# Chapter 7

# NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION



### Introduction

A partial differential equation is a differential equation involving more than one independent variable.

Partial differential equations arise in the study of many blanches of engineering: *e.g.*, in heat flow problems; fluid dynamics; electrical potential distribution; Electro-magnetic theory; study of diffusion of matter; analysis of torsion in a bar subject to twisting; etc.

Most of these problems can be formulated as second order partial differential equation with the highest order of derivatives being the second.

## 7.1 Classification of Partial Differential Equations of the Second Order

The most general form of linear partial differential equation of the second order in two independent variable is;

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 y}{\partial x \partial y} + C\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad (1)$$

where, A, B, C, D, E, F and G are functions of x and y only. u is dependent variable and x and y two independent variables.

An equation of the form;

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 y}{\partial x \partial y} + C\frac{\partial^2 u}{\partial x^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad (2)$$

In which the terms involving second order partial derivatives alone are linear is known as quasi-linear P.D.E. (Partial Differential Equation) of second order.

There are basically two numerical techniques, namely finite-difference method and finite-element method that can be used to solve partial differential equation. We will discuss here the application of finite-difference methods only, which are based on formulae for approximating the first and second derivatives of a function.

The equation (1) or, (2) is said to be of;

i) the elliptic type, of $B^2 - 4AC < 0$
ii) the parabolic type, if $B^2 - 4AC = 0$ and
iii) the hyperbolic type, if $B^2 - 4AC > 0$

The frequently occurring canonical forms of P.D.E.'s are;

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial x \partial y} = 0$ (Laplace Equation). Here, $A = 1, B = 0, C = 1$
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial x \partial y} = -f(x, y)$ (Poisson equation)

   Here, $A = 1, B = 0, C = 1$ and $f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = f(x, y)$
3. $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ (One-dimensional heat flow equation)
4. $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ (One-dimensional wave equation)

It can be easily verified that equations (1) and (2) are of the elliptic type. Equation (3) is of the parabolic type and equation (4) is of hyperbolic type.

Consider some examples;

i) $\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial x^2} - 2\left(\frac{\partial u}{\partial x}\right) = 0$

or, $u_{xx} - 2xu_{xy} + x^2u_{yy} - 2u_y = 0$

Here,

$A = 1, B = -2x, C = x^2$

$B^2 - 4AC = -2x - 4 \times 1 \times x^2 = 0$; for all x and y

Therefore, the equation is parabolic at all points.

ii) $x^2u_{xx} - y^2u_{yy} = 0$

Here,

$A = x^2, B = 0, C = -y^2$

$B^2 - 4AC = 0 + 4x^2y^2 > 0$; for all x and y $(\neq 0)$

Therefore, the equation is hyperbolic at all points.

iii) $u_{xx} - xu_{yy} = 0$

Here,

$$A = 1, B = 0, C = -x$$

$$B^2 - 4AC = 0 + 4x = 4x$$

Therefore, the equation is elliptic, parabolic or, hyperbolic, according as $x < 0, x = 0$ or $x > 0$ respectively.

iv) $u_{xx} - 4u_{yy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$

Here,

$$A = 1, B = 4, C = x^2 + 4y^2$$

$$B^2 - 4AC = 16 - 4(x^2 + 4y^2) = 4(4 - x^2 - 4y^2)$$

Therefore, the equation is elliptic, if $4 - x^2 - 4y^2 < 0$

*i.e.*, $x^2 + 4y^2 > 4$

*i.e.*, $\frac{x^2}{4} + \frac{y^2}{1} > 1$

*i.e.*, outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1.$

The gives P.D.E. is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ and hyperbolic inside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1.$

## 7.2 Numerical Solution of Laplace Equation

In practical life, it is derived to find solution of the Laplace equation over a square or rectangular region whose sides will be parallel to the co-ordinate axes. The boundary values *i.e.*, value of u will be given at all on some specific points on the boundaries. The mathematical solution of Laplace equation will give the value of $u(x, y)$ at every $(x, y)$ of the given region.

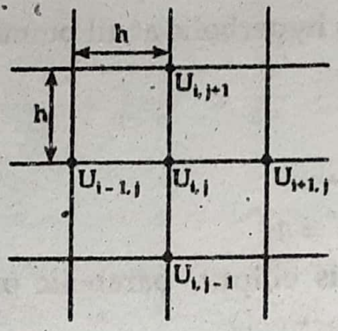
When the mathematical solution is not available, the given region in the xy-plane is divided into a network or lattice of squares by drawing line parallel to the co-ordinates axes at intervals of h. The points of intersection of their lines are called lattice points mesh points or gird points.

The numerical solution of equation (1) or equivalently equation (4) will give values of u only at gird paints.

### Standard Five Point Formula

The value of u at any interior gird point is the average of the values of u at the four nearest gird points (two of which lying on the vertical line and other two on the horizontal line passing through it.)

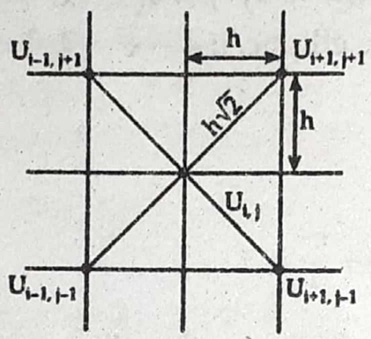
$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$$




**Diagonal Five Point Formula**

The value of u at any interior gird point is the average of the values of u at the four neighbouring diagonal mesh point.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j-1} + u_{i-1,j+1}\right]$$




**Choice between Standard and Diagonal Five Point Formula**

The standard formula is to be preferred to the diagonal formula as the error in the latter is four times that in the former.

Choice between them should be done so that we get less number of unknown variables whose value we have to guess zero for more accuracy.

**Solution of Laplace Equation [Standard Five Point Formula with Iterative Method]**

**Laplace Equations**

In two dimensions, the general linear partial differential equation of the second order is given in form of;

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad (1)$$

In equation (1); when A = 1, B = 0, C = 1 and $f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$; then,

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \text{or,}\quad &u_{xx} - u_{yy} = 0 \\ \text{or,}\quad &\nabla^2 u = 0 \end{aligned}\right\} \text{This is Laplace equation.}$$

Laplace equation is an example of elliptical partial differential equation which arises in steady-state flow and potential problems.

## Difference Equation Corresponding Laplace Equation

The Laplace equation is;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

Assuming that the step-sizes for x and y are each equal to and that $x_i = x_0 + ih$ and $y_j = y_0 + jh$ and denoting $u(x_i, y_j)$ by $u_{i,j}$ the finite difference approximations for particle derivatives of $u(x, y)$ are given by;

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i, y_j)} = \frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} \qquad (2)$$

and, $$\left(\frac{\partial^2 u}{\partial y^2}\right)_{(x_i, y_j)} = \frac{(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{h^2} \qquad (3)$$

Using (2) and (3) in equation (1); we get,

$$\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} = \frac{(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{h^2} = 0$$

or, $$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

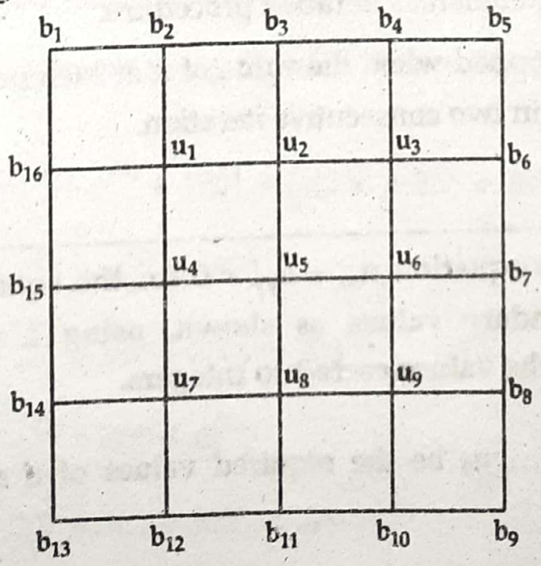
Equation (4) in the finite difference equation corresponding to Laplace equation, which true for the point $(x_i, y_j)$.

## Liebmann's Iteration Process

It is only a modified version of Gauss-Seidel method which is used to solve the equations formed by standard and diagonal five point equation to find unknown variables.

Let us consider a square region and it is required to find numerical solution of $\nabla^2 u = 0$.

Let, boundary values be $b_1, b_2, \ldots\ldots\ldots, b_{16}$. Let the values required (interior gird point) be $u_1, u_2, \ldots\ldots\ldots, u_9$.



Though we can assume initial values of unknown variables to zero but to obtain solution quickly, we find non-zero initial values as explained below.

**Step 1:**

Find the innermost (central) grid point.

$$u_5^{(0)} = \frac{1}{4}(b_{15} + b_7 + b_{11} + b_3) \qquad \text{[By standard formula]}$$

**Step 2:**

Find the corner grid point ($u_1$, $u_3$, $u_7$, $u_9$)

$$\left.\begin{aligned} u_1^{(0)} &= \frac{1}{4}(b_3 + b_{15} + b_1 + u_5^{(0)}) \\ u_3^{(0)} &= \frac{1}{4}(b_5 + u_5^{(0)} + b_3 + b_7) \\ u_7^{(0)} &= \frac{1}{4}(u_5^{(0)} + b_{13} + b_{15} + b_{11}) \\ u_9^{(0)} &= \frac{1}{4}(b_7 + b_{11} + u_5^{(0)} + b_9) \end{aligned}\right\} \text{[by diagonal formula]}$$

**Step 3:**

Finally we find remaining $u_2$, $u_4$, $u_6$, $u_8$ by using the standard and formula.

*i.e.,*
$$u_2^{(0)} = \frac{1}{4}(u_1^{(0)} + u_1^{(0)} + b_3 + u_5^{(0)})$$
$$u_4^{(0)} = \frac{1}{4}(b_{15} + u_5^{(0)} + u_1^{(0)} + u_7^{(0)})$$
$$u_6^{(0)} = \frac{1}{4}(u_5^{(0)} + b_7 + u_3^{(0)} + u_9^{(0)})$$
$$u_8^{(0)} = \frac{1}{4}(u_7^{(0)} + u_9^{(0)} + u_5^{(0)} + b_{11})$$

The values of $u_i^{(0)}$ thus obtained are only crude approximations. To improve their accuracy we precede with an iteration process *i.e.,* we compute the values of $u_1^{(i)}, u_2^{(i)}, \ldots\ldots\ldots, u_9^{(i)}$ in that order by using the latest available values at the neighboring points and using standard formula. This is known as Liebmann's iteration procedure.

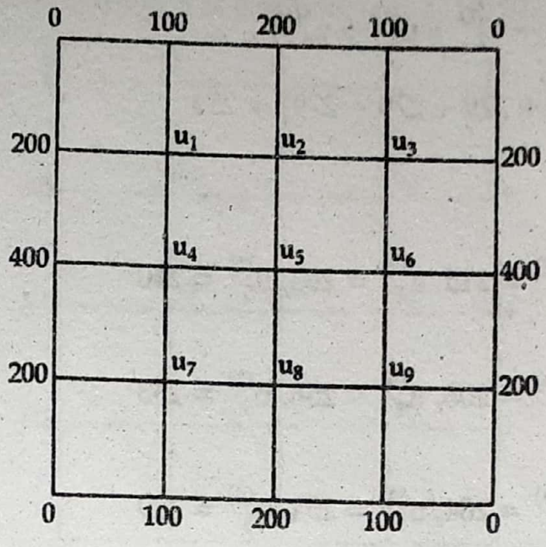
This iteration is stopped when the value of u at each point are equal to the required accuracy in two consecutive iteration.

**Example 7.1**

**Solve the Laplace equation $u_{xx} + u_{yy} = 0$ for the square mesh in figure below with boundary values as shown, using Liebmann's iteration procedure obtain the values correct to integers.**

**Solution:**

Let $u_1, u_2, u_3, \ldots\ldots\ldots, u_9$ be the required values of u at the interior grid points.




**Step 1: Check for symmetry**

The given boundary values are symmetrical with respect to middle most horizontal and vertical lines. Therefore, values of u at the interior grid are also symmetrical.

*i.e.,* $u_1 = u_2 = u_9 = u_7$

$u_2 = u_8$

$u_4 = u_6$

**Step 2: Number of unknown variables**

$u_1, u_2, u_4, u_5$

**Step 3: Initial values of unknown variables**

$$u_5^{(0)} = \frac{1}{4}(400 + 400 + 200 + 200) = 300 \qquad \text{Standard formula}$$

$$u_1^{(0)} = \frac{1}{4}(0 + 400 + 200 + 400) = 225 \qquad \text{Diagonal formula}$$

$$u_2^{(0)} = \frac{1}{4}(225 + 225 + 200 + 300) = 238$$

$$u_4^{(0)} = \frac{1}{4}(400 + 300 + 225 + 225) = 288$$

**Step 4: Apply iterative process using standard formula**

**Iteration 1**

$$u_1^{(1)} = \frac{1}{4}(200 + u_2^{(0)} + u_4^{(0)} + 100) = \frac{1}{4}(200 + 238 + 288 + 100) = 207$$

$$u_2^{(1)} = \frac{1}{4}(u_1^{(1)} + u_1^{(1)} + u_5^{(0)} + 200)$$

$$= \frac{1}{4}(207 + 207 + 300 + 200) = 229 \qquad [\because u_3 = u_1]$$

$$u_4^{(1)} = \frac{1}{4}(u_1^{(1)} + u_1^{(1)} + 400 + u_5^{(0)})$$

$$= \frac{1}{4}(207 + 207 + 400 + 300) = 279 \qquad [\because u_7 = u_1]$$

$$u_5^{(1)} = \frac{1}{4}\left(u_2^{(1)} + u_2^{(1)} + u_4^{(1)} + u_4^{(1)}\right) \qquad \left[\because \begin{array}{l} u_6 = u_4 \\ u_8 = u_2 \end{array}\right]$$

$$= \frac{1}{4}(229 + 229 + 279 + 279) = 254$$

Similarly,

**Iteration 2**

$$u_1^{(2)} = 202,\ u_2^{(2)} = 215,\ u_4^{(2)} = 265,\ u_5^{(2)} = 240$$

**Iteration 3**

$$u_1^{(3)} = 195,\ u_2^{(3)} = 208,\ u_4^{(3)} = 258,\ u_5^{(3)} = 233$$

**Iteration 4**

$$u_1^{(4)} = 192,\ u_2^{(4)} = 204,\ u_4^{(4)} = 254,\ u_5^{(4)} = 229$$

**Iteration 5**

$$u_1^{(5)} = 190,\ u_2^{(5)} = 202,\ u_4^{(5)} = 252,\ u_5^{(5)} = 227$$

**Iteration 6**

$$u_1^{(6)} = 189,\ u_2^{(6)} = 201,\ u_4^{(6)} = 251,\ u_5^{(6)} = 226$$

**Iteration 7**

$$u_1^{(7)} = 188,\ u_2^{(7)} = 201,\ u_4^{(7)} = 251,\ u_5^{(7)} = 226$$

Since, value of u in the sixth and seventh iteration is mostly equal and convergence has occurred correct to integer.
So, the required solution is;

$$u_1 = u_3 = u_7 = u_9 = 188$$
$$u_2 = u_8 = 201$$
$$u_4 = u_6 = 251$$
$$u_5 = 226$$

**Example 7.2**

**Given that $u(x, y)$ satisfies the equation $\nabla^2 u = 0$ and the boundary conditions $u(x, 0) = 0$, $u(x, 4) = 8 + 2x$, $u(0, y) = \frac{1}{2}y^2$ and $u(4, y) = y^2$, find the values of $u(i, j)$, $i = 1, 2, 3$, $j = 1, 2, 3$ correct to two decimals by Liebmann's iteration method.**

**Solution:**

**Step 1:**

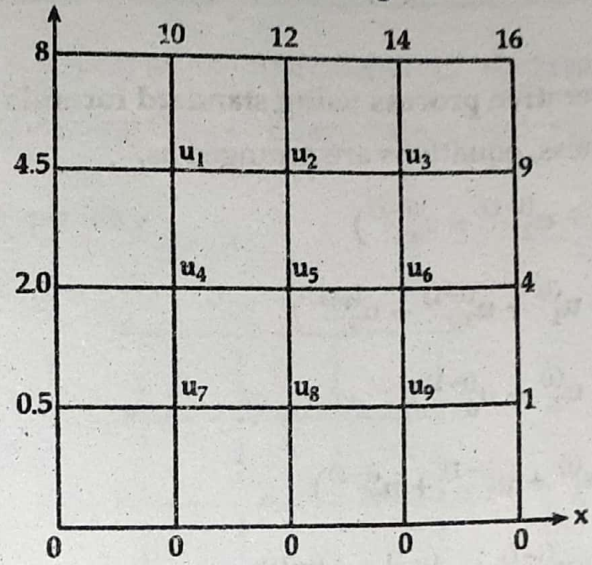
Here, $u(i, j)$, $i = 1, 2, 3$, $j = 1, 2, 3$ specify that we have 9 interior unknown grid points. Let them be $u_1, u_2, u_3, \ldots\ldots\ldots, u_9$.

Now, from boundary constraints;

$$u(x, 0) = 0 \text{ and } u(x, 4)$$

$$u(0, y) = \frac{y^2}{2} \text{ and } u(4, y)$$

specifies that we have gird of 4 × 4 with origin at (0, 0).




**Step 2: Find boundary values**

$$u(1, 0) = u(2, 0) = u(3, 0) = u(4, 0) = 0$$

$$u(0, 1) = \frac{1}{2}$$

$$u(0, 2) = \frac{(2)^2}{2} = 2$$

$$u(0, 3) = \frac{(3)^2}{2} = 4.5$$

$$u(0, 4) = \frac{(4)^2}{2} = 8$$

$$u(1, 4) = 8 + 2 \times 1 = 10$$

$$u(2, 4) = 8 + 2 \times 2 = 12$$

$$u(3, 4) = 8 + 2 \times 3 = 14$$

$$u(4, 4) = 8 + 2 \times 4 = 16$$

$$u(4, 1) = (1)^2 = 1$$

$$u(4, 2) = (2)^2 = 4$$

$$u(4, 3) = (3)^2 = 9$$

$$u(4, 4) = (4)^2 = 16$$

Substitute the value in grid above.

**Step 3: Check for symmetry**

No any symmetry

**Step 4: Number of unknowns**

$u_1, u_2, \ldots\ldots\ldots, u_9$

**Step 5: Initial values of unknown variables**

By same process as in previous question;

$$u_5^{(0)} = 4.5, u_1^{(0)} = 6.63, u_3^{(0)} = 9.13, u_7^{(0)} = 1.63$$

$u_9^{(0)} = 2.13, u_2^{(0)} = 8.07, u_4^{(0)} = 3.69, u_6^{(0)} = 4.95$

$u_8^{(0)} = 2.07$

**Step 6: Applying iterative process using standard formula**

For the sake of easiness, equations are arranged as;

$$u_1^{(i)} = \frac{1}{4}\left(14.5 + u_2^{(i-1)} + u_4^{(i-1)}\right)$$

$$u_2^{(i)} = \frac{1}{4}\left(12 + u_1^{(i)} + u_3^{(i-1)} + u_5^{(i-1)}\right)$$

$$u_3^{(i)} = \frac{1}{4}\left(23 + u_2^{(i)} + u_6^{(i-1)}\right)$$

$$u_4^{(i)} = \frac{1}{4}\left(2 + u_1^{(i)} + u_5^{(i-1)} + u_7^{(i-1)}\right)$$

$$u_6^{(i)} = \frac{1}{4}\left(u_2^{(i)} + u_4^{(i-1)} + u_6^{(i-1)} + u_8^{(i-1)}\right)$$

$$u_7^{(i)} = \frac{1}{4}\left(0.5 + u_4^{(i)} + u_8^{(i-1)}\right)$$

$$u_8^{(i)} = \frac{1}{4}\left(u_5^{(i)} + u_7^{(i)} + u_7^{(i-1)}\right)$$

$$u_9^{(i)} = \frac{1}{4}\left(1 + u_6^{(i)} + u_8^{(i)}\right)$$

**After iteration 1**

$u_1^{(1)} = 6.57, u_2^{(1)} = 8.05, u_3^{(1)} = 9.0, u_4^{(1)} = 3.68, u_5^{(1)} = 4.69, u_6^{(1)} = 4.96,$

$u_7^{(1)} = 1.56, u_8^{(1)} = 2.10, u_9^{(1)} = 2.02$

**After iteration 2**

$u_1^{(2)} = 6.56, u_2^{(2)} = 8.06, u_3^{(2)} = 9.01, u_4^{(2)} = 3.70, u_5^{(2)} = 4.71, u_6^{(2)} = 4.94,$

$u_7^{(2)} = 1.58, u_8^{(2)} = 2.08, u_9^{(2)} = 2.01$

**After iteration 3**

$u_1^{(3)} = 6.57, u_2^{(3)} = 8.07, u_3^{(3)} = 9.00, u_4^{(3)} = 3.71, u_5^{(3)} = 4.70, u_6^{(3)} = 4.93,$

$u_7^{(3)} = 1.58, u_8^{(3)} = 2.07, u_9^{(3)} = 2.0$

**After iteration 4**

$u_1^{(4)} = 6.57, u_2^{(4)} = 8.07, u_3^{(4)} = 9.00, u_4^{(4)} = 3.71, u_5^{(4)} = 4.70, u_6^{(4)} = 4.93,$

$u_7^{(4)} = 1.57, u_8^{(4)} = 2.07, u_9^{(4)} = 2.0$

As convergence has occurred at most of the gird points (correct to 2 decimals), the values are;

$u_1 = 6.57, u_2 = 8.07, u_3 = 9.00, u_4 = 3.71, u_5 = 4.70, u_6 = 4.93,$

$u_7 = 1.57, u_8 = 2.07, u_9 = 2.00$

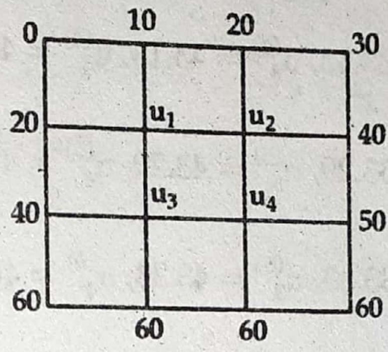
**Example 7.3**

**Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} = 0$ correct to two places of decimals, at the nodal points of square grid using the boundary values indicated.**

**Solution:**

**Step 1: Check for symmetry**

No any symmetry




**Step 2: Number of unknown**

$u_1, u_2, u_3, u_4$

**Step 3: Initial value of unknown variables**

Let us assume, $u_4^{(0)} = 0$

$$u_1^{(0)} = \frac{1}{4}\left(0 + u_4^{(0)} + 20 + 40\right) \qquad \text{[By diagonal formula]}$$

or, $u_1^{(0)} = \frac{1}{4}(0 + 0 + 20 + 40) = 15$

$$u_2^{(0)} = \frac{1}{4}\left(u_1^{(0)} + 40 + u_4^{(0)} + 20\right) \qquad \text{[By standard formula]}$$

or, $u_2^{(0)} = \frac{1}{4}(15 + 40 + 0 + 20) = 18.75$

$$u_3^{(0)} = \frac{1}{4}\left(40 + u_4^{(0)} + 60 + u_1^{(0)}\right) \qquad \text{[By standard formula]}$$

or, $u_3^{(0)} = \frac{1}{4}(40 + 0 + 60 + 15) = 28.75$

**Step 4: Applying iterative process using standard from first we find general equations**

$$u_1^{(i)} = \frac{1}{4}\left(20 + u_2^{(i-1)} + u_3^{(i-1)} + 10\right) = \frac{1}{4}\left(30 + u_2^{(i-1)} + u_3^{(i-1)}\right)$$

Similarly,

$$u_2^{(i)} = \frac{1}{4}\left(60 + u_1^{(i)} + u_4^{(i-1)}\right)$$

$$u_3^{(i)} = \frac{1}{4}\left(100 + u_1^{(i)} + u_4^{(i-1)}\right)$$

$$u_4^{(i)} = \frac{1}{4}\left(110 + u_2^{(i)} + u_3^{(i)}\right)$$

Now, apply iteration

After iteration 1

$$u_1^{(1)} = 19.38,\ u_2^{(1)} = 19.85,\ u_3^{(1)} = 29.85,\ u_4^{(1)} = 3$$

After iteration 2

$$u_1^{(2)} = 19.93,\ u_2^{(2)} = 19.85,\ u_3^{(2)} = 39.97,\ u_4^{(2)} = 44.95$$

After iteration 3

$$u_1^{(3)} = 24.99,\ u_2^{(3)} = 32.49,\ u_3^{(3)} = 42.49,\ u_4^{(3)} = 46.25$$

After iteration 4

$$u_1^{(4)} = 26.25,\ u_2^{(4)} = 33.13,\ u_3^{(4)} = 43.13,\ u_4^{(4)} = 46.57$$

After iteration 5

$$u_1^{(5)} = 26.57,\ u_2^{(5)} = 33.29,\ u_3^{(5)} = 43.29,\ u_4^{(5)} = 46.65$$

After iteration 6

$$u_1^{(6)} = 26.65,\ u_2^{(6)} = 33.33,\ u_3^{(6)} = 43.33,\ u_4^{(6)} = 46.67$$

After iteration 7

$$u_1^{(7)} = 26.67,\ u_2^{(7)} = 33.33,\ u_3^{(7)} = 43.33,\ u_4^{(7)} = 46.67$$

After iteration 8

$$u_1^{(8)} = 26.67,\ u_2^{(8)} = 33.33,\ u_3^{(8)} = 43.33,\ u_4^{(8)} = 46.67$$

Convergence has occurred at all the points.

Therefore, the required solution is;

$$u_1 = 26.67,\ u_1 = 33.33,\ u_1 = 43.33,\ u_1 = 46.67$$

## Example 7.4

**Solve $u_{xx} + u_{yy} = 0$ for the following mesh with boundary rules as shown. Iterate until the maximum difference between successive values at any gird point less than 0.001.**

**Solution:**

**Step 1: Check for symmetry**

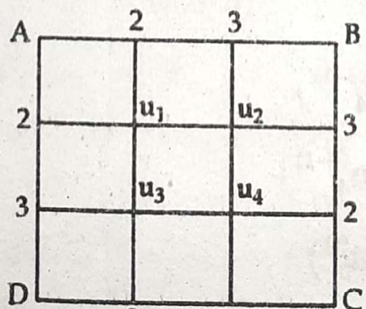
Boundary values are symmetrical about diagonal AC and BD.

*i.e.,* $u_1 = u_4$

and, $u_2 = u_3$

**Step 2: Number of unknown variables**

$u_1, u_2$



**Step 3: Initial values of unknown variables**

Let, $u_2^{(0)} = 0$

$$u_1^{(0)} = \frac{1}{4}(2 + u_1^{(0)} + 2 + u_3^{(0)}) = \frac{1}{4} \times 4 = 1 \qquad \text{[By standard formula]}$$

**Step 4: Apply iterative process using standard formula,**

First, we find general formula,

$$u_1^{(i)} = \frac{1}{4}(2 + u_2^{(i-1)} + u_3^{(i-1)} + 2) = \frac{1}{4}(4 + u_2^{(i-1)} + u_3^{(i-1)})$$

and, $$u_2^{(i)} = \frac{1}{4}(u_1^{(i-1)} + 3 + u_4^{(i-1)} + 3) = \frac{1}{4}(6 + u_1^{(i-1)} + u_4^{(i-1)})$$

Now, apply iteration; we have,

**After iteration 1**

$$u_1^{(1)} = 1, u_2^{(1)} = 2$$

**After iteration 2**

$$u_1^{(2)} = 2, u_2^{(2)} = 2.5$$

**After iteration 3**

$$u_1^{(3)} = 2.25, u_2^{(3)} = 2.625$$

**After iteration 4**

$$u_1^{(4)} = 2.313, u_2^{(4)} = 2.657$$

**After iteration 5**

$$u_1^{(5)} = 2.329, u_2^{(5)} = 2.665$$

**After iteration 6**

$$u_1^{(6)} = 2.333, u_2^{(6)} = 2.667$$

**After iteration 7**

$$u_1^{(7)} = 2.333, u_2^{(7)} = 2.667$$

Since, $u_1^{(6)} = u_1^{(7)}$ and $u_2^{(6)} = u_2^{(7)}$ convergence has occurred.

The required solution is;

$$u_1 = u_4 = 2.333$$

$$u_2 = u_3 = 2.667$$

## 7.3 Solution of Poisson's Equation (Finite Difference Approximation)

In two dimensions, the general linear partial differential equation of the second order is given by;

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

In the above equation, when $A = 1, B = 0, C = 1$ and $F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = f(x, y)$; then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

or, $\nabla^2 u = f(x, y)$

The above equation is called Poisson's equation.

It's method of solution is similar to that of the Laplace equation. So, assuming that the step sizes for x and y is each equal to h and that $x_i = x_0 + ih$, $y_j = y_0 + jh$ and denoting $u(x_i, y_j)$ by $u_{i,j}$, the finite difference formula for solving poison's equation takes the form as;

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(x_i, y_j)\right]$$

By applying the above formula to each grid point in the domain of consideration, we will get a system of linear equations in terms of $u_{i,j}$. These equations may be solved either by any of the elimination methods or by any iteration techniques as done in solving Laplace's equation.

**Example 7.5**

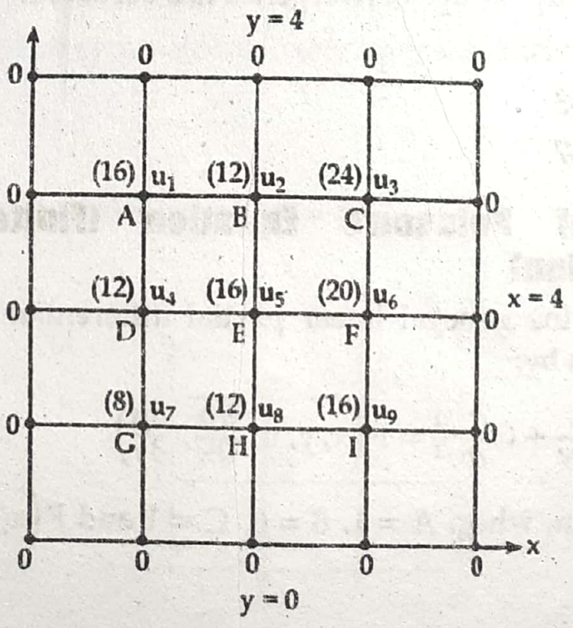
**Solve the Poisson equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -4(x + y)$ over the square mesh with sides $x = 0$, $y = 0$, $x = 4$ and $y = 4$ with $u = 0$ on the boundary, taking the mesh length equal to 1 unit.**

**Solution:**

The difference equation corresponding to poison equation $\nabla^2 u = -f(x, y)$ is;

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(x_i, y_j)\right]$$

This formula tells that when we find 'u' at any point we have to add the four adjacent values of 'u' and the value of $h^2 f(x_i, y_j)$ corresponding to the concerned point and divide the sum by 4.



Here,

$$h = 1$$

and, $f(x_i, y_j) = 4(x_i + y_j)$

Also,

$$x_0 = 0$$

and, $y_0 = 0$ (co-ordinates of centre)

so, $x_i = x_0 + ih = 0 + i \times 1 = i$

and, $y_i = y_0 + jh = 0 + j \times 1$

Equation at points A, B, C, ........., I will be as;

$A(i = 1, j = 3)$

so, $u_1 = \frac{1}{4}[0 + u_2 + u_4 + 0 - (1)^2 \times 4(1 + 3)]$

i.e., $u_1^{(i)} = 4 + \frac{1}{4}[u_2^{(i-1)} + u_4^{(i-1)}]$

$B(i = 2, j = 3)$

so, $u_2 = \frac{1}{4}[u_1 + u_3 + u_5 + 0 + (1)^2 \times 4(2 + 3)]$

i.e., $u_2^{(i)} = 5 + \frac{1}{4}[u_1^{(i-1)} + u_3^{(i-1)} + u_5^{(i-1)}]$

$C(i = 3, j = 3)$

so, $u_3 = \frac{1}{4}[u_2 + 0 + u_6 + 0 + (1)^2 \times 4(3 + 3)]$

i.e., $u_3^{(i)} = 6 + \frac{1}{4}[u_2^{(i)} + u_6^{(i-1)}]$

$D(i = 1, j = 2)$

so, $u_4 = \frac{1}{4}[0 + u_5 + u_7 + u_1 + (1)^2 \times 4(1 + 2)]$

i.e., $u_4^{(i)} = 3 + \frac{1}{4}[u_1^{(i)} + u_5^{(i-1)} + u_7^{(i-1)}]$

$E(i = 2, j = 2)$

so, $u_5 = \frac{1}{4}[u_4 + u_6 + u_2 + u_8 + (1)^2 \times 4(2 + 2)]$

i.e., $u_5^{(i)} = 4 + \frac{1}{4}[u_2^{(i)} + u_4^{(i)} + u_6^{(i-1)} + u_8^{(i-1)}]$

$F(i = 3, j = 2)$

so, $u_6 = \frac{1}{4}[u_5 + 0 + u_9 + u_3 + (1)^2 \times 4(3 + 2)]$

i.e., $u_6^{(i)} = 5 + \frac{1}{4}[u_3^{(i)} + u_5^{(i)} + u_9^{(i-1)}]$

$G(i = 1, j = 1)$

so, $u_7 = \frac{1}{4}[0 + u_8 + 0 + u_4 + (1)^2 \times 4(1+1)]$

i.e., $u_7^{(i)} = 2 + \frac{1}{4}[u_4^{(i)} + u_8^{(i-1)}]$

$H(i = 2, j = 1)$

so, $u_8 = \frac{1}{4}[u_7 + u_9 + 0 + u_5 + (1)^2 \times 4(2+1)]$

i.e., $u_8^{(i)} = 3 + \frac{1}{4}[u_5^{(i)} + u_7^{(i)} + u_9^{(i-1)}]$

$I(i = 3, j = 1)$

so, $u_9 = \frac{1}{4}[u_8 + 0 + 0 + u_6 + (1)^2 \times 4(3+1)]$

i.e., $u_9^{(i)} = 4 + \frac{1}{4}[u_6^{(i)} + u_8^{(i)}]$

Neither the application of standard formula nor that of diagonal formula will be non-zero values for u's at the grid point, we simply assume that $u_r^{(0)} = 0, r = 1, 2, \ldots\ldots\ldots, 9$ and proceed with iterations.

**First iteration**

$$u_1^{(1)} = 4 + \frac{1}{4}[0 + 0] = 4$$

$$u_2^{(1)} = 5 + \frac{1}{4}[4 + 0 + 0] = 6$$

$$u_3^{(1)} = 6 + \frac{1}{4}[6 + 0] = 7.5$$

$$u_4^{(1)} = 3 + \frac{1}{4}[4 + 0 + 0] = 4$$

$$u_5^{(1)} = 4 + \frac{1}{4}[6 + 4 + 0 + 0] = 6.5$$

$$u_6^{(1)} = 5 + \frac{1}{4}[7.5 + 6.5 + 0] = 8.5$$

$$u_7^{(1)} = 2 + \frac{1}{4}[4 + 0] = 3$$

$$u_8^{(1)} = 3 + \frac{1}{4}[6.5 + 3 + 0] = 5.38$$

$$u_9^{(1)} = 4 + \frac{1}{4}[8.5 + 5.38] = 7.47$$

**Second iteration**

$u_1^{(2)} = 6.5$ $\quad u_2^{(2)} = 10.13$ $\quad u_3^{(2)} = 10.66$

$u_4^{(2)} = 7.0$ $\quad u_5^{(2)} = 11.75$ $\quad u_6^{(2)} = 12.47$

$u_7^{(2)} = 5.10$ $\quad u_8^{(2)} = 9.08$ $\quad u_9^{(2)} = 9.39$

**Third iteration**

$u_1^{(3)} = 8.28$ $\quad u_2^{(3)} = 12.67$ $\quad u_3^{(3)} = 12.29$

$u_4^{(3)} = 9.28$ $\quad u_5^{(3)} = 14.88$ $\quad u_6^{(3)} = 14.14$

$u_7^{(3)} = 6.59$ $\quad u_8^{(3)} = 10.72$ $\quad u_9^{(3)} = 10.22$

**Fourth iteration**

$u_1^{(4)} = 9.49$ $\quad u_2^{(4)} = 14.17$ $\quad u_3^{(4)} = 13.08$

$u_4^{(4)} = 10.74$ $\quad u_5^{(4)} = 16.44$ $\quad u_6^{(4)} = 14.94$

$u_7^{(4)} = 7.37$ $\quad u_8^{(4)} = 11.11$ $\quad u_9^{(4)} = 10.61$

**Fifth iteration**

$u_1^{(5)} = 10.23$ $\quad u_2^{(5)} = 14.94$ $\quad u_3^{(5)} = 13.47$

$u_4^{(5)} = 11.51$ $\quad u_5^{(5)} = 17.23$ $\quad u_6^{(5)} = 15.33$

$u_7^{(5)} = 7.76$ $\quad u_8^{(5)} = 11.90$ $\quad u_9^{(5)} = 10.81$

## Example 7.6

**Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$, given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = \frac{1}{3}$**

**Solution:**

The finite difference formula for the Poisson's equation $\nabla^2 u = -f(x, y)$ is;

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(x_i, y_j)\right] \quad (1)$$

Here,

$$h = \frac{1}{3}$$

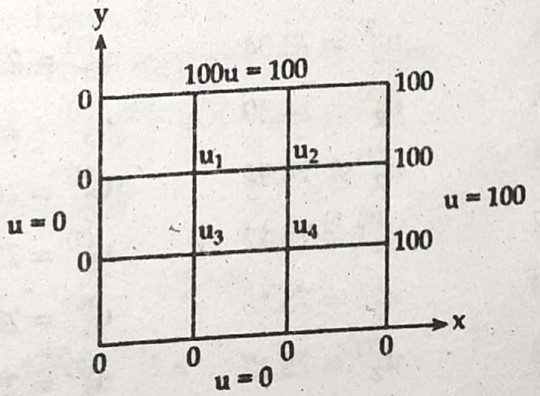
$$f(x_i, y_i) = 81 x_i y_i$$

$$x_i = x_0 + ih = 0 + ih$$

$$y_i = y_0 + jh = 0 + jh = jh$$

so, $$h^2 f(x_i, y_i) = h^2 \times 81 \times ih \times jh = h^4 \times 81 \times (ij) = \left(\frac{1}{3}\right)^4 \times 81 \times (ij)$$

$$= ij$$



Since, the partial differential equation and the boundary conditions are not altered when x and y are interchanged, the values of u will be symmetric about the line y = x. Hence, $u_u = u_1$

**Note**

In the previous example also, we could have used this idea of symmetry about the line y = x and noted that $u_1 = u_9$, $u_2 = u_6$ and $u_4 = u_8$

Equation (1) can be written as algorithms given below for successive iterations.

For $u_1 \left(i = \frac{1}{3}, j = \frac{2}{3}\right)$

$$u_1^{(i)} = \frac{1}{4}\left[0 + u_2^{(i-1)} + u_3^{(i-1)} + 100 + \frac{1}{3} \times \frac{2}{3}\right]$$

$$= \frac{1}{18} + \frac{1}{4}\left[u_2^{(i-1)} + u_3^{(i-1)} + 100\right]$$

For $u_2 \left(i = \frac{2}{3}, j = \frac{2}{3}\right)$

$$u_1^{(i)} = \frac{1}{4}\left[u_1^{(i)} + 100 + u_4^{(i-1)} + 100 + \frac{2}{3} \times \frac{2}{3}\right]$$

$$= \frac{1}{9} + \frac{1}{4}\left[u_1^{(i)} + u_4^{(i-1)} + 200\right]$$

$$= \frac{1}{9} + \frac{1}{4}\left[2u_1^{(i)} + 200\right] \qquad [\because u_1 = u_4]$$

For $u_3 \left(i = \frac{1}{3}, j = \frac{1}{3}\right)$

$$u_3^{(i)} = \frac{1}{4}\left[0 + u_4 + 0 + u_1 + \frac{1}{3} \times \frac{1}{3}\right]$$

$$= \frac{1}{36} + \frac{1}{2}u_1^{(i)} \qquad [\because u_1 = u_4]$$

Let us assume that $u_2^0 = u_3^0 = 0$; so,

$$u_1^{(1)} = 25.06$$

$$u_2^{(1)} = \frac{1}{9} + \frac{1}{4}(2 \times 25.06 + 200) = 62.64$$

$$u_3^{(1)} = \frac{1}{36} + \frac{1}{2} \times 25.06 = 12.56$$

| | | |
|---|---|---|
| $u_1^{(2)} = 43.86$ | $u_2^{(2)} = 72.04$ | $u_3^{(2)} = 21.96$ |
| $u_1^{(3)} = 48.56$ | $u_2^{(3)} = 74.39$ | $u_3^{(3)} = 24.31$ |
| $u_1^{(4)} = 49.73$ | $u_2^{(4)} = 74.98$ | $u_3^{(4)} = 24.89$ |
| $u_1^{(5)} = 50.02$ | $u_2^{(5)} = 75.12$ | $u_3^{(5)} = 25.04$ |
| $u_1^{(6)} = 50.10$ | $u_2^{(6)} = 75.16$ | $u_3^{(6)} = 25.08$ |
| $u_1^{(7)} = 50.12$ | $u_2^{(7)} = 75.17$ | $u_3^{(7)} = 25.09$ |

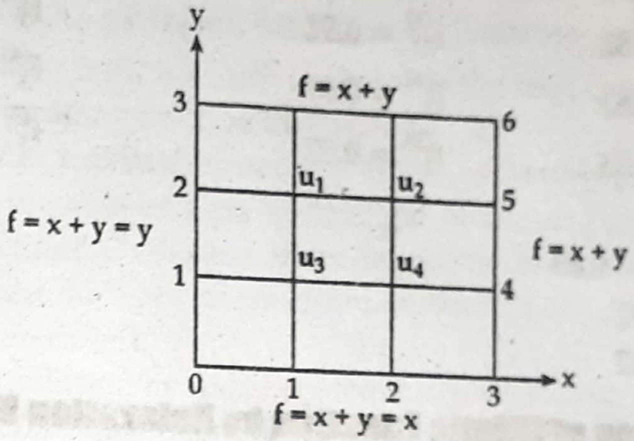
Hence,

$u_1 = 50.1$ $\quad u_2 = 75.1$ $\quad u_3 = 25.0$

**Example 7.7**

**Solve the Poisson equation $\nabla^2 f - 2xy = 0$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = x + y$ on the boundary. Take $h = 1$**

**Solution:**




The boundary point's values are found using $f = x + y$

$$x_i = 0 + hi = i$$

$$y_i = 0 + hj = j$$

$$h^2 f(x_i, y_i) = h^2 \times 2 \times (x_i)(y_i) = 1^2 \times 2 \times i \times j = 2ij$$

The finite difference formula for me position's equation;

$$\nabla^2 u - f(x, y) = 0$$

or, $$\nabla^2 u = f(x, y)$$

$$\nabla^2 u = g(x, y) \text{ will be as;}$$

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(x_i, y_i)]$$

For the given Poisson's equation it will be as;

$$f_{i,j} = \frac{1}{4}[f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1} - 2ij] \quad (1)$$

Since, the partial differential equation and the boundary conditions are not altered when x and y are interchanged, the values of 'f' will be symmetric about me line $y = x$. Hence, $f_1 = f_4$

Equation (1) can be written in algorithm form as;

$$f_1^{(i)} = \frac{1}{4}\left[2 + f_2^{(i-1)} + f_3^{(i-1)} + 4 - 2 \times 1 \times 2\right] \quad [\because i = 1, j = 2]$$

or, $$f_1^{(i)} = \frac{1}{2} + \frac{1}{4}\left[f_2^{(i-1)} + f_3^{(i-1)}\right]$$

$$f_2^{(i)} = \frac{1}{4}\left[f_1^{(i)} + 5 + f_4^{(i-1)} + 5 - 2 \times 2 \times 2\right] \quad [\because i = 2, j = 2]$$

or, $$f_2^{(i)} = \frac{1}{2} + \frac{1}{4} \times 2f_2^{(i)} = \frac{1}{2}\left[1 + f_1^{(i)}\right] \quad [\because f_1 = f_4]$$

$$f_3^{(i)} = \frac{1}{4}\left[1 + f_4^{(i-1)} + 1 + f_1^{(i-1)} - 2 \times 1 \times 1\right] \qquad [\because i = 1, j = 1]$$

or, $$f_3^{(i)} = \frac{1}{2} f_1^{(i)} \qquad [\because f_1 = f_4]$$

Let as assume $f_2^0 = f_3^0 = 0$; so,

$$f_1^{(1)} = 0.5 \qquad f_2^{(1)} = \frac{1}{2}[1 + 0.5] = 0.75 \qquad f_3^{(1)} = 0.25$$

$$f_1^{(2)} = 0.75 \qquad f_2^{(2)} = 0.88 \qquad f_3^{(2)} = 0.38$$

$$f_1^{(3)} = 0.82 \qquad f_2^{(3)} = 0.91 \qquad f_3^{(3)} = 0.41$$

$$f_1^{(4)} = 0.83 \qquad f_2^{(4)} = 0.92 \qquad f_3^{(4)} = 0.42$$

$$f_1^{(5)} = 0.83 \qquad f_2^{(5)} = 0.92 \qquad f_3^{(5)} = 0.42$$

Hence,

$$f_1 = f_4 = 0.83$$

$$f_2 = 0.92$$

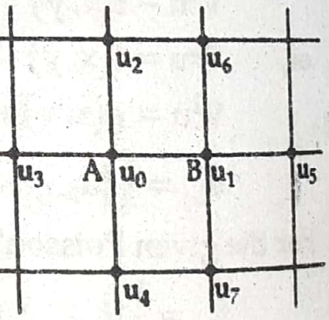
and, $$f_3 = 0.42$$

## 7.4 Solution of Elliptic Equation by Relaxation Method

Consider the Laplace equation;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

We take a square region and divide it into a square net of mesh sign h. Let the value of u at A be $u_0$ and its values at the four adjacent points be $u_1$, $u_2$, $u_3$, $u_4$ figure (i); then,




$$\frac{\partial^2 u}{\partial x^2} = \frac{u_1 + u_3 - 2u_0}{h^2}$$

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{u_2 + u_4 - 2u_0}{h^2}$$

If (1) is satisfied at A; then,

$$\frac{u_1 + u_3 - 2u_0}{h^2} + \frac{u_2 + u_4 - 2u_0}{h^2} = 0$$

or, $$u_1 + u_2 + u_3 + u_4 - 4u_0 = 0$$

If $r_0$ be the residual (discrepancy) at the mesh point A; then,

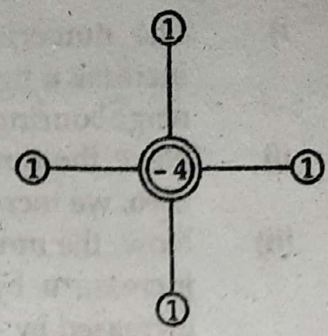
$$r_0 = u_1 + u_2 + u_3 + u_4 - 4u_0 \qquad (2)$$

Similarly, the residual at the point B, is given by;

$$r_1 = u_0 + u_5 + u_6 + u_7 - 4u_1 \text{ and so on.} \qquad (3)$$

The main aim of the relaxation process is to reduce all the residuals to zero by making them as small as possible step by step. We, therefore, try to adjust the value of 'u' at an internal mesh point so as to make the residual there at zero. But when the value of 'u' is changing at a mesh point, the values of the residuals at the neighbouring interior points will also be changed, If $u_0$ is given an increment 1; then,

i) (2) shows that $r_0$ is changed by $-4$.
ii) (3) shows that $r_1$ is change by 1.




*i.e.*, if the value of the function is increase by 1 at a mesh point shown by a double ring, then the residual at that point is decreased by 'u' while the residuals at the adjacent interior points (shown by a single ring), get increased each by 1. The relation pattern is shown in figure (ii).

## Working Procedure to Solve an Equation by Relation Method

1. Write down by trial, the initial value u at the interior mesh points by diagonal averaging or cross averaging.
2. Calculate the residual at each of these points by (2) above. If we apply this formula at a point near the boundary, one or more end points get chopped off since there are no residuals at the boundary.
3. Write the residuals at a mesh-point on the right of this point and the value of u on its left.
4. Obtain the solution by reducing the residuals to zero, one by one, by giving suitable increments to 'u' and using figure (ii). At each step, we reduce the numerically largest residual to zero and record the increment of 'u' on the left (below the earlier value thereat) and the modified residual on the right (below the earlier residual).
5. When a round of relaxation is completed, the value of u and its increments are added at each point. Using these values, calculated all the residual of mesh. If some of the recalculated residual are large, liquidate these again.
6. Stop the relaxation process, when the current values of the residuals are quite small. The solution will be the current value of u at each of the nodes.

**Example 7.8**

Solve by relaxation method, the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ inside the square bounded by the lines $x = 0$, $x = 4$, $y = 0$, $y = 4$ given that $u = x^2y^2$ on the boundary.

**Solution:**

Taking $h = 1$, we find u on the boundary from $u = x^2y^2$. The initial value of u at the nine mesh-points are estimated be 24, 56, 104; 16, 32, 56; 8, 16, 24 as shown on the left of the point in figure (i).

$\therefore$ Residual at A *i.e.*, $r_A = 0 + 56 + 16 + 16 - 4 \times 24 = -8$

Similarly,

$r_B = 0, r_C = -16, r_D = 0, r_E = 16, r_F = 0, r_G = 0, r_H = 0, r_I = -8$

i) The numerically largest residual is 16 at E. To liquidate it, we increase u by 4 so that the residual becomes zero and the residuals at neighbouring nodes get increased by 4.

ii) Next, the numerically largest residuals is – 16 at C. To reduce it to zero, we increase u by – 4.

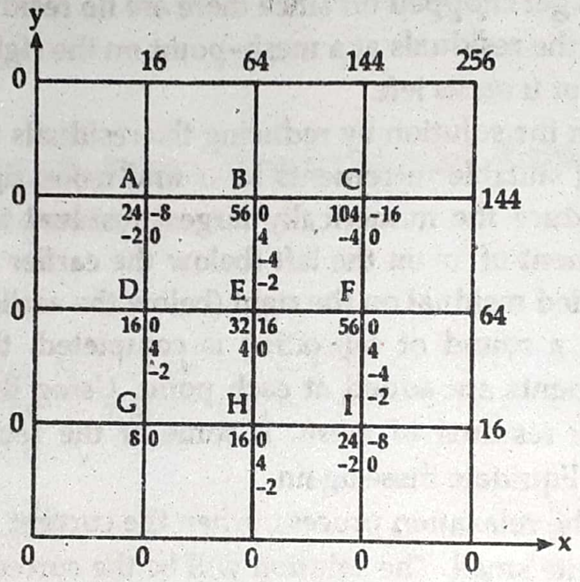
iii) Now, the numerically largest residual is – 8 at A. To liquidate it, we increase u by – 2 so that the residuals at the adjacent nodes are increased by – 2.

iv) Finally, the largest residual is – 8 at I. To liquidate it, we increase u by – 2 so that the residuals at the adjacent points are increased by – 2.

v) The numerically largest current residual being 2, we stop the relaxation process. Hence the final values of u are;

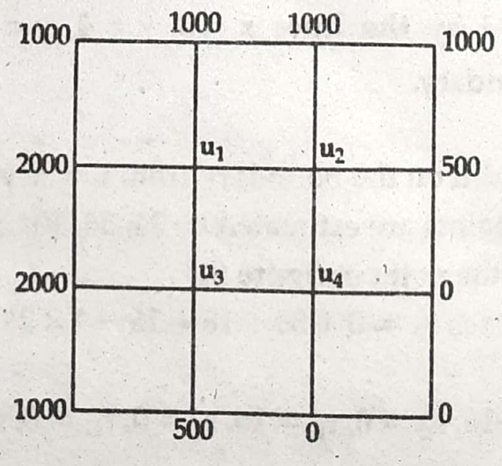
$u_A = 22,\quad u_B = 56,\quad u_C = 100,\quad u_D = 16,$

$u_E = 36,\quad u_F = 56,\quad u_G = 8,\quad u_H = 16,\quad u_I = 22$




## Example 7.9

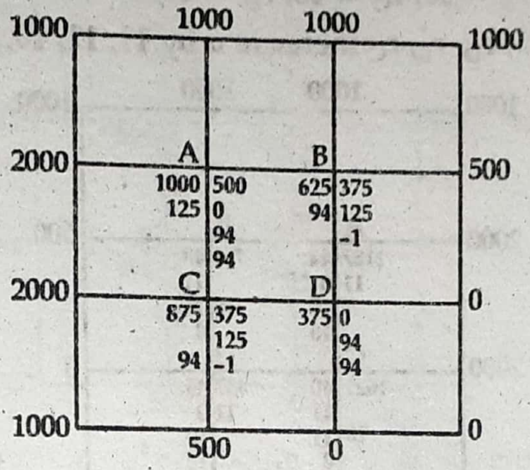
Given the values of u(x, y) on the boundary of the square in the figure below, evaluate the function of u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ the pivotal points of this figure by relaxation method.



**Solution:**

i) The initial value of u A, B, C and D are estimated to be 1000, 625, 875 and 375 [figure (i)]

$\therefore \quad r_A = 500,\ r_B = 375,\ r_C = 375,\ r_D = 0$



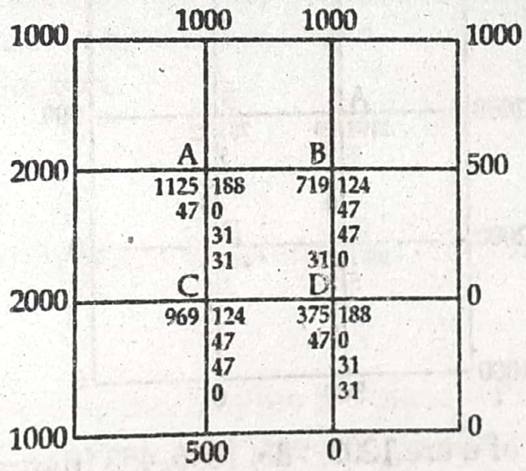

To liquidate $r_A$, increase u by 125

To liquidate $r_B$, increase u by 94

To liquidate $r_C$, increase u by 94

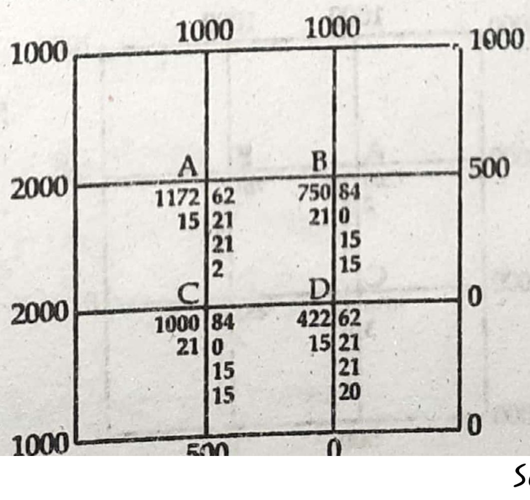
ii) Modified value of u are 1125, 719, 969, 375 [figure (ii)]

$\therefore \quad r_A = 500,\ r_B = 375,\ r_C = 375,\ r_D = 0$




To liquidate $r_A, r_D, r_B, r_C$ increase u by 47, 47, 31, 31 in turn.

iii) Revised values of u are 1172, 750, 1000, 422 [figure (iii)]



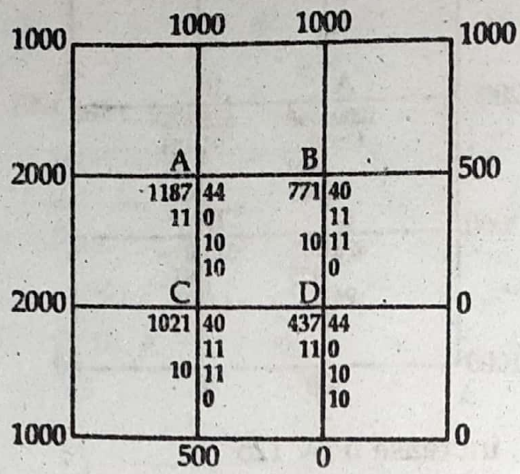
$\therefore \quad r_A = 62, r_B = 84, r_C = 84, r_D = 62$

To liquidate $r_B, r_C, r_A, r_D$ increase u by 21, 21, 15, 15 respectively.

iv) Improved values of u are 1187, 771, 1021, 437 [figure (iv)]

$\therefore \quad r_A = 44, r_B = 40, r_C = 40, r_D = 44$

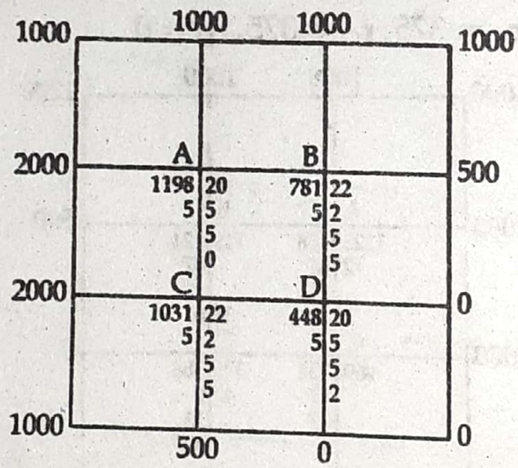
To liquidate $r_A, r_D, r_B, r_C$ increase u by 11, 11, 10, 10 respectively.




v) Modified values of u are 1198, 781, 1031, 448 [figure (v)]

$\therefore \quad r_A = 20, r_B = 22, r_C = 22, r_D = 20$

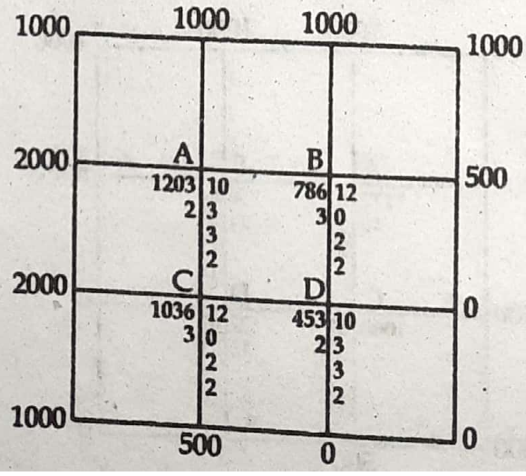
To liquidate $r_B, r_C, r_A, r_D$ increase u by 5, 5, 5, 5 respectively.




vi) Revised values of u are 1203, 786, 1036, 453 [figure (vi)]

$\therefore \quad r_A = 10, r_B = 12, r_C = 12, r_D = 10$

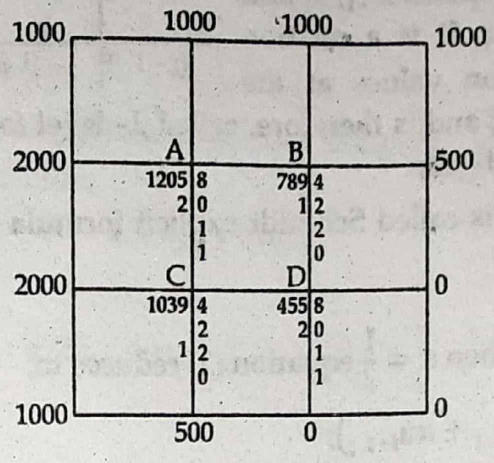
To liquidate $r_B, r_C, r_A, r_D$ increase u by 3, 3, 2, 2 respectively.



vii) Improved values of u are 1205, 789, 1039, 455 [figure (vii)]

$\therefore$ $r_A = 8, r_B = 4, r_C = 4, r_D = 8$

To liquidate $r_A, r_D, r_B, r_C$ increase u by 2, 2, 1, 1 respectively.




viii) Finally the current residual being 1, 0, 0, 1, we stop the relation process. Hence, the values of u at A, B, C, D are 1207, 790, 1040, 457.

## 7.5 Solution of One Dimension Heat Equation

The one-dimension heat conduction equation $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function u(x, t) which is defined for values of x form 0 to L and for values of time t from 0 to $\infty$.

One dimension heat equation;

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

where, $C^2 = \frac{k}{sp}$ is the diffusivity of the substance (cm²/sec.).

### Schmidt Method

Let us consider a rectangular mesh in the x–t plane with shaking h along x dilution and k along time t direction. Demoting mesh point $(x, t) = (ih, jk)$ as simple i, j; we have,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

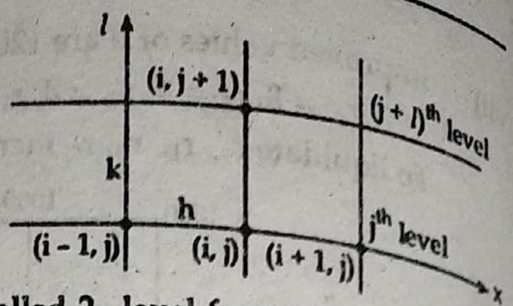
Substituting there in (1); we obtain,

$$u_{i,j+1} - u_{i,j} = \frac{kC^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \qquad (2)$$

where, $\alpha = \frac{kC^2}{h^2}$ is the mesh retro parameter.

This formula enables us to determine the value of 'u' at the $(i, j+1) + h$ mesh point its terms of the known function value at the point $x_{i-1}$, $x_i$ and $x_{i+1}$ at the instant $t_j$. It is a relation between the function values at the two levels $j+1$ and $j$ and is therefore, called 2- level formula. In schematic form (2) is shown in figure.




Hence, equation (2) is called Schmidt explicit formula which is valid only for $0 < \alpha \leq \frac{1}{2}$.

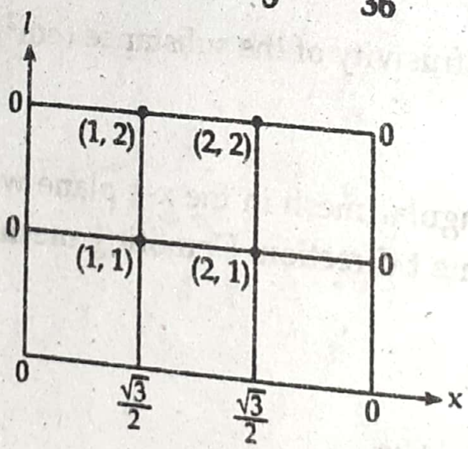
Also, in particular when $\alpha = \frac{1}{2}$ equation (2) reduces to;

$$u_{i, j+1} = \frac{1}{2}(u_{i-1, j} + \alpha u_{i+1, j});$$

which shown that the value of u at $x_i$ at time $t_{j+1}$ is the mean of the u-values at $x_{i-1}$ and $x_{i+1}$ at time $t_j$. This solution, known as Bendre-Schmidt reoccurrence relation gives the value of 'u' at the interval mesh point with the help of boundary and.

**Example 7.10**

**Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subjected to the condition $u(x, 0) = \sin \pi x$, $0 \leq x \leq 1$, $u(0, t) = u(1, t) = 0$, use Schmidt method. Carryout computation for two levels taking $h = \frac{1}{3}$, $k = \frac{1}{36}$**




**Solution:**

Here,

$$C^2 = 1, h = \frac{1}{3}, k = \frac{1}{36}$$

so that;

$$\alpha = \frac{kC^2}{h^2} = \frac{1}{4}$$

Also,

$$u_{1,0} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

and, $u_{1,0} = \sin\frac{2\pi}{3} = \sqrt{\frac{3}{2}}$

and, all boundary values all zero as shown in figure.

We know Schmidt formula;

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

becomes;

$$u_{i,j+1} = \frac{1}{4}(u_{i-1,j} + 2u_{i,j} + u_{i+1,j})$$

For i = 1, 2; j = 0

$$u_{1,1} = \frac{1}{4}(u_{0,0} + 2u_{1,0} + u_{2,0}) = \frac{1}{4}\left(0 + 2 \times \frac{\sqrt{3}}{2} + \sqrt{\frac{3}{2}}\right) = 0.65$$

$$u_{2,1} = \frac{1}{4}(u_{1,0} + 2u_{2,0} + u_{3,0}) = \frac{1}{4}\left(\sqrt{\frac{3}{2}} + 2 \times \frac{\sqrt{3}}{2} + 0\right) = 0.65$$

For i = 1, 2; j = 1

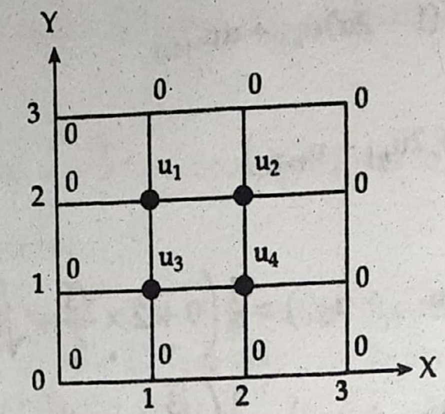
$$u_{1,2} = \frac{1}{4}(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49$$

## 7.6 EXAMINATION PROBLEMS

**1.** **Solve the equation $\Delta^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0$ y, $x = 3 = y$ with $u = 0$ on the boundary and mesh length $= 1$.** **[2071 Chaitra]**

**Solution:**




Here, the standard 5-point formula is:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - uu_{i,j} = -10\,(i^2 + j^2 + 10)$$

**For $u_1$ (i = 1, j = 2)**

$$0 + u_2 + u_3 - 4u_1 = -10\,(1^2 + 2^2 + 10)$$

or, $\quad u_2 + u_3 - 4u_1 = -150$

or, $\quad u_2 + u_3 + 150 = 4u_1$

$\therefore \quad u_1 = \frac{1}{4}(u_2 + u_3 + 150) \qquad (1)$

**For, $u_2$ (i = 2, j = 2)**

$$u_1 + 0 + 0 + u_4 - 4u_2 = -10\,(2^2 + 2^2 + 10)$$

or, $\quad u_1 + u_4 - 4u_2 = -180$

or, $\quad u_1 + u_4 + 180 = 4u_2$

$\therefore \quad u_2 = \frac{1}{4}(u_1 + u_4 + 180) \qquad (2)$

**For $u_3$ (i = 1, j = 1)**

$$0 + u_4 + u_1 + 0 = 4u_3 = -10(1^2 + 1^2 + 10)$$

or, $\quad u_4 + u_1 - 4u_3 = -120$

or, $\quad u_1 + u_4 + 120 = 4u_3$

$\therefore \quad u_3 = \frac{1}{4}(u_1 + u_4 + 120) \qquad (3)$

**For, $u_4$ (i = 2, j = 2)**

$$u_3 + 0 + u_2 + 0 - 4u_4 = -10(2^2 + 1^2 + 10)$$

or, $\quad u_2 + u_3 - 4u_4 = -150$

or, $\quad u_2 + u_3 + 150 = 4u_4$

$$\therefore \quad u_4 = \frac{1}{4}(u_2 + u_3 + 150) \qquad (4)$$

Solving equation (1) and (4), we get,

$$u_4 = u_1$$

$$\therefore \quad u_1 = \frac{1}{4}(u_2 + u_3 + 150)$$

$$u_2 = \frac{1}{4}(u_1 + u_1 + 180) = \frac{1}{2}(u_1 + 90) \ (\because u_4 = u_1)$$

$$u_3 = \frac{1}{4}(u_1 + u_1 + 120) = \frac{1}{2}(u_1 + 60) \ (\because u_4 = u_1)$$

Now,

**1st iteration,**

Let, $u_2 = u_3 = 0$

$$u_1^{(1)} = \frac{1}{4}(0 + 0 + 150) = 37.5$$

$$u_2^{(1)} = \frac{1}{2}(37.5 + 90) = 63.75$$

$$u_3^{(1)} = \frac{1}{2}(37.5 + 60) = 48.75$$

**2nd iteration,**

$$u_1^{(2)} = \frac{1}{4}(63.75 + 48.75 + 150) = 65.625$$

$$u_2^{(2)} = \frac{1}{2}(65.65 + 90) = 77.8125$$

$$u_3^{(2)} = \frac{1}{2}(65.65 + 60) = 62.8125$$

**3rd iteration,**

$$u_1^{(3)} = \frac{1}{4}(77.8125 + 62.8125 + 150) = 72.656$$

$$u_2^{(3)} = \frac{1}{2}(65.65 + 90) = 77.8125$$

$$u_3^{(3)} = \frac{1}{2}(65.65 + 60) = 62.8125$$

**4th iteration,**

$$u_1^{(4)} = \frac{1}{4}(81.328 + 66.328 + 150) = 74.424$$

$$u_2^{(4)} = \frac{1}{2}(74.414 + 90) = 82.207$$

$$u_3^{(4)} = \frac{1}{2}(-14.414 + 60) = 67.207$$

**5th iteration,**

$$u_1^{(5)} = \frac{1}{4}(82.207 + 67.207 + 150) = 74.854$$

$$u_2^{(5)} = \frac{1}{2}(74.854 + 90) = 82.427 \approx 82.43$$

$$u_3^{(5)} = \frac{1}{2}(74.854 + 60) = 67.427 \approx 67.43$$

**6th iteration,**

$$u_1^{(6)} = \frac{1}{4}(82.43 + 67.43 + 150) = 74.97$$

$$u_2^{(6)} = \frac{1}{2}(74.97 + 90) = 82.49$$

$$u_3^{(6)} = \frac{1}{2}(74.97 + 60) = 67.49$$

**7th iteration,**

$$u_1^{(7)} = \frac{1}{4}(82.49 + 67.49 + 150) = 74.99$$

$$u_2^{(7)} = \frac{1}{2}(74.99 + 90) = 82.49$$

$$u_3^{(7)} = \frac{1}{2}(74.99 + 60) = 67.49$$

Here, the values of iteration 6 and 7 are almost same.

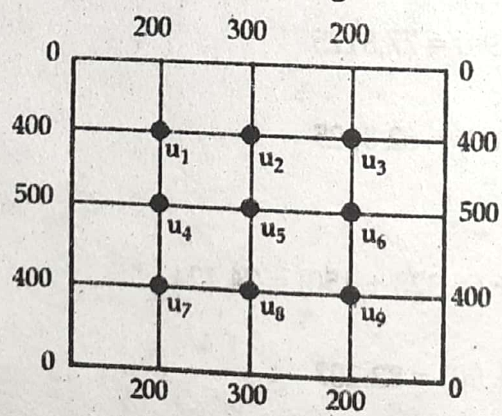
$\therefore$ $u_1 = 74.99$

$u_2 = 82.49$

$u_3 = 67.49$

$u_4 = u_1 = 74.99$

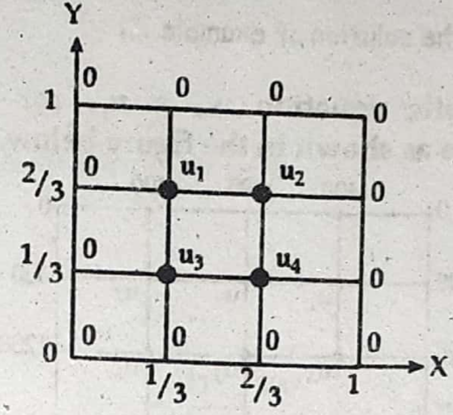
2. **Solve the Laplace equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in the figure.**




[2072 Ashwin]

Solution: Proceed same as the solution of example 7.1

3. Solve Poisson's equation $u_{xx} + u_{yy} = 729x^2y^2$ over the square domain $0 \le x \le 1, 0 \le y \le 1$ with step size $h = \frac{1}{3}$ with $u = 0$ on the boundary. [2072 Chaitra]




**Solution:**

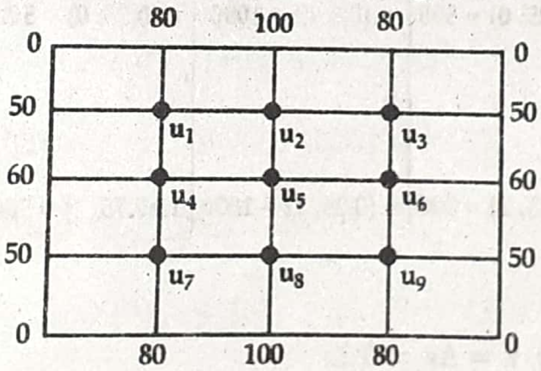
Here,

$$h = \frac{1}{3}$$

Further,

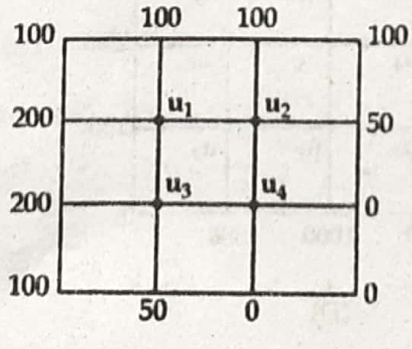
Proceed same as the solution of example 7.6

4. Solve the Laplace equations $u_{xx} + U_{yy} = 0$, over the square grid with boundary conditions as shown in the figure. [2073 Shrawan]




**Solution:** See the solution of example 7.1

5. Find the values of u (x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal point of the square region with boundary conditions as shown below. [2073 Magh]



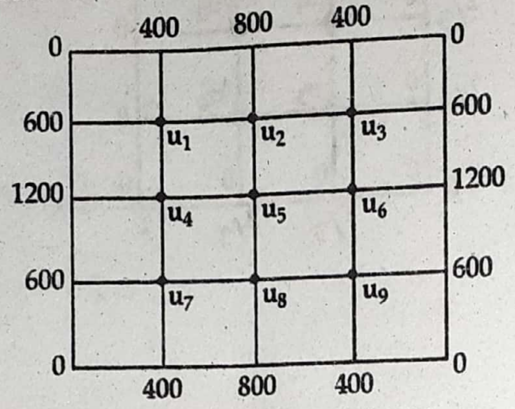
**Solution:**
Here,

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Now,

Proceed same as the solution of example 7.1

6. **Solve the elliptic equation $u_{xx} + u_{yy}$ for square mesh with boundary values as shown in the figure below.** **[2074 Bhadra]**



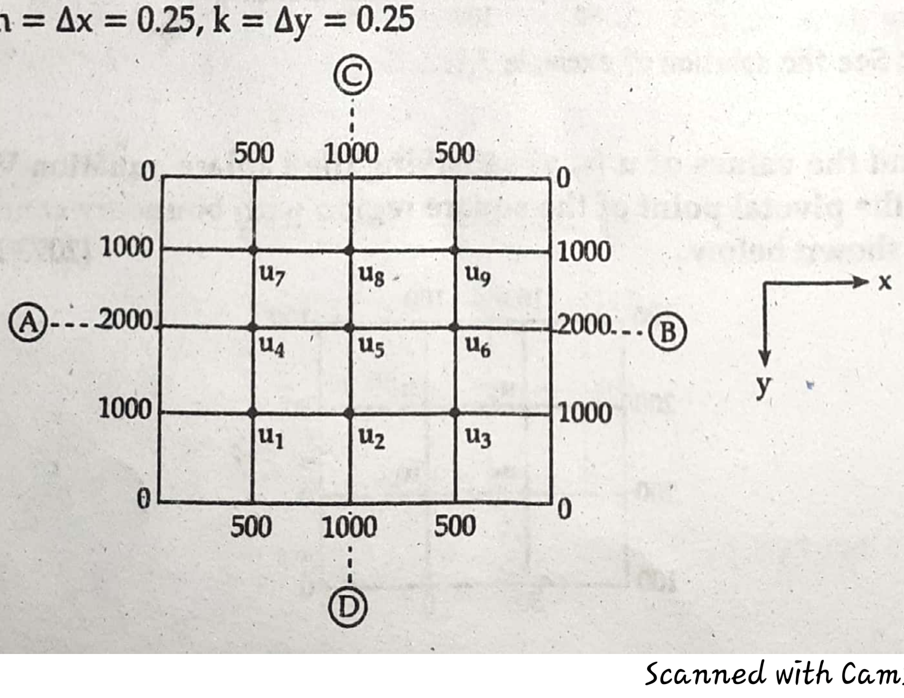

**Solution:** Proceed same as the solution of example 7.1

7. **Solve the elliptical equation (Laplace) $\mu_{xx} + \mu_{yy} = 0$ for the square mesh $0 \le x \le 1, 0 \le y \le 1$; where, $h = \Delta x = 0.25$ and $k = \Delta y = 0.25$ with the following boundary conditions.** **[2074 Ashwin]**

| | | | | |
|---|---|---|---|---|
| $u(0, 0) = 0$ | $u(0.25, 0) = 500$ | $u(0.5, 0) = 1000$ | $u(0.75, 0) = 500$ | $u(1, 0) = 0$ |
| $u(0, 0.25) = 1000$ | | | | $u(1, 0.25) = 1000$ |
| $u(0, 0.5) = 2000$ | | | | $u(1, 5) = 2000$ |
| $u(0, 0.75) = 1000$ | | | | $u(1, 0.75) = 1000$ |
| $u(0, 1) = 0$ | $u(0.25, 1) = 500$ | $u(0.25, 1) = 1000$ | $u(0.75, 1) = 500$ | $u(1, 1) = 0$ |

**Solution:**
Here,

$$h = \Delta x = 0.25, \; k = \Delta y = 0.25$$



Finite difference equation for $\mu_{xx} + \mu_{yy} = 0$ is;

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$

Using symmetric property about AB;

$$u_1 = u_7$$
$$u_2 = u_8$$

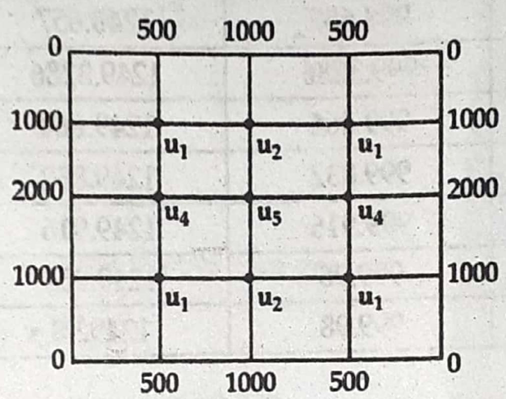
and, $u_3 = u_9$

Using symmetric property about CD;

$$u_1 = u_3$$
$$u_4 = u_6$$

and, $u_7 = u_9$

Hence,




Now, guess values are;

$$u_1 = \frac{1}{4}(1000 + 500 + 500 + 1000) = 750$$

$$u_2 = \frac{1}{4}(750 + 1000 + 1000 + 1000) = 937.5$$

$$u_4 = \frac{1}{4}(750 + 750 + 2000 + 2000) = 1375$$

$$u_5 = \frac{1}{4}(937.5 + 937.5 + 1375 + 1375) = 921.875$$

Again,

$$u_1 = \frac{1}{4}(1000 + u_4 + u_2 + 500)$$
$$= 0.25(1500 + u_2 + u_4) \quad (1)$$

$$u_2 = \frac{1}{4}(u_5 + 1000 + 2u_1)$$
$$= 0.25(1000 + u_5 + 2u_1) \quad (2)$$

$$u_4 = \frac{1}{4}(2000 + u_5 + 2u_1)$$
$$= 0.25(2000 + u_5 + 2u_1) \quad (3)$$

$$u_5 = \frac{1}{4}(2u_4 + 2u_2)$$
$$= 0.5(u_2 + u_4) \quad (4)$$

Using Gauss Siedal method; we have,

| n | $u_1$ (A) | $u_2$ (B) | $u_4$ (C) | $u_5$ (D) |
|---|---|---|---|---|
| | A = 0.25 (1500 + B + C) | B = 0.25 (1500 + D + 2A) | C = 0.25 (2000 + D + 2A) | D = 0.5 (B + C) |
| 0 | 750 | 937.5 | 1375 | 921.875 |
| 1 | 953.125 | 957.0312 | 1207.031 | 1082.031 |
| 2 | 916.0156 | 978.5156 | 1228.515 | 1103.515 |
| 3 | 926.757 | 989.2578 | 1239.257 | 1114.257 |
| 4 | 932.1289 | 994.629 | 1244.629 | 1119.629 |
| 5 | 934.814 | 997.3144 | 1247.314 | 1122.314 |
| 6 | 936.157 | 998.657 | 1248.657 | 1123.657 |
| 7 | 936.8286 | 999.3286 | 1249.3286 | 1124.3286 |
| 8 | 937.164 | 999.664 | 1249.664 | 1124.664 |
| 9 | 937.332 | 999.832 | 1249.832 | 1124.832 |
| 10 | 937.416 | 999.916 | 1249.916 | 1124.916 |
| 11 | 937.458 | 999.96 | 1249.958 | 1124.95 |
| 12 | 937.48 | 999.98 | 1249.98 | 1124.98 |

Hence,

$u_1 = 937.5$

$u_2 = 1000$

$u_4 = 1250$

and, $u_5 = 1125$

8. **Derive recurrence formula for 1-D heat equation $u_t = c^2U_{xx}$. Using it to solve the heat equation $U_t = 0.5U_{xx}$, $0 \le x \le 5$, $0 \le t \le 4$ with boundary conditions $U(x, 0) = xe^x \cdot (5 - x)$, $U(0, t) = 0 = U(5, t)$. Taking h = 1** **[2074 Chaitra]**

**Solution:**

**For the first part**

See the definition part 7.5

**For the second part**

We have,

$$U_t = 0.5U_{xx} \quad (1)$$

Comparing equation (1) with standard equation $u_t = c^2U_{xx}$; we get,

$c^2 = 0.5$

or, $c = 0.707$

and, $h = 1$ (given)

Now,

$$\frac{1}{2} = \frac{kc^2}{h^2}$$

or, $$\frac{1}{2} = \frac{k \times (0.707)^2}{(1)^2}$$

$\therefore$ $k = 1$

**Boundary condition**

i) $u(0, t) = 0$

$u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4} = 0$

ii) $u(5, t) = 0$

$u(5, 0) = u(5, 1) = u(5, 2) = u(5, 3) = u(5, 4) = 0$

iii) $u(x, 0) = xe^x(5 - x)$

$u(1, 0) = 1 \cdot e^1(5 - 1) = 10.873$

$u(2, 0) = 44.334$

$u(3, 0) = 120.51$

$u(4, 0) = 218.39$

$u(5, 0) = 0$

When j = 0, from recurrence relation; we get,

$$u_{i,0} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

$$\therefore \quad u_{1,0} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{0 + 44.334}{2} = 22.167$$

$$u_{2,0} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{10.873 + 120.51}{2} = 65.6915$$

Similarly, other calculations were made and we fill table as shown below:

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 10.873 | 44.334 | 120.51 | 218.39 | 0 |
| 1 | 0 | 22.167 | 65.6915 | 131.362 | 60.255 | 0 |
| 2 | 0 | 32.8457 | 76.7645 | 62.973 | 65.681 | 0 |
| 3 | 0 | 38.382 | 47.907 | 71.222 | 31.4865 | 0 |
| 4 | 0 | 23.9535 | 54.802 | 39.696 | 35.611 | 0 |

9. **Derive Bender-Schmidt recurrence formula for solving 1-D heat equation $u_t = C^2u_{xx}$ and use it to solve boundary value problem $u_t = u_{xx}$ under the consideration $u(0, t) = u(1, t) = 0$ and $u(x, 0) =$ $\sin \pi x$ up to t = 5 sec. Take h = 0.2** **[2075 Bhadra]**

**Solution:**

**For the first part**

Any function can be expanded by Taylor's series as;

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \dots\dots\dots$$

Now, forward difference formula for $\frac{\partial u}{\partial t}$ is;

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

where, k is the step size for time t.

and, $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Finite difference equation for 1-D heat equation is;

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

or, $$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{C^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

or, $$u_{i,j+1} = \frac{kC^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + u_{i,j}$$

or, $$u_{i,j+1} = \lambda(u_{i-1,j} + u_{i+1,j}) + (1 - 2\lambda)u_{i,j} \qquad \left[\because \lambda = k\frac{C^2}{h^2}\right]$$

In Bender Schmidt method, we choose h and k so that $\lambda = \frac{1}{2}$; we get,

$$u_{i,j+1} = \left(\frac{u_{i-1,j} + u_{i+1,j}}{2}\right)$$

It is called Bender-Schmidt recurrence relation.

**For the second part**

We have,

$$u_t = u_{xx} \qquad (1)$$

Comparing equation (1) with standard equation $u_t = C^2 u_{xx}$; we get,

$$C = 1$$

$$h = 0.2 \text{ (given)}$$

Now,

$$\frac{1}{2} = \frac{kC^2}{h^2}$$

or, $$\frac{1}{2} = \frac{k \times 1^2}{(0.2)^2}$$

$\therefore$ $k = 0.02$

**Boundary condition**

i) $u(0, t) = 0$

$$u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4} = u_{0,5} = 0$$

ii) $u(1, t) = 0$

$$u(5, j) = 0 \qquad [\because x = ih = 0.2i]$$

$$u_{5,1} = u_{5,2} = u_{5,3} = u_{4,4} = u_{5,5} = 0$$

iii) $u(x, 0) = \sin(\pi x) = \sin(\pi \cdot ih)$

$$u(0, 0) = \sin(\pi \cdot 0) = 0$$
$$u(1, 0) = \sin(\pi \times 1 \times 0.2) = 0.01$$
$$u(2, 0) = \sin(\pi \times 2 \times 0.2) = 0.022$$
$$u(3, 0) = \sin(\pi \times 3 \times 0.2) = 0.0329$$
$$u(4, 0) = \sin(\pi \times 4 \times 0.2) = 0.0438$$

When j = 0, form recurrence relation; we get,

$$u_{i,0} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

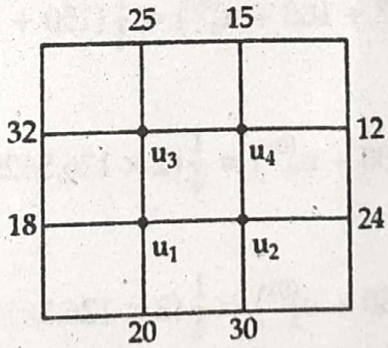
or, $$u_{1,0} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{0 + 0.022}{2} = 0.011$$

$$u_{2,0} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{0.01 + 0.0329}{2} = 0.02145$$

Similarly, other calculations were made and we fill table as shown below:

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.01 | 0.022 | 0.0329 | 0.0438 | 0 |
| 1 | 0 | 0.011 | 0.02145 | 0.0329 | 0.01645 | 0 |
| 2 | 0 | 0.0107 | 0.02195 | 0.0189 | 0.01645 | 0 |
| 3 | 0 | 0.0109 | 0.0148 | 0.0192 | 0.00945 | 0 |
| 4 | 0 | 0.0074 | 0.01505 | 0.012 | 0.0096 | 0 |
| 5 | 0 | 0.00752 | 0.0097 | 0.0123 | 0.006 | 0 |

10. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary condition as shown in the figure attached. [2075 Chaitra]



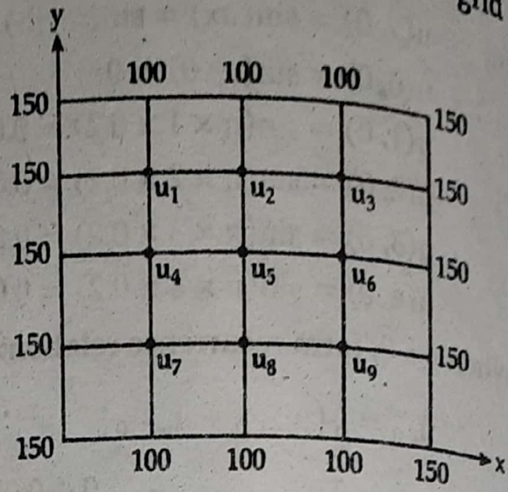

Solution: Proceed as the solution of Q. no. 7.1

11. Solve $U_{xx} + U_{yy} = 0$ for the square mesh bounded by $0 \le x \le 4$; $0 \le y \le 4$ and the boundary conditions $u(0, y) = 150$, $u(4, y) = 150$, $u(x, 0) = 100$, $u(x, 4) = 100$; $0 \le x \le 4$; $0 \le y \le 4$. Find the values of $u(i, j)$, $i = 1, 2, 3$ correct to 3 places decimals. [2076 Ashwin]

Solution:

Let $u_1, u_2, u_3, ........., u_9$ be the required values of u at the interior grid points.

From the boundary conditions, the boundary values will be as shown in the figure.




**Step I : Check for symmetry**

The given boundary values are symmetrical with respect to middle most horizontal and vertical lines. Therefore, the values of u at the interior grid are also symmetrical.

*i.e.,* $u_1 = u_3 = u_7 = u_9$

$u_2 = u_8$

$u_4 = u_6$

**Step II : Numbers of unknown variables**

$u_1, u_2, u_4$ and $u_5$

**Step III : Initial values of unknown variables**

$$u_5^{(0)} = \frac{1}{4}(150 + 150 + 100 + 100) = 125 \text{ Standard formula}$$

$$u_1^{(0)} = \frac{1}{4}(150 + 125 + 100 + 150) = 131.25 \text{ Diagonal formula}$$

$$u_2^{(0)} = \frac{1}{4}(131.25 + 131.25 + 100 + 125) = 121.875 \text{ Standard formula}$$

$$u_4^{(0)} = \frac{1}{4}(150 + 125 + 131.25 + 131.25) = 134.375 \text{ Standard formula}$$

**Step IV : Applying iterative process using standard formula**

**Iterative 1**

$$u_1^{(1)} = \frac{1}{4}(150 + u_2^{(0)} + 100 + u_4^{(0)}) = \frac{1}{4}(150 + 121.875 + 100 + 134.375)$$
$$= 126.5625$$

$$u_2^{(1)} = \frac{1}{4}(2u_1^{(1)} + 100 + u_5^{(0)}) = \frac{1}{4}(2 \times 126.5625 + 100 + 125)$$
$$= 119.5312$$

$$u_4^{(1)} = \frac{1}{4}(2u_1^{(1)} + 150 + u_5^{(0)}) = \frac{1}{4}(2 \times 126.5625 + 150 + 125)$$
$$= 132.0312$$

$$u_5^{(1)} = \frac{1}{4}(2u_2^{(1)} + 2u_4^{(1)}) = \frac{1}{4}(2 \times 119.5312 + 2 \times 132.0312)$$
$$= 125.7812$$

Now, precede next iteration as above up to two iteration values get same.