Chapter 4

INTERPOLATION

Introduction

Consider a single valued continuous function $y = f(x)$ defined over $[a, b]$ where $f(x)$ is known explicitly. It is easy to find the values of 'y' for a given set of values of 'x' in $[a, b]$, *i.e.*, it is possible to get information of all the points (x, y) where $a \leq x \leq b$.

But the converse is not so easy. That is, using only the points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n); where, $a \leq x_i \leq b$, $i = 0, 1, 2, \ldots\ldots\ldots, n$, it is not so easy to find the relation between x and y in the form $y = f(x)$ explicitly. That is one of the problem we face in numerical differentiation or integration.

Now; we have, first to find a simpler function, say $h(x)$, such that $f(x)$ and $h(x)$ agree at the given set of points and accept the value of $h(x)$ as the required value of $f(x)$ at some point x in between a and b. Such a process is called interpolation.

4.1 Newton's Interpolation (Forward, Backward)

There are two types of Newton's interpolation method and they are:

i) Newton's divided difference interpolation method

Consider a set of $(n+1)$ tabulated values $(x_0, y_0), (x_1, y_1), \ldots\ldots\ldots, (x_n, y_n)$ of a function $y = f(x)$ whose explicit nature is not known.

Let $P_n(x)$ be an interpolating polynomial of the function $y = f(x)$. Such that $P_n(x)$ agree at the given tabulated values then $P_n(x)$ can be written as;

$$P_n(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots\ldots (x - x_{n-1})a_n \quad (1)$$

where $a_0, a_1, \ldots\ldots\ldots, a_n$ are constants to be determined.

Substituting successively; we get,

At (x_0, y_0);

$$P_n(x_0) = a_0 + 0 = a_0 = y_0 = f[x_0]$$

At (x_1, y_1);

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) + 0 = y_1$$

or, $a_1(x_1 - x_0) = y_1 - a_0$

But, $a_0 = y_0$

so, $a_1(x_1 - x_0) = y_1 - y_0$

or, $a_1 = \dfrac{y_1 - y_0}{x_1 - x_0} = f[x_0, x_1]$

At (x_2, y_2);

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) + 0 = y_2$$

or, $a_2(x_2 - x_0)(x_2 - x_1) = y_2 - a_0 - a_1(x_2 - x_0)$

or, $a_2(x_2 - x_0)(x_2 - x_1) = y_2 - a_0 - \dfrac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)$

whereas, $a_0 = y_0$ and $a_1 = \dfrac{y_1 - y_0}{x_1 - x_0}$.

or, $a_2 = \dfrac{(y_2 - y_0) - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$

or, $a_2 = \dfrac{\dfrac{y_2 - y_0}{x_2 - x_0} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_1}$

$\therefore \quad a_2 = f[x_0, x_1, x_2]$

Similarly,

$$a_n = f[x_0, x_1, x_2, \ldots\ldots\ldots, x_n]$$

where, a_1 represent first divided difference.

a_2 represent second divided difference and so on.

Now, substituting the value of $a_0, a_1, a_2, \ldots\ldots, a_n$ in the equation (2); we get,

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots\ldots$$
$$+ f[x_0, x_1, x_2, \ldots\ldots\ldots, x_n](x - x_0)(x - x_1)\ldots\ldots\ldots(x_n - x_{n-1})$$

This can be written as;

$$P_n(x) = \sum_{i=1}^{n} f[x_0, \ldots\ldots, x_i] \prod_{j=0}^{n-1} (x - x_j)$$

This is called Newton's divided difference interpolation.
The divided difference table is:
(For fourth order polynomial function)

x	y	First divided difference (Δy)	Second divided difference ($\Delta^2 y$)	Third divided difference ($\Delta^3 y$)	Fourth divided difference ($\Delta^4 y$)
x_0	y_0				
		$\frac{y_1 - y_0}{x_1 - x_0} = \Delta y_0$			
x_1	y_1		$\frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \Delta^2 y_0$		
		$\frac{y_2 - y_1}{x_2 - x_1} = \Delta y_1$		$\frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0} = \Delta^3 y_0$	
x_2	y_2		$\frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \Delta^2 y_1$		$\frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4 - x_0} = \Delta^4 y_0$
		$\frac{y_3 - y_2}{x_3 - x_2} = \Delta y_2$		$\frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1} = \Delta^3 y_1$	
x_3	y_3		$\frac{\Delta y_3 - \Delta y_2}{x_4 - x_2} = \Delta^2 y_2$		
		$\frac{y_4 - y_3}{x_4 - x_3} = \Delta y_3$			
x_4	y_4				

Hence, the constant 1 a_0, a_1, a_2, a_3 and a_4 are;

$$a_0 = y_0$$

$$a_1 = \Delta y_0$$

$$a_2 = \Delta^2 y_0$$

$$a_3 = \Delta^3 y_0$$

and, $a_4 = \Delta^4 y_0$

Example 4.1

Using Newton's divided difference interpolation method; find the value of f(6) and f(9).

Given:

x	5	7	11	13	17
f(x) = y	150	392	1452	2366	5202

Solution:

From the given;

$$x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13 \text{ and } x_4 = 17$$

$$y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366 \text{ and } y_4 = 5202$$

so, polynomial function of fourth order is;

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + a_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) \quad (1)$$

where, a_0, a_1, a_2, a_3 and a_4 are constants to be determined.

whose values are:

$$a_0 = y_0$$

$$a_1 = \Delta y_0$$

$$a_2 = \Delta^2 y_0$$

$$a_3 = \Delta^3 y_0$$

and, $a_4 = \Delta^4 y_0$

For finding Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ and $\Delta^4 y_0$ we have to develop the divided difference table for fourth order.

x	y	First divided difference (Δy)	Second divided difference ($\Delta^2 y$)	Third divided difference ($\Delta^3 y$)	Fourth divided difference ($\Delta^4 y$)
5	150				
		$\frac{392-150}{7-5} = 121$			
7	392		$\frac{265-121}{11-5} = 24$		
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$	
11	1452		$\frac{457-265}{13-7} = 32$		$\frac{1-1}{17-5} = 0$
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$	
13	2366		$\frac{709-457}{17-11} = 42$		
		$\frac{5202-2366}{17-13} = 709$			
17	5202				

Hence, from divided difference table a bores

$$a_0 = y_0 = 150$$

$$a_1 = \Delta y_0 = 121$$

$$a_2 = \Delta^2 y_0 = 24$$

$$a_3 = \Delta^3 y_0 = 1$$

and, $a_4 = \Delta^4 y_0 = 0$

Now, substituting the values of a_0, a_1, a_2, a_3 and a_4 in the equation (1); we get, the polynomial function of 4^{th} order.

$$f(x) = 150 + 121(x-5) + 24(x-5)(x-7) + 1(x-5)(x-7)(x-11) + 0(x-5)(x-7)(x-11)(x-13)$$

or, $$f(x) = 150 + 121(x-5) + 24(x-5)(x-7) + (x-5)(x-7)(x-11) \quad (2)$$

Now, the value of f(6) can be obtained by putting $x = 6$

$$f(6) = 150 + 121(6-5) + 24(6-5)(6-7) + (6-5)(6-7)(6-11)$$

$\therefore$ $f(6) = 252$

and also for f(9), Taking $x = 9$

$$f(9) = 150 + 121(9-5) + 24(9-5)(9-7) + (9-5)(9-7)(9-11)$$

$\therefore$ $f(9) = 810$

Example 4.2

Determine f(x) as a polynomial in x for the following given data:

x	-4	-1	0	2	5
f(x)	1245	33	5	9	1335

Solution:

From the given data above,

$x_0 = -4$ $\quad$ $x_1 = -1$

$x_2 = 0$ $\quad$ $x_3 = 2$

and, $x_4 = 5$

$y_0 = 1245$ $\quad$ $y_1 = 33$,

$y_2 = 5$ $\quad$ $y_3 = 9$

and, $y_4 = 1335$

so, the polynomial function in x will be of fourth order.

Let the polynomial function of fourth order is;

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) \quad (1)$$

where, a_0, a_1, a_2, a_3 and a_4 are constants to be determined and whose valves are:

$a_0 = y_0$

$a_1 = \Delta y_0$

$a_2 = \Delta^2 y_0$

$a_3 = \Delta^3 y_0$

and, $a_4 = \Delta^4 y_0$

and, hence for finding these above value we have to develop the divided difference table for fourth order polynomial function.

x	y	First divided difference (Δy)	Second divided difference ($\Delta^2 y$)	Third divided difference ($\Delta^3 y$)	Fourth divided difference ($\Delta^4 y$)
-4	1245				
		$\frac{33-1245}{-1-(-4)} = -404$			
-1	33		$\frac{-28-(-404)}{0-(-4)} = 94$		
		$\frac{5-33}{0-(-1)} = -28$		$\frac{10-94}{2-(-4)} = -14$	
0	5		$\frac{2-(-28)}{2-(-1)} = 10$		$\frac{13-(-14)}{5-(-4)}$ $= 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5-(-1)} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

Hence, from divided difference table above,

$$a_0 = y_0 = 1245$$

$$a_1 = \Delta y_0 = -404$$

$$a_2 = \Delta^2 y_0 = 94$$

$$a_3 = \Delta^3 y_0 = -14$$

and, $a_4 = \Delta^4 y_0 = 3$

Now, putting these values of a_0, a_1, a_2, a_3 and a_4 in the equation (1) and also the values of x_0, x_1, x_2 and x_3.

$$f(x) = 1245 + (-404)[x-(-4)] + 94[x-(-4)][x-(-1)] + (-14)[x-(-4)][x-(-1)](x-0) + 3[x-(-4)][x-(-1)](x-0)(x-2)$$

or, $$f(x) = 1245 - 404x - 1616 + 94(x^2+5x+4) - 14x(x^2+5x+4) + 3x(x-2)(x^2+5x+4)$$

or, $$f(x) = 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x(x^3+3x^2-6x-8)$$

or, $$f(x) = -14x^3 - 24x^2 + 10x + 5 + 3x^4 + 9x^3 - 18x^2 - 24x$$

$\therefore$ $$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Example 4.3

From the following data given below by using the Newton's divided difference formula find the value of 'm'.

x	1	2	4	5	6
f(x) = y	14	15	5	m	9

Solution:

From the given data above, the value of m is f(5) *i.e.*, value of m is value of m is value of polynomial function at x = 5.

so, for finding the value of m, at first we have to generate the polynomial function.

so, assume

$$x_0 = 1, x_1 = 2, x_2 = 4 \text{ and } x_3 = 6$$

$$y_0 = 14, y_1 = 15, y_2 = 5 \text{ and } y_3 = 9$$

The polynomial function will be of third order.

so, let the polynomial function is:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \quad (1)$$

The divided difference table is:

x	y	First divided difference (Δy)	Second divided difference ($\Delta^2 y$)	Third divided difference ($\Delta^3 y$)
1	14			
		$\frac{15-15}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{\frac{7}{4}-(-2)}{6-1} = \frac{3}{4}$
4	5		$\frac{2-(-5)}{6-2} = \frac{7}{4}$	
		$\frac{9-5}{6-4} = 2$		
6	9			

Hence, from the tables; we have,

$$a_0 = y_0 = 14$$

$$a_1 = \Delta y_0 = 1$$

$$a_2 = \Delta^2 y_0 = -2$$

and, $a_3 = \Delta^3 y_0 = \frac{3}{4}$

Putting these values along with the values of x_0, x_1, x_2 and x_3 in the equation (1).

$$f(x) = 14 + 1(x-1) + (-2)(x-1)(x-2) + \frac{3}{4}(x-1)(x-2)(x-4)$$

Now, for values of m, take x = 5

or, $f(5) = 14 + 1(5-1) + (-2)(5-1)(5-2) + \frac{3}{4}(5-1)(5-2)(5-4)$

$\therefore$ $f(5) = 3$

Hence, value of m = f(5) = 3.

Example 4.4

Find the value of f(8) and f(15) from the given data by using Newton's divided difference formula:

x	4	5	7	10	11	13
f(x) = y	48	100	294	900	1210	2028

Solution:

From the given data above;

$x_0 = 4$ $\quad$ $x_1 = 5$

$x_2 = 7$ $\quad$ $x_3 = 10$

$x_4 = 11$ $\quad$ and, $x_5 = 13$

$y_0 = 48$ $\quad$ $y_1 = 100$

$y_2 = 294$ $\quad$ $y_3 = 900$

$y_4 = 1210$ $\quad$ and, $y_5 = 2028$

Since, number of point is six so the polynomial function in x will be of fifth order,

Let the polynomial function of fifth order is;

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) + a_5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4) \quad (1)$$

where, a_0, a_1, a_2, a_3, a_4 and a_5 are constants to be determined and whose values are:

$a_0 = y_0$

$a_1 = \Delta y_0$

$a_2 = \Delta^2 y_0$

$a_3 = \Delta^3 y_0$

$a_4 = \Delta^4 y_0$

and, $a_5 = \Delta^5 y_0$

and, hence for finding these above value. We have to develop the divided difference table for fifth order polynomial function.

x	y	First divided difference (Δy)	Second divided difference ($\Delta^2 y$)	Third divided difference ($\Delta^3 y$)	Fourth divided difference ($\Delta^4 y$)	Fifth divided difference ($\Delta^5 y$)
4	48					
		$\frac{100-48}{5-4} = 52$				
5	100		$\frac{97-52}{7-4} = 15$			
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$		
7	294		$\frac{202-97}{10-5} = 21$		$\frac{1-1}{11-4} = 0$	
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$		$\frac{0-0}{13-4} = 0$
10	900		$\frac{310-202}{11-7} = 27$		$\frac{1-1}{13-5} = 0$	
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$		
11	1210		$\frac{409-310}{13-10} = 33$			
		$\frac{2028-1210}{13-11} = 409$				
13	2028					

Hence, from the table above,

$$a_0 = y_0 = 48$$
$$a_1 = \Delta y_0 = 52$$
$$a_2 = \Delta^2 y_0 = 15$$
$$a_3 = \Delta^3 y_0 = 1$$
$$a_4 = \Delta^4 y_0 = 0$$

and, $a_5 = \Delta^5 y_0 = 0$

Putting these values along with the valves of x_0, x_1, x_2, x_3 and x_4 in the equation (2); we have,

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + 1(x-4)(x-5)(x-7) + 0 + 0 \quad (2)$$

Now, for f(8), putting the value of x = 8 in the equation (2); we have,

$$f(8) = 48 + 52(8-4) + 15(8-4)(8-5) + (8-4)(8-5)(8-7)$$

or, $f(8) = 448$

and, also for f(15), putting the value of $x = 15$ in the equation (2); we get,

$$f(15) = 48 + 52(15-4) + 15(15-4)(15-5) + (15-4)(15-5)(15-7)$$

or, $f(15) = 3150$

ii) Newton's simple difference interpolation method

This method is also called "Gregory-Newton's forward interpolation formula."

Statement:

If $y_0, y_1, y_2, \ldots\ldots, y_n$ are the valves of $y = f(x)$ corresponding to equidistant valves of $x = x_0, x_1, x_2, \ldots\ldots, x_n$; where, $x_i - x_{i-1} = h$ for $i = 1, 2, 3, \ldots\ldots, n$; then,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots + \frac{u(u-1)\ldots\ldots(u-\overline{n-1})}{n!}\Delta^n y_0$$

where, $u = \dfrac{x - x_0}{h}$

Proof:

Since, $(n+1)$ valves of y are given, we can assume y to be a polynomial in x of the n^{th} degree.

Let, $y = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots\ldots + a_n(x-x_0)(x-x_1) \ldots\ldots(x-x_{n-1})$ (1)

When, $x = x_0$, $y = y_0$; using this in the equation (1); we get,

$$y_0 = a_0$$

or, $a_0 = y_0$ (2)

When $x = x_1$, $y = y_1$; using this and equation (2) in the equation (1); we have,

$$y_1 = a_0 + a_1(x_1 - x_0) + 0$$

or, $y_1 - a_0 = a_1(x_1 - x_0)$

or, $a_1(x_1 - x_0) = y_1 - y_0$ $[\because a_0 = y_0]$

or, $a_1(x_1 - x_0) = \Delta y_0$

or, $a_1 = \dfrac{\Delta y_0}{x_1 - x_0}$

But, $x_1 - x_0 = h$

$\therefore$ $a_1 = \dfrac{\Delta y_0}{h}$ (3)

When $x = x_2$, $y = y_2$; using this in the equation (1); we have,

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$= y_0 + \frac{\Delta y_0}{h}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \text{[By equation (2) and (3)]}$$

But; we have,

h = Interval of x

or, $\quad h = x_2 - x_1 = x_1 - x_0$

so, $\quad y_2 = y_0 + \dfrac{\Delta y_0}{h}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

or, $\quad y_2 = y_0 + \dfrac{\Delta y_0}{h}[(x_2 - x_1) + (x_1 - x_0)] + a_2[(x_2 - x_1) + (x_1 - x_0)](x_2 - x_1)$

or, $\quad y_2 = y_0 + \dfrac{\Delta y_0}{h}(h + h) + a_2(h + h)h$

or, $\quad y_2 = y_0 + \dfrac{\Delta y_0}{h} \times 2h + a_2 \times 2h^2$

or, $\quad a_2 \times 2h^2 = y_2 - y_0 - 2\Delta y_0 = (y_2 - y_1) + (y_1 - y_0) - 2\Delta y_0$

$$= \Delta y_1 + \Delta y_0 - 2\Delta y_0$$

$$= \Delta y_1 - \Delta y_0$$

$$= \Delta^2 y_0$$

$\therefore \quad a_2 = \dfrac{\Delta^2 y_0}{2!\,h^2}$

Similarly, proceeding, we can find that,

$$a_3 = \frac{\Delta^3 y_0}{3!\,h^3}, \ldots\ldots\ldots, a_n = \frac{\Delta^n y_0}{n!\,h^n} \qquad (4)$$

Using these values of $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ in the equation (1); we have,

$$y = y_0 + (x - x_0)\frac{\Delta y_0}{1!\,h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_0}{2!\,h^2} + \ldots\ldots\ldots + (x - x_0)(x - x_1)\ldots\ldots\ldots(x - x_{n-1})\frac{\Delta^n y_0}{n!\,h^n} \qquad (5)$$

Since, $u = \dfrac{x - x_0}{h}$, $x - x_0 = uh$

Now,

$$x - x_1 = (x - x_0) - (x_1 - x_0) = uh - h = (u - 1)h$$

$$x - x_2 = (x - x_0) - (x_2 - x_0)$$

$$= uh - [(x_2 - x_1) + (x_1 - x_0)]$$

$$= uh - (h + h)$$

$$= uh - 2h$$

$\therefore \quad x - x_2 = (u - 2)h$ and so on.

and, $\quad x - x_{n-1} = (u - \overline{n-1})h$

Using these valves in the equation (5); we have,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots + \frac{u(u-1)\ldots\ldots(u - \overline{n-1})}{n!}\Delta^n y_0$$

Note

1. Since the formula derived involves the forward differences of y at y_0, it is called Newton's forward interpolation formula.

2. If only 2 valves of y, namely y_0 and y_1 corresponding to $x = x_0$ and x_1 are given, the above formula takes the form

$$y = y_0 + \frac{x - x_0}{h}(y_1 - y_0)$$

i.e., $$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0);$$

which is called linear interpolation formula.

3. If three valves of y, namely y_0, y_1 and y_2 corresponding to $x = x_0, x_1$ and x_2 are given, then Newton's forward interpolation formula is called parabolic interpolation formula.

ii) Newton's backward interpolation method

This method is also called "Gregory-Newton's backward interpolation formula."

Statement:

If $y_0, y_1, y_2, \ldots\ldots, y_n$ are the valves of $y = f(x)$ corresponding to equidistant valves of $x_0, x_1, x_2, \ldots\ldots, x_n$; where, $x_i - x_{n-1} = h$, for $i = 1, 2, \ldots\ldots, n_j$;

Then,

$$y = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_0 + \ldots\ldots + \frac{u(u+1)\ldots\ldots(u+n-1)}{n!}\nabla^n y_n$$

where, $u = \dfrac{x - x_n}{h}$

Proof:

Since, $(n + 1)$ valves of y are given, we can assume y as a polynomial in x of the n^{th} degree in the following form.

$$y = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \ldots\ldots + a_n(x - x_n)(x - x_{n-1})\ldots\ldots(x - x_1) \quad (1)$$

When, $x = x_n$, $y = y_n$ using this in the equation (1); we get,

$$y_n = a_0$$

or, $$a_0 = y_n \quad (2)$$

When $x = x_{n-1}$, $y - y_{n-1}$ using this equation (2) in the equation (1); we have,

$$y_{n-1} = a_0 + a_1(x_{n-1} - x_n)$$

or, $$a_1(x_{n-1} - x_n) = y_{n-1} - a_0$$

or, $$a_1(x_{n-1} - x_n) = y_{n-1} - y_n$$

or, $$a_1 = \frac{y_{n-1} - y_n}{x_{n-1} - x_n}$$

or, $$a_1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{1}{h}\nabla y_n \quad (3)$$

where, h = Interval of $x = x_n - x_{n-1}$

$$\nabla y_n = y_n - y_{n-1}$$

When $x = x_{n-2}$, $y = y_{n-2}$ using this in the equation (1); we get,

$$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$
$$= y_n + a_1[(x_{n-2} - x_{n-1}) + (x_{n-1} - x_n)] + a_2[(x_{n-2} - x_{n-1}) + (x_{n-1} - x_n)](x_{n-2} - x_{n-1})$$
$$= y_n + a_1(-h - h) + a_2(-h - h)(-h)$$
$$= y_n + a_1(-2h) + a_2(2h^2)$$

or, $a_2(2h^2) = a_1 \times 2h + y_{n-2} - y_n$

or, $a_2(2h^2) = \frac{1}{h}\nabla y_n \times 2h + y_{n-2} - y_n$

or, $a_2(2h^2) = 2\nabla y_n + (y_{n-2} - y_{n-1}) + (y_{n-1} - y_n)$

$$= 2\nabla y_n + \nabla y_{n-1} - \nabla y_n$$
$$[\because \nabla y_n = y_n - y_{n-1} \text{ and } \nabla y_{n-1} = y_{n-1} - y_{n-2}]$$

or, $a_2(2h^2) = \nabla y_n - \nabla y_{n-1} = \nabla^2 y_n$ $\quad [\because \nabla^2 y_n = \nabla y_n - \nabla y_{n-1}]$

or, $a_2 = \dfrac{\nabla^2 y_n}{2!\, h^2}$

Similarly, proceeding we can find that;

$$a_3 = \frac{\nabla^3 y_n}{3!\, h^3}, \ldots\ldots\ldots, a_n = \frac{\nabla^n y_n}{n!\, h^n} \tag{4}$$

Using these valves of $a_0, a_1, \ldots\ldots\ldots, a_n$ in the equation (1); we have,

$$y = y_n + (x - x_n)\frac{\nabla y_n}{1!\, h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{2!\, h^2} + \ldots\ldots\ldots + (x - x_n)(x - x_{n-1})\ldots\ldots\ldots(x - x_1)\frac{\nabla^n y_n}{n!\, h^n} \tag{5}$$

Since, $u = \dfrac{x - x_n}{h}$; we have, $x - x_n = uh$

Now,

$$x_n - x_{n-1} = (x - x_n) + (x_n - x_{n-1}) = uh + h = (u + 1)h$$

Also,

$$x - x_{n-2} = (x - x_n) + (x_n - x_{n-2})$$
$$= uh + (x_n - x_{n-1}) + (x_{n-1} - x_{n-2})$$
$$= uh + h + h$$
$$= (u + 2)h$$

and so on.

$$x - x_{n-1} = (u + n - 1)h$$

Using these valves in the equation (5); we have,

$$y = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \ldots\ldots + \frac{u(u+1)\ldots\ldots(u+n-1)}{n!}\nabla^n y_n$$

Note

1. Since this formula involves the backward difference of y at y_n, it is called Newton's backward interpolation formula.

2. Thought it is usually suggested that forward formula should be used for interpolating or extrapolating y at points near x_0 and the backward formula should be used for interpolating or extrapolating y at point near x_n, it is not necessary. Either of the formulas can be used for interpolating or extrapolating y at any point in fact the interpolating polynomials of f(x) that occur in R.H.S.'s of both the forward and backward formula are identical.

Errors in Newton's Interpolation Formula

We give below the errors in Newton's interpolation formula, without proof.

i) Error in Newton's forward interpolation formula

$$\frac{u(u-1)(u-2)\,\ldots\ldots\,(u-n)}{(n+1)!}\left(\Delta^{n+1}y\right)_x = C'$$

where, $x_0 < C < x_n$ and $u = \dfrac{x - x_0}{h}$

ii) Error in Newton's backward interpolation formula

$$\frac{u(u+1)(u+2)\,\ldots\ldots\,(u+n)}{(n+1)!}\left(\nabla^{n+1}y\right)_x = C$$

where, $x_0 < C < x_n$ and $u = \dfrac{x - x_n}{h}$

The simple difference table for Newton's forward and Newton's backward interpolating method are given below:

Simple difference table for Newton's forward interpolation

x	y	First simple difference (Δy)	Second simple difference ($\Delta^2 y$)	Third simple difference ($\Delta^3 y$)
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
$x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$x_0 + 2h$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
$x_0 + 3h$	y_3			

where, h = Interval of $x = x_i - x_{i-1}$

Simple difference table for Newton's backward interpolation

x	y	First simple difference (Δy)	Second simple difference ($\Delta^2 y$)	Third simple difference ($\Delta^3 y$)
x_0	y_0			
		$\nabla y_0 = y_1 - y_0$		
$x_0 + h$	y_1		$\nabla^2 y_0 = \nabla y_1 - \nabla y_0$	

		$\nabla y_1 = y_2 - y_1$		$\nabla^3 y_0 = \nabla^2 y_1 - \nabla^2 y_0$
$x_0 + 2h$	y_2		$\nabla^2 y_1 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_3 - y_2$		
$x_0 + 3h$	y_3			

Example 4.5

Find the value of function x = 0.16 from the following tabulated function

x	0.1	0.2	0.3	0.4
y = f(x)	1.005	1.020	1.045	1.081

Solution:

From given table above

Forward

x	0.1	0.2	0.3	0.4
y = f(x)	1.005	1.020	1.045	1.081

As we know that; we have, to find the value of function at x = 0.16 which lies in between 0.1 and 0.2 and also the point x = 0.16 lies near the starting point (x_0) so we use the Newton's forward interpolation method.

From Newton's forward interpolation formula; we have,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots\ldots(u-\overline{n-1})}{n!}\Delta^n y_0 \quad (1)$$

where, $u = \dfrac{x - x_0}{h}$

and, h = Interval of $x = x_i - x_{i-1}$

At first finding the value of y_0, Δy_0, $\Delta^2 y_0$, etc. from the simple difference table for Newton's forward interpolation method

Simple difference table for Newton's forward interpolation

x	y	First simple difference (Δy)	Second simple difference ($\Delta^2 y$)	Third simple difference ($\Delta^3 y$)
0.1	1.005			
		$\Delta y_0 = 1.020 - 1.005 = 0.015$		
0.2	1.020		$\Delta^2 y_0 = 0.025 - 0.015 = 0.01$	
		$\Delta y_1 = 1.045 - 1.020 = 0.025$		$\Delta^3 y_0 = 0.011 - 0.01 = 0.001$
0.3	1.045		$\Delta^2 y_1 = 0.036 - 0.025 = 0.011$	

		$\Delta y_2 = 1.081 - 1.045$ $= 0.036$		
0.4	1.081			

Hence, from above table; we have,

$$y_0 = 1.005$$
$$\Delta y_0 = 0.015$$
$$\Delta^2 y_0 = 0.01$$
$$\Delta^3 y_0 = 0.001$$

Also, from given table, $x_0 = 0.1, x_1 = 0.2$ and $x = 0.16$

$$h = x_1 - x_0 = 0.2 - 0.1 = 0.1$$

and, $u = \frac{x - x_0}{h} = \frac{0.16 - 0.1}{0.1} = 0.6$

Now, putting these valves in the equation (1); we get,

$$y(3) = 1.005 + \frac{0.6}{1!} \times 0.015 + \frac{0.6(0.6-1)}{2!} \times 0.01 + \frac{0.6(0.6-1)(0.6-2)}{3!} \times 0.001$$

$\therefore \quad y(3) = 1.0128$

Hence, value of function at $x = 0.16$ is;

$$y(3) = 1.0128$$

Example 4.6

From the following given data which given data which gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = Height	100	150	200	250	300	350	400
y = Distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

i) $x = 218$ ft.

ii) $x = 410$ ft.

Solution:

From the given table,

x = Height	100	150	200	250	300	350	400
y = Distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Since, we have to find the value of y at $x = 218$ ft., in this case, the point $x = 218$ ft. lies between 200 and 250 and also this point lies near the starting point (x_0) so we use forward interpolation method and (ii) at $x = 410$ ft. which lies near the point (x_n) so in this case we use backward interpolation formula.

From Newton's forward interpolation formula; we have,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \quad (1)$$

Also, from Newton's backward in interpolation formula; we have,

$$y = y_n + \frac{u}{1!}\nabla y_0 + \frac{u(u+1)}{2!}\nabla^2 y_0 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_0 + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_0 + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_0 + \frac{u(u+1)(u+2)(u+3)(u+4)(u+5)}{6!}\nabla^6 y_0 \quad (2)$$

where, $u = \frac{x - x_0}{h}$.

and, h = Interval of $x = x_i - x_{i-1}$.

So, for finding y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, ... and also for finding y_n, ∇y_n, $\nabla^2 y_n$, $\nabla^3 y_n$, ... we have to generate simple difference table for both Newton's forward and backward interpolation method.

Simple difference table for both Newton's forward and backward

x	y	First s.d. Δy or ∇y	Second s.d. $\Delta^2 y$ or $\nabla^2 y$	Third s.d. $\Delta^3 y$ or $\nabla^3 y$	Fourth s.d. $\Delta^4 y$ or $\nabla^4 y$	Fifth s.d. $\Delta^5 y$ or $\nabla^5 y$	Sixth s.d. $\Delta^6 y$ or $\nabla^5 y$
100	10.63						
		2.40					
150	13.03		- 0.39				
		2.01		0.15			
200	15.04		- 0.24		- 0.07		
		1.77		0.08		0.02	
250	16.81		- 0.16		- 0.05		0.02
		1.61		0.03		0.04	
300	18.42		- 0.13		- 0.01		
		1.48		0.02			
350	19.90		- 0.11				
		1.37					
400	21.27						

i) At $x = 218$ ft.

$y = ?$

Hence, from above simple difference table; we have,

Data required for forward interpolation method:

As we have to find value of y at $x = 218$ ft.

so, we take,

$x_0 = 200$

$x_1 = 250$

$y_0 = 15.04$

$\Delta y_0 = 1.77$

$\Delta^2 y_0 = -0.16$

$\Delta^3 y_0 = 0.03$

$\Delta^4 y_0 = -0.01$

$h = \text{Interval of } x = x_1 - x_0 = 250 - 200 = 50$

and, $u = \frac{x - x_0}{h} = \frac{218 - 200}{50} = 0.36$

Now, putting above these values in the equation (1); we get,

$$y(218) = 15.04 + \frac{0.36}{1!} \times 1.77 + \frac{0.36(0.36-1)}{2!} \times (-0.16) + \frac{0.36(0.36-1)(0.36-2)}{3!} \times (0.03) + \frac{0.36(0.36-1)(0.36-2)(0.36-3)}{4!} \times (-0.01)$$

or, $y(218) = 15.6979$

Hence, at $x = 218$ ft.;

$y = 15.6979$ nautical miles

ii) At $x = 410$ ft.

Since the point $x = 410$ lies near the end of the table so, we use Newton's backward interpolation method

Hence, data required for backward interpolation,

so, taking $x_n = 400$,

As we have $x = 410$

and, $h = \text{Interval of } x = 50$

Therefore, in this case;

$u = \frac{x - x_n}{h} = \frac{410 - 400}{50} = 0.2$

Also,

$y_n = 21.27$

$\nabla y_n = 1.37$

$\nabla^2 y_n = -0.11$

$\nabla^3 y_n = 0.02$

$\nabla^4 y_n = -0.01$

$\nabla^5 y_n = 0.04$

and, $\nabla^6 y_n = 0.02$

Putting these values in the equation (2); we get,

$$y(410) = 21.27 + \frac{0.2}{1!} \times 1.37 + \frac{0.2(0.2+1)}{2!} \times (-0.11) + \frac{0.2(0.2+1)(0.2+2)}{3!} \times (0.03) + \frac{0.2(0.2+1)(0.2+2)(0.2+3)}{4!} \times (-0.01) + \frac{0.2(0.2+1)(0.2+2)(0.2+3)(0.2+4)}{5!} \times (0.04) + \frac{0.2(0.2+1)(0.2+2)(0.2+3)(0.2+4)(0.2+5)}{6!} \times (0.02)$$

or, $y(410) = 21.5352$

Hence, at x = 410 ft.;

$y = 21.5352$ nautical miles

Example 4.7

If $y(10) = 35.3$, $y(15) = 32.4$, $y(20) = 29.2$, $y(25) = 26.1$, $y(30) = 23.2$ and $y(35) = 20.5$ find $y(12)$ using

i) Newton's forward interpolation formula and

ii) Newton's backward interpolation formula

Solution:

From the given;

$y(10) = 35.3$

$y(15) = 32.4$

$y(20) = 29.2$

$y(25) = 26.1$

$y(30) = 23.2$

and, $y(35) = 20.5$

Then; we have to find $y(12)$

Since, we have to find the value of $y(10)$ by using both Newton's forward and backward formula.

So, from Newton's forward formula; we have,

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 \quad (1)$$

and, from Newton's Backward formula; we have,

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_n \quad (2)$$

where for, forward formula,

$$u = \frac{x - x_0}{h}$$

and, h = Interval of x

and, for backward formula; we have,

$$u = \frac{x - x_n}{h}$$

and, h = Interval of x

Now,

At first, finding the valves of y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ and $\Delta^5 y_0$ and also the values of y_n, ∇y_n, $\nabla^2 y_n$, $\nabla^3 y_n$, $\nabla^4 y_n$ and $\nabla^5 y_n$ by generating the simple difference table for forward and backward interpolation.

Simple difference table for both Newton's forward and backward

x	y	First s.d. Δy or ∇y	Second s.d. $\Delta^2 y$ or $\nabla^2 y$	Third s.d. $\Delta^3 y$ or $\nabla^3 y$	Fourth s.d. $\Delta^4 y$ or $\nabla^4 y$	Fifth s.d. $\Delta^5 y$ or $\nabla^5 y$
10	35.3					
		= 32.4 − 35.3 = −2.9				
15	32.4		= −3.2 − (−2.9) = −0.3			
		= 29.2 − 32.4 = −3.2		= 0.1 − (−0.3) = 0.4		
20	29.2		= −3.1 − (−3.2) = 0.1		= 0.1 − 0.4 = −0.3	
		= 26.1 − 29.2 = −3.1		= 0.2 − 0.1 = 0.1		= −0.1 − (−0.3) = 0.2
25	26.1		= −2.9 − (−3.1) = 0.2		= 0 − 0.1 = −0.1	
		= 23.2 − 26.1 = −2.9		= 0.2 − 0.2 = 0		
30	23.2		= −2.7 − (−2.9) = 0.2			
		= 20.5 − 23.2 = −2.7				
35	20.5					

i) By using Newton's forward interpolation formula,

Since, $x = 12$ lies near starting point of the table so, in forward interpolation formula.

We take,

$$x_0 = 10$$

$$y_0 = 35.3$$

$\Delta y_0 = -2.9$

$\Delta^2 y_0 = -0.3$

$\Delta^3 y_0 = 0.4$

$\Delta^4 y_0 = -0.3$

and, $\Delta^5 y_0 = 0.2$

$\therefore$ h = Interval of x

$= 15 - 10 = 5$

and, $u = \dfrac{x - x_0}{h} = \dfrac{12 - 10}{5} = 0.4$

Now, putting these values in the equation (1); we get,

$$y(12) = 35.3 + \frac{0.4}{1!}(-2.9) + \frac{0.4(0.4-1)}{2!}(-0.3) + \frac{0.4(0.4-1)(0.4-2)}{3!}(0.4) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-0.3) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{5!}(0.2)$$

$\therefore$ $y(12) = 34.22$

ii) By using Newton's backward formula,

Here,

$x = 12$

We take,

$x_n = 35$

$y_n = 20.5$

and, also

$\nabla y_n = -2.7$

$\nabla^2 y_n = 0.2$

$\nabla^3 y_n = 0$

$\nabla^4 y_n = -0.1$

and, $\nabla^5 y_n = 0.2$

$\therefore$ h = Interval of x = 15 − 10 = 5

and, $u = \dfrac{x - x_n}{h} = \dfrac{12 - 35}{5} = -4.6$

Now, putting these values in the equation (2); we get,

$$y(12) = 20.5 + \frac{-4.6}{1!}(-2.7) + \frac{(-4.6)(-4.6+1)}{2!}(0.2) + \frac{(-4.6)(-4.6+1)(-4.6+2)}{3!}(0) + \frac{(-4.6)(-4.6+1)(-4.6+2)(-4.6+3)}{4!}(-0.1)$$

$$+\frac{(-4.6)(-4.6+1)(-4.6+2)(-4.6+3)(-4.6+4)}{5!}(0.2)$$

$\therefore \quad y(12) = 34.220064 = 34.22$

Example 4.8

The population in a town in the census is as given in the data. Estimate the population in the year 1996 using Newton's (i) forward interpolation and (ii) backward interpolation formula.

Year (x)	1961	1971	1881	1991	2001
Population (in 1000's)	46	66	81	93	101

Solution:

From the given above;

Year (x)	1961	1971	1881	1991	2001
Population (in 1000's)	46	66	81	93	101

Hence, we have to find the valve of 'y' at x = 1996

We have to use both Newton's forward and backward interpolation formula.

So, from Newton's forward formula; we have,

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \quad (1)$$

where, $u = \frac{x - x_0}{h}$ and h = Interval of x

and, from Newton's backward formula; we have,

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n \quad (2)$$

where, $u = \frac{x - x_n}{h}$ and h = Interval of x

At first, finding the values of y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ and $\Delta^4 y_0$ also the values of y_n, ∇y_n, $\nabla^2 y_n$, $\nabla^3 y_n$ and $\nabla^4 y_n$ by generating the simple difference table for forward and backward interpolation.

Simple difference table for both Newton's forward and backward interpolation

x	y	First s.d. Δy or ∇y	Second s.d. $\Delta^2 y$ or $\nabla^2 y$	Third s.d. $\Delta^3 y$ or $\nabla^3 y$	Fourth s.d. $\Delta^4 y$ or $\nabla^4 y$
1961	46				
		= 66 − 46 = 20			

1971	66		$= 15 - 20$ $= -5$		
		$= 81 - 66 = 15$		$= -3 - (-5)$ $= 2$	
1981	81		$= 12 - 15$ $= -3$		$= -1 - 2$ $= -3$
		$= 93 - 81 = 12$		$= -4 - (-3)$ $= -1$	
1991	93		$= 8 - 12$ $= -4$		
		$= 101 - 93 = 8$			
2001	101				

i) By using Newton's forward interpolation formula; we have,

$x_0 = 1961$

$y_0 = 46$

$\Delta y_0 = 20$

$\Delta^2 y_0 = -5$

$\Delta^3 y_0 = 2$

and, $\Delta^4 y_0 = -3$

Also

$h = \text{Interval of } x = 1971 - 1961 = 10$

and, $x = 1996$

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{1996 - 1961}{10} = 3.5$$

Putting these values in the equation (1); we get,

$$y(1996) = 46 + \frac{3.5}{1!}(20) + \frac{3.5(3.5-1)}{2!}(-5) + \frac{3.5(3.5-1)(3.5-2)}{3!}(2) + \frac{3.5(3.5-1)(3.5-2)(3.5-3)}{4!}(-3)$$

$$\therefore \quad y(1996) = 97.6796875$$

Hence, population of town in the year 1886 is 97.6796875 (in thousands)

ii) By using Newton's backward interpolation formula; we have,

Here,

$x = 1996$

and, $x_n = 2001$

$y_n = 101$

Also,

$$\nabla y_n = 8$$
$$\nabla^2 y_n = -4$$
$$\nabla^3 y_n = -1$$
$$\nabla^4 y_n = -3$$

$\therefore$ h = Interval of x = 1971 − 1961 = 10

and, $u = \frac{x - x_n}{h} = \frac{1996 - 2001}{10} = -0.5$

Putting these values in the equation (2); we get,

$$y(1996) = 101 + \frac{-0.5}{1!}(8) + \frac{-0.5(-0.5+1)}{2!}(-4) + \frac{-0.5(-0.5+1)(-0.5+2)}{3!}(-1) + \frac{-0.5(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(-3)$$

$\therefore$ $y(1996) = 97.6796875$

Hence, population of town in the year 1996 is 97.6796875 (in thousands)

Example 4.9

Find the value of $e^{-0.75}$ and $e^{-2.25}$ from the following data using both Newton's forward and backward formula.

x	1.00	1.2.5	1.50	1.75	2.00
$y = e^{-x}$	0.3679	0.2865	0.2231	0.1738	0.1353

Solution:

From the given above,

x	1.00	1.2.5	1.50	1.75	2.00
$y = e^{-x}$	0.3679	0.2865	0.2231	0.1738	0.1353

Here; we have, to find the value of $e^{-0.75}$ and $e^{-2.25}$ *i.e.,* value of y at x = 0.75 and 2.25.

We have to use both Newton's forward and backward interpolation formula; we have,

so, from Newton's forward formula; we have,

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \quad (1)$$

where, $u = \frac{x - x_0}{h}$

and, h = Interval of x

and, also from Newton's backward formula; we have,

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n \quad (2)$$

where, $u = \frac{x - x_n}{h}$

and, h = Interval of x

Here, at first; we have, to find the values of y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ and $\Delta^4 y_0$ also the values of y_n, ∇y_n, $\nabla^2 y_n$, $\nabla^3 y_n$ and $\nabla^4 y_n$ by generating the simple difference table for forward and backward interpolation.

Simple difference table for both Newton's forward and backward interpolation

x	y	First s.d. Δy or ∇y	Second s.d. $\Delta^2 y$ or $\nabla^2 y$	Third s.d. $\Delta^3 y$ or $\nabla^3 y$	Fourth s.d. $\Delta^4 y$ or $\nabla^4 y$
1.00	0.3679				
		– 0.0814			
1.25	0.2865		0.0180		
		– 0.01634		– 0.0039	
1.50	0.2231		0.0141		0.0006
		– 0.0493		– 0.0033	
1.75	0.1738		0.0108		
		– 0.0385			
2.0	0.1353				

i) By using Newton's forward interpolation formula,

a) For x = 0.75

$x_0 = 1.00$

$y_0 = 0.3679$

$\Delta y_0 = -0.0814$

$\Delta^2 y_0 = 0.0180$

$\Delta^3 y_0 = -0.0039$

and, $\Delta^4 y_0 = 0.0006$

and, also,

h = Interval of $x = 1.25 - 1 = 0.25$

$$u = \frac{x - x_0}{h} = \frac{0.75 - 1.00}{0.25} = -1$$

Putting these values in the equation (1); we get,

$$y(0.75) = 0.3679 + \frac{-1}{1!}(-0.0814) + \frac{(-1)(-1-1)}{2!}(0.0180)$$

$$+\frac{(-1)(-1-1)(-1-2)}{3!}(-0.0039)$$
$$+\frac{(-1)(-1-1)(-1-2)(-1-3)}{4!}(0.0006)$$

∴ $y(0.75) = 0.4718$

Hence,

$$e^{-0.75} = 0.4718$$

b) For $x = 2.25$

$$u = \frac{x - x_n}{h} = \frac{2.25 - 1.00}{0.25} = 5$$

Again, putting these above values in the equation (1); we get,

$$y(2.25) = 0.3679 + \frac{5}{1!}(-0.0814) + \frac{5(5-1)}{2!}(0.0180)$$
$$+\frac{5(5-1)(5-2)}{3!}(-0.0039) + \frac{5(5-1)(5-2)(5-3)}{4!}(0.0006)$$

∴ $y(2.25) = 0.1049$

∴ $e^{-2.25} = 0.1049$

ii) By using Newton's backward interpolation formula; we get,

a) For $x = 0.75$;

$x_n = 2.00$

$y_n = 0.1353$

$\nabla y_n = -0.0385$

$\nabla^2 y_n = 0.0108$

$\nabla^3 y_n = -0.0033$

and, $\nabla^4 y_n = 0.0006$

$h = \text{Interval of } x = 0.25$

$$u = \frac{x - x_n}{h} = \frac{0.75 - 2.00}{0.25} = -5$$

Putting these values in the equation (2); we get,

$$y(0.75) = 0.1353 + \frac{-5}{1!}(-0.0385) + \frac{(-5)(-5+1)}{2!}(0.0180)$$
$$+\frac{(-5)(-5+1)(-5+2)}{3!}(-0.0033)$$
$$+\frac{(-5)(-5+1)(-5+2)(-5+3)}{4!}(0.0006)$$

∴ $y(0.75) = 0.4718$

∴ $e^{-0.75} = 0.4718$

b) For $x = 2.25$

$$u = \frac{x - x_n}{h} = \frac{2.25 - 2.00}{0.25} = 1$$

Again, putting these values in the equation (2); we get,

$$y(2.25) = 0.1353 + \frac{1}{1!}(-0.0385) + \frac{1(1+1)}{2!}(0.0180)$$

$$+\frac{1(1+1)(1+2)}{3!}(-0.0033) + \frac{1(1+1)(1+2)(1+3)}{4!}(0.0006)$$

$\therefore \quad y(2.25) = 0.1049$

$\therefore \quad e^{-2.25} = 0.1049$

Note

From the above example it is obvious that either of the two formulas may be used to interpolate or extrapolate 'y' corresponding to any value of x, whatever be its position.

Example 4.10

Find the interpolation polynomial for y from the following data, using both Newton's forward and backward formulae.

x	4	6	8	10
y	1	3	8	16

Solution:

From given data above;

x	4	6	8	10
y	1	3	8	16

At first, generating the simple difference table for both Newton's forward and backward formula; we have,

x	y	First s.d. Δy or ∇y	Second s.d. $\Delta^2 y$ or $\nabla^2 y$	Third s.d. $\Delta^3 y$ or $\nabla^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

i) By Newton's forward formula; we have,

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \qquad (1)$$

where, $u = \dfrac{x - x_0}{h}$

and, $h = \text{Interval of } x = 2$

$x_0 = 4$

$$y_0 = 1$$
$$\Delta y_0 = 2$$
$$\Delta^2 y_0 = 3$$
and, $\Delta^3 y_0 = 0$

$\therefore \quad u = \dfrac{x - x_0}{h} = \dfrac{x-4}{2}$

Putting these valves in the equation (1); we have,

$$y(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)\left(\frac{x-4}{2}-2\right)}{3!}(0)$$

or, $y(x) = 1 + x - 4 + \dfrac{(x-4)(x-4-2)}{2\times 2\times 2} \times 3$

or, $y(x) = 1 + x - 4 + \frac{3}{8} \times (x-4)(x-6)$

or, $y(x) = 1 + (x-4)\left[1 + \frac{3}{8}(x-6)\right]$

or, $y(x) = \dfrac{1 + (x-4)(8 + 3x - 18)}{8}$

or, $y(x) = \frac{1}{8}[8 + (x-4)(3x-10)]$

or, $y(x) = \frac{1}{8}(8 + 3x^2 - 10x - 12x + 40)$

$\therefore \quad y(x) = \frac{1}{8}(3x^2 - 22x + 48);$

which is the required interpolating polynomial for 'y'.

ii) By Newton's backward formula; we have,

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n \qquad (2)$$

where, $u = \dfrac{x - x_n}{h}$

where, $x_n = 10$

and, $h = 2$

and, $y_n = 16$

$\therefore \quad u = \dfrac{x-10}{h}$

$$\nabla y_n = 8$$
$$\nabla^2 y_n = 3$$
and, $\nabla^3 y_n = 0$

Now, putting these values in the equation (2); we have,

$$\therefore \quad y(x) = 16 + \frac{\left(\frac{x-10}{2}\right)}{1!}(8) + \frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2}+1\right)}{2!}(3) + \frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2}+1\right)\left(\frac{x-10}{2}+2\right)}{3!}(0)$$

$$\text{or,} \quad y(x) = 16 + 4(x-10) + \frac{3(x-10)(x-10-2)}{2\times 2\times 2}$$

$$\text{or,} \quad y(x) = 16 + (x-10)\left[4 + \frac{3(x-8)}{8}\right]$$

$$\text{or,} \quad y(x) = 16 + \left[\frac{(x-8)(32-3x-24)}{8}\right]$$

$$\text{or,} \quad y(x) = \frac{1}{8}[128 + (x-10)(3x+8)]$$

$$\text{or,} \quad y(x) = \frac{1}{8}(128 + 3x^2 + 8x - 30x - 80)$$

$$\therefore \quad y(x) = \frac{1}{8}(3x^2 - 22x + 48);$$

which is the required interpolating polynomial for 'y'.

4.2 Central Difference

Interpolation: Sterling's Formula, Bessel's Formula

As we have seen above that The Newton's forward interpolation formula contains the leading term y_0 and the leading forward differences, namely Δy_0, $\Delta^2 y_0$, etc. It assumes that y_0 is the leading value of 'y' in the table of values. Similarly, the Newton's backward difference formula contains the last value of 'y' and the leading backward differences, namely y_n, ∇y_n, $\nabla^2 y_n$, etc. However, we may assume that y_0 occurs at or near the centre of the table of values, in which case the values of 'y' preceding y_0 will be y_{n-1}, y_{n-2}, y_{n-3}, etc. and the values of 'y' following y_0 will be y_1, y_2, y_3, etc.

The following five interpolation formulae usually called central difference interpolation formulae are derived on the assumption that y_0 occurs at or near the centre of the table of values of y and that the forward difference at y_0, y_{-1}, y_{-2}, etc. are available.

i) Gauss's forward interpolation formula
ii) Gauss's backward interpolation formula
iii) Laplace-Everett formula
iv) Sterling y formula
v) Bessel's formula

According to our syllabus; we have to study about only two formulas in detail which are;

i) Sterling's formula
ii) Bessel's formula

These about two formulas are described in detail below:

i) Sterling's formula

From Gauss's forward formula; we have,

$$y = y(x) = y_0 + \left[\binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1}\right] + \left[\binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2}\right] + \dots\dots\dots \quad (1)$$

From Gauss's backward formula; we have,

$$y = y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \left[\binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2}\right] + \left[\binom{u+2}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-3}\right] + \dots\dots\dots \quad (2)$$

If we get 'y' as the average of the value of y given by Gauss's forward and backward interpolation formula; we get, Sterling's interpolation formula.

So, adding equation (1) and (2) such that the differences of the same order grouped and dividing by 2; we have,

$$y = y(x) = y_0 + \binom{u}{1}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \left[\binom{u}{2} + \binom{u+2}{4}\right]\frac{\Delta^2 y_{-1}}{2} + \binom{u+1}{3}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left[\binom{u}{4} + \binom{u+2}{4}\right]\frac{\Delta^4 y_{-2}}{2} + \dots\dots\dots$$

i.e.,
$$y = y(x) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2 - 1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2 - 1^2)}{4!}\Delta^4 y_{-2} + \dots\dots\dots;$$

which is Sterling's formula.

Note

1. Sterling's formula contains even order differences in the central line and the average the same order lying on the lines just above and just below the central line as indicated below.

$$y_0 \longrightarrow \begin{pmatrix}\Delta y_{-1} \\ \Delta y_0\end{pmatrix} \longrightarrow \Delta^2 y_{-1} \longrightarrow \begin{pmatrix}\Delta^3 y_{-2} \\ \Delta^3 y_{-1}\end{pmatrix} \longrightarrow \Delta^4 y_{-2} \longrightarrow \begin{pmatrix}\Delta^5 y_{-3} \\ \Delta^5 y_{-2}\end{pmatrix} \longrightarrow$$

2. While applying the Sterling's formula, (x_0, y_0) is so chosen that $-\frac{1}{2} < u < \frac{1}{2}$; where, $u = \frac{x - x_0}{h}$

We will get better results if $-\frac{1}{4} < u < \frac{1}{4}$

ii) Bessel's formula

From Gauss's forward formula; we have,

$$y = y(x) = y_0 + \left[\binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1}\right] + \left[\binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2}\right] + \ldots\ldots\ldots \quad (1)$$

Now,

$$\Delta y_0 = y_1 - y_0$$

$$y_0 = y_1 - \Delta y_0 \quad (2)$$

Similarly,

$$y_{-1} = y_0 - \Delta y_{-1}$$

$$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \text{ (On operating by } \Delta^2) \quad (3)$$

Also,

$$y_{-2} = y_{-1} - \Delta y_{-2}$$

$$\therefore \quad \Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2} \text{ (On operating by } \Delta^4) \quad (4)$$

and so on.

Splitting the alternate term in (1) into two equal parts, (1) can be written as;

$$y = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + u\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{u(u+1)(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{u(u+1)(u-1)(u-2)}{5!}\Delta^4 y_{-2} + \ldots\ldots\ldots \quad (5)$$

Using equations (2), (3) and (4) for the second split parts in (5); we have,

$$y = \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + u\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{u(u-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{u(u+1)(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{u(u+1)(u-1)(u-2)}{4!}(\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \ldots\ldots\ldots$$

or, $$\left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{u(u-1)}{2!}\left(-\frac{1}{2} + \frac{u+1}{3}\right)\Delta^3 y_{-1} + \ldots\ldots\ldots$$

i.e., $$y = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots\ldots; \quad (6)$$

which Bessel's formula.

Note

1. Bessel's formula contains odd order differences lying on the line just below the central line and the average of the even order differences of the same order lying on the lines corresponding to y_0 and y_1 as indicated.

$$\begin{pmatrix} y_0 \\ \\ y_1 \end{pmatrix} \begin{matrix} \longrightarrow \\ \rightarrow \Delta y_0 \rightarrow \\ \longrightarrow \end{matrix} \begin{pmatrix} \Delta^2 y_{-1} \\ \\ \Delta^2 y_0 \end{pmatrix} \begin{matrix} \longrightarrow \\ \rightarrow \Delta^3 y_{-1} \rightarrow \\ \longrightarrow \end{matrix} \begin{pmatrix} \Delta^4 y_{-2} \\ \\ \Delta^4 y_{-1} \end{pmatrix} \begin{matrix} \rightarrow \text{Central } (y_0) \text{ line} \\ \rightarrow \\ \rightarrow y_1 \text{ line} \end{matrix}$$

2. As Bessel's formula is only a modified version of Gauss's forward formula, (x_0, y_0) is so fixed that $0 < u < 1$. However, we will get better results if $\frac{1}{4} < u < \frac{3}{4}$.

3. If $u = \frac{1}{2}$ the coefficients if odd order differences in Bessel's formula become zero. Putting $u = \frac{1}{2}$ in equation (6), Bessel's formula becomes;

$$y = \left(\frac{y_0 + y_1}{2}\right) - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3}{128}\left(\frac{\Delta^2 y_{-2} + \Delta^2 y_{-1}}{2}\right) - \frac{5}{1024}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) \quad (7)$$

4. Formula (7) which is a particular case of Bessel's formula is suitable for interpolating at the mid-point of any interval, considered as (x_0, x_1). Hence, the formula is known as formula for interpolating to halves.

5. Formula (7) contains only even order differences.

Example 4.11

By using the sterling's formula, find the value of tan 89°26' the following table below gives the value of tan x:

x	89°21'	89°23'	89°25'	89°27'	89°29'
tan x	88.14	92.21	98.22	104.17	110.90

Solution:

From the given,

x	89°21'	89°23'	89°25'	89°27'	89°29'
tan x	88.14	92.21	98.22	104.17	110.90

At first generating the simple difference tables

x	y = tan x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
89°21'	88.14				
		4.77			

89°23'	92.21		0.54		
		$\boxed{5.31}$		$\boxed{0.10}$	
89°25'	$\boxed{98.22}$		$\boxed{0.64}$		$\boxed{0.04}$
		$\boxed{5.95}$		$\boxed{0.14}$	
89°27'	104.17		0.78		
		6.73			
89°29'	110.90				

From Sterling's formula; we have,

$$y = y(x) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2 - 1^2)}{3!}$$
$$\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2 - 1^2)}{4!}\Delta^4 y_{-2} + \text{.........} \quad (1)$$

where, $u = \frac{x - x_0}{h}$.

Since, $-\frac{1}{2} < u < \frac{1}{2}$, we choose $x_0 = 89°25'$

When $x_0 = 89°25'$;

$$u = \frac{89°26' - 89°25'}{2'} = \frac{1'}{2'} = 0.5$$

Though u is not strictly less than 0.5 any other choice of x_0 will make u lie outside $\left(-\frac{1}{2}, \frac{1}{2}\right)$, of course, we may choose $x_0 = 89°27'$, in which case $u = 0.5$.

But the choice of $x_0 = 89°25'$ will enable us to make use of more number of differences than $x_0 = 89°27'$.

The odd differences whose average is to be taken are enclosed in big rectangular boxes, while the even differences on the central line are enclosed in small rectangular boxes. Using value of $u = 0.5$ and the relevant values of the differences form the table above in (1); we have,

$$y(x = 89°26') = \tan 89°26'$$
$$= 98.22 + 0.5\left(\frac{5.31 + 5.95}{2}\right) + \frac{0.25}{2} \times 0.64 + \frac{0.5(0.25 - 1)}{6}$$
$$\left(\frac{0.10 + 0.14}{2}\right) + \frac{0.25(0.25 - 1)}{24} \times 0.04$$
$$= 98.22 + 0.5 \times 5.63 + \frac{0.25 \times 0.64}{2} + \frac{0.5 \times (-0.75) \times 0.12}{6}$$
$$+ \frac{0.25 \times (-0.75) \times 0.04}{24}$$

$$= 98.22 + 2.815 + 0.08 - 0.0075 - 0.0003125$$
$$= 101.1071875$$
$$\cong 101.11$$

$\therefore \quad y(x = 89°26') = \tan 89°26' = 101.11$

Example 4.12

Find the value of f(1.12) and f(1.13) using Sterling's formula form the following table:

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
f(x)	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

Solution:

From the given table:

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
f(x)	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

At first generating the simple difference table:

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.00	1.00000						
		0.0247					
1.05	1.02470		– 0.00059				
		$\boxed{0.02411}$		$\boxed{0.00005}$			
1.10	$\boxed{1.04881}$		$\boxed{-0.00054}$		$\boxed{-0.00002}$		
		$\boxed{0.02357}$		$\boxed{0.00003}$		0.00001	
1.15	1.07238		– 0.00051		– 0.00001		– 0.00002
		0.02306		0.00004		– 0.00001	
1.20	1.09544		– 0.00047		– 0.00002		
		0.02259		0.00002			
1.25	1.11803		– 0.00045				
		0.02214					
1.30	1.14017						

i) We have to find f(1.12), *i.e.*, $y(x = 1.12)$; so, we choose $x_0 = 1.10$ so that;

$$u = \frac{1.12 - 1.10}{0.05} = 0.4 < 0.5$$

Using this value of u = 0.4 and the relevant values of the differences form the table above in sterling's formula given by step (1) for the example (1); we have,

$$y(x = 1.12) = 1.04881 + 0.4\left(\frac{0.02411 + 0.02357}{2}\right) + \frac{0.16}{2}$$

$$\times(-0.00054) + \frac{0.4(0.16-1)}{6}\left(\frac{0.00005+0.00003}{2}\right)$$
$$+\frac{0.4(0.16-1)}{24}(-0.00002)$$

or, $y(x = 1.12) = 1.04881 + 0.009536 - 0.0000432 - 0.00000224$
$+0.000000112$

or, $y(x = 1.12) = 1.058300672 \cong 1.05830$

$\therefore$ $y(x = 1.12) = 1.05830$

ii) Again, to find f(1.13), *i.e.*, $y(x = 1.13)$ we cannot choose $x_0 = 1.10$ as

$$u = \frac{x - x_0}{h} = \frac{1.13 - 1.10}{0.05} = 0.6 > 0.5$$

Hence to find $y(x = 1.13)$, we choose $x_0 = 1.10$, so that;

$$u = \frac{1.13 - 1.15}{0.05} = -0.4;$$

which lies between -0.5 and 0.5.

Using the value of $u = -0.4$ and the relevant values of the differences form above of the differences form above table (not enclosed by rectangles) in sterling's formula; we have,

$$y(x = 1.13) = 1.07238 + (-0.4)\left(\frac{0.02357 + 0.02306}{2}\right) + \frac{0.16}{2}$$
$$(-0.00051) + \frac{(-0.4)(-0.84)}{6}\left(\frac{0.00003 + 0.00004}{2}\right) + \frac{0.16(-0.84)}{24}$$
$$(-0.00001) + \frac{(-0.4)(-0.84)(-3.84)}{120} \times 0 + \frac{(-0.16)(-0.84)(-3.84)}{720}$$
$$(-0.00002)$$

or, $y(x = 1.13) = 1.07238 - 0.009326 - 0.0000408 + 0.00000196$
$+ 0.000000056 + 0 - 0.000000014$

or, $y(x = 1.13) = 1.063015202 \cong 1.06302$

$\therefore$ $f(1.13) = 1.06302$

Example 4.13

From the following data given below, find the value of f(2.73) interpolation formula.

x	2.5	2.6	2.7	2.8	2.9	3.0
f(x)	0.4938	0.4953	0.4965	0.4974	0.4981	0.4987

Solution:

From the given data:

x	2.5	2.6	2.7	2.8	2.9	3.0
f(x)	0.4938	0.4953	0.4965	0.4974	0.4981	0.4987

Here, we have to find the value of f(2.73), so we chose $x_0 = 2.7$, so that;

$$u = \frac{x - x_0}{h}$$

or, $$u = \frac{2.73 - 2.7}{0.1} = 0.3;$$ which lies between 0.25 and 0.75.

Hence, Bessel's formula is best suited.

Now, generating the simple difference table:

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	0.4938					
		0.0015				
2.6	0.4953		– 0.0003			
		0.0012		0.0000		
2.7	$\boxed{0.4965}$		$\boxed{-0.0003}$		$\boxed{0.0001}$	
		$\boxed{0.0009}$		$\boxed{0.0001}$		– 0.0001
2.8	$\boxed{0.4974}$		$\boxed{-0.0002}$		$\boxed{0.0000}$	
		0.0007		0.0001		
2.9	0.4981		– 0.0001			
		0.0006				
3.0	0.4987					

From Bessel's formula; we have,

$$y = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!} \times \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}$$
$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$
$$+ \frac{\left(u - \frac{1}{2}\right)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \ldots\ldots \quad (1)$$

The even order differences whose average is to be taken are enclosed in big rectangular boxes, while the odd order differences to be included in Bessel's formula are enclose in small rectangular boxes.

Using the value of u = 0.3 and the relevant values of the differences form table above in (1); we have,

$$y(x = 2.73) = \left(\frac{0.4965 + 0.4974}{2}\right) + (0.3 - 0.5)(0.0009) + \frac{(0.3)(-0.7)}{2}$$
$$\left(\frac{-0.0003 - 0.0002}{2}\right) + \frac{(-0.2)(0.3)(-0.7)}{6}(0.0001)$$
$$+ \frac{(1.3)(0.3)(-0.7)(-1.7)}{24}\left(\frac{0.0001 + 0.0000}{2}\right)$$
$$+ \frac{(-0.2)(1.3)(0.3)(-0.7)(-1.7)}{120}(-0.0001)$$

or, $y(x = 2.73) = 0.49695 - 0.00018 + 0.00002625 + 0.0000007$

$+0.000000966 + 0.000000077$

or, $y(x = 2.73) = 0.496797993 \cong 0.4968$

$\therefore$ $f(2.73) = 0.4968$

Example 4.14

Given that sin(0.1) = 0.0998, sin(0.2) = 0.1986, sin(0.3) = 0.255, sin(0.4) = 0.3894 and sin(0.5) = 0.4794, find sin(0.35) and sin(0.37) using Bessel's formula.

Solution:

From the given data;

$\sin(0.1) = 0.0998$

$\sin(0.2) = 0.1986$

$\sin(0.3) = 0.255$

$\sin(0.4) = 0.3894$

$\sin(0.5) = 0.4794$

At first, generating the simple difference table:

x	y = sin x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	0.0998				
		0.0988			
0.2	0.1986		- 0.0019		
		0.0969		- 0.0011	
0.3	$\boxed{0.2955}$		$\boxed{-0.0030}$		$\boxed{0.0002}$
		$\boxed{0.0939}$		$\boxed{-0.0009}$	
0.4	$\boxed{0.3894}$		$\boxed{-0.0039}$		
		0.0900			
0.5	0.4794				

i) To find the value of $\sin(0.35) = y(x = 0.35)$, we choose $x_0 = 0.30$; so that,

$$u = \frac{x - x_0}{h} = \frac{0.35 - 0.30}{0.1} = 0.5$$

Hence Bessel's modified formula for interpolating to halves may be used:

$$y = \left(\frac{y_0 + y_1}{2}\right) - \frac{1}{8}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3}{128}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) \quad \ldots\ldots (1)$$

Using the relevant values of the differences from the table above in (1); we have,

$$y(x = 0.35) = \left(\frac{0.2955 + 0.3894}{2}\right) - \frac{1}{8}\left(\frac{-0.0030 - 0.0039}{2}\right)$$

or, $y(x = 0.35) = 0.34245 + 0.00043125 = 0.34288125 \cong 0.3429$

$\therefore$ $\sin(0.35) = 0.3429$

ii) To find value of $\sin(0.37) = y(x = 0.37)$ we choose again $x_0 = 0.30$ that;

$$u = \frac{x - x_0}{h} = \frac{0.37 - 0.30}{0.1} = 0.7;$$

which lies between 0.25 and 0.75.

Hence, Bessel's formula is best suited for interpolating, using $u = 0.7$ and the relevant values of the difference form above table in Bessel's formula (1) given in example (3); we have,

$$y(x = 0.37) = \left(\frac{0.2955 + 0.3894}{2}\right) + (0.7 - 0.5)(0.0939) + \frac{(0.7)(-0.3)}{2}$$

$$\left(\frac{-0.0030 - 0.0039}{2}\right) + \frac{(0.2)(0.7)(-0.3)}{6}(-0.0009)$$

or, $y(x = 0.37) = 0.34245 + 0.01878 + 0.00036225$

or, $y(x = 0.37) = 0.36159855 \cong 0.3616$

$\therefore$ $\sin(0.37) = 0.3616$

4.3 Lagrange's Interpolation

Lagrange's Interpolation Formula for Unequal Intervals

Statement

If $y_0, y_1, \ldots\ldots, y_n$ are the values of a function $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$ which are not necessarily equally spaced; then,

$$y = f(x)$$

$$= \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_1 - x_2)\ldots(x_1 - x_n)}y_1$$

$$+\ldots\ldots + \frac{(x - x_0)(x - x_2)\ldots(x - x_{n-1})}{(x_n - x_1)(x_n - x_2)\ldots(x_n - x_{n-1})}y_n$$

Proof

Since, $(n + 1)$ values of $y = f(x)$ are given, we can approximate $f(x)$ by a polynomial in n^{th} of the nth degree.

We assume $f(x)$ as the sum of $(n + 1)$ terms each of which is a polynomial of the n^{th} degree as given below:

$$y = a_0(x - x_1)(x - x_2)\ldots\ldots(x - x_n) + a_1(x - x_0)(x - x_2)\ldots\ldots(x - x_n)$$

$$+\ldots\ldots + a_i(x - x_0)(x - x_1)\ldots\ldots(x - x_{i-1})(x - x_{i+1})\ldots\ldots(x - x_n)$$

$$+\ldots\ldots + a_n(x - x_0)(x - x_1)\ldots\ldots(x_n - x_{n-1}) \quad (1)$$

Note

The factor $(x - x_i)$ is not to be included in the term with coefficient ... in the equation (1).

1. Equation (1) is satisfied by the given pairs of values $(x_0, y_0), (x_1, y_1), \ldots\ldots\ldots, (x_n, y_n)$

i.e., when $x = x_0, y = y_0$

Using this in (1); we have,

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \ldots\ldots\ldots (x_0 - x_n)$$

$$\therefore \quad a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \ldots\ldots\ldots (x_0 - x_n)}$$

Similarly, putting $x = x_0$ and $y = y_0$ in (1); we have,

$$y_1 = a_1(x_1 - x_0)(x_1 - x_2) \ldots\ldots\ldots (x_1 - x_n)$$

$$\therefore \quad a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots\ldots (x_1 - x_n)}$$

In the same manner,

Putting $x = x_2, y = y_2$; we can get,

$$y_2 = a_2(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \ldots\ldots\ldots (x_2 - x_n)$$

$$\therefore \quad a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \ldots\ldots\ldots (x_2 - x_n)}$$

and, so on, finally putting $x = x_n$ and $y = y_n$; we can get,

$$y_n = a_n(x_n - x_0)(x_n - x_1) \ldots\ldots\ldots (x_n - x_{n-1})$$

$$\therefore \quad a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \ldots\ldots\ldots (x_n - x_{n-1})}$$

Using these values of $a_0, a_1, \ldots\ldots\ldots, a_n$ in (1); we get, langrage's interpolation formula in the required form given in the statement.

$$y = a_0(x - x_1)(x - x_2) \ldots\ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots\ldots (x - x_n) + \ldots\ldots\ldots + a_i(x - x_0)(x - x_1) \ldots\ldots (x - x_{i-1}) (x - x_{i+1}) \ldots\ldots (x - x_n) + \ldots\ldots\ldots + a_n(x - x_0)(x - x_1) \ldots\ldots (x_n - x_{n-1})$$

or, $$y = \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_1 - x_2) \ldots (x_1 - x_n)} y_1 + \ldots\ldots\ldots + \frac{(x - x_0)(x - x_2) \ldots (x - x_{n-1})}{(x_n - x_1)(x_n - x_2) \ldots (x_n - x_{n-1})} y_n;$$

which is the required Lagrange's Interpolation formula for unequal intervals.

Alternative Forms of Lagrange's Formula

1. Dividing both sides of the Lagrange's interpolation formula by $(x - x_0)(x - x_1) \ldots\ldots (x - x_n)$, it takes the following form:

$$\frac{y}{(x - x_0)(x - x_1) \ldots (x - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} \frac{1}{(x - x_0)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)} \frac{1}{(x - x_1)} + \ldots\ldots\ldots$$

$$+\frac{y_n}{(x_n-x_0)(x_n-x_2)\ldots(x_n-x_{n-1})}\frac{1}{(x-x_n)}$$

2. Yet another form of Lagrange's interpolation formula is;

$$y=\sum_{r=0}^{n}\frac{\phi(x)}{\phi'(x_r)}\frac{y_r}{x-x_r},$$

where, $\phi(x)=\prod_{r=0}^{n}(x-x_r)=(x-x_0)(x-x_1)\ldots\ldots(x-x_n)$.

The derivation is as follows:

$$\phi(x)=(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)$$

$$\phi'(x)=(x-x_1)(x-x_2)\ldots\ldots(x-x_n)+(x-x_0)(x-x_2)\ldots\ldots$$
$$(x-x_n)+(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})$$

By the extension of product formula of differentiation; we have,

$\therefore\quad \phi'(x_0)=(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)$

= Denominator of the first term in Lagrange's interpolation formula

Similarly, $\phi'(x_1)$, $\phi'(x_2)$, ..., $\phi'(x_n)$ are the denominators of the second, third, ..., $(n+1)^{th}$ terms in the Lagrange's interpolation formula.

Using $\phi(x)$ and the values of $\phi(x_0)$, $\phi(x_1)$, etc. in Lagrange's interpolating formula; we have,

$$y=\frac{\phi(x)}{x-x_0}\frac{y_0}{\phi'(x_0)}+\frac{\phi(x)}{x-x_1}\frac{y_1}{\phi'(x_1)}+\ldots\ldots\ldots+\frac{\phi(x)}{x-x_n}\frac{y_n}{\phi'(x_n)}$$

i.e., $$y=\sum_{r=0}^{n}\frac{\phi(x)}{\phi'(x_r)}\frac{y_r}{x-x_r}$$

where, $\phi(x)=\prod_{r=0}^{n}(x-x_r)$.

3. In the Lagrange's interpolation formula derived earlier; we have, treated 'y' as the dependent variable that was expressed as a function of the independent variable x. If we treat x (dependent variable) as a function of y (independent variable), then Lagrange's interpolation formula can be put as;

$$x=\frac{(y-y_1)(y-y_2)\ldots\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots\ldots(y_0-y_n)}x_0$$
$$+\frac{(y-y_0)(y-y_2)\ldots\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots\ldots(y_1-y_n)}x_1+\ldots\ldots\ldots$$
$$+\frac{(y-y_0)(y-y_1)\ldots\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)\ldots\ldots\ldots(y_n-y_{n-1})}x_n$$

This relation is sometimes referred to as Lagrange's inverse interpolation formula.

Example 4.15

By using Lagrange's methods determine the percentage number of patients over 40 years using the following data given below:

Age over 40 years (x):	30	35	45	55
% number of patients (y):	148	96	68	34.

Solution:

From the given data;

Age over 40 years (x):	30	35	45	55
% number of patients (y):	148	96	68	34

From Lagrange's formula; we have,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_2)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \quad (1)$$

where, $x_0 = 30, \quad y_0 = 148$

$x_1 = 35, \quad y_1 = 96$

$x_2 = 45, \quad y_2 = 68$

$x_3 = 55, \quad y_3 = 34$

and, also;

$x = 40$

Putting these above values in (1); we get,

$$[y]_{x=40} = \frac{(40-35)(40-45)(40-55)}{(30-35)(30-45)(30-55)} \times 148 + \frac{(40-30)(40-45)(40-55)}{(35-30)(35-45)(35-55)}$$
$$\times 96 + \frac{(40-30)(40-45)(40-55)}{(45-30)(45-35)(45-55)} \times 68$$
$$+ \frac{(40-30)(40-35)(40-45)}{(55-30)(55-35)(55-45)} \times 34$$

or, $\quad [y]_{x=40} = -\frac{148}{5} + 72 + 34 - \frac{17}{10}$

$\therefore \quad [y]_{x=40} = 74.70$

Hence, percentage number of patients over 40 years is 74.70.

Example 4.16

By using Lagrange's interpolation formula, find f(x) and $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$, $f(7) = 128$ and hence find f(5) and f(6).

Solution:

From the given;

$f(1) = 2 = y_0$

$x_0 = 1$

$f(2) = 4 = y_1$

or, $x_1 = 2$

$f(3) = 8 = y_2$

or, $x_2 = 3$

$f(4) = 16 = y_3$

or, $x_3 = 4$

and, $f(7) = 128 = y_4$

or, $x_4 = 7$

Then, we have to find f(5) and f(6) *i.e.*, value of function at x = 5 and x = 6.

From Lagrange's interpolation formula; we have,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_2)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4 \quad (1)$$

Now, putting the values of x_0, y_0, x_1, y_1, x_2, y_2, x_3, y_3 and x_4, y_4 in the equation (1); we have,

or, $$f(x) = \frac{(x-2)(x-3)(x-4)(x-7)}{(1-2)(1-3)(1-4)(1-7)} \times 2 + \frac{(x-1)(x-3)(x-4)(x-7)}{(2-1)(2-3)(2-4)(2-7)} \times 4 + \frac{(x-1)(x-2)(x-4)(x-7)}{(3-1)(3-2)(3-4)(3-7)} \times 8 + \frac{(x-1)(x-2)(x-3)(x-7)}{(4-1)(4-2)(4-3)(4-7)} \times 16 + \frac{(x-1)(x-2)(x-3)(x-4)}{(7-1)(7-2)(7-3)(7-4)} \times 128$$

or, $$f(x) = \frac{1}{18}[(x^2-5x+6)(x^2-11x+28)] - \frac{2}{5}[(x^2-4x+3)(x^2-11x+28)] + [(x^2-3x+2)(x^2-11x+28)] - \frac{8}{9}[(x^2-3x+2)(x^2-10x+21)] + \frac{16}{45}[(x^2-3x+2)(x^2-7x+12)]$$

or, $$f(x) = \frac{1}{18}(x^4 - 11x^3 + 28x^2 - 5x^3 + 55x^2 - 140x + 6x^2 - 66x + 168) - \frac{2}{5}(x^4 - 11x^3 + 28x^2 - 4x^3 + 44x^2 - 112x + 3x^2 - 33x + 84) + (x^4 - 11x^3 + 28x^2 - 3x^3 + 33x^2 - 84x + 2x^2 - 22x + 56) - \frac{8}{9}(x^4 - 10x^3 + 21x^2 - 3x^3 + 30x^2 - 63x + 2x^2 - 20x + 42) - \frac{16}{45}(x^4 - 7x^3 + 12x^2 - 3x^3 + 21x^2 - 36x + 2x^2 - 14x + 24)$$

or, $f(x) = \frac{1}{18}(x^4 - 16x^3 + 89x^2 - 20+x + 168) - \frac{2}{5}(x^4 - 15x^3 + 75x^2$

$-145x + 84) + (x^4 - 14x^3 + 63x^2 - 106x + 56) - \frac{8}{9}(x^4 - 13x^3$

$+53x^2 - 83x + 42) - \frac{16}{45}(x^4 - 10x^3 + 35x^2 - 50x + 24)$

or, $f(x) = \frac{11}{90}x^4 - \frac{8}{9}x^3 + \frac{59}{18}x^2 - \frac{31}{9}x + \frac{44}{15}$

Hence,

$$f(x) = \frac{1}{90}(11x^4 - 80x^3 + 295x^2 - 310x + 264); \quad (2)$$

which is the required function.

Now, for finding the value of f(5) and f(6); putting x = 5 in (2); we get,

$$f(5) = \frac{1}{90}[11 \times (5)^4 - 80 \times (5)^3 + 295 \times (5)^2 - 310 \times 5 + 264]$$

$\therefore$ $f(5) = 32.933$

Again, when x = 6;

$$f(6) = \frac{1}{90}[11 \times (6)^4 - 80 \times (6)^3 + 295 \times (6)^2 - 310 \times 6 + 264]$$

$\therefore$ $f(6) = 66.67$

Example 4.17

Find the equation of the cubic curve that passes through the points (–1, – 8), (0, 3), (2, 1) and (3, 2) using Lagrange's interpolation formula.

Solution:

From the given;

x:	- 1	0	2	3
y:	- 8	3	1	2

We have from Lagrange's interpolation formula; we have,

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}y_1$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}y_3 \quad (1)$$

Also, we have from given,

$x_0 = -1$, $y_0 = -8$

$x_1 = 0$, $y_1 = 3$

$x_2 = 2$, $y_2 = 1$

$x_3 = 3$, $y_3 = 2$

Now, putting these values in (1); we get,

$$f(x) = y = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{[x-(-1)](x-2)(x-3)}{[0-(-1)](0-2)(0-3)}(3)$$

$$+\frac{[x-(-1)](x-0)(x-3)}{[2-(-1)](2-0)(2-3)}(1)+\frac{[x-(-1)](x-0)(x-2)}{[3-(-1)](3-0)(3-2)}(2)$$

or, $$y=\frac{2}{3}x(x^2-5x+6)+\frac{1}{2}(x+1)(x^2-5x+6)+\left(-\frac{1}{6}\right)(x+1)(x^2-3x)$$
$$+\frac{1}{6}(x+1)(x^2-2x)$$

or, $$y=\frac{2}{3}(x^3-5x^2+6)+\frac{1}{2}(x^3-5x^2+6x+x^2-5x+6)+\left(-\frac{1}{6}\right)$$
$$(x^3-3x^2+x^2-3x)+\frac{1}{6}(x^3-2x^2+x^2-2x)$$

or, $$y=\frac{2}{3}x^3-\frac{10}{3}x^2+4x+\frac{1}{2}x^3-2x^2+\frac{1}{2}x+3-\frac{1}{6}x^3+\frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6}x^3$$
$$-\frac{1}{6}x^2-\frac{1}{3}x$$

or, $$y=\frac{7}{6}x^3-\frac{31}{6}x^2+\frac{14}{3}x+3$$

Therefore, the equation of the required cubic curve is;

$$6y = 7x^3 - 31x^2 + 28x + 18$$

Example 4.18

By using Lagrange's interpolation formula to find the value of x when y = 20 and when y = 40, using the following data:

x:	1	2	3	4
y:	1	8	27	64

Solution:

From the given data;

x:	1	2	3	4
y:	1	8	27	64

Hence, since the values of x corresponding to certain values of y are required we will use Lagrange's inverse interpolation formula *i.e.,* we will express x as a polynomial in y, as follows:

$$x=\frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$
$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_2)(y_2-y_3)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3 \quad (1)$$

Also; we have, from given,

$y_0 = 1,$ $\quad x_0 = 1$

$y_1 = 8,$ $\quad x_1 = 2$

$y_2 = 27,$ $\quad x_2 = 3$

and, $y_3 = 64,$ $\quad x_3 = 4$

Now, putting these values in (1); we get,
Also, when y = 20

$$x(y=20)=\frac{(20-8)(20-27)(20-64)}{(1-8)(1-27)(1-64)}(1)+\frac{(20-1)(20-27)(20-64)}{(8-1)(8-27)(8-64)}(2)+\frac{(20-1)(20-27)(20-64)}{(27-1)(27-8)(27-64)}(3)+\frac{(20-1)(20-8)(20-27)}{(64-1)(64-8)(64-27)}(4)$$

or, $x(y=20)=-\frac{88}{273}+\frac{11}{7}+\frac{792}{481}-\frac{33}{77}$

$\therefore$ $x(y=20)=2.8468$

Again, when y = 40; then,
From equation (1); we get,

$$x(y=40)=\frac{(40-8)(40-27)(40-64)}{(1-8)(1-27)(1-64)}(1)+\frac{(40-1)(40-27)(40-64)}{(8-1)(8-27)(8-64)}(2)+\frac{(40-1)(40-8)(40-64)}{(27-1)(27-8)(27-64)}(3)+\frac{(40-1)(40-8)(40-27)}{(64-1)(64-8)(64-27)}(4)$$

or, $x(y=40)=\frac{128}{147}-\frac{3042}{931}+\frac{3456}{703}+\frac{2704}{5439}$

$\therefore$ $x(y=40)=3.0165$

4.4 Least Square Method of Fitting Linear and Nonlinear Curve for Discrete and Continuous Function

Introduction

In many branches of applied mathematics, it is required to express a given data. Obtained from observations, in the form of a law connecting the two variables involved, such a law inferred by some scheme, is known as the empirical law. *For example;* it may connect the length and the temperature of a metal bar. At various temperatures the length of bar is measure. Then, by one of the methods explained below a law is obtained that represents the relationship existing between temperature and length for observed values. This relation can then be used to predict length at an arbitrary temperature.

Scatter Diagram

To find a relationship between the set of paired observations x and y (say), We plot their corresponding values on the graph, taking one of the variables along the x-axis and other along the y-axis *i.e.,* (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). The resulting diagram showing a collection of dots is called a scatter diagram. A smooth curve that approximates the above set of points is known as the approximate curve.

Curve Fitting

Several equations of different types can be obtained approximately. But the problem is to find the equation of the curve of 'best fit' which may be most suitable for pre ducting the unknown values. The process of finding such an equation of 'best fit' is known as curve-fitting.

If there are n pairs of observed values then it is possible to fit the given data to an equation that contains 'n' arbitrary constants for we can solve 'n' simultaneous equations for n unknowns. If it were desired to obtain an equation representing these data but having less than 'n' arbitrary constants, then we can have recourse to any of the four methods: Graphical, method of group averages and methods of moments. The graphical method and the method of averages fail to give the method of averages fail to give the values of the unknown constraints uniquely and accurately, while the other methods do. The method of least squares is probably the best to fit a unique curve to a given data. It is widely used in applications and can be easily implemented o a computer.

Laws Reducible to the Linear Law

We give below some of the laws in common use, indicating the way these can be reduced to the linear form by suitable substitutions:

1. When the law is $y = mx^n + C$

   Taking $x^n = X$ and $y = Y$, the above law becomes,

   $$Y = mX^n + C$$

2. When the law is $y = ax^n$

   Taking logarithms of both sides, it becomes;

   $$\log_{10} y = \log_{10} a + n \log_{10} x$$

   Putting, $\log_{10} y = Y$ and $\log_{10} x = X$, also $\log_{10} x = \text{Constant (C)}$ it reduces to the form;

   $$Y = C + nX$$

   or, $Y = nX + C$

3. When the law is $y = ax^n + b \log x$

   Dividing on both sides by $\log x$, then it becomes;

   $$\frac{y}{\log x} = a\frac{x^n}{\log x} + b$$

   Putting $\frac{y}{\log x} = Y$ and $\frac{x^n}{\log x} = X$

   Then the above law becomes;

   $$Y = aX + b$$

4. When the law is $y = ae^{bx}$

   Taking logarithms on both the sides; it becomes,

$$\log_{10} y = \log_{10} a + (b \log_{10} e)x$$

Putting $\log_{10} y = Y$ and $x = X$

Then, it takes the form;

$$Y = mX + C$$

where, $m = b\log_{10} e$

and, $C = \log_{10} a$

5. When the law is $xy = ax + by$

Dividing by 'x' on both the sides; we have,

$$y = a + b\left(\frac{y}{x}\right)$$

Putting $\frac{y}{x} = X$ and $y = Y_1$

It reduces to the form;

$$Y = bX + a$$

Least Square Method of Fitting Linear Curve (A straight line)

Let the equation of the best fitting straight line for the given data represented by the points $(x_r, y_r)(r = 1, 2, \ldots\ldots\ldots, n)$ be $Y = ax + b$ (1)

Since, the equation (1) represents the best fitting straight line; the plotted points either lie on it or close to it. Hence, the sum of the squares of the residuals of the plotted points can be assumed to be the least. This is the principle of least squares. Thus,

$$S = \sum_{r=1}^{n} \{y_r - (ax_r + b)\}^2 \text{ is least.} \tag{2}$$

Therefore, the necessary conditions for (2) are $\frac{\partial S}{\partial a} = 0$ and $\frac{\partial S}{\partial b} = 0$ (by treating S as a function of the two varying quantities a and b)

Differentiating S partially with respect to a and equating it to zero; we have,

$$\sum_{r=1}^{n} 2\{y_r - (ax_r + b)\}(-x_r) = 0$$

i.e., $$a\sum_{r=1}^{n} x_r^2 + b\sum_{r=1}^{n} x_r = \sum_{r=1}^{n} x_r y_r \tag{3}$$

Differentiating S partially with respect to 'b' and equating it to zero; we have,

$$\sum_{r=1}^{n} 2\{y_r - (ax_r + b)\}(-1) = 0$$

i.e., $$\sum_{r=1}^{n} x_r + nb = \sum_{r=1}^{n} y_r \quad [\because \textstyle\sum_{r=1}^{n} b = nb] \tag{4}$$

Solving equations (3) and (4) simultaneously; we get, the values of 'a' and 'b' and hence the equation of the best fitting straight line.

Note

Dropping the suffix and rearranging the equations (3) and (4) they can be written as;

$$a\sum x + nb = \sum y \quad (5)$$

and, $$a\sum x^2 + b\sum x = \sum xy \quad (6)$$

Equations (5) and (6) are called normal equations.

(This term should not be confused with the equation of the normal curve in probability theory/statistics).

Equation (5) can be obtained by simply attaching $\sum$ to each of the terms in the equation $y = ax + b$ and nothing that $\sum b = nb$.

Equation (6) can be obtained by multiplying each of the terms in $y = ax + b$ by x and then attaching $\sum$ to each of the terms.

Least Square Method of Fitting Non–linear Curve

(A second degree parabola)

Let the equation of the best fitting parabola for the given data represented by the points $(x_r, y_r)(r = 1, 2, \ldots\ldots\ldots, n)$ be;

$$y = ax^2 + bx + C \quad (1)$$

By the principle of least squares,

$$S = \sum_{r=1}^{n} \{y_r - (ax_r^2 + bx_r + C)\}^2 \text{ is least.} \quad (2)$$

The necessary conditions for (2) are $\frac{\partial S}{\partial a} = 0$, $\frac{\partial S}{\partial b} = 0$ and $\frac{\partial S}{\partial c} = 0$ (treating 'S' as a function of a, b, c)

i.e., $$\sum_{r=1}^{n} 2\{y_r - (ax_r^2 + bx_r + C)\}^2(-x_r^2) = 0 \quad (3)$$

$$\sum_{r=1}^{n} 2\{y_r - (ax_r^2 + bx_r + C)\}^2(-x_r) = 0 \quad (4)$$

and, $$\sum_{r=1}^{n} 2\{y_r - (ax_r^2 + bx_r + C)\}^2(-1) = 0 \quad (5)$$

Rewriting equations (3), (4) and (5); we get,

$$a\sum x_r^4 + b\sum x_r^3 + C\sum x_r^2 = \sum x_r^2 y_r \quad (4a)$$

$$a\sum x_r^3 + b\sum x_r^2 + C\sum x_r = \sum x_r y_r \quad (5a)$$

$$a\sum x_r^2 + b\sum x_r + xC = \sum y_r \quad (6a)$$

Dropping the suffix r and rearranging the equations (4a), (5a) and (6a); we get, the following normal equations:

$$a\sum x^2 + b\sum x + nC = \sum y \quad (7)$$

$$a\sum x^3 + b\sum x^2 + C\sum x = \sum xy \quad (8)$$
$$a\sum x^4 + b\sum x^3 + C\sum x^2 = \sum x^2y \quad (9)$$

Solving equations (7), (8) and (9) simultaneously; we get, the values of a, b, c and hence, the equation of the best fitting parabola.

Note

1. Equation (7) can be obtained by simply attaching $\sum$ to each of the terms in $y = ax^2 + bx + C$ and nothing that $\sum C = nC$ equation (8) can be obtained by multiplying equation (1) by x through and then attaching $\sum$ to each of the terms. Equation (9) can be obtained by multiplying equation (1) by x^2 throughout and then attaching $\sum$ to each of the terms.
2. From the above discussion it is clear that the method of least squares can be used to fit the curve $y = f(x)$ directly, only if $f(x)$ is a polynomial in x.

Least Square Regression

It is the technique to minimize the sum of squares of error of individual error.

$$S = \sum_{r=1}^{n} \{y_r - (ax_r + b)\}^2$$

- For the equation $y = ax + b$, we can solve by,

$$\begin{bmatrix} n & \sum x_r \\ \sum x_r & \sum x_r^2 \end{bmatrix}\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} \sum y_r \\ \sum x_r y_r \end{bmatrix}$$

$$\therefore \quad a = \frac{n\sum(x_r y_r) - \Sigma x_r \sum y_r}{n\sum x_r^2 - (\Sigma x_r)^2}$$

$$\text{and, } b = \frac{\sum y_r - b\Sigma x_r}{n}$$

- For quadratic equation $y = ax^2 + bx + C$

$$\begin{bmatrix} n & \sum x & \sum x^2 \\ \sum x & \sum x^2 & \sum x^3 \\ \sum x^2 & \sum x^3 & \sum x^4 \end{bmatrix}\begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \sum x^2y \end{bmatrix}$$

Example 4.19

Fit a straight line y = bx + a to the following set of data:

x:	1	2	3	4	5
y:	3	4	5	6	8

Solution:

From the given;

We have to fit a straight line ($y = bx + a$) for the following set of data:

x:	1	2	3	4	5
y:	3	4	5	6	8

Here, at first we have to find the value of constants 'a' and b.

Also, we have to fit straight line $y = bx + a$ introducing $\sum$ on both sides of equation (1)

$$\sum y = \sum bx + \sum a \quad (2)$$

or, $$\sum y = b\sum x + na \quad [\because \sum a = na]$$

$$na + b\sum x = \sum y \quad (3)$$

Again, multiplying equation (1) by 'x' on both sides and also introducing $\sum$ on both sides; we get,

$$\sum xy = \sum bx^2 + \sum ax$$

or, $$\sum xy = b\sum x^2 + a\sum x$$

or, $$a\sum x + b\sum x^2 = \sum xy \quad (4)$$

Now, at first, finding the value of $\sum x$, $\sum y$, $\sum x^2$ and $\sum xy$

x	y	x^2	xy
1	3	1	3
2	4	4	8
3	5	9	15
4	6	16	24
5	8	25	40
$\sum x = 15$	$\sum y = 26$	$\sum x^2 = 55$	$\sum xy = 90$

Also,

$$x = 5$$

Now, putting these values in the equation, (3) and (4) and solving them;

$$5a + 15b = 26 \quad (5)$$

$$15a + 55b = 90 \quad (6)$$

By solving; we get,

$$a = 1.6$$

and, $$b = 1.2$$

Hence, $y = 1.2x + 1.6$ is the required straight line.

Example 4.20

Fit a quadratic curve $ax^2 + bx + C$ for the following set order:

x:	1	2	3	5
y:	-1	2	9	55

Solution:

From given we have to fit a quadratic curve $(ax^2 + bx + C)$ for the following set of orders:

x:	1	2	3	5
y:	- 1	2	9	55

Here, to fit a quadratic curve $(ax^2 + bx + C)$ at first; we have, to find the value of a, b and c.

Let, $y = ax^2 + bx + C$ (1)

At first, introducing $\sum$ on both sides of equation (1); we get,

$$\sum y = \sum ax^2 + \sum bx + \sum C$$

or, $\sum y = a\sum x^2 + b\sum x + nc \quad [\because \sum C = nc]$ (2)

Again, multiplying equation (1) by x and introducing $\sum$ on both sides; we get,

$$\sum xy = \sum ax^3 + \sum bx^2 + \sum cx$$

or, $\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$ (3)

Again, multiplying equation (1) by x^2 on both sides and introducing $\sum$ on both sides; we have,

$$\sum x^2y = \sum ax^4 + \sum bx^3 + \sum x^2c$$

or, $\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2$ (4)

Now finding the value of $\sum x$, $\sum x^2$, $\sum x^4$, $\sum xy$, $\sum x^2y$

x	y	x^2	x^3	x^4	xy	x^2y
1	- 1	1	1	1	- 1	- 1
2	2	4	8	16	4	8
3	9	9	27	81	27	81
5	55	25	125	625	275	1375
$\sum x = -11$	$\sum y = 65$	$\sum x^2 = 39$	$\sum x^3 = 161$	$\sum x^4 = 723$	$\sum xy = 305$	$\sum x^2y = 1463$

Here,

$n = 4$

Now, putting these above values in the equation (2), (3) and (4); we have,

$65 = 39a + 11b + 4C$ (5)

$305 = 161a + 39b + 11C$ (6)

$1463 = 723a + 161b + 39C$ (7)

Now, solving equation (5), (6) and (7) by using calculator;

$a = 4.27$

$b = -11.82$

and, $c = 7.09$

Hence, the required quadratic curve of the best fitting line is;

$$y = 4.27x^2 - 11.82x + 7.09$$

Example 4.21

Fit a curve of the form $y = \frac{x}{ax - b}$ for the data given below by the method of least squares:

x:	2	4	6	8	10
y:	8.8	13.87	17.0	18.9	20.4

Solution:

From the given data;

x:	2	4	6	8	10
y:	8.8	13.87	17.0	18.9	20.4

Here; we have, to fit the curves;

$$y = \frac{x}{ax - b}; \text{ which is not linear.} \quad (1)$$

The above equation (1) can be written as;

$$\frac{1}{y} = \frac{ax - b}{x}$$

or, $$\frac{1}{y} = a + b\left(\frac{1}{x}\right)$$

Putting $X = \frac{1}{x}$ and $Y = \frac{1}{y}$ the equation take the linear form;

$$Y = bX + a \quad (2)$$

Now, introducing Σ to each of the term in the equation (2),

$$\Sigma Y = \Sigma bX + \Sigma a$$

or, $$\Sigma Y = b\Sigma X + na \quad [\because \Sigma a = na] \quad (3)$$

Again, multiplying equation (2) by X throughout and then attaching Σ to each of the terms;

$$\Sigma XY = \sum bX^2 + \Sigma aX$$

or, $$\Sigma XY = b\sum X^2 + a\Sigma X \quad (4)$$

The required values of $\Sigma X, \Sigma Y, \Sigma XY$ and $\sum X^2$ are computed in the table below:

x	y	$X = \frac{1}{x}$	$Y = \frac{1}{y}$	X^2	XY
2	8.8	0.5	0.1136	0.25	0.0568
4	13.7	0.25	0.0730	0.0625	0.0183
6	17.0	0.1667	0.0588	0.0278	0.0098
8	18.9	0.125	0.0529	0.0156	0.0066
10	20.4	0.1	0.0490	0.01	0.0049
		$\Sigma X = 1.1417$	$\Sigma Y = 0.3473$	$\sum X^2 = 0.3659$	$\Sigma XY = 0.0964$

$n = 5$

Using these above values in the equation (3) and (4); we have,

$$0.3473 = 1.1417b + 5a \tag{5}$$

and, $$0.0964 = 0.3659b + 1.1417a \tag{6}$$

Now, solving these above equation (5) and (6); we get,

$$D = \begin{vmatrix} 1.1417 & 5 \\ 0.3659 & 1.1417 \end{vmatrix} = -0.5260$$

$$D_1 = \begin{vmatrix} 0.3473 & 5 \\ 0.0964 & 1.1417 \end{vmatrix} = -0.0855$$

$$D_2 = \begin{vmatrix} 1.1417 & 0.3473 \\ 0.3659 & 0.0964 \end{vmatrix} = -0.0170$$

$$\therefore \quad b = \frac{D_1}{D} = \frac{-0.0855}{-0.5260} = 0.1625$$

and, $$a = \frac{D_2}{D} = \frac{-0.0170}{-0.5260} = 0.0323$$

Using these values of 'a' and 'b' in the equation (1); we get, the required relations;

$$y = \frac{x}{0.0323x + 0.1625}$$

Example 4.22

Determine the values of 'a' and 'b' so that the equation $Q = ah^b$ best fits the following data by the method of least squares:

h:	25	20	12	9	7	5
q:	0.22	0.20	0.15	0.13	0.12	0.10

Solution:

From the given data;

h:	25	20	12	9	7	5
q:	0.22	0.20	0.15	0.13	0.12	0.10

The given relation is;

$$Q = ah^b; \tag{1}$$

which is not linear.

Now, converting relation (1) into linear form by taking logarithms (to the base 10);

$$\log_{10} Q = \log_{10}(ah^b) = \log_{10} a + \log_{10} h^b$$

or, $\log_{10} Q = \log_{10} a + b\log_{10} h$

Now putting $\log_{10} Q = Y$ and $\log_{10} a = A$ then, above equation becomes;

$$Y = A + bx \tag{2}$$

Now, introducing $\sum$ to each of the term in the equation (2);

$$\sum A + \sum bx = \sum Y$$

or, $nA + b\sum x = \sum Y \qquad [\because \sum A = nA]$ (3)

Again, multiplying equation (2) by X throughout and then attaching $\sum$ to each of the terms:

$$\sum AX + \sum bX^2 + \sum XY$$

$$A\sum X + b\sum X^2 + \sum XY \tag{4}$$

The required values of $\sum X$, $\sum Y$, $\sum X^2$ and $\sum XY$ are computed in the table below:

h	Q	$X = \log_{10} h$	$X = \log_{10} h$	X^2	XY
25	0.22	1.3979	– 0.6575	1.9541	– 0.9191
20	0.20	1.3010	– 0.6990	1.6926	– 0.9094
12	0.15	1.0792	– 0.8239	1.1647	– 0.8892
9	0.13	0.9542	– 0.8861	0.9105	– 0.8455
7	0.12	0.8451	– 0.9208	0.7142	– 0.7782
5	0.10	0.6990	– 1.0000	0.4886	– 0.6990
		$\sum X = 6.2764$	$\sum Y = -4.9873$	$\sum X^2 = 6.9247$	$\sum XY = -5.0404$

Also,

$$n = 6$$

Now, using these relevant values in (3) and (4); we have,

$$6A + 6.2764b = -4.9873 \tag{5}$$

$$6.2764A + 6.9247b = --5.0404 \tag{6}$$

Solving equation (5) and (6); we get,

$$D = \begin{vmatrix} 6 & 6.2764 \\ 6.2764 & 6.9247 \end{vmatrix} = 2.1550$$

$$D_1 = \begin{vmatrix} -4.9873 & 6.2764 \\ -5.0404 & 6.9247 \end{vmatrix} = -2.9000$$

$$D_2 = \begin{vmatrix} 6 & -4.9873 \\ 6.2764 & -5.0404 \end{vmatrix} = 1.0599$$

$\therefore \quad A = \frac{D_1}{D} = \frac{-2.9000}{2.1550} = -1.3457$

and, $\quad b = \frac{D_2}{D} = \frac{1.0599}{2.1550} = 0.4918$

Since, $A = \log_{10} a$

or, $\quad a = 10^A = 10^{-1.3457} = 0.0451$

Using these values of 'a' and 'b' in the equation (1), the required relation for best fit is;

$$Q = 0.0451(h)^{0.4918}$$

Example 4.23

Find the equation of the best fitting straight line for the following data, using the method of least squares and assuming that a is the independent variable Estimate the value of y when x = 50

x:	0	5	10	15	20	25	30
y:	10	14	19	25	31	36	39

Solution:

From the given:

We have to find the equation of the best fitting straight line for the following data;

x:	0	5	10	15	20	25	30
y:	10	14	19	25	31	36	39

Here, it is also given that x is independent variable and also we have to find the value of y when x = 50.

Important note

In all the problems considered so far, the values of the independent variable (as given after transformation) were not equally spaced. Very often the value of the independent variable will be equally spaced, *viz.,* in arithmetic progression in such situation, if we make a suitable change of origin and scale, *viz.,* if we make the transformation of the form $u = \frac{x-a}{c}$, the numerical work involved in fitting curves will be considerably reduced.

First we observe that if we put $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$ in the equation $y = f(x)$ in x of some degree, the transformed equation will be $v = \phi(u)$; where, $\phi(u)$ will also be a polynomial in u of the same degree as $f(x)$.

For example;

Let us put $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$ in $y = Ax + B$. The transformed equation is $dv + b = A(cu + a) + B$.

i.e., $v = \frac{Ac}{d}u + \frac{Aa + B - b}{d}$

i.e., $v = \alpha u + \beta$

To simplify the numerical work to the maximum extent, the values of the new origin 'a' and the new scale 'c' are chosen as follows:

i) If the number of values of x given is odd, the middlemost value of x is chosen as 'a' and the common difference of the A.P. is chosen as 'C'.

ii) If the number of values of x given is even, the common average of the two middle most values of x is chosen as 'a' and half the common difference of the A.P. is chosen as 'c'.

The values of the dependent variable will not be generally in A.P. We may have to be satisfied with a change of origin only: *viz,* if it is felt necessary we can choose the average or a value near the average of the extreme values of y as 'b' and d = 1 in $v = \frac{y-b}{d}$.

Since from given, the number of values of x is odd (*i.e.,* n = 7).

So, the middlemost value of x is chosen as 'a' and the common difference of the A.P. is chosen as 'C'.

i.e., a = 15

and, Common difference (C) = 5 − 0 = 5

and, also, we can choose the average or a value near the average of the extreme values of y as 'b' so;

Put, $u = \frac{x-15}{c}$

and, $v = y - 25$

Since, the required relation between x and y is linear, the relation between 'u' and u will also be linear.

Let it be;

$$v = au + b \tag{1}$$

Now, introducing $\sum$ to each of the term in the equation (1);

$$\sum v = \sum au + \sum b \qquad [\because \sum b = nb]$$

$$\therefore \quad a\sum u + nb = \sum v \tag{2}$$

Again, multiplying equation (1) by 'u' throughout and then attaching $\sum$ to each of the terms,

$$\sum uv = \sum au^2 + \sum bu$$

$$\text{or,} \quad a\sum u^2 + b\sum u = \sum uv \tag{3}$$

The required values of $\sum u$, $\sum v$, $\sum u^2$ and $\sum uv$ are computed in the table below:

x	y	$u = \frac{x-15}{c}$	$v = y - 25$	u^2	uv
0	10	- 3	- 15	9	45
5	14	- 2	- 11	4	22
10	19	- 1	- 6	1	6
15	25	0	0	0	0
20	31	1	6	1	6
25	36	2	11	4	22
30	39	3	14	9	42
		$\sum u = 0$	$\sum v = -1$	$\sum u^2 = 28$	$\sum uv = 143$

$n = 7$

Using these above values in the equation (2) and (3); we get,

$$a \times 0 + 7b = -1$$

or, $b = -\frac{1}{7} = -0.1429$

$$28a + b \times 0 = 143$$

or, $a = 5.1071$

Using these value of 'a' and 'b' in (1); we get,

$$v = 5.1071u - 0.1429$$

i.e., $y - 25 = 5.1071\left(\frac{x-15}{5}\right) - 0.1429$ [Using the transformation]

or, $y - 25 = 1.0214x - 15.3214 - 0.1429$

or, $y = 1.0214x + 25 - 15.4643$

or, $y = 1.0214x + 9.5357$;

which is the required equation of the best fitting straight line.

Again, when $x = 50$ then value of y is given by;

$$[y]_{x=50} = 1.0214 \times 50 + 9.5357$$

$\therefore$ $[y]_{x=50} = 60.6057$

Example 4.24

Fit a curve of the form $y = ab^x$ to the following data, by the method of least squares;

x:	1	2	3	4	5	6	7	8
y:	15.3	20.5	27.4	36.6	49.1	65.6	87.8	117.6

Solution:

Since, the relation is $y = ab^x$ which is not linear, so the convert the given relation into a linear relation by taking logarithms (to the base 10) on both the sides.

i.e., $\log_{10} y = \log_{10}(ab^x)$

or, $\log_{10} y = \log_{10} a + \log_{10} b^x$

or, $\log_{10} y = \log_{10} a + x \log_{10} b$

Now, putting $\log_{10} y = Y$, $\log_{10} a = A$ and $\log_{10} b = B$

So, the above relation becomes;

$$Y = A + Bx$$

Since, from given the number of values of 'x' is even (*i.e.,* n = 8) so the common average of the two middlemost values of 'x' is chosen as 'a' and half the common difference of the A.P. is chosen as 'c'.

Here common difference is 1.

So, we put,

$$u = \frac{x - 4.5}{0.5} \qquad \left[\because u = \frac{x - a}{c}\right]$$

$\left[\text{Since, for even, } a = \frac{4+5}{2} = 4.5 \text{ and } C = \frac{\text{Common difference}}{2} = \frac{1}{2} = 0.5\right]$

Then, the relation between u and y is also linear. Let it be;

$$Y = C + Du \qquad (2)$$

Now, introducing Σ to each of the term in the equation (2); $[\because \Sigma C = nc]$

$$\Sigma Y = \Sigma C + \Sigma Du \qquad (3)$$

or, $nc + D\Sigma u = \Sigma Y$

Again, multiplying equation (2) by 'u' throughout and then attaching Σ to each of the terms:

$$\Sigma Yu = \Sigma Cu + \Sigma Du^2 \qquad (4)$$

or, $C\Sigma u + D\Sigma u^2 = \Sigma uY$

The required values of Σu, Σu^2, ΣY and ΣYu are computed in the table below:

x	y	$u = \frac{x-4.5}{0.5}$	$Y = \log_{10} y$	u^2	uY
1	15.3	– 7	1.1847	49	– 8.2929
2	20.5	– 5	1.3117	25	– 6.5585
3	27.4	– 3	1.4377	9	– 4.3131
4	36.6	– 1	1.5635	1	– 1.5635
5	49.1	1	1.6911	1	1.6911
6	65.6	3	1.8169	9	5.4507
7	87.8	5	1.9435	25	9.7175
8	117.6	7	2.0704	49	14.4928
		$\Sigma u = 0$	$\Sigma Y = 13.0195$	$\Sigma u^2 = 168$	$\Sigma uY = 10.6241$

Here,

$$n = 8$$

Using these values in the equation (3) and (4); we get,

$$8C + D \times 0 = 13.0195$$

or, $C = 1.6274$

and, $C \times 0 + 168D = 10.6241$

or, $D = 0.0632$

Using these values of 'C' and 'D' in the equation (2); we get,

$$Y = 1.6274 + 0.0632\left(\frac{x - 4.5}{0.5}\right) \qquad \left[\because u = \frac{x - 4.5}{0.5}\right]$$

or, $Y = 1.6274 + 0.1264x - 0.5688$

or, $Y = 1.0586 + 0.1264x$ (5)

Comparing (1) and (5); we get,

$$A = 1.0586$$

and, $B = 0.1264$

but, $A = \log_{10} a$

or, $a = 10^A = (10)^{1.0586} = 11.4445$

Also,

$$B = \log_{10} b$$

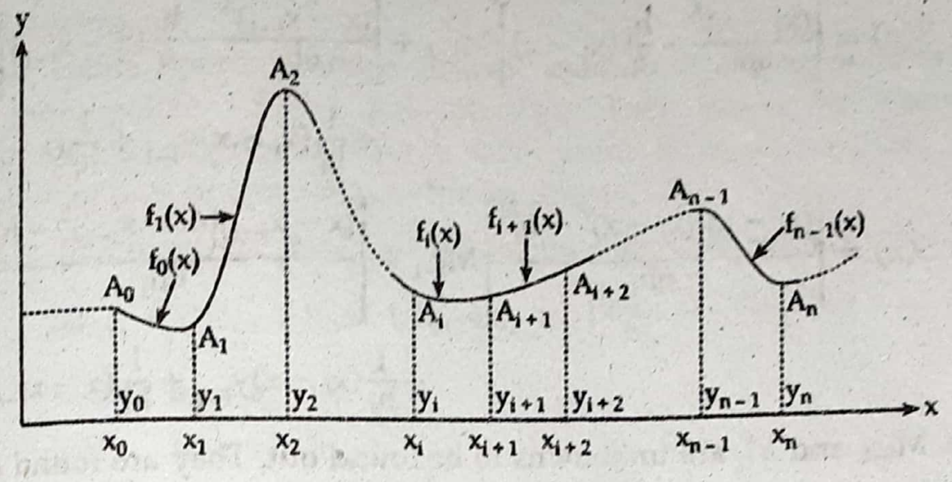
or, $b = 10^B = (10)^{0.1264} = 1.3378$

Therefore, the required relation is $Y = 11.4445 + 1.3378^x$.

4.5 Spline Interpolation (Cubic Spline)

Spline Interpolation

In the above interpolation methods a single polynomial has been fitted to the tabulated points. If the given set of points belongs to the polynomial, then this method works well, otherwise the results are rough approximation only. If we draw lines through every two closest points, the resulting graph will not be smooth. Similarly we may draw a quadratic curve through points A_i, A_{i+1} and another quadratic curve through A_{i+1}, A_{i+2}, such that the slopes of two quadratic curves match at A_{i+1}. The resulting curve looks better but is not quite smooth. We can ensure this by drawing a cubic curve through A_i, A_{i+1} and another cubic through A_{i+1}, A_{i+2} such that the slopes and curvatures of the two curves match at A_{i+1}. Such a curve is called a "cubic spline". We may use polynomials of higher order but the resulting graph is not better. As such, cubic splines are commonly used.



Cubic Spline Interpolation

Another piecewise cubic polynomial of interest is the cubic spline function or polynomial which is defined as follows:

A cubic spline function $S(x)$ with respect to the points $x_0, x_1, \ldots\ldots, x_n$ is a polynomial of degree three in each interval (x_{i-1}, x_i); $i = 1, 2, \ldots\ldots, n$ such that $S(x)$, $S'(x)$ and $S''(x)$ are continuous in (x_0, x_n).

We shall now derive a formula $S(x)$. Since $S(x)$ is a cubic polynomial in $x_{i-1} \le x \le x_i$ is a linear expression in that interval.

Then, by Lagrange's interpolation formula; we have,

$$S''(x) = \frac{x_i - x}{x_i - x_{i-1}} S''(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} S''(x_i) \tag{1}$$

Integrating equation (1) twice with respect to x; we get,

$$S(x) = \frac{(x_i - x)^3}{6h_i}(M_{i-1}) + \frac{(x - x_{i-1})^3}{6h_i}(M_i) \tag{2}$$

where, $x - x_{i-1} = h_i$, $M_i = S''(x_i)$ and C_1, C_2 are constants of integration.

Since, $S(x)$ has to agree with $y = f(x)$ at points x_{i-1} and x_i, we have $S(x_{i-1}) = y_{i-1}$ and $S(x_i) = y_i$, using these conditions in equation (2); we have,

$$\frac{h_i^2}{6} M_{i-1} + C_1 x_{i-1} + C_2 = y_{i-1} \tag{3}$$

and, $$\frac{h_i^2}{6} M_i + C_1 x_i + C_2 = y_i \tag{4}$$

Solving equations (3) and (4) for C_1 and C_2; we get,

$$C_1 = \frac{(y_i - y_{i-1})}{h_i} - \frac{(M_i - M_{i-1})}{6} h_i \tag{5}$$

and, $$C_2 = \frac{(x_i y_{i-1} - x_{i-1} y_i)}{h_i} - \frac{1}{6}(x_i M_{i-1} - x_{i-1} M_i) h_i \tag{6}$$

Using equations (5) and (6) in (2) we have,

$$S(x) = \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} M_i + \frac{(y_i - y_{i-1})x}{h_i} - \frac{(M_i - M_{i-1})}{6} h_i x + \frac{(x_i y_{i-1} - x_{i-1} y_i)}{h_i} - \frac{1}{6}(x_i M_{i-1} - x_{i-1} M_i) h_i \tag{7}$$

or, $$S(x) = \left[\frac{(x_i - x)^3}{6h_i} - \frac{h_i}{6}(x_i - x)\right]M_{i-1} + \left[\frac{(x - x_{i-1})^3}{6h_i} - \frac{h_i}{6}(x - x_{i-1})\right]M_i + \frac{1}{h_i}(x_i - x)y_{i-1} + \frac{1}{h_i}(x - x_{i-1})y_i$$

i.e., $$S(x) = \left[\frac{(x_i - x)\{(x_i - x)^2 - h_i^2\}}{6h_i}\right]M_{i-1} + \left[\frac{(x - x_{i-1})\{(x - x_{i-1})^2 - h_i^2\}}{6h_i}\right]M_i + \frac{1}{h_i}(x_i - x)y_{i-1} + \frac{1}{h_i}(x - x_{i-1})y_i \quad (8)$$

where, M_{i-1} and M_i are unknowns to be found out. They are found out by using the condition of continuity of $S'(x)$ at point x_i, *viz;* by using the condition $S'(x_i - \varepsilon) = S'(x_i - \varepsilon)$ as $\varepsilon \to 0$. (9)

Differentiating equation (7) with respect to x; we have,

$$S(x) = \frac{(x_i - x)^2}{2h_i}M_{i-1} + \frac{(x - x_{i-1})^2}{2h_i}M_i - \frac{(M_i - M_{i-1})h_i}{6} + \frac{(y_i - y_{i-1})}{h_i}; \quad (10)$$

valid for $x < x_i$ *i.e.*, for $x = x_i - \varepsilon$

and, $$S'(x) = -\frac{(x_{i-1} - x)^2}{2h_{i+1}}M_i + \frac{(x - x_i)^2}{2h_{i+1}}M_{i+1} - \frac{(M_{i+1} - M_i)h_{i+1}}{6} + \frac{(y_{i+1} - y_i)}{h_{i+1}}; \quad (11)$$

valid for $x > x_i$ *i.e.*, for $x = x_i + \varepsilon$

From equations (9), (10) and (11); we get,

$$\frac{h_i}{6}M_{i-1} + \frac{h_i}{3}M_i + \frac{1}{h_i}(y_i - y_{i-1}) = -\frac{h_{i+1}}{3}M_i - \frac{h_{i+1}}{6}M_{i+1} + \frac{1}{h_{i+1}}(y_{i+1} - y_i)$$

i.e., $$\frac{h_i}{6}M_{i-1} + \left(\frac{h_i + h_{i+1}}{3}\right)M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{1}{h_{i+1}}(y_{i+1} - y_i) - \frac{1}{h_i}(y_i - y_{i-1}) \quad (12)$$

The $(n-1)$ equations in (12) along with the conditions $M_0 = 0$ and $M_n = 0$ provide $(n+1)$ equations in the $(n+1)$ unknowns $M_0, M_1, \ldots\ldots\ldots, M_n$ and hence they can be found.

Deduction

When $h_i = h$ for all i, S(x) is given by;

$$S(x) = \frac{1}{6h}[(x_i - x)^3M_{i-1} + (x - x_{i-1})^3M_i] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right)$$

where, $M_{i-1} + 4M_i + M_{i-1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$; $i = 1, 2, \ldots\ldots\ldots, (n-1)$

and, $M_0 = M_n = 0$

Conclusion

Cubic spline approximations provide continuity and smoothness at the common points of the sub-intervals. They are more useful than interpolating polynomials for a large number of data points, because number of data points, because higher degree.

Hence, finally we use the following formula for cubic spline interpolation:

i) $$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right] \quad (1)$$

where, $a_0 = a_n = 6.$

and, $h_i = x_i - x_{i-1}$

ii) $$S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1}U_i) \quad (2)$$

where, $U_i = x - x_i$.

Note

1. This method is applicable for uniformly or non-uniformly distributed for any value of x.
2. Number of coefficient is equal to number of points.
3. Evaluate (1) by straight form $i = 1$ onwards a number of times equal to number of unknown coefficients.
4. Now, we evaluate (2) at i = a value given by position of interval.

Example 4.25

Estimate the functional value of 'f' at $x = 7$ using cubic splines form the following data given below:

x:	4	9	16
y:	2	3	4

Solution:

From the given data:

x:	4	9	16
y:	2	3	4

Here, we have to use cubic splines and we have for cubic splines,

$$a_0 = a_n = 0 \text{ (i.e., first and last values are zero)}$$

From cubic spline formula:

At $i = 1$

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right] \quad (1)$$

where, $a_0 = a_n = 0$

and, $h_i = x_i - x_{i-1}$

or, $h_1 = x_1 - x_0 = 9 - 4 = 5$

Also,

$$h_2 = x_2 - x_1 = 16 - 9 = 7$$

Now, putting these values in (1); we have,

$$5 \times 0 + 2a_1(5 + 7) + 7 \times 0 = 6\left(\frac{4-3}{7} - \frac{3-2}{5}\right)$$

or, $$24a_1 = 6\left(\frac{1}{7} - \frac{1}{5}\right)$$

or, $$a_1 = -\frac{1}{70} = -0.01428$$

Again; we have to estimate the function value of 'f' at x = 7, *i.e.*, we have to find S(7) at i = 1

From above formula (2) for cubic spline,

$$S_1(x) = \frac{a_0}{6h_1}(h_1^2U_1 - U_1^3) + \frac{a_1}{6h_1}(U_0^3 - h_1^2U_0) + \frac{1}{h_1}(f_1U_0 - f_0U_1) \quad (2)$$

where, $U_i = x - x_i = x - 4$

and, $U_1 = x - x_1 = x - 9$

$f_0 = 2$

$f_1 = 3$

$h_1 = 5$

$h_2 = 7$

Also,

$a_0 = a_2 = 0$

and, $a_1 = -0.01428$

Now, putting these values in (2); we get,

$$S_1(x) = 0 + \frac{(-0.01428)}{6 \times 5}[(x-4)^3 - (5)^2(x-4)] + \frac{1}{5}[3(x-4) - 2(x-9)]$$

Now, at x = 7

$$S_1(7) = -\frac{1}{2100}[(7-4)^3 - 25(7-4)] + \frac{1}{5}[3(7-4) - 2(7-9)]$$

or, $$S_1(7) = +\frac{4}{175} + \frac{13}{5}$$

∴ $$S_1(7) = 2.6228$$

Example 4.26

Obtain the cubic spline for the following data:

x:	0	1	2	3
y:	2	-6	-8	2

Solution:

From the given data:

x:	0	1	2	3
y:	2	-6	-8	2

Here, we have to use cubic splines and for cubic splines; we have, first value and last value are zero.

i.e., $a_0 = a_n = 0$

From cubic spline formula; we have,

At i = 1;

$$h_1a_0 + 2a_1(h_1 + h_2) + h_2a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right] \quad (1)$$

At i = 2;

$$h_2a_1 + 2a_2(h_2 + h_3) + h_3a_3 = 6\left[\frac{f_3 - f_2}{h_3} - \frac{f_2 - f_1}{h_2}\right] \quad (2)$$

where, $a_0 = a_3 = 0$ [Since, first and last value are zero]

Also,

$$h_i = x_i - x_{i-1}$$
$$h_1 = x_1 - x_0 = 1 - 0 = 1$$
$$h_2 = x_2 - x_1 = 2 - 1 = 1$$
$$h_3 = x_3 - x_2 = 3 - 2 = 1$$

Also,

$$f_0 = 2, f_1 = -6, f_2 = -8$$

and, $f_3 = 2$

Now, putting these values in the equation (1) and (2); we get,

$$0 + 2a_1(1 + 1) + a_2 = 6\left[\frac{-8 - (-6)}{1} - \frac{-6 - 2}{1}\right]$$

or, $4a_1 + a_2 = 6(-2 + 8)$

∴ $4a_1 + a_2 = 36$ (3)

and, $a_1 + 2a_2(1 + 1) + 0 = 6\left[\frac{2 - (-8)}{1} - \frac{-8 - (-6)}{1}\right]$

or, $a_1 + 4a_2 = 6(10 + 2)$

or, $a_1 + 4a_2 = 72$ (4)

Solving (3) and (4); we get,

$$a_1 = 4.8$$

and, $a_2 = 16.8$

Again, from second formula of cubic spline interpolation; we have,

At i = 1;

$$S_1(x) = \frac{a_0}{6h_1}(h_1^2U_1 - U_1^3) + \frac{a_1}{6h_1}(U_0^3 - h_1^2U_0) + \frac{1}{h_1}(f_1U_0 - f_0U_1) \quad (5)$$

where, $U_i = x - x_i$

∴ $U_0 = x - x_0 = x - 0 = x$

and, $U_1 = x - x_1 = x - 1$

Again, putting these above values in (5); we get,

$$S_1(x) = 0 + \frac{4.8}{6 \times 1}\{x^3 - (1)^2 x\} + \frac{1}{1}\{-6x - 2(x - 1)\}$$

$$= 0 + 0.8(x^3 - x) + (-6x - 2x + 2)$$

$$= 0.8x^3 - 0.8x - 8x + 2$$

$\therefore$ $S_1(x) = 0.8x^3 - 8.8x + 2$; which is the required cubic spline.

4.6 EXAMINATION PROBLEMS

1. **Find the best fit curve in the form of $y = a + bx + cx^2$ using least square approximation from the following discrete data.[2071 Bhadra]**

X	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Solution:

From the given we have to fit a quadratic curve $(a + bx + cx^2)$ for following set of orders.

X	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Here, to fit a quadratic curve $(a + bx + cx^2)$ *i.e.*, $(cx^2 + bx + a)$ at first; we have to find the value of c, b and a.

Let, $Y = cx^2 + bx + a$ (i)

At first, introducing Σ on both side of equation (i); we get,

$\Sigma y = \Sigma cx^2 + \Sigma bx + \Sigma a$

or, $\Sigma y = c\Sigma x^2 + b\Sigma x + na$ [$\because \Sigma a = na$] (ii)

Again multiplying equation (i) by x and introducing Σ on both sides; we get,

$\Sigma xy = \Sigma cx^3 + \Sigma bx^2 + \Sigma ax$

$\Sigma xy = c\Sigma x^3 + b\Sigma x^2 + a\Sigma x$ (iii)

Again, multiplying equation (i) by x^2 on both sides and introducing Σ on both sides; we have,

$\Sigma x^2y = \Sigma cx^4 + \Sigma bx^3 + \Sigma ax^2$

or, $\Sigma x^2y = c\Sigma x^4 + b\Sigma x^3 + a\Sigma x^2$ (iv)

Now, finding the value of Σx, Σx^2, Σx^4, Σxy, Σx^2y, Σx^3;

x	y	x^2	x^3	x^4	xy	x^2y
1.0	1.1	1	1	1	1.1	1.1
1.5	1.3	2.25	3.375	5.062	1.95	2.925
2	1.6	4	8	16	3.2	6.4
2.5	2.0	6.25	15.625	39.062	5	12.5
3	2.7	9	27	81	8.1	24.3
3.5	3.4	12.25	42.875	150.062	11.9	41.65
4	4.1	16	64	256	16.4	65.6
$\Sigma x =$ 17.5	$\Sigma y =$ 16.2	$\Sigma x^2 =$ 50.75	$\Sigma x^3 =$ 161.87	$\Sigma x^4 =$ 548.186	$\Sigma xy =$ 47.65	$\Sigma x^2y =$ 154.475

Here,

$n = 7$

Now, putting these above values in equation (ii), (iii) and (iv); we have,

$$16.2 = 50.75c + 17.5b + 7a$$

$$47.65 = 161.875c + 50.75b + a(17.5) \quad \text{(v)}$$

$$154.475 = 548.186c + 161.875b + 50.75a \quad \text{(vi)}$$

On solving equation (v), (vi) and (vii) by using calculator; we get, (vii)

$c = 0.2529$

$b = -0.19$

and, $a = 1.036$

Hence, the required quadratic curve of the best fitting line as;

$$y = 0.2529x^2 - 0.19x + 1.036$$

2. Find the Lagrange's Interpolation formula to find the value of y when x = 3.0 from the following table. **[2071 Bhadra]**

x	3.2	2.7	1.0	4.8	5.6
y	22.0	17.8	14.2	38.3	51.7

Solution:

From the given data:

x	3.2	2.7	1.0	4.8	5.6
y	22.0	17.8	14.2	38.3	51.7

Here, we have from Lagrange's interpolation formula; we have,

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} y_3 + \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_3)(x_4 - x_3)} \times y_4$$

where, $x_0 = 3.2$ $\quad y_0 = 22$

$x_1 = 2.7$ $\quad y_1 = 17.8$

$x_2 = 1.0$ $\quad y_2 = 14.2$

$x_3 = 4.8$ $\quad y_3 = 38.3$

$x_4 = 5.6$ $\quad y_4 = 51.7$

$x = 3.0$

Also,

$(x - x_0) = (3 - 3.2) = -0.2$

$(x - x_1) = 3 - 2.7 = 0.3$

$(x - x_2) = 3 - 1.0 = 2$
$(x - x_3) = 3 - 4.8 = -1.8$
$(x - x_4) = 3 - 5.6 = -2.6$

Now, substituting the above values in equation; we have,

$$y(3) = \frac{0.3 \times 2 \times (-1.8) \times (-2.6)}{(3.2 - 2.7)(3.2 - 1.0)(3.2 - 4.8)(3.2 - 5.6)} \times 22$$
$$+ \frac{(0.2) \times 2 \times (-1.8) \times (-2.6)}{(2.7 - 3.2)(2.7 - 1.0)(2.7 - 4.8)(2.7 - 5.6)} \times 17.8$$
$$+ \frac{-0.2 \times 0.3 \times (-1.8)(-2.6)}{(1 - 3.2)(1 - 2.7)(1 - 4.8)(1 - 5.6)} \times 14.2$$
$$+ \frac{-0.2 \times 0.3 \times 2 \times (-2.6)}{(4.8 - 3.2)(4.8 - 2.7)(4.8 - 1)(4.8 - 5.6)} \times 38.3$$
$$+ \frac{-0.2 \times 0.3 \times 2 \times (-1.8)}{(5.6 - 3.2)(5.6 - 2.7)(5.6 - 1)(5.6 - 4.8)} \times 51.7$$

$= 14.625 + 6.437 - 0.6099 - 1.1698 + 0.436$
$= 19.718$

Hence, y (3) = 19.718.

3. **From the following table estimate f(1.6) using Newton's forward interpolation method.** **[2071 Chaitra]**

x	1	1.4	1.8	202
f (x)	3.49	4.82	5.96	6.5

Solution:

From the given table,

x	1	1.4	1.8	202
f(x)	3.49	4.82	5.96	6.5

Here, we have to find the value of f (1.6) by using Newton's forward interpolation formula; we have,

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad (1)$$

where, $u = \dfrac{X - X_0}{h}$

X = Value at which interpolation is to be found

X_0 = Initial value relative to X

h = Interval of X

At first, finding the value of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ .... etc. by generating the Newton's forward interpolation table:

x	y = f (x)	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3.49			
		4.82 – 3.49 = 1.33		
1.4	4.82		1.14 – 1.33 = – 0.19	
		5.96 – 4.82 = 1.14		– 0.6 – (0.19) = – 0.41
1.8	5.96		0.54 – 1.14 = – 0.6	
		6.5 – 5.96 = 0.54		
2.2	6.5			

Since we have to find the value of f (x) at X = 1.6

So take,

$X_0 = 1.4$

$Y_0 = 4.82$

$\Delta y_0 = 1.14$

$\Delta^2 y_0 = -0.6$

and, h = Interval of X = 1.4 – 1 = 0.4

$$\therefore \quad u = \frac{X - X_0}{h} = \frac{1.6 - 1.4}{0.4}$$

$$\therefore \quad u = 0.5$$

Now putting these values in equation (i); we get,

$$y\,(X = 1.6) = 4.82 + \frac{0.5}{1!}(1.14) + \frac{0.5\,(0.5-1)}{2!}(-0.6)$$

$$\therefore \quad y\,(x = 1.6) = 5.465$$

Hence, f (1.6) = 5.465

4. Using appropriate Newton's interpolation technique, estimate y (15) and y (85) from the following data: [2072 Ashwin]

x	10	30	50	70	90
y	34	36	45	23	36

Solution:

From the given table;

x	10	30	50	70	90
y	34	36	45	23	36

Since, we have to find the value of y at x = 15 lies between 10 and 30 and also this point lies near the starting point (x_0) so, we use forward interpolation method and (ii) at x = 85 which lies near point (x_n) so in this case we use backward interpolation formula.

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \quad (i)$$

Also, from Newton's backward interpolation formula; we have,

$$y = y_n + \frac{u}{1!}\Delta y_n + \frac{u(u+1)}{2!}\Delta^2 y_n + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\Delta^4 y_n + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\Delta^5 y_n + \frac{u(u+1)(u+2)(u+3)(u+4)(u+5)}{6!}\Delta^6 y_n$$

where, $u = \frac{x - x_0}{h}$

and, h = Interval of $x = x_i - x_{i-1}$

So, for finding y_0, Δy_0, $\Delta^2 y_0$, $\Delta y_3 y_0$, ……… and also for finding y_n, Δy_n, $\Delta^2 y_n$, $\Delta^3 y_n$, ……… we have to generate simple difference table for both Newton's forward and backward interpolation method.

Simple difference table for both forward and backward

x	y	First s.d. or Δy	Second s.d. $\Delta^2 y$ or $\Delta^2 y$	Third s.d. $\Delta^3 y$ or $\Delta^3 y$	Fourth s.d. $\Delta_4 y$ or $\Delta_4 y$
10	34				
		22			
30	56		– 33		
		– 11		22	
50	45		– 11		24
		– 22		46	
70	23		35		
		13			
90	36				

i) At, $x = 15$

$y(15) = ?$

Hence, from above table; we have, data required for forward interpolation method;

As, we have to find value of y at $x = 15$

so, we take,

$x_0 = 10$

$x_1 = 30$

$y_0 = 34$

$\Delta y_0 = 22$

$\Delta^2 y_0 = -33$

$\Delta^3 y_0 = 22$

$\Delta^4 y_0 = 24$

h = Interval of $x = x_1 - x_0 = 30 - 10 = 20$

and, $u = \frac{x - x_0}{h} = \frac{15 - 10}{20} = 0.25$

Now, putting above these values in the equation (i); we get,

$$y(15) = 34 + 0.25 \times 22 + \frac{0.25(0.25-1) \times (-33)}{2!}$$

$$+ \frac{0.25(0.25-1) \times (0.25-2)}{3!} + \frac{0.25(0.25-1) \times (0.25-2)(0.25-3)}{4!}$$

$$= 34 + 5.5 + 3.093 + 1.203 - 0.902$$

$$= 42.894$$

ii) For y (85)

At, x = 85

Since, the point x = 85 lies near the end of the table so, we use Newton's backward interpolation method.

Hence, data required for backward interpolation; so taking $x_n = 90$

$x = 85$

$y_n = 36$

Also,

$\Delta y_n = 13$

$\Delta^2 y_n = 35$

$\Delta^3 y_n = 46$

$\Delta^4 y_n = 24$

$\therefore$ $h = 90 - 70 = 20$

and, $u = \frac{x - x_n}{h} = \frac{85 - 90}{20} = -0.25$

Putting these values in equation (ii); we get,

$$y(85) = 36 + (-0.25) \times 13 + \frac{(-0.25)(-0.25+1)}{2} \times 35$$

$$+ \frac{(-0.25)(-0.25+1)(-0.25+2) \times 46}{3!} \times 46$$

$$+ \frac{(-0.25)(-0.25+1)(-0.25+2)(-0.25+3)}{4!} \times 24 + 0 + 0$$

$$= 36 - 0.75 - 3.281 - 2.515 - 0.902$$

$$= 28.552$$

5. Using the least square method determine the exponential fit of the form $y = ae^{bx}$ for the following data: [2072 Magh]

x	0	1	2	3	4	5
y	1.5	2.5	3.5	5	7.5	11.25

Solution:
From the given data:

x	0	1	2	3	4	5
y	1.5	2.5	3.5	5	7.5	11.25

The given relation is;

$$y = ae^{bx}; \qquad \text{(i)}$$

which is not linear.

Now, converting relation (i) into linear form by taking logarithms (to the base 10); we have,

$$\log_{10}(y) = \log_{10}(ae^{bx})$$
$$= \log_{10}(a) + \log_{10} e^{bx}$$
$$\log_{10}(y) = \log_{10} a + bx \log_{10} e$$

Comparing with equation; we have,

$$Y = A + BX \qquad \text{(ii)}$$
$$Y = \log_{10} y$$
$$A = \log_{10} a$$

and, $B = b \log_{10} e$

Introducing Σ to each term in equation (ii); we have,

$$\Sigma Y = \Sigma A + \Sigma BX$$
$$\Sigma Y = nA + B\Sigma X \; [\because \Sigma A = nA] \qquad \text{(iii)}$$

Again, multiplying equation (ii) by x throughout and then attaining Σ; we have,

$$\Sigma XY = \Sigma XA + \Sigma BX^2$$
$$\Sigma XY = A\Sigma X + B\Sigma X^2 \qquad \text{(iv)}$$

The required value of Σx, Σy, Σx^2 and Σxy are computed in the table below:

x	y	$y = \log_{10}(y)$	x^2	xy
0	1.5	0.1760	0	0
1	2.5	0.397	1	0.397
2	3.5	0.544	4	1.088
3	5.0	0.698	9	2.094
4	7.5	0.875	16	3.5
5	11.25	1.0511	25	5.255
$\Sigma x = 15$	$\Sigma y = 31.25$	$\Sigma y = 3.7411$	$\Sigma x^2 = 55$	$\Sigma xy = 12.334$

Also,

$$n = 6$$

Now, using these values in equation (iii) and (iv); we have,

$$3.7411 = 6 \times A + B \times 15$$

and, $12.334 = 15A + 55B$

Using calculator; we have,

$$A = 0.19762$$

$$B = 0.17035$$

Also, we have,

$$A = \log_{10} a$$

i.e., $a = 10^A$

$\therefore$ $a = 10^{0.19762} - 1.5762$

Also,

$$B = b \log_{10} e$$

or, $B = \log_{10} (e^b)$

or, $e^b = 10^B$

$\therefore$ $b = \log_e (10^B) = \log_e (10^{0.17035}) = 0.392245$

Hence, the required quadratic curve is $y = 1.5762\, e^{0.392245x}$

6. **Compute y (6) from the following data using cubic Spline Interpolation.** [2072 Magh]

x	0	1	2	3	4
y	1.5	2.5	3.5	5	7.5

Solution:

From the given data; we have,

x	0	1	2	3	4
y	1.5	2.5	3.5	5	7.5

Here, we have to use cubic splines and we have for cubic splines,

$a_0 = a_n = 0$ (*i.e.,* first and last value are zero)

From cubic spline formula; we have,

At i = 1

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left[\frac{L_2 - L_1}{h_2} - \frac{L_1 - L_0}{h_1}\right]$$

where, $a_0 = a_n = 0$

and, $h_1 = x_i + x_{i-1}$

or, $h_1 = x_1 - x_0 = 3 - 1 = 2$

At i = 2,

$$h_2a_1 + 2a_2(h_2 + h_3) + h_3a_3 = 6\left[\frac{L_3 - L_2}{h_3} - \frac{L_2 - L_1}{h_2}\right]$$

$h_2 = x_2 - x_1 = 5 - 3 = 2$

$h_3 = x_3 - x_2 = 7 - 5 = 2$

$h_4 = x_4 - x_3 = 9 - 7 = 2$

At $i = 3$,

$$h_3a_2 + 2a_3(h_3 + h_4) + h_4a_4 = 6\left[\frac{L_4 - L_3}{h_4} - \frac{L_3 - L_2}{h_3}\right]$$

Also,

$L_0 = 3$

$L_1 = 5$

$L_2 = 4$

$L_3 = 2$

$L_4 = 3$

Now, putting this value in equation (i), (ii) and (iii); we get,

$$2 \times 0 + 2a_1(2 + 2) + 2a_2 = 6\left[\frac{4-5}{2} - \frac{5-3}{2}\right]$$

or, $8a_1 + 2a_2 = 3[-1 - 2]$

or, $8a_1 + 2a_2 = -9$ (iv)

and, $2a_1 + 2a_2(2 + 2) + 2a_3 = 6\left[\frac{2-4}{2} - \frac{4-5}{2}\right]$

or, $2a_1 + 8a_2 + 2a_3 = 3[-2 - (-1)]$

or, $2a_1 + 8a_2 = 2a_3 = -3$ (v)

Also,

$$2a_2 + 2a_3(2 + 2) + 0 = 6\left[\frac{3-2}{2} - \frac{2-4}{2}\right]$$

or, $2a_2 + 8a_3 = 3[1 - (2)]$

or, $2a_2 + 8a_3 = 9$ (vi)

On solving equations (iv), (v) and (vi); we get,

$a_1 = -1.0178$

$a_2 = -0.4285$

$a_3 = 1.2321$

At target point $x = 6$ *i.e.*, $(5 \le x \le 7)$

$\therefore \quad i = 3$

Again, from second formula of cubic spline interpolation; we have,

At $i = 3$,

$$y_3(x) = \frac{a_2}{6h_3}\left(h_3^2 U_3 - U_3^3\right) + \frac{a_3}{6h_3}\left(U_2^3 - h_3^2 U_2\right) + \frac{1}{h_3}(L_3U_2 - L_2U_3) \quad \text{(vii)}$$

where, $U_i = x - x_i$

$U_2 = x - x_2 = 6 - 5 = 1$

$U_3 = x - x_3 = 6 - 1 = -1$

Again, putting these above value in equation (vii); we get,

$$y_3(6) = \frac{-0.4285}{6\times 2}[2^2\times(-1) - (-1)^3] + \frac{1.2321}{6\times 2}[1^3 - 2^2\times 1] + \frac{1}{2}[2\times 1 - 4\times(-1)]$$

$= 2.7991$

Hence, y (6) = 2.7991.

7. **Fit the following set of data to a curve of the form $y = ab^x$**

x	1.0	1.5	2.0	2.5	3	3.5	4.0
y	8.2	5.2	3.1	2.5	1.7	1.6	1.4

[2073 Magh]

Solution: Initiate same as example 4.24

8. **Estimate y (4.5) from the following data using natural cubic spline interpolation technique.** [2073 Magh]

x	1	3	5	7	9
y	10	12	11	13	9

Solution:

From the given data:

x	1	3	5	7	9
y	10	12	11	13	9

Here,

$h = 2$

and $n = 4$

Also,

$i = 1, 2, 3$

Here, we have to use cubic splines and for cubic splines; we have to, first value and last value are zero.

$q_0 = a_n = 0$

i.e., $a_4 = 0$

From cubic spline formula; we have,

At $i = 1$,

$$h_1a_0 + 2a_1(h_1 + h_2) + h_2a_2 = 6\left[\frac{L_2 - L_1}{h_2} - \frac{L_1 - L_0}{h_1}\right]$$

At i = 2,

$$h_1a_1 + 2a_2(h_2 + h_3) + h_3a_3 = 6\left[\frac{L_3 - L_2}{h_3} - \frac{L_2 - L_1}{h_2}\right] \quad \text{(ii)}$$

At i = 3,

$$h_3a_2 + 2a_3(h_3 + h_4) + h_4a_4 = 6\left[\frac{L_4 - L_3}{h_4} - \frac{L_3 - L_2}{h_3}\right] \quad \text{(iii)}$$

Also,

$h_i = X_i - X_i - 1$

$h_1 = 3 - 1 = 2$

$h_2 = 5 - 3 = 2$

$h_3 = 7 - 5 = 2$

$h_4 = 9 - 7 = 2$

Again,

$L_0 = 10$

$L_1 = 12$

$L_2 = 11$

$L_3 = 13$

$L_4 = 9$

Now, putting these values in equation (i), (ii) and (iii); we get;

$$2 \times 0 + 2 \times a_1(2 + 2) + 2 \times a_2 = 6\left[\frac{11 - 12}{2} - \frac{12 - 10}{2}\right]$$

or, $8a_1 + 2a_2 = 6\left[\frac{-1}{2} - 1\right]$

or, $8a_1 + 2a_2 = -9$ (iv)

and, $2a_1 + 2a_2(2 + 2) + 2 \times a_3 = 6\left[\frac{13 - 11}{2} - \frac{11 - 12}{2}\right]$

or, $2a_1 + 8a_2 + 2a_3 = 3\,[2 - (-1)]$

or, $2a_1 + 8a_2 + 2a_3 = 9$ (v)

Also,

$$2a_2 + 2a_3(2 + 2) + 0 = 6\left[\frac{9 - 13}{2} - \frac{13 - 11}{2}\right]$$

or, $2a_2 + 8a_3 = 3\,(-4 - 2)$

or, $2a_2 + 8a_3 = -18$ (vi)

On solving equation (iv), (v) and (vi); we get,

$a_1 = -1.6875$

$a_2 = 2.25$

$a_3 = -2.8125$

At target point X = 4.5, *i.e.*, $(3 \leq x \leq 5)$

$\therefore \quad i = 2$

Again from second formula of cubic spline interpolation; we have,

At i = 2,

$$S_2(x) = \frac{a_1}{6h_2}\left(h_2^2 U_2 - U_2^3\right) + \frac{a_2}{6h_2}\left(U_1^3 - h_2^2 U_1\right) + \frac{1}{h_2}(L_2U_2 - L_1U_2) \quad \text{(vii)}$$

where, $U_i = X - X_i$

$\therefore \quad U_0 = X - X_0$

$= X - 1 = 4.5 - 1 = 3.5$

and, $U_1 = x - x_1$

$= x - 3 = 4.5 - 3$

$= 1.5$

$U_2 = x - x_2$

$= x - 5 = 4.5 - 5$

$= -0.5$

Again putting these above value in equation (vii); we get,

$$\therefore \quad S_2(4.5) = \frac{(-1.6875)}{6 \times 2}[2^2(-0.5) - (-0.5)^3] + \frac{2.25}{6 \times 2}[1.5^3 - 2^2 \times 1.5] + \frac{1}{2}[11 \times 1.5 - 12 \times (-0.5)]$$

$= 11.0215$

9. **From the following data, compute:** **[2074 Bhadra]**

a) **y (3) using Newton's forward interpolation formula**

b) **y (6.4) using Sterling's formula**

x	2	4	6	8	10	12
y	5.2	4.2	3.1	3.5	6.2	7.3

Solution:

a) **y (3) using Newton's forward interpolation formula**

Using Newton's forward interpolation formula; we have,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots\ldots(u-\overline{n-1})}{n!}\Delta^n y_0 \quad (1)$$

where, $u = \dfrac{x - x_0}{h}$

$h = \text{Interval of } x = x_i - x_{i-1}$

At first, finding y_0, Δy_0, $\Delta^2 y_0$, etc. from the simple difference table for Newton's forward interpolation method;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	5.2					
		– 1				
4	4.2		– 0.1			
		– 1.1		1.6		
6	3.1		1.5		– 0.8	
		0.4		0.8		– 3.9
8	3.5		2.3		– 4.7	
		2.7		– 3.9		
10	6.2		– 1.6			
		1.1				
12	7.3					

Hence, from above table; we have,

$$y_0 = 5.2$$
$$\Delta y_0 = -1$$
$$\Delta^2 y_0 = -0.1$$
$$\Delta^3 y_0 = 1.6$$
$$\Delta^4 y_0 = -0.8$$
$$\Delta^5 y_0 = -3.9$$

Also, from the given table; we have,

$$h = x_1 - x_0 = 4 - 2 = 2$$

and, $u = \frac{x - x_0}{h} = \frac{3-2}{2} = 0.5$

Putting these values in equation (1); we get,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0$$

or, $$y(3) = 5.2 + \frac{0.5}{1!}(-1) + \frac{0.5(0.5-1)}{2!}(-0.1) + \frac{0.5(0.5-1)(0.5-2)}{3!}(1.6) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(-0.8) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{5!}(-3.9)$$

or, $y(3) = 5.2 - 0.5 + 0.0125 + 0.1 + 0.03125 - 0.1066$

∴ $y(3) = 4.73715$

b) By using Sterling formula; we have to find $y(6.4)$,

By using same table above, proceed same as the solution of example 4.12

10. Write pseudo code to fix a given set of data to second degree polynomial ($y = ax^2 + bx + C$) using the least square method.

[2074 Ashwin]

Solution:

The process of finding an equation of best fit is known as curve fitting. For a second degree equation; we have,

$$y = ax^2 + bx + C$$

Normal equations are;

$$\sum y = a\sum x^2 + b\sum x + nC$$

$$\sum xy = a\sum x^3 + b\sum x^2 + C\sum x$$

$$\sum x^2y = a\sum x^4 + b\sum x^3 + C\sum x^2$$

Pseudo code

i) Input 'n' number of data.

ii) For i = 1 to n; input x_i, y_i

iii) Initialize all sum to zero (0)

iv) For i = 1 to n;

sum x = sum x + x_i

sum x2 = sum x_2 + x_ix_i

sum x3 = sum x_3 + $x_ix_ix_i$

sum x4 = sum x_4 + $x_ix_ix_ix_i$

sum xy = sum xy + x_iy_i

sum x2y = sum x2y + $x_ix_iy_i$

sum y = y_i + sum y

v) Assign values as;

$a_{11} = a_{22} = a_{33}$ = sum x2

$a_{12} = a_{23}$ = sum x

a_{24} = sum xy

a_{34} = sum x2y

a_{31} = sum x4

$a_{21} = a_{32}$ = sum x3

a_{14} = sum y

a_{13} = n

vi) For j = 1 to 3

For i = 1 to 3

if $i \neq j$

$$r = \frac{a_{ij}}{a_{jj}}$$

For k = 1 to 4

$$a_{ik} = a_{ik} - r \times a_{jk}$$

vii) For i = 1 to 3

$$r = \frac{a_{i4}}{a_{ii}}$$

viii) Print values of a, b, c as x_1, x_2, x_3

11. **Fit the following data to the curve $y = ax^b$ using least square method.** **[2074 Ashwin]**

x	350	400	500	600
y	61	26	7	2.6

Solution:

The given equation is;

$$y = ax^b \quad (1)$$

Taking log on both sides; we get,

$$\log_e y = \log_e a + b \log_e x$$

Let, $Y = A + bx$ (2)

where, $Y = \log_e y$

$A = \log_e a$

$X = \log_e x$

Now,

x	y	$X = \log_e x$	$Y = \log_e y$	XY	X^2
350	61	5.85	4.11	24.08	34.31
400	26	5.99	3.26	19.52	35.89
500	7	6.21	1.94	12.09	38.62
600	2.6	6.39	0.95	6.11	40.92
		24.46	10.27	61.807	149.75

The normal form of the equation (2) are;

$$\sum Y = nA + b\sum X$$

or, $10.27 = 4A + 24.46b$ (3)

and, $\sum XY = A\sum X + b\sum X^2$

or, $61.80 = 24.46A + 149.75b$ (4)

Solving equation (3) and (4); we get,

$A = 37.13$

$b = -5.65$

since, $\log_e a = A$

or, $a = e^A = e^{37.13} = 1.33 \times 10^{16}$

Hence, the required curve fitting equation is $y = 1.33 \times 10^{16} x^{-5.65}$.

12. **Fit the curve of the form $y = a\log_e x + b$ to the following data sets.** **[2074 Chaitra]**

x	2	3	4	5	6	7
y	5.45	6.26	6.84	7.29	7.66	7.96

Solution:

The given equation is;

$$y = a\log_e x + b \quad (1)$$

Let, $y = ax + b; \quad (2)$

where, $x = \log_e x$.

Now, the normal form of equation (2) are;

$$\sum y = nb + a\sum X \quad (3)$$

$$\sum Xy = b\sum X + a\sum X^2 \quad (4)$$

x	y	$X = \log_e x$	X^2	Xy
2	5.45	0.69	0.48	3.77
3	6.26	1.09	1.20	6.87
4	6.84	1.38	1.92	9.48
5	7.29	1.60	2.59	11.73
6	7.66	1.79	3.21	13.72
7	7.96	1.94	3.78	15.49
	41.46	8.52	13.19	61.08

Substituting the value in equation (3) and (4); we have,

$$41.46 = 6b + 8.52a \quad (5)$$

and, $61.08 = 8.52b + 13.19a \quad (6)$

Solving (5) and (6); we get,

$b = 4.04$

$a = 2.02$

Substituting the value of a and b in the equation (1); we get,

$y = 2.02\log_e x + 4.04$; which is the required curve fitting equation.

13. **Approximate y(2) and y(10) using appropriate interpolation formula from the following data:** **[2074 Chaitra]**

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution:

The difference table is as under;

x	y	Δ	Δ^2	Δ^3	Δ^4
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

Using forward interpolation formula for y(2); we have,

$x_0 = 3$

$x = 2$

$h = 1$

$\therefore \quad p = \frac{x - x_0}{h} = \frac{2-3}{1} = -1$

$$\therefore \quad y_2 = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 4.8 + (-1) \times 3.6 + \frac{(-1)(-2)}{2} \times 2.5 + \frac{(-1)(-2)(-3)}{6} \times 0.5$$

$$= 3.2$$

Using backward interpolation for y(10); we have,

$x_n = 9$

$x = 10$

$h = 1$

$$\therefore \quad y_{10} = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$= 73.9 + 1 \times 21.1 + \frac{1 \times 2}{2} \times 4.5 + \frac{1 \times 2 \times 3}{6} \times 0.5$$

$$= 100$$

14. Estimate y (6.5) using natural cubic spline interpolation technique from the following data. [2075 Baishakh]

x	3	5	7	9	11
y	8	10	9	12	5

Solution:

From the given table:

x	3	5	7	9	11
y	8	10	9	12	5

Here,

$h = 2$

$n = 4$

Also,

$i = 1, 2, 3$

Here we have to use cubic splines and for cubic splines; we have,

First value and last values are zero.

i.e., a_0

$a_n = a_4 = 0$

From cubic spline formula; we have,

At $i = 1$,

$$h_1a_0 + 2a_1(h_1 + h_2) + h_2a_2 = 6\left[\frac{L_2 - L_1}{h_2} - \frac{L_1 - L_0}{h_1}\right] \tag{1}$$

At $i = 2$,

$$h_2a_1 + 2a_2(h_2 + h_3) + h_3a_3 = 6\left[\frac{L_3 - L_2}{h_3} - \frac{L_2 - L_1}{h_2}\right] \tag{2}$$

At $i = 3$,

$$h_3a_2 + 2a_3(h_3 + h_4) + h_4a_4 = 6\left[\frac{L_4 - L_3}{h_4} - \frac{L_3 - L_2}{h_3}\right] \tag{3}$$

Also,

$h_i = x_i - x_{i-1}$

$h_1 = 5 - 3 = 2$

$h_2 = 7 - 5 = 2$

$h_3 = 9 - 7 = 2$

$h_4 = 11 - 9 = 2$

Also,

$L_0 = 8$

$L_1 = 10$

$L_2 = 9$

$L_3 = 12$

$L_4 = 5$

Now, putting these values in equation (1), (2) and (3); we get,

$$2 \times 2a_1 (2 + 2) + 2a_2 = 6\left[\frac{9-10}{2} - \frac{10-8}{2}\right]$$

or, $8a_1 + 2a_2 = 3(-1-2)$

or, $8a_1 + 2a_2 = -9$ (4)

and, $$2a_1 + 2a_2 (2+2) + 2a_3 = 6\left[\frac{12-9}{2} - \frac{9-10}{2}\right]$$

or, $2a_1 + 8a_2 + 2a_3 = 3[3-(-1)]$

or, $2a_1 + 8a_1 + 2a_3 = 12$ (5)

Also,

$$2a_2 + 2a_3 (2+2) + 0 = 6\left[\frac{5-12}{2} - \frac{12-9}{2}\right]$$

or, $2a_2 + 8a_3 = 3[-7-3]$

or, $2a_2 + 8a_3 = -30$ (6)

On solving equation (4), (5) and (6); we have,

$a_1 = -1.9018$

$a_2 = 3.1071$

$a_3 = -4.5267$

At target point x = 6.5; which lies between 5 and 7.

$\therefore$ $i = 2$

From second formula of cubic spline interpolation; we have,

At i = 2,

$$y_2(x) = \frac{a_2}{6h_2}\left(h_2^2 U_2 - U_2^3\right) + \frac{a_2}{6h_2}\left(U_1^3 - h_1^2 U_1\right) + \frac{1}{h_2}(L_2U_1 - L_1U_2) \quad (7)$$

where, $U_i = x - x_i$

$\therefore$ $U_0 = x - x_0$

$= x - 3 = 6.5 - 3 = 3.5$

$U_1 = x - x_1$

$= x - 5 = 6.5 - 5 = 1.5$

$U_2 = x - x_2$

$= x - 7 = 6.5 - 7 = -0.5$

Putting the value in equation (7); we have,

$$\therefore \quad y_2(6.5) = \frac{-1.9018}{6 \times 2}[2^2(-0.5) - (-0.5)^3] + \frac{3.1071}{6 \times 2}[(1.5)^3 - 2^2(1.5)] + \frac{1}{2}[9 \times 1.5 - 10(-0.5)]$$

$= 8.8674$

Hence, y (6.5) = 8.8674.

15. Fit the curve $y = ax^b$ to the following data. **[2075 Baishakh]**

4	5	7	10	11	13
48	100	294	900	1210	2028

Solution:

From the given data;

4	5	7	10	11	13
48	100	294	900	1210	2028

The given relation is;

$$y = ax^b; \quad (1)$$

which is not linear

Now, converting relation (1) into linear form by taking Logarithms (to the base 10)

$$\log_{10}(Y) = \log(ax^b) \quad (2)$$

$$\log_{10}(Y) = \log_{10}(a) + \log(x^b)$$

$$\log_{10}(y) = \log_{10}(a) + b\log(x)$$

Comparing it with;

$$Y = A + bx \quad (3)$$

$$Y = \log_{10}(y)$$

$$A = \log_{10}(a)$$

$$X = \log_{10}(x)$$

The normal equation is;

$$\Sigma Y = nA + b\Sigma x \quad (4)$$

$$\Sigma XY = A\Sigma X + b\Sigma x^2 \quad (5)$$

X	Y	X = log x	Y = log y	X^2	XY
1	8.2	0	0.9138	0	0
1.5	5.2	0.1761	0.716	0.031	0.1261
2.0	3.1	0.3010	0.4913	0.0906	0.1478
2.5	2.5	0.3979	0.3979	0.1583	0.1583
3.0	1.7	0.4771	0.2304	0.2276	0.1099
3.5	1.6	0.5440	0.2041	0.296	0.1110
4.0	1.4	0.6020	0.1461	0.3624	0.0879
		$\Sigma X = 2.4981$	$\Sigma Y = 3.0996$	$\Sigma X^2 = 1.169$	$\Sigma XY = 0.742$

Substituting equation (4) and (5) and n = 7; we have,

$$3.0996 = 7 \times A + b \times 2.4981$$

and, $$0.741 = 2.4981A + 1.1659b$$

Solving and using calculator; we have,

$A = 0.9177$

and, $b = -1.3307$

Substituting value in equation (iii); we have,

$Y = 0.9177 - 1.3307X$

or, $\log_{10}(y) = 0.9177 - 1.3307 \log_{10}(x)$

or, $\log_{10}(y) + 1.3307 \log_{10}(x) - 0.9177$

or, $\log_{10}(y) + \log_{10}(x^{1.3307}) + 0.9177$

or, $\log_{10}(x_{1.3307}y) = 0.9177$

$\therefore$ $x^{1.3307}\,y = 10^{0.9177}$

$yx^{1.3307} = 8.2737$

$\therefore$ $y = \dfrac{8.2737}{x^{1.3307}}$

$\therefore$ $y = 8.2737\,x^{-1.3307}$; which is the equation.

16. Fit the following set of data to a curve of the form $y = ae^{bx}$

[2075 Ashwin]

x	2	3	4	5	6	7
y	15.1	10.2	7.8	5.5	3.8	1.7

Solution:

The given equation is;

$y = ae^{bx}$ (1)

Taking log on both sides; we get,

$\log_e y = \log_e(ae^{bx})$

or, $\log_e y = \log_e a + bx \log_e e$

or, $\log_e y = \log_e a + bx$ $[\because \log_e e = 1]$

or, $Y = A + bx$ (2)

where, $Y = \log_e y$

$A = \log_e a$

Now,

x	y	$Y = \log_e y$	xy	x^2
2	15.1	2.71	5.42	4
3	10.2	2.32	6.96	9
4	7.8	2.05	8.2	16
5	5.5	1.70	8.5	25
6	3.8	1.33	7.98	36

7	1.7	0.53	3.71	49
27		10.64	40.77	139

The normal form of equation (2) are;

$$\sum Y = nA + b\sum x$$

or, $10.64 = 6A + 27b$ (3)

and, $\sum xy = A\sum x + b\sum x^2$

or, $40.77 = 27A + 139b$ (4)

Solving equation (3) and (4); we get,

$$A = 3.60$$

$$b = -0.406$$

since, $\log_e a = A$

or, $a = e^A = e^{3.6} = 36.6$

Hence, the required curved fitting equation is $y = 36.6e^{-0.406x}$.

17. **Using cubic spline interpolation technique, estimate the value of y(4) from the following data:** **[2075 Ashwin]**

x	1	3	5	7
y	1.56	-0.43	-16.90	6.10

Solution:

Here, n = 3 and 'x' is equally spaced *i.e.*, h = 2. Let, $m_0, m_1, \ldots\ldots, m_n$ be the second order derivatives at $x = x_0, x_1, \ldots\ldots\ldots, x_n$. Thus, we have,

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{bmatrix} = \frac{6}{h^2}\begin{bmatrix} \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

For natural spline: $m_0 = m_n = 0$

so, equation becomes;

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \frac{6}{(2)^2}\begin{bmatrix} \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

Now,

y	Δy	$\Delta^2 y$
1.56		
	-1.99	
-0.43		-14.48
	-16.47	
-16.90		39.47
	23	
6.10		

so, the substituting the value of $\Delta^2 y_1$ and $\Delta^2 y_2$; we get,

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = 1.5\begin{bmatrix} -14.48 \\ 39.47 \end{bmatrix}$$

On solving; we get,

$$m_1 = -9.74$$
$$m_2 = 17.23$$

Since, 4 lies between x_1 and x_2;

$$\therefore \quad y(4) = \frac{m_1}{6}\left[b(x - x_2) - \frac{(x - x_2)^3}{h}\right] - \frac{m_2}{6}\left[h(x - x_1) - \frac{(x - x_1)^3}{h}\right] + \frac{y_2(x - x_1) - y_1(x - x_2)}{h}$$

$$= -1.62(-2 + 0.5) - 2.87\left(2 - \frac{1}{2}\right) + [(-16.90 \times 1) - (-0.43)(-1)]$$

$$= 2.43 - 4.305 - 17.33 = -19.205$$

18. **Using the cubic spline techniques, estimate f(4) from the following data.** **[2075 Ashwin]**

x	1	3	5	7	9
f (x)	1.5	- 0.4	- 6.9	6.1	6.4

Solution:

Here, n = 4 and 'x' is equally spaced *i.e.*, h = 2. Let, m_0, m_1, ..., m_n be the second order derivatives at $x = x_0, x_1, \ldots\ldots\ldots, x_n$. Thus, we have,

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \frac{6}{h^2}\begin{bmatrix} \Delta^2 y_0 \\ \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

For natural spline,

$$m_0 = m_n = 0$$

Now,

y	Δy	$\Delta^2 y$
1.5		
	- 1.9	
- 0.4		- 4.6
	- 6.5	
- 6.9		19.5
	13	
6.1		- 12.7
	0.3	
6.4		

Thus, the above equation becomes;

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \frac{6}{4} \begin{bmatrix} -4.6 \\ 19.5 \\ -12.7 \end{bmatrix}$$

On solving; we get,

$m_1 = -4.27$

$m_2 = 10.21$

and, $m_3 = -7.31$

Since, 4 lies between x_1 and x_2;

$$\therefore \quad f(4) = \frac{m_1}{6}\left[h(x - x_2) - \frac{(x - x_2)^3}{h}\right] - \frac{m_2}{6}\left[h(x - x_1) - \frac{(x - x_1)^3}{h}\right] + y_2(x - x_1) - y_1(x - x_2)$$

$$= -0.71\left(-2 + \frac{1}{2}\right) - 1.70\left(2 - \frac{1}{2}\right) + [(-0.90 \times 1) - (-0.4)(-1)]$$

$$= 1.065 - 2.55 - 1.3 = -2.785$$

19. Derive an expression to evaluate first derivative from Newton's backward interpolation formula and evaluate $\frac{dy}{dx}$ at x = 8 from the following table. **[2075 Ashwin]**

x	0	2	4	6	8
y	0	- 0.7553	- 11.2151	34.2867	- 8.3226

Solution:

The difference table is as under;

x	y	Δ	Δ^2	Δ^3	Δ^4
0	0				
		- 0.7553			
2	- 0.7553		- 9.7045		
		- 10.4598		65.6661	
4	- 11.2151		55.9616		- 209.7388
		45.5018		- 144.07271	
6	34.2867		-88.11111		
		- 42.6093			
8	- 8.3226				

Using Newton backward interpolation formula; we have,

$x_n = 8$

$h = 2$

$$\therefore \quad \left(\frac{dy}{dx}\right)_8 = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n\right)$$

$$= \frac{1}{2}\left[-42.6093 + \frac{1}{2}(-88.1111) + \frac{1}{3}(-144.07271) + \frac{1}{4}(-209.7388)\right]$$

$$= -93.5619$$

20. **Using the method of least squares, fit the following set of data to a curve of the form $y = a\log_e x + b$.** [2075 Chaitra]

x	0.5	1.0	1.5	2.0	2.5	3.0
y	3.7	5.3	5.8	6.6	6.9	7.5

Solution:

The given equation is; (1)

$$y = a\log_e x + b \quad (2)$$

Let, $y = aX + b$

where, $X = \log_e x$.

Now,

x	2	$X = \log_e x$	X^2	xy
0.5	3.7	-0.69	0.48	-2.56
1	5.3	0	0	0
1.5	5.8	0.40	0.16	2.35
2	6.6	0.69	0.48	4.57
2.5	6.9	0.91	0.83	6.32
3	7.5	1.10	1.20	8.24
	35.8	2.42	3.17	18.92

The normal form of the equation (2) are;

$$\sum y = nb + a\sum X \quad (3)$$

or, $35.8 = 6b + 2.42a$

and, $\sum Xy = b\sum X + a\sum X^2$ (4)

or, $18.92 = 2.42b + 3.17a$

Solving equation (3) and (4); we get,

$b = 5.14$

$a = 2.04$

Hence, the required curve fitting equation is $y = 2.04\log_e x + 5.14$.

21. **Using finite difference table, show that the following data satisfies a cubic polynomial.** [2076 Ashwin]

x	0	1	2	3	4
y	-8	0	26	88	204

Solution:
Using Newton's forward table for finite difference; we have,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	−8				
		8			
1	0		18		
		26		18	
2	26		36		0
		62		18	
3	88		54		
		116			
4	204				

We take;

$$h = 1$$

$$x_0 = 0$$

and, $p = \dfrac{x-0}{h} = x$

Using Newton's forward interpolation formula for finite difference; we get,

$$f(x) = f(0) + \frac{x}{1!}\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0) + \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 f(0) \quad (1)$$

$$= 1 + x(8) + \frac{x(x-1)}{2}\times 18 + \frac{x(x-1)(x-2)}{6}\times 18 + \frac{x(x-1)(x-2)(x-3)}{24}\times 0$$

$$= 1 + 8x + 9x^2 - 9x + (3x^2 - 3x)(x-2)$$

$$= 3x^3 + 5x + 1$$

Hence, $f(x) = 3x^3 + 5x + 1$ is cubic polynomial.

22. Interpolate y(24) from the following data using natural cubic spline. [2076 Ashwin]

x	10	15	20	25	30
y	22	31	28	25	26

Solution:
Using formula; we have,

$$M_{i-1} + 4M_i + 4M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \quad (1)$$

$$i = 1$$

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2)$$

Putting $h = 5$ and $y_0 = 22, y_1 = 31, y_2 = 28$

$$M_0 + 4M_1 + M_2 = \frac{6}{25}(22 - 2 \times 31 + 28)$$

$$\therefore \quad M_0 + 4M_1 + M_2 = \frac{-72}{25} \qquad (2)$$

Again, putting $i = 2$ in the equation (1); we get,

$$M_1 + 4M_2 + M_3 = \frac{6}{25}(y_1 - 2y_2 + y_3) = \frac{6}{25}(31 - 2 \times 28 + 25)$$

$$\therefore \quad M_1 + 4M_2 + M_3 = 0 \qquad (3)$$

Again, putting $i = 3$ in the equation (1); we get,

$$M_2 + 4M_3 + M_4 = \frac{6}{25}(y_2 - 2y_3 + y_4) = \frac{6}{25}(28 - 2 \times 25 + 26)$$

$$\therefore \quad M_2 + 4M_3 + M_4 = \frac{24}{25} \qquad (4)$$

For equal interval;

$$M_0 = M_n = 0 \ (i.e., M_0 = M_4 = 0)$$

From equations (2), (3) and (4); we have,

$$4M_1 + M_2 = \frac{-72}{25}$$

$$M_1 + 4M_2 + M_3 = 0$$

$$M_2 + 4M_3 = \frac{24}{25}$$

Solving above these three equations; we get,

$$M_1 = \frac{-132}{175}, M_2 = \frac{24}{175} \text{ and } M_3 = \frac{36}{175}$$

Now, the cubic spline in $(x_i \le x \le x_{i+1})$ is;

$$f(x) = \frac{1}{6h}\{(x_{i+1} - x)^3M_i + (x - x_i)^3M_{i+1}\} + \left(\frac{x_{i+1} - x}{h}\right)\left(y_i - \frac{h^2}{6}M_i\right)$$
$$+ \left(\frac{x - x_i}{h}\right)\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right) \qquad (5)$$

Putting $i = 2$ in the equation (5); we have,
Since, 24 lies between 20 and 25;

$$f(x) = \frac{1}{6 \times 5}\{(x_3 - x)^3M_2 + (x - x_2)^3M_3\} + \left(\frac{x_3 - x}{5}\right)\left(y_2 - \frac{25}{6}M_2\right)$$
$$+ \left(\frac{x - x_2}{h}\right)\left(y_3 - \frac{25}{6}M_3\right)$$

$$\therefore \quad f(24) = 25.2434$$

23. Using the least square method, fit a second-order polynomial $y = ax^2 + bx + c$ to the following set of data: [2076 Ashwin]

x	1.0	1.50	2.0	2.5
y	0.75	1.25	1.45	1.25

Solution:

We have,

$$y = ax^2 + bx + c$$

Let, normal equations are;

$$\sum y = a\sum x^2 + b\sum x + nc$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x \quad (1)$$

$$\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2 \quad (2)$$

(3)

Now, calculating the values of $\sum x$, $\sum x^2$, $\sum x^3$, $\sum x^4$, $\sum y$, $\sum xy$ and $\sum x^2y$ by table below:

x	y	xy	x^2	x^2y	x^3	x^4
1	0.75	0.75	1	0.75	1	1
1.50	1.25	1.875	2.25	2.8125	3.375	5.06
2.0	1.45	2.9	4	5.8	8	16
2.50	1.25	3.125	6.25	7.81	15.62	39.06
$\sum x = 7$	$\sum y = 4.7$	$\sum xy = 8.65$	$\sum x^2 = 13.5$	$\sum x^2y = 17.175$	$\sum x^3 = 28$	$\sum x^4 = 61.125$

Now, putting these values in the equation (1), (2) and (3); we get,

$$4.7 = 13.5a + 7b + 4c \quad (4)$$

$$8.65 = 28a + 13.5b + 7c \quad (5)$$

$$17.175 = 61.125a + 28b + 4c \quad (6)$$

Solving the equations (4), (5) and (6); we get,

$$a = -0.7$$

$$b = 2.79$$

$$c = -1.345$$

Hence, the required equation is $y = -0.7x^2 + 2.79x - 1.345$.

24. A rod is rotating in a plane. The following table gives the angle (radian) through which the rod has turned for a various of time 't' seconds. **[2076 Ashwin]**

t	1.0	1.2	1.4	1.6	1.8	1.9	2.0
θ	2.10	2.31	2.52	2.85	3.34	3.95	4.31

Calculate the angular velocity of rod when t = 1.1 sec.

Solution:

Re-arranging the given data:

t	1.0	1.2	1.4	1.6	1.8	2.0
θ	2.10	2.31	2.52	2.85	3.24	4.31

Now, making the Newton's forward difference table; we have,

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
1.0	2.10					
		0.21				
1.2	2.31		0			
		0.21		0.12		
1.4	2.52		0.12		- 0.18	
		0.33		- 0.06		0.86
1.6	2.85		0.06		0.68	
		0.39		0.62		
1.8	3.24		0.68			
		1.07				
2.0	4.31					

Now,

$$h = 0.2$$

We have to calculate angular velocity;

$$\omega = \left(\frac{d\theta}{dt}\right)_{at\ t=1.1}$$

so, $$\left(\frac{d\theta}{dt}\right)_{t=1.1} = \frac{1}{h}\left(\theta_0 - \frac{1}{2}\Delta^2\theta_0 + \frac{1}{3}\Delta^3\theta_0 - \frac{1}{4}\Delta^4\theta_0 + \frac{1}{5}\Delta^5\theta_0\right)$$

$$= \frac{1}{0.2}\left(0.21 - \frac{1}{2}\times 0 + \frac{1}{3}\times 0.12 - \frac{1}{4}\times(-0.18) + \frac{1}{5}\times 0.86\right)$$

$$= 2.335 \text{ radian/sec.}$$

$\therefore$ $$\omega = \left(\frac{d\theta}{dt}\right)_{t=1.1} = 2.335 \text{ radian/sec.}$$