Chapter 1

INTRODUCTION, APPROXIMATION AND ERRORS OF COMPUTATION

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1.1 Introduction, Importance of Numerical Methods

Numerical Methods are techniques by which mathematical problems are formulated so that they can be solve with arithmetic and logical operations. It is an area of mathematics and computer science to solve the problems of applied mathematics, physics and real world problems numerically. Numerical method deals in finding the exact solution or at least approximate solution to the required significant digits of higher mathematics through basic arithmetic operations.

With the advancement of science, technology and engineering field, the importance of numerical method has also increased significantly. Especially in this modern era where computer programmings are used for computation, the basics of numerical methods are used widely to solve daily mathematical problems. The importance of Numerical methods can be listed as;

i) The types of problems that can be addressed greatly increases due to the use of numerical methods.

ii) The use of numerical methods reduces time and effort to great extent.

iii) Numerical methods allow to solve problems with different complexities which, otherwise, would have limitation of capabilities.

iv) Numerical methods help to understand and use computer programming.

v) Numerical method helps to reinforce the understanding of mathematics.

1.2 Approximation and Errors in Computation

The exact solution of general arithmetic and physical problems whose well defined unique solution exists can be computed using the arithmetic and algebraic rules. But there are complicated problems too whose exact solution cannot be found due to their complexities and the result is always approximated value close enough to the exact value. The approximation of the result to the exact value is dependent on the numbers of significant digits involved in calculation and accuracy desired.

When approximation is made while computing any problem, it is natural that the problems possess some errors. Error is the difference between the true result and the calculated result. Based on the source and characteristics, errors can be classified as:

i) Inherent error

The error due to the limitations of measuring device, methodology, scale and uncertainties is known as inherent error. The inherit errors are:

a) Date error

The error obtained while reading, observations is known as data error.

b) Conversion error

Conversion errors are the errors accumulated during the conversion of unit, scale, etc of the field observation.

ii) Numerical error

The errors obtained during the arithmetic and algebraic calculation process is known as numerical errors. Numerical errors are:

a) Round off error

Rounding off a number within n significant digits means confining the number up to a digit place by increasing, decreasing or unaltering the n^{th} digit.

b) Truncation error

It is not possible to make consideration of every term of an infinite series or even of a very long series. For such problems, we eliminate all the terms after a specific number of terms which is termed as truncation error.

c) Blunders

Blunders are errors due to negligence, carelessness, hurry, lack of cross check, etc.

1.3 Taylor's Series

Taylor's series is a representative form of a function as the sum of infinite values of the functions derivatives at a single point. The Taylor series can be used to approximate the value of a function by using a finite number of terms. The polynomial formed by taking some initial terms of the Taylor series is called a Taylor polynomial. The degree of accuracy of the functional value depends on the number of terms used in the Taylor polynomial. The quantitative estimate of error can be obtained by the use of such an approximation.

Example of Taylor's series

The Taylor series of functions f(x) at a point 'a' is given by;

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots\dots\dots$$

Thus, Taylor series for log (x) at some point $a = x_0$ is given by:

$$\log x_0 + \frac{1}{x_0}(x - x_0) - \frac{1}{x_0^2}\frac{(x-x_0)^2}{2} + \dots\dots\dots$$

1.4 Newton's Finite differences (Forward, backward, control difference, divided difference)

Newton's finite difference is the study of change in dependent variable of a function due to a finite change in independent variable of that function. The finite change may be an equal change or unequal changes. Based on the path or system set out to obtain difference, Newton's finite differences are classified as forward, backward, central and divided difference.

i) Newton's forward difference

If the equidistant variables of x produces output $y = y_0, y_1, y_2, y_3, \dots\dots, y_n$ respectively for the function $y = f(x)$ and the finite differences are represented as $y_1 - y_0, y_2 - y_1, \dots\dots\dots, y_n - y_{n-1}$, then the difference is known as first forward difference if the differences are represented as $\Delta y_0, \Delta y_1, \Delta y_2, \dots\dots\dots, \Delta y_n$ where, Δ is the forward difference operator.

The general term of first forward difference is;

$$\Delta y_i = y_{i+1} - y_i$$

Similarly, the general term of second forward difference is;

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) = y_{i+2} - 2y_{i+2} + y_i$$

The general term of third forward difference is;

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i = (y_{i+3} - 2y_{i+2} + y_{i+1}) - (y_{i+1} - 2y_{i+2} + y_i)$$
$$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

Thus, the general term of any n^{th} forward difference is;

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$$

The forward differences are shown in the table below:

Value of x	value of y	1st difference Δy	2nd difference $\Delta^2 y$	3rd difference $\Delta^3 y$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
			$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
$x_0 + h$	y_1			
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
			$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\vdots$
$x_0 + 2h$	y_2			
		$\Delta y_2 = y_3 - y_2$	$\vdots$	$\vdots$
$x_0 + 3h$	y_3	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

ii) Newton's backward difference

If the equidistant variables of x produces output $y = y_0, y_1, y_2, y_3, \ldots\ldots, y_n$ respectively for the function $y = f(x)$ and the finite differences are represented as $y_1 - y_0,\ y_2 - y_1,\ \ldots\ldots\ldots,\ y_n - y_{n-1}$ then the difference is known as first backward difference if the differences are represented as $\nabla y_1, \nabla y_2, \nabla y_3, \ldots\ldots\ldots, \nabla y_n$ where, ∇ is the backward difference operator.

The general term of first backward difference is:

$$\nabla y_i = y_i - y_{i-1}$$

Similarly, the general term of second backward difference is;

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}$$

The general term of third backward difference is;

$$\nabla^3 y_i = \nabla^2 y_i - \nabla^2 y_{i-1}$$

Thus, the general term of any n^{th} backward difference is;

$$\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}$$

The backward differences are shown in the table below:

Value of x	value of y	1st difference ∇y	2nd difference $\nabla^2 y$	3rd difference $\nabla^3 y$
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
$x_0 + h$	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
$x_0 + 2h$	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		$\vdots$
$x_0 + 3h$	y_3	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

iii) Newton's central difference

If the equidistant variables of x produces output $y = y_0, y_1, y_2, y_3, \ldots\ldots, y_n$ respectively for the function $y = f(x)$ and the finite differences are represented as $y_1 - y_0, y_2 - y_1; y_3 - y_2, \ldots\ldots\ldots, y_n - y_{n-1}$ then the difference is known as first central difference if the differences are represented as $\delta y_{\frac{1}{2}}$, $\delta y_{\frac{3}{2}}, \delta y_{\frac{5}{2}}, \ldots\ldots\ldots, \delta y_{\frac{n-1}{2}}$ where, δ is the central difference operator.

The general term of first central difference is;

$$\delta y_{\frac{i-1}{2}} = y_i - y_{i-1}$$

Similarly, the general term of second central difference is;

$$\delta^2 y_i = \delta y_{\frac{i-1}{2}} - \delta y_{\frac{i-1}{2}-1}$$

The general term of third central difference is;

$$\delta^3 y_{\frac{i-1}{2}} = \delta^2 y_i - \delta^2 y_{i-1}$$

and, so on …….

The central differences are shown in the table below;

Value of x	value of y	1st difference δy	2nd difference $\delta^2 y$	3rd difference $\delta^3 y$
x_0	y_0			
		$\delta y_{\frac{1}{2}} = y_1 - y_0$		
$x_0 + h$	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$	
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$
$x_0 + 2h$	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$	
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\vdots$
$x_0 + 3h$	y_3	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

iv) Newton's divided difference

If the variables $x_0, x_1, x_2, x_3, \ldots\ldots, x_n$ produces output $y_0, y_1, y_2, y_3, \ldots\ldots, y_n$ respectively for the function $y = f(x)$ and $P_n(x)$ be an interpolating polynomial of the function $y = f(x)$, then $P_n(x)$ can be written as;

$$P_n(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots(x - x_{n-1})a_n$$

where, $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants.

Substituting points successively in the interpolating function; we get,

At (x_0, y_0);

$$P_n(x_0) = a_0 + 0 = y_0 = f(x_0)$$

At (x_1, y_1);

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) + 0 = y_1$$

or, $$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = f[x_1, x_0]$$

At (x_2, y_2);

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) + 0 = y_2$$

or, $$a_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_2 - y_0}{x_1 - x_0}}{x_2 - x_0} = f[x_0, x_1, x_2]$$

Thus,

$$a_n = f[x_0, x_1, x_2, \ldots\ldots, x_n]$$

Here, a_1 represents first divided difference, a_2 represents second divided difference, and so on. The divided differences are shown in the table below:

Value of x	value of y	1st difference Δy	2nd difference $\Delta^2 y$	3rd difference $\Delta^3 y$
x_0	y_0			
		$\frac{y_1 - y_0}{x_1 - x_0} = \Delta y_0$		
x_1	y_1		$\frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \Delta^2 y_0$	
		$\frac{y_2 - y_1}{x_2 - x_1} = \Delta y_1$		$\frac{\Delta^2 y_2 - \Delta^2 y_0}{x_3 - x_0} = \Delta^3 y_0$
x_2	y_2		$\frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \Delta^2 y_1$	⋮
		$\frac{y_3 - y_2}{x_3 - x_2} = \Delta y_2$	⋮	⋮
x_3	y_3	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮

1.5 Difference Operators, Shift Operators, Differential Operators

Difference Operators

Difference operators are the operators that perform the operation of calculating finite differences. The different difference operators with their notations are listed below:

i) Forward difference (Δ)
ii) Backward difference (∇)
iii) Central difference (δ).

Shift Operators

Shift operators involves the operation of increasing the argument x by finite difference h and is represented by E such that;

$$Ef(x) = f(x + h)$$

For any higher order of shift operator n, the operation is given by;

$$E^n f(x) = f(x + nh)$$

Differential Operators

Differential operators are the operators that involve the function of differentiation. The most common notations for differential operator for taking the first derivative with respect to a variable x include $\frac{d}{dx}$, D, D_x, D_x^n, $f^n(x)$.

For higher n^{th} order derivatives, the operator can be written as;

$$\frac{d^n}{dx^n}, D^n, D_x^n, f^n(x)$$

1.6 Uses and Importance of Computer Programming in Numerical Methods

The field of numerical method is finding the result of numerical problems. Many of the daily numerical problems can be solved using general analytical methods. But with the advancement of science, technology and engineering, the general analytical methods could not solve all the problems of computation. Even the problems that can be solved by using general analytical method could not meet desired level of accuracy and promptness.

Hence, it became necessary to enhance the method of computation. With the invention of high speed computers and development of sophisticated computer programmer, their use in the solution of numerical problems started. The use of computer and programming in numerical methods could easily solve the complex scientific and engineering problems that required the modern dimensions of mathematics like differentiation, integration, iteration, etc.

The importances of computer programming in numerical methods are as follows:

i) Numerical problems can be solved with higher degree of accuracy and precision.
ii) Very large and complex calculations can be done in very short duration of time.
iii) The problems that require iteration multiple times can also be easily calculated.
iv) The desired degree of accuracy with approximation of errors can be made.
v) The results can be saved and compared for future use.
vi) The calculation is free from human errors like negligence, manipulation, blunders, etc.
vii) The use of computer programming reduces the effort and cost invested in calculation.

1.7 EXAMINATION PROBLEMS

1. **Discuss the difference between absolute and relative error with examples.** **[2072 Ashwin, 2073 Bhadra]**

Solution:

The difference between absolute and relative error are as follows:

	Absolute error	Relative error
i)	Absolute error is a Δx value (+ or – value); where, x is a variable; it is the physical error in a measurement. It is also known as actual error in a measurement. In other words, it is the difference between true value and their experimental value. $\boxed{\text{Absolute error : Actual value} - \text{Measures value}}$	Relative error is the ratio of absolute error (Δx) to the measured value (x). It is expresses either as a percentage (percentage error) or as a fraction (fractional uncertainty). Relative error: $\dfrac{\text{Absolute error}}{\text{Experimental value}} \times 100$
ii)	It has the same units as the measured value.	It can be expressed as a fraction or as a percentage.
iii)	A driver's speedometer says his car is going 60 mph when it's actually going 62 mph. The absolute error of his speedometer is 62 mph–60 mph = 2 mph.	The relative error of the measurement is $\dfrac{2\text{ mph}}{60\text{ mph}} = 0.333$ or 3.33%.

2. **Construct divided difference table from the following data.**

x	1	2	4	5	6
y	14	15	5	6	19

[2073 Shrawan]

Solution:

The divided difference table is;

x	y	1^{st} difference (Δ)	2^{nd} difference (Δ^2)	3^{rd} difference (Δ^3)	4^{th} difference (Δ^4)
1	14	$\dfrac{5-14}{2-1} = 1$			
2	15		$\dfrac{-5-1}{4-1} = -2$		
		-5		$\dfrac{2+2}{5-1} = 1$	

4	5		2		$\frac{1-1}{6-1}=0$
		1		1	
5	6		6		
		13			
6	19				

and, the function representing the table is;

$$f(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + a_3 (x - x_0)(x - x_1)(x - x_2) + a_4 (x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

3. **Discuss the importance of numerical methods is science and engineering.** [2073 Magh, 2073 Chaitra]

Solution: See the definition part

4. **Discuss the significance of numerical methods in the field of science and engineering in modern day context.** [2074 Bhadra]

Solution: See the definition part

5. **Explain the importance of numerical methods in the field of science and engineering.** [2075 Ashwin]

Solution: See the definition part

6. **Discuss the advantages and limitations in solving mathematical problems by numerical techniques rather than analytically.** [2075 Chaitra]

Solution:

Numerical techniques are important tool to mathematical problems. In spite of their importance, they have their own advantages and limitations, which is listed below.

Advantages

i) They are often easy to use and simple to understand.

ii) Their use reduces time and effort to great extent.

iii) Problems with different complexities can be solved which otherwise would be impossible for the analytical method.

iv) They can be applied in different branches of mathematics and types of problems that can be addressed greatly increases.

Limitations

i) They are not as elegant as analytic solutions.

ii) In most of the cases, the answer is an approximation and an exact value may not be clear.

iii) They do not provide any insight into generalizations.

iv) The solution of certain problems faces stability condition problem and problem in desired accuracy.