_3 = 9$

$x_1 + 7x_2 - x_3 = 14$

Solution:

The above set of equations can be written in the matrix form [A][X] = [B].

where, $[A] = \begin{bmatrix} 1 & -3 & 8 \\ 5 & 1 & 2 \\ 1 & 7 & -1 \end{bmatrix}$

$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

and, $[B] = \begin{bmatrix} 3 \\ 9 \\ 14 \end{bmatrix}$

The augmented matrix for the above set of equations can be written as;

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -3 & 8 & 3 \\ 5 & 1 & 2 & 9 \\ 1 & 7 & -1 & 14 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 1 & 2 & 9 \\ 1 & -3 & 8 & 3 \\ 1 & 7 & -1 & 14 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - \frac{1}{5}R_1$, $R_3 \leftarrow R_3 - \frac{1}{5}R_1$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 1 & 2 & 9 \\ 0 & -3.2 & 7.6 & 1.2 \\ 0 & 6.8 & -1.4 & 12.2 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 1 & 2 & 9 \\ 0 & 6.8 & -1.4 & 12.2 \\ 0 & -3.2 & 7.6 & 1.2 \end{array}\right]$$

Applying $R_3 \leftarrow R_3 + \frac{1}{2.125}R_2$; we get,

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 1 & 2 & 9 \\ 0 & 6.8 & -1.4 & 12.2 \\ 0 & 0 & 6.941 & 6.941 \end{array}\right]$$

Now, applying backward substitution; we have,

$6.941x_3 = 6.941$

$\therefore \quad x_3 = 1$

and, $6.8x_2 - 1.4x_3 = 12.2$

or, $6.8x_2 - 1.4 = 12.2$

$\therefore \quad x_2 = 2$

and, $5x_1 + x_2 + 2x_3 = 9$

or, $5x_1 + 2 + 2 = 9$

$\therefore \quad x_1 = 1$

Thus, the required solution is;

$$[X] = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

16. Obtain the dominant eigen value and its corresponding eigen vector of the following matrix using power method. [2075 Chaitra]

$$\begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$

Solution:

Let, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

We have,

At iteration 1;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 0.5 \\ 0.125 \\ 1 \end{bmatrix}$$

At iteration 2;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.125 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3.125 \\ 4 \end{bmatrix}$$

$$= 5\begin{bmatrix} 1 \\ 0.625 \\ 0.8 \end{bmatrix}$$

At iteration 3;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.625 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 6.7 \\ 11.025 \\ 9.8 \end{bmatrix}$$

$$= 11.025\begin{bmatrix} 0.607 \\ 1 \\ 0.888 \end{bmatrix}$$

At iteration 4;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.607 \\ 1 \\ 0.888 \end{bmatrix} = \begin{bmatrix} 8.159 \\ 10.532 \\ 11.316 \end{bmatrix}$$

$$= 11.316\begin{bmatrix} 0.721 \\ 0.930 \\ 1 \end{bmatrix}$$

At iteration 5;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.721 \\ 0.930 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.441 \\ 11.814 \\ 11.324 \end{bmatrix} = 11.814\begin{bmatrix} 0.714 \\ 1 \\ 0.958 \end{bmatrix}$$

At iteration 6;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.714 \\ 1 \\ 0.958 \end{bmatrix} = \begin{bmatrix} 8.548 \\ 11.52 \\ 11.814 \end{bmatrix}$$

$$= 11.814 \begin{bmatrix} 0.723 \\ 0.975 \\ 1 \end{bmatrix}$$

At iteration 7;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.723 \\ 0.975 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.623 \\ 11.867 \\ 11.692 \end{bmatrix}$$

$$= 11.867 \begin{bmatrix} 0.726 \\ 1 \\ 0.985 \end{bmatrix}$$

At iteration 8;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.726 \\ 1 \\ 0.985 \end{bmatrix} = \begin{bmatrix} 8.666 \\ 11.784 \\ 11.889 \end{bmatrix} = 11.889 \begin{bmatrix} 0.728 \\ 0.991 \\ 1 \end{bmatrix}$$

At iteration 9;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.728 \\ 0.991 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.692 \\ 11.902 \\ 11.84 \end{bmatrix} = 11.902 \begin{bmatrix} 0.730 \\ 1 \\ 0.994 \end{bmatrix}$$

At iteration 10;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.730 \\ 1 \\ 0.994 \end{bmatrix} = \begin{bmatrix} 8.706 \\ 11.872 \\ 11.914 \end{bmatrix} = 11.914 \begin{bmatrix} 0.730 \\ 0.996 \\ 1 \end{bmatrix}$$

At iteration 11;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.730 \\ 0.996 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.714 \\ 11.916 \\ 11.888 \end{bmatrix} = 11.916 \begin{bmatrix} 0.731 \\ 1 \\ 0.997 \end{bmatrix}$$

At iteration 12;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.731 \\ 1 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 8.719 \\ 11.9 \\ 11.921 \end{bmatrix} = 11.921 \begin{bmatrix} 0.731 \\ 0.998 \\ 1 \end{bmatrix}$$

At iteration 13;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.731 \\ 0.998 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.723 \\ 11.922 \\ 11.908 \end{bmatrix} = 11.922 \begin{bmatrix} 0.731 \\ 1 \\ 0.998 \end{bmatrix}$$

At iteration 14;

$$Y = [A][X] = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} 0.731 \\ 1 \\ 0.998 \end{bmatrix} = \begin{bmatrix} 8.723 \\ 11.908 \\ 11.922 \end{bmatrix} = 11.922 \begin{bmatrix} 0.731 \\ 0.998 \\ 1 \end{bmatrix}$$

Since, the further iterations given approximately same values, so we stop the iteration here. The highest value of scalar is the value in iteration (14), so we take its value as the solution.

Thus, Largest eigen value = 11.922

$$\text{Corresponding eigen vector} = \begin{bmatrix} 0.731 \\ 0.998 \\ 1 \end{bmatrix}$$

17. Compute the inverse of the following matrix using the Gauss-Jordan method. **[2076 Ashwin]**

$$\begin{bmatrix} 9 & 9 & 8 \\ 7 & 8 & 7 \\ 6 & 8 & 8 \end{bmatrix}$$

Solution:

The given matrix is;

$$[A] = \begin{bmatrix} 9 & 9 & 8 \\ 7 & 8 & 7 \\ 6 & 8 & 8 \end{bmatrix}$$

The augmented matrix for the above matrix is;

$$[A|I] = \begin{bmatrix} 9 & 9 & 8 \\ 7 & 8 & 7 \\ 6 & 8 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \leftarrow R_1 - \frac{8}{6}R_3$; we get,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1.66 & -2.66 & 1 & 0 & -1.33 \\ 7 & 8 & 7 & -7 & 1 & 9.31 \\ 6 & 8 & 8 & -6 & 0 & 8.98 \end{array}\right]$$

Applying $R_1 \leftarrow R_2 - \frac{18.62}{17.96}R_3$; we get,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1.66 & -2.66 & 1 & 0 & -1.33 \\ 0 & 1 & 0.77 & -0.77 & 1 & 0 \\ 0 & 17.96 & 23.96 & -6 & 0 & 8.98 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 + 1.66R_2$, $R_3 \leftarrow R_3 - 17.96R_2$; we get,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1.38 & 0.27 & 1.66 & -1.33 \\ 0 & 1 & 0.77 & -0.77 & 1 & 0 \\ 0 & 0 & 10.13 & 7.82 & -17.96 & 8.98 \end{array}\right]$$

Applying $R_3 \leftarrow R_1 + \frac{R_3}{10.13}$; we get,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1.38 & 0.27 & 1.66 & -1.33 \\ 0 & 1 & 0.77 & -0.77 & 1 & 0 \\ 0 & 0 & 1 & 0.77 & -1.77 & 0.88 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 + 1.38R_3$, $R_2 \leftarrow R_2 - 0.77R_3$; we get,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.33 & -0.78 & -0.11 \\ 0 & 1 & 0 & -1.36 & 2.36 & -0.67 \\ 0 & 0 & 1 & 0.77 & -1.77 & 0.88 \end{array}\right]$$

This is in the form $[I][A^{-1}]$.

Thus,

$$[A^{-1}] = \begin{bmatrix} 1.33 & -0.78 & -0.11 \\ -1.36 & 2.36 & -0.67 \\ 0.77 & -1.77 & 0.88 \end{bmatrix}$$

18. Find the largest Eigen value and corresponding Eigen vector of the matrix $\begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$. **[2076 Ashwin]**

Solution: See the solution of Q. no. 5