4.10 Testing Numbers for Randomness

A sequence of random numbers is considered to be random, if;

(*i*) The numbers are uniformly distributed, that is every number has an equal chance of occurrence.

(*ii*) The numbers are not serially autocorrelated. This means that there is no correlation between adjacent pairs or numbers, or that the appearance of one number does not influence the appearance of next number. The sequence 1, 3, 5, 7, 9, ... or 1, 4, 2, 6, 3, 1, 4, 7, 5, ... or 1, 3, 5, 2, 4, 6, 3, 5, 9, ... are serially correlated.

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Uniformity Test

The two different methods are available to test for uniformity.

1). The Kolmogorov Smirnov Test (KS-Test)

2). Chi-Square Test

KS-Test : The test procedure follows these steps:

step 1: Rank the data from smallest to largest.

$$R(1) \leq R(2) \leq \ldots\ldots \leq R(N)$$

step 2: Calculate:

$$D^{+} = \max_{1 \leq i \leq N} \left\{ \frac{i}{N} - R(i) \right\}$$

$$D^{-} = \max_{1 \leq i \leq N} \left\{ R(i) - \frac{i-1}{N} \right\}$$

step 3: calculate:

$$D = \max(D^{+}, D^{-})$$

step 4: Determine the critical value, D_α from standard table for the specific significant level α and the given sample size N.

step 5: If the sample statistics D is greater than the critical value D_α, the null hypothesis that the data area sample from a uniform distribution is rejected. If $D < D_\alpha$ then it is accepted.

4.11 Uniformity Test

The test of uniformity or frequency test is a basic test that should always be performed to validate a random number generator. Two frequency tests are available. They are, Kolmogorov-Smirnov test and the Chi-Squared test. Both of these tests compare the generated random numbers with the theoretical uniform distribution.

The Kolmogorov-Smirnov Test

This test compares the continuous cdf, $F(x)$ of the uniform distribution to the empirical cdf, $S_N(x)$, of the sample of N random numbers. The largest absolute deviation between $F(x)$ and $S_N(x)$ is determined and is compared with the critical value, which is available as function of N in Appendix Table A-6, for various levels of significance. The procedure of employing the Kolmogorov-Smirnov uniformity test is clearly illustrated in the next example.

Example 4.2. The Kolmogorov-Smirnov test is to be performed to test the uniformity of following random numbers with a level of significance of $\alpha = 0.05$.

.24, .89, .11, .61, .23, .86, .41, .64, .50, .65

The calculations of the test are given in Table 4.1. The top row of the table lists the given random numbers R_i ($i = 1, N$) in the ascending order. Here $N = 10$. In the second row, the numbers are computed from the empirical distribution, i.e., i/N values are listed. In the third row deviation, $\frac{i}{N} - R_i$ is computed maximum of which gives D^+, while in the fourth row, the deviation $R_i - \frac{(i-1)}{N}$ is computed, the maximum of which given D^-.

The largest deviation, $D = \max(D^+, D^-)$.

From the table, $D^+ = 0.15$, $D^- = 0.13$ giving the largest deviation $D = 0.15$.

The critical value of D obtained from Appendix Table A-6 for $\alpha = 0.05$ and $N = 10$ is 0.410. Since the computed value 0.15 is less than the critical value, the given random numbers are uniform at 95% level of significance. At $\alpha = 0.01$, critical values is 0.368, which again is more than 0.15, hence, the given random numbers are uniform even at 99% level of significance.

Table 4.1

R_i	.11	.23.	.24	.41	.50	.61	.64	.65	.86	.89
i/N	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.0
$i/N - R_i$	—	—	.06	—	00	—	.06	.15	.04	.11
$R_i - (i-1)/N$	.11	.13	.04	.11	.10	.11	.04	—	.06	—

TABLE A–6 : KOLMOGOROV–SMIRNOV CRITICAL VALUES

Degrees of Freedom *(N)*	$D_{0.10}$	$D_{0.05}$	$D_{0.01}$
1	0.950	0.975	0.995
2	0.776	0.842	0.929
3	0.642	0.708	0.828
4	0.564	0.624	0.733
5	0.510	0.565	0.669
6	0.470	0.521	0.618
7	0.438	0.486	0.577
8	0.411	0.457	0.543
9	0.388	0.432	0.514
10	0.368	0.410	0.490
11	0.352	0.391	0.468
12	0.338	0.375	0.450
13	0.325	0.361	0.433
14	0.314	0.349	0.418
15	0.304	0.338	0.404
16	0.295	0.328	0.392
17	0.286	0.318	0.381
18	0.278	0.309	0.371
19	0.272	0.301	0.363
20	0.264	0.294	0.356
25	0.24	0.27	0.32
30	0.22	0.24	0.29
35	0.21	0.23	0.27
Over 35	$\frac{1.22}{\sqrt{N}}$	$\frac{1.36}{\sqrt{N}}$	$\frac{1.63}{\sqrt{N}}$

Q Suppose that the given five numbers 0.44, 0.81, 0.14, 0.05, 0.93, were generated and it is desired to perform a test for uniformity using KS-test with a level of significance α is 0.05.

⇒ Given $N=5$

$\alpha = 0.05$

$i = 1, 2, 3, 4, 5.$ $(1 \le i \le N)$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$R(i)$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
$D^+ = i/N - R(i)$	0.15	0.26	0.16		0.07
$D^- = R(i) - \left(\frac{i-1}{N}\right)$	0.05		0.04	0.21	0.13

$D = \max(D^+, D^-)$

$= \max(0.26, 0.21)$

$D = 0.26$

From standard table value of $D_{0.05} = 0.565$

Since $D < D_\alpha$ i.e $0.26 < 0.565$ the number distributed is accepted.

4.12 Chi-Squared Test

The Chi-Squared test uses the sample statistic

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed number in the ith class, E_i is the expected number in the ith class and n is the number of classes. For the uniform distribution, E_i, the expected number in each class is given by

$$E_i = \frac{N}{n}$$

for equally spaced classes, where N is the total number of observations. It can be shown that the sampling distribution of χ_0^2 is approximately the chi-square distribution with $n - 1$ degrees of freedom.

... square is more uniformly distributed.

Example 4.3. The two-digit random numbers generated by a multiplicative congruential method are given below. Determine Chi-square. Is it acceptable at 95% confidence level?

36,	91,	51,	02,	54,	06,	58,	06,	58,	02,	54,	01,	48,	97,	43,
22,	83,	25,	79,	95,	42,	87,	73,	17,	02,	42,	95,	38,	79,	29,
65,	09,	55,	97,	39,	83,	31,	77,	17,	62,	03,	49,	90,	37,	13,
17,	58,	11,	51,	92,	33,	78,	21,	66,	09,	54,	49,	90,	35,	84,
26,	74,	22,	62,	12,	90,	36,	83,	32,	75,	31,	94,	34,	87,	40,
07,	58,	05,	56,	22,	58,	77,	71,	10,	73,	23,	57,	13,	36,	89,
22,	68,	02,	44,	99,	27,	81,	26,	85.						

Solution: The given 100 random numbers can be divided into 10 classes as given below:

Class	*Count*	*Frequency*	*Diff*	$(Diff)^2$
$0 < r \leq 10$	*************	13	3	9
$10 < r \leq 20$	********	8	2	4
$20 < r \leq 30$	*********	9	1	1
$30 < r \leq 40$	*************	13	3	9
$40 < r \leq 50$	*******	7	3	9
$50 < r \leq 60$	*************	13	3	9
$60 < r \leq 70$	*****	5	5	25
$70 < r \leq 80$	************	12	2	4
$80 < r \leq 90$	************	12	2	4
$90 < r \leq 100$	********	8	2	4
				78

$$\text{Chi-square} = \frac{78}{10} = 7.8$$

For 9 (10 – 1) degrees of freedom, at 95% confidence level, the acceptable value of Chi-square is 16.9, which is more than 7.8. Hence, the given set of random numbers is acceptable, so far as its uniformity in distribution is concerned.