# Chapter 6

# SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS



## Introduction

Mathematical methods have been developed for solving certain types of ordinary differential equations. They are dealt with in textbooks on calculus and differential equations. But by such methods, solutions of only few differential equations can be obtained in finite form or closed form. Differential equations which one encounters in practical problems in the fields of engineering and science cannot either be solved at all by the classical methods or possess solutions that are to be obtained by such difficult and cumbersome procedures that they are not worth the trouble.

*For example;* the equation $\frac{dy}{dx} = x^2 + y^2$ has no elementary solution.

When the classical methods are not of any help to solve differential equations, we shall go in for numerical methods to solve them. We know that the general solution of a differential equation of the nth order has n arbitrary constants. Hence, to find the particular solution of an $n^{th}$ order equation we require n conditions. Usually these 'n' conditions will be provided by the values of the dependent variable or its derivatives variable. It all the n conditions are specified only for the initial value of x (the independent variable), then the problem is called an initial value problem.

*For example;* the problem of solving the differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$, subjected to the conditions $y(0) = a$ and $y'(0) = b$ is an initial value problem.

If the conditions are specified for two or more values of x, then the problem is called a boundary value problem.

*For example;* Solving the differential equation $y''(x) + a(x)y'(x) + b(x)y(x) = c(x)$, subjected to the conditions $y(x_0) = k_1$ and $y(x_n) = k_2$ is a boundary value problem.

### Numerical Solution of a Differential Equation

Numerical solution of a differential equation satisfying the given initial or boundary conditions consists in finding the values $y_1, y_2, y_3, \ldots\ldots\ldots$ of the dependent variable y corresponding to the pivotal (equally spaced) values $x_1, x_2, x_3, \ldots\ldots\ldots$ of the independent variable x so that the pairs of values $(x_i, y_i)$ satisfy approximately a particular solution of the differential equation. A solution of this type is known as a point-wise solution.

There exist many numerical methods for finding approximate solutions of differential equations (both initial and boundary value problems). In this chapter, we shall deal with the most important of the available methods, first for solving initial value problems and then the boundary value problems.

Numerical solutions of differential equations (initial value problems) are obtained in one of the following two forms:

i) An approximate series for y in ascending powers of x or $x - x_0$, from which the values of y corresponding to specified value of x can be obtained by direct substitution.

ii) Approximate values of y corresponding to only specified values of x.

The methods of Picard and Taylor series belongs the first category, while the methods of Runge-kutta, Euler, Milne and Adam-Bashforth belong to the second category.

## 6.1 Euler's and Modified Euler's Method

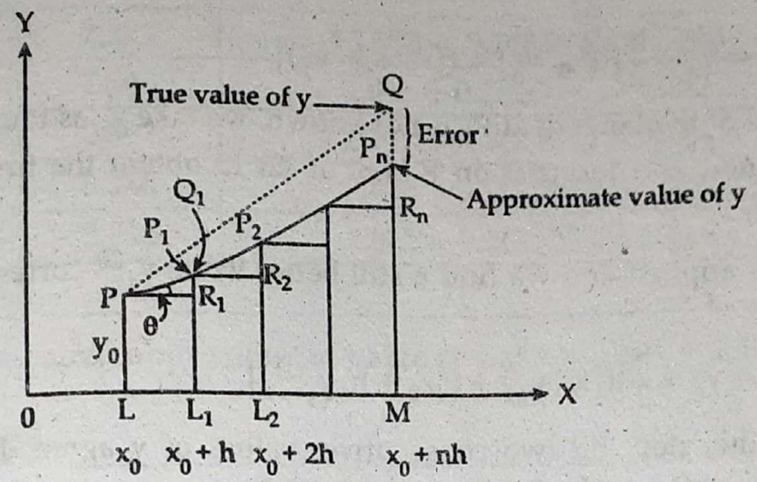
### Euler's Method

Consider the equation;

$$\frac{dy}{dx} = f(x, y); \quad (1)$$

given that $y(x_0) = y_0$ its curve of solution through $P(x_0, y_0)$ is shown dotted in figure (a) below.

Now, we have to find the ordinate of any other point Q on this curve.




Let us divide LM into n sub-intervals each of width h at $L_1$, $L_2$, ......... so that 'h' is quite small:

In the interval $LL_1$, we approximate the curve by the tangent at P. If the ordinate through $L_1$ meets this tangent in $P_1(x_0 + h, y_1)$; then,

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

or, $$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_P$$

or, $$y_1 = y_0 + hf(x_0, y_0)$$

Let $P_1Q_1$ be the curve of solution of (1) through $P_1$ and let its tangent at $P_1$ meet the ordinate through $L_2$ in $P_2(x_0 + 2h, y_2)$. Then,

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

Repeating this process n time, we finally reach on an approximation $MP_n$ of MQ given by;

$$y_n = y_{n-1} + hf[x_0 + (n - L)h, y_{n-1}]$$

This is Euler's method of finding an approximate solution of (1).

## Modified Euler's Method

In the Euler's method, the curve of solution in the interval $LL_1$ is a approximated by tangent at P (shown in figure (a) above) such that at $P_1$; we have, (1)

$$y_1 = y_0 + hf(x_0, y_0)$$

Then the slope of the curve of solution through $P_1$

i.e.,, $$\left(\frac{dy}{dx}\right)P_1 = f(x_0 + h, y_1);$$

is computed and the tangent at $P_1$ to $P_1Q_1$ is drawn meeting the ordinate through $L_2$ in $P_2(x_0 + 2h, y_2)$.

Now, we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and $P_1$ *i.e.,,*

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] \quad (2)$$

As the slope of the tangent at $P_1$ is not known, we take $y_1$ as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value of $y_1^{(1)}$.

Again, (2) is applied and we find a still better value $y_1^{(2)}$ corresponding to $L_1$ as;

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive, values of y agree. This is then taken as the starting point for the next interval $L_1L_2$.

Once $y_1$ is obtained to desired degree of accuracy, y corresponding to $L_2$ is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and, a better approximation $y_2^{(1)}$ is obtained from (2);

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until $y_2$ becomes stationary. Then we proceed to calculate $y_3$ above and so on.

This is the modified Euler's method which gives great improvement in accuracy over the original method.

**Example 6.1**

**Using Euler's method, find an approximate value of y corresponding to $x = L$ given that $\frac{dy}{dx} = x + y$ and $y = L$ when $x = 0$**

**Solution:**

We take,

$n = 10$

and, $h = 0.1$; which is sufficiently small.

The various calculations are arranged as follows:

| x | y | $\frac{dy}{dx} = x + y$ | New y = Old y + $0.1\left(\frac{dy}{dx}\right)$ $\left[\text{as } y_1 = y_0 + h\left(\frac{dy}{dx}\right)\right]$ |
|---|---|---|---|
| 0.0 | 1.00 | 1.00 | 1.00 + 0.1(1.00) = 1.10 |
| 0.1 | 1.10 | 1.20 | 1.10 + 0.1(1.20) = 1.22 |
| 0.2 | 1.22 | 1.42 | 1.22 + 0.1(1.42) = 1.362 |
| 0.3 | 1.362 | 1.662 | 1.362 + 0.1(1.662) = 1.5282 |
| 0.4 | 1.5282 | 1.9282 | 1.5282 + 0.1(1.9282) = 1.721 |

| | | | |
|---|---|---|---|
| 0.5 | 1.721 | 2.221 | 1.1.721 + 0.1(2.221) = 1.943 |
| 0.6 | 1.943 | 2.543 | 1.943 + 0.1(2.543) = 2.197 |
| 0.7 | 2.197 | 2.897 | 2.197 + 0.1(2.897) = 2.486 |
| 0.8 | 2.486 | 3.286 | 2.486 + 0.1(3.286) = 2.815 |
| 0.9 | 2.815 | 3.715 | 2.815 + 0.1(3.715) = 3.186 |
| 1.0 | 3.186 | | |

Thus, the required approximated value of y at x = 1 is y = 3.186.

### Example 6.2

**Using modified Euler's method, find an approximate value of y when x = 0.3 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0**

**Solution:**

Here, we have to find an approximate value of y when x = 0.3 so take h = 0.1 which is sufficiently small and given condition is $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

The various calculations are arranged as follows in the table:

| x | $x + y = \frac{dy}{dx}$ | Mean slope | Old y + h(Mean slope) = New y |
|---|---|---|---|
| 0.0 | 0 + 1 = 1 | – | 1.0 + 0.1(1.00) = 1.10 |
| 0.1 | 0.1 + 1.10 = 1.2 | $\frac{1+1.2}{2} = 1.1$ | 1.0 + 0.1(1.1) = 1.11 |
| 0.1 | 0.1 + 1.11 = 1.21 | $\frac{1+1.21}{2} = 1.105$ | 1.0 + 0.1(1.105) = $\underline{1.1105}$ |
| 0.1 | 0.1 + 1.1105 = 1.2105 | $\frac{1+1.2105}{2} = 1.105$ | 1.0 + 0.1(1.105) = $\underline{1.1105}$ |
| 0.1 | 0.2105 | – | 1.1105 + 0.1(1.2105) = 1.2315 |
| 0.2 | 0.2 + 1.2315 = 1.4315 | $\frac{1.2105+1.4315}{2}$ | 1.1105 + 0.1(1.321) = 1.2426 |
| 0.2 | 0.2 + 1.2426 = 1.4426 | $\frac{1.2105+1.4426}{2}$ | 1.1105 + 0.1(1.3265) = $\underline{1.2432}$ |
| 0.2 | 0.2 + 1.2432 = 1.4432 | $\frac{1.2105+1.4432}{2}$ | 1.1105 + 0.1(1.3268) = $\underline{1.2432}$ |
| 0.2 | 1.4432 | – | 1.2432 + 0.1(1.4432) = 1.3875 |
| 0.3 | 0.3 + 1.3875 = 1.6875 | $\frac{1.4432+1.6875}{2}$ | 1.2432 + 0.1(1.5654) = 1.3997 |
| 0.3 | 0.3 + 1.3997 = 1.6997 | $\frac{1.4432+1.6997}{2}$ | 1.2432 + 0.1(1.5715) = 1.4003 |
| 0.3 | 0.3 + 1.4003 = 1.7003 | $\frac{1.4432+1.7003}{2}$ | 1.2432 + 0.1(1.5718) = $\underline{1.4004}$ |
| 0.3 | 0.3 + 1.4004 = 1.7004 | $\frac{1.4432+1.7004}{2}$ | 1.2432 + 0.1(1.5718) = $\underline{1.4004}$ |

Hence, an approximate value of y when x = 0.3 is y = 1.4004.

## 6.2 Runge Kutta Methods for First and Second Order Ordinary Differential Equations

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However, there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution up to the term in $h^r$, where, r differs from method to method and is called the order of that method.

### First Order Runge-Kutta Method ($R - K_1$ Method)

We have, from Euler's method;

$$y_1 = y_0 + hf(x_0, y_0)$$

or, $$y_1 = y_0 + hy'_0 \qquad [\because y' = f(x, y)]$$

Expanding by Taylor's series;

$$y_1 = y(x_0 + h)$$

$$\therefore \quad y_1 = y_0 + hy'_0 + \frac{h^2}{2}y''_0 + \ldots\ldots\ldots$$

If follows that the Euler's method agrees with the Taylor's series solution up to the term in h.

Hence, Euler's method is the Runge-Kutta method of the first order.

### Second Order Runge-Kutta Method ($R - K_2$ Method)

We have, from modified Euler's method;

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad (1)$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ on the right hand side of (1); we obtain,

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)] \qquad (2)$$

where, $f_0 = f(x_0, y_0)$.

Expanding L.H.S. by Taylor's series; we get,

$$y_1 = y(x_0 + h)$$

or, $$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \ldots\ldots\ldots \qquad (3)$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series for a function of two variables, (2) gives;

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + 0(h^2)^*\right\}\right]$$

where, $0(h^2)^*$ = Term containing second and higher powers of h and is read as order of $h^2$.

or, $y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + 0(h^3)\right]$

or, $y_1 = y_0 + hf_0 + \frac{h^2}{2}f_0' + 0(h^3)$ $\left[\because \frac{\partial f(x, y)}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right]$

$\therefore$ $y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + 0(h^3)$ (4)

Comparing (3) and (4) it follows that the modified Euler's method agrees with the Taylor's series solution up to the term in $h^2$.

Hence, the modified Euler's method is the Runge-Kutta method of the second order.

Therefore, the second order Runge-Kutta formula is;

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

where, $K_1 = hf(x_0, y_0)$

and, $K_2 = hf(x_0 + h, y_0 + K_1)$

### Third-Order Runge-Kutta Method ($R - K_3$ Method)

It can be seen that Runge's method agrees with the Taylor's series solution up to the term in $h^3$.

As Runge's method is the Runge-Kutta method of the third order.

The Third order Runge-Kutta formula is;

$$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

where, $K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1\right)$

and, $K_3 = hf(x_0 + h, y_0 + K')$

where, $K' = hf(x_0 + h, y_0 + K_1)$

### Fourth Order Runge-Kutta Method ($R - K_4$ Method)

This method is most commonly used and is often referred to as Runge-Kutta method only.

**Working rule**

For finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is as follows:

***Calculate successively***

$K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1\right)$

$$K_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2\right)$$

and, $K_4 = hf(x_0 + h, y_0 + K_3)$

*Finally compute*

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4);$$

which gives the required approximate value as $y_1 = y_0 + K$.

**Note**

That k is the weighted mean of $K_1$, $K_2$, $K_3$ and $K_4$.

**Example 6.3**

**From given data $\frac{dy}{dx} = y - x$; where, y(0) = 2 find y(0.1) and y(0.2) correct to four decimal places.**

**Solution:**

From the given,

$$\frac{dy}{dx} = y - x;$$

where, $y(0) = 2$.

Then, we have to find $y(0.1)$ and $y(0.2)$

i) From Runge-Kutta second order formula:

Take $h = 0.1$

Then,

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

where, $K_1 = hf(x_0, y_0) = 0.1 \times 2 = 0.2$

and, $K_2 = hf(x_0 + h, y_0 + K_1)$

Here, we have,

When $x_0 = 0$ then, $y_0 = 2$

so, $K_2 = 0.1f(0 + 0.1, 2 + 0.2) = 0.1f(0.1, 22)$

or, $K_2 = 0.1(2.2 - 0.1)$ $\left[\because \frac{dy}{dx} = y - x\right]$

or, $K_2 = 0.1 \times 2.1 = 0.21$

$\therefore$ $y_1 = y(0.1) = y_0 + \frac{1}{2}(K_1 + K_2) = 2 + \frac{1}{2}(0.2 + 0.21) = 2.2050$

Again, to determine $y_2 = y(0.2)$

We take, $x_0 = 0.1$ and $y_0 = 2.2050$ with $h = 0.1$

Then,

$$K_1 = hf(x_0, y_0) = 0.1 \times f(0.1, 2.2050) = 0.1 \times (2.2050 - 0.1)$$

or, $K_1 = 0.2105$

$K_2 = hf(x_0 + h, y_0 + K_1)$

or, $K_2 = 0.1f(0.1 + 0.1, 2.2050 + 0.2105) = 0.1f(0.2, 2.4155)$

or, $K_2 = 0.1(2.4155 - 0.2) = 0.22155$

$\therefore$ $y_2 = y(0.2) = y_0 + \frac{1}{2}(K_1 + K_2) = 2.2050 + \frac{1}{2}(0.2105 + 0.22155)$

or, $y_2 = 2.2050 + 0.216025 = 2.4210$

ii) From Runge–Kutta fourth order formula;

Here,

$x_0 = 0$

and, $y_0 = 2$

We have to find $y(0.1)$ and $y(0.2)$

so, take $h = 0.1$

We have,

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where, $K_1 = hf(x_0, y_0) = 0.1 \times 2 = 0.2.$

$K_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1\right) = 0.1f\left(0 + \frac{0.1}{2}, 2 + \frac{0.2}{2}\right)$

or, $K_2 = 0.1f(0.05, 2.1) = 0.1 \times (2.1 - 0.05) = 0.205$

Also,

$K_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2\right) = 0.1f\left(0 + \frac{0.1}{2}, 2 + \frac{0.205}{2}\right)$

or, $K_3 = 0.1f(0.05, 2.1025) = 0.1 \times (2.1025 - 0.05) = 0.20525$

and, $K_4 = hf(x_0 + h, y_0 + K_3) = 0.1f(0 + 0.1, 2 + 0.20525)$

or, $K_4 = 0.1f(0.1, 2.20525) = 0.1 \times (2.20525 - 0.1) = 0.21053$

$\therefore$ $y(0.2) = 2 + \frac{1}{6}(0.2 + 2 \times 0.205 + 2 \times 0.20525 + 0.21053)$

or, $y(0.2) = 2.2052$

Again, we have to determine $y_2 = y(0.2)$

so, take $x_0 = 0.1, y_0 = 2.2052$ and $h = 0.1$; then,

$K_1 = hf(x_0, y_0) = 0.1f(0.1, 2.2052) = 0.1 \times (2.2052 - 0.1)$

or, $K_1 = 0.21052$

$K_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1\right) = 0.1f\left(0 + \frac{0.1}{2}, 2.2052 + \frac{0.21052}{2}\right)$

or, $K_2 = 0.1f(0.15, 2.31046) = 0.1 \times (2.31046 - 0.15) = 0.216046$

$K_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2\right) = 0.1f\left(0 + \frac{0.1}{2}, 2.2052 + \frac{0.216046}{2}\right)$

or, $K_3 = 0.1f(0.15, 2.313223) = 0.1 \times (2.313223 - 0.15) = 0.2163223$

and, $K_4 = hf(x_0 + h, y_0 + K_3) = 0.1f(0.1 + 0.1, 2.2052 + 0.2163223)$

or, $K_4 = 0.1 \times (2.4215223 - 0.2) = 0.22215$

Now,

$$y_2 = y(0.2)$$

$$= 2.2052 + \frac{1}{6}(0.21052 + 2 \times 0.216046 + 2 \times 0.2163223 + 0.22215)$$

or, $y_2 = y(0.2) = 2.4214$

**Example 6.4**

**Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$ and $0.4$.**

**Solution:**

From the given; we have,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

with $x_0 = 0$ and $y_0 = 1$

**To find y(0.2)**

Here, $x_0 = 0$, $y_0 = 1$ and take $h = 0.4 - 0.2 = 0.2$

From fourth order Runge-Kutta formula:

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where, $K_1 = hf(x_0, y_0) = 0.2f(0, 1) = 0.2 \times 1 = 0.2000.$

$$K_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1\right) = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2000}{2}\right)$$

or, $$K_2 = 0.2f(0.1, 1.1) = 0.2 \times \left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right] = 0.19672$$

$$K_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2\right) = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right)$$

or, $$K_3 = 0.2f(0.1, 1.09836) = 0.2 \times \left[\frac{(1.09836)^2 - (0.1)^2}{(1.09836)^2 + (0.1)^2}\right] = 0.1967$$

and, $K_4 = hf(x_0 + h, y_0 + K_3) = 0.2f(0 + 0.2, 1 + 0.1967)$

or, $$K_4 = 0.2f(0.2, 1.1967) = 0.2 \times \left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right] = 0.1891$$

Now,

$$y(0.2) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}(0.2000 + 2 \times 0.19672 + 2 \times 0.1967 + 0.1897)$$

$$= 1 + 0.196$$

$\therefore \quad y(0.2) = 1.196$

Again, to find y(0.4)

Here, take $x_0 = 0.2$, $y_1 = 1.196$ and $h = 0.2$

Now,

$$y(0.4) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where, $K_1 = hf(x_1, y_1) = 0.2f(0.2, 1.196) = 0.2 \times \left[\frac{(1.196)^2 - (0.2)^2}{(1.196)^2 + (0.2)^2}\right] = 0.1891$

$$K_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_1\right) = 0.2f\left(0 + \frac{0.2}{2}, 1.196 + \frac{0.1891}{2}\right)$$

or, $K_2 = 0.2f(0.3, 1.2906) = 0.2 \times \left[\frac{(1.2906)^2 - (0.3)^2}{(1.2906)^2 + (0.3)^2}\right] = 0.1795$

$$K_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_2\right) = 0.2f\left(0.2 + \frac{0.2}{2}, 1.196 + \frac{0.1795}{2}\right)$$

or, $K_3 = 0.2f(0.3, 1.2858) = 0.2 \times \left[\frac{(1.2858)^2 - (0.3)^2}{(1.2858)^2 + (0.3)^2}\right] = 0.1793$

and, $K_4 = hf(x_1 + h, y_1 + K_3) = 0.2f(0.2 + 0.2, 1.196 + 0.1793)$

or, $K_4 = 0.2f(0.4, 1.3753) = 0.2 \times \left[\frac{(1.3753)^2 - (0.4)^2}{(1.3753)^2 + (0.4)^2}\right] = 0.1688$

Now,

$$y(0.4) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= y_1 + \frac{1}{6}[0.1891 + (2 \times 0.1795) + (2 \times 0.1793) + 0.1688]$$

$\therefore \quad y(0.4) = 1.3752$

**Runge–Kutta Formulas for the Solutions of the Simultaneous Equations of the First Order of the Form**

$\frac{dy}{dx} = f_1(x, y, z)$, $\frac{dz}{dx} = f_2(x, y, z)$, $y(x_0) = y_0$, $z(x_0) = z_0$ are;

$$y(x + h) = y(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

and, $z(x + h) = z(x) + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

where, $K_1 = hf_1(x, y, z)$

$$l_1 = hf_2(x, y, z)$$

$$K_2 = hf_1\left(x + \frac{h}{2}, y + \frac{K_1}{2}, z + \frac{l_1}{2}\right)$$

$$l_2 = hf_2\left(x + \frac{h}{2}, y + \frac{K_1}{2}, z + \frac{l_1}{2}\right)$$

$$K_3 = hf_1\left(x + \frac{h}{2}, y + \frac{K_2}{2}, z + \frac{l_2}{2}\right)$$

$$l_3 = hf_2\left(x + \frac{h}{2}, y + \frac{K_2}{2}, z + \frac{l_2}{2}\right)$$

and, $K_4 = hf_1(x + h, y + K_3, z + l_3)$

$$l_4 = hf_2(x + h, y + K_3, z + l_3)$$

## Example 6.5

**Solve the simultaneous differential equations $\frac{dy}{dx} = 2y + z$; $\frac{dz}{dx} = y - 3z$; $y(0) = 0$; $y(0) = 0.5$ for $y(0.1)$ and $z(0.1)$ using Runge-Kutta method of the fourth order.**

**Solution:**

From the given,

$$\frac{dy}{dx} = 2y + z; \frac{dz}{dx} = y - 3z; y(0) = 0; y(0) = 0.5$$

We have to find $y(0.1)$ and $z(0.1)$

Here,

$$f_1(x, y, z) = 2y + z$$

and; $f_1(x, y, z) = y - 3z$

Also,

$$x = 0, y = y(0) = 0, z = z(0) = 0.5$$

Take $h = 0.4$

From fourth order Runge-Kutta method;

For $y(0.1)$

The formula is;

$$y(x + h) = y(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

and, for $z(0.1)$

$$z(x + h) = z(x) + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

At first calculating $K_1, K_2, K_3, K_4$ and $l_1, l_2, l_3, l_4$

$\therefore$ $K_1 = hf_1(x, y, z) = 0.1f_1(0, 0, 0.5) = 0.1(2 \times 0 + 0.5) = 0.05$

$$l_1 = hf_2(x, y, z) = 0.1f_2(0, 0, 0.5) = 0.1(0 - 3 + 0.5) = -0.15$$

$$K_2 = hf_1\left(x + \frac{h}{2}, y + \frac{K_1}{2}, z + \frac{l_1}{2}\right)$$

$$= 0.1f_1\left[0 + \frac{0.1}{2}, 0 + \frac{0.05}{2}, 0.5 + \frac{(-0.15)}{2}\right]$$

$$= 0.1f_1(0.05, 0.025, 0.425)$$

$$= 0.1(2 \times 0.025 + 0.425) = 0.0475$$

$$l_2 = hf_2\left(x + \frac{h}{2}, y + \frac{K_1}{2}, z + \frac{l_1}{2}\right)$$

$$= 0.1f_2\left[0 + \frac{0.1}{2}, 0 + \frac{0.05}{2}, 0.5 + \frac{(-0.15)}{2}\right]$$

$$= 0.1f_2(0.05, 0.025, 0.425)$$

$$= 0.1(0.025 - 3 \times 0.425) = -0.125$$

$$K_3 = hf_1\left(x + \frac{h}{2}, y + \frac{K_2}{2}, z + \frac{l_2}{2}\right)$$

$$= 0.1f_1\left[0 + \frac{0.1}{2}, 0 + \frac{0.0475}{2}, 0.5 + \frac{(-0.125)}{2}\right]$$

$$= 0.1f_1(0.05, 0.02375, 0.4375)$$

$$= 0.1(2 \times 0.02375 + 0.4375) = 0.0485$$

$$l_3 = hf_2\left(x + \frac{h}{2}, y + \frac{K_2}{2}, z + \frac{l_2}{2}\right)$$

$$= 0.1f_2\left[0 + \frac{0.1}{2}, 0 + \frac{0.0475}{2}, 0.5 + \frac{(-0.125)}{2}\right]$$

$$= 0.1f_2(0.05, 0.02375, 0.4375)$$

$$= 0.1(0.02375 - 3 \times 0.4375) = -0.128875$$

and, $K_4 = hf_1(x + h, y + K_3, z + l_3)$

$$= 0.1f_1[0 + 0.1, 0 + 0.0485, 0.5 - 0.128875]$$

$$= 0.1f_1(0.1, 0.0485, 0.371125)$$

$$= 0.1(2 \times 0.0485 + 0.371125) = 0.0468125$$

$$l_4 = hf_2(x + h, y + K_3, z + l_3)$$

$$= 0.1f_2[0 + 0.1, 0 + 0.0485, 0.5 - 0.128875]$$

$$= 0.1f_2(0.1, 0.0485, 0.371125)$$

$$= 0.1(0.0485 - 3 \times 0.371125) = -0.1064875$$

Now,

$$y(x + h) = y(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

or, $y(0 + 0.1) = y(0) + \frac{1}{6}(0.05 + 2 \times 0.0475 + 2 \times 0.485 + 0.0468125)$

or, $y(0.1) = 0 + 0.048135 = 0.0481$

Also,

$$z(x + h) = z(x) + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

or, $z(0 + 0.1) = z(0) + \frac{1}{6}[-0.05 + 2 \times (-0.125) + 2 \times (-0.128875)$

$+(-0.1064875)]$

or, $z(0 + 0.1) = 0.5 + (-0.127372) = 0.3726$

**Example 6.6**

**Write a computer oriented algorithm and the corresponding c program to solve the differential equation $\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$ at the pivotal points, by using (i) simple (ii) improve and (iii) modified Euler's methods.** **[2072 Chaitra]**

**Solution:**

1. Algorithm 1.
2. Define $f(x, y)$ [the R.H.S. function of the differential equation is $y' = f(x, y)$]
3. Read $x$, $y_s$, $h$ [the initial values of $x$, $y$ and the step size $h$ of $x$]
4. $y_i \leftarrow y_m \leftarrow y_s [y_i = y_m = y_s]$
5. For $i = 1(1)$ to 10, do till (13)
6. $y_s \leftarrow y_s + hf(x, y_s)$
7. $x_n \leftarrow x + h$
8. $y_i \leftarrow y_i + \frac{h}{2}[f(x, y_i) + f\{x_n, y_i + hf(x_n, y_i)\}]$
9. $y_m \leftarrow y_m + h\left[\left(x + \frac{h}{2}\right), y_m + \frac{h}{2}f(x, y_m)\right]$
10. $y_e \leftarrow$ "the exact solution of the equation to be supplied with $x$ replaced by $x_n$"
11. Write $x_n$, $y_s$, $y_i$, $y_m$
12. $x \leftarrow x_n$
13. Next i
14. End

```
|| Program-1 ||
|| Euler's Methods ||
# include <stdio. h>
# include <math.h>
# define f(x, y) (3*exp(x)+2*y)
void main ( )
        { int i;
        float h, xn, x, ys, yi, ym, ye;
        clrscr ( );
        print f ("Enter the initial value of x:\n");
        scan f ("%f", &x);
        print f ("Enter the initial value of y:\n" );
        scan f ("%f", &y);
```

```
print f ("Enter the step size h:\n");
scan f ("%f", &h);
print f ("...................................................\n");
Print f ("Euler's method for solving y' = 3 * exp (x) + 2 * y\n");
Printf ("\tx\ty simple/ty Improved/ty modified/ty exact/n");
printf ("...................................................\n");
yi = ym = ys;
for (i = 1; I < 11; i++)
{ ys = ys + h * f (x, yx);
  xn = x + h;
  xn = yi + (h\2) * (f(x, yi) + f(xn, yi + h * f (x, yi)));
  ym = ym + h * f(x + h\2, ym + (h\2) * f(x, ym));
  ye = 3 * (exp (2 * xn) - exp (xn);
print f (" %f\t%f\t%f\t%f\t%f\n", xn, ys, yi, ym, ye);
x = xn;
}
getch ( );
}
```

**Remarks**

1. In algorithm, step 3 reads the initial value of y and assumed it as the initial value $y_s$ for Euler's simple method and in step 4 it is assumed as $y_i$ and $y_m$ the initial values for the Euler's improved and modified methods.
2. Steps 5 to 13 compute the values of y by all the three methods and the exact value of y at $x_0 + h$, $x_0 + 2h$, ........., $x_0 + 10h$ and print them in such tabular form that the values can be easily compared.

**Example 6.7**

**Write a computer oriented algorithm and the corresponding C program to solve that differential equation $\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$ at the specified pivotal points, by using Runge-Kutta method of the fourth order.**

**[2073 Shrawan]**

**Solution:**

1. Algorithm 2
2. Define f(x, y) [= R.H.S. function of the differential equation $y' = f(x, y)$]
3. Read x, y, h [the initial values of x, y and the step size h to x]
4. For i = 1 (1) to 10, do till (13)
5. $K_1 \leftarrow hf(x, y)$

6. $K_2 \leftarrow hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$
7. $K_3 \leftarrow hf\left(x+\frac{h}{2}, y+\frac{K_2}{2}\right)$
8. $x \leftarrow x+h$
9. $K_4 \leftarrow hf(x, y+K_3)$
10. $y \leftarrow y+\frac{K_1+2K_2+2K_3+K_4}{6}$
11. $y_m$ ← " the mathematical solution of the equation to be supplied".
12. Write x, y, $y_m$
13. Next i
14. End.

```
|| Program - 2 ||
|| Runge-Kutta Method-Fourth order ||
# include < stdio.h>
# include <conio.h>
# define f(x, y) (3 × x + (y\2))
void main ( )
    { int i;
    float h, K1, K2, K3, K4, x, y, ym;
    clrscr ( );
    printf ("Enter the initial value of x:\n");
    scanf ("%f" & x);
    printf ("Enter the initial value of y:\n");
    scanf ("%f", & y);
    printf ("Enter the initial value of h : \n");
    scanf ("%f", & h);
    prinff (" \in Runge Kutta method-solution of y' = 3x + (y\2)\n");
    printf ("i\tx\ty\tExact y\n");
    for (i = 1; i < 11; i ++)
        { K1 = h * f (x, y);
        K2 = h * f ((x + h/2), (y + K1/2));
        K3 = h * f ((x + h/2), (y + K2/2));
        x = x + h;
        K4 = h * f (x, (y + K3));
        y = y + (K1 + 2K2 + 2K3 + K4)/6;
        ym = 13 * exp (x/2) - 6 * x - 12;
        printf ("%d\t%f\t%f\t%f\n", i, x,y,ym);
    }
```

```
    getch ();
}
```

**Remarks**

1. In algorithm step 3 reads the initial values $x_0$ and $y_0$ and assumes them as x, y.
2. Step 4 to 13 compute the values of y at the points $x_0 + h$, $x_0 + 2h$, ........, $x_0 + 10h$ using the Runge-Kutta method of the fourth order and also the exact values of y. at these points.

**Example 6.8**

**Write a computer oriented algorithm and the corresponding C program to solve the simultaneous differential equations $\frac{dy}{dx} = f(x, y, z)$, $\frac{dz}{dx} = g(x, y, z)$, $y(x_0) = y_0$, $z(x_0) = z_0$ at the specified pivotal points, by using Runge-Kutta method of the fourth order.**

**Solution:**

1. Algorithm 3
2. $f(x, y, z) \leftarrow$ 'Function to be supplied'
3. $g(x, y, z) \leftarrow$ 'Function to be supplied'
4. Read $x_0, y_0, z_0, h$
5. For i = 0 (1) to 9, do till (18)
6. $K_1 \leftarrow hf(x_1, y_1, z_1)$
7. $l_1 \leftarrow hg(x_i, y_i, z_i)$
8. $K_2 \leftarrow hf\left(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}, z_i + \frac{l_1}{2}\right)$
9. $l_2 \leftarrow hg\left(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}, z_i + \frac{l_1}{2}\right)$
10. $K_3 \leftarrow hf\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}, z_i + \frac{l_2}{2}\right)$
11. $l_3 \leftarrow hg\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}, z_i + \frac{l_2}{2}\right)$
12. $K_4 \leftarrow hf(x_i + h, y_i + K_3, z_i + l_3)$
13. $l_4 \leftarrow hg(x_i + h, y_i + K_3, z_i + l_3)$
14. $y_{i+1} \leftarrow y_i + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$
15. $z_{i+1} \leftarrow z_i + \frac{l_1 + 2l_2 + 2l_3 + l_4}{6}$
16. $x_{i+1} \leftarrow x_i + h$
17. Write $x_{i+1}, y_{i+1}, z_{i+1}$
18. Next i
19. End.

```
|| program - 3||
|| Runge-Kutta Method-simultaneous equations ||
# include < stdio.h>
# include < conio.h>
# include < math.h>
# define f(x, y, z) (x * z + 1)
# define g(x, y, z) (-x * y)
void main ()
{ int i, j;
        float h, K1, K2, K3, K4, l1, l2, l3, l4, x [10], y [10], z [10];
        clrscr ();
        printf ("Enter the initial values of x, y, n : \n");
        scanf ("%f%f%f", & x [0], & y [0], & z [0]);
        printf ("Enter the step size h:\n");
        scanf ("%f", & h);
        printf ("\n Runge Kutta method solution of equations\n");
        printf (y' = xz + 1 and z' = -xy \n");
        printf ("\tx\t\ty\t\tz\t\n");
        for (i = 0; i < 10; i ++)
        { K1 = h * f (x [i], y [i], z [i]);
          l1 = h * g (x [i], y [i], z [i]);
          K2 = h * f ((x [i] + h/2), (y [i] + K1/2), (z [i] + l1/2));
          l2 = h * g ((x [i] + h/2), (y [i] + K1/2), (z [i] + l1/2));
          K3 = h * f ((x [i] + h/2), (y [i] + K2/2), (z [i] + l2/2));
          l3 = h * g ((x [i] + h/2), (y [i] + K2/2), (z [i] + l2/2));
          K4 = h * f ((x [i] + h), (y [i] + K3), (z [i] + l3));
          l4 = h * g ((x [i] + h), (y [i] + K3), (z [i] + l3));
          y [i + 1] = y [i] + (K1 + 2 * K2 + 2 * K3 + K4)/6;
          z [i + 1] = z [i] + (l1 + 2 * l2 + 2 * l3 + l4)/6;
          x [i + 1] = x [i] + h;
          print f("%f\t%f\t%f\n", x [i + 1], y [i + 1], z [i + 1]);
          }
          getch ();
}
```

**Remarks**

1. In algorithm step 6 to 15 are the usual steps in the Runge-Kutta formula of fourth order for the solution of simultaneous equations.
2. Step 5 to 18 compute the values of $y_1$, $y_2$, ........., $y_{10}$ and $z_1$, $z_2$, ........., $z_{10}$ and print them.

### Runge–Kutta Formula for the Solution of a Second Order Differential Equation

The Runge-Kutta formulas for the solution of the second order differential equation of the form $y'' = (x, y, y'); y(x_0) = y_0, y'(x_0) = y'_0$.

i) $y(x+h) = y(x) + h\left[y'(x) + \frac{1}{6}(K_1 + K_2 + K_3)\right]$ and

ii) $y'(x+h) = y'(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

where, $K_1 = hf(x, y, y')$

$$K_2 \leftarrow hf\left(x + \frac{h}{2}, y + \frac{h}{2}y' + \frac{h}{8}K_1, y' + \frac{K_1}{2}\right)$$

$$K_3 \leftarrow hf\left(x + \frac{h}{2}, y + \frac{h}{2}y' + \frac{h}{8}K_1, y' + \frac{K_2}{2}\right)$$

and, $K_4 \leftarrow hf\left(x + h, y + hy' + \frac{h}{2}K_3, y' + K_3\right)$

As, the above formulas are difficult to remember and apply in problems, usually a second order differential equation is converted into a pair of simultaneous differential equations of the first order and then Runge-Kutta formulas for the solution of simultaneous equations are applied to solve the given problems.

**Example 6.9**

Solve the equation $\frac{d^2y}{dx^2} = xy^2$ and $y(0) = 1, y'(0) = 0$ for $y(0.2)$ by Runge-Kutta method of the fourth order.

**Solution:**

Here, in this problem, $f(x, y, y') = xy^2$

For f(0.2);

$x = 0, y = 1, y' = 0$

Take $h = 0.2$;

We haves form Runge-Kutta fourth order formula for second order differential equation is;

$$y(x+h) = y(x) + h\left[y'(x) + \frac{1}{6}(K_1 + K_2 + K_3)\right] \quad (1)$$

and, $$y'(x+h) = y'(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \quad (2)$$

where, $K_1 = hf(x, y, y') = 0.2f(0, 1, 0) = 0.2 \times 0 \times (1)^2 = 0$

$[\because f(x, y, y') = xy^2]$

$$K_2 = hf\left(x + \frac{h}{2}, y + \frac{h}{2}y' + \frac{h}{8}K_1, y' + \frac{K_1}{2}\right)$$

$$= 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2} \times 0 + \frac{0.2}{8} \times 0, 0 + \frac{0}{2}\right)$$

$$= 0.2f(0.1, 1 + 0) = 0.2 \times 0.1 \times (1)^2 = 0.02$$

$$K_3 = hf\left(x + \frac{h}{2}, y + \frac{h}{2}y' + \frac{h}{8}K_1, y' + \frac{K_2}{2}\right)$$

$$= 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2} \times 0 + \frac{0.2}{8} \times 0, 0 + \frac{0.2}{2}\right)$$

$$= 0.2f(0.1, 1 + 0.01) = 0.2 \times 0.1 \times (1)^2 = 0.02$$

and, $K_4 = hf\left(x + h, y + hy' + \frac{h}{2}K_3, y' + K_3\right)$

$$= 0.2f\left(0 + 0.2, 1 + 0.2 \times 0 + \frac{0.2}{8} \times 0.2, 0 + 0.2\right)$$

$$= 0.2f(0.2, 1.002 + 0.02) = 0.2 \times 0.2 \times (1.002)^2 = 0.04016$$

Now, putting these values of $K_1$, $K_2$, $K_3$ and $K_4$ in (1) and (2); we have,

$$y(0 + 0.2) = y(0) + 0.2\left[y'(0) + \frac{1}{6}(K_1 + K_2 + K_3)\right]$$

or, $y(0.2) = 1 + 0.2\left[0 + \frac{1}{6}(0 + 0.02 + 0.02)\right]$

∴ $y(0.2) = 1.0013$

and, $y'(0 + 0.2) = y'(0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

or, $y'(0.2) = 0 + \frac{1}{6}(0 + 2 \times 0.02 + 2 \times 0.02 + 0.04016)$

∴ $y'(0.2) = 0.02$

Now, again to find $y(0.4)$

Take $x = 0.2$, $y = y(0.2) = 1.0013$, $y' = y'(0.2) = 0.02$ and $h = 0.2$

Then,

$$K_1 = hf(x, y, y') = 0.2f(0.2, 1.0013, 0.02) = 0.2 \times 0.2 \times (1.0013)^2$$

$$= 0.04010$$

$$K_2 = hf\left(x + \frac{h}{2}, y + \frac{h}{2}y' + \frac{h}{8}K_1, y' + \frac{K_1}{2}\right)$$

$$= 0.2f\left(0.2 + \frac{0.2}{2}, 1.0013 + \frac{0.2}{2} \times 0.02 + \frac{0.2}{8} \times 0.04013, 0.02 + \frac{0.04010}{2}\right)$$

$$= 0.2f(0.3, 1.00430 + 0.04005) = 0.2 \times 0.3 \times (1.0043)^2 = 0.06052$$

$$K_3 = hf\left(x + \frac{h}{2}, y + \frac{h}{2}y' + \frac{h}{8}K_1, y' + \frac{K_2}{2}\right)$$

$$= 0.2f\left(0.2 + \frac{0.2}{2}, 1.0013 + \frac{0.2}{2} \times 0.02 + \frac{0.2}{8} \times 0.04010, 0.02 + \frac{0.06052}{2}\right)$$

$$= 0.2f(0.3, 1.0043 + 0.05026) = 0.2 \times 0.3 \times (1.0043)^2 = 0.06052$$

Here, we are not interested in $y'(0.4)$ so, value of $K_4$ is not required.

Now,

$$y(0.2 + 0.2) = y(0.2) + 0.2\left[0.02 + \frac{1}{6}(0.04010 + 0.06052 + 0.06052)\right]$$

or, $y(0.4) = 1.0013 + 0.009371$

∴ $y(0.4) = 1.0107$

## Example 6.10

**Write a computer oriented algorithm and corresponding c program to solve the second order differential equation $\frac{d^2y}{dx^2} = f(x, y, y'),\ y(x_0) = y_0,\ y'(x_0) = y'_0$ at the specified pivotal points, by using Runge-Kutta method of the Fourth-order.**

**Solution:**

1. Algorithm 4
2. $f(x, y, P) \leftarrow$ 'Function to be supplied'
3. Read $x_0, y_0, P_0, h\ [P_0 = y'_0]$
4. For $i = 0\ (1)$ to $(9)$ do till $(13)$
5. $K_1 \leftarrow hf(x_i, y_i, P_i)$
6. $K_2 \leftarrow hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}P_i + \frac{h}{8}K_1, P_i + \frac{K_1}{2}\right)$
7. $K_3 \leftarrow hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}P_i + \frac{h}{8}K_1, P_i + \frac{K_2}{2}\right)$
8. $K_4 \leftarrow hf\left(x_i + h, y_i + hP_i + \frac{h}{2}K_3, P_i + K_3\right)$
9. $y_{i+1} \leftarrow y_i + h\left[P_i + \left(\frac{K_1 + K_2 + K_3}{6}\right)\right]$
10. $P_{i+1} \leftarrow P_i + \left[P_i + \left(\frac{K_1 + 2K_2 + 2K_3 + K_4}{6}\right)\right]$
11. $x_{i+1} \leftarrow x_i + h$
12. Write $x_{i+1}, y_{i+1}, P_{i+1}$
13. Next i
14. End.

```
|| Program - 4 ||
|| Runge-Kutta Method- Second order equation
# include < stdio.h>
# include < math.h>
# define f (x, y, P) (pow (x, 2) * P + 2 * x * y + 1)
void main ( )
{ int i;
    float h, K1, K2, K3, K4, x [10], y [10], P [10];
    clrscr ( );
    printf ("Enter the initial values of x, y, P (= y') : \n";
    scanf ("%f%f%f", & x [0], & y [0], & P [0]);
    printf ("Enter the step size h : \n");
    scanf ("%f", & h);
    printf ("\n");
```

```
        printf ("Runge Kutta method-SECOND ORDER EQUATION \n");
        printf ("solution of the equation y" = (x * x)y' + 2xy + L\n");
        printf ("x\t\t  y\t\t  P\n");
        for (i = 0; i < 10; i ++)
        { K1 = h * f (x [i], y [i], P [i]);
          K2 = h * f ((x [i] + h/2), (y [i] + ((h/2) * P [i]), + (h/8) * K1),
                                                            (P [i] + K1/2));
          K3 = h * f ((x [i] + h/2), (y [i] + ((h/2) * P [i]) + (h/8) * K1),
                                                            (P [i] + K2/2));
          K4 = h * f ((x [i] + h), (y [i] + h * P [i] + (h/2) * K3), (P [i] + K2));
          y [i + 1] = y [i] + h (P [i]) + (K1 + K2 + K3)/6);
          P [i + 1] = P [i] + (K1 + 2 * K2 + 2 * K3 + K4)/6;
          x [i + 1] = x [i] + h;
          printf ("% %f\t%f\t%f\n", x [i + 1], y [i + 1], P [i + 1]);
          }
          getch ( );
}
```

**Remarks**

1. In algorithm, steps 5 to 10 are the usual steps in the Runge-Kutta formula of the fourth order for the solution of a second order equation.
2. Steps 4 to 13 compute the values of $y_1, y_2, \ldots\ldots\ldots, y_{10}$ and $P_1, P_2, \ldots\ldots\ldots, P_{10}$ and print them.

## 6.3 Solution of Boundary Value Problem by Finite Difference Method and Shooting Method

### Boundary Value Problem

When the differential equation is to be solved satisfying the conditions specified at the end points of an interval, the problem is called a boundary value problem. Boundary value problems for ordinary differential equations occur in many branches of science and engineering. In many practical situations, analytical solutions are not available for boundary value problems and hence we resort to numerical methods.

We shall discuss two-point linear boundary value problems of the following types:

i) $\dfrac{d^2y}{dx^2} + \lambda(x)\left(\dfrac{dy}{dx}\right) + \mu(x)y = \gamma(x)$ with conditions $y(x_0) = a,\ y(x_n) = b$

ii) $\dfrac{d^4y}{dx^4} + \lambda(x)y = \mu(x)y$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$

There exist two numerical methods for solving such Boundary value problems. The first one is known as the "finite difference method" which makes use of finite difference equivalents of derivatives. The second is called the "Shutting method" which makes use of the techniques for solving initial value problems.

The finite-difference approximations to the various derivations are derived as under:

It $y(x_n)$ and its derivatives are single-valued continuous functions of x then by Taylor's expression; we have,

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots\ldots \quad (1)$$

and, $$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \ldots\ldots \quad (2)$$

Equation (1) gives;

$$y'(x) = \frac{1}{h}[y(x+h) - y(x)] - \frac{h}{2}y''(x) - \ldots\ldots$$

*i.e.,,* $$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + 0(h);$$

which is the forward difference approximation of $y'(x)$ with an error of the order h.

Similarly, (2) gives:

$$y'(x) = \frac{1}{h}[y(x) - y(x-h)] + 0(h);$$

which is the backward difference approximation of $y'(x)$ with an error of the order h.

Subtracting (2) from (1); we obtain,

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + 0(h^2);$$

which is the central-difference approximation of $y'(x)$ with an error of the order $h^2$ clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximation and hence should be preferred.

Adding (1) and (2); we get,

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + 0(h^2);$$

which is the central difference approximation of $y''(x)$. Similarly, we can derive central difference approximations to higher derivatives.

Hence, the working expressions for the central difference approximations to the first four derivatives of y are as under:

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$$

$$y_i'' = \frac{1}{h^2}(y_{i+1} + 2y_i - y_{i-1})$$

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 2y_{i-1} + y_{i-2})$$

The accuracy of this method depends on the size of the sub-interval 'h' and also on the order of approximation. As we reduce h, accuracy improves but the number of equations to be solved also increases.

**Example 6.11**

**Determine values of y at the pivotal points of the interval (0, 1) if y satisfies the boundary values problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y'''(0) = y''(1) = 0$. (Take n = 3)**

**Solution:**

From the given; we have to determine the values of y at the pivotal points of the interval (0, 1) and we have to take n = 3

So take,

Interval of x *i.e.*, $h = \frac{1}{3}$

so, $x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = \frac{2}{3}$ and $x_3 = 1$

The corresponding y-values are;

$y_0 (= 0),\ y_1,\ y_2,\ y_3 (= 0)$

Also, from given,

$y'''(0) = 0$ and $y''(1) = 0$

The boundary value problem is;

$y^{iv} + 81y = 81x^2$

Now, replacing $y^{iv}$ by its central difference approximation, (as seen in theory portion),

The differential equation becomes;

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 2y_{i-1} + y_{i-2}) + 81y = 81x_i^2$$

or, $$\frac{1}{\left(\frac{1}{3}\right)^4}(y_{i+2} - 4y_{i+1} + 6y_i - 2y_{i-1} + y_{i-2}) + 81y = 81x_i^2 \qquad \left[\text{As } h = \frac{1}{3}\right]$$

or, $$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2$$

At i = 1;

$$y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = x_1^2 \qquad (1)$$

At i = 2

$$y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = x_2^2 \qquad (2)$$

Using, $y_0 - y_3 = 0$, $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$ in (1) and (2); we have,

$$0 - 4y_2 + 7y_1 - 4 \times 0 + y_{-1} = \frac{1}{9}$$

or, $$-4y_2 + 7y_1 + y_{-1} = \frac{1}{9} \quad (3)$$

Also,

$$y_4 - 0 + 7y_2 + 4y_1 + 0 = \frac{4}{9}$$

or, $$y_4 + 7y_2 + 4y_1 = \frac{4}{9} \quad (4)$$

Regarding the conditions $y_0'' = y_3'' = 0$, we know that;

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$

$$y_i'' = \frac{1}{\left(\frac{1}{3}\right)^2}(y_1 - 2y_0 + y_{-1})$$

$$0 = 9(y_1 - 2 \times 0 + y_{-1}) \quad \left[\begin{array}{ll} \because & y_0'' = 0 \\ \text{and,} & y_0 = 0 \end{array}\right]$$

or, $$y_{-1} = -y_1 \quad (5)$$

At $i = 3$

$$y_3'' = \frac{1}{\left(\frac{1}{3}\right)^2}(y_4 - 2y_3 + y_2)$$

or, $$0 = 9(y_4 - 2 \times 0 + y_2) \quad \left[\begin{array}{ll} \because & y_3'' = 0 \\ \text{and,} & y_3 = 0 \end{array}\right]$$

or, $$y_4 = -y_2 \quad (6)$$

Using (5) in (3); the equation becomes,

$$-4y_2 + 7y_1 + (-y_1) = \frac{1}{9}$$

or, $$-4y_2 + 6y_1 = \frac{1}{9} \quad (7)$$

Also, using (6) in (4) the equation becomes;

$$-y_2 + 7y_2 - 4y_1 = \frac{4}{9}$$

or, $$6y_2 - 4y_1 = \frac{4}{9} \quad (8)$$

Solving (7) and (8); we get,

$$y_2 = \frac{7}{45}$$

and, $$y_1 = \frac{11}{90}$$

$$\therefore \quad y_1 = y\left(\frac{1}{3}\right) = \frac{11}{90} = 0.1222$$

and, $$y_2 = y\left(\frac{2}{3}\right) = \frac{7}{45} = 0.1556$$

**Example 6.12**

**The deflection of a beam is governed by the equation $\frac{d^4y}{dx^4} + 81y = \phi(x)$; where, $\phi(x)$ is given by the table:**

| x | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 |
|---|---|---|---|
| $\phi(x)$ | 81 | 162 | 243 |

**and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.**

**Solution:**

Here, from the given boundary condition:

$$y(0) = y'(0) = y''(1) = y'''(1) = 0$$

Since, we have to use three

Sub-intervals so take $h = \frac{1}{3}$

and, the pivotal points are $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$ and $x_3 = 1$

The corresponding y-values are;

$$y_0(= 0), y_1, y_2, y_3$$

The given differential equation is;

$$\frac{d^4y}{dx^4} + 81y = \phi(x)$$

Replacing $\frac{d^4y}{dx^4}$ *i.e.,,* $y^{iv}$ its central difference approximation, the differential equation becomes;

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 2y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At i = 1

$$\frac{1}{h^4}(y_3 - 4y_2 + 6y_i - 4y_0 + y_{-1}) + 81y_1 = \phi(x_1)$$

Now, using $x_1 = \frac{1}{3}$, $y_0 = 0$ and $h = \frac{1}{3}$

$$\frac{1}{\left(\frac{1}{3}\right)^4}(y_3 - 4y_2 + 6y_i - 4 \times 0 + y_{-1}) + 81y_1 = \phi\left(\frac{1}{3}\right)$$

or, $\quad y_3 - 4y_2 + 6y_i + y_{-1} + y_1 = 81 \times \frac{1}{81}$ $\quad \left[\because \phi\left(\frac{1}{3}\right) = 81\right]$

or, $\quad y_3 - 4y_2 + 7y_i + y_{-1} = 1$ $\quad$ (1)

At i = 2

$$\frac{1}{h^4}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 81y_2 = \phi(x_2)$$

Now, using $h = \frac{1}{3}, y_0 = 0$

$$x_2 = \frac{2}{3}$$

and, $\phi\left(\frac{2}{3}\right) = 162$

$$\frac{1}{\left(\frac{1}{3}\right)^4}(y_4 - 4y_3 + 6y_2 - 4y_1 + 0) + 81y_2 = 162$$

or, $y_4 - 4y_3 + 6y_2 - 4y_1 + y_2 = 2$

or, $y_4 - 4y_3 + 7y_2 - 4y_1 = 2$ (2)

At i = 3

$$\frac{1}{h^4}(y_5 - 4y_4 + 6y_3 - 4y_2 + y_1) + 81y_3 = \phi(x_3)$$

Using $h = \frac{1}{3}, x_3 = 1, \phi(1) = 243$

$$\frac{1}{\left(\frac{1}{3}\right)^4}(y_5 - 4y_4 + 6y_3 - 4y_2 + y_1) + 81y_3 = 243$$

or, $y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 + y_3 = 3$

or, $y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3$ (3)

Since, $y_i = \frac{1}{2h}(y_{i+1} - y_{i-1})$

For i = 0;

$$y'_0 = \frac{1}{2h}(y_1 - y_{-1})$$

But, $y'_0 = 0$;

so, $0 = \frac{1}{2h}(y_1 - y_{-1})$

or, $y_{-1} = y_1$ (4)

Also,

$$y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

For i = 3

$$y''_3 = \frac{1}{h^2}(y_4 - 2y_3 + y_2)$$

But, $y''_3 = 0$;

so, $0 = \frac{1}{h^2}(y_4 - 2y_3 + y_2)$

or, $y_4 = 2y_3 - y_2$ (5)

Also,

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$$

For i = 3

$$y_3''' = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$$

But, $y_3'' = 0;$

so, $0 = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$

or, $y_5 = 2y_4 - 2y_2 + y_1$

or, $y_5 = 2(2y_3 - y_2) - 2y_2 + y_1$ [From (5)]

or, $y_5 = 4y_3 - 2y_2 - 2y_2 + y_1$

$\therefore$ $y_5 = 4y_3 - 4y_2 + y_1$ (6)

From (1) and (4); we have,

$$4y_3 - 4y_2 + 7y_1 + y_1 = 1$$

$\therefore$ $y_3 - 4y_2 + 8y_1 = 1$ (7)

From (2) and (5); we have,

$$2y_3 - y_2 + 4y_3 + 7y_2 - 4y_1 = 2$$

or, $-2y_3 + 6y_2 - 4y_1 = 2$ (8)

Again, from (3) and (6); we have,

$$4y_3 - 4y_2 + y_1 - 4(2y_3 - y_2) + 7y_3 - 4y_2 + y_1 = 3$$

or, $3y_3 - 4y_2 + 2y_1 = 3$ (9)

Solving (7), (8) and (9); we have,

$$y_3 = \frac{37}{13}$$

$$y_2 = \frac{22}{13}$$

and, $y_1 = \frac{8}{13}$

Hence,

$$y_1 = y\left(\frac{1}{3}\right) = \frac{8}{13} = 0.6154$$

$$y_2 = y\left(\frac{2}{3}\right) = \frac{22}{13} = 1.6923$$

and, $y_3 = y(1) = \frac{37}{13} = 2.8462$

## Shooting Method

In this method, the given boundary value problem is first transformed to an initial value problem. Then this initial value problem is solved by Taylor's series method or Runge-Kutta method, etc. Finally the given boundary value problem is solved. The approach in this method is quite simple.

Consider the boundary value problem:

$$y''(x) = y(x),\ y(x) = A,\ y(b) = B \tag{1}$$

One condition is $y(a) = A$ and let us assume that $y'(a) = m$ which represent the slope. We start with two initial guesses for m, and then find the corresponding value of $y(b)$ using any initial value method.

Let the two guesses be $m_0$, $m_1$ so that the corresponding values of $y(b)$ are $y(m_0, b)$ and $y(m_1, b)$. Assuming that the values of m and $y(b)$ are linearly related, we obtain a better approximation $m_2$ for m from the relation.

$$\frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

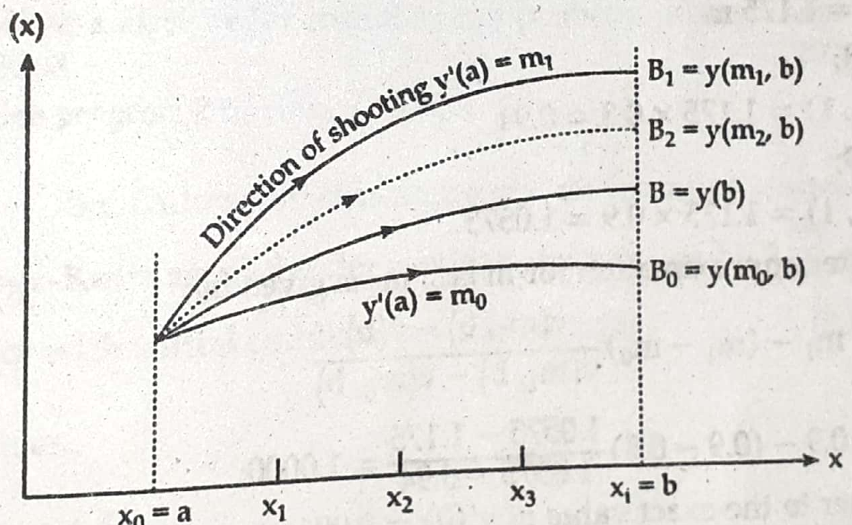
or, $$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, b) - y(b)}{y(m_1, b) - y(m_0, b)} \tag{2}$$

We now solve the initial value problem:

$$y''(x) = y(x),\ y(a) = A,\ y'(a) = m_2$$

and, obtain the solution $y(m_2, b)$

To obtain a better approximation $m_3$ for m, we again use the linear relation (2) with $[m_1, y(m_1, b)]$ and $[m_2, y(m_2, b)]$. This process is repeated until the value of $y(m_1, b)$ agrees with $y(b)$ to desired accuracy.




This method resembles an artillery problem and as such is called the shooting method. The speed of convergence in this method depends on our initial choice of two guesses for m. However, the shooting method is quite slow in practice. Also this method is quite tedious to apply to higher order boundary value problems.

**Example 6.13**

**Using the shooting method, solve the boundary value problem:**

$y''(x) = y(x)$, $y(0) = 0$ and $y(1) = 1.17$

**Solution:**

From the given;

$$y'''(x) = y(x), y(0) = 0 \text{ and } y(1) = 1.17$$

Let the initial guesses for $y'(0) = m$ be $m_0 = 0.8$ and $m_1 = 0.9$.

Then, $y'''(x) = y(x), y(0) = 0$ gives;

$$y'(0) = m$$
$$y''(0) = y(0) = 0$$
$$y'''(0) = y'(0) = m$$
$$y^{iv}(0) = y''(0) = 0$$
$$y^{v}(0) = y'''(0) = m$$
$$y^{iv}(0) = y^{iv}(0) = 0$$

and so on.

Putting these values in the Taylor's series; we have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots\ldots\ldots$$

$$= 0 + x \times m + \frac{x^2}{2} \times 0 + \frac{x^3}{6} \times m + 0 + \frac{x^5}{120} \times m + 0 + \frac{x^7}{5040} + \ldots\ldots$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots\ldots\ldots\right)$$

Putting $x = h$

$$y(1) = m(1 + 0.1667 + 0.00833 + 0.000198 + \ldots\ldots\ldots)$$

$$\therefore \quad y(1) = 1.175\,m$$

For $m_0 = 0.8$;

$$y(m_0, 1) = 1.175 \times 0.8 = 0.94$$

For $m_0 = 0.9$;

$$y(m_1, 1) = 1.175 \times 0.9 = 1.0575$$

Hence, a better approximation for m *i.e.,,* $m_2$ is given by;

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, b) - y(b)}{y(m_1, b) - y(m_0, b)}$$

or, $$m_2 = 0.9 - (0.9 - 0.8)\frac{1.0575 - 1.175}{1.0575 - 0.94} = 1.0000;$$

which is closer to the exact value of $y'(0) = 0.996$.

Now, we solve the initial value problem:

$$y''(x) = y(x), y(0) = 0 \text{ and } y'(0) = m_2$$

Taylor's series solution is given by;

$$y(m_2, 1) = 1.175m_2 = 1.1759$$

Hence, the solution at $x = 1$ is $y = 1.176$, which is close to the exact value of $y(1) = 1.17$.

## 6.4 EXAMINATION PROBLEMS

1. **Write pseudo-code to solve an initial value problem (first order ordinary differential equation) using the Range-Kutta fourth order method.** [2071, Chaitra]

Solution: See program 2 of theory part

2. **Write pseudo-code to solve a first order differential equation using Euler's method.** [2072 Chaitra]

Solution: See program 1 of theory portion of this chapter

3. **Write a program in C / C + + / FORTRAN to solve a second order ordinary differential equation (initial value problem) using the Runge-Kutta fourth order method.** [2073 Magh]

Solution: See program 4 of theory part

4. **Write pseudo-code to solve a first order differential equation using R–K4 method.** [2073 Shrawan]

Solution: See program 2 of theory portion

5. **Write a program in any high level language (C / C ++ / FORTRAN) to solve a first order initial value problem using classical RK – 4 method.** [2073 Shrawan]

Solution: See program 2 of theory portion

6. **Solve the following simultaneous differential equations using Runge-Kutta second order method at x = 0.1 and 0.2 $\frac{dy}{dx} = xz + 1$; $\frac{dz}{dx} = -xy$ with initial conditions y (0) = 0, z (0) = 1** [2073 Bhadra]

Solution:

From the given,

$$\frac{dy}{dx} = xz + 1$$

$$\frac{dz}{dx} = -xy$$

With initial conditions, y (0) = 0 and z (0) = 1

i.e., $x_0 = 0,\ y_0 = z_0 = 1$

We have to find y(0.1), y(0.2), z(0.1) and z(0.2)

Assume,

$$\frac{dy}{dx} = f_1(x, y, z) = xz + 1$$

and, $$\frac{dz}{dx} = f_2(x, y, z) = -xy$$

**By using Runge-Kutta second order formula;**

$$y_1(x) - y_0(x) + \frac{1}{2}(k_1 + k_2)$$

and, $z_1(x) = z_0(x) + \frac{1}{2}(\ell_1 + \ell_2)$

where, $k_1 = h.f_1(x, y, z)$

$\ell_1 = h.f_2(x_0, y_0, z_z)$

and, $k_2 = h.f_1(x_0, h_0, y_0 + k_1, z_0 + \ell_1)$

$\ell_2 = h.f_2(x_0, h_1, y_0 + k_1, z_0 + \ell_1)$

For y(0.1) and z(0.1)

Take, h = 0.1

$k_1\ h \times f_1(x_0, y_0, z_0)$

$= 0.1 \times f_1(0, 0, 1)$

$= 0.1 \times (0 \times 1 + 1)$ $\quad [\because f_1(x_0, y_0, z_0) = xz + 1]$

$\therefore \quad k_1 = 0.1$

$\ell_1 = h \times f_2(x_0, y_0, z_0)$

$= 0.1 \times f_2(0, 0, 1)$

$= 0.1 \times (-0 \times 0)$ $\quad [\because f_2(x_0, y_0, z_0) = -xy]$

$\therefore \quad \ell_1 = 0$

$k_2 = hf_2(x_0 + h_1 y_0 + k_1, z_0 + \ell_1)$

$= 0.1 \times f_1(0 + 0.4, 0 + 0.1, 1 + 0)$

$k_2 = 0.1 \times f_1(0.1, 0.1, 1)$

$k_2 = 0.1 \times (0.1 \times 1 + 1]$

$\therefore \quad x_2 = 0.11$

$\ell_2 = hf_2(x_0 + h, y_0 + k_1, z_1 + \ell_1]$

$= 0.1 \times f_2(0 + 0.1, 0 + 0.1, 1 + 0$

$= 0.1 \times f_2(0.1, 0.1, 1)$

$= 0.1 \times (-0.1 \times 0.1)$

$\therefore \quad \ell_2 = -0.001$

$\therefore \quad y(0.1) = y(0) + \frac{1}{2}(k_1 + k_2) = 0 + (1,2)(0.1 + 0.11)$

$\therefore \quad y(0.1) = 0.105$

Also,

$$z(0.1) = z(0) + \frac{1}{2}(\ell_1 + \ell_2) = 1 + \frac{1}{2}(0 - 0.001)$$

$\therefore \quad z(0.1) = 0.995$

Again, for y(0.2) and z(0.2)

Take, $y_0 = 0.1$, $y_0 = 0.105$, $z_0 = 0.9995$

Also, take h = 0.1;

$$k_1 = h \times f_1 (x_0, y_0, z_0)$$
$$= 0.1 \times f_1 (0.1, 0.105, 0.9995$$
$$= 0.1 \times (0.1 \times 0.9995 + 1)$$

$\therefore$ $k_1 = 0.2099$

$$\ell_1 = h \times f_2 (x_0, y_0, z_0)$$
$$= 0.1 \times f_2 (0.1, 0.105, 0.9995)$$
$$= 0.1 \times (-0.1 \times 0.105$$

$\therefore$ $\ell_1 = 0.00105$

$$k_2 = h \times f_1 (x_0 + h, y_0 + k_1, z_0 + \ell_1)$$
$$= 0.1 \times f_1 (0.1 + 0.1, 0.105 + 0.2099, 0.9995 - 0.00105)$$
$$= 0.1 \times f_1 (0.2, 0.3149, 0.9984)$$
$$= 0.1 \times (0.2 \times 0.9984 + 1)$$

$\therefore$ $k_2 = 0.1199$

Also,

$$\ell_2 = h \times f_2 (x_0 + h, y_0 + k_1, z_0 + \ell_1)$$
$$= 0.1 \times f_2 (0.1 + 0.1, 0.105 + 0.2099, 0.9995 - 0.00105)$$
$$= 0.1 \times f_2 (0.2, 0.3149, 0.9984)$$
$$= 0.1 \times (-0.2 \times 0.3149)$$

$\therefore$ $\ell_2 = -0.006298$

Now,

$$y(0.2) = 0.105 + \frac{1}{2}(k_1 + k_2) = 0.105 + \frac{1}{2}(0.2099 + 0.1199)$$

$\therefore$ $y(0.2) = 0.2699$

and, $z(0.2) = 0.9995 + \frac{1}{2}(\ell_1 + \ell_2) = 0.9995 + \frac{1}{2}(0.00105 + (-0.006298)$

$\therefore$ $z(0.2) = 0.9958$

7. Solve the following boundary value problem using the finite difference method by dividing the interval into four sub-intervals. [2074, Bhadra]

$y'' = e^x + 2y' - y$; $y(0) = 1.5$ and $y(20) = 2.5$

**Solution:**

From the given,

$$y'' = e^x + 2y' - y;\ y(0) = 1.5 \text{ and } y(20) = 2.5$$

Since, we have to solve the given boundary value problem by finite difference method by dividing the interval into four sub-interval

i.e., interval is (0, 2) so take n = 4

$$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1.0,\ x_3 = 1.5 \text{ and } x_4 = 2$$

The corresponding y = values are;

$y_0$ (= 1.5), $y_1$, $y_2$, $y_3$, $y_4$ (= 2.5)

Now, the boundary value problem is;

$$y'' = e^x + 2y' - y \quad (1)$$

Replacing y'' and y' by its central difference approximate (from theory)

The differential equation becomes,

$$\frac{1}{h^2}(y_{i+1} - 2y_{i-1}) = e^{xi} + 2\left\{\frac{1}{2h}(y_{i+1} - y_{i-1})\right\} - y_i$$

as, h = 0.5

so, $\frac{1}{(0.5)^2}(y_{i+1} - 2y_i + y_{i-1}) = e^x + \frac{1}{0.5}(y_{i+1} - y_{i-1}) - y_i$

$$4y_{i+1} - 8y_i + 4y_{i-1} = e^x + 2y_{i+1} - 2y_{i-1} - y_i$$

or, $2y_{i+1} - 7y_{i-1} + 6y_{i-1}\ e^{xi}$ (2)

At i = 1 in equation (2); we have,

$$2y_2 - 7y_1 + 6y_0 = e^{x1}$$

since, $y_0 = 1.5$ and $x_1 = 0.5$

$$2y_2 - 7y_1 + 6 \times 1.5 = e^{0.5}$$

or, $2y_2\ 7y_1 = -7.3512$ (3)

Again, at i = 2 in equation (2); we have,

$$2y_3 - 7y_2 + 6y_1 = e^{x2}$$

since, $x_2 = 1.0$

so, $2y_3 - 7y_2 + 6y_1 = e^1$

or, $2y_3 - 7y_2 + 6y_1 = 2.7183$ (4)

Again, at i = 3 in equation (2); we have,

$$2y_4 - 7y_3 + 6y_2 = e^{x3}$$

since, $y_4 = 2.5$ and $x_3 = 1.5$

so, $2 \times 2.5 - 7y_3 + 6y_2 = 3^{1.5}$

or, $-7y_3 + 6y_2 = -0.5183$ (5)

Now, solving equation (3), (4) and (5); we get,

$y_3 = 0.9694$

$y_0 = 1.0446$

and, $y_1 = 1.3486$

$\therefore$ $y(0.5) = y_1 = 1.3486$

$y(1.0) = y_2 = 1.0446$

and, $y(1.5) = y_3 = 0.9694$

8. Solve $y' = 4e^{0.8x} - 0.5y$; subject to initial condition y(0) = for y(0.5) and y(1.0) using Runge-Kutta 2nd order method. [2074 Bhadra]

**Solution:**

From the given;

$$y' = 4e^{0.8x} - 0.5y$$

where, $y(0) = 2$

and, we have to find $y(0.5)$ and $y(1.0)$

Take, $h = 0.5$

Then, from Runge-Kutta 2nd order formula

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) \qquad (1)$$

where, $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$

For $y(0.5)$ $x_0 = 0$ and $y_0 = 2$

$k_1 = hf(x_0, y_0) = 0.5 \times f(0, 2) = 0.5 \times (4e^{0.8 \times 0} - 0.5 \times 2)$

$\therefore$ $k_1 = 1.5$

Again,

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.5 \times f(0 + .5, 2 + 1.5) = 0.5 \times f(0.5, 3.5)$

$= 0.5 \times (4e^{0.8 \times 0.5} - 0.5 \times 3.5)$

or, $k_2 = 2.1086$

so, $y_1 = y(0.5) = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(1.5 + 2.1086)$

$\therefore$ $y(0.5) = 3.8043$

Again, for $y(1.0)$

Taken, $x_0 = 0.5$ and $y_0 = 3.8043$

With $h = 0.5$

Now,

$k_1 = hf(x_0, y_0) = 0.5 \times f(0.5, 3.8043) = 0.5 \times (4e^{0.8 \times 0.5} - 0.5 \times 3.8043)$

$\therefore$ $x_1 = 2.0325$

Also,

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.5 \times f(0.5 + 0.5, 3.8043 + 2.0625$

$= 0.5 \times f(1.0, 5.8368)$

$= 0.5 \times (4e^{0.8 \times 1.0} - 0.5 \times 5.8368)$

$\therefore$ $k_2 = 2.9919$

so, $y_2 = y(1.0) = y_0 + \frac{1}{2}(k_1 + k_2) = 3.8043 + \frac{1}{2}(2.0325 + 2.9919)$

$\therefore$ $y(1.0) = 6.3165$

9. **Solve the following boundary value problem using shooting method employing Euler's formula taking a step size of 0.25. $y'' = x - y + y'$ subject to boundary conditions $y(0) = 2$ and $y(1) = 3$** **[2074 Ashwin]**

**Solution:**

Let $y' = z$ then the given equation becomes;

$$z' = x - y + z$$

and, $y' = z$

With conditions;

$$y(0) = 2$$

and, $y(1) = 3$

| x | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| y | 2 | $y_1$ | $y_2$ | $y_3$ | 3 |

**Step 1**

$x_0 = 0$

$y_0 = 2$

$h = 0.25$

$z_0 = 2$ (let)

$y_{n+1} = y_n + hz_n$ $[\because z_n = y']$

and, $z_{n+1} = z_n + hz'$ $[\because z' = x - y + z]$

$y_1 = y_0 + hz_0 = 2 + 0.25 \times 2 = 2.5$

and, $z_1 = z_0 + h(x_0 - y_0 + z_0) = 2 + 0.25 \times (0 - 2 + 2) = 2$

$y_2 = y_1 + hz_1 = 2.5 + 0.25 \times 2 = 3$

and, $z_2 = z_1 + h(x_1 - y_1 + z_1) = 2 + 0.25 \times (0.25 - 2.5 + 2) = 1.9375$

$y_3 = y_2 + hz_2 = 3 + 0.25 \times 1.9375 = 3.4844$

and, $z_3 = z_2 + h(x_2 - y_2 + z_2) = 1.9375 + 0.25 \times (0.25 - 3 + 1.9375)$

$= 1.7969$

$y_4 = y_3 + hz_3 = 3.4844 + 0.25 \times 1.7969 = 3.9336$

$\therefore$ $y_4 = 3.9336 > 3$

**Step 2**

$x_0 = 0, y_0 = 2, h = 0.25, z_0 = 1.4$ (let)

$y_1 = y_0 + hz_0 = 2 + 0.25 \times 1.4 = 2.35$

and, $z_1 = z_0 + h(x_0 - y_0 + z_0) = 1.4 + 0.25 \times (0 - 2 + 1.4) = 1.25$

$y_2 = y_1 + hz_1 = 2.35 + 0.25 \times 1.25 = 2.6625$

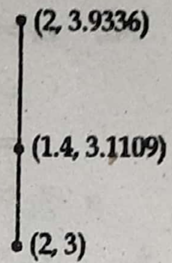
and, $z_2 = z_1 + h(x_1 - y_1 + z_1) = 1.25 + 0.25 \times (0.25 - 2.35 + 1.4) = 1.0375$

$y_3 = y_2 + hz_2 = 2.6625 + 0.25 \times 1.0375 = 2.9219$

and, $z_3 = z_2 + h(x_2 - y_2 + z_2) = 1.0375 + 0.25 \times (0.5 - 2.6625 + 1.0375)$
$= 0.7563$
$y_4 = y_3 + hz_3 = 2.9219 + 0.25 \times 0.7563 = 3.1109$
$\therefore \quad y_4 = 3.1109 > 3$

**Step 3**

Let (2, 3.9336), (1.4, 3.1109) and (2, 3) be collinear points.




$$\frac{z-2}{3-3.9336} = \frac{2-1.4}{3.9336-3.1109}$$

$\therefore \quad z = 1.3191$

**Step 4**

$x_0 = 0, y_0 = 2, h = 0.25, z_0 = 1.3191$
$y_1 = y_0 + hz_0 = 2 + 0.25 \times 1.3191 = 2.3298$
and, $z_1 = z_0 + h(x_0 - y_0 + z_0) = 1.3191 + 0.25 \times (0 - 2 + 1.3191) = 1.1489$
$y_2 = y_1 + hz_1 = 2.3298 + 0.25 \times 1.1489 = 2.6170$
$z_2 = z_1 + h(x_1 - y_1 + z_1) = 1.1489 + 0.25 \times (0.25 - 2.3298 + 1.1489)$
$= 0.9162$
$y_3 = y_2 + hz_2 = 2.6170 + 0.25 \times 0.9162 = 2.84605$
$z_3 = z_2 + h(x_2 - y_2 + z_2) = 0.9162 + 0.25 \times (0.5 - 2.6170 + 0.9162)$
$= 0.616$
$y_4 = y_3 + hz_3 = 2.84605 + 0.25 \times 0.616 = 3.00005$

10. Solve the following boundary value problem using shooting method. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$; with $y(1) = 1$, $y(2) = 5$. Taking $h = 0.25$ [2074 Chaitra]

**Solution:**

Let $y' = z$ then above equation will become;

$$z' - 2z + y = e^x \qquad \left[z = \frac{dy}{dx} = y'\right]$$

with $y(1) = 1, y(2) = 5$

| | | | | | |
|---|---|---|---|---|---|
| x | 1 | 1.25 | 1.5 | 1.75 | 2 |
| y | 1 | $y_1$ | $y_2$ | $y_3$ | 5 |

**Step 1**

$x_0 = 1, y_0 = 1, z_0 = 2$ (let)
$y_{n+1} = y_n + hz_n \qquad [y' = z]$
and, $z_{n+1} = z_n + hz' \qquad [z' = 2z - y + e^x]$
$y_1 = y_0 + hz_0 = 1 + 0.25 \times 2 = 1.5$

and, $z_1 = z_0 + h(2z_0 - y_0 + e^x) = 2 + 0.25 \times (2 \times 2 - 1 + e^1) = 3.429$

$y_2 = y_1 + hz_1 = 1.5 + 0.25 \times 3.429 = 2.357$

and, $z_2 = z_1 + h(2z_1 - y_1 + e^x) = 3.429 + 0.25 \times (2 \times 3.429 - 1.5 + e^{1.25})$

$= 5.6419$

$y_3 = y_2 + hz_2 = 2.357 + 0.25 \times 5.6419 = 3.7678$

and, $z_3 = z_2 + h(2z_2 - y_2 + e^{x_2})$

$= 5.6419 + 0.25 \times (2 \times 5.6419 - 2.357 + e^{1.5})$

$= 8.9939$

$y_4 = y_3 + hz_3 = 3.7678 + 0.25 \times 8.9939 = 6.016$

$\therefore$ $y_4 = 6.016 > 5$

**Step 2**

$x_0 = 1, y_0 = 1, z_0 = 1$ (let)

$y_1 = y_0 + hz_0 = 1 + 0.25 \times 1 = 1.25$

and, $z_1 = z_0 + h(2z_0 - y_0 + e^{x_0}) = 1 + 0.25 \times (2 \times 1 - 1 + e^1) = 1.929$

$y_2 = y_1 + hz_1 = 1.25 + 0.25 \times 1.929 = 1.732$

and, $z_2 = z_1 + h(2z_1 - y_1 + e^{x_1}) = 1.929 + 0.25 \times (2 \times 1.929 - 1.25 + e^{1.25})$

$= 3.4544$

$y_3 = y_2 + hz_2 = 1.732 + 0.25 \times 3.454 = 2.596$

and, $z_3 = z_2 + h(2z_2 - y_2 + e^{x_2})$

$= 1.454 + 0.25 \times (2 \times 3.454 - 1.732 + e^{1.5})$

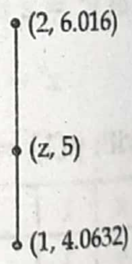
$= 5.8689$

$y_4 = y_3 + hz_3 = 2.596 + 0.25 \times 5.8689 = 4.0632$

$\therefore$ $y_4 = 4.0632 < 5$

**Step 3**

Let (2, 6.016), (1, 4.0632) and (z, 5) be collinear points;




$$\frac{2-z}{2-1} = \frac{6.016 - 5}{6.016 - 4.0632}$$

$\therefore$ $z = 1.4797$

**Step 4**

$x_0 = 1, y_0 = 1, z_0 = 1.4797$

$y_1 = y_0 + hz_0 = 1 + 0.25 \times 1.4797 = 1.3699$

$y_2 = y_1 + hz_1 = 1.3699 + 0.25 \times 2.6491 = 2.0322$

$y_3 = y_2 + hz_2 = 3.1581$

$y_4 = y_3 + hz_3 = 5.00016$

and, $z_1 = z_0 + h(2z_0 - y_0 + e^{x_0}) = 1.4797 + 0.25 \times (2 \times 1.4797 - 1 + e^1)$

$= 1.929$

$z_2 = z_1 + h(2z_1 - y_1 + e^{x_1})$

$= 2.6491 + 0.25 \times (2 \times 2.6491 - 1.3699 + e^{1.25})$

$= 4.5037$

$z_3 = z_2 + h(2z_2 - y_2 + e^{x_2}) = 7.3680$

| | | | | | |
|---|---|---|---|---|---|
| x | 1 | 1.25 | 1.5 | 1.75 | 2 |
| y | 1 | 1.3699 | 2.0322 | 3.1581 | 5 |

**Code in calculator**

$A : B\,C : D = B + 0.25 \times C : E = C + 0.25 \times (2C - B + e^{A}) : A = A + 0.25 : B = D : C = E + \boxed{\text{Calc}}$

**11.** **Write a program in any high level language (C/C++ FORTRAN) to solve the second order differential equation using classical RK-4 method.** **[2074 Chaitra]**

Solution: Proceed same as the solution of Q. no. 3

**12.** **Using finite difference method, solve the following BVP: $y'' - 3y' + 2y = 2$, $y(0) = 1$, $y(1) = 4$ in interval [0, 1]. Take $h = 0.25$** **[2075 Bhadra]**

Solution:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |
|---|---|---|---|---|---|
| x | 0 | 0.25 | 0.5 | 0.75 | 1 |
| y | 1 | $y_2$ | $y_3$ | $y_4$ | 4 |

The given differential equation is;

$$y'' - 3y' + 2y = 2 \quad (1)$$

We have,

$$y'' = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}$$

and, $y' = \dfrac{y_{i+1} - y_{i-1}}{2h}$

Then, equation (1) becomes;

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - 3\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + 2y_i = 2$$

or, $y_{i+1} + y_{i-1} - 2y_i - 1.5h(y_{i+1} - y_{i-1}) + 2h^2 y_i = 2h^2$

or, $(1 - 1.5h)y_{i+1} + (2h^2 - 2)y_i + (1 + 1.5h)y_{i-1} = 2h^2$

We have,

$h = 0.25;$

so, $(1 - 1.5 \times 0.25)y_{i+1} + [2 \times (0.25)^2 - 2]y_i + (1 + 1.5 \times 0.25)y_{i-1} = 2 \times (0.25)^2$

$\therefore \quad 0.625y_{i+1} - 1.875y_i + 1.375y_{i-1} = 0.125$ (2)

i) When i = 1,

$x_1 = 0$

and, $y_1 = 1$

ii) When i = 2;

$x_2 = 0.25$

$0.625y_3 - 1.875y_2 + 1.375y_1 = 0.125$

or, $0.625y_3 - 1.875y_2 = -1.25$ [$\because y_1 = 1$] (3)

iii) When i = 3;

$x_3 = 0.50$

$0.625y_4 - 1.875y_3 + 1.375y_2 = 0.125$ (4)

iv) When i = 4, $x_4 = 0.75$ [$y_5 = 4$]

$0.625y_5 - 1.875y_4 + 1.375y_3 = 0.125$

or, $0.625 \times 4 - 1.875y_4 + 1.375y_3 = 0.125$

or, $-1.875y_4 + 1.375y_3 = -2.375$ (5)

Solving equation (3), (4) and (5); we get,

$y_2 = 1.2174$

$y_3 = 1.6521$

$y_4 = 2.4782$

| | | | | | |
|---|---|---|---|---|---|
| x | 0 | 0.25 | 0.5 | 0.75 | 1 |
| y | 1 | 1.2174 | 1.6521 | 2.4782 | 4 |
| | 1 | 2 | 3 | 4 | 5 |

13. Write a program in any high level language (C/C++ FORTRAN) to solve second order differential equation using classical RK-4 method. [2075 Bhadra]

Solution:

**Source code in C**

```
# include <stdio.h>
   Float F (float x, float y, float z)
   {
      return z;
   }
   Float g (float x, float y, float z)
   {
      return (x * z * z - y * y);
```

```
}
int main ()
{
    Float x0, y0, h, xn, x, y, z, k1, k2, k3, k4, k, l1, l2; l3, l4, l;
Print ("Enter values of x0, y0, z0");
Scanf ("% f % f % f, & x0, y0, z0");
Printf ("Enter the values of xn and h:");
Scanf ("% f % f", & xn, h);
    x = x0;
    y = y0;
    z = z0;
    while (x! = xn)
{
    k1 = h * F (x0, y0, z0);
    l1 = h * g (x0, y0, z0);
    k2 = h * F (x0 + h/2, y0 + k1/2, z0 + l1/2);
    l2 = h * g (x0 + h/2, y0 + k1/2, z0 + l1/2);
    k3 = h * F (x0 + h/2, y0 + k2/2, z0 + l2/2);
    l3 = h * g (x0 + h/2, y0 + k2/2, z0 + l2/2);
    k4 = h * F (x0 + h, y0 + k3, z0 + l3);
    l4 = h * g (x0 + h, y0 + k3, z0 + l3);
    k = (k1 + 2k2 + 2k3 + k4)/6;
    l = (l1 + 2l2 + 2l3 + l4)/6;
    x = x + h;
    y = y + k;
    z = z + l;
Printf ("when x = % 0.4f y = % 0.4f z = % 0.4f", x, y, z);
    }
        return 0;
}
```

**14.** **Using Euler method, solve $\frac{dy}{dx} = \frac{y+x}{y-x}$ with $y = 1$ at $x = 0$ for $x = 0.1, h = 0.02$** **[2075 Chaitra]**

**Solution:**

**Here,**

$x_0 = 0, h = 0.02, y_0 = 1$

| n | x | y | Slope (m) $= \frac{y+x}{y-x}$ | $y_{new} = y_{old} + h \times m$ |
|---|---|---|---|---|
| 0 | 0 | 1 | $m_0 = \frac{1+0}{1-0} = 1$ | $y_1 = y_0 + h \times m_0$ $= 1 + 0.2 \times 1$ $= 1.2$ |
| 1 | 0.2 | 1.2 | $m_1 = \frac{1.2+0.2}{1.2-0.2} = 1.4$ | $y_2 = 1.2 + 1.4 \times 0.2 = 1.48$ |
| 2 | 0.4 | 1.48 | $m_2 = \frac{1.48+0.4}{1.48-0.4} = 1.74$ | $y_3 = 1.48 + 1.74 \times 0.2$ $= 1.828$ |
| 3 | 0.6 | 1.828 | $m_3 = \frac{1.828+0.6}{1.828-0.6} = 1.977$ | $y_4 = 1.828 + 1.977 \times 0.2$ $= 2.223$ |
| 4 | 0.8 | 2.223 | $m_4 = \frac{2.223+0.8}{2.223+0.8} = 2.124$ | $y_5 = 2.223 + 2.124 \times 0.2$ $= 2.6478$ |
| 5 | 1 | 2.6478 | 2.2137 | $y_6 = 2.6478 + 2.2137 \times 0.2$ $= 3.09054$ |

**15.** **Solve the following boundary value problem using finite difference method taking a step size of 0.5. $y'' + 2y' + y = 3x^2$ subject to boundary conditions $y(0) = 5$, $y(2) = 4$** **[2075 Chaitra]**

**Solution:**

The given differential equation is $y'' + 2y' + y = 3x^2$ and $h = 0.5$

| x | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| y | 5 | $y_2$ | $y_3$ | $y_4$ | 4 |

We have,

$$y'' = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}$$

and, $y' = \frac{y_{i+1} - y_{i-1}}{2h}$

Hence, above equation becomes;

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} + 2\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + y_i = 3x_i^2$$

or, $y_{i+1} + y_{i-1} - 2y_i + hy_{i+1} - hy_{i-1} + h^2 y_i = 3h^2 x_i^2$

or, $(1 + h)y_{i+1} + (h^2 - 2)y_i + (1 - h)y_i = 3h^2 x_i^2$

We have,

$$h = 0.5$$

so, above equation becomes;

$$(1 + 0.5)y_{i+1} + [(0.5)^2 - 2]y_i + (1 - 0.5)y_{i-1} = 3(0.5)^2 x_i^2$$

or, $1.5y_{i+1} - 1.75y_i + 0.5y_{i-1} = 0.75x_i^2$ (1)

i) When $i = 1$, $x_1 = 0$, $y_1 = 5$

ii) When $i = 2$, $x_2 = 0.5$

$$1.5y_3 - 1.75y_2 + 0.5y_1 = 0.75x_2^2$$

or, $1.5y_3 - 1.75y_2 + 0.5 \times 5 = 0.75 \times (0.5)^2$

∴ $1.5y_3 - 1.75y_2 = -2.3125$ (2)

iii) When $i = 3$, $x_3 = 1$

$$1.5y_4 - 1.75y_3 + 0.5y_2 = 0.75x_3^2$$

or, $1.5y_4 - 1.75y_3 + 0.5y_2 = 0.75 \times (1)^2$

∴ $1.5y_4 - 1.75y_3 + 0.5y_2 = 0.75$ (3)

iv) When $i = 4$, $x_4 = 1.5$, $y_5 = 4$

$$1.5y_5 - 1.75y_4 + 0.5y_3 = 0.75x_4^2$$

or, $1.5 \times 4 - 1.75y_4 + 0.5y_3 = 0.75 \times (1.5)^2$

∴ $-1.75y_4 + 0.5y_3 = -4.3125$ (4)

Solving equations (2), (3) and (4); we get,

$$y_2 = 4.7843$$
$$y_3 = 4.04$$
$$y_4 = 3.618$$

14. **Solve the differential equation $y'' + xy' - y = x$; $y(0) = 1$; $y(1) = 0$ using finite difference method by dividing four sub-intervals.**

**[2076 Ashwin]**

**Solution:**

Dividing the interval (0, 1) into four sub-intervals so that $h = 0.25$ and the pivot points are at $x_0 = 0$.

$$x_1 = 0.25$$
$$x_2 = 0.5$$
$$x_3 = 0.75$$

and, $x_4 = 1$

The given equation $y'' + xy' - y = x$ is approximate as;

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + x_i\left\{\frac{1}{2h}(y_{i+1} - y_{i-1})\right\} - y_i = x_i$$

or, $\frac{1}{(0.25)^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{x_i}{2 \times 0.25}(y_{i+1} - y_{i-1}) - y_i = x_i$

or, $16y_{i-1} - 32y_i + 16y_{i-1} + 2x_i y_{i+1} - 2x_i y_{i-1} - y_i = x_i$

or, $16y_{i-1} - 33y_i + 16y_{i-1} + 2x_i y_{i+1} - 2x_i y_{i-1} = x_i$ (1)

Using i = 1 in the equation (1); we have,

$$16y_2 - 33y_1 + 16y_0 + 2x_1y_2 - 2x_1y_0 = x_1$$

Again, using $y_0 = 1$ and $y_4 = 0$ (from given); we have,

or, $16y_2 - 33y_1 + 16 \times 1 + 2x_1y_2 - 2x_1 \times 1 = x_1$

Again using the value of $x_1 = 0.25$; we have,

or, $16y_2 - 33y_1 + 16 + 2 \times 0.25y_2 - 2 \times 0.25 = 0.25$

or, $16y_2 - 33y_1 + 15.25 = 0$ (2)

Again using i = 2 in the equation (1); we have,

$$16y_3 - 33y_2 + 16y_1 + 2x_2y_3 - 2x_2y_1 = x_2$$

Using the values of $x_2$; we have,

$$16y_3 - 33y_2 + 16y_1 + 2 \times 0.5y_3 - 2 \times 0.5y_1 = 0.5$$

or, $17y_3 - 33y_2 + 15y_1 - 0.5 = 0$ (3)

Using i = 3 in the equation (1); we have,

$$16y_4 - 33y_3 + 16y_2 + 2x_3y_4 - 2x_3y_2 = x_3$$

Putting the values of $y_4$ and $x_3$; we have,

$$16 \times 0 - 33y_3 + 16y_2 + 2 \times 0.75 \times 0 - 2 \times 0.75y_2 = 0.75$$

or, $-33y_3 + 16y_2 - 1.5y_2 = 0.75$

or, $-33y_3 + 14.5y_2 - 0.75 = 0$ (4)

Now, solving equation (2), (3) and (4); we get,

$y_1 = 0.629$

$y_2 = 0.335$

$y_3 = 0.124$

Hence,

$y(0.25) = 0.629$

$y(0.5) = 0.335$

and, $y(0.75) = 0.124$

15. **Write the pseudo-code for solving a first order ordinary differential equation using Runge-Kutta fourth order method.**

**[2076 Ashwin]**

**Solution:** See the algorithm of example 6.7