EQUALIZING GROUP DELAY
Filter Type I Design

Extraordinary Efforts

EE603: Applications of Digital Signal Processing

Report

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1 | Introduction

We are given a M value that gives most of the specifications of the filter. The Band Specifications: All frequencies are in kiloHertz (kHz). There are two groups of frequency bands, which will be used in specifying the filters ahead. For each group of frequency bands, we pass the argument, which is an integer ranging from 0 to 10. The frequency band in each group is specified according to the argument.
Group I of Frequency Bands: The frequency band in this group is (40 + 5D) to (70 + 5D)
Group II of Frequency Bands: The frequency band in this group is (170 + 5D) to (200 + 5D).

$$M = 11Q + R$$

The frequency band from Group I can be obtained by passing the argument D = Q and the frequency band from Group II can be obtained by passing the argument D = R.
For given $\mathbf{M = 11}$, we get $\mathbf{Q = 1}$ and $\mathbf{R = 0}$.
An analog signal is bandlimited to 300 kHz. It is ideally sampled, with a sampling rate of 650 kHz. We wish to build a series of discrete time filters, as described below, to extract specific frequency bands of an analog signal or to suppress specific frequency bands of the signal.
(i) For all filters, the passband AND stopband tolerances are 0.15 in magnitude. That is, the filter magnitude response (note: NOT magnitude squared) must lie between 1.15 and 0.85 in the passband; and between 0 and 0.15 in the stopband. For the IIR Filter, the passband magnitude response must lie between 1 and 0.85.
(ii) For bandpass filters, the transition bands are 5 kHz on either side of each passband. For bandstop filters, the transition bands are 5 kHz on either side of each stopband.

Group I frequency band range is from 45 to 75
Group II frequency band range is from 170 to 200
Transition band for Group I frequency band is 40 to 45 and 75 to 80
Transition band for Group II frequency band is 165 to 170 and 200 to 205

2 | Butterworth Filter Approximation

The Butterworth filter is characterized by its maximally flat magnitude response in the passband. The core of the Butterworth approximation is defining the magnitude squared response of the analog low-pass filter (LPF).
For a rational transfer function $H_{\text{analog, LPF}}(s)$, the magnitude squared response can be expressed as:

$$|H_{\text{analog, LPF}}(j\Omega)|^2 = H_{\text{analog, LPF}}(j\Omega)H_{\text{analog, LPF}}(-j\Omega)$$

2.1 | Magnitude Squared Function

The magnitude squared response of an Nth-order analog Butterworth low-pass filter is given by:

$$|H_{\text{analog, LPF}}(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

By replacing $j\Omega$ with s, we get:

$$H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

The magnitude squared function is specifically defined to be maximally flat at $\Omega = 0$:

$$|H_{\text{analog, LPF}}(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

2.2 | Poles of the Butterworth Filter

The poles of $H(s)H(-s)$ are found by solving the denominator equation:

$$1 + \left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

$$\left(\frac{s}{\Omega_c}\right)^{2N} = -1 = e^{j(2k+1)\pi} \quad \text{for } k \in \mathbb{Z}$$

This gives the pole locations:

$$s_k = \Omega_c e^{j\frac{(2k+1)\pi}{2N}} \quad \text{for } k = 0, 1, 2, ..., 2N-1$$

These $2N$ poles are uniformly distributed on a circle of radius Ω_c in the s-plane. For the filter to be stable, only the poles in the Left-Half Plane (LHP) are chosen for the transfer function $H(s)$. A pole on the imaginary axis (the order circle) results in an unstable system.

3 | Design Methodology

The design follows these steps:

1. **Digital to Analog Transformation:** Convert digital frequency specifications to analog using the Bilinear Transform.
2. **Bandpass to Low-Pass Transformation:** Convert analog bandpass specs to a low-pass prototype.
3. **Low-Pass Prototype Design:** Design an analog Butterworth low-pass filter.
4. **Transform Back:** Convert the LPF to an analog BPF and then to the final digital filter.

4 | Filter Design Specifications

Given that $\delta_1 = 0.15$ and $\delta_2 = 0.15$. We have multiple passbands and stopbands:

- $\Omega_{sI1} = 45$
- $\Omega_{sI2} = 75$
- $\Omega_{sII1} = 170$
- $\Omega_{sII2} = 200$
- $\Omega_{pI1} = 40$
- $\Omega_{pI2} = 80$
- $\Omega_{pII1} = 165$
- $\Omega_{pII2} = 205$

The Filter Type IV is an IIR Butterworth Filter. I am using two Bandpass filters in parallel and subtracting them from A low pass filter.

$$H_{\text{final}}(e^{j\omega}) = H_{\text{Bandstop1}}(e^{j\omega}) \cdot H_{\text{Bandstop2}}(e^{j\omega})$$

$$h_{\text{final}}[n] = h_{\text{Bandstop1}}[n] * h_{\text{Bandstop2}}[n]$$

4.1 | Individual Filter Calculation

We're cascading two bandstop filters to meet our final design specs, and one of the key requirements is that the overall system stays within a tolerance of 0.15. Since the filters are applied in sequence, their responses multiply which means any deviation from ideal behavior in each filter can stack up. To manage this, I've decided to set the tolerance for each individual filter to 0.08.
Technically, we'd need each filter to stay within about 0.075 to guarantee the combined result fits inside the 0.15 limit. But after some trial and error, I found that 0.08 works well in practice — it gives us a bit more flexibility during design and still keeps the final output within spec. So far, the results look promising and the filters seem to be holding up against the requirements.

5 | Procedure

5.1 | Frequency Table

We first list the frequencies which form the passband and the stopband edges. We then convert it to normalised frequency to make it independent of the sampling rate.

$$\omega = \frac{2\pi f}{f_s}$$

The table for the bands is given below: Note : All the bands are monotonic.

Category	Un-normalised Frequency (kHz) [f]	Normalised Frequency (rad/s) [ω]
Stopband for Group 1	0-40	0.0-0.3866
Passband for Group 1	45-75	0.4349-0.7249
Stopband Intermediate	80-165	0.77331-1.5949
Passband for Group 2	170-200	1.643-1.933
Stopband for Group 2	205-325	1.9816-3.1416

Table 5.1: Frequency Table with Normalized Frequencies

5.2 | Target Filter

The filter for which we aim is the following:

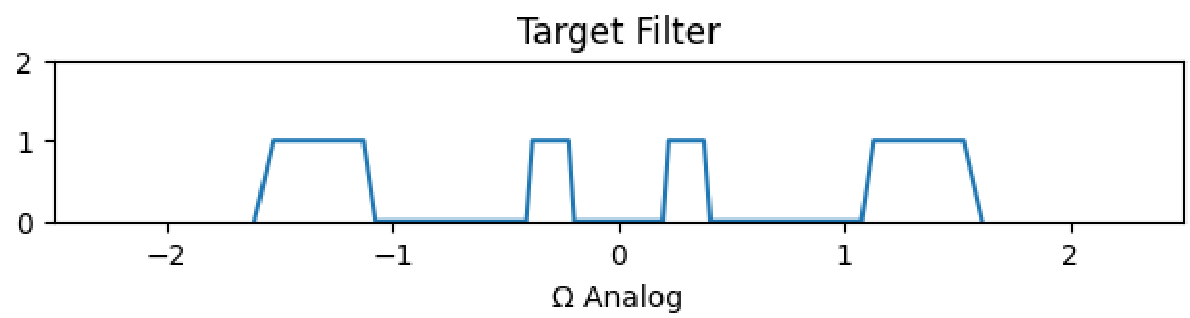

Figure 5.1: Target Filter

We achieve this filter by having two parallel bandpass filters.

1. To achieve a multi-bandpass filter, the task can be divided into designing two separate bandpass filters.
2. The final multi-bandpass filter can then be obtained by cascading or implementing a parallel combination of the two filters. Since the
3. My approach :Two Bandpass Filters in parallel

4. So for the individual bandpass filters we use tolerance $= 0.075$ initially, and then maybe slighlty increase the tolerance values until the filter satisfies the conditons.

5. We implement the above target filter using parallel of two bandpass filters, we name them BP1 and BP2.

$$H_{\text{final}}(e^{j\omega}) = H_{\text{BP1}}(e^{j\omega}) + H_{\text{BP2}}(e^{j\omega})$$

$$h_{\text{final}}[n] = h_{\text{BP1}}[n] + h_{\text{BP2}}[n]$$

5.3 | Designing BandPass Filters BPF1 and BPF2

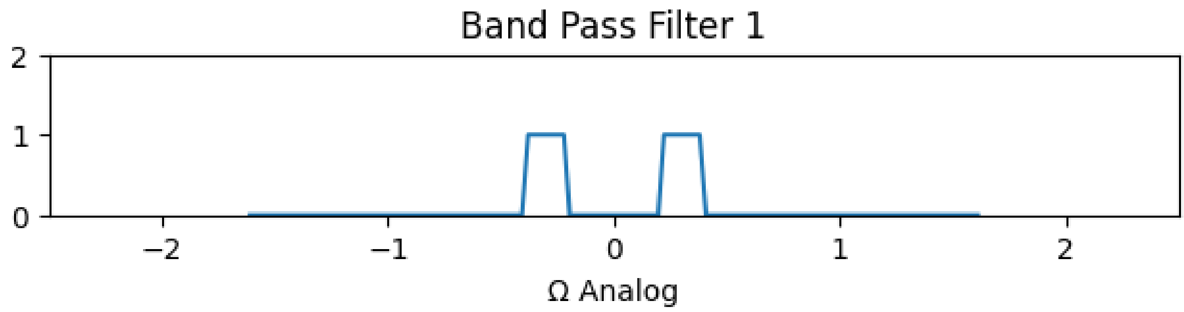

Figure 5.2: BandPass Filter BPF1 Target

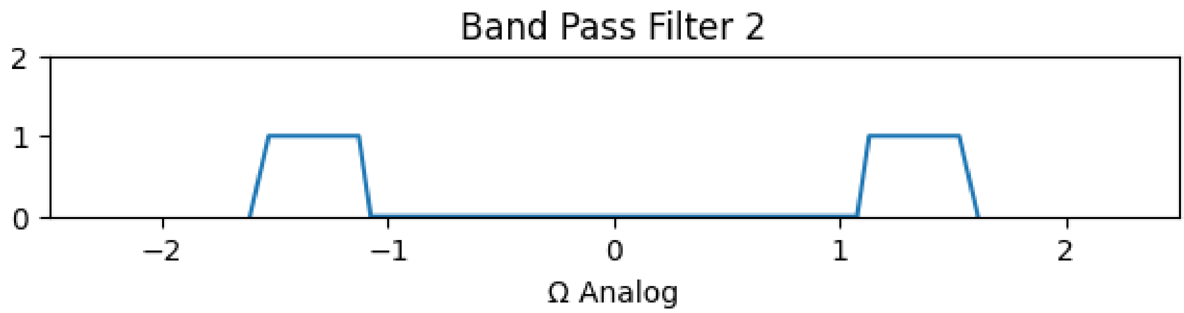

Figure 5.3: BandPass Filter BPF12Target

1. We first design the lowpassfilter : LP1 and LP2.
2. We then use the low pass filter and apply the bandpass filter transformation to obtain the bandpass filter.
3. Finally we add the two filters to get the final filter.

6 | Bandpass Filter 1 Design (45–75 kHz)

6.1 | Analog Prototype Design

The first bandpass filter is designed for the passband of 45 kHz to 75 kHz.

$$\Omega_{p1} = 0.221 \text{ rad/s}, \quad \Omega_{p2} = 0.379 \text{ rad/s}$$
$$\Omega_{s1} = 0.196 \text{ rad/s}, \quad \Omega_{s2} = 0.407 \text{ rad/s}$$

The transformation parameters are:

$$\Omega_0^2 = \Omega_{p1}\Omega_{p2} = 0.0838$$
$$B = \Omega_{p2} - \Omega_{p1} = 0.158$$

This maps the bandpass stopband edges to the prototype low-pass filter stopband edge:

$$\Omega_{LS} = \min\left(\frac{B\Omega_{s1}}{\Omega_{s1}^2 - \Omega_0^2}, \frac{B\Omega_{s2}}{\Omega_{s2}^2 - \Omega_0^2}\right) = 1.272$$

We choose the low pass filter passband edge as:

$$\Omega_{LP} = 1$$

The Butterworth filter order is:

$$N \geq \lceil \frac{ln(\frac{D_2}{D_1})}{2 \cdot ln(\frac{\Omega_{LPstop}}{\Omega_{LPpass}})} \rceil$$

With a tolerance of 0.08, the minimum order is $N_{\min} = 15$. I chose $N = 19$ to ensure specifications are satisfied. We chose the tolerance values as $\delta = 0.075$ for getting a total tolerance of 0.15. On multiple trials, i figured out that $\delta 0.08$ as works well and gives less complexity.

$$D_1 = \frac{1}{(1-\delta)^2} - 1$$

$$D_2 = \frac{1}{\delta^2} - 1$$

For a value of $\delta = 0.08$, we get D_1=0.1815 and D_2=155.25.
The cutoff frequency is:

$$\Omega_c = 1.0799 \text{ rad/s}$$

6.2 | Poles of the correspoding Low Pass Filter

$$s_k = \Omega_c e^{j\frac{(2k+1)\pi}{2N}} = 1.0799 e^{j\frac{(2k+1)\pi}{38}} \quad \text{for } k = 0, 1, 2, ..., 2N-1$$

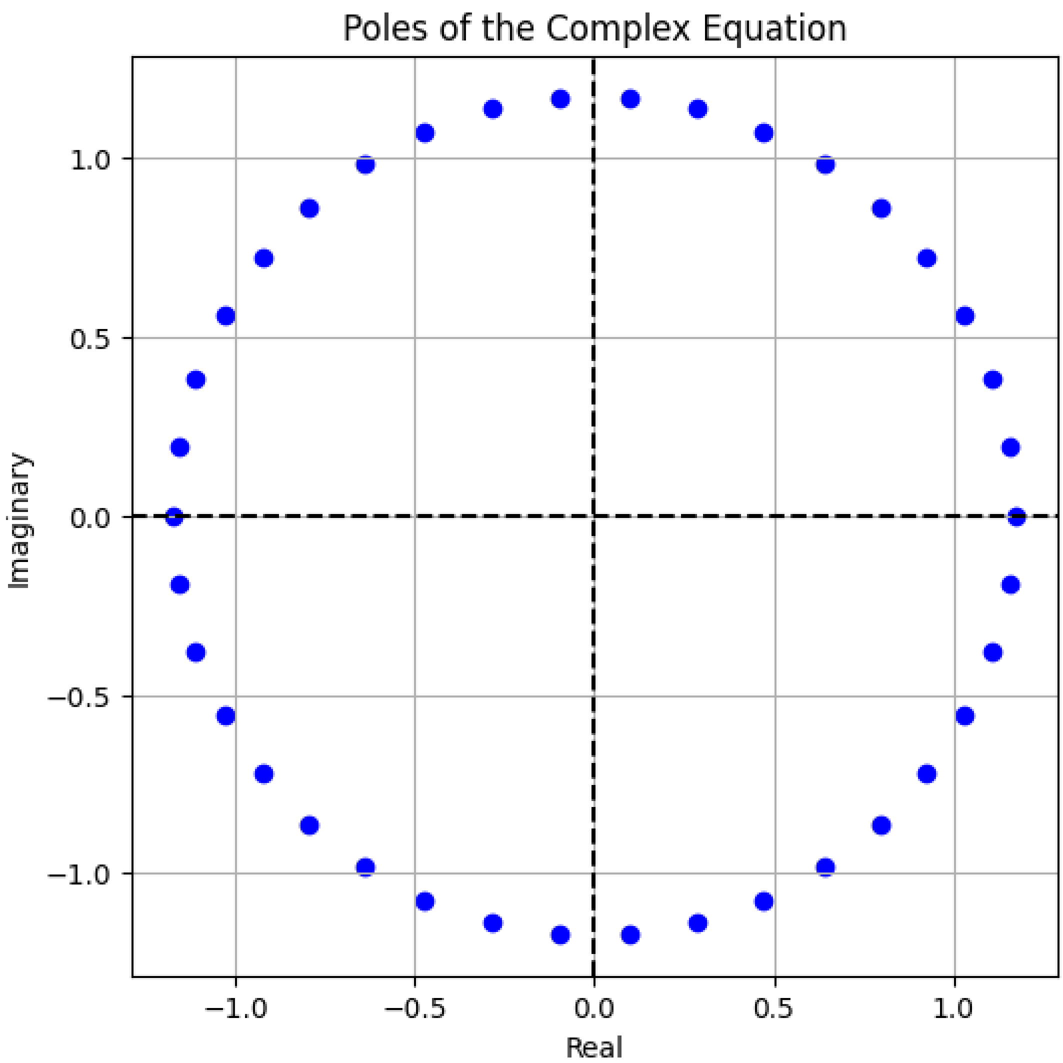

Figure 6.1: Poles of the Low Pass Filter

6.3 | Low-Pass Filter Transfer Functions

We only choose the poles in the left half plane for keeping the filter stable.

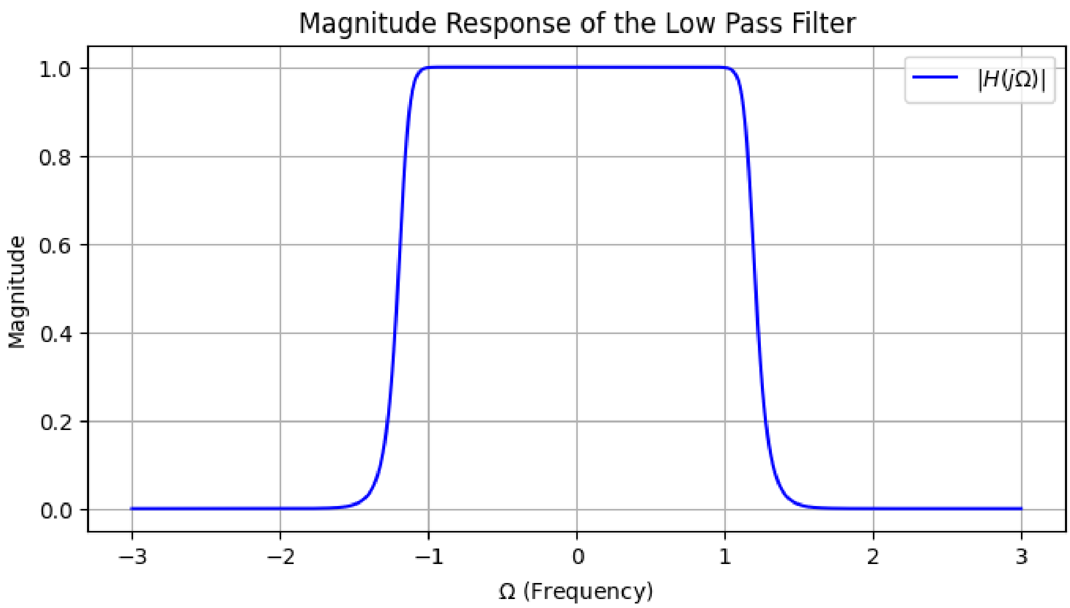

Figure 6.2: Magnitude response of the Low Pass Filter

6.4 | Band-Pass Filter Transfer Function

Applying the transformation

$$s \to \frac{Bs}{s^2 + \Omega_0^2}$$

yields the first bandpass filter:

$$H_{BP1,\text{analog}}(s)$$

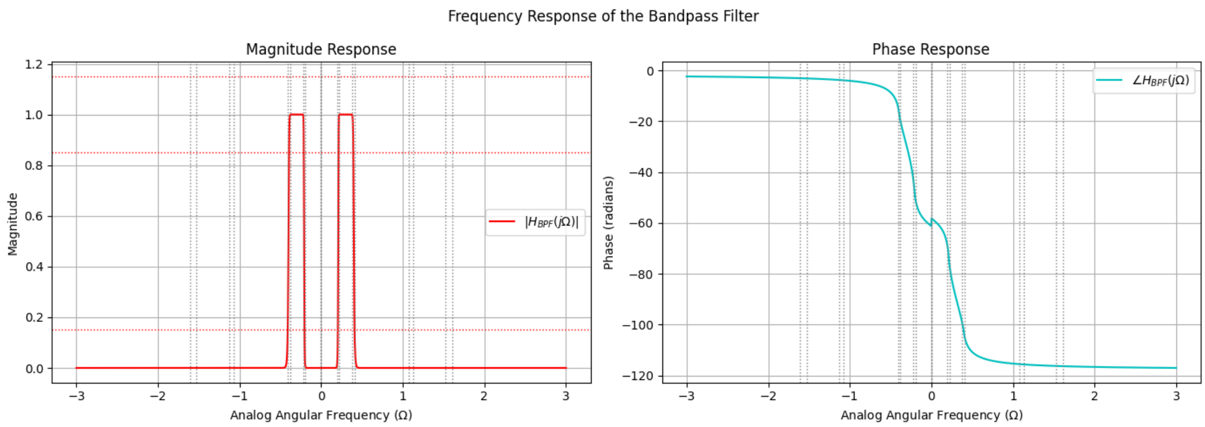

Figure 6.3: Frequency Response of BPF1

7 | Bandpass Filter 2 Design (170–200 kHz)

7.1 | Analog Prototype Design

The second bandpass filter is designed for the passband of 170 kHz to 200 kHz.

$$\Omega_{p1} = 1.075 \text{ rad/s}, \quad \Omega_{p2} = 1.449 \text{ rad/s}$$
$$\Omega_{s1} = 1.024 \text{ rad/s}, \quad \Omega_{s2} = 1.526 \text{ rad/s}$$

Transformation parameters:

$$\Omega_0^2 = \Omega_{p1}\Omega_{p2} = 1.558$$
$$B = \Omega_{p2} - \Omega_{p1} = 0.373$$

The prototype stopband edge:

$$\Omega_{LS} = 1.328$$

With tolerance 0.08, the minimum order is $N_{\text{min}} = 12$. We choose $N = 19$ to maintain consistency. The Butterworth filter order is:

$$N \geq \lceil \frac{ln(\frac{D_2}{D_1})}{2 \cdot ln(\frac{\Omega_{LPstop}}{\Omega_{LPpass}})} \rceil$$

With a tolerance of 0.08, the minimum order is $N_{\text{min}} = 15$. I chose $N = 19$ to ensure specifications are satisfied. We chose the tolerance values as $\delta = 0.075$ for getting a total tolerance of 0.15. On multiple trials, i figured out that $\delta 0.08$ as works well and gives less complexity.

$$D_1 = \frac{1}{(1-\delta)^2} - 1$$

$$D_2 = \frac{1}{\delta^2} - 1$$

For a value of $\delta = 0.08$, we get D_1=0.1815 and D_2=155.25.
The cutoff frequency is:

$$\Omega_c = 1.1046 \text{ rad/s}$$

7.2 | Poles of the correspoding Low Pass Filter

$$s_k = \Omega_c e^{j\frac{(2k+1)\pi}{2N}} = 1.0799 e^{j\frac{(2k+1)\pi}{38}} \quad \text{for } k = 0, 1, 2, ..., 2N-1$$

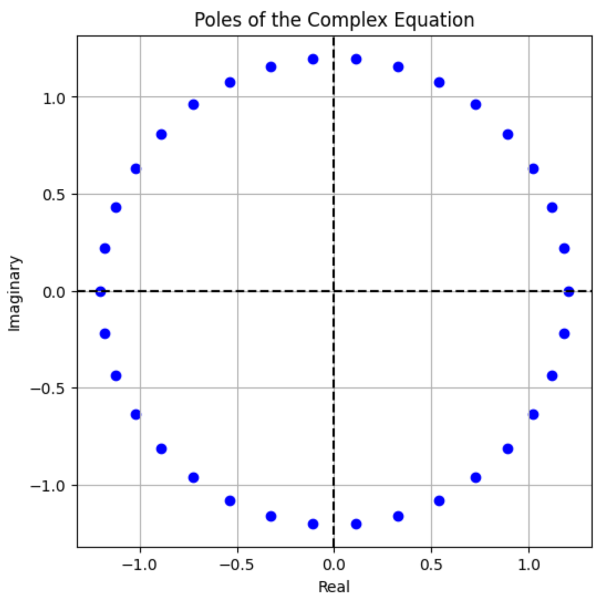

Figure 7.1: Poles of the Low Pass Filter

7.3 | Low-Pass Filter Transfer Functions

We only choose the poles in the left half plane for keeping the filter stable.

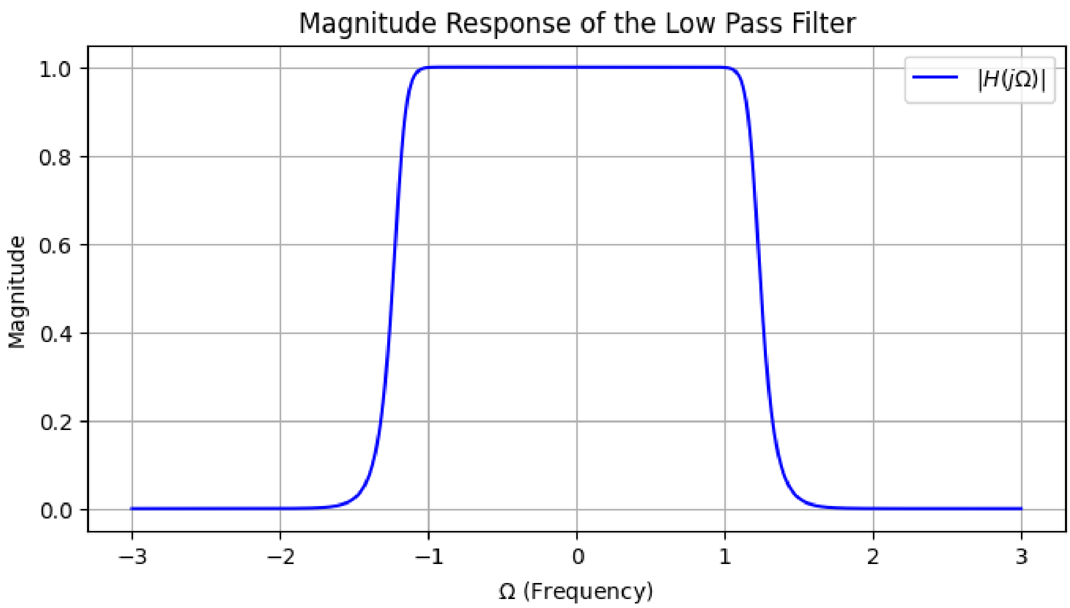

Figure 7.2: Magnitude response of the Low Pass Filter

7.4 | Band-Pass Filter Transfer Function

Applying the transformation

$$s \rightarrow \frac{Bs}{s^2 + \Omega_0^2}$$

yields the first bandpass filter:

$$H_{BP2,\text{analog}}(s)$$

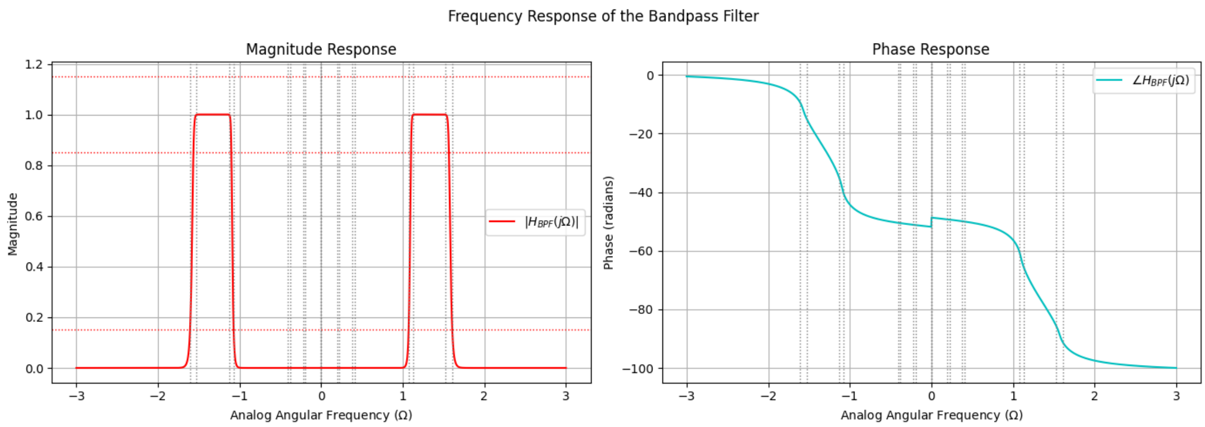

Figure 7.3: Frequency Response of BPF2

8 | Final Analog and Digital Filter

8.1 | Final MultiBandPass filter

The final analog filter is formed by parallel addition:

$$H_{\text{final,analog}}(s) = H_{BP1,\text{analog}}(s) + H_{BP2,\text{analog}}(s)$$

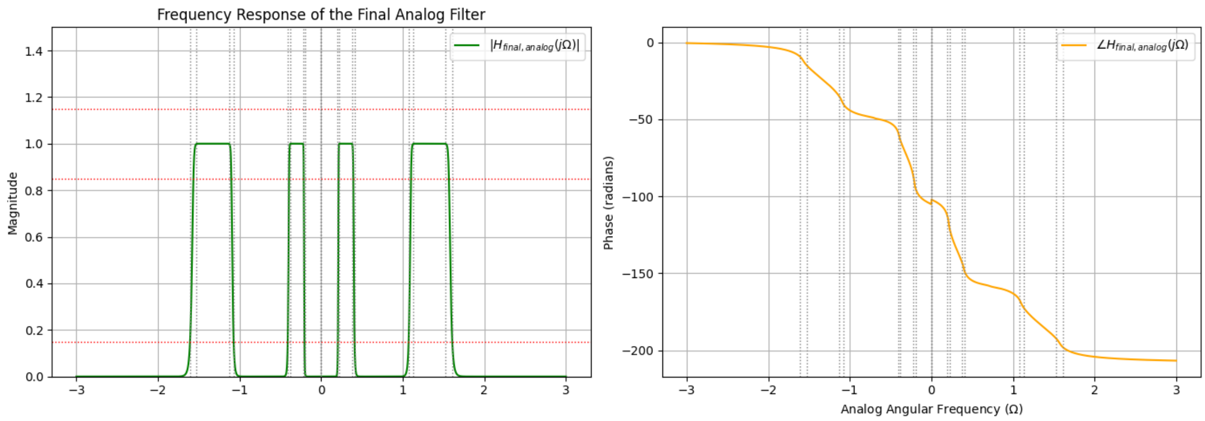

Figure 8.1: Final MultiBandPass Filter

8.2 | Final Digital Multiband Pass Filter

Using the bilinear transformation

$$s = \frac{z+1}{z-1}$$

we obtain the digital filter:

$$H_{\text{final}}(z)$$

The frequency response confirms that all passband and stopband specifications are satisfied.

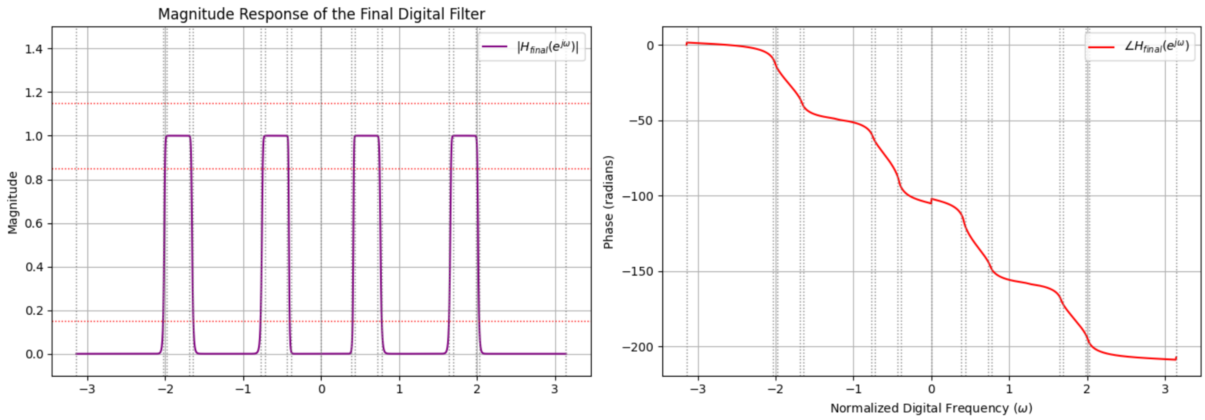

Figure 8.2: Final Digital BandPass Filter

8.3 | The Analog MultiBandPass Filter Equation

$$H_{\text{final,analog}}(\Omega) = \frac{31.8267218127685}{D_1(s)} + \frac{20.7236840657809}{D_2(s)},$$

$$
\begin{aligned}
D_1(s) = {} & 31.8267218127684 - 1.89182003396127 \times 10^{-13}\,\mathrm{i} \\
& + 2.67744235632394\,(321.237443382501 - 1.79767312147305 \times 10^{-12}\,\mathrm{i})\,\frac{s^2 + 1.5577820980695}{s} \\
& + 17.3961713325808\,(1621.1769411565 - 8.58335624798201 \times 10^{-12}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^2}{s^2} \\
& + 72.5571999396629\,(5429.33089172435 - 2.75122147286311 \times 10^{-11}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^3}{s^3} \\
& + 302.626777032507\,(13510.5663786548 - 6.09361450187862 \times 10^{-11}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^4}{s^4} \\
& + 1262.2174815627\,(26514.8751289415 - 1.01863406598568 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^5}{s^5} \\
& + 5264.54726308422\,(42523.0407600806 - 1.60071067512035 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^6}{s^6} \\
& + 21957.7515682434\,(56991.0580265068 - 2.07364792004228 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^7}{s^7} \\
& + 91582.9661770826\,(64744.8811057935 - 2.11002770811319 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^8}{s^8} \\
& + 381980.808359407\,(62877.6360616886 - 1.60071067512035 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^9}{s^9} \\
& + 1593192.97076248\,(52408.5199006129 - 1.12777343019843 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{10}}{s^{10}} \\
& + 6645003.58797797\,(37490.6098322537 - 1.01863406598568 \times 10^{-10}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{11}}{s^{11}} \\
& + 27715457.8852476\,(22926.3630222424 - 5.45696821063757 \times 10^{-11}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{12}}{s^{12}} \\
& + 115597620.922079\,(11884.0408520738 - 2.09183781407773 \times 10^{-11}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{13}}{s^{13}} \\
& + 482142853.932695\,(5148.0266631799 - 7.04858393874019 \times 10^{-12}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{14}}{s^{14}} \\
& + 2010956019.19922\,(1822.3682277145 - 2.04636307898909 \times 10^{-12}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{15}}{s^{15}} \\
& + 8387439693.79265\,(508.768562982595 - 5.82645043323282 \times 10^{-13}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{16}}{s^{16}} \\
& + 34982935452.2742\,(105.539675277558 - 9.59232693276135 \times 10^{-14}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{17}}{s^{17}} \\
& + 145909337954.906\,(14.5285701483358 - 5.77315972805081 \times 10^{-15}\,\mathrm{i})\,\frac{(0.641938305260578\,s^2 + 1)^{18}}{s^{18}} \\
& + 608569138844.378\,\frac{(0.641938305260578\,s^2 + 1)^{19}}{s^{19}}.
\end{aligned}
$$

$$
\begin{aligned}
D_2(s) = {} & 20.7236840657809 - 1.16795462190566 \times 10^{-13}\,\mathrm{i} \\
& + 6.31873675861677\,(213.947783530439 - 1.15107923193136 \times 10^{-12}\,\mathrm{i})\,\frac{s^2 + 0.0838106395680178}{s} \\
& + 39.9264342246948\,(1104.38023307761 - 5.79802872380242 \times 10^{-12}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^2}{s^2} \\
& + 252.284627576073\,(3783.04139546229 - 1.8644641386345 \times 10^{-11}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^3}{s^3} \\
& + 1594.12014989888\,(9628.86093627869 - 4.04725142288953 \times 10^{-11}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^4}{s^4} \\
& + 10072.8255888177\,(19328.4719415621 - 7.45785655453801 \times 10^{-11}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^5}{s^5} \\
& + 63647.5333111981\,(31705.8078888453 - 1.14596332423389 \times 10^{-10}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^6}{s^6} \\
& + 402172.008328753\,(43463.8113284656 - 1.4915713109076 \times 10^{-10}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^7}{s^7} \\
& + 2541219.05231362\,(50504.8517848037 - 1.70985003933311 \times 10^{-10}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^8}{s^8} \\
& + 16057294.2375513\,(50168.422468691 - 1.45519152283669 \times 10^{-10}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^9}{s^9} \\
& + 101461815.342741\,(42770.3407316563 - 1.09139364212751 \times 10^{-10}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{10}}{s^{10}} \\
& + 641110502.202164\,(31294.6354096487 - 7.6397554948926 \times 10^{-11}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{11}}{s^{11}} \\
& + 4051008496.60007\,(19574.430358726 - 4.72937244921923 \times 10^{-11}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{12}}{s^{12}} \\
& + 25597256296.9357\,(10378.2631790752 - 2.00088834390044 \times 10^{-11}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{13}}{s^{13}} \\
& + 161742324283.182\,(4598.41229062072 - 6.3664629124105 \times 10^{-12}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{14}}{s^{14}} \\
& + 1022007169872.26\,(1664.98310423263 - 1.81898940354586 \times 10^{-12}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{15}}{s^{15}} \\
& + 6457794271841.73\,(475.445279724173 - 4.12114786740858 \times 10^{-13}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{16}}{s^{16}} \\
& + 40805102045071.1\,(100.879425773163 - 3.5527136788005 \times 10^{-14}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{17}}{s^{17}} \\
& + 257836698231299.0\,(14.2041842971121 - 3.10862446895044 \times 10^{-15}\,\mathrm{i})\,\frac{(s^2 + 0.0838106395680178)^{18}}{s^{18}} \\
& + 1.62920222283449 \times 10^{15}\,\frac{(s^2 + 0.0838106395680178)^{19}}{s^{19}}.
\end{aligned}
$$

8.4 | The Digital Multibandpass Filter

$$H_{\text{final}}(z) = \frac{31.8267218127685}{E_1(z)} + \frac{20.7236840657809}{E_2(z)},$$

$$\begin{aligned}
E_1(z) = {} & 31.8267218127684 - 1.89182003396127 \times 10^{-13}\,\mathrm{i} \\
& + 2.67744235632394\,(321.237443382501 - 1.79767312147305 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)\left(\frac{(z-1)^2}{(z+1)^2} + 1.5577820980695\right)}{(z-1)} \\
& + 17.3961713325808\,(1621.1769411565 - 8.58335624798201 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)^2\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^2}{(z-1)^2} \\
& + 72.5571999396629\,(5429.33089172435 - 2.75122147286311 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^3\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^3}{(z-1)^3} \\
& + 302.626777032507\,(13510.5663786548 - 6.09361450187862 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^4\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^4}{(z-1)^4} \\
& + 1262.2174815627\,(26514.8751289415 - 1.01863406598568 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^5\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^5}{(z-1)^5} \\
& + 5264.54726308422\,(42523.0407600806 - 1.60071067512035 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^6\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^6}{(z-1)^6} \\
& + 21957.7515682434\,(56991.0580265068 - 2.07364792004228 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^7\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^7}{(z-1)^7} \\
& + 91582.9661770826\,(64744.8811057935 - 2.11002770811319 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^8\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^8}{(z-1)^8} \\
& + 381980.808359407\,(62877.6360616886 - 1.60071067512035 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^9\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^9}{(z-1)^9} \\
& + 1593192.97076248\,(52408.5199006129 - 1.12777343019843 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^{10}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{10}}{(z-1)^{10}} \\
& + 6645003.58797797\,(37490.6098322537 - 1.01863406598568 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^{11}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{11}}{(z-1)^{11}} \\
& + 27715457.8852476\,(22926.3630222424 - 5.45696821063757 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^{12}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{12}}{(z-1)^{12}} \\
& + 115597620.922079\,(11884.0408520738 - 2.09183781407773 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^{13}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{13}}{(z-1)^{13}} \\
& + 482142853.932695\,(5148.0266631799 - 7.04858393874019 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)^{14}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{14}}{(z-1)^{14}} \\
& + 2010956019.19922\,(1822.3682277145 - 2.04636307898909 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)^{15}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{15}}{(z-1)^{15}} \\
& + 8387439693.79265\,(508.768562982595 - 5.82645043323282 \times 10^{-13}\,\mathrm{i})\,\frac{(z+1)^{16}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{16}}{(z-1)^{16}} \\
& + 34982935452.2742\,(105.539675277558 - 9.59232693276135 \times 10^{-14}\,\mathrm{i})\,\frac{(z+1)^{17}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{17}}{(z-1)^{17}} \\
& + 145909337954.906\,(14.5285701483358 - 5.77315972805081 \times 10^{-15}\,\mathrm{i})\,\frac{(z+1)^{18}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{18}}{(z-1)^{18}} \\
& + 608569138844.378\,\frac{(z+1)^{19}\left(0.641938305260578\frac{(z-1)^2}{(z+1)^2} + 1\right)^{19}}{(z-1)^{19}}.
\end{aligned}$$

$$
\begin{aligned}
E_2(z) = {} & 20.7236840657809 - 1.16795462190566 \times 10^{-13}\,\mathrm{i} \\
& + 6.31873675861677\,(213.947783530439 - 1.15107923193136 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)}{(z-1)} \\
& + 39.9264342246948\,(1104.38023307761 - 5.79802872380242 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)^2\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^2}{(z-1)^2} \\
& + 252.284627576073\,(3783.04139546229 - 1.8644641386345 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^3\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^3}{(z-1)^3} \\
& + 1594.12014989888\,(9628.86093627869 - 4.04725142288953 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^4\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^4}{(z-1)^4} \\
& + 10072.8255888177\,(19328.4719415621 - 7.45785655453801 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^5\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^5}{(z-1)^5} \\
& + 63647.5333111981\,(31705.8078888453 - 1.14596332423389 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^6\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^6}{(z-1)^6} \\
& + 402172.008328753\,(43463.8113284656 - 1.4915713109076 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^7\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^7}{(z-1)^7} \\
& + 2541219.05231362\,(50504.8517848037 - 1.70985003933311 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^8\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^8}{(z-1)^8} \\
& + 16057294.2375513\,(50168.422468691 - 1.45519152283669 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^9\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^9}{(z-1)^9} \\
& + 101461815.342741\,(42770.3407316563 - 1.09139364212751 \times 10^{-10}\,\mathrm{i})\,\frac{(z+1)^{10}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{10}}{(z-1)^{10}} \\
& + 641110502.202164\,(31294.6354096487 - 7.6397554948926 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^{11}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{11}}{(z-1)^{11}} \\
& + 4051008496.60007\,(19574.430358726 - 4.72937244921923 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^{12}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{12}}{(z-1)^{12}} \\
& + 25597256296.9357\,(10378.2631790752 - 2.00088834390044 \times 10^{-11}\,\mathrm{i})\,\frac{(z+1)^{13}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{13}}{(z-1)^{13}} \\
& + 161742324283.182\,(4598.41229062072 - 6.3664629124105 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)^{14}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{14}}{(z-1)^{14}} \\
& + 1022007169872.26\,(1664.98310423263 - 1.81898940354586 \times 10^{-12}\,\mathrm{i})\,\frac{(z+1)^{15}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{15}}{(z-1)^{15}} \\
& + 6457794271841.73\,(475.445279724173 - 4.12114786740858 \times 10^{-13}\,\mathrm{i})\,\frac{(z+1)^{16}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{16}}{(z-1)^{16}} \\
& + 40805102045071.1\,(100.879425773163 - 3.5527136788005 \times 10^{-14}\,\mathrm{i})\,\frac{(z+1)^{17}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{17}}{(z-1)^{17}} \\
& + 257836698231299.0\,(14.2041842971121 - 3.10862446895044 \times 10^{-15}\,\mathrm{i})\,\frac{(z+1)^{18}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{18}}{(z-1)^{18}} \\
& + 1.62920222283449 \times 10^{15}\,\frac{(z+1)^{19}\left(\frac{(z-1)^2}{(z+1)^2} + 0.0838106395680178\right)^{19}}{(z-1)^{19}}.
\end{aligned}
$$

9 | Evidence of Correctness

Table 9.1: Frequency Response and Magnitude Data

Index	Un-normalised Freq. (kHz) [f]	Analog Angular Freq. (rad/s) [Ω]	Normalised Freq. (rad/s) [ω]	Magnitude
0	-205	-1.981620	-1.526371	0.099652
1	-200	-1.933288	-1.448750	0.999507
2	-170	-1.643295	-1.075259	0.999507
3	-165	-1.594962	-1.024463	0.143038
4	-80	-0.773315	-0.407153	0.149182
5	-75	-0.724983	-0.379250	0.998838
6	-45	-0.434990	-0.220990	0.998838
7	-40	-0.386658	-0.195774	0.014081
8	40	0.386658	0.195774	0.014081
9	45	0.434990	0.220990	0.998838
10	75	0.724983	0.379250	0.998838
11	80	0.773315	0.407153	0.149182
12	165	1.594962	1.024463	0.143038
13	170	1.643295	1.075259	0.999507
14	200	1.933288	1.448750	0.999507
15	205	1.981620	1.526371	0.099652

We see that all the band restrictions are respected.

10 | Group And Phase Delay

10.1 | Plot

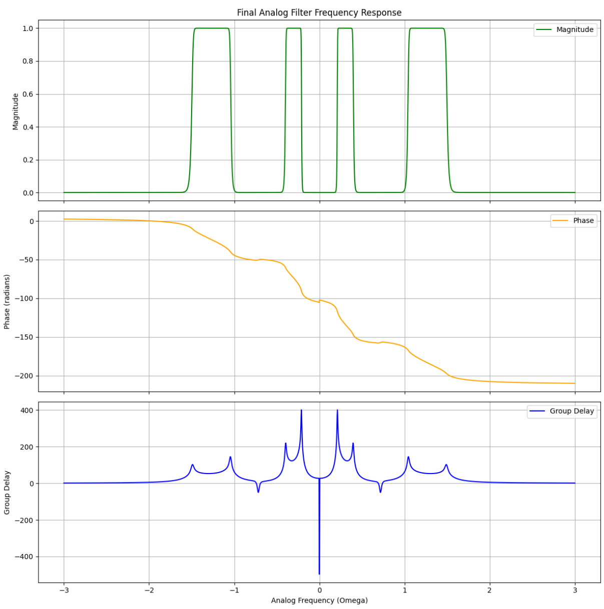

Figure 10.1: Filter Characteristics

10.2 | Analysis

For a filter to pass a signal without distorting its waveform, the group delay should be as constant (flat) as possible across the passband.

10.2.1 | Group Delay T_g

1. **Primary Inference:** The group delay is not constant across the passbands. Instead, it exhibits a distinct parabolic or "U" shape within each passband.

2. **Effect on Signal:** This variation means that different frequency components within the signal's envelope will be delayed by slightly different amounts. This results in phase distortion (also known as dispersion). For example, in the first passband, frequencies at the edges are delayed by 40 samples, while frequencies at the center are delayed by only 25 samples.

3. **Stopband Behavior:** In the stopbands, the group delay is highly erratic and varies wildly. This is expected and irrelevant to performance, as the signal components at these frequencies are heavily attenuated by the filter anyway.

4. **Latency:** The filter introduces a significant latency (delay) to the signal, averaging around 25-40 samples in the passband regions.

10.2.2 | Phase Delay T_p

1. Inference: The phase delay, which represents the delay of individual sinusoidal frequencies, is also highly non-uniform across the passbands.

2. We notice that the phase delay is almost constant in the stopband region and somewhat linear in the passband region.

3. Relevance: While phase delay is a valid metric, group delay is a more critical measure for complex signals (like audio, data packets, etc.) because it describes the delay of the signal's information-carrying envelope. The high variability of the phase delay further reinforces that this is not a linear-phase filter.

11 | Extraordinary Effort : Group Delay Equalization for a Dual-Bandpass Filter

11.1 | The Problem: Non-Constant Group Delay

The initial filter was designed to meet specific magnitude response requirements. However, a side effect of its high order is a highly non-linear phase response. This results in a non-constant group delay across the passbands, as seen in the original filter's response plot.

11.2 | Why solve this?

When the delay is not constant, different parts of the signal are delayed by different amounts. This causes phase distortion, which degrades the shape and integrity of the signal waveform passing through the filter.
The Goal: The objective was to design a corrective filter that, when cascaded with the original, would produce a single, constant group delay across both passbands without affecting the magnitude response.

11.3 | The Solution: A Cascaded Dual-Band All-Pass Equalizer

The chosen solution was to use an all-pass equalizer. This is a special type of filter that has a perfectly flat magnitude response (a gain of 1) but introduces a variable phase shift.
Why it Works: When cascaded, the magnitudes multiply ($H_{total} = H_{original}$Ö1) and the group delays add ($GD_{total} = GD_{original} + GD_{equalizer}$). We can design an equalizer whose group delay has the inverse shape of the original's, making their sum flat.

11.4 | Why a Complex Equalizer?

1. High Order: A simple 2nd-order all-pass filter is insufficient to correct a complex 19th-order filter. Therefore, a high-order equalizer, built by cascading multiple 2nd-order sections, was required.

2. Dual-Band: The filter has two distinct passbands. A single equalizer cannot correct both simultaneously. The solution was to design two independent, high-order equalizers—one for each passband—and cascade them together.

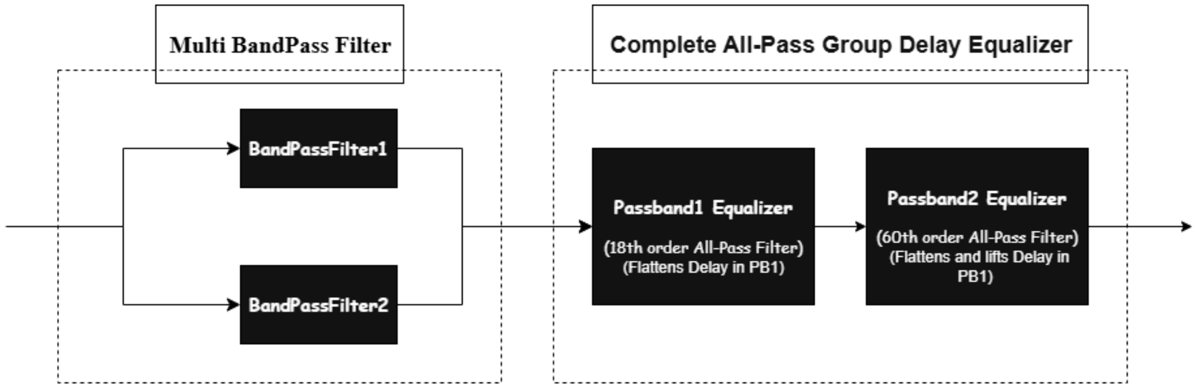

Figure 11.1: Block Diagram for obtaining piecewise constant Group Delay

This is the equation of the All-Pass Filter :

$$H_{ap}(s) = \frac{s^2 - \frac{\omega_0}{Q}s + \omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Where:

- s is the complex frequency variable ($s = \sigma + j\omega$).
- ω_0 is the natural frequency (or center frequency), which you tuned to target specific areas of the group delay curve.
- Q is the quality factor, which you tuned to control the "sharpness" or "strength" of the phase correction.

11.5 | The Process: Systematic Multi-Stage Optimization

To achieve the flattest possible response, a systematic, automated optimization was performed in three stages using a grid search methodology.

11.5.1 | Stage 1: Flattening Passband 1

First, we focused exclusively on Passband 1. A 2D grid search was conducted to find the optimal combination of the equalizer's Q-factor (controlling the "shape" of the correction) and a k-shift parameter (controlling the "tilt"). The search minimized the peak-to-peak group delay variation within the band. The result of the grid search was as follows:

```
    --- Starting 2D Grid Search for 'Q' and 'k' ---
For Q = 3.45, best k is 0.0328 with variation = 23.6399
For Q = 3.48, best k is 0.0328 with variation = 23.3655
For Q = 3.50, best k is 0.0318 with variation = 22.7855
For Q = 3.53, best k is 0.0318 with variation = 22.3921
For Q = 3.56, best k is 0.0318 with variation = 22.6369
For Q = 3.58, best k is 0.0308 with variation = 23.0915
For Q = 3.61, best k is 0.0308 with variation = 22.8899
For Q = 3.64, best k is 0.0308 with variation = 22.8120
For Q = 3.66, best k is 0.0308 with variation = 23.1130
For Q = 3.69, best k is 0.0308 with variation = 23.7212

--- Grid Search Finished in 53.74 seconds ---

Overall Best Q found: 3.53
Overall Best k found: 0.0318
Lowest variation achieved: 22.3921
```

- Optimal Q1 found: 3.53
- Optimal k1 found: 0.0318

- The plot shows the group delays, where the blue colored curve is the original and the dotted green color curve is the group delay after the all-pass filter. We see that we obtain a much more constant group delay and the zoomed in version of this is shown in the second plot of Figure 11.2

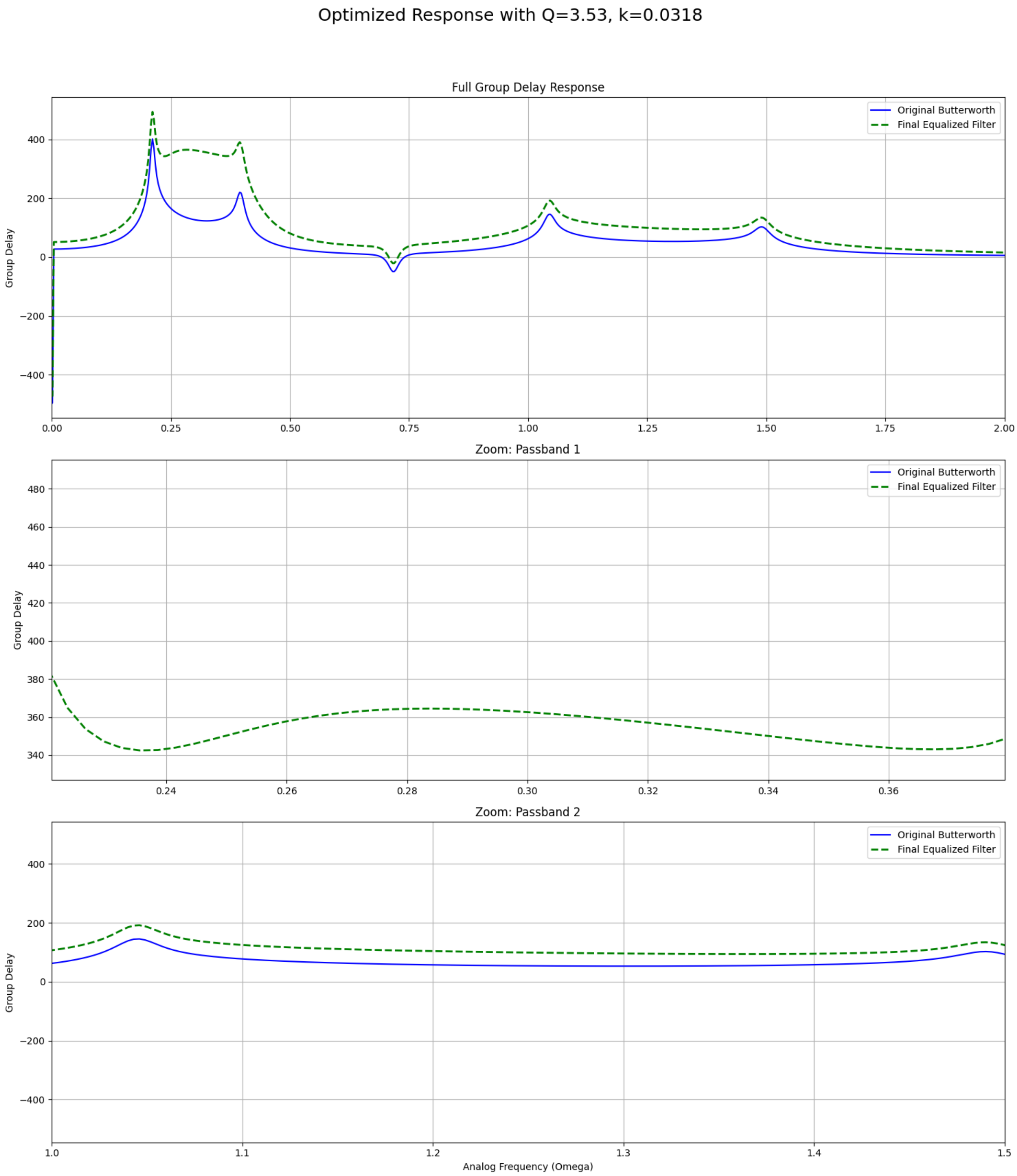

Figure 11.2: Delay flattened for Passband1; the second graph is a zoom in into the region

11.5.2 | Stage 2: Flattening Passband 2

With the first equalizer's parameters locked, the same 2D grid search was performed for Passband 2.

- Optimal Q2 found: 6.30
- Optimal k2 found: 0.0490

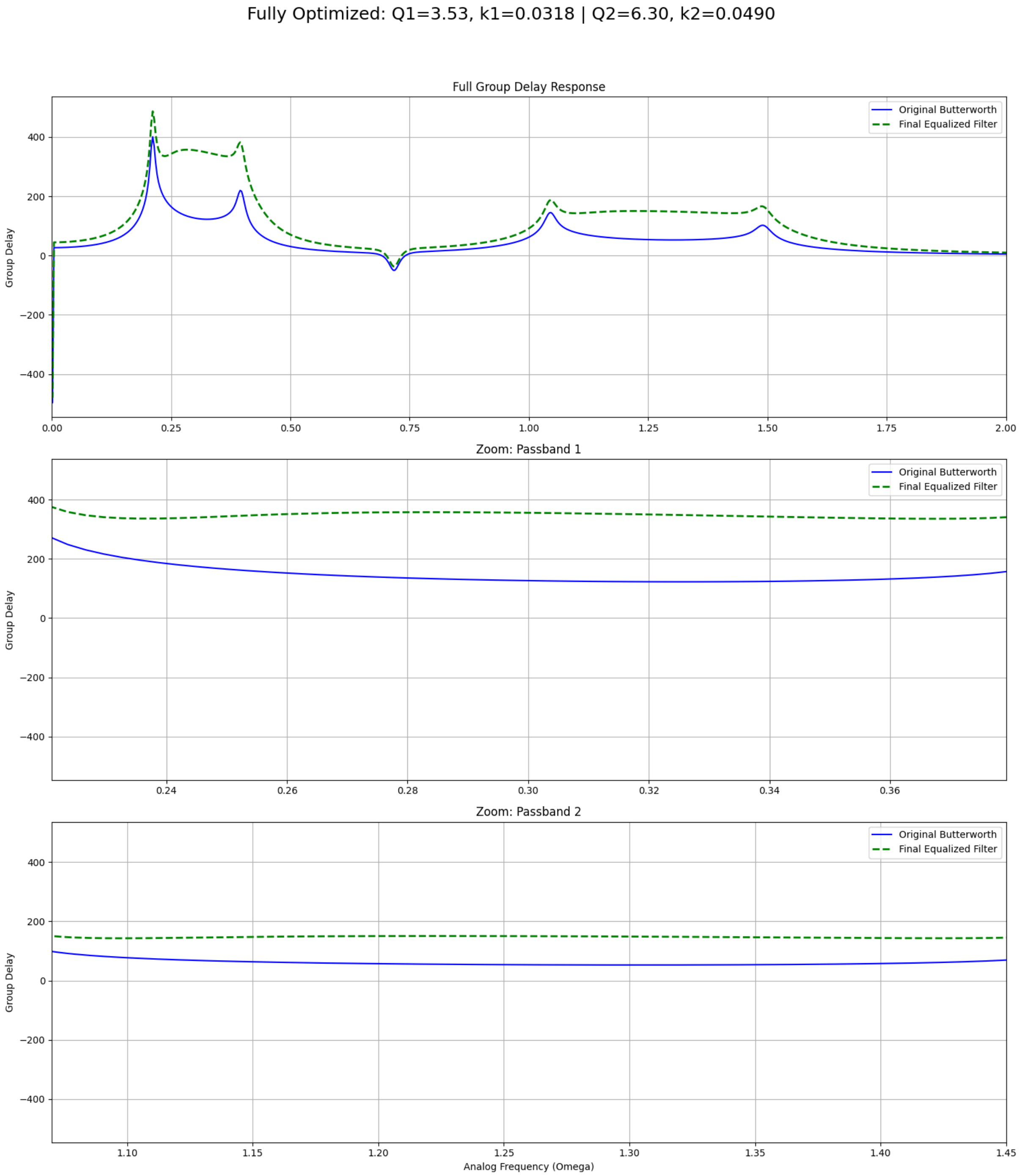

Figure 11.3: Delay flattened for both Passbands; the second and third graphs are zoom-ins into the two passbands

11.5.3 | Stage 3: Matching the Delay Levels

The previous stage resulted in two flat passbands at different delay levels. This will result in shiting more of one passband and less of the other which is problematic. So we want to have the constant of delay of both the passbands the same. The final optimization was to make these levels equal. It was observed that Passband 2 had a lower average delay. A 1D grid search was performed to find the optimal number of extra all-pass sections to add only to the Passband 2 equalizer. Each section "lifts" the delay of that band. The search minimized the difference between the average delays of the two bands.

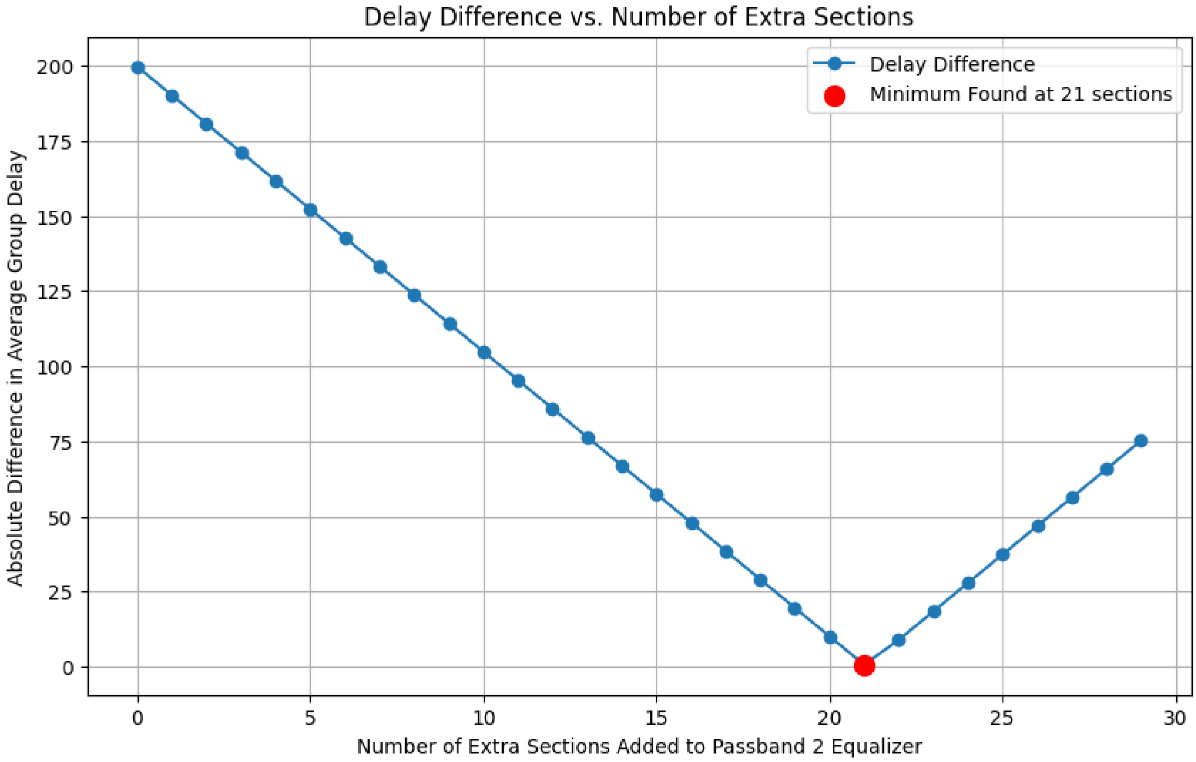

Figure 11.4: Checking the number of extra all-pass filters required to get equal group delay

11.6 | Final Results and Analysis

The multi-stage grid search yielded a complete set of optimized parameters, resulting in a final filter with a nearly constant group delay across both operational bands.

11.7 | Why was the Stopband Not Equalized?

Why was the Stopband not equalised, why not one whole constant group delay? Group delay is a measure of signal distortion. The stopband's function is to eliminate signal components in its frequency range. Since these signals are attenuated to near-zero, their phase and delay are irrelevant—there is no signal left to distort. Equalizing the stopband would add unnecessary complexity for no practical benefit. The focus is exclusively on preserving the integrity of the signals that pass through the filter. Also getting the whole thing constant would have required a huge number of allpass filters, which would serve the same purpose in the end.

12 | Conclusion

A multi-bandpass IIR filter was successfully designed using the Butterworth approximation and the bilinear transformation method. Two separate bandpass filters in parallel were employed to achieve the required specifications for the two distinct passbands. The final digital filter exhibits the desired monotonic passband and stopband behavior and satisfies all tolerance requirements. Then the group delay was analysed and it was made an approximately piecewise constant group delay filter.

13 | Reviews and Reviewers

- **Name and Roll Number of the reviewer** : Jatin Kumar 22B3922
  Group Number : 1
  Review Comments :
  I have reviewed the filter design assignment of Prajwal Nayak, Roll Number 22B4246. The filter number assigned to him is 11. Following are my comments on his assignment:

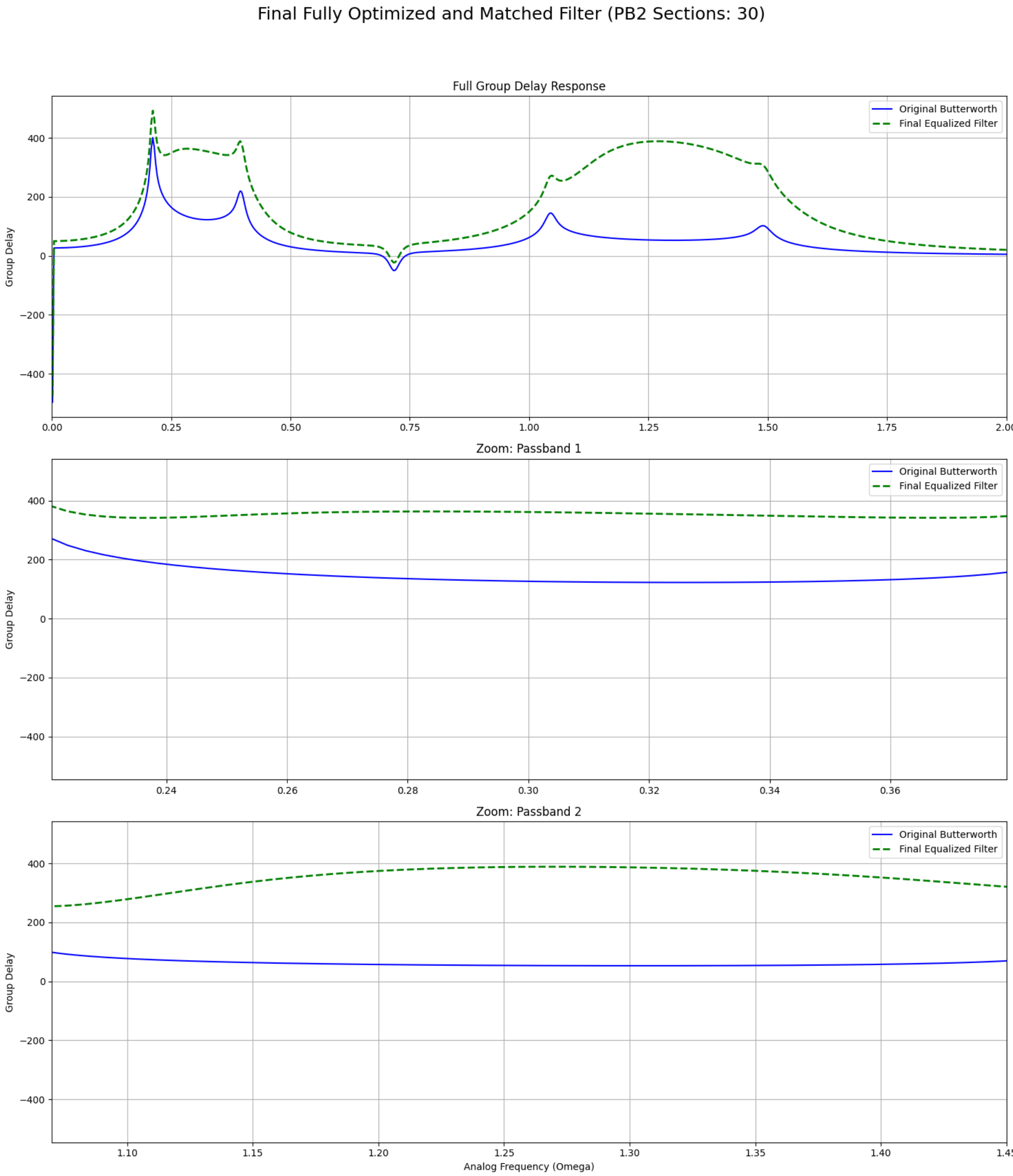

Figure 11.5: Final approximately piecewise constant group delay

He has correctly implemented the filter, and the response are in accordance to the expected frequency response. He has also correctly used the butterworth approximations to implement the IIR designs for each filter. He has included all the code, results and their plots in the report.

- **Name and Roll Number of the reviewer** : Sarvadnya Purkar 22B4232
  Group Number : 1
  Review Comments : It looks fine and complete.