A Beautiful Journey in Introductory Calculus

With applications and analysis

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Pre-Calculus (Setting the Scene)

This module begins by looking at the different kinds of numbers that fall on the real number line, decimal expansions and approximations, then continues with an exploration of manipulation of equations and inequalities, of sign diagrams and the use of the Cartesian plane.

Learning Objectives

- recognise different types of numbers and their representations
- manipulate expressions, equations and inequalities algebraically and relate these to geometric interpretations
- develop fluency with the Cartesian plane and equations of lines and curves

1. Introduction to the Course

1.1 Welcome and Introduction to Module 1

Greetings and welcome to the foundational course on Calculus. This program is crafted to swiftly acquaint you with the fundamental ideas and methods of calculus, a discipline integral to the sciences, engineering, and various fields in business and economics. The course unfolds over five comprehensive modules, each bolstered by instructional notes, formative exercises to build concept mastery, and answer key to gauge your achievement level.

The journey begins with the first two modules focused on precalculus, laying the groundwork and equipping you with an array of significant functions. In this context, a function is defined as a rule or mechanism that establishes links between disparate physical quantities or measurements, which, in this course, are consistently represented by real numbers. These initial modules supplied the essential foundations required for the subsequent course content. While precalculus may present an intimidating challenge, particularly if you're unaccustomed to its concepts and notational language, it's crucial to immerse yourself in the exercises. They may initially seem daunting, similar to deciphering a foreign tongue, but engaging with them is vital. They are designed not only to pique your interest but also to aid in cultivating a deep-seated familiarity and proficiency with the central concepts, thereby ensuring a seamless progression to the next stages of the course. Remember, in the realm of mathematics, practice is irreplaceable. The exercises, meticulously curated to facilitate your learning, are your best tool for honing focus and skill.

Modules three and four delve into the concepts of limits and derivatives. Limits serve as a mathematical bridge, connecting tangible entities—commonly perceived as real numbers—to fascinating operations that flirt with the concept of infinity. Consider the optical illusion experienced when looking down a straight road, where the parallel sides seem to converge at a distant point on the horizon.

Figure 1.1: Parallel lines seem to converge at a point.

This visual confluence is an illustration of a limit point. By integrating limits with ratios or rates of change—specifically, certain types of fractions—we arrive at derivatives, forming the bedrock of differential calculus. The methodologies of differential calculus have unlocked profound applications and insights, igniting a surge of scientific advancement since their inception in the 17th century.

The allure of space exploration, the dream of voyaging to distant stars, was shared by Isaac Newton in the 17th century. With unparalleled imaginative power, rivalled only by Einstein centuries later, Newton calculated the escape velocity necessary for a rocket to break free from Earth's gravitational pull—11 kilometers per second—using only the nascent theory of calculus and ancient astronomical data. As you advance through this course, you'll have the opportunity to deconstruct Newton's reasoning, culminating in the final module.

The concluding module, module five, introduces integral calculus, which employs limiting processes to quantify areas. The fundamental theorem of calculus, a 17th-century discovery made independently by Newton and Leibniz, uncovers a striking correlation between areas delineated by curves and the gradients of their tangents. This revelation has profound implications, leading seamlessly to the domain of differential equations and their solutions, which permeate all of science.

We invite you to embark on this enlightening journey through calculus, a testament to human creativity and one of the towering achievements of intellectual history. May your sense of wonder remain undiminished as you encounter surprises at every turn in this course, and that you obtain great benefit from the videos, and the practice and challenges provided by the many exercises. So, let's begin.

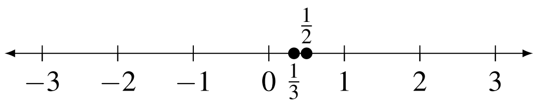
[0]Image 1.1 taken from : https://www.sei.org/wp-content/uploads/2022/04/gettyimages-1168227195-1-1373x916.jpg

2. Numbers and their representations

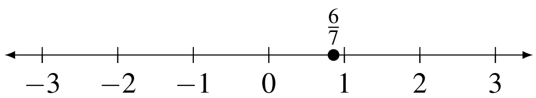
2.1 Real line, decimals and significant figures

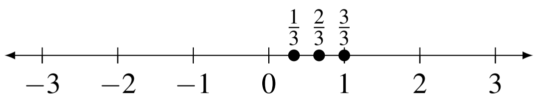
2.1.1 Real numbers and decimals in number line

In this session, we delve into the real number line, examining the diverse types of numbers it encompasses and their decimal representations. We'll also cover approximations, significant figures, and scientific notation. Envision every number discussed in this course as positioned on the real number line, which extends indefinitely in both directions. At its heart is zero, with positive integers to the right and negative integers to the left, known as integers. Scattered between these integers are myriad other numbers. For instance, nestled between zero and one is the fraction one-half, denoted by the decimal 0.5.

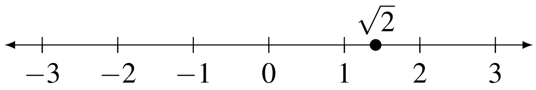


As we proceed, we encounter one-third of the way between zero and one, the fraction one-third, which when divided continuously by three, results in an endless string of threes 0.3333333...., represented as $0.\overline{3}$. Nearing one, there's the fraction six-sevenths, which upon division yields the decimal (0.857142), with these digits repeating up to infinity, expressed as $0.\overline{857142}$.

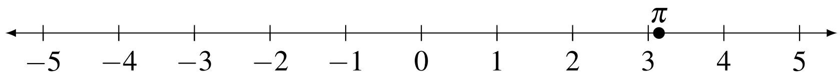




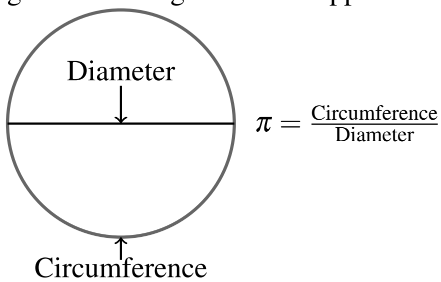
Further exploration reveals that one-third in decimal form is $0.\overline{3}$, leading us to deduce that two-thirds should be $0.\overline{6}$, and intriguingly, three-thirds equates to one, or $0.\overline{9}$. This demonstrates that as we extend the decimal expansion of one, denoted by $0.\overline{9}$, and truncate it at any point, we inch ever closer to one. This concept of approximation and its behavior at the limit is a cornerstone of calculus, recurring throughout this course.



Another intriguing number on the real line is the square root of two, $\sqrt{2}$, with a non-repeating decimal expansion beginning with (1.41421). Unlike the previous examples, $\sqrt{2}$ is not a fraction of integers and is deemed irrational. This concept will be further explored in our subsequent session on the Pythagorean Theorem. The square root of two is defined as a number whose square equals two, meaning that a square with side lengths of $\sqrt{2}$ will have an area of exactly two square units.



Additionally, we have the critical number π, slightly greater than three, with a decimal expansion of 3.14159 and continuing indefinitely. Similar to the square root of two, the decimal expansion of pi is infinite and non-repeating, which classifies it as an irrational number. To fully grasp this concept, one must utilize the comprehensive tools developed in calculus. π is fundamentally associated with circles; it is defined as the ratio of a circle's perimeter to its diameter, a ratio that astonishingly remains constant for all circles, irrespective of their size. This universal constancy is a result of geometric principles involving similar triangles and the application of limits.



In practical scenarios, we use finite decimal expansions. For example, calculators display a limited number of digits. The abstract number one-third, represented as $0.\overline{3}$, is truncated for practical use. Truncating one-third to three decimal places yields approximately 0.333. Similarly, two-thirds, $0.\overline{6}$, becomes 0.667 when rounded to three decimal places.

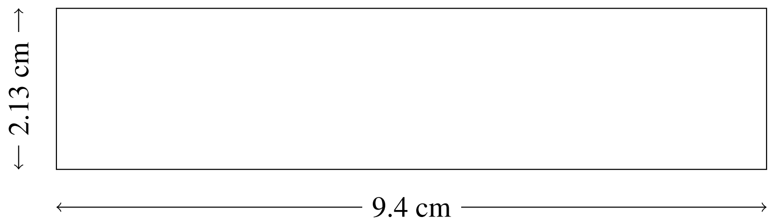
Consider six-sevenths, $0.\overline{857142}$. Truncated to three decimal places, it's 0.857 , and at one decimal place, we round up to 0.9 . These examples illustrate varying levels of approximation accuracy for six-sevenths.

2.1.2 Approximations in Arithmetic Operations of Real Numbers

The real number line permits arithmetic operations, and there are specific rules for handling approximations. When adding or subtracting, the result should be quoted to the least precise decimal place. For multiplication or division, the result should be quoted to the least number of significant figures.

For instance, adding 9.4 cm (measured to one decimal place) to 2.13 cm (measured to two decimal places) results in 11.53 cm. However, we report this as 11.5 cm, adhering to the precision of the least accurate measurement.

Similarly, with subtraction, if we remove a 2.13 cm segment from a 9.4 cm length, the remainder is 7.27 cm. Yet, we round to 7.3 cm due to the initial measurement's one decimal place precision.



In multiplication, consider a rectangle 9.4 cm wide and 2.13 cm high. The area, 9.4×2.13 , calculates to 20.022 cm^2. However, we round to the least significant figures used, resulting in an area of approximately 20 cm^2. To express this with precision, it's written as 2.0×10 cm^2, indicating two significant figures.

If we consider Earth's radius, 6,370,000 meters may seem precise, but only three figures are significant. Using scientific notation, we write 6.37×10^6 meters, clarifying that only the first three digits are significant.

These principles are elaborated upon in the further course materials, and if you require further exercises, please do head to the exercise problems.

2.1.3 Quiz

Question 1

Evaluate the following expression:

$$\frac{1}{2} + \frac{2}{3} = \ldots$$

(a) $\frac{5}{6}$
(b) $\frac{3}{5}$
(c) $\frac{7}{6}$
(d) $\frac{1}{6}$

Question 2

Simplify the following expression as a single fraction:

$$\frac{a}{b} \times \frac{c}{d}$$

(a) $\frac{ac}{bd}$
(b) $\frac{ad}{bc}$
(c) $\frac{a+c}{b+d}$
(d) $\frac{abc}{d}$

Question 3

Simplify the following expression as a single fraction:

$$\frac{a}{b} \div \frac{c}{d}$$

(a) $\frac{ad}{bc}$
(b) $\frac{ac}{bd}$
(c) $\frac{a+c}{b+d}$
(d) $\frac{a}{b}$

Question 4

Find the numerical value of $\sqrt[3]{0.125}$ and express your answer in decimal form.

(a) 0.05
(b) 0.25
(c) 0.5
(d) 0.125

Question 5

The volume of a cube is 1.331 cm^3. What is the side length of the cube in cm?

(a) 1.1 cm
(b) 1.3 cm
(c) 1.5 cm
(d) 2.0 cm

Question 6

The side length of a square is 10 cm and the side length of a rectangle is 10 cm and 15 cm respectively. What is the ratio of the area of the square to the area of the rectangle?

(a) 1 : 1.5
(b) 1 : 1
(c) 1 : 2
(d) 2 : 1

Question 7

A cyclist covers a distance of 120 km in 5 hours. Find the speed of the cyclist in km per hour.

(a) 20 km/hr
(b) 24 km/hr
(c) 30 km/hr
(d) 25 km/hr

Question 8

A rectangle is measured having side lengths 9.64 cm, correct to two decimal places, and 3.313 cm, correct to three decimal places. Estimate the area of the rectangle.

(a) 31.9373 cm^2
(b) 31.94 cm^2
(c) 31.9 cm^2
(d) 32.0 cm^2
(e) 31.937 cm^2

Question 9

A rectangle is measured having area 9.64 cm^2, correct to two decimal places. One of the sides of the rectangle has length 3.313 cm, correct to three decimal places. Estimate the length of the other side of the rectangle.

(a) 2.9 cm
(b) 2.90 cm
(c) 2.91 cm
(d) 2.9097 cm
(e) 2.910 cm

Question 10

The radius of the moon is estimated to be 1,740,000 m, correct to 4 significant figures. Rewrite this estimate of the radius using scientific notation.

(a) 1.7400×10^6 m
(b) 1.740000×10^6 m
(c) 1.7×10^6 m
(d) 1.740×10^6 m
(e) 1.74×10^6 m

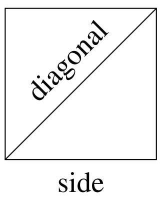
Answers

The answers will be revealed at the end of the module.

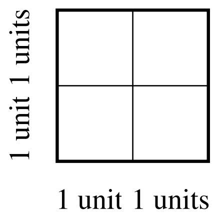
2.2 The theorem of Pythagoras and the properties of the square root of 2

2.2.1 The theorem of Pythagoras

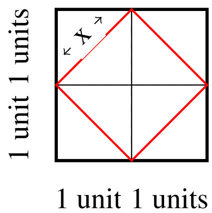
In this section, we delve into the renowned Pythagorean Theorem, a fundamental concept in geometry that you may recall from your school days. We explore its historical significance, tracing back to the era of Babylonian Mathematics. The theorem is used to demonstrate a geometric interpretation of the square root of two, a number of profound importance in mathematical history. We establish that the square root of two is irrational, meaning it cannot be represented as a fraction—a fact well-known to the ancient Greeks.



We start our exploration with a unit square, whose sides are one unit in length. To travel from one corner to the opposite, the shortest route is along the diagonal.



This section illustrates a Babylonian method where we create a larger square by arranging four unit squares, leading to a new square with side lengths of two units. By connecting the midpoints of this larger square, we form another square inside, with side length x , which is the diagonal of the original unit square. The area of this inner square is x^2 , which we deduce to be two units, indicating that x is the square root of two.

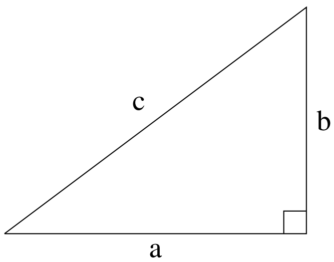


Area of Big Square = 4 sq. units
Area of Red Square = 2 sq. units = x^2
Thus, side of Red Square = x = $\sqrt{2}$

This exploration continues with a creative twist, where the inner square is imprinted onto clay and baked, resulting in a tablet that serves as the oldest known proof of the Pythagorean Theorem, predating Pythagoras himself by a millennium.

Figure 2.1: Tablet showing oldest proof of Pythagorean Theorem

This leads to a discussion of an isosceles triangle with sides of one unit and its scaled versions. By examining these triangles, we reveal the theorem's broader implications: for any right-angled triangle, the square of the hypotenuse c^2 is equal to the sum of the squares of the other two sides a^2+b^2 .



Pythagoras Theorem:
$c^2 = a^2 + b^2$
$c = \sqrt{a^2+b^2}$

In a right angle triangle, expressions involving square roots, known as surd expressions, typically do not simplify. However, there are exceptions, such as the 3-4-5 triangle, where the sides neatly fit the theorem's equation. This is a rare occurrence, as surd expressions usually remain unsimplified. For instance, the diagonal of a unit square, which is the square root of two, cannot be expressed as a fraction. This limitation led to a significant realization in the development of mathematics: rational numbers, or fractions, are not sufficient to represent certain distances, such as the square root of two.

2.2.2 $\sqrt{2}$ is irrational.

Theorem 2.1 — $\sqrt{2}$ is irrational.

$$\sqrt{2} \neq \frac{p}{q} \text{ where p and q are integers \& q} \neq 0$$

To prove this, the we employ a method akin to a criminal investigation, where we assume the square root of two is a fraction and look for contradictions. If we let a and b be integers with no common divisors and set $\sqrt{2} = \frac{a}{b}$, squaring both sides leads to $2b^2 = a^2$. This implies that a is even, and if we let a = 2c , then $b^2 = 2c^2$, suggesting b is also even. This contradicts our initial assumption that $\frac{a}{b}$ is in its simplest form, thus proving that the square root of two is not a fraction.

[0]Image 2.1 taken from MOOC

2.2.3 Quiz

Question 1

A right-angled triangle has shorter side lengths exactly 6 cm and 8 cm respectively. Find the exact length of the hypotenuse.

(a) $3\sqrt{10}$ cm
(b) 11 cm
(c) 14 cm
(d) 10 cm
(e) $10\sqrt{2}$ cm

Question 2

The length of the hypotenuse of a right-angled triangle is exactly 15 cm. One of the shorter side lengths is exactly 9 cm. Find the exact length of the remaining side.

(a) 6 cm
(b) 13 cm
(c) 12 cm
(d) $\sqrt{154}$ cm
(e) $\sqrt{6}$ cm

Question 3

A right-angled triangle has shorter side lengths exactly 5 cm and 12 cm respectively. Find the exact length of the hypotenuse.

(a) $\sqrt{159}$ cm
(b) $\sqrt{17}$ cm
(c) 13 cm
(d) 14 cm
(e) 17 cm

Question 4

The length of the hypotenuse of a right-angled triangle is exactly 20 cm. One of the shorter side lengths is exactly 10 cm. Find the exact length of the remaining side.

(a) 12 cm
(b) $10\sqrt{3}$ cm
(c) $10\sqrt{2}$ cm
(d) 15 cm
(d) 10 cm

Question 5

A right-angled triangle has shorter side lengths 4.54 cm and 7.08 cm respectively, correct to 2 decimal places. Estimate the length of the hypotenuse, rounding off your answer to 2 decimal places.

(a) 8.58 cm
(b) 8.90 cm
(c) 73.61 cm
(d) 8.57 cm
(d) 8.89 cm

Question 6

The length of the hypotenuse of a right-angled triangle is 9.4 cm, correct to 1 decimal place. One of the shorter side lengths is 3.2 cm, correct to 1 decimal place. Estimate the length of the remaining

side, rounding your answer to 1 decimal place.
(a) 78.1 cm
(b) 8.7 cm
(c) 8.9 cm
(d) 6.2 cm
(e) 8.8 cm

Question 7

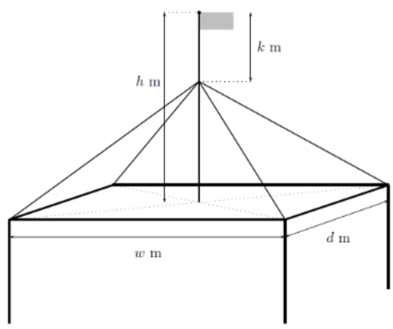
A right-angled triangle has shorter side lengths exactly $a^2 - b^2$ and $2ab$ units respectively, where a and b are positive real numbers such that a is greater than b. Find an exact expression for the length of the hypotenuse (in appropriate units).
(a) $(a+b)^2$
(b) $(a-b)^2$
(c) $\sqrt{a^4+4a^2b^2-b^4}$
(d) a^2+b^2
(e) $\sqrt{a^2+2ab-b^2}$

Question 8

A ladder rests against a wall. The top of the ladder touches the wall at height 12 meters. The length of the ladder is 4 meters longer than the distance from the base of the ladder to the wall. Find the length of the ladder.
(a) 16 m
(b) 12 m
(c) 18 m
(d) 24 m
(e) 20 m

Question 9

A vertical flag pole of height h meters is erected exactly in the middle of the flat roof of a building. The roof is rectangular of width w meters and depth d meters. The flag pole is stabilized by cables that join the corners of the rooftop to the flag pole at a point k meters below the top of the flagpole. Let l m be the total length of cable required to stabilize the flag pole. Find a correct expression for l in terms of w, d, h, and k.



(a) $4\sqrt{w^2+d^2+(h-k)^2}$
(b) $4\sqrt{\frac{w^2}{2}+(h-k)^2}$
(c) $4\sqrt{\frac{w^2+d^2}{4}+(h-k)^2}$
(d) $4\sqrt{w^2+d^2+h^2+k^2}$
(e) $4\sqrt{(w+d)^2+(h-k)^2}$

Question 10

Consider the diagram of the previous question where the flagpole has been stabilized using a total of 50 meters of cable ($l = 50$), the rooftop is 12 meters wide ($w = 12$), 10 meters in depth ($d = 10$), and the point where the cables are fastened to the flagpole is 3 meters from the top of the flagpole ($k = 3$). Estimate the height h of the flagpole to the nearest meter.

(a) 12 m
(b) 11 m
(c) 13 m
(d) 14 m
(e) 10 m

2.3 Algebraic expressions, surds and approximations

2.3.1 Algebraic Expressions

In this section, we're going to explore the fascinating world of mathematical expressions and their algebraic manipulations, which can lead to some unexpected and intriguing results.

I'll start by sharing a personal story from my childhood that left a lasting impression on me and fueled my passion for mathematics. Unlike some who may have had negative experiences with math early on, I was fortunate to have an excellent teacher. One day, he walked into our classroom and challenged us to think of a secret number between one and ten. He then guided us through a series of steps: double the number, add four, halve it, and then subtract the original number. To our amazement, we all ended up thinking of the number two.

$$\frac{2x+4}{2} - x = x + 2 - x = 2$$

The teacher explained this phenomenon by constructing an algebraic expression that simplified to two, providing my first glimpse into the power of mathematical proof.

2.3.2 Surds

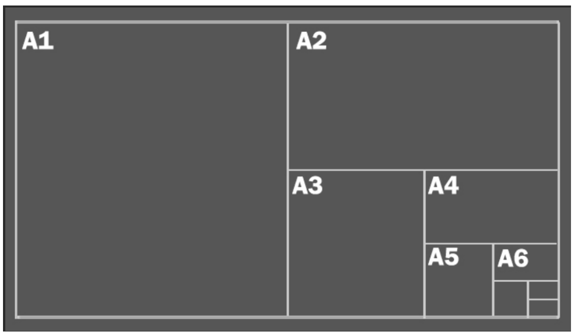


Figure 2.2: Papers in the A-series

Next, I'd like to discuss something as commonplace as sheets of paper, specifically those in the A series. Take an A4 sheet, for example; it's twice the size of an A5, which in turn is twice the size of an A6, and so on. Despite their size differences, all these sheets are similar rectangles, meaning they share the same proportions. We can prove this by drawing lines along their diagonals—if the diagonals align, the rectangles are indeed similar.

[0]Image 2.2 taken from http://www.victoriana.com/a2-paper-size-in-meters-k.html

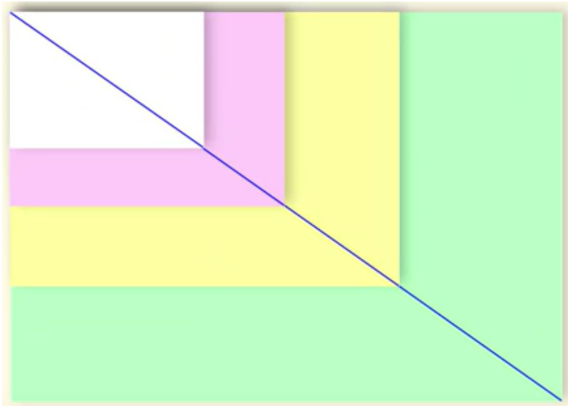
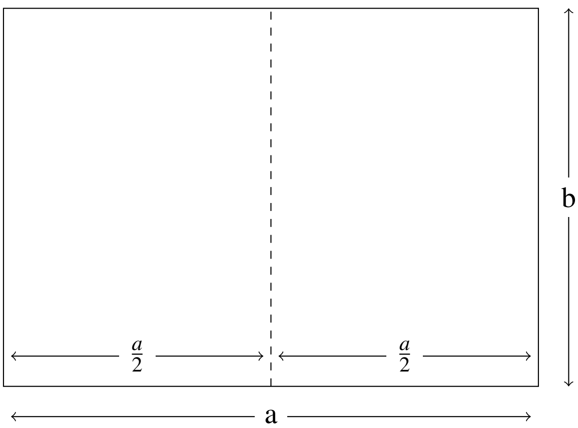


Figure 2.3: Papers in the A-series are similar

This concept extends to the slopes of these diagonals, which are essential in calculus. By examining an A4 sheet and its division into A5 sheets, we can determine that the ratio of the longer side to the shorter side is the square root of two.



Know that:
1) Area of A4 is double of A5
2) $\frac{Length}{Breadth}$ of a A series paper are same

Let x be that ratio

$$\frac{a}{b} = x = \frac{b}{\frac{a}{2}}$$

$$x = \frac{b}{\frac{a}{2}} = \frac{2}{\frac{a}{b}} = \frac{2}{x}$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

Through algebraic manipulation, we find that this ratio, represented by the variable x, satisfies the equation $x^2 = 2$, leading us to conclude that x is the square root of two. This discovery is particularly fascinating because the square root of two is an irrational number, as we've discussed previously.

2.3.3 Approximations

Finally, let's look at how calculators begin the decimal expansion of the square root of two. By employing algebraic manipulation, we can remove the integer part of the square root of two and focus on the decimal expansion. There's a clever way to find it.

$$\sqrt{2} = 1.414213562.....$$

$$\sqrt{2} - 1 = 0.414213562....$$

Inverting numbers, or finding their reciprocals, is a common practice in mathematics. When we play with the reciprocal of $\sqrt{2} - 1$, we encounter a surd expression. Simplifying this expression

[0]Image 2.3 taken from MOOC

using a technique called rationalizing the denominator, we find that the reciprocal of $\sqrt{2}-1$ is actually $\sqrt{2}+1$.

$$\frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$$

Now, let's employ a small trick.

$$\sqrt{2}-1 = \frac{1}{\sqrt{2}+1} = \frac{1}{2+(\sqrt{2}-1)}$$

This leads to a recursive formula, reminiscent of self-referential images, like the man on the porridge packet holding a packet with the same image.

Figure 2.4: Self-referential Image

By defining Y as $\sqrt{2}-1$, we establish a self-referential equation,

$$Y = \frac{1}{2+Y},$$

which can be expanded into a continued fraction.

$$Y = \frac{1}{2+Y} = \frac{1}{2+\frac{1}{2+Y}} = \frac{1}{2+\frac{1}{2+\frac{1}{2+Y}}} = \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+Y}}}} = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}}}}}}}$$

Truncating this fraction provides us with an approximate value for $\sqrt{2}$,

$$Y \approx \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}} = 0.41428$$

Thus, $\sqrt{2}-1 \approx 0.41428$

$\sqrt{2} \approx 1.41428$

demonstrating the effectiveness of continued fractions in approximating irrational numbers without the need for calculators.

2.4 Quiz

Question 1

Which one of the algebraic expressions below corresponds to the instructions of the following mind-reading exercise? Think of a number x between 1 and 10. Triple it. Add 30. Double what you have. Now divide by three. Subtract 10. Halve what you have. Subtract the number at the you first started with. You are now thinking of the number 5.

(a) $\frac{\frac{3x+30x2}{3}-10}{2}-x$
(b) $\frac{\frac{3x+30x2}{3}-x}{2}-10$
(c) $\frac{\frac{2x+30x2}{3}-10}{3}-x$
(d) $\frac{\frac{2x+30x3}{3}-10}{2}-x$
(e) $\frac{\frac{3x-10x2}{3}+30}{2}-x$

Question 2

Simplify the following expression: $\frac{1}{1+\frac{1}{1+1}}$

(a) $\frac{3}{4}$
(b) $\frac{3}{5}$
(c) $\frac{5}{3}$
(d) $\frac{2}{5}$
(e) $\frac{1}{5}$

Question 3

Simplify the following expression: $\frac{1}{2-\frac{1}{2}}$

(a) -1
(b) $\frac{1}{8}$
(c) $-\frac{1}{8}$
(d) 1
(e) $\frac{1}{2}$

Question 4

Simplify the following expression (which becomes an excellent approximation of $\sqrt{2}$):

$$1+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}$$

(a) $\frac{241}{170}$
(b) $\frac{239}{168}$
(c) $\frac{240}{170}$
(d) $\frac{238}{169}$
(e) $\frac{239}{169}$

Question 5

Solve for x given that

$$x=\frac{a}{b}=\frac{a}{b/\sqrt{3}}$$

where a and b are some given positive real numbers.

(a) $x=9$

(b) $x = 3$
(c) $x = \frac{1}{\sqrt{3}}$
(d) $x = \frac{1}{3}$
(e) $x = \sqrt{3}$

Question 6

Solve for x given that

$$x = \frac{2a}{b} = \frac{b}{a/\sqrt{2}}$$

where a and b are some given positive real numbers.
(a) $x = 2$
(b) $x = \frac{1}{2}$
(c) $x = \frac{1}{\sqrt{2}}$
(d) $x = 4$
(e) $x = \sqrt{2}$

Question 7

Simplify the following expression:

$$\frac{1}{2-\sqrt{3}} =$$

(a) $4-\sqrt{3}$
(b) $4+\sqrt{3}$
(c) $-2+\sqrt{3}$
(d) $2+\sqrt{3}$
(e) $-2-\sqrt{3}$

Question 8

Simplify the following expression:

$$\frac{\sqrt{5}+1}{\sqrt{5}-1} =$$

(a) $\frac{6-\sqrt{5}}{2}$
(b) $\frac{3-\sqrt{5}}{2}$
(c) $\frac{3+\sqrt{5}}{2}$
(d) $\frac{6+\sqrt{5}}{2}$
(e) $\frac{\sqrt{5}-3}{2}$

Question 9

Which one of the expressions below is equivalent to the expression

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$$

where a and b are distinct positive real numbers?
(a) $\frac{a-b+2\sqrt{ab}}{a-b}$
(b) $\frac{a+b-2\sqrt{ab}}{a-b}$

(c) $\frac{a+b+2\sqrt{ab}}{a-b}$
(d) $\frac{a-b-2\sqrt{ab}}{a-b}$
(e) $\frac{a+b}{a-b}$

Question 10

Which one of the following statements is false?
(a) $1-\sqrt{2}$ is irrational.
(b) $(1+\sqrt{2})/(1-\sqrt{2})$ is irrational.
(c) $1+\sqrt{2}$ is irrational.
(d) $(1-\sqrt{2})(\sqrt{2}-1)$ is irrational.
(e) $(1+\sqrt{2})^2$ is irrational.

3. Equations, Inequalities and Solution Sets

3.1 Equations and Inequalities

3.1.1 Equations

Welcome to our another math journey! In this section, we're unraveling the mysteries of equations and inequalities. Equations are like a perfectly balanced scale, where each side is equal. Inequalities, on the other hand, tell us how numbers aren't equal but rather more or less than each other.

■ **Example 3.1** Solve : 2x+1=7 ■

Let's start with a simple equation: 2x + 1 = 7. Our goal is to find out what x is. We do this by isolating x on one side. Subtracting 1 from both sides, we get 2x = 6. Dividing by 2 on both sides, we find x = 3. You can check your work by plugging x back into the original equation to see if it holds true.

Now, let's complicate things a bit. Consider an equation where x appears multiple times and is mixed with fractions.

■ **Example 3.2** Solve: $\frac{x}{4} = \frac{3x}{5} - \frac{9-x}{10}$ ■

$$Or, \frac{x}{4} = \frac{6x-(9-x)}{10}$$

$$Or, \frac{x}{4} = \frac{7x-9}{10}$$

Multiplying both sides by 40,

$$Or, 10x = 28x - 36$$

$$Or, 36 = 18x$$

$$Or, x = 2$$

By finding a common denominator and simplifying, we can isolate x to find its value. It's like gathering apples; we collect all terms involving x and simplify until x stands alone.

■ **Example 3.3** Solve: (x-1)(x-2)=0 ■

Moving on to a special case: an equation with zero on one side and a product of factors on the other, like (x - 1)(x - 2) = 0. This tells us that at least one of the factors must be zero. It's a fundamental property of multiplication—if the product is zero, at least one of the multiplicands must be zero. This leads us to two possible solutions for x: 1 & 2.
A student once approached this differently by expanding the equation and completing the square, a technique that rewrites expressions to reveal perfect squares.

$$(x-1)(x-2)=0$$

$$x^2-3x+2=0$$

$$\underbrace{x^2-2.x.\frac{3}{2}+(\frac{3}{2})^2}_{\text{Complete Square}}-(\frac{3}{2})^2+2=0$$

$$(x-\frac{3}{2})^2-(\frac{1}{2})^2=0 \quad [(a+b)^2=a^2+2ab+b^2||a=x||b=-\frac{3}{2}]$$

$$(x-\frac{3}{2})^2=(\frac{1}{2})^2$$

$$x-\frac{3}{2}=\pm\frac{1}{2}$$

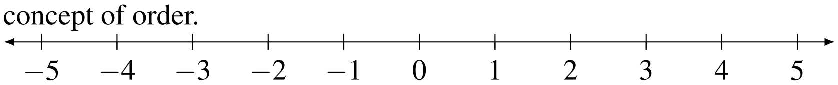
$$x=\frac{3}{2}\pm\frac{1}{2}$$

$$x=1,2$$

This method also isolates x and provides the same solutions, showcasing that even with little mathematical background, creative thinking can lead to significant discoveries.

3.1.2 Inequalities

Finally, let's talk about inequalities. They require an understanding of the number line and the concept of order.



If a is to the left of b on the number line, we say a is less than b, written as $a < b$. The reverse is also true; if b is to the right of a, then a is less than b, or $b > a$. We also have symbols for 'less than or equal to' as $\leq$ and 'greater than or equal to' as $\geq$.

Solving inequalities is similar to solving equations,

■ **Example 3.4** Solve:

$$2x-1>3$$

$$Sol^n:$$

$$2x>4$$

$$x>2$$

■

Next, here's a catch while solving the inequalities: multiplying or dividing by a negative number flips the inequality. For example, if we multiply both sides of $1-x<3x+6$ by -1, the inequality changes direction, becoming $x-1>-3x-6$. After simplifying, we find $x>-\frac{5}{4}$.

In our next section, we'll delve into solution sets and interval notation, which provide a convenient way to represent the range of solutions for real numbers. Be sure to go through exercises to solidify your understanding. Thanks for reading, and I can't wait to continue this mathematical adventure with you!

3.1.3 Practice Quiz

Question 1

Solve for all real numbers x given that

$$\frac{3x-2}{7}=10.$$

(a) $x=24$
(b) $x=20$
(c) $x=22$
(d) $x=26$
(e) $x=\frac{76}{3}$

Question 2

Solve for all real numbers x given that

$$\frac{1-x}{2}=\frac{3}{4}+5x.$$

(a) $x=\frac{1}{21}$
(b) $x=\frac{1}{10}$
(c) $x=\frac{1}{20}$
(d) $x=-\frac{1}{21}$

Question 3

Which one of the following statements is false?
(a) $-1<-2$
(b) $2\geq -1$
(c) 1 ± 2
(d) $1<2$
(e) $2\geq 1$

Question 4

Which one of the following statements is true?
(a) $-1.1>-1.01$
(b) $-2.35<-2.34$
(c) $0\geq 0.01$
(d) $35.61<34.62$
(e) $1.1<1.01$

Question 5

Which one of the following statements is false?

(a) $\frac{3}{2} < \frac{7}{5}$
(b) $\frac{12}{29} \geq \frac{29}{70}$
(c) $\frac{70}{169} > \frac{29}{70}$
(d) $\frac{12}{29} < \frac{5}{12}$
(e) $\frac{3}{2} > \frac{7}{5}$

Question 6

Solve for all real numbers x given that

$$\frac{3x-2}{7} > 7.$$

(a) $x > 1$
(b) $x < 17$
(c) $x > 17$
(d) $x < 19$
(e) $x > 19$

Question 7

Solve for all real numbers x given that

$$\frac{9-3x}{7} \leq -7.$$

(a) $x \geq 17$
(b) $x \leq 1$
(c) $x \leq 19$
(d) $x \leq 17$
(e) $x \geq 19$

Question 8

Solve for all real numbers x given that

$$2-3x \leq 4+9x.$$

(a) $x \leq -\frac{1}{3}$
(b) $x \geq -\frac{1}{3}$
(c) $x \leq -\frac{1}{6}$
(d) $x \geq -\frac{1}{6}$
(e) $x \leq \frac{1}{6}$

Question 9

Solve for all real numbers x given that

$$(x+2)(x-5) = 0.$$

(a) $x = 2, -5$
(b) $x = 2, 5$
(c) $x = -2, -5$
(d) $x = -2, 5$
(e) $x = \frac{5}{2}$

Question 10

Observe that the product of two real numbers is negative if and only if one of the numbers is positive and the other is negative. Hence solve for all real numbers x given that

$$(x+2)(x-5) < 0.$$

(a) $x < 5$
(b) $x > 5$
(c) $x < -2$
(d) $-2 < x < 5$
(e) $x > -2$

Answers

The answers will be revealed at the end of the module.

Question 1

Which one of the algebraic expressions below corresponds to the instructions of the following mind-reading exercise? Think of a number x between 1 and 10. Triple it. Add 30. Double what you have. Now divide by three. Subtract 10. Halve what you have. Subtract the number at the you first started with. You are now thinking of the number 5.

(a) $\frac{\frac{3x+30\text{x}2}{3}-10}{2}-x$
(b) $\frac{\frac{3x+30\text{x}2}{3}-x}{2}-10$
(c) $\frac{\frac{2x+30\text{x}2}{3}-10}{3}-x$
(d) $\frac{\frac{2x+30\text{x}3}{3}-10}{2}-x$
(e) $\frac{\frac{3x-10\text{x}2}{3}+30}{2}-x$

Question 2

Simplify the following expression: $\frac{1}{1+\frac{1}{1+1}}$

(a) $\frac{3}{4}$
(b) $\frac{3}{5}$
(c) $\frac{5}{3}$
(d) $\frac{2}{5}$
(e) $\frac{1}{5}$

Question 3

Simplify the following expression: $\frac{1}{2-\frac{1}{2}}$

(a) -1
(b) $\frac{1}{8}$
(c) $-\frac{1}{8}$
(d) 1
(e) $\frac{1}{2}$

Question 4

Simplify the following expression (which becomes an excellent approximation of $\sqrt{2}$):

$$1+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}$$

(a) $\frac{241}{170}$

(b) $\frac{239}{168}$
(c) $\frac{240}{170}$
(d) $\frac{238}{169}$
(e) $\frac{239}{169}$

Question 5

Solve for x given that

$$x = \frac{a}{b} = \frac{a}{b/\sqrt{3}}$$

where a and b are some given positive real numbers.

(a) $x = 9$
(b) $x = 3$
(c) $x = \frac{1}{\sqrt{3}}$
(d) $x = \frac{1}{3}$
(e) $x = \sqrt{3}$

Question 6

Solve for x given that

$$x = \frac{2a}{b} = \frac{b}{a/\sqrt{2}}$$

where a and b are some given positive real numbers.

(a) $x = 2$
(b) $x = \frac{1}{2}$
(c) $x = \frac{1}{\sqrt{2}}$
(d) $x = 4$
(e) $x = \sqrt{2}$

Question 7

Simplify the following expression:

$$\frac{1}{2-\sqrt{3}} =$$

(a) $4-\sqrt{3}$
(b) $4+\sqrt{3}$
(c) $-2+\sqrt{3}$
(d) $2+\sqrt{3}$
(e) $-2-\sqrt{3}$

Question 8

Simplify the following expression:

$$\frac{\sqrt{5}+1}{\sqrt{5}-1} =$$

(a) $\frac{6-\sqrt{5}}{2}$
(b) $\frac{3-\sqrt{5}}{2}$
(c) $\frac{3+\sqrt{5}}{2}$
(d) $\frac{6+\sqrt{5}}{2}$
(e) $\frac{\sqrt{5}-3}{2}$

Question 9

Which one of the expressions below is equivalent to the expression

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$$

where a and b are distinct positive real numbers?

(a) $\frac{a-b+2\sqrt{ab}}{a-b}$
(b) $\frac{a+b-2\sqrt{ab}}{a-b}$
(c) $\frac{a+b+2\sqrt{ab}}{a-b}$
(d) $\frac{a-b-2\sqrt{ab}}{a-b}$
(e) $\frac{a+b}{a-b}$

Question 10

Which one of the following statements is false?

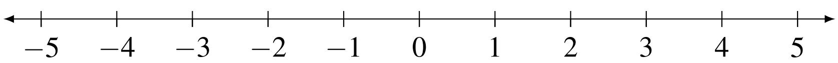
(a) $1-\sqrt{2}$ is irrational.
(b) $(1+\sqrt{2})/(1-\sqrt{2})$ is irrational.
(c) $1+\sqrt{2}$ is irrational.
(d) $(1-\sqrt{2})(\sqrt{2}-1)$ is irrational.
(e) $(1+\sqrt{2})^2$ is irrational.

3.2 Sign Diagrams, Solution sets and intervals

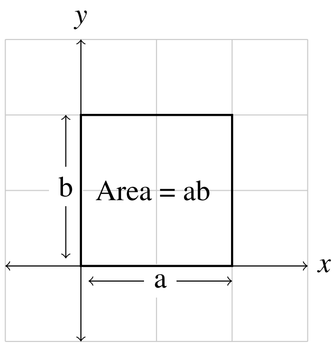
3.2.1 Negative and Positive Areas

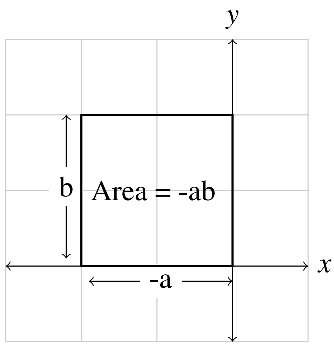
In the section, we delve into Sign Diagrams, which are tools for identifying where mathematical expressions are positive or negative. We discuss solution sets, which describe groups of real numbers that meet certain conditions, often linked with sign diagrams and interval notation.

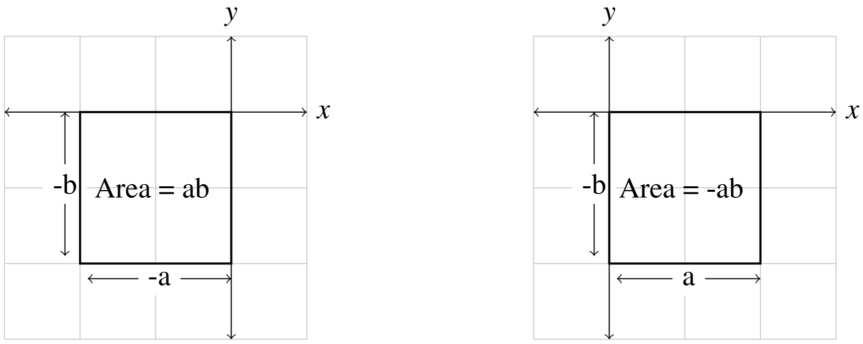
Our conversation starts with the concepts of positive and negative as they apply to real numbers, divided by zero on the real line.



We also visit the idea of interpreting the product of two numbers, ab , as the area of a rectangle, emphasizing that directionality introduces the concepts of positive and negative areas, which have real-world applications like displacement and in areas calculated by integration.







The figures explains that the product ab is positive if a and b are both positive or both negative, and negative if they have opposite signs. For example, $2 \times 3 = 6$, and $(-2) \times (-3) = 6$, showing that the product of two negatives is positive.

In the previous section, we saw that $(x-1) \times (x-2) = 0$ when x equals one or two. Along with that information, understanding when expressions are positive or negative is more crucial in calculus, as it relates to the slopes of curves. Sign diagrams are introduced as a concise way to represent these situations, which will be explained using examples.

3.2.2 Sign Diagrams

In the second part, we illustrate how to create a sign diagram, using it to determine the sign of an expression as x varies. The diagram includes a line representing the real number line and marks the points where the expression equals zero.

■ **Example 3.5** Create a sign diagram for the product (x-1)(x-2). ■

x		1		2	
(x-1)(x-2)	$+$	0	$-$	0	$+$

For example, if x is greater than two, both factors (x - 1) and (x - 2) are positive, making the product positive. Between one and two, the product is negative, and to the left of one, it's positive again.

Now, let's move to a more complex expression, $(x+1) \times (x-3)$, and builds a sign diagram for it, showing how the sign changes depending on the value of x .

■ **Example 3.6** Create a sign diagram for (x+1)(x-3). ■

x		-1		3	
(x+1)(x-3)	$+$	0	$-$	0	$+$

In this one, after you have marked the points where the product becomes 0, we trying checking the sign of the product in each interval. In the interval where $x > 3$, each of the (x+1)(x-3) are postive making the entire product positive. In between -1 and 3, (x+1) is positive but (x-3) is negative making the product negative. Next, when $x < -1$, both (x+1) and (x-3) are negative making the product positive. With all this information, we can construct the sign diagram as presented above.

Next, we will discuss solution sets, which are collections of numbers that satisfy a certain condition, such as being positive or negative. These sets are captured using sign diagrams, providing a clear visual representation of the behavior of expressions on the real number line. This concept is particularly important in calculus, where it aids in understanding the physical interpretations of areas under curves and the increasing-decreasing property of a function. The use of sign diagrams and solution sets offers a clear and concise method for analyzing mathematical expressions in advanced studies.

3.2.3 Solution Sets

A solution set is a group of numbers that fulfill a specific condition, such as solving an equation or inequality.

■ **Example 3.7** Use set notation to express solution of (x+1)(x-3) = 0 ■

For instance, the solution set for the equation (x + 1)(x - 3) = 0 is $\{-1,3\}$, indicating that both -1 and 3 are solutions. The curly braces represent a set, and the comma separates the individual elements.

■ **Example 3.8** Use set notation to express solution of (x+1)(x-3)>0 ■

When we consider inequalities, like $(x+1)(x-3) > 0$, we use set notation to express the solution. This can be written as $\{x \in \mathbb{R} | x < -1 \text{ or } x > 3\}$, where:

The curly braces $\{\}$ denote a set.
The symbol $\in$ means "is an element of."
$\mathbb{R}$ represents all real numbers.
The vertical line | (or sometimes a colon :) means "such that."

The solution set derived from the sign diagram shows that the expression is positive when x is less than -1 or greater than 3. Conversely, the expression is negative when x is between -1 and 3, not including the endpoints.

3.2.4 Interval Notation

Interval notation offers a concise way to describe ranges of numbers. For example:

$(3,\infty)$ represents all numbers greater than 3, not including 3 itself.
$[a,b]$ includes all numbers between a and b, including the endpoints.
(a,b) includes all numbers between a and b, excluding the endpoints.
Mixed brackets, like $[a,b)$ or $(a,b]$, include one endpoint but exclude the other.

For infinite intervals:
$(-\infty,a)$ or $(-\infty,a]$ represent all numbers less than a, with the latter including a.
(b,∞) or $[b,\infty)$ represent all numbers greater than b, with the latter including b.
To express the entire set of real numbers, we use $(-\infty,\infty)$.

■ **Example 3.9** Use interval notation to express solution of (x+1)(x-3)>0 ■

Applying interval notation to the sign diagram for $(x+1)(x-3) > 0$, we find:

The expression is negative between -1 and 3, denoted as (-1, 3). The expression is positive for $x < -1$ or $x > 3$, represented by two intervals: $(-\infty,-1)$ and $(3,\infty)$, combined with a union symbol ($\cup$).Thus, the solution in interval notation for the given inequality will be $(-\infty,-1) \cup (3,\infty)$.

This notation helps us succinctly describe sets of numbers and their relationships on the real number line. It's a powerful tool in mathematics for conveying complex ideas in a simple form.

If you want to be master in these ideas, there are exercise problems for you to try which can help enhance your understanding.

3.2.5 Practice Quiz

Question 1

Find the solution set for the equation $x+2=1$.

(a) -1
(b) $\{1\}$
(c) 1
(d) $\{1,-1\}$
(e) $\{-1\}$

Question 2

Find the solution set for the equation $(x+1)(x+2)=0$.

(a) $1, 2$
(b) $\{-1\}$
(c) $\{-2\}$
(d) $-1,-2$
(e) $\{-2,-1\}$

Question 3

Find the solution set for the equation $(x+1)(x+2)=12$.

(a) $\{-5,-2\}$
(b) $\{-5,2\}$
(c) $\{-2,5\}$
(d) $\{10,11\}$
(e) -5,2

Question 4

Construct the sign diagram for the expression $(x-1)(x-3)$.

(a)

x		1		3	
(x-1)(x-3)	$+$	0	$-$	0	$+$

(b)

x		-3		-1	
(x-1)(x-3)	$+$	0	$-$	0	$+$

(c)

x		-3		-1	
(x-1)(x-3)	$-$	0	$+$	0	$-$

(d)

x		1		3	
(x-1)(x-3)	$-$	0	$+$	0	$-$

Question 5

Construct the sign diagram for the expression $x(x-3)$.

(a)

x		-1		0	
x(x-1)	$-$	0	$+$	0	$-$

(b)

x		0		1	
x(x-1)	$-$	0	$+$	0	$-$

(c)

x		0		1	
x(x-1)	$+$	0	$-$	0	$+$

(d)

x		-1		0	
x(x-1)	$+$	0	$-$	0	$+$

Question 6

Construct the sign diagram for the expression $x(x-1)(x+2)$.

(a)

x		-1		0		2	
x(x-1)(x + 2)	$+$	0	$-$	0	$+$	0	$-$

(b)

x		-1		0		2	
x(x-1)(x + 2)	$-$	0	$+$	0	$-$	0	$+$

(c)

x		-2		0		1	
x(x-1)(x + 2)	$-$	0	$+$	0	$-$	0	$+$

(d)

x		-2		0		1	
x(x-1)(x + 2)	$+$	0	$-$	0	$+$	0	$-$

Question 7

Use interval notation to find an expression for the solution set of the following inequality:

$$(x-1)(x-3) < 0.$$

(a) $[1,3)$
(b) $(-\infty,3)$
(c) $(-\infty,1)$
(d) $(1,\infty)$
(e) $(1,3]$

Question 8

Use interval notation to find an expression for the solution set of the following inequality:

$$x(x-1) \geq 0.$$

(a) $[1,\infty)$
(b) $(-\infty,0]$
(c) $(-\infty,-1] \cup [0,\infty)$
(d) $(-\infty,0] \cup [1,\infty)$
(e) $[0,1]$

Question 9

Use interval notation to find an expression for the solution set of the following inequality:

$$x(x-1)(x+2) > 0.$$

(a) $(1,\infty)$
(b) $(-2,0)$
(c) $(-2,0) \cup (1,\infty)$
(d) $(0,\infty)$
(e) $(-\infty,-2) \cup (0,1)$

Question 10

Use interval notation to find an expression for the solution set of the following inequality:

$$x(x-1)(x+2) \leq 0.$$

(a) $[-2,0] \cup [0,1]$
(b) $[0,1]$
(c) $(-\infty,-2] \cup [0,1]$
(d) $(-\infty,-2]$
(e) $(1,\infty]$

Answers

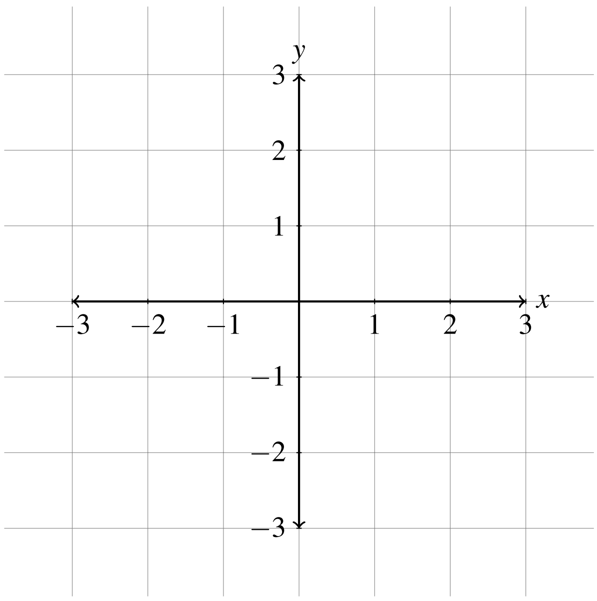
The answers will be revealed at the end of the module.

4. The Cartesian plane and distance

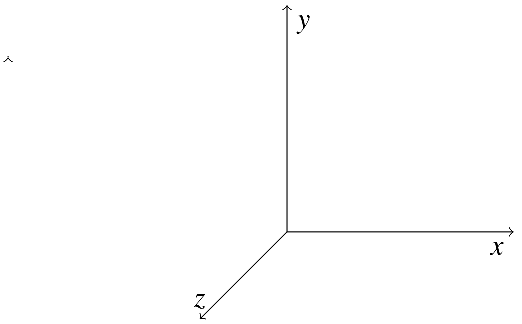
4.1 Coordinate Systems

4.1.1 An Introduction to Coordinate Planes

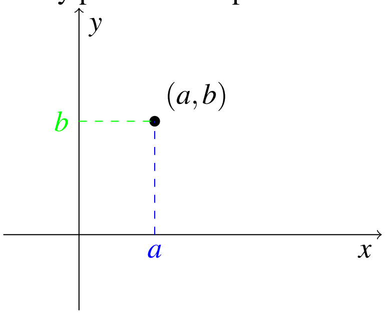
In this section, we will dive into the realm of mathematics where the Cartesian plane emerges as a pivotal concept. This two-dimensional framework, conceptualized by René Descartes, is defined by the intersection of the X and Y axes at the origin. It's a brilliant tool for analyzing phenomena within two-dimensional spaces.



Descartes' ingenuity didn't stop there; he introduced a third dimension with the Z axis, propelling us into the modeling of three-dimensional spaces.

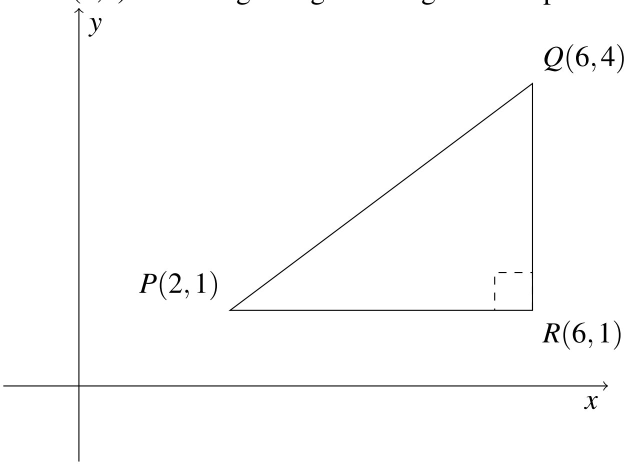


This leap is reminiscent of the creative problem-solving approach termed 'lateral thinking,' introduced by Edward de Bono. Descartes exemplified this innovative thinking, which led to his invention of the Cartesian plane. He envisioned not just one, but two real lines intersecting perpendicularly at zero, a truly lateral concept. The horizontal line became known as the X-axis, and the vertical one as the Y-axis. The origin, denoted by 'O,' serves as the reference point for this system. Every point on this plane is identified by coordinates (x, y), which act as a numerical address.



4.1.2 Distance Formula

By extending this plane infinitely in all directions, we can plot points and form shapes, such as triangles, and calculate distances using Pythagoras' theorem. For example, the points P(2,1), Q(6,4), and R(6,1) form a right-angled triangle on the plane.



The distance between these points can be determined by the differences in their coordinates, illustrating the practical application of the Cartesian plane.

Distance $PR = 6 - 2 = 4$

Distance $QR = 4 - 1 = 3$

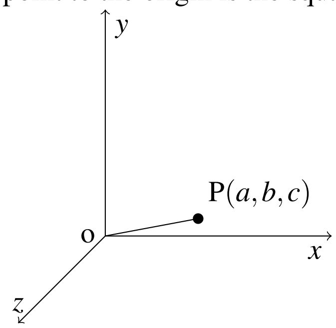
Distance $PQ = \sqrt{PR^2 + QR^2} = \sqrt{(6-2)^2 + (4-1)^2} = \sqrt{4^2 + 3^2} = 5$

Observe that, the above diagram also proves a wonderful result,

Theorem 4.1 — Distance Formula.

The distance between two points $P(a_1, b_1)$ and $Q(a_2, b_2)$ is $\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$ (4.1)

As we transition from two-dimensional to three-dimensional thinking, we add the Z axis to our Cartesian plane. Now, points in space have three coordinates (x, y, z), and the distance from any point to the origin is the square root of the sum of the squares of its coordinates.



$$|OP| = \sqrt{(a-0)^2 + (b-0)^2 + (c-0)^2} = \sqrt{a^2 + b^2 + c^2}$$

This extension of Pythagoras' theorem into three dimensions allows us to measure distances in space, further demonstrating the versatility of Cartesian coordinates.

4.1.3 Practice Quiz

Question 1

Recall that the distance (in appropriate units) between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the xy-plane is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Find the distance (in appropriate units) between the points $P(1, 3)$ and $Q(2, 1)$.

(a) $\sqrt{2}$
(b) $\sqrt{5}$
(c) $\sqrt{3}$
(d) 4
(e) 2

Question 2

Find the distance (in appropriate units) between the points $P(-2, 4)$ and $Q(5, -7)$ in the xy-plane.

(a) $\sqrt{58}$
(b) $\sqrt{18}$
(c) 13
(d) $\sqrt{6}$
(e) $\sqrt{170}$

Question 3

You are told that the points $P(1, 2)$, $Q(5, 6)$, and $R(a, 2)$ form a right-angled triangle in the xy-plane, where the right angle occurs at the point R. Find the real number a.

(a) 2
(b) 5
(c) 6
(d) 4

(e) 3

Question 4

The points $P(0,2)$, $Q(4,5)$, and $R(4,2)$ form a right-angled triangle in the xy-plane. Find the length of the hypotenuse (in appropriate units).

(a) 4
(b) $\sqrt{13}$
(c) $\sqrt{7}$
(d) 5
(e) 3

Question 5

It is a fact that the midpoint of the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the xy-plane has the following coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Find the coordinates of the midpoint of the line segment joining $P(-2,4)$ and $Q(4,-2)$.

(a) (1,1)
(b) (0,0)
(c) (3,3)
(d) (-3,3)
(e) (2,2)

Question 6

Consider the points $P(1,-2)$, $Q(2,1)$ and $R(0,1)$ in the plane. Which one of the following statements is true?

(a) P is the midpoint of the line joining Q to R.
(b) Q is closer to R than to P.
(c) P is closer to R than to Q.
(d) P is closer to Q than to R.
(e) Q is closer to P than to R.

Question 7

Recall that the distance (in appropriate units) from a point $P(a,b,c)$ to the origin $O(0,0,0)$ in space is

$$\sqrt{a^2 + b^2 + c^2}.$$

Find the distance (in appropriate units) from the point $P(1,-2,3)$ to the origin.

(a) 4
(b) $\sqrt{11}$
(c) $\sqrt{6}$
(d) 6
(e) $\sqrt{17}$

Question 8

It is a fact following from the Theorem of Pythagoras that the distance (in appropriate units) between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Find the distance (in appropriate units) between the points $P(1,-2,3)$ and $Q(3,1,-3)$.

(a) $\sqrt{71}$
(b) 7
(c) $\sqrt{13}$
(d) 11
(e) 13

Question 9

The midpoint of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space has the following coordinates:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Find the coordinates of the midpoint of the line segment joining $P(1, -2, -3)$ and $Q(3, 2, 7)$.
(a) (2, 0.5, 2)
(b) (4, 0.4, 4)
(c) (2.1, 0)
(d) (4, 0.2)
(e) (2, 0, 2)

Question 10

Consider the points $P(1, 0, 2)$, $Q(3, 1, -2)$ and $R(-1, -1, 6)$ in space. Which one of the following statements is true?
(a) P is the midpoint of the line joining Q to R.
(b) Q is closer to R than to P.
(c) P is closer to Q than to R.
(d) R is closer to Q than to P.
(e) P is closer to R than to Q.

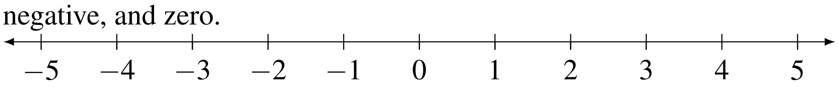
Answers

The answers will be revealed at the end of the module.

4.2 Distance and Absolute Value

4.2.1 Defining distances with absolute value

In our previous section, we explored the measurement of distances between points in a two-dimensional plane and extended our journey into three-dimensional space, employing Pythagoras' theorem to navigate through right-angled triangles. Today, let's take a step back and concentrate on the one-dimensional real line. Previously, our illustrations predominantly featured non-negative numbers. Now, we'll introduce a framework that meticulously handles all real numbers—positive, negative, and zero.



We'll delve into the concept of absolute value, also known as the magnitude of a real number, which precisely encapsulates the idea of distance on the real line. Consider a real number x . The absolute value or magnitude of x , represented by |x| , is defined as follows:

|x| =

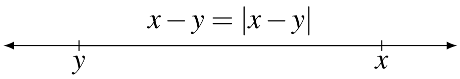
$$\begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

This definition aligns with the intuitive understanding of distance: if x is positive or zero, the distance from zero is x itself; if x is negative, the distance is the positive counterpart, -x .
$|1| = 1$

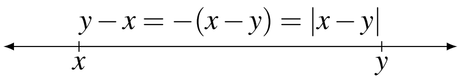
$$|-1| = 1$$

For instance, both 1 and -1 are one unit away from zero, hence their absolute value is 1. Similarly, 2 and -2 are two units from zero, making their absolute value 2. The same logic applies to 1.5 and -1.5, which are 1.5 units from zero, resulting in an absolute value of 1.5. Naturally, the absolute value of zero is zero.

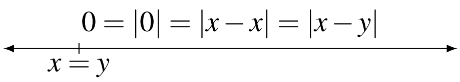
An important and practical fact is that the distance between any two numbers on the real line is the absolute value of their difference. This holds true regardless of the order in which the difference is taken.



For example, if x is to the right of y on the number line, the distance between them is x - y , which is the absolute value of x - y since it's positive.



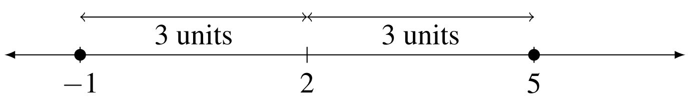
Conversely, if x is to the left of y , the distance is y - x , which is the positive form of x - y , and thus the absolute value of x - y . If x and y are the same, their distance is zero, which is the absolute value of x - y .



4.2.2 Problems involving Absolute Value

Let's apply this knowledge to solve an equation.

■ **Example 4.1** Find all values of x that are exactly three units away from 2. ■



Visualizing this on the number line, there are two possibilities: moving to the right, we find 5; moving to the left, we find -1. Therefore, the solution set comprises two numbers: -1 and 5.

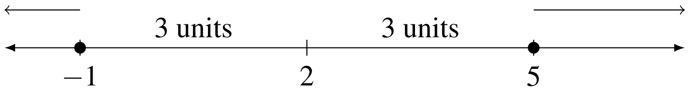
■ **Example 4.2** Solve: |x-2| = 3 ■

Either $x - 2 = 3$ or $2 - x = 3$

$$x = 5 \text{ or } -1$$

Expanding on this exercise, let's transform the equation into an inequality by replacing the equals sign with a greater-than sign.

■ **Example 4.3** Solve: |x-2| > 3 ■



We're now looking for values of x whose distance from 2 is more than three units. The solution set becomes the union of two intervals: one extending indefinitely to the left from -1 (excluding -1) and the other extending indefinitely to the right from 5 (excluding 5).

While a geometric approach is often the most intuitive for solving such problems, it's not always feasible. In such cases, a robust algebraic technique is invaluable. The accompanying exercises provide several examples. Please review them thoroughly and attempt the exercises at your own pace. I eagerly await our next section.

4.2.3 Practice Quiz

Question 1

Find the absolute value (magnitude) of -5.

(a) $\frac{5}{2}$
(b) 0
(c) 5
(d) -5
(e) 10

Question 2

Find the distance between the points -6 and -2 on the real number line.

(a) 2
(b) -8
(c) 4
(d) -4
(e) 8

Question 3

Find the distance between the points 6 and -2 on the real number line.

(a) -8
(b) 8
(c) 2
(d) 4
(e) 3

Question 4

Find the solution set for the following equation: $|x-1| = 4$

(a) $\{1,4\}$
(b) $\{-3\}$
(c) $\{3\}$
(d) $\{3,5\}$
(e) {-3, 5}

Question 5

Find the solution set for the following equation: $|x+1| = 3$

(a) $\{-1,3\}$
(b) $\{2\}$
(c) $\{-4\}$

(d) $\{-4,2\}$
(e) $\{2,4\}$

Question 6

Find the solution set for the following equation: $|x-2| = |x-4|$

(a) $\{-2,0\}$
(b) $\{3\}$
(c) $\{4,-2\}$
(d) $\{1\}$
(e) $\{-3\}$

Question 7

Find the solution set for the following equation: $|x-2| = |x+4|$

(a) $\{1\}$
(b) $\{-2,4\}$
(c) $\{4,-2\}$
(d) $\{-1\}$
(e) $\{3\}$

Question 8

Find the solution set for the following inequality, using interval notation: $|x-2| \le 7$

(a) $[5,9]$
(b) $[-5,9]$
(c) $[-9,-5]$
(d) $[-9,5]$
(e) $(-\infty,9]$

Question 9

Find the solution set for the following inequality, using interval notation: $|x-2| > 7$

(a) $(-\infty,-5) \cup (9,\infty)$
(b) $(9,\infty)$
(c) $(5,9)$
(d) $(-\infty,-5)$
(e) $(-\infty,-9) \cup (3,\infty)$

Question 10

Find the solution set for the following inequality, using interval notation: $|x-2| \ge |x+6|$

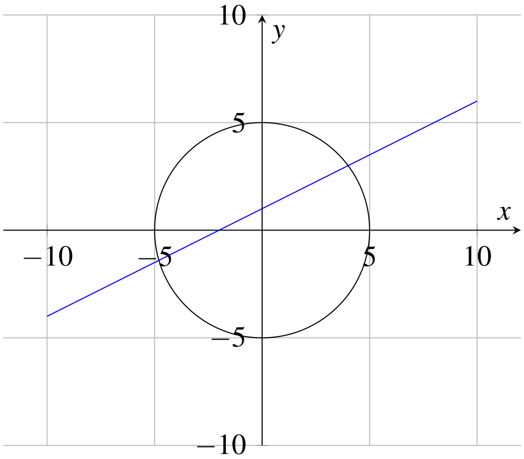
(a) $(-\infty,-2]$
(b) $(-\infty,2]$
(c) $[2,\infty)$
(d) $(-\infty,-4]$
(e) $[-2,\infty)$

Answers

The answers will be revealed at the end of the module.

4.3 Lines and Circles in the plane

At this point, we are ready to take a look at the concepts of lines and circles in the Cartesian plane, their equations, and the mathematical principles that govern them.



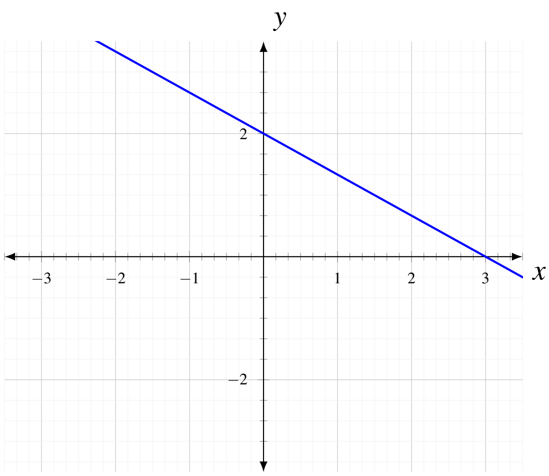
The Cartesian plane serves as the stage for the exploration of fundamental geometric shapes: lines and circles. These shapes are not only central to geometry but also to calculus. Lines, characterized by their simplicity, are integral to calculus, which seeks to simplify complex processes into basic arithmetic. This simplification is the essence of calculus—deeply understanding simple things like lines.

Circles represent cycles and periodic behavior, which are omnipresent in our lives, from the daily cycle of day and night to the rhythmic beating of our hearts. The mathematics of circles is essential for understanding these repetitive patterns.

4.3.1 Equations of Lines

Lines in the plane are described by the general equation $ax + by = c$, where x and y are variables representing coordinates on the plane, and a , b , and c are constants. This form is widely used by mathematicians due to its ability to be generalized for more complex applications.

■ **Example 4.4** Sketch the line : 2x+3y=6 ■



For instance, the line $2x + 3y = 6$ can be visualized by finding two points that lie on it. When $x = 0$, the equation simplifies to $3y = 6$, yielding $y = 2$ as the y-intercept. Similarly, setting $y = 0$ gives us $2x = 6$, and $x = 3$ as the x-intercept. By connecting these points, we graph the line.

Now, we have just used a general principle : A line is determined by two points.

4.3.2 Slope of a Line

The slope of a line, denoted by m , is the measure of its steepness and is calculated as the ratio of the vertical change (rise) to the horizontal change (run). Using the coordinates of two points on the line, $P(x_1,y_1)$ and $Q(x_2,y_2)$, the slope is given by $m = \frac{y_2-y_1}{x_2-x_1}$.

■ **Example 4.5** Find the slope of the line 2x+3y=6.
We have already found out that it passes through (0,2) and (3,0).
Thus, taking $(x_1,y_1) = (0,2)$ and $(x_2,y_2) = (3,0)$, we get

$$m = \frac{0-2}{3-0} = \frac{-2}{3}$$

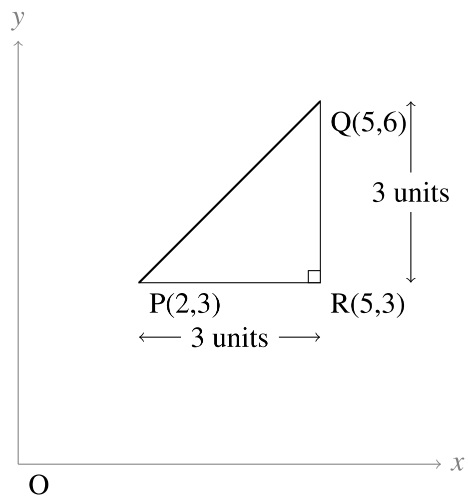
■

But what does our answer convey ?
It says that for every 3 units in positive direction of x axis, we are going -2 units in y-axis.

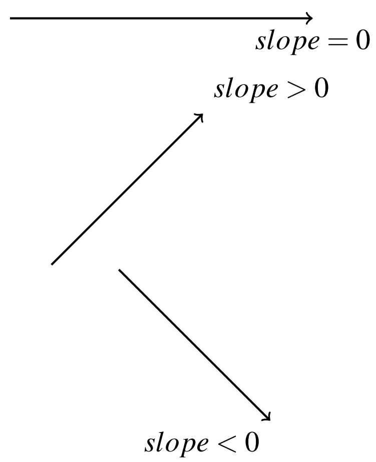
■ **Example 4.6** Q. Find the slope of the line passing through P(2,3) and Q(5,6).

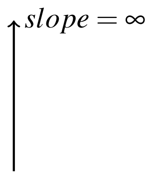
$$\text{Slope} = \frac{6-3}{5-2} = \frac{3}{3} = 1$$

■



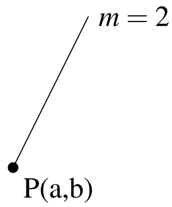
4.3.2.1 Possible Slopes of a Line





4.3.3 Equation of line when a point in the line and slope is known

In many cases, we will know the tiltation or slope of a line and a point from which the line will pass. In such cases, we can clearly see that there will be a single line defined by these conditions.



To find the equation of the given line in such case, We will try to find the locus [1] of a point (x,y) lying on that line.
Now, we have two points, P(a,b) and (x,y) and the slope is m. Using the formula of slope, we have

$$m = \frac{y-b}{x-a}$$

$$y - b = m(x-a)$$

$$y = m(x-a) + b$$

which is the required equation of line in point-slope form.

■ **Example 4.7** Find the equation of line passing through P(2,3) and Q(7,13).
Solution:

$$m = \frac{13-3}{7-2}$$

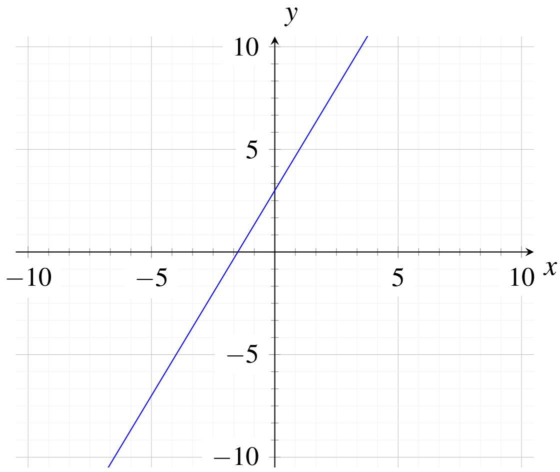
$$m = \frac{10}{5} = 2$$

Take P(2,3) as the point for using point slope form.
(y-b)=m(x-a)
(y-3)=2(x-2)
y=2x-1 is the required equation of line.

After finding the equation, substitute P and Q coordinates in the equation of line one by one and check if they satisfy the equation. ■

[1] Locus is just the path of a point (x,y) satisfying some given conditions

4.3.4 Equation of Line when the slope and y-intercept is given



In the picture given above, say that the line cuts the y-axis at c. Then, c is known as the y-intercept. If the slope of line is m, since we know it passes through (0,c), using point slope form, the equation of line will be

$$y = m(x-0)+c$$
$$y = mx+c.$$

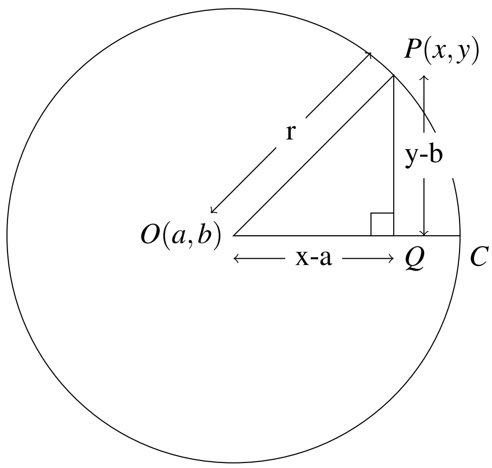
which is the required eqaution of line in slope-intercept form.

Now, you see that y=ax+b is actually a line in 2D plane with ’a’ as slope of the line and ’b’ as y-intercept of line. What do you think will be $y = ax^2+bx+c$? This equation gives a parabola which we will dicuss in the next part: Functions.

4.3.5 Equation of Plane in 3D and beyond

Just like ax+by=c represents the line in 2D surface, ax+by+cz=d represents a plane surface in 3D space. ax+by+cz+dw=e represents a 3D space in 4D. More of this is left for readers to explore.

4.3.6 Circles and Pythagoras’ Theorem

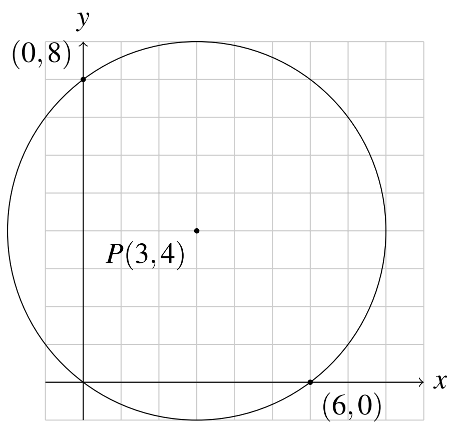


Moving on to circles, we invoke Pythagoras’ theorem to derive their equations. A circle centered at point O(a, b) with radius r has an equation based on any point P(x, y) on its circumference. The relationship between the coordinates of O and P and the radius r is expressed as $(x-a)^2+(y-b)^2 = r^2$, encapsulating the circle’s geometry.

Theorem 4.2 — Standard Equation of Circle. The equation of a circle with centre at (a,b) and radius r is $(x-a)^2+(y-b)^2=r^2$

■ **Example 4.8** Find the equation of circle with centre at (3,4) and radius 5 units. ■

For example, a circle centered at O(3, 4) with a radius of 5 units has the equation $(x-3)^2+(y-4)^2=25$.



Questions such as whether the circle intersects the axes can be answered using this equation. At the y-axis (x = 0), the equation simplifies to $(y-4)^2=16$, revealing the intersection points at (0, 0) and (0, 8) . Similarly, for the x-axis (y = 0),the equation simplifies to $(x-3)^2=9$ letting us know the intersections are at (0, 0) and (6, 0) .

Through this examination, we've seen how the algebraic equations of lines and circles capture their geometric properties. These equations allow us to analyze gradients(slopes), intercepts, and the application of Pythagoras' theorem to circles. The interplay between algebra and geometry is a powerful tool in understanding the world around us.

Please attempt the exercises to have a solid understanding of the materials learnt.

4.3.7 Practice Quiz

Question 1

Find the slope of the line that passes through $P(2,5)$ and $Q(4,9)$.

(a) -2
(b) 4
(c) 1
(d) -4
(e) 2

Question 2

Find the slope of the line that passes through $P(-2,5)$ and $Q(4,-9)$.

(a) $\frac{7}{3}$
(b) -2
(c) $-\frac{2}{3}$
(d) $\frac{-7}{3}$
(e) $\frac{2}{3}$

Question 3

An equation of the line passing through the points $P(2,5)$ and $Q(4,9)$ in the xy-plane is which one of the following?

(a) $y = 2x - 1$
(b) $y = -2x + 9$
(c) $y = 4x - 3$
(d) $y = -2x - 1$
(e) $y = 2x + 1$

Question 4

An equation of the line passing through the points $P(-1,1)$ and $Q(0,2)$ in the xy-plane is which one of the following?

(a) $x + y = 2$
(b) $x - y = 2$
(c) $x + y = 0$
(d) $-x + y = 2$
(e) $x - y = 0$

Question 5

An equation of the line passing through the points $P(2,0)$ and $Q(8,3)$ in the xy-plane is which one of the following?

(a) $y = \frac{x+2}{3}$
(b) $y = \frac{x-2}{3}$
(c) $y = \frac{x-2}{2}$
(d) $y = 2x + 2$
(e) $y = \frac{x+2}{2}$

Question 6

Which one of the following equations describes the circle in the xy-plane with centre $P(2,1)$ and radius 3 units?

(a) $(x-2)^2 + (y-1)^2 = 9$
(b) $(x-2)^2 + (y-1)^2 = 3$
(c) $(x+2)^2 + (y+1)^2 = 3$
(d) $(x-1)^2 + (y-2)^2 = 9$
(e) $(x+2)^2 + (y+1)^2 = 9$

Question 7

Which one of the following equations describes the circle in the xy-plane with centre $P(3,-4)$ and radius 2 units?

(a) $(x-3)^2 + (y+4)^2 = 4$
(b) $(x+3)^2 + (y+4)^2 = 2$
(c) $(x+4)^2 + (y-3)^2 = 4$
(d) $(x+3)^2 + (y-4)^2 = 4$
(e) $(x-4)^2 + (y-3)^2 = 2$

Question 8

Which one of the following points lies on the circle centered at the origin with radius 13?

(a) $(-4,9)$
(b) $(8,5)$
(c) $(5,11)$
(d) $(-5,12)$

(e) $(9,4)$

Question 9

At which one of the following points does the circle with equation $(x+3)^2+(y-4)^2=16$ touch the x-axis?

(a) $(1,4)$
(b) $(1,0)$
(c) $(3,0)$
(d) $(-3,0)$
(e) $(-4,0)$

Question 10

The line $x+y=1$ intersects the circle $(x-2)^2+(y+1)^2=8$ at which two points?

(a) $(1,0)$ and $(-3,4)$
(b) $(2,-1)$ and $(-1,2)$
(c) $(0,1)$ and $(-3,4)$
(d) $(0,1)$ and $(4,-3)$
(e) $(1,0)$ and $(4,-3)$

Answers

The answers will be revealed at the end of the module.

5. Assessment

5.1 Module Quiz

Question 1

Find $f(0)$ when f is the function given by the rule $f(x) = \frac{x-2}{x-1}$.

(a) 0
(b) 1
(c) -1
(d) 2
(e) -2

Question 2

Consider the following polynomial: $p(x) = x^2 + x - 12$.

Which one of the following is correct?

(a) $p(-4) = 0$
(b) $p(-6) = 0$
(c) $p(-1) = 0$
(d) $p(4) = 0$
(e) $p(-3) = 0$

Question 3

Suppose that f and g are functions with the rules $f(x) = 7x - 4$ and $g(x) = x^2$.

Find $(f \circ g)(x) = f(g(x))$.

(a) $7x^2 - 4$
(b) $7x^2 - 16$
(c) $7x^4 - 16$
(d) $(7x - 4)^2$
(e) $49x^2 - 4$

Question 4

Suppose that f and g are functions with the rules $f(x) = 4x - 7$ and $g(x) = x^2$.

Find $(g \circ f)(x) = g(f(x))$.

(a) $16x^2 - 7$
(b) $4x^2 - 7$

(c) $4x^2+49$
(d) $(4x-7)^2$
(e) $4x^2-49$

Question 5

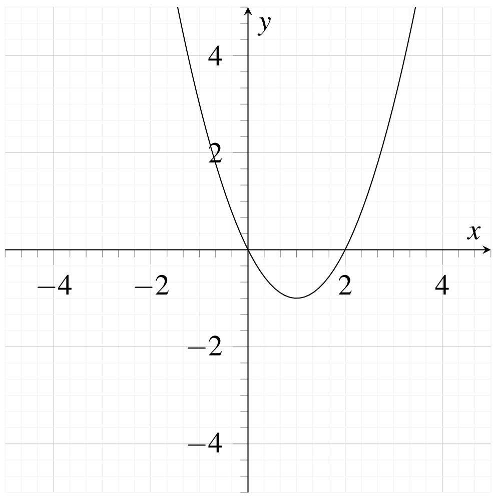
Solve for x given that $x^2+x-6=0$.
(a) $x=2,3$
(b) $x=-2,3$
(c) $x=-5,6$
(d) $x=2,-3$
(e) $x=-2,-3$

Question 6

Solve for x given that $x^2+x-1=0$.
(a) $x=\frac{3}{2},\frac{1}{2}$
(b) $x=\frac{3}{2},\frac{-1}{2}$
(c) $x=\frac{-1\pm\sqrt{5}}{2}$
(d) $x=\frac{1+\sqrt{3}}{2}$
(e) $x=\frac{1\pm\sqrt{5}}{2}$

Question 7

Which one of the functions below corresponds to the following parabola?



(a) $y=x^2+2x-1$
(b) $y=x(x-2)$
(c) $y=x^2-1$
(d) $y=x^2-2x+1$
(e) $y=x(x+2)$

Question 8

Find the range of the function f with the following rule:

$$f(x)=x^2+2x+3$$

(a) range $=[5,\infty)$
(b) range $=[4,\infty)$

(c) range $= [2, \infty)$
(d) range $= [1, \infty)$
(e) range $= [3, \infty)$

Question 9

Find the domain and range of the function f with the following rule:

$$f(x) = 1 + \sqrt{x-1}$$

(a) domain = $[1, \infty)$, range = $[0, \infty)$
(b) domain = $[1, \infty)$, range = $(-\infty, 1]$
(c) domain = $[1, \infty)$, range = $[1, \infty)$
(d) domain = $(-\infty, 1]$, range = $[1, \infty)$
(e) domain = $[-1, \infty)$, range = $[1, \infty)$

Question 10

Suppose that $f(x) = 7x - 2$ for all $x \in \mathbb{R}$. Find $f^{-1}(x)$.
(a) $\frac{2-x}{7}$
(b) $\frac{x-2}{7}$
(c) $\frac{1}{7x-2}$
(d) $\frac{x+2}{7}$
(e) $\frac{7}{x} + 2$

Question 11

Express the angle $\frac{3\pi}{2}$ radians in degrees.
(a) 210°
(b) 270°
(c) 350°
(d) 240°
(e) 300°

Question 12

Which one of the following is the exact value of $\sin\left(\frac{5\pi}{4}\right)$?
(a) $\frac{1}{\sqrt{2}}$
(b) 1
(c) -1
(d) $-\frac{1}{2}$
(e) $-\frac{1}{\sqrt{2}}$

Question 13

Express the angle 120° in radians.
(a) $\frac{3\pi}{4}$ radians
(b) π radians
(c) $\frac{7\pi}{6}$ radians
(d) $\frac{2\pi}{3}$ radians
(e) $\frac{5\pi}{3}$ radians

Question 14

Which one of the following is the exact value of $\tan(330^\circ)$?
(a) $\sqrt{3}$

(b) -1
(c) $-\frac{1}{\sqrt{2}}$
(d) 1
(e) $\frac{1}{\sqrt{3}}$

Question 15

Suppose that θ is an acute angle such that $\tan(\theta) = 2$. Find $\sin(\theta)$.
(a) $\frac{1}{\sqrt{3}}$
(b) $\frac{2}{\sqrt{3}}$
(c) $\frac{1}{3}$
(d) $\frac{3}{\sqrt{5}}$
(e) $\frac{2}{\sqrt{5}}$

Question 16

A kite is attached to the ground by a piece of string of length 35 meters. The kite is flying 14 meters directly above the ground.

Which one of the following is a good estimate for the angle θ of inclination of the piece of string with the ground?
(a) $\sin^{-1}\left(\frac{3}{5}\right)$
(b) $\cos^{-1}\left(\frac{2}{5}\right)$
(c) $\sin^{-1}\left(\frac{2}{5}\right)$
(d) $\cos^{-1}\left(\frac{3}{5}\right)$
(e) $\tan^{-1}\left(\frac{1}{6}\right)$

Question 17

Which one of the following expressions is equivalent to $\left(y^4x^{-3}\right)^{-2}$?
(a) $-\frac{y^8}{x^6}$
(b) $\frac{1}{y^8x^6}$
(c) $\frac{x^5}{y^2}$
(d) $\frac{y^2}{x^5}$
(e) $\frac{x^6}{y^8}$

Question 18

Which one of the following expressions is equivalent to $e^x e^{x^2}$?
(a) $e^{x(2-x)}$
(b) $e^{(x(1+x)}$
(c) e^{2x^2}
(d) $e^{x(2+x)}$
(e) e^{x^3}

Question 19

Solve for x when $3^x = 5$.
(a) $x = \frac{\ln 5}{\ln 3}$
(b) $x = \frac{\ln 3}{\ln 5}$
(c) $x = \frac{\ln 5}{3}$
(d) $x = \ln\left(\frac{5}{3}\right)$
(e) $x = \ln\left(\frac{3}{5}\right)$

Question 20

We have a sample of 100 g of cesium. The half-life of cesium is 30 years. How much of the sample will remain after 100 years (to the nearest tenth of a gram)?

(a) 9.8 g
(b) 9.7 g
(c) 9.6 g
(d) 9.5 g
(e) 9.9 g

6. Answer Key

6.1 Real Line, Decimals and Significant Figures

Answers

1 (c)
2 (d)
3 (b)
4 (e)
5 (c)
6 (e)
7 (d)
8 (b)
9 (b)
10 (d)

6.2 The Theorem of Pythagoras and Properties of the Square Root of 2

6.2.1 Quiz

Answers

1 (d)
2 (c)
3 (c)
4 (b)
5 (b)
6 (d)
7 (d)
8 (e)
9 (a)
10 (b)

6.3 Algebraic Expressions, Surds and Approximations

Answers

1 (a)
2 (a)
3 (d)
4 (e)
5 (e)
6 (d)
7 (d)
8 (c)
9 (c)
10 (d)

6.4 Equations and Inequalities

Answers

1 (a)
2 (b)
3 (a)
4 (b)
5 (e)
6 (c)
7 (b)
8 (b)
9 (b)
10 (d)

6.5 Sign Diagrams, Solution Sets and Intervals

Answers

1 (a)
2 (e)
3 (b)
4 (d)
5 (b)
6 (c)
7 (e)
8 (d)
9 (c)
10 (c)

6.6 Coordinate Systems

Answers

1 (b)
2 (e)
3 (b)
4 (d)
5 (b)
6 (c)
7 (e)
8 (a)
9 (e)
10 (c)

6.7 Distance and Absolute Value

Answers

1 (c)
2 (e)
3 (b)
4 (e)
5 (d)
6 (b)
7 (d)
8 (b)
9 (a)
10 (a)

6.8 Lines and Circles in the Plane

Answers

1 (e)
2 (d)
3 (a)
4 (d)
5 (b)
6 (a)
7 (a)
8 (d)
9 (d)
10 (e)

6.9 Module Quiz

Answers

1 (d)
2 (e)
3 (c)
4 (e)
5 (d)
6 (b)
7 (d)
8 (a)
9 (a)
10 (e)
11 (b)
12 (a)

13 (b)
14 (b)
15 (c)
16 (c)
17 (e)
18 (b)
19 (b)
20 (d)

II Functions (Useful Repertoire)

This module introduces the notion of a function which captures precisely ways in which different quantities or measurements are linked together. The module covers quadratic, cubic and general power and polynomial functions; exponential and logarithmic functions; and trigonometric functions related to the mathematics of periodic behaviour. We create new functions using composition and inversion and look at how to move backwards and forwards between quantities algebraically, as well as visually, with transformations in the xy-plane.

Learning Objectives

- develop fluency with:
- parabolas, quadratic equations and the quadratic formula
- functions, their features and associated terminology, including domain, range, graph, composite and inverse
- exponential and logarithmic functions and applications, such as exponential growth and decay
- circular functions, sine, cosine, tangent, their inverses and applications

7. Introduction

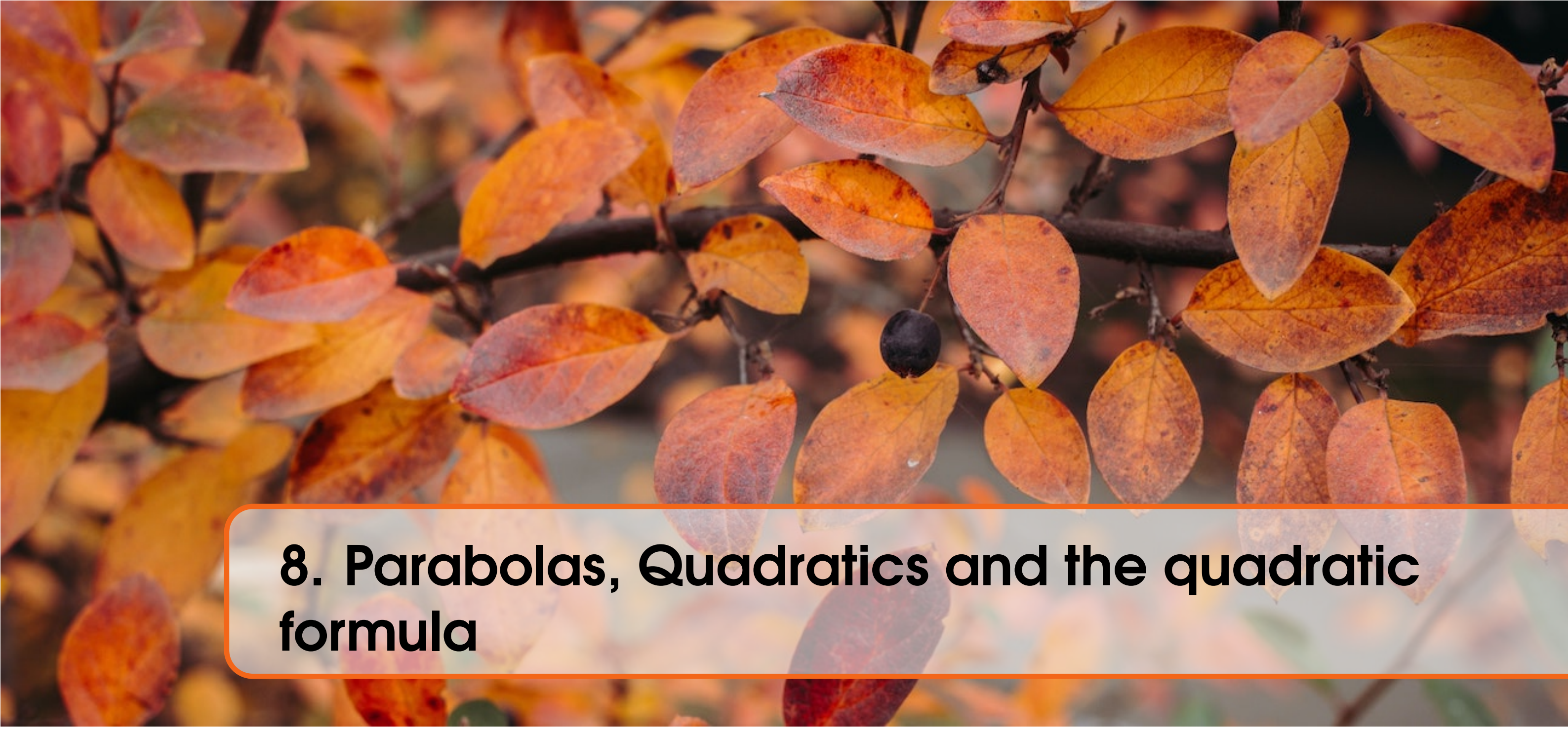
7.1 Introduction to Module 2

Welcome back. Having completed the initial module, you've gained substantial experience with real numbers and the art of algebraic manipulation, particularly within the realms of equations, inequalities, and the estimations involved in the Pythagorean Theorem. Your journey also took you through the Cartesian xy-plane, enhancing your ability to conceptualize the interplay between various quantities.

As we venture into the subsequent module, we delve into the concept of functions—a cornerstone of mathematics that encapsulates the intricate connections between diverse quantities and measurements. Through precise rules and procedures, functions offer a window into the complex interrelations and nuances, which you'll explore by plotting them on the xy-plane.

In the upcoming series of sections, we'll expand our mathematical toolkit to include an array of functions: from quadratic, cubic, and other power and polynomial functions to the exponential and logarithmic functions that characterize growth and decay, as well as the trigonometric functions that underpin the study of periodic phenomena. Moreover, you'll acquire techniques for crafting new functions via composition and inversion, and learn to navigate the relationship between quantities—not only algebraically but also visually, by examining the effects of transformations on the xy-plane.

It is our hope that you'll find this content engaging and thought-provoking, that the materials prove to be a valuable resource, and that the extensive practice and challenges presented by the exercises enrich your learning experience. I eagerly anticipate your ongoing engagement and participation.



8. Parabolas, Quadratics and the quadratic formula

8.1 Parabolas and Quadratics

8.1.1 Parabolas and their transformation

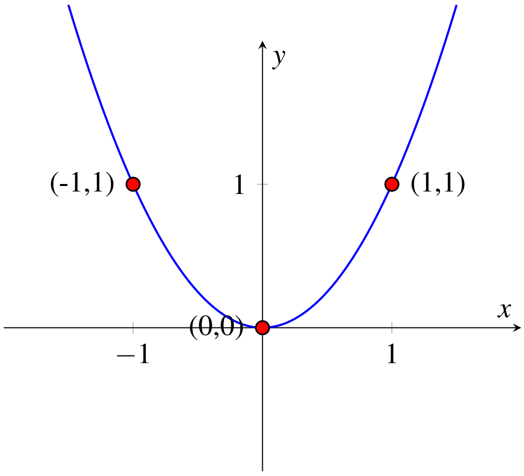
In this section, we delve into the realm of quadratic expressions, demonstrating their visual and geometric interpretations through the lens of parabolas. Building upon our previous discussion on linear equations, particularly the slope-intercept form y = mx + k , we now explore a more complex structure. The linear expression mx + k involves basic arithmetic operations—multiplication and addition. These operations underpin the simplicity of linear equations, designed for ease of use.

$$\text{Linear equation :} y = ax + b$$

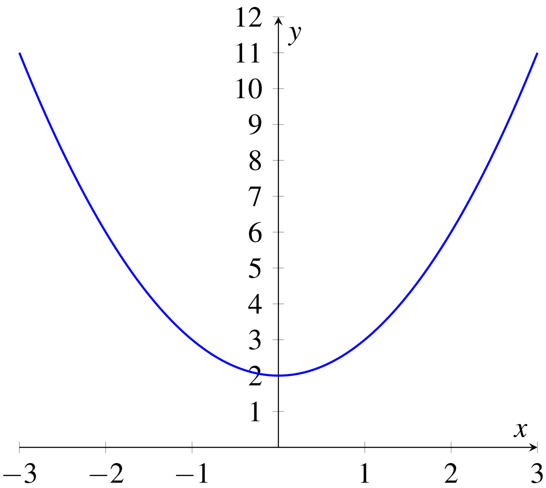
$$\text{Quadratic equation :} y = ax^2 + bx + c$$

As we venture into increased complexity, we introduce an additional term involving x^2 and a new constant, leading to the formation of a quadratic expression. The standard form of a quadratic is y = ax^2 + bx + c , where a , b , and c are constants, and x and y are variables. But also note that 'a' must be non-zero; otherwise, we revert to the linear case. The term 'quadratic' originates from the Latin word for 'square,' reflecting the four sides of a square and the area represented by x^2 .

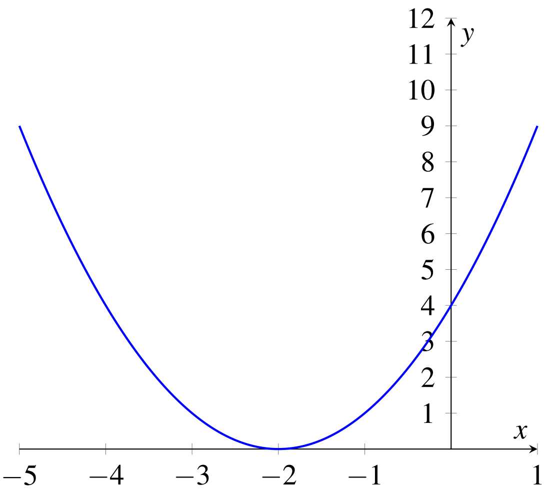
Consider the simple equation y = x^2 , where a = 1 and b and c are zero. Plotting points like (0,0), (1,1), and (-1,1), which satisfy y=x^2, on the Cartesian plane, we connect them with a smooth curve to reveal a parabola.



This curve is pivotal in mathematics. For instance, with a = 1, b = 0, and c = 2, the equation y = x^2 + 2 represents a parabola shifted upward by two units.



When you juxtapose the original parabola y = x^2 with its variant y = x^2 + 4x + 4 , a fascinating shift occurs. The latter, a perfect square, can be expressed as $(x+2)^2$, essentially x^2 with x replaced by x + 2 . This shift explains why the curve moves two units to the left.

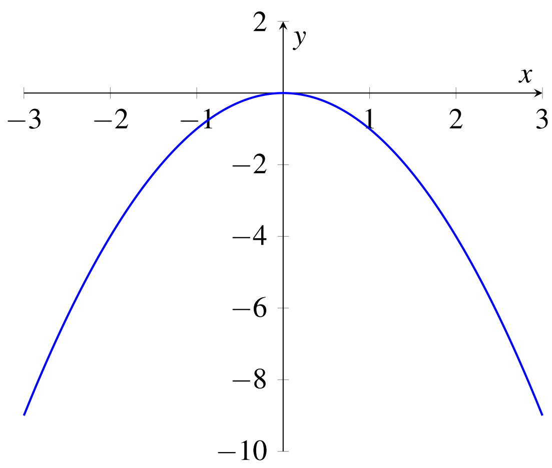


Every quadratic parabola, described by y = ax^2 + bx + c , results from stretching, shifting, or flipping the simplest parabola y = x^2. These transformations are not immediately apparent but can be seen by completing the square.

$$y = ax^2 + bx + c$$
$$y = a(x^2 + \frac{b}{a}x + \frac{c}{a})$$
$$y = a(x^2 + 2.x.\frac{b}{2a} + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a})$$
$$y = a(x + \frac{b}{2a})^2 - (\frac{b^2 - 4ac}{4a})$$
$$(y + \frac{b^2 - 4ac}{4a}) = a(x + \frac{b}{2a})^2$$

Now, how do you get this parabola by some transformations on y=x^2. Move the curve of y=x^2 to the negative x - axis by $\frac{b}{2a}$ and down by $\frac{b^2-4ac}{4a}$. Then, stretch the curve by a factor of 'a'. This way, you will obtain the curve of $y = ax^2 + bx + c$ from y=x^2.

But what if a is negative? How do you stretch by a factor of negative number. If a is negative, the parabola inverts, as if reflected across the x-axis. For example, y = $-x^2$ yields an inverted parabola.



Algebraic manipulations, such as completing the square, reveal the transformations behind these curves. The apex of the parabola, where it changes direction, is crucial for practical applications.

8.1.2 Apex of the Parabola

Let us also discuss how to find the apex of the parabola.
We earlier saw that y=$ax^2 + bx + c$ can be written as

$$(y + \frac{b^2 - 4ac}{4a}) = a(x + \frac{b}{2a})^2 \tag{8.1}$$

Now, for a parabola in this form, when a is positive, y value gets to its minimum at vertex. Thus, to see for what x values, y becomes minimum in eqn. 8.1, we notice that $(x + \frac{b}{2a})^2$ need to approach its minimum value. And the minimum of a square is 0 which occurs when $x = -\frac{b}{2a}$. At $x = -\frac{b}{2a}$, y

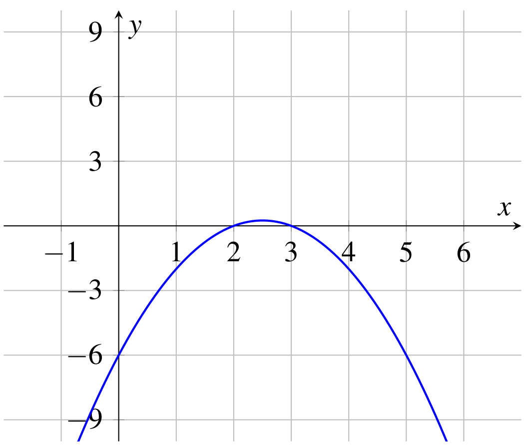
$= \frac{4ac-b^2}{4a}$. Thus, the vertex of the parabola is $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$.
By similar argument, even when a is negative, vertex of the parabola is $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$.

There are myriad perspectives from which to view these curves. With practice, one can swiftly discern characteristics like the direction of a parabola based on the sign of x^2's coefficient and its x-axis intersections through zero factorization.

■ **Example 8.1** Use the tools we have developed till now to describe how the parabola $y = -x^2 + 5x - 6$ should look like.
Solution:
Firstly, since the coefficient of x^2 is negative, we can know that the parabola is open to downwards.
Now if you try to factorize the equation, you get y=-(x-2)(x-3). For y to be 0, either x=2 or x=3.
Thus, the x intercepts of parabola are x=2 and 3.
Now, notice that comparing the equation with the standard equation of parabola $y = ax^2 + bx + c$, we get a=-1, b=5, c=-6. Since a is negative, we can say that the parabola is inverted, scaled by a factor of -1.
Next, we would want find the vertex of the parabola. We know that vertex of parabola is given by $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$. Substituting the values, we get

$$(-\frac{5}{2\times(-1)}, \frac{4\times(-1)\times(-6)-5^2}{4\times(-1)}) = (\frac{5}{2}, \frac{1}{4})$$

We have now collected these facts about the parabola $y = -x^2 + 5x - 6$
i) Inverted
ii) crosses x-axis at 2 and 3
iii) Vertex at $(\frac{5}{2}, \frac{1}{4})$
Based on this, we might be able to draw a rough sketch of the required parabola as follows:



■

I want you to use these facts to make predictions about many different parabolas. And then plot them point by point. In this way, you can see how accurate your predictions become.
If you ever get internet, also explore desmos.com to plot any kind of graph.

Parabolas have intriguing properties, particularly the apex—the turning point. To provide a fuller picture, we will explore the quadratic formula in an upcoming section.

We've laid a solid foundation, introducing quadratic expressions and their corresponding parabolic curves. We've also shown the interplay between various parabolas through shifts and inversions.

Please tackle the exercises at your leisure. Thank you for tuning in, and I eagerly anticipate our next session.

8.1.3 Practice Quiz

Question 1

Evaluate the quadratic expression

$$y = x^2 - 3x + 4$$

when $x = 2$.

(a) 1
(b) 2
(c) 0
(d) -2
(e) -1

Question 2

Evaluate the quadratic expression

$$y = 1 - 2x - 3x^2$$

when $x = -1$.

(a) -4
(b) -2
(c) 3
(d) 2
(e) 0

Question 3

The parabola $y = x^2 + 4$ is obtained from the parabola $y = x^2$ by which one of the following shifts?

(a) upwards by 2 units
(b) downwards by 4 units
(c) sideways right by 2 units
(d) upwards by 4 units
(e) sideways left by 2 units

Question 4

The parabola $y = x^2 - 4x + 4$ is obtained from the parabola $y = x^2$ by which one of the following shifts?

(a) sideways right by 2 units
(b) downwards by 4 units
(c) upwards by 4 units
(d) sideways left by 2 units
(e) upwards by 2 units

Question 5

If the equation $x^2 - 4x + 1 = 0$ holds for some real number x, then the expression $(x-2)^2$ must evaluate to which one of the following?

(a) 1
(b) 4
(c) 5
(d) -3
(e) 3

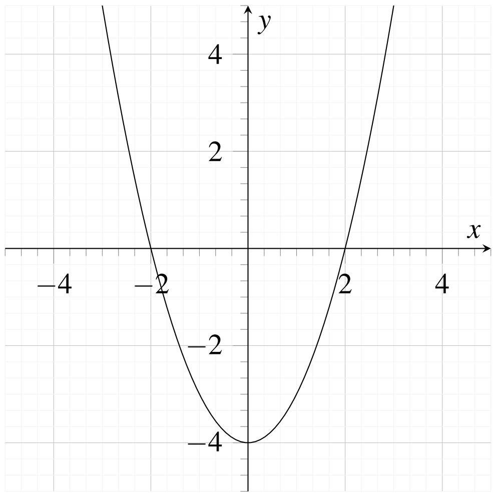
Question 6

The parabola $y = x^2 - 2x + 4$ is obtained from the parabola $y = x^2$ by which one of the following sequences of moves?

(a) movement to the left 1 unit followed by movement upwards by 3 units
(b) movement upwards by 3 units followed by movement to the left 1 unit
(c) movement to the right 1 unit followed by movement upwards by 3 units
(d) movement downwards by 3 units followed by movement to the right 1 unit
(e) movement upwards by 4 units followed by movement to the left 2 units

Question 7

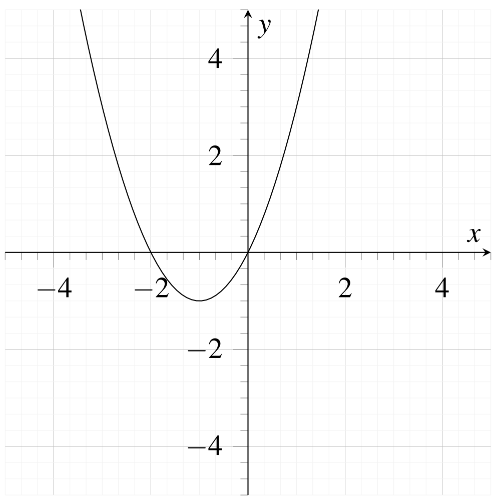
Which one of the equations below corresponds to the following parabola?



(a) $y = x^2 + 4$
(b) $y = (x-2)^2$
(c) $y = (x+2)^2$
(d) $y = x^2 + 2x - 4$
(e) $y = x^2 - 4$

Question 8

Which one of the equations below corresponds to the following parabola?

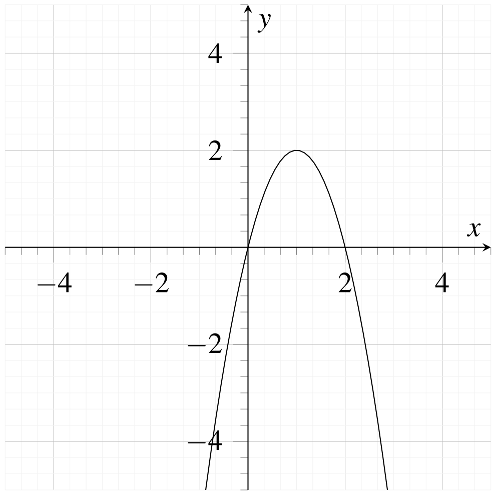


(a) $y = x(x-2)$

(b) $y = x^2 - 2x + 1$
(c) $y = x(x+2)$
(d) $y = x^2 + 2x - 1$
(e) $y = (x-1)^2 - 1$

Question 9

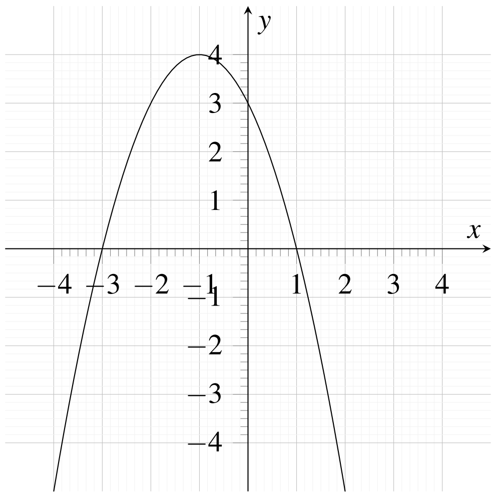
Which one of the equations below corresponds to the following parabola?



(a) $y = -2(x+1)^2 + 2$
(b) $y = -(x-1)^2 + 2$
(c) $y = 2(x-1)^2 + 2$
(d) $y = 2x(2-x)$
(e) $y = x(2-x)$

Question 10

Which one of the equations below corresponds to the following parabola?



(a) $y = (x-1)(x+3)$
(b) $y = (x-1)^2 + 4$
(c) $y = -(x+1)^2 + 4$
(d) $y = -(x-1)^2 + 4$
(e) $y = -(x-1)(x+3)$

Answers

The answers will be revealed at the end of the module.

8.2 The Quadratic formula

8.2.1 "Completing Square" Method

"In this section, we're going to delve into the intricacies of the quadratic formula, a fundamental tool in solving any quadratic equation. We'll explore the origins of this formula, which is derived from a method you might remember from earlier lessons called completing the square. We start with a quadratic equation that seems simple but is actually tied to some profound and far-reaching ideas that extend even beyond the field of mathematics, which I will reveal as we progress.

$$x^2 - x - 1 = 0 \tag{8.2}$$

The equation we're examining doesn't lend itself easily to factorization, prompting us to seek an algebraic trick to isolate x . It's noteworthy that the coefficient of x is negative one, a detail that proves to be significant in our upcoming discussion. Our initial step is to add one to both sides of the equation, a strategic move towards isolating x .

$$x^2 - x = 1$$

$$\underbrace{x^2 + 2.x.(-\frac{1}{2}) + (-\frac{1}{2})^2}_{\text{Perfect Square}} = 1 + (-\frac{1}{2})^2$$

Following this, we add the square of minus one-half, which is the square of half the coefficient of x , transforming the left-hand side into the square of $(x - \frac{1}{2})$, while the right-hand side simplifies to $(\frac{5}{4})$. This maneuver leverages the algebraic identity $(x+k)^2 = x^2 + 2xk + k^2$, where in our case, k is minus one-half, and thus 2k equates to minus one, the coefficient of x in our original equation. This leaves us with

$$(x - \frac{1}{2})^2 = \frac{5}{4}$$

, where x is now singularly represented, in contrast to its dual presence in the original equation 8.2. To solve, we take the square roots of both sides,

$$(x - \frac{1}{2}) = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$$

then add one-half to each, arriving at

$$x = \frac{1 \pm \sqrt{5}}{2}$$

which neatly solves the equation.

8.2.2 Quadratic Formula

Now, let's generalize this approach to tackle any quadratic equation in the form $ax^2 + bx + c = 0$. We start by subtracting c from both sides,

$$ax^2 + bx = -c$$

and for convenience, we divide everything by a to ensure the coefficient of x^2 is one.

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

We then rewrite the left-hand side to emphasize the coefficient of x as $\frac{b}{a}$.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Continuing, we add the square of half the coefficient of x , which is $\frac{b}{2a}$, to both sides, resulting in the left-hand side becoming a perfect square,

$$\underbrace{x^2 + 2.x.\frac{b}{2a} + (\frac{b}{2a})^2}_{\text{Perfect Square}} = -\frac{c}{a} + (\frac{b}{2a})^2$$

specifically $(x + \frac{b}{2a})^2$, while the right-hand side sums up to $\frac{b^2-4ac}{4a^2}$.

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

As a result, we have an equation where x appears only once. We proceed by taking the square root of both sides,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

The denominator of the RHS simplifies to 2a , since the square root of four is two, and the square root of a^2 is a if a is positive, or -a if a is negative. However, in the latter case, the minus sign can be absorbed into the $\pm$ in the numerator.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

So, we subtract $\frac{b}{2a}$ from both sides of the equation and express it as a single fraction with 2a in the denominator.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{8.3}$$

The result 8.3 is the well-known quadratic formula. It's important to note that the formula involves a surd expression, which is only meaningful if the content under the square root sign is non-negative, meaning it must be positive or zero, as the real number system does not permit square roots of negative numbers. In advanced mathematics, you'll encounter complex numbers, which allow for the consideration of square roots of negative numbers, making the quadratic formula applicable even if $b^2 - 4ac$ is negative.

This formula proves to be incredibly useful and potent, but for the scope of this course, we won't concern ourselves with complex numbers. It's always prudent to test a general formula against specific, simpler cases where the answers are known or can be easily determined by other methods. Revisiting the equation we previously solved by ad hoc methods, we now apply the quadratic formula where a = 1 and b and c are both minus one.

■ **Example 8.2** Solve using Quadratic formula: $x^2 - x - 1 = 0$
Solution: Notice that a=1,b=-1,c=-1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1 \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2 \times 1}$$

■

The solution simplifies to $\frac{1 \pm \sqrt{5}}{2}$, aligning with our earlier findings, which is quite satisfying. Let's consider another example that we can solve by factorization.

■ **Example 8.3** Solve using Quadratic formula: $x^2 + 4x - 12 = 0$ ■

Noting that the left-hand side factorizes as (x + 6)(x - 2) , leading to x + 6 = 0 or x - 2 = 0 , and thus x = -6 or x = 2 . However, applying the quadratic formula, we find $x = \frac{-4 \pm \sqrt{4^2 - 4(-12)}}{2}$, which simplifies to $x = -2 \pm 4$, yielding x = 2 or x = -6 , consistent with the solutions obtained through factorization, which is also quite pleasing.

8.2.3 Golden Ratio (ϕ)

Let's return to the very first quadratic equation we examined today, which in fact is related to something known as the golden ratio, an important proportion traditionally utilized by artists for selecting canvas sizes.

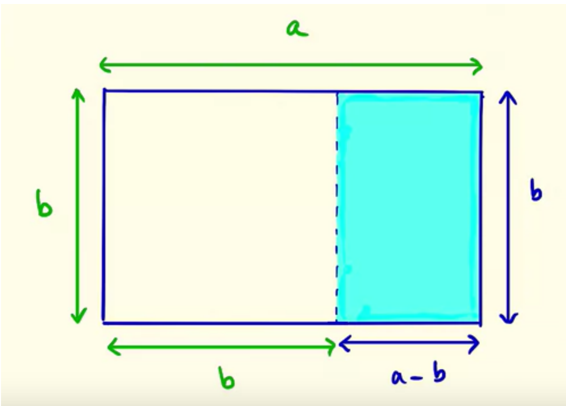


Figure 8.1: Rectangle whose side lengths are in Golden Ratio

$$\frac{a}{b} = \frac{b}{a - b}$$

The associated rectangle has a unique property: when you remove the square formed by the shortest side, the remaining section, colored blue in our diagram, is another rectangle with the same proportions as the original, but now oriented vertically. This same proportion is known as the Golden Ratio. This differs from the A-series of paper we studied in an earlier section 2.3.2 Surds, where dividing the rectangle in half yielded two similar rectangles and the propotion was $\sqrt{2}$.

[0]Image 8.1 taken from MOOC

The golden ratio is the fraction obtained by dividing the longer side length by the shorter side length of the original rectangle. Let's denote the longer side length as 'a' and the shorter side length as 'b' . The square we remove has side lengths of 'b' , and the remaining blue rectangle has a longer side length of 'b' and a shorter side length of 'a - b' .

We'll assign the golden ratio the symbol X , representing the unknown quantity we aim to find. Thus, $X = \frac{a}{b}$ for the larger rectangle and $\frac{b}{a-b}$ for the smaller one.

$$x = \frac{a}{b} = \frac{b}{a-b}$$

Dividing the top and bottom of second ratio by b , we arrive at $\frac{1}{\frac{a}{b}-1}$, which simplifies to $\frac{1}{X-1}$.

$$X = \frac{1}{X-1}$$

To determine X , we transform this into a quadratic equation by multiplying both sides by X - 1 ,

$$X(X-1) = 1$$

expanding the left-hand side, and then subtracting one from both sides.

$$X^2 - X - 1 = 0$$

This yields the same equation we started with, but with X instead of x . Solving this equation gives us two solutions, $(\frac{1\pm\sqrt{5}}{2})$. However, the solution involving the minus sign, $\frac{1-\sqrt{5}}{2}$, is negative, and since the golden ratio must be positive, we conclude that it is $\frac{1+\sqrt{5}}{2}$.

Entering this into a calculator, we can begin to write out its decimal expansion. Because it includes $\sqrt{5}$, this number is irrational, meaning its decimal expansion is non-recurring. In a previous lesson, we explored continued fractions while investigating the properties of $\sqrt{2}$, resembling a packet of oats in image 2.4 that recursively refers to itself ad infinitum. We can apply a similar approach to the golden ratio, which has an expansion that starts with $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$, and so on.

$$x^2 - x - 1 = 0$$

$$x^2 = x + 1$$

$$x = 1 + \frac{1}{x}$$

$$x = 1 + \frac{1}{1+\frac{1}{x}}$$

$$x = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{x}}}$$

$$x = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}}}$$

This expansion is particularly elegant, utilizing only the number one and representing, in a sense, the simplest non-trivial continued fraction imaginable. The golden ratio is also referred to as the divine proportion and is highly esteemed in the art world.

For instance, Leonardo da Vinci's Mona Lisa, arguably the most renowned painting in history, features the divine proportion in various aspects throughout the artwork.

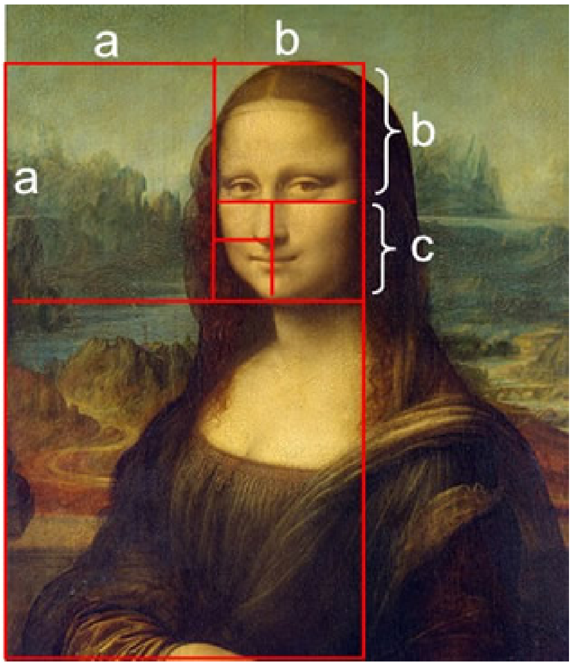


Figure 8.2: Golden Ratio in Monalisa by Da Vinci

Da Vinci, both a scientist and an artist, was adept at replicating anatomical proportions as they naturally occur.

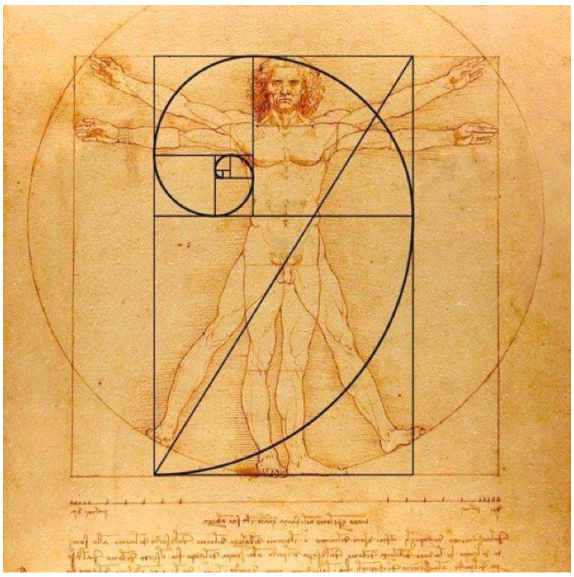


Figure 8.3: Divine Proportion / Golden Ratio in the art of Da-Vinci

Once you become cognizant of the divine proportion, you'll start to notice its prevalence everywhere.

[0]Image 8.2 taken from https://mathbitsnotebook.com/Geometry/Modeling/MDDesignPractice.html
[0]Image 8.3 taken from https://scottwilsonarchitect.com/

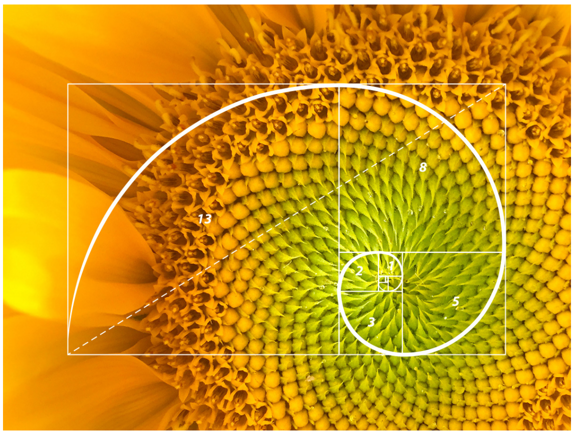


Figure 8.4: Golden Ratio Spiral in nature

Its natural occurrence is linked to the Fibonacci Sequence, which holds a pivotal place in mathematics and computational theory. These topics and their connections to the golden ratio will become more apparent as you delve deeper into mathematics beyond this course.

Today, we've traversed a considerable amount of territory. We've established the quadratic formula and its derivation via completing the square, applied it to several examples, including one of the most celebrated uses of a quadratic equation: finding a surd expression for the golden ratio. There are additional details in the exercises. Please take the time to tackle the exercises. Thank you for watching, and I eagerly anticipate our next encounter.

8.2.4 Practice Quiz

Question 1

Find the values of x that satisfy the following quadratic equation:

$$x^2 - 7x + 12 = 0$$

(a) $-3, -4$
(b) $3, -4$
(c) $-3, 4$
(d) $2, 5$
(e) $3, 4$

Question 2

Find the values of x that satisfy the following quadratic equation:

$$6x^2 + x - 2 = 0$$

(a) $-\frac{1}{6}, \frac{1}{2}$
(b) $\frac{1}{6}, \frac{1}{2}$
(c) $-\frac{1}{3}, \frac{1}{3}$
(d) $\frac{1}{3}, -\frac{1}{3}$
(e) $\frac{1}{3}, \frac{1}{3}$

Question 3

Find the values of x that satisfy the following quadratic equation:

$$2x^2 + 3x - 1 = 0$$

(a) $\frac{3\pm\sqrt{7}}{4}$
(b) $1, \frac{1}{2}$
(c) $-1, -\frac{1}{2}$
(d) $\frac{-3\pm\sqrt{7}}{4}$
(e) 4

Question 4

Find the value or values of x that satisfy the following quadratic equation:

$$4x^2 - 12x + 9 = 0$$

(a) ± 3
(b) $\pm\frac{3\pm\sqrt{3}}{2}$
(c) $\pm 3\sqrt{3}$
(d) 3
(e) ± 3

Question 5

Find the value or values of x that satisfy the following equation:

$$\frac{1}{x-1} + \frac{1}{x} = 3$$

(a) $\frac{3\pm\sqrt{13}}{10}$
(b) $\frac{3}{4}$
(c) $\frac{5\pm\sqrt{13}}{10}$
(d) 2
(e) $\pm\frac{3\sqrt{13}}{10}$

Question 6

Which one of the following quadratic equations has $x = -2$ and $x = 5$ as the two solutions?
(a) $x^2 - 3x - 10 = 0$
(b) $x^2 + 3x - 10 = 0$
(c) $x^2 - 7x - 10 = 0$
(d) $x^2 - 7x + 10 = 0$
(e) $x^2 + 3x + 10 = 0$

Question 7

Which one of the following quadratic equations has $x = 2 \pm \sqrt{3}$ as the two solutions?
(a) $x^2 + 4x - 1 = 0$
(b) $x^2 - 4x - 1 = 0$
(c) $x^2 - 2x - 1 = 0$
(d) $x^2 - 2x + 1 = 0$
(e) $x^2 - 4x + 1 = 0$

Question 8

Which one of the following quadratic equations has $x = \frac{5}{6}$ as the unique solution?
(a) $x^2 - 5x + 6 = 0$
(b) $36x^2 - 25 = 0$
(c) $36x^2 - 60x + 25 = 0$
(d) $36x^2 - 360x + 25 = 0$
(e) $x^2 - 6x + 5 = 0$

Question 9

Which one of the following quadratic equations has no solution?
(a) $x^2 - x - 1 = 0$
(b) $x^2 - x + 1 = 0$
(c) $x^2 - 2x + 1 = 0$
(d) $x^2 + 2x + 1 = 0$
(e) $x^2 + 2x - 1 = 0$

Question 10

The following equation defines a circle in the xy-plane:

$$x^2 - 4x + y^2 + 6y - 12 = 0$$

Apply the technique of completing the square to a quadratic involving x, and also to a quadratic involving y, to find its centre and radius.
(a) centre $(2, -3)$, radius 4
(b) centre $(2, -3)$, radius 5
(c) centre $(-2, 3)$, radius 5
(d) centre $(-3, 2)$, radius 5
(e) centre $(2, 3)$, radius 4

Answers

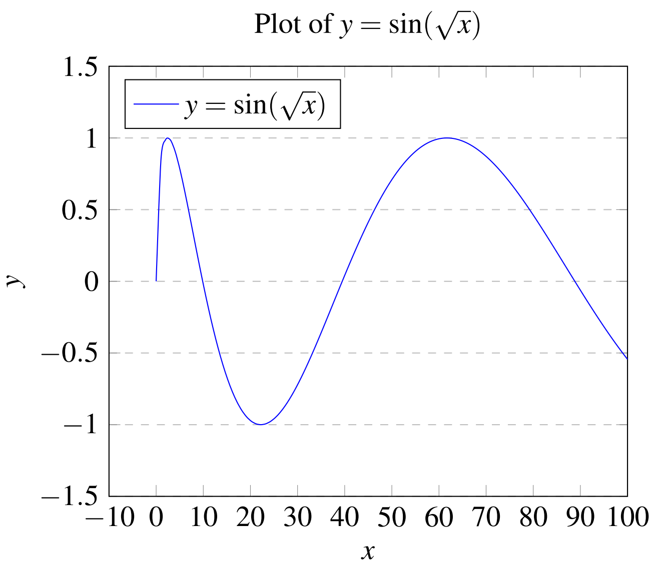
The answers will be revealed at the end of the module.

9. Functions, Composition and Inversion

9.1 Functions as rules, with domain, range and graph

9.1.1 Domain and Range of a Function

In this section, we're going to unpack the concept of functions, which are essential mathematical constructs for defining precise relationships between varying quantities. Functions are particularly crucial when it comes to making predictions or modeling real-world phenomena. They are typically represented by formulas and are only applicable for certain values, which introduces us to the concept of the domain—the set of all valid inputs for a function. Correspondingly, the range is the set of all possible outputs that a function can generate.



In the above function, you can see that only the non negative real numbers are the valid inputs. Thus, the domain of the function is $[0, \infty)$.
For range, also notice that the possible output of the function is also always between -1 and 1. Thus, the range of the function is $[-1, 1]$.
The Cartesian plane serves as a visual aid to illustrate these relationships, with the graph of a function offering a powerful pictorial representation that can provide significant insight. With

practice, you'll become adept at sketching or even mentally visualizing these graphs.

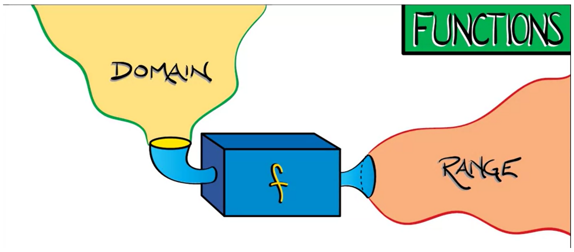


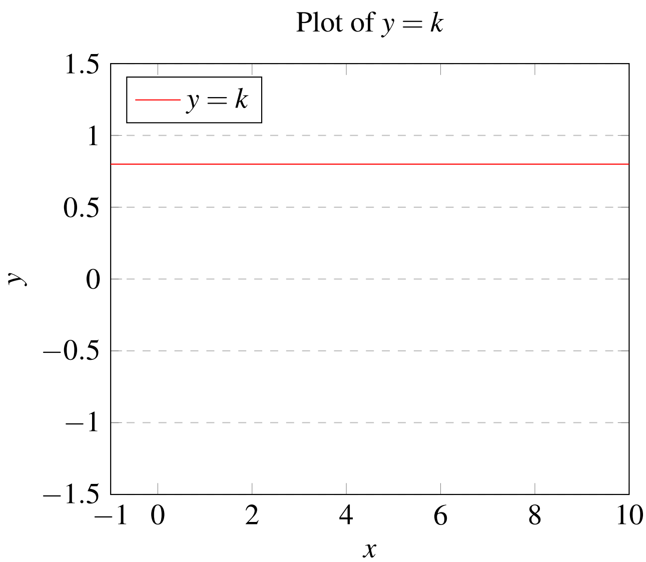
Figure 9.1: Function as a machine

A function, denoted as f , is a rule that assigns to each input x (usually a real number) an output, also typically a real number, denoted by f(x) . You can think of f as a machine that processes numbers: you input x , and it outputs f(x) .

9.1.2 Linear function

For instance, consider a linear function defined by the rule f(x) = mx + k , where m and k are constants. This function takes an input x , multiplies it by m , and then adds k . Graphically, this equation represents a straight line on the xy -plane, with m as the slope and k as the y-intercept. If we choose m = 2 and k = 3 , then f(x) = 2x + 3 .

Evaluating this function is straightforward. For example, f(0) would be $2\times 0+3$, which equals 3. Similarly, f(1) equals 5, and f(6) equals 15. You can apply the same process to other functions with different constants to see how the outputs vary.



The domain of a function is the set of all real numbers that are valid inputs for the function. For a linear function, as long as the slope m is not zero, the domain is all real numbers, and so is the range. However, if m = 0 , the function simplifies to y = k , and the range is just the single number k , since the output will always be the same regardless of the input.

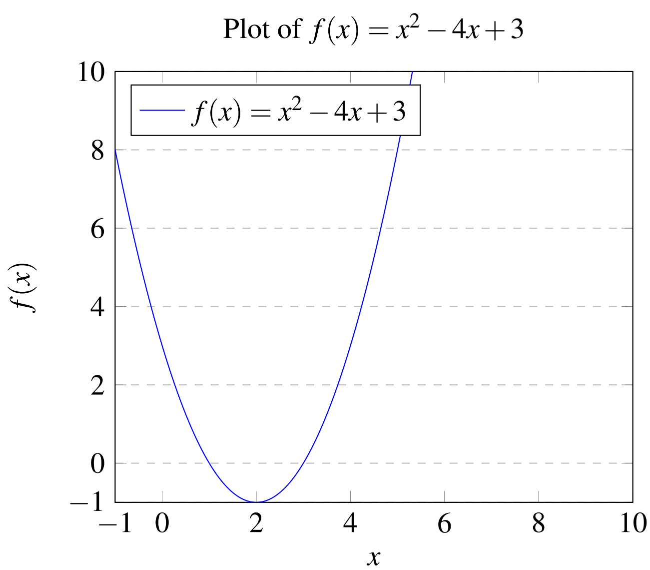
[0]Image taken from MOOC (Single Variable Calculus by Professor Ghrist)

The graph of a function f consists of all the points (x, y) on the Cartesian plane where y = f(x) . For a linear function with a non-zero slope, the graph is a straight line, and the range covers all real numbers. If the slope is zero, the graph is a horizontal line at y = k , and the range is just k .

9.1.3 Quadratic Function

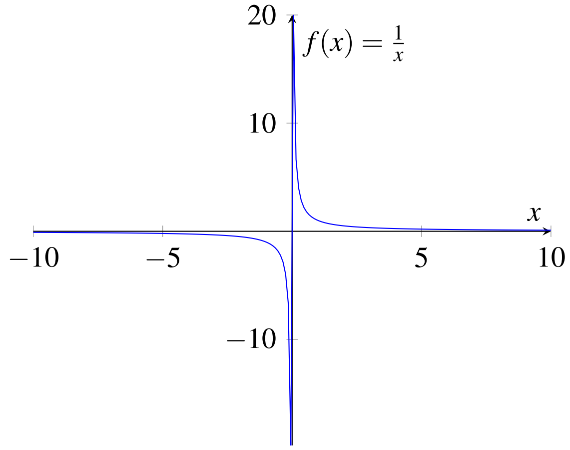
Now, let's consider a quadratic function, such as $f(x) = x^2 - 4x + 3$ By evaluating this function at various inputs, we find that f(0) = 3 , f(1) = 0 , and so on. The domain of any quadratic function is all real numbers. To determine the range, we can complete the square, revealing that f(x) can be rewritten as $(x-2)^2 - 1$, which is always greater than or equal to -1. Thus, the range is all real numbers greater than or equal to -1.

The equation can be factorized as f(x)=(x-3)(x-1).The graph of this quadratic function is a parabola that intersects the x -axis at x = 3 and x = 1 , with its vertex at (2, -1) , indicating the minimum value of the function. The vertex can be found using the formula $(-\frac{b}{2a}, \frac{4ac-b^2}{4a}) = (-\frac{-4}{2}, \frac{4\times3-(-4)^2}{4}) = (2,-1)$



9.1.4 Hyperbola

Another important family of functions includes hyperbolas, exemplified by the reciprocal function $f(x) = \frac{1}{x}$.



The graph of this function forms two disconnected branches, one in the positive quadrant and one in the negative quadrant, with the x - and y - axes serving as asymptotes.[1] The domain of the reciprocal function is all non-zero real numbers, and the range is the same.
Mathematically, we represent this set in any of the following ways:

$$\{x \in \mathbb{R} | x \neq 0\}$$

$$R \backslash \{0\}$$

$$(-\infty, 0) \cup (0, \infty)$$

In summary, we've discussed functions as rules that produce outputs from inputs, the domain as the set of permissible inputs, the range as the set of outputs produced, and the graph of the function as a set of points (x, y) in the Cartesian plane. We've illustrated these concepts using linear functions, whose graphs are lines; quadratic functions, whose graphs are parabolas; and the reciprocal function, whose graph is a hyperbola with two branches and asymptotic behavior.

Please review the material and when you feel ready, try your hand at the exercises. Thank you for engaging with this material, and I look forward to our continued journey in mathematics.

9.1.5 Practice Quiz

Question 1

Find the value $f(0)$ when f is the function given by the following rule:

$$f(x) = 4x - 7$$

(a) 0
(b) 4
(c) -7
(d) -3
(e) 3

[1] An asymptote is a straight line that a curve approaches as it heads towards infinity. The distance between the curve and the asymptote tends to zero, and they never intersect at any finite point. Asymptotes can be horizontal, vertical, or oblique (slant), depending on their orientation relative to the curve.

Question 2

Find the value $f(3)$ when f is the function given by the following rule:

$$f(x) = 2x^2 - 3x + 7$$

(a) 25
(b) 7
(c) 34
(d) 22
(e) 16

Question 3

Find the value $f(-1)$ when f is the function given by the following rule:

$$f(x) = \frac{x-1}{x-2}$$

(a) 2
(b) 0
(c) $\frac{3}{2}$
(d) -1
(e) 1

Question 4

Consider the following rule for a function f:

$$f(x) = \frac{x}{x-2}$$

For which input x is the rule undefined?
(a) 2
(b) -2
(c) 1
(d) -1
(e) 0

Question 5

Find the domain and range of the function f given by the following rule:

$$f(x) = x^2 - 1$$

(a) domain = $[-1, \infty)$, range = $\mathbb{R}$
(b) domain = $\mathbb{R}$, range = $\mathbb{R}$
(c) domain = $\mathbb{R}$, range = $[-1, \infty)$
(d) domain = $\mathbb{R}$, range = $(-\infty, -1]$
(e) domain = $\mathbb{R}$, range = $[-1, 1]$

Question 6

Find the domain and range of the function f given by the following rule:

$$f(x) = 1 - x^2$$

(a) domain = $\mathbb{R}$, range = $[-1, 1]$
(b) domain = $\mathbb{R}$, range = $[-1, \infty)$
(c) domain = $\mathbb{R}$, range = $[1, \infty)$
(d) domain = $\mathbb{R}$, range = $(-\infty, 1]$
(e) domain = $(-\infty, 1]$, range = $\mathbb{R}$

Question 7

Find the range of the function f with the following rule:

$$f(x) = x^2 + 2x + 5$$

(a) $(-\infty, \infty)$
(b) $[-1, \infty)$
(c) $[4, \infty)$
(d) $[3, \infty)$
(e) $[5, \infty)$

Question 8

Find the range of the function f with the following rule:

$$f(x) = 5 - 2x - x^2$$

(a) range = $(-\infty, 5]$
(b) range = $(-\infty, 3]$
(c) range = $(-\infty, 6]$
(d) range = $(-\infty, 4]$
(e) range = $(-\infty, 2]$

Question 9

Find the domain and range of the function f given by the following rule:

$$f(x) = 2 + \sqrt{x-1}$$

(a) domain = $[2, \infty)$, range = $[1, \infty)$
(b) domain = $(-\infty, 1]$, range = $(2, \infty)$
(c) domain = $[1, \infty)$, range = $[0, \infty)$
(d) domain = $[1, \infty)$, range = $[2, \infty)$
(e) domain = $[2, \infty)$, range = $(-\infty, 1]$

Question 10

Find the domain and range of the function f given by the following rule:

$$f(x) = 2 + \frac{1}{x}$$

(a) domain = $\mathbb{R} \setminus \{1\}$, range = $\mathbb{R} \setminus \{0\}$
(b) domain = $\mathbb{R} \setminus \{1\}$, range = $\mathbb{R} \setminus \{2\}$
(c) domain = $\mathbb{R} \setminus \{2\}$, range = $\mathbb{R} \setminus \{1\}$
(d) domain = $(-\infty, 1) \cup (1, \infty)$, range = $(2, \infty)$
(e) domain = $(1, \infty)$, range = $(2, \infty)$

Answers

The answers will be revealed at the end of the module.

9.1.6 Practice Quiz

Question 1

Rewrite $\sqrt[3]{441}$ as a fractional power:

(a) $441^{\frac{1}{2}}$

(b) $441^{\frac{1}{3}}$
(c) $441^{\frac{1}{4}}$
(d) $21^{\frac{1}{3}}$
(e) 441^{3}

Question 2

Simplify the following expression:

$$8^{\frac{4}{3}} \cdot 4^{\frac{1}{2}} \cdot 2^{\frac{5}{6}}$$

(a) 4
(b) 16
(c) 8
(d) 32
(e) 64

Question 3

Simplify the following expression:

$$\frac{3^{\frac{1}{3}} \cdot 3^{\frac{1}{2}}}{3^{\frac{1}{6}}}$$

(a) $\sqrt{3}$
(b) 9
(c) 1
(d) 3
(e) $3\sqrt{3}$

Question 4

Find the degree of the following polynomial:

$$p(x) = 4 + 2x - 3x^2 - 7x^3$$

(a) 4
(b) 2
(c) 0
(d) -3

Question 5

Consider the following polynomial:

$$p(x) = x^2 + x - 6$$

Which one of the following is correct?

(a) $p(3) = 0$
(b) $p(-1) = 0$
(c) $p(0) = 0$
(d) $p(-2) = 0$
(e) $p(-6) = 0$

Question 6

Consider the following polynomial:

$$p(x) = x^2 + x - 6$$

Find the polynomial $q(x)$ such that $p(x) = (x-2)q(x)$.
(a) $q(x) = x - 4$
(b) $q(x) = x - 6$
(c) $q(x) = x - 3$
(d) $q(x) = x - 2$
(e) $q(x) = x + 3$
(f) $q(x) = x + 6$

Question 7

Consider the following polynomial:

$$p(x) = 3x^2 - 14x - 5$$

Find the polynomial $q(x)$ such that $p(x) = (x-5)q(x)$.
(a) $q(x) = 3x + 9$
(b) $q(x) = 3x - 9$
(c) $q(x) = 3x + \frac{5}{3}$
(d) $q(x) = 3x + 1$
(e) $q(x) = 3x - 1$

Question 8

Consider the following polynomial:

$$p(x) = 2x^3 - 3x^2 - 11x + 6$$

Find the polynomial $q(x)$ such that $p(x) = (x+2)q(x)$.
(a) $q(x) = 2x^2 - 7x + 3$
(b) $q(x) = 2x^2 + 7x + 3$
(c) $q(x) = 2x^2 - 5x + 3$
(d) $q(x) = 2x^2 + 5x - 3$
(e) $q(x) = 2x^2 - 5x - 3$

Question 9

Consider the following polynomial:

$$p(x) = 2x^3 - 3x^2 - 11x + 6$$

Find the polynomial $q(x)$ such that $p(x) = (x-3)q(x)$.
(a) $q(x) = 2x^2 + 3x + 2$
(b) $q(x) = 2x^2 - 3x + 2$
(c) $q(x) = 2x^2 - 5x + 2$
(d) $q(x) = 2x^2 + 5x + 2$
(e) $q(x) = 2x^2 - 3x - 2$

Question 10

Find the solution set for the following equation:

$$2x^3 - 3x^2 - 11x + 6 = 0$$

(a) $\{1,2,-3\}$
(b) $\{\frac{1}{2},2,-3\}$
(c) $\{1,-2,3\}$
(d) $\{2,-3,-1\}$
(e) $\{-1,2,-3\}$

Answers

The answers will be revealed at the end of the module.

9.2 Polynomial and Power functions

9.2.1 Power Functions

In this lesson, we explore polynomial functions, which include linear and quadratic functions as subsets, and power functions that emphasize specific powers of x. We start with basic powers and gradually expand to complex concepts. A power function is defined as $y = x^n$, where x is the independent variable, and n is a constant exponent that determines the power of x, resulting in the output $y = x^n$.

We begin with simple cases where n is a positive integer, leading to x^n being the product of x multiplied by itself n times, termed the nth power of x. For example, $n = 1$ gives us the identity function $y = x$, which returns the input unchanged. With $n = 2$, we have the square function $f(x) = x^2$, and for $n = 3$, the cube function, which calculates the volume of a cube with side x.

$$f(x) = x^n$$

$$y = x^n$$

$$y^{\frac{1}{n}} = (x^n)^{\frac{1}{n}}$$

$$x = y^{\frac{1}{n}}$$

Reversing x and y ,

$$y = x^{\frac{1}{n}}$$

$$f^{-1}(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$$

Reversing the nth power function yields the nth root function, denoted by a root sign with a small n, indicating y is the nth root of x.

$$\sqrt[2]{x^2} = x$$

$$\sqrt[3]{x^3} = x$$

The square root, for $n = 2$, reverses the square function, while the cube root, for $n = 3$, reverses the cube function. This concept is known as function inversion in mathematics, and we'll explore inverse functions in more detail in a subsequent lesson.

We introduce a convenient notation for roots using the reciprocal of n as $\frac{1}{n}$, allowing us to express the nth root of x as $x^{\frac{1}{n}}$, using the reciprocal in the exponent. This reveals that nth root functions are power functions with reciprocals of integers as exponents. For instance, $\sqrt{2}$ becomes $2^{\frac{1}{2}}$.

9.2.2 Polynomial Functions

Polynomial functions, denoted as $p(x)$, are formed by adding constant multiples of non-negative integer powers of x. A general polynomial function looks like :

$$P(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + ... + a_nx^n$$

The degree of the polynomial is the highest power of x present, and the coefficients are real constants.

Degree of $x^3 + 3x^2 + 6$ is 3

Polynomials are constructed using basic arithmetic operations and are essential in mathematics due to their simplicity and the ease of working with them.

9.2.2.1 Types of Polynomial Expression

Constant Expression : a
Linear Expression : $a + bx$
Quadratic Expression: $a + bx + cx^2$
Cubic Expression: $a + bx + cx^2 + dx^3$
Bi-Quadratic Expression: $a + bx + cx^2 + dx^3 + ex^4$
Quintic Expression: $a + bx + cx^2 + dx^3 + ex^4 + fx^5$
There are many more too which you can explore by yourself.

9.2.2.2 Factor Theorem

Factorization of polynomials is crucial for solving equations. If $p(x)$ is a polynomial of degree n and $p(\lambda) = 0$ for some real number λ, then $x - \lambda$ is a factor of $p(x)$,[2] and we can reduce the degree of the polynomial by one through factorization. This is demonstrated using polynomial long division, a method that systematically subtracts multiples of a known factor from the polynomial until it is fully factored.

9.2.2.3 Polynomial Long Division

■ **Example 9.1** Divide $x^3 + 6x^2 + 6x + 7$ by $x + 1$ using Polynomial long division and find out the quotient and remainder.

$$\begin{array}{r} x^2 + 5x + 1 \\ x+1 \overline{)\; x^3 + 6x^2 + 6x + 7} \\ \underline{-x^3 \;\; -x^2 \qquad\qquad} \\ 5x^2 + 6x \quad\; \\ \underline{-5x^2 - 5x \quad\;} \\ x + 7 \\ \underline{-x - 1} \\ 6 \end{array}$$

Here is the explanation of steps involved:

i) Set up the division: Place $x^3 + 6x^2 + 6x + 7$ inside the division symbol and $x + 1$ outside.
ii) Divide the first term: Divide x^3 by x to get x^2. Write x^2 above the division symbol.

[2]Officially known as the factor theorem, you can get the idea in this way. Say f(x) = (x-a)(x-b)(x-c)(x-d). Now, if you try to find f(a), f(a) = (a-a)(a-b)(a-c)(a-d) = 0 . Thus, you now get the feel that if f(a)=0 , it is because it should have contained (x-a) in its factor due to which it reduced to 0

iii) Multiply and subtract: Multiply x^2 by $x+1$ to get x^3+x^2. Subtract this from the dividend to get $5x^2+6x+7$. (Though 7 is not shown, it's meant to be there.)

iv) Repeat the process: Divide $5x^2$ by x to get $5x$. Write $5x$ next to x^2 above the division symbol.

v) Multiply and subtract again: Multiply $5x$ by $x+1$ to get $5x^2+5x$. Subtract this from the current dividend to get $x+7$.

vi) Final division: Divide x by x to get 1. Write 1 next to $5x$ above the division symbol.

vii) Last multiplication and subtraction: Multiply 1 by $x+1$ to get $x+1$. Subtract this from the current dividend to get the remainder 6.

viii) Write the quotient and remainder: The quotient is x^2+5x+1 and the remainder is 6.

■

9.2.2.4 Factorization of a cubic expression

For example, if $p(x) = x^3-6x^2+11x-6$ and we discover that $p(3)=0$, then $x-3$ is a factor. Using long division, we can find that $p(x) = (x-3)(x^2-3x+2)$,

$$
\begin{array}{r|l}
 & x^2 \quad -3x+2 \\ \hline
x-3 & x^3-6x^2+11x-6 \\
 & -x^3+3x^2 \\ \cline{2-2}
 & \quad -3x^2+11x \\
 & \quad 3x^2 \quad -9x \\ \cline{2-2}
 & \qquad 2x-6 \\
 & \qquad -2x+6 \\ \cline{2-2}
 & \qquad\quad 0
\end{array}
$$

and further factorization reveals $p(x) = (x-3)(x-1)(x-2)$. This complete factorization allows us to solve $p(x)=0$ by setting each factor equal to zero, yielding the solutions $x=3$, $x=1$, and $x=2$. This solution set can be mathematically written as $\{1,2,3\}$.

In summary, we've covered power functions, including fractional powers, and polynomial functions, which are sums of constant multiples of powers of x. We've also discussed the importance of factorizing polynomials and the technique of polynomial long division. These concepts lay a solid foundation for future calculus studies. Please review the material and practice the exercises to master these techniques. Thank you for reading, and I look forward to our next encounter.

9.3 Composite functions

9.3.1 Introduction to Composite Functions

In this educational series, we're venturing into the realm of function composition, a mathematical technique that allows us to construct intricate functions from more elementary ones. This process is comparable to the way a chef combines basic ingredients to create a complex dish or how a chemist assembles atoms into elaborate molecules. Our focus is on two particular composite functions known as $f \circ g$ and $g \circ f$, which are derived from the functions f and g with the rules described in the paragraph below.

The composite function $f \circ g$, often referred to as 'fog', is defined by the rule $y = f(g(x))$. This means that we first apply the function g to the input x, and then the function f is applied to the result of $g(x)$. The small 'o' symbol between f and g signifies the composition operation. It's crucial to understand that the sequence in which these functions are applied is of utmost importance, as altering the order can lead to a completely different result. This is analogous to the order in which we put on socks and shoes; reversing the order would yield a different outcome.

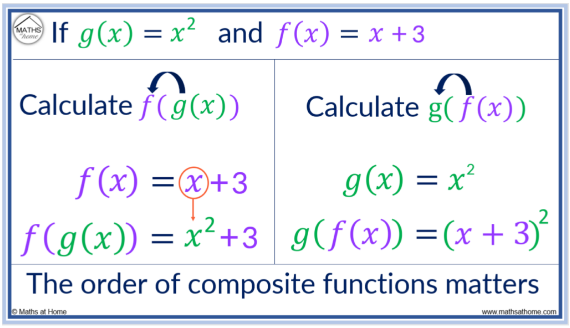


Figure 9.2: Order of Composition of function matters

When we reverse the order of composition to form $g \circ f$, or 'gof', the rule becomes $y = g(f(x))$, indicating that f is applied to the input x first, followed by g being applied to the result of $f(x)$. Through an example, we've demonstrated that $f \circ g$ and $g \circ f$ are distinct functions, each with its own set of rules and outcomes.

$$f(x) = \frac{1}{x} \qquad g(x) = x^2 + 1$$

$$f \circ g(x) = f(g(x)) = f(x^2+1) = \frac{1}{x^2+1}$$

$$g \circ f(x) = g(f(x)) = g(\frac{1}{x}) = (\frac{1}{x})^2 + 1 = \frac{1}{x^2} + 1$$

A particularly intriguing case arises when the composition of functions results in the identity function, which essentially means the output is identical to the input. This occurs when the functions involved are inverses of each other, a rare and significant phenomenon in mathematics.

$$f(x) = x^2 \qquad g(x) = \sqrt{x}$$

$$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

$$g \circ f(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = x$$

9.3.2 Computing the composition of functions

■ **Example 9.2** Q. Consider the functions:

$$f(x) = 2x - 3 \qquad g(x) = x^2 \qquad h(x) = \frac{x+3}{2}$$

Find the rules for $f \circ g$, $g \circ f$, $f \circ h$, $h \circ f$, $g \circ h$, $h \circ g$.

$$f \circ g(x) = f(g(x)) = f(x^2) = 2x^2 - 3$$

$$g \circ f(x) = g(f(x)) = g(2x-3) = (2x-3)^2$$

$$f \circ h(x) = f(h(x)) = f(\frac{x+3}{2}) = 2(\frac{x+3}{2}) - 3 = (x+3) - 3 = x$$

$$h \circ f(x) = h(f(x)) = h(2x-3) = \frac{(2x-3)+3}{2} = \frac{2x}{2} = x$$

[2]Image 9.2 taken from https://mathsathome.com/wp-content/uploads/2021/10/the-order-of-composite-functions-matters-1024x581.png

$$g \circ h(x) = g(h(x)) = g(\frac{x+3}{2}) = (\frac{x+3}{2})^2$$

$$h \circ g(x) = h(g(x)) = h(x^2) = \frac{x^2+3}{2}$$

■

We've been fortunate to encounter such an example where functions f and h undo each other's operations, resulting in the identity function regardless of the order of composition. The identity function is known as the "lazy function", and f and h functions are said to be inverses of each other.

The arithmetic of functions, as introduced through function composition, is a fascinating and complex area of study that extends to the cutting edge of modern mathematical research. It's a domain so rich and intricate that it remains a topic of active investigation. Alan Turing, a renowned mathematician and the subject of the film “The Imitation Game”, is celebrated for his abstract and ingenious conceptualization of computation, akin to Newton's contemplation of escape velocities prior to the existence of rockets.

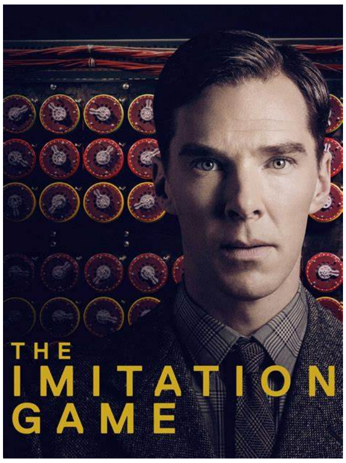


Figure 9.3: The Imitation Game and Alan Turing

Turing's work on Turing machines, which can be researched online, provides a formal framework demonstrating the inherent complexity of function arithmetic, even when limited to processing simple counting numbers like 1, 2, and 3. This complexity is so profound that it precludes the possibility of a mechanical device capable of performing such arithmetic, a groundbreaking and somewhat surprising revelation in the mathematical community during the first half of the 20th century.

In today's section, we've introduced the concept of composite functions and provided simple examples to illustrate their properties, including the special case where functions are inverses of each other. The next section will delve deeper into the concept of inverse functions and their applications. I encourage you to engage with the exercises to enhance your understanding of these concepts. Thank you for your attention, and I eagerly anticipate our next session together where we will continue to explore the fascinating world of functions and their arithmetic.

[2] Image 9.3 taken from https://www.rottentomatoes.com/m/theimitationgame/reviews

9.3.3 Practice Quiz

Question 1

Suppose that f and g are functions with the rules

$$f(x) = 2x - 3 \quad \text{and} \quad g(x) = x^3.$$

Find $(f \circ g)(x) = f(g(x))$.

(a) $8x^3 - 27$
(b) $2x^3 - 27$
(c) $2x^3 - 3$
(d) $(2x - 3)^3$
(e) $6x^3 - 3$

Question 2

Suppose that f and g are functions with the rules

$$f(x) = 2x - 3 \quad \text{and} \quad g(x) = x^3.$$

Find $(g \circ f)(x) = g(f(x))$.

(a) $2x^3 - 27$
(b) $6x^3 - 3$
(c) $2x^3 - 3$
(d) $(2x - 3)^3$
(e) $8x^3 - 27$

Question 3

Suppose that f is a function with the rule

$$f(x) = 2x - 3.$$

Find $(f \circ f)(x) = f(f(x))$.

(a) $4x - 3$
(b) $(2x - 3)^2$
(c) $4x - 9$
(d) $4x^2 - 3$

Question 4

Suppose that f is a function with the rule

$$f(x) = \frac{2x - 1}{3}.$$

Find $(f \circ f)(x) = f(f(x))$.

(a) $\frac{4x-6}{9}$
(b) $\frac{4x^2-1}{9}$
(c) $\left(\frac{2x-1}{3}\right)^2$
(d) $\frac{4x-3}{9}$
(e) $\frac{4x-5}{9}$

Question 5

Suppose that f, g, and h are functions with the rules

$$f(x) = x+1, \quad g(x) = 2x, \quad \text{and} \quad h(x) = x-1.$$

Find $(f \circ g \circ h)(x) = f(g(h(x)))$.

(a) $2x-1$
(b) $2x$
(c) $2x+1$
(d) $2x^2-2$
(e) $2x^2-1$

Question 6

Suppose that f, g, and h are functions with the rules

$$f(x) = 2x, \quad g(x) = x+1, \quad \text{and} \quad h(x) = \frac{x}{2}.$$

Find $(f \circ g \circ h)(x) = f(g(h(x)))$.

(a) $x+1$
(b) x^3+1
(c) $x+\frac{1}{2}$
(d) x^3+x
(e) $x+2$

Question 7

Suppose that f is a function with the rule

$$f(x) = \frac{1}{1-x}.$$

For $x \in \mathbb{R} \setminus \{0,1\}$, find $(f \circ f)(x)$.

(a) $\frac{1-x}{x}$
(b) $\frac{x}{x-1}$
(c) $\frac{x-1}{x}$
(d) $\frac{1-x^2}{x}$
(e) $\frac{x}{1-x}$

Question 8

Suppose that f and g are functions with the rules

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad g(x) = \frac{1}{x}.$$

For $x \in \mathbb{R} \setminus \{0,1\}$, find $(g \circ f)(x)$.

(a) $\frac{x}{1-x}$
(b) $\frac{1-x}{x}$
(c) $1-x$
(d) $\frac{1}{x}$
(e) $\frac{x-1}{x}$

Question 9

Suppose that f and g are functions with the rules

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad g(x) = \frac{1}{x}.$$

For $x \in \mathbb{R} \setminus \{0,1\}$, find $(f \circ g)(x)$.

(a) $\frac{x-1}{x}$
(b) $\frac{x}{1-x}$
(c) $1-x$
(d) $\frac{1-x}{x}$
(e) $\frac{1}{x}$

Question 10

Suppose that f is a function with the rule

$$f(x) = \frac{1}{1-x}.$$

For $x \in \mathbb{R} \setminus \{0,1\}$, find $(f \circ f \circ f)(x)$.

(a) $\frac{2-x}{1-2x}$
(b) $\frac{1}{(x-1)^2}$
(c) x
(d) $\frac{1}{1-x^3}$
(e) $\frac{x-2}{1-2x}$

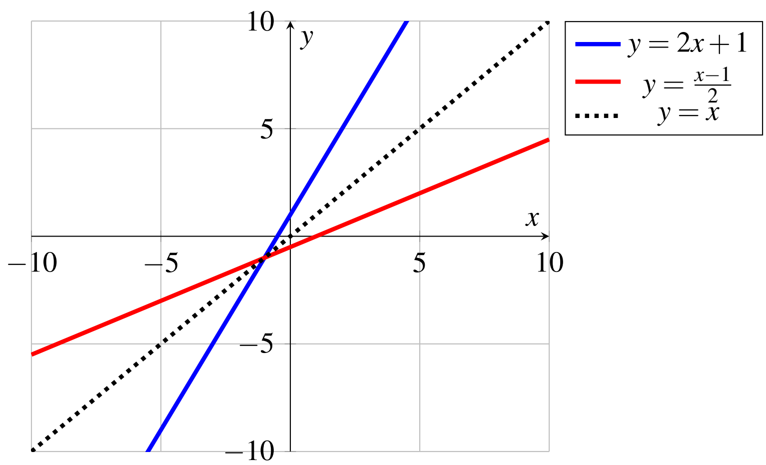
Answers

The answers will be revealed at the end of the module.

9.4 Inverse functions

9.4.1 Introduction to Inverse Function

In this section, let's revisit the concepts of inverse functions, function graphs, and the vertical and horizontal line tests from the beginning, providing a comprehensive understanding of the material.

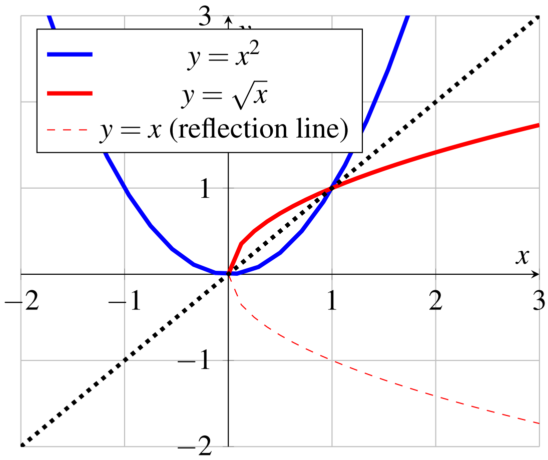


Inverse functions are mathematical operations that reverse the effect of the original functions. To understand an inverse function, imagine a process where the roles of the input and output are interchanged.

■ **Example 9.3 — An Analogy.** .
Say x be input and y be output. And the funtion be f(x) = 2x+1. which takes input to output
y = 2x+1
$x = \frac{y-1}{2}$
Interchanging input and output or (y and x),
$y = \frac{x-1}{2}$
$f^{-1}(x) = \frac{x-1}{2}$ which takes output to input.
Thus, $f^{-1}(f(x))$ takes input to output and then the resultant output to input. Thus,$f^{-1}(f(x))$ always gives the input value itself which in this case is x. Thus, $f^{-1}(f(x)) = x$

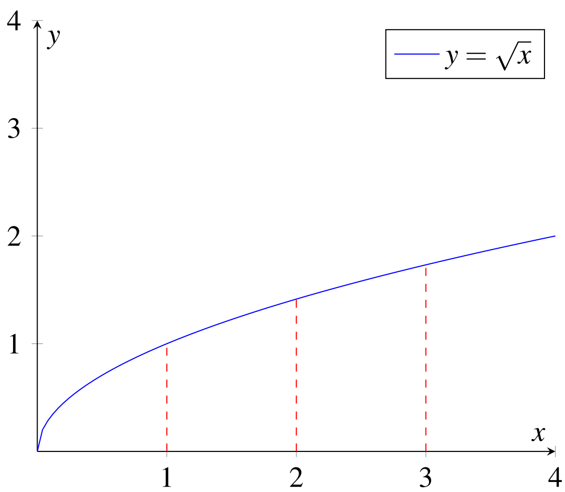
■

This analogy of interchanging x and y is also captured by the action of reflecting a point across the line $y = x$ in the Cartesian plane. Thus, reflecting a function about y=x gives the inverse of a funtion.

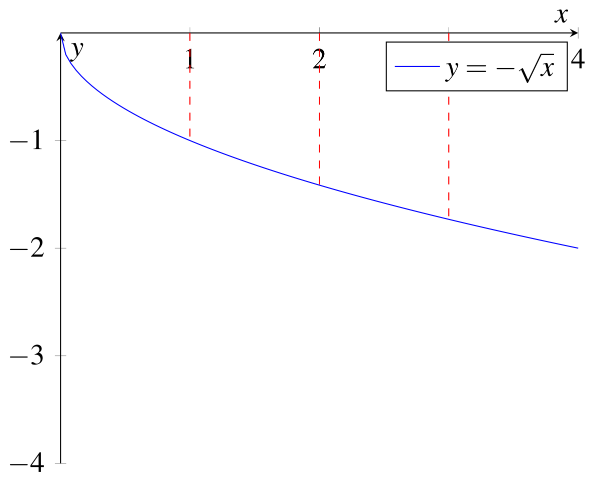


For example, if we have a function f represented by $y = x^2$, which graphs as a parabola, its inverse would be the function that 'undoes' the squaring, which is the square root function g, represented by $y = \sqrt{x}$. The graph of g is also a parabola but oriented sideways, and we only consider the top half to represent the positive square root. If we wanted the negative square root, we could define another function h with $h(x) = -\sqrt{x}$, which would give us the bottom half of the sideways parabola.

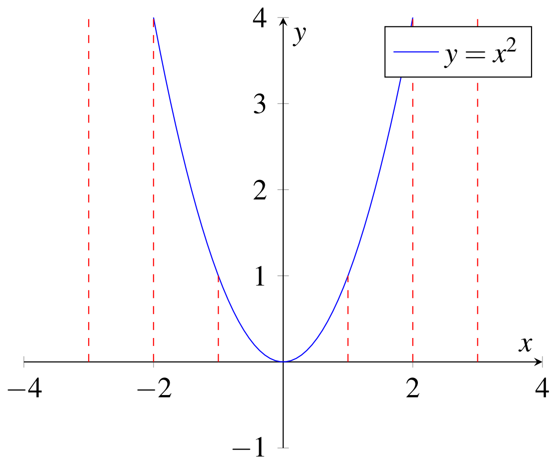
9.4.2 Vertical line test for a function



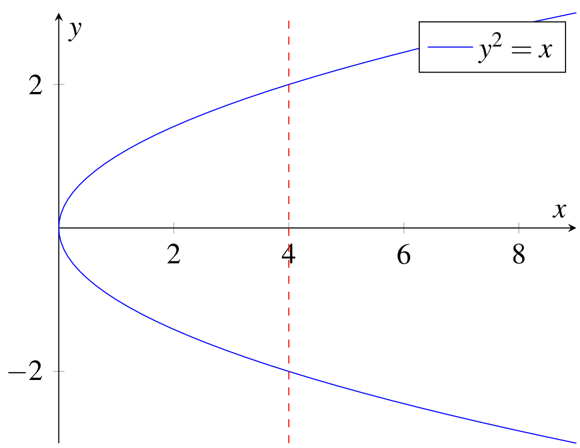
Vertical Line Test - I (passed)



Vertical Line Test - II (passed)



Vertical Line Test - III (passed)

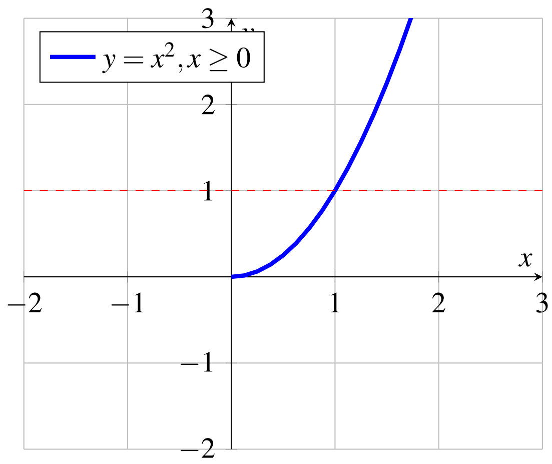


Vertical Line Test - IV (failed)

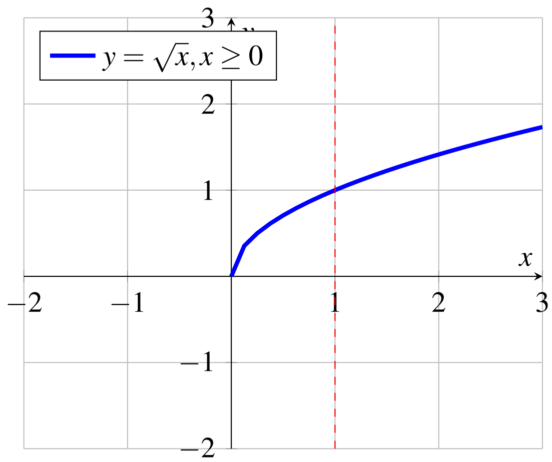
The vertical line test is a way to determine if a relation presented in a graph is a function or not. A graph passes this test if no vertical line intersects the graph at more than one point. **This ensures that each input x has at most one output y, avoiding ambiguity.** The graph of $y = x^2$ passes this test, but the full sideways parabola representing $x = y^2$ does not, except at the origin. To resolve this, we consider only the positive or negative square root, each of which passes the vertical line test on its own.

9.4.3 Horizontal Line Test for an invertible function

The horizontal line test is used to determine if a function is invertible. If every horizontal line intersects the graph at most once, the function can be inverted. The standard parabola $y = x^2$ fails this test, but by restricting the domain to non-negative reals ($x \geq 0$), we get the right-hand half of the parabola, which passes the test.

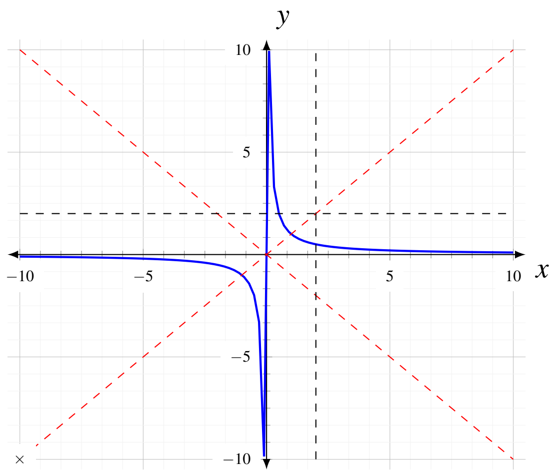


The reason horizontal line test works is because if a function passes horizontal line test, then it's inverse will pass vertical line test assuring that the inverse is also a function.



A relation that passes both the vertical and horizontal line tests is invertible function, meaning we can move back and forth between inputs and outputs without confusion. Inverting a function involves interchanging the roles of x and y, which can be visualized by reflecting the graph across the line $y = x$. This reflection interchanges the horizontal and vertical axes, as demonstrated with the half parabola for non-negative x only. Reflecting this graph gives us the positive square root function.

The notation for an inverse function is f^{-1}, which is related to reciprocal notation but comes from the composition of functions and the role of the identity function in function arithmetic. The rules for f and its inverse are closely linked: $y = f(x)$ if and only if $x = f^{-1}(y)$. If f takes x to y, then f^{-1} brings y back to x, and vice versa.



For the function $h(x) = \frac{1}{x}$, whose graph is a hyperbola with two branches, the domain and range exclude zero. The graph satisfies both line tests and is symmetrical about the line $y = x$, so reflecting it across this line results in the same graph, indicating that h is its own inverse. Algebraically, $h(h(x)$ = h$(\frac{1}{x}) = x$, confirming that h undoes itself.

9.4.4 Computing Inverse of a function algebraically

To algebraically find the inverse of a function, we interchange x and y in the function's equation and solve for y. For a function f with a complicated rule, such as $x \mapsto \sqrt{x+2} - \frac{1}{3}$, we set y equal to this expression and manipulate it to express x in terms of y.

$$f(x) = \sqrt{x+2} - \frac{1}{3}$$

$$y = \sqrt{x+2} - \frac{1}{3}$$

$$y = \sqrt{x+2} - \frac{1}{3}$$

$$y + \frac{1}{3} = \sqrt{x+2}$$

$$\frac{3y+1}{3} = \sqrt{x+2}$$

$$\frac{(3y+1)^2}{9} - 2 = x$$

$$x = \frac{(3y+1)^2}{9} - 2$$

$$f^{-1}(x) = \frac{(3x+1)^2}{9} - 2 = (x + \frac{1}{3})^2 - 2$$

The result is the inverse function $f^{-1}(x) = \frac{(3x+1)^2}{9} - 2$. We verify that this inverse function undoes the action of f by checking that $f^{-1}(f(x))$ returns the original input x.

$$f^{-1}(f(x))$$

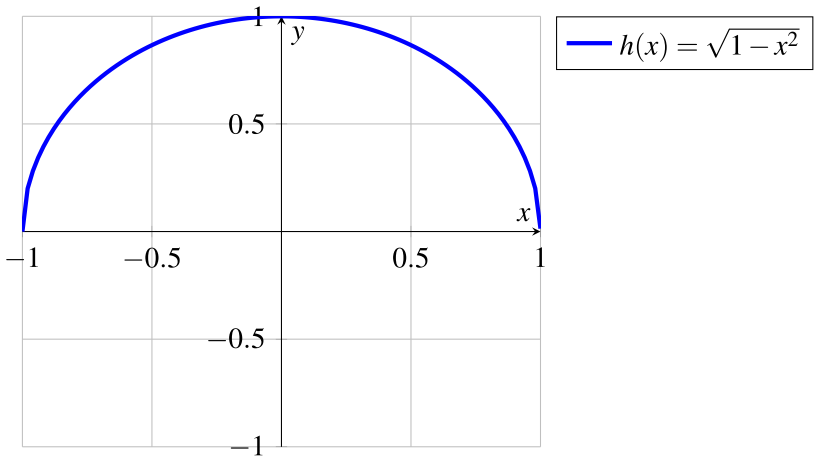
$$f^{-1}(\sqrt{x+2} - \frac{1}{3})$$

$$(\sqrt{x+2} - \frac{1}{3} + \frac{1}{3})^2 - 2$$

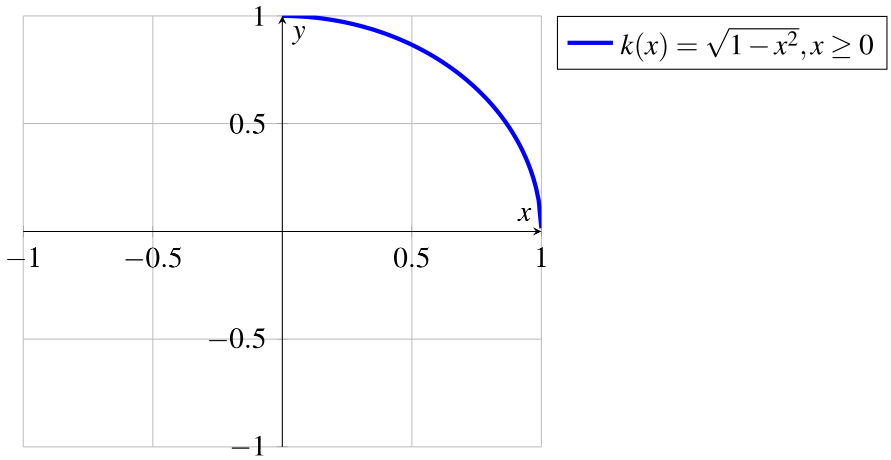
$$x + 2 - 2$$

$$x$$

Using the unit circle as an example, we see that to satisfy the vertical line test and create a function, we can take either the upper or lower semicircle. However, neither satisfies the horizontal line test. By restricting the domain to $0 \le x \le 1$, we get a quarter-circle that passes both tests. The function k representing this quarter-circle is defined by $k(x) = \sqrt{1 - x^2}$, and its inverse is found by reflecting the graph across the line $y = x$, resulting in k being its own inverse.



Satisfies the Vertical Line Test but not Horizontal line test



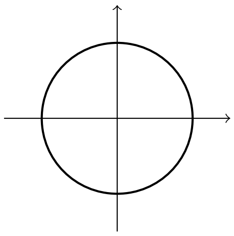
Satisfies both the Vertical Line Test and Horizontal line test

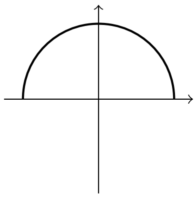
We've covered a lot of ground, discussing how to determine if a function is invertible, how to geometrically find an inverse by reflecting a graph, and how to algebraically derive the inverse function. Please refer to exercises for more details and to reinforce these concepts, Thank you for engaging with this material, and I look forward to our next lesson together.

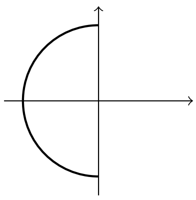
9.4.5 Practice Quiz

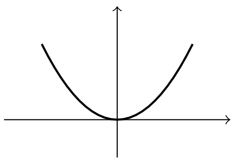
Question 1

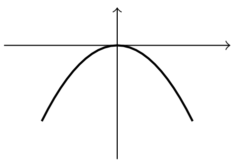
Which one of the following curves satisfies the vertical line test, and therefore is the graph of a function?

(a) 

(b) 

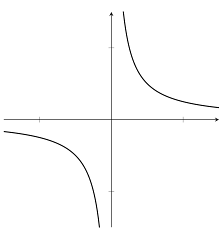
(c) 

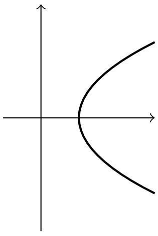
(d) 

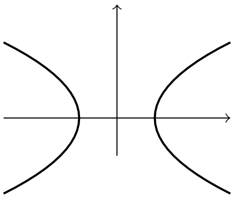
(e) 

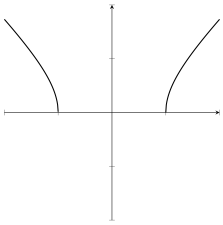
Question 2

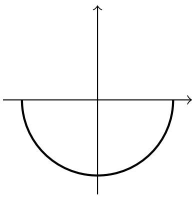
Which one of the following curves satisfies both the vertical and horizontal line tests, and therefore is the graph of an invertible function?

(a) 

(b) 

(c) 

(d) 

(e) 

Question 3

Suppose that $f(x) = x^2$ for $x \geq 0$. Find $f^{-1}(x)$.

(a) $-\sqrt{x}$
(b) 1
(c) $\frac{x^2}{9}$
(d) $\sqrt{x}$
(e) 2x

Question 4

Suppose that $f(x) = x^2$ for $x \leq 0$. Find $f^{-1}(x)$.

(a) $-\frac{1}{x^2}$
(b) $\frac{x}{2}$
(c) $-\sqrt{x}$
(d) $\sqrt{x}$
(e) -2x

Question 5

Suppose that $f(x) = 3x - 2$ for all $x \in \mathbb{R}$. Find $f^{-1}(x)$.

(a) $\frac{x-2}{3}$
(b) $\frac{x+2}{3}$
(c) $\frac{3x-2}{3}$
(d) $\frac{2-x}{3}$
(e) $\frac{x}{3} + 2$

Question 6

Suppose that $f(x) = \frac{1-2x}{4}$ for all $x \in \mathbb{R}$. Find $f^{-1}(x)$.

(a) $\frac{4-2x}{1}$
(b) $\frac{4x-1}{2}$
(c) $\frac{1+4z}{2}$
(d) $\frac{1+4x}{2}$

Question 7

Suppose that $f(x) = (x-2)^2 + 1$ for $x \geq 2$. Find $f^{-1}(x)$.
(a) $(x-1)^2 + 2$
(b) $2 + \sqrt{x-1}$
(c) $-1 + \sqrt{x+2}$
(d) $1 + \sqrt{x-2}$
(e) $2 - \sqrt{x-1}$

Question 8

Suppose that $f(x) = (x-2)^2 + 1$ for $x \leq 2$. Find $f^{-1}(x)$.
(a) $2 + \sqrt{x-1}$
(b) $1 + \sqrt{x-2}$
(c) $-1 + \sqrt{x+2}$
(d) $2 - \sqrt{x-1}$
(e) $(x-1)^2 + 2$

Question 9

Suppose that $f(x) = \frac{1}{1-x}$ for $x \neq 1$. Find $f^{-1}(x)$.
(a) $\frac{x-1}{x}$
(b) $x - 1$
(c) $\frac{1}{x}$
(d) $1 - x$
(e) $\frac{x}{1-x}$

Question 10

Suppose that $f(x) = \sqrt{1-x^2}$ for $0 \leq x \leq 1$. Find $f^{-1}(x)$.
(a) $\sqrt{1-x}$
(b) $\sqrt{1-x^2}$
(c) $(1-x)^2$
(d) $\sqrt{x^2-1}$
(e) $-\sqrt{x-1}$

10. Exponential and Logarithmic functions

10.1 The exponential function

Welcome to the next section on exponential functions, a key concept in mathematics that leads to the understanding of exponential growth and introduces us to Euler's Number, e. This journey will expand our mathematical understanding from basic operations to the powerful realm of exponentiation.

10.1.1 The Basics of Exponentiation

Let's start with the basics. For any real number x and any positive integer n, the expression x^n signifies x multiplied by itself n times. For example:
$x^1 = x$
$x^2 = x \cdot x$
$x^3 = x \cdot x \cdot x$
... and so on. When x is a specific number, such as 2, we have:
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
... continuing in this pattern.

10.1.2 Exponential Laws

Exponential laws are crucial for simplifying expressions and understanding the behavior of powers. They state that:
To multiply terms with the same base, add their exponents. $2^3 \times 2^4 = 2^{3+4} = 2^7$
To raise a power to another power, multiply the exponents. $(2^3)^4 = 2^{3\times 4} = 2^{12}$
To divide terms with the same base, subtract the exponents. $\frac{2^3}{2^4} = 2^{3-4} = 2^{-1}$
To multiply terms with different bases but the same exponent, multiply the bases. $2^4 \times 3^4 = (2 \times 3)^4 = 6^4$
These laws are easily verified with integer exponents and become intuitive with practice.

10.1.3 Zero and Negative Powers

The concept of exponentiation also extends to zero and negative integers. By definition:
i) $x^0 = 1$

which you can see in this way: $1 = \frac{x^a}{x^a} = x^{a-a} = x^0$
ii) $x^{-n} = \frac{1}{x^n}$
which you can see in this way: $\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n}$
For instance, with $x = 2$:
$2^0 = 1$
$2^{-1} = \frac{1}{2}$
$2^{-2} = \frac{1}{4}$
$2^{-3} = \frac{1}{8}$
... and this pattern continues.

10.1.4 Fractional Powers

We can further extend our understanding to fractional powers. If x is a positive real number and n is a positive integer, $x^{\frac{1}{n}}$ is defined as the positive nth root of x. This is consistent with our exponential laws, as $\frac{1}{n} \times n = 1$, and $x^1 = x$.

$$\sqrt[n]{x^n} = (x^n)^{\frac{1}{n}} = x^{n \times \frac{1}{n}} = x^1 = x$$

For any positive fraction $\frac{m}{n}$, where m and n are positive integers, $x^{\frac{m}{n}}$ can be found by either taking the (nth root of x) raised to the mth power or by raising x to the mth power and then taking the nth root.

$$x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$$

■ **Example 10.1** Simplify:

$$\frac{2^{2.5} \times 3^{2.5}}{\sqrt{6}}$$

$$= \frac{2^{2.5} \times 3^{2.5}}{(2 \times 3)^{0.5}} = \frac{2^{2.5} \times 3^{2.5}}{2^{0.5} \times 3^{0.5}} = 2^{2.5-0.5} \times 3^{2.5-0.5} = 2^2 \times 3^2 = 36$$

■

10.1.5 Introducing the Exponential Function

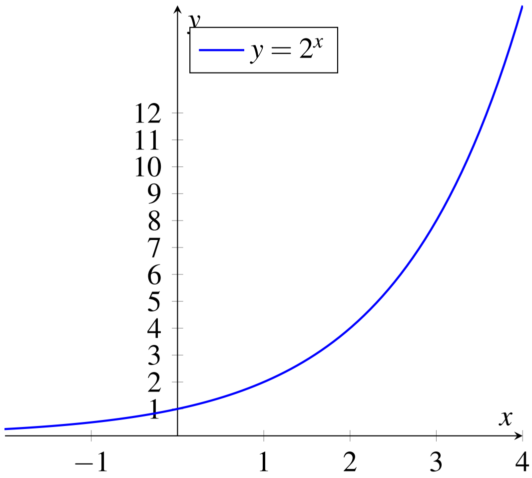
We now work towards introducing the exponential function. But first recall the following form of power of function.

$$y = x^n$$

Notice that the base x is variable and exponential n is constant.
Now, we change our point of view completely. Remember the power of lateral thinking to create what we call exponential function.
We now consider the base as a constant and power as the variable.

$$y = f(x) = a^x$$

For eg: $y = 2^x, y = 3^x, y = 0.5^x, y = e^x$
where e is euler's number, approximately equal to 2.718 . e is irrational number discovered by Leonhard Euler in 18th century. More about e is discussed below.

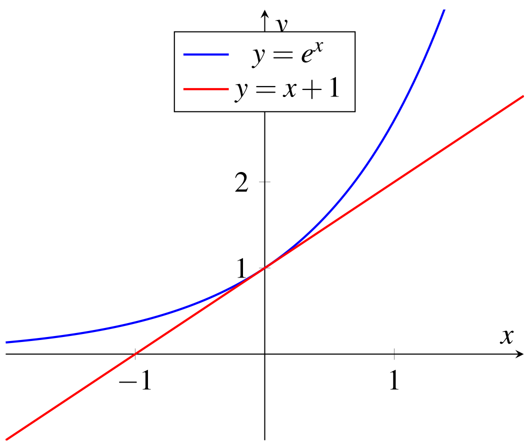


The curve gets very steep, very quickly as you move to the right.

Also know that, if we just want to plot the y-value for slightly more x values say 6, 7 or 8 , we have to go upto 256 which is a way too big number. This brings us to the concept of exponential explosion.

10.1.6 Exponential Explosion and Euler's Number

The concept of exponential explosion, often associated with phenomena like population growth, illustrates the rapid increase of a quantity. To visualize this, consider a graph where a small increase in the x-axis requires a disproportionately large extension of the y-axis due to the steepness of the exponential curve.



Now, let's focus on Euler's Number, e, a unique and important constant in mathematics. When we plot the function $y = e^x$ and the line $y = x + 1$, we find that the line is tangent to the curve at the point where $x = 0$. This is no coincidence; e is specifically chosen so that the slope of the curve $y = e^x$ at the y-intercept is exactly 1. This property has profound implications in calculus, particularly in the study of derivatives and differential equations.

One of the most fascinating aspects of the function $y = e^x$ is its self-replicating property. In calculus, we learn that the derivative of e^x is e^x itself. This self-reference is not just a mathematical curiosity; it's a fundamental property that underpins much of the theory in differential equations.

Today, we've covered a lot of ground. We've discussed the importance of changing perspectives, from considering the variable as the base in power functions to viewing it as the exponent in exponential functions. We've seen an example of exponential explosion and delved into the special properties of Euler's Number, e. As you continue to study these concepts, remember to practice with the exercises provided, and I'm confident you'll gain a deeper understanding of these fundamental mathematical principles.

Thank you for engaging with this material, and I eagerly anticipate our next journey together.

10.1.7 Practice Quiz

Question 1

Evaluate 2^7.

(a) 14
(b) 49
(c) 128
(d) 256
(e) 64

Question 2

Evaluate 3^{-3}.

(a) -9
(b) -27
(c) $-\frac{1}{27}$
(d) $\frac{1}{27}$
(e) $\frac{1}{3}$

Question 3

Evaluate $16^{\frac{3}{4}}$.

(a) 8
(b) 12
(c) $\frac{1}{8}$
(d) $\frac{1}{64}$
(e) 64

Question 4

Evaluate $16^{-\frac{3}{4}}$.

(a) $\frac{1}{64}$
(b) $\frac{1}{8}$
(c) -12
(d) 1
(e) $-\frac{8}{3}$

Question 5

Evaluate $27^{\frac{1}{3}} \times 81^{-\frac{3}{4}}$.

(a) 1
(b) 3
(c) 9
(d) 1
(e) $\frac{1}{9}$

Question 6

Evaluate $\frac{3^{2.5}\times 4^{2.5}}{\sqrt{12}}$.

(a) 12
(b) 1728
(c) 24
(d) 576
(e) 144

Question 7

Which one of the following expressions is equivalent to $\left(x^4y^{-3}\right)^{-2}$?

(a) $\frac{x^2}{y^6}$
(b) $\frac{y^6}{x^8}$
(c) x^8y^6
(d) $\frac{1}{x^8y^6}$
(e) $\frac{y^6}{x^2}$

Question 8

Which one of the following expressions is equivalent to $3^{-3}3^x$?

(a) 3^{x-3}
(b) 6^{x-3}
(c) 9^x
(d) 3^x
(e) 9^{x-3}

Question 9

Which one of the following expressions is equivalent to $(0.2)^{-z}$?

(a) 2^{-z}
(b) 5^{-z}
(c) 5^z
(d) 2^z
(e) 5^z

Question 10

Which one of the following expressions is equivalent to $e^{3x}e^{x^2-4}$?

(a) $e^{(x+4)(x+1)}$
(b) $e^{(x-4)(x-1)}$
(c) $e^{(x+1)(x-4)}$
(d) e^{2x^2-4}
(e) $e^{(x-1)(x-4)}$

Answers

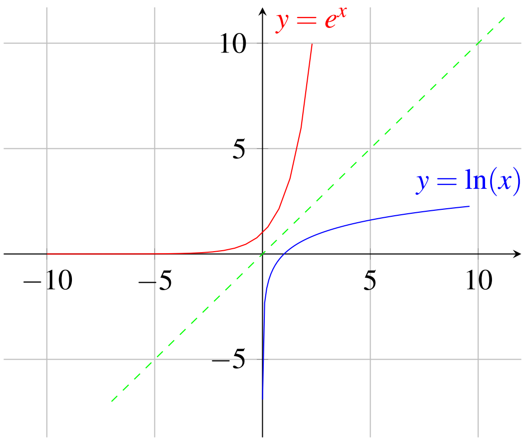
The answers will be revealed at the end of the module.

10.2 The Logarithmic function

In this section, let's delve deeper into the fascinating interplay between exponential and logarithmic functions, ensuring clarity and detail in every step of our exploration.

10.2.1 Understanding the Inverse Relationship

The journey begins with the exponential function $y = e^x$, where e is Euler's number, approximately 2.71828. This function represents continuous growth, with its rate of increase directly proportional to its current value.[1] The graph of $y = e^x$ is a curve that rises sharply as x increases.



To find the inverse of this function, we reflect its graph across the line $y = x$. This reflection process swaps the x and y values of every point on the curve, effectively reversing the function's operation. The result is the natural logarithm, denoted as $y = \ln(x)$. The term 'ln' is an abbreviation whose origin is not definitively known but is speculated to be derived from the French word 'naturel', meaning natural.

10.2.2 Domains and Ranges of Logarithmic Function

The domain of $y = e^x$ is the set of all real numbers, denoted as $\mathbb{R}$, because for every real number x, there is a corresponding y value. The range, however, is the set of positive real numbers, $\mathbb{R}^+$, since e^x is always positive.

When we invert the function to obtain $y = \ln(x)$, the domain and range switch roles. The domain of the natural logarithm is $\mathbb{R}^+$, reflecting the fact that you can only take the logarithm of positive numbers. The range becomes $\mathbb{R}$, indicating that the output of the natural logarithm can be any real number.

10.2.3 Function Inversion and Properties

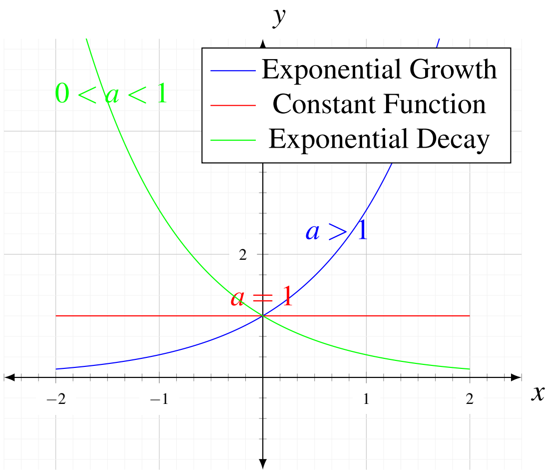
Inverting functions interchanges their domain and range, and this is evident in the relationship between e^x and $\ln(x)$. These functions are inverses of each other, meaning they 'undo' each other's effects. For instance, $\ln(e^x) = x$ for any real x, and $e^{\ln(x)} = x$ for any positive x.

10.2.4 Logarithms with Different Bases

Moving beyond the natural logarithm, we consider exponential functions with other bases a, where a is a positive real number. By exploiting the property $a = e^{\ln(a)}$, we can express a^x in terms of the natural exponential function: $a^x = (e^{\ln(a)})^x = e^{\ln(a)\cdot x}$. This expression shows that the graph of $y = a^x$ is a rescaled version of $y = e^x$, with the rescaling factor being $\ln(a)$.

[1] This is evident from the fact that derivative of e^x is e^x itself.

Now, if try to reflect $y = a^x$ about y=x, you get its inverse function which is written as $y = \log_a(x)$ which is spoken as " logarithm of x base a". This implies that $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$.



. Graph. $y = a^x$ for different values of a

For bases $a > 1$, the curve $y = a^x$ exhibits growth similar to $y = e^x$, but at a different rate. When $a = 1$, the function $y = 1^x$ is constant and uninteresting. For $0 < a < 1$, the function represents exponential decay, which we'll explore further in another section.

10.2.5 Laws of Logarithm

Logarithms follow specific laws that mirror the properties of exponential functions. These laws will (in upcoming sections) transform complex arithmetic operations into simpler ones:

10.2.5.1 Product Law

The logarithm of a product is equal to the sum of the logarithms of the factors: $\log_a(xy) = \log_a(x) + \log_a(y)$. If you are interested in proving this, raise both sides to power of a and use the property that $a^{\log_a(x)} = x$. This will make the proof obvious. Use similar approach for the laws that follow.

10.2.5.2 Quotient Law

The logarithm of a quotient is equal to the difference of the logarithms of the numerator and denominator: $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$.

10.2.5.3 Power Law

The logarithm of a power is equal to the exponent times the logarithm of the base: $\log_a(x^k) = k \cdot \log_a(x)$. These laws simplify multiplication into addition, division into subtraction, and exponentiation into multiplication, making arithmetic operations more manageable.

10.2.6 Practical Application of Logarithm

Let's apply these laws to simplify $\log_{10}(\sqrt{1000})$. Recognizing that $\sqrt{1000} = 1000^{1/2}$ and $1000 = 10^3$, we can use the logarithmic laws to find that $\log_{10}(1000^{1/2}) = \frac{1}{2} \cdot \log_{10}(10^3) = \frac{1}{2} \cdot 3 = \frac{3}{2}$, which simplifies to 1.5.

10.2.7 General Exponential Expressions

For any positive a and real number b, the expression a^b can be rewritten using natural logarithms and exponentials: $a^b = e^{b \cdot \ln(a)}$. This remarkable relationship allows us to reduce general exponential expressions to the mathematics of Euler's number.

10.2.8 Calculating Logarithms to Any Base

A useful fact is that for any positive base a and positive real number x, we have $x = a^{\log_a(x)}$. Taking natural logarithms of both sides and applying logarithmic properties,

$$\ln(x) = \ln(a^{\log_a(x)})$$

$$\ln(x) = \log_a(x)\ln(a)$$

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

we arrive at the formula $\log_a(x) = \frac{\ln(x)}{\ln(a)}$. This means we can calculate logarithms to any base using only natural logarithms and division.

In summary, we've thoroughly examined the inverse nature of exponential and logarithmic functions, their domains and ranges, and the laws that govern logarithms. We've also demonstrated how to simplify complex arithmetic using logarithmic laws and explored the generalization of exponential expressions. Please attempt the exercises to solidify your understanding. Thank you for your attention, and we eagerly anticipate your next steps in the world of mathematics.

10.2.9 Practice Quiz

Question 1

Evaluate $\log_2 128$.

(a) 8
(b) 6
(c) 9
(d) 5
(e) 7

Question 2

Evaluate $\log_{10}\sqrt{100,000}$.

(a) $\frac{7}{2}$
(b) 3
(c) 4
(d) $\frac{5}{2}$
(e) 5

Question 3

Simplify the expression $\ln(e^x)$.

(a) x
(b) $\frac{1}{x}$
(c) 1
(d) $-\frac{1}{x}$
(e) $-x$

Question 4

Simplify the expression $\ln(e^{-x})$.

(a) $-\frac{1}{x}$
(b) x
(c) $-x$
(d) 1
(e) $\frac{1}{x}$

Question 5

Simplify the expression $e^{-\ln x}$ for $x > 0$.

(a) $-x$
(b) x
(c) 0
(d) $\frac{1}{x}$
(e) 1

Question 6

Simplify the expression $e^{x \ln a}$ for $a > 0$.

(a) a^x
(b) x^a
(c) a^{-x}
(d) ax
(e) x

Question 7

Which one of the following expressions is equivalent to $\ln\left(\frac{ab^2}{c^3}\right)$ for $a, b, c > 0$?

(a) $\ln a + 2\ln b - 3\ln c$
(b) $\ln a + \ln(2b) - \ln(3c)$
(c) $\ln a + 2\ln(2b) - 3\ln c$
(d) $\ln a + 2\ln b - \ln(3c)$
(e) $2\ln(ab) - \ln(3c)$

Question 8

Which one of the following expressions is equivalent to $\log_2 x$ for $x \geq 0$?

(a) $\ln(2x)$
(b) $\ln\left(\frac{x}{2}\right)$
(c) $(\ln x)(\ln 2)$
(d) $\ln 2 \ln x$
(e) $\ln x \ln 2$

Question 9

Solve for x when $3^x = 10$.

(a) $x = \ln\left(\frac{10}{3}\right)$
(b) $x = 10^3$
(c) $x = \frac{\ln 3}{\ln 10}$
(d) $x = \ln\left(\frac{3}{10}\right)$
(e) $x = \frac{\ln 10}{\ln 3}$

Question 10

Solve for x when $6^x = e^{x+1}$.

(a) $x = \frac{1}{1-\ln 6}$
(b) $x = \frac{1}{1+6\ln x}$
(c) $x = \frac{1}{-1+\ln 6}$
(d) $x = 1+\ln 6$
(e) $x = -1+\ln 6$

Answers

The answers will be revealed at the end of the module.

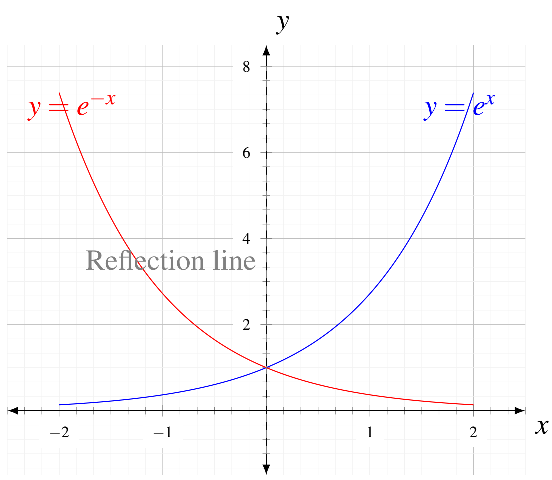
10.3 Exponential Growth and Decay

10.3.1 Exponential Growth and Decay

In this section, we use exponential functions to model exponential growth and decay, and explore some contrasting examples in applications.
Exponential growth and decay are two sides of the same coin in the realm of mathematics. They describe processes where a quantity increases or decreases at a rate proportional to its current value. Exponential growth and decay are modeled by the equation $y = Ae^{kx}$ for some constants A and k.

Exponential Growth: This phenomenon is modeled by the function $y = a^x$, where a is the base greater than 1. The classic example is the natural exponential function $y = e^x$, where e is Euler's number, approximately 2.71828. This curve represents rapid and unbounded growth, as seen in unchecked populations, financial investments, and, famously, in the story of Aladdin's rice challenge.

Exponential Decay: In contrast, exponential decay is characterized by a base a between 0 and 1, leading to the function $y = a^x$ approaching zero as x increases. It's exemplified by the natural decay function $y = e^{-x} = (1/e)^x$, which models the reduction in the amount of a radioactive substance over time, the cooling of an object, or the decrease in air pressure with altitude. Notice that the value of 1/e is less than 1 which makes it a decay equation. Also notice that $y = e^{-x}$ is just the reflection of $y = e^x$ about y axis.

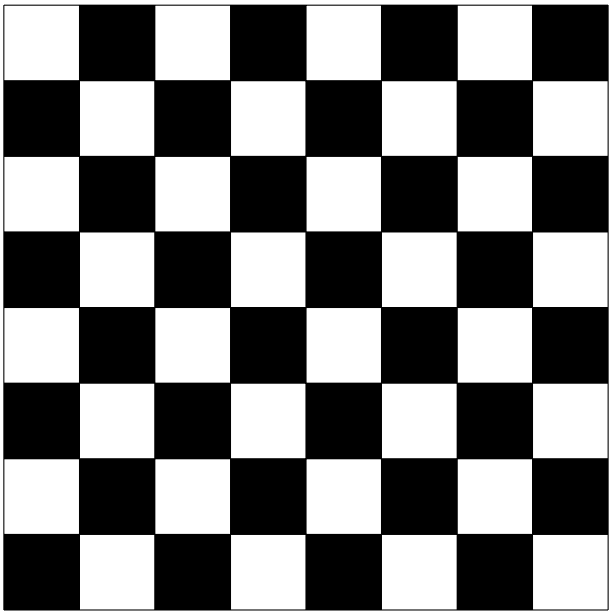


10.3.2 Power of Exponential growth and decay

Gather around, for I have a tale to weave, one that stretches from the bustling streets of Agrabah to the lofty heights of Mexico City, and it's all about the mysterious and powerful force of exponential functions.

10.3.2.1 The Tale of Aladdin's Rice Challenge

Once upon a time, in a land of sultans and genies, young Aladdin faced a seemingly simple challenge. The Sultan, wary of Aladdin's intentions with Princess Jasmine, devised a task that appeared trivial at first glance. "Aladdin," he said, "you must place one grain of rice on the first square of my grand courtyard chessboard on the first day. On the second day, two grains, and on the third, four. Continue to double the grains each day, and if you succeed for 64 days, you shall have my blessing to marry my daughter."



Aladdin agreed, unaware of the mathematical trap laid before him. The task was an embodiment of exponential growth, where each square represented the next power of two. The formula $n(x) = 2^{x-1}$ captured the essence of his challenge, with $n(x)$ being the number of grains on day x.

■ **Example 10.2** What is the first day X, when the number of grains of rice exceeds the number of grains of sand on Bondi Beach ? (Number of Grains of sand on Bondi Beach is 2×10^5 ?)
We want,

$$2^{x-1} > 2 \times 10^{15}$$

Taking ln in both sides

$$(x-1)\ln(2) > \ln(2) + 15\ln(10)$$

$$(x-1) > \frac{\ln(2) + 15\ln(10)}{\ln(2)}$$

$$x - 1 > 50.82$$

$$x > 51.82$$

■

Little did Aladdin know that by the 52nd day, he would need more grains of rice than the grains of sand on the vast Bondi Beach, a number estimated to be around 2×10^{15}. Such is the deceptive nature of exponential growth, a force not even a genie could easily contend with.

10.3.2.2 The High Altitude Dilemma

Figure 10.1: Summer Olympics 1968 held in Mexico

Our story now takes us to the year 1968, to the high-altitude city of Mexico City, where athletes from around the world gathered for the Summer Olympics. The air was thin, and breaths were short, for the air pressure at this great height was a mere fraction of that at sea level. Scientists tell us that air pressure decreases by about 0.4% every 30 meters above sea level, a perfect example of exponential decay.

■ **Example 10.3** Q. Find the percentage reduction in air pressure as you move from Sea level to Mexico City which has an altitude of 2237 metres.
Fact: Air pressure decays exponentially at about 0.4% per 30 meters.
Solution:

Let y = y(x) = air pressure at altitude x metres above sea level

Using exponential decay model,

$$y = Ae^{kx} \text{ for some constants A and k} \tag{10.1}$$

At sea level, x = 0 and the corresponding air pressure is y(0). Substituting this in equation 10.1

$$y(0) = Ae^0$$

which gives $A = y(0)$.

Thus,

$$y(x) = y(0)e^{kx} \tag{10.2}$$

Now, to find k , we know that at height of 30 meters, air pressure is (1-0.004) of air pressure at sea level,

$$i.e. y(30) = 0.996y(0). \tag{10.3}$$

Substituting x=30 in eqn 10.2 and then using eqn 10.3, we get,

[1]Image 10.1 taken from https://globalsportmatters.com/1968-mexico-city-olympics/2018/10/31/1968-olympics-iconic-moments/

$y(30) = y(0)e^{30k}$

$0.996y(0) = y(0)e^{30k}$

$0.996 = e^{30k}$

$30k = \ln(0.996)$

$k = \frac{\ln(0.996)}{30}$

$k = -0.0001336$

Using k = -0.0001336 in eqn 10.2, we get

$y(x) = y(0)e^{-0.000136k}$

At 2237 meters,

$y(2237) = y(0)e^{-0.000136\times 2237}$

$y(2237) = 0.7377y(0) = 73.77\%$ of $y(0)$

Thus, this shows that there is 26.23% reduction of air pressure as you move from sea level to Mexico city which has an altitude of 2237 metres. ■

In this modelling, we used the function $y = Ae^{kx}$, where A is the air pressure at sea level, and k is a constant that captures the rate of decay. Through careful calculations, we found that the air pressure at the altitude of Mexico City is about 74% of that at sea level, leading to a 26% reduction. This significant drop explained the athletes' struggle to perform as they would at lower altitudes.

So, what can we learn from Aladdin's ordeal and the Olympians' plight? Exponential functions are not mere numbers on a page; they are powerful forces that shape our world in ways both wondrous and daunting. They teach us that what starts as a grain of rice can fill a kingdom, and what breathes life into our lungs can dwindle to a whisper in the clouds. Pay heed to these tales, for they remind us to respect the exponential, whether it be in grains of rice or breaths of air.

And lastly, don't eat food left out in the sun if there's any risk of flies and bacteria playing games with exponential explosion .

10.3.2.3 Exponential Growth in Populations

Exponential growth is not limited to hypothetical scenarios. It's prevalent in population dynamics and the spread of diseases. A small number of bacteria on food can multiply rapidly if left unchecked, leading to an exponential explosion in their population. This growth can overwhelm the body's defenses if the contaminated food is consumed.

These examples highlight the pervasive nature of exponential functions in our world, from ancient myths to modern-day challenges. They serve as a powerful reminder of the exponential impact of seemingly small factors over time.

10.3.3 Practice Quiz

Question 1

Find the smallest positive integer n such that $2^n > 1,000,000$.

(a) 21
(b) 19
(c) 20
(d) 18
(e) 17

Question 2

Find the smallest positive integer n such that $2^{-n} < 10^{-9}$.

(a) 31
(b) 30
(c) 32
(d) 28
(e) 29

Question 3

We have a sample of 10 g of caesium. The half-life of caesium is 30 years. How much of the sample will remain after 60 years (to the nearest tenth of a gram)?

(a) 2.6 g
(b) 2.5 g
(c) 2.8 g
(d) 2.7 g
(e) 2.9 g

Question 4

We have a sample of 10 g of caesium. The half-life of caesium is 30 years. How much of the sample will remain after 80 years (to the nearest tenth of a gram)?

(a) 1.3 g
(b) 1.4 g
(c) 1.2 g
(d) 1.6 g
(e) 1.5 g

Question 5

We have a sample of 10 g of caesium. The half-life of caesium is 30 years. Estimate the least number of years (in whole numbers) for the sample to decay to less than 1 gram.

(a) 103 years
(b) 99 years
(c) 101 years
(d) 100 years
(e) 102 years

Question 6

An animal skull still has 20 per cent of the carbon-14 that was present when the animal died. The half-life of carbon-14 is 5730 years. Estimate the age of the skull to the nearest thousand years.

(a) 11,000 years
(b) 12,000 years
(c) 15,000 years
(d) 14,000 years
(e) 13,000 years

Question 7

You are given that air pressure decays exponentially near the surface of the earth at about 0.4 per cent for each 30 metres increase in altitude. Estimate the percentage reduction in air pressure (to the nearest per cent) in moving from sea-level to an altitude of 1,000 metres.

(a) 14 per cent
(b) 11 per cent
(c) 12 per cent

(d) 13 per cent
(e) 10 per cent

Question 8

The Australian population was estimated at about 22.3 million in 2010 and growing annually at about 1.7 per cent. Use this information to estimate what the population might become in 2050 (to the nearest million).

(a) 40 million
(b) 43 million
(c) 44 million
(d) 42 million
(e) 41 million

Question 9

The Australian population was estimated at about 22.3 million in 2010 and growing annually at about 1.7 per cent. Use this information to estimate the first year in which one might expect the population to exceed 50 million throughout the entire year.

(a) the year 2062
(b) the year 2060
(c) the year 2058
(d) the year 2059
(e) the year 2061

Question 10

The Australian population was estimated at about 22.3 million in 2010 and growing annually at about 1.7 per cent. Use this information to go backwards in time to estimate what the population might have been in 1910 (to the nearest million). (Out of interest later, using an internet search, you might want to look up the actual population in 1910, for comparison with your mathematical prediction.)

(a) 4 million
(b) 5 million
(c) 6 million
(d) 3 million
(e) 7 million

Answers

The answers will be revealed at the end of the module.

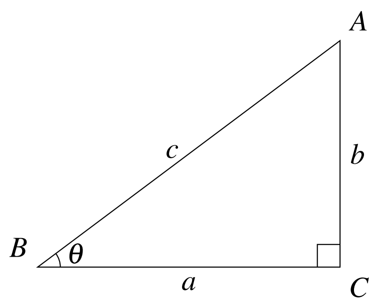
11. Circular Functions and Trigonometry

11.1 Sine, Cosine and Tangent

Welcome to our another journey in the world of trigonometry, the mathematical study that delves into the relationships between the angles and sides of triangles. In this session, we will introduce the sine (sin), cosine (cos), and tangent (tan) of an acute angle—key concepts that form the bedrock of trigonometry and lead us to the broader realm of circular functions.

11.1.1 Introduction to the trigonometric ratios

We begin with a right-angled triangle, where we have sides a, b, and c with c being the hypotenuse. When we draw an acute angle θ between sides a (adjacent to θ) and c (hypotenuse), we set the stage for defining our trigonometric functions:



11.1.1.1 Labelling sides in a Right angled Triangle

We can name the sides in the Right angled triangle based on the reference angle θ. Side 'b' is called as "opposite" because it lies opposite to reference angle theta. Side 'c' is called as 'hypotenuse' because hypotenuse is defined as the longest side of a right angled triangle. Side 'b' is called as 'adjacent' because it lies just adjacent to reference angle θ.

11.1.1.2 Definition of Trigonometric Ratios

$\sin(\theta)$ is the ratio of the opposite side b to the hypotenuse c.

$$\sin(\theta) = \frac{opposite(b)}{hypotenuse(c)}$$

$\cos(\theta)$ is the ratio of the adjacent side a to the hypotenuse c.

$$\cos(\theta) = \frac{adjacent(a)}{hypotenuse(c)}$$

$\tan(\theta)$ is the ratio of the opposite side b to the adjacent side a.

$$\tan(\theta) = \frac{opposite(b)}{adjacent(a)}$$

These functions are often abbreviated for convenience to sin, cos, and tan, respectively.

11.1.2 Trigonometric Identities

11.1.2.1 The Pythagorean Identity

Invoking the Pythagorean theorem, $a^2 + b^2 = c^2$, and dividing through by c^2,

$$(\frac{a}{c})^2 + (\frac{b}{c})^2 = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

we arrive at the elegant identity $\cos^2(\theta) + \sin^2(\theta) = 1$, known as the Pythagorean identity. This fundamental relationship will later be connected to the properties of circles.

11.1.2.2 Complementary Angle Identity

From triangle ABC, you can notice that

$$A + B = 90^\circ$$

Also,

$$\sin(A) = \frac{a}{c}$$

$$\cos(B) = \frac{a}{c}$$

This shows that,

$$\sin(A) = \cos(B)$$

But A is 90-B,

$$\sin(90 - B) = \cos(B)$$

Writing B as θ, we state the identity

$$\sin(90 - \theta) = \cos(\theta)$$

11.1.2.3 Ratio Identity

We have,

$$\sin(\theta) = \frac{b}{c}$$
$$\cos(\theta) = \frac{a}{c}$$

If we take the ratio of sin and cos,

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} = \tan(\theta)$$

Thus,

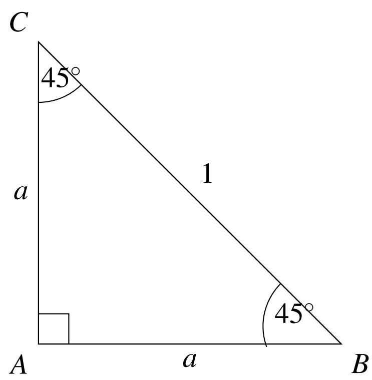
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

11.1.3 Special Angles and Their Trigonometric Values

It's time to work out some common trigonometric values.

11.1.3.1 Trigonometric Ratios for 45 degrees

Consider a right angle isosceles triangle with hypotenuse 1 and legs of length a.



Since it is a right angled triangle, we can use pythagorean theorem to find 'a'.

$$a^2 + a^2 = 1$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \frac{1}{\sqrt{2}}$$

Now, if we try to take 45° angle as reference angle and find the trigonometric ratios, we get

$$\sin(45^\circ) = \frac{a}{1} = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}$$

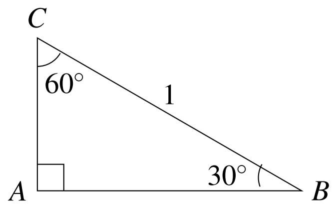
$$\cos(45^\circ) = \frac{a}{1} = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}$$

$$\tan(45^\circ) = \frac{a}{a} = 1$$

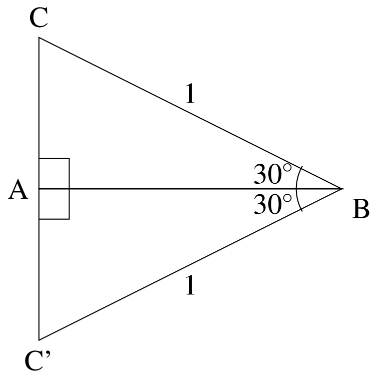
Remember these values. It will be helpful in your real life someday.

11.1.3.2 Trigonometric Ratios for 30 and 60 degrees

Next, consider a right angled triangle with hypotenuse 1 and the angle theta equal to 30 degrees.



Similar to what we did previously, we will first try to figure out what AC and AB will be. Then, find the trigonometric ratios.
Let me disclose that AC = $\frac{1}{2}$. To see how, reflect the triangle ABC about the side AB.



Now, we see that all angles of triangle CBC' are 60 degrees. Thus, it is an equilateral triangle.
All sides of equilateral triangle are equal. This implies CC' = 1.
Now, since AC' is just reflection of AC, both are equal which implies $AC = \frac{1}{2}$ just like I stated earlier.
Using pythagoras theorem in triangle ABC, we get

$$AC^2 + AB^2 = BC^2$$

$$(\frac{1}{2})^2 + AB^2 = 1^2$$

$$AB^2 = 1 - \frac{1}{4}$$

$$AB^2 = \frac{3}{4}$$

$$AB = \frac{\sqrt{3}}{2}$$

Now, let's try to find trigonometric ratios taking 30° as reference angle.

$$\sin(30°) = \frac{AC}{BC} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\cos(30°) = \frac{AB}{BC} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\tan(30°) = \frac{AC}{AB} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Now, taking angle C which is 60° as reference angle,

$$\sin(60^\circ) = \frac{AB}{BC} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\cos(60^\circ) = \frac{AC}{BC} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\tan(60^\circ) = \frac{AB}{AC} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

11.1.3.3 Trigonometric ratios for 0 and 90 degrees

There are also trigonometric ratios for 0 and 90 degrees though you might not be able to construct a right angled triangle for these reference angles. It's time for you to get creative with these angles which I want you to ponder yourself. I will just give the trigonometric values.

$$\sin(0^\circ) = 0 \qquad \sin(90^\circ) = 1$$

$$\cos(0^\circ) = 1 \qquad \sin(90^\circ) = 1$$

$$\tan(0^\circ) = 0 \qquad \tan(90^\circ) = \frac{1}{0}$$

11.1.3.4 Table of trigonometric Values

Now, we summarize all the trigonometric values we learnt in the table below:

	0°	30°	45°	60°	90°
SIN	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
COS	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
TAN	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not Defined

Figure 11.1: Table of Trigonometric Values

11.1.3.5 Some Remarks on the trigonometric values

We've examined special cases where θ is 45, 30, and 60 degrees, revealing natural surd expressions involving $\sqrt{2}$ and $\sqrt{3}$. For instance, in an isosceles right-angled triangle (where $\theta = 45^\circ$), both a and b are equal to $\frac{1}{\sqrt{2}}$, leading to $\sin(45^\circ) = \cos(45^\circ) = \frac{1}{\sqrt{2}}$ and $\tan(45^\circ) = 1$.

[0]Image 11.2 fromhttps://www.cuemath.com/trigonometry/trigonometric-table/

When $\theta = 30^\circ$, we deduced that the opposite side is half the hypotenuse, and the adjacent side, by Pythagorean theorem, is $\frac{\sqrt{3}}{2}$. This gives us $\sin(30^\circ) = \frac{1}{2}$, $\cos(30^\circ) = \frac{\sqrt{3}}{2}$, and $\tan(30^\circ) = \frac{1}{\sqrt{3}}$.

For $\theta = 60^\circ$, we simply use the same triangle to find $\sin(60^\circ) = \frac{\sqrt{3}}{2}$, $\cos(60^\circ) = \frac{1}{2}$, and $\tan(60^\circ) = \sqrt{3}$.

In advanced trigonometry, if you are trying to find the trigonometric ratios for non-special angles, the result might involve more complicated surd expressions but we won't need them. Typically it suffices to use your calculator for most angles.

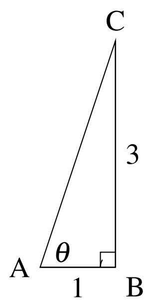
Another thing I would like to remark is the verification of complementary angle identity.

$$\frac{\sqrt{3}}{2} = \sin(60^\circ) = \cos(90^\circ - 60^\circ) = \cos(30^\circ)$$

Though we have taken the measure of angles in degree in this section, the more natural system of angle measurement in calculus is radian which we will discuss in the next section.

11.1.4 Trigonometry Without Known Angles

We also tackle a scenario where we could determine trigonometric values from a single known ratio without the actual angle value. For example, given $\tan(\theta) = 3$, we can see that this is satisfied in a right angled triangle where opposite = 3 and adjacent = 1 because $\tan(\theta) = \frac{opposite}{adjacent}$.



Now, using the pythagorean Theorem in triangle ABC, we get

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{1+9} = \sqrt{10}$$

We can now deduce that $\sin(\theta) = \frac{3}{\sqrt{10}}$ and $\cos(\theta) = \frac{1}{\sqrt{10}}$.

You can confirm the correctness of your answer using pythagorean identity.

$$\sin^2(\theta) + \cos^2(\theta) = \left(\frac{3}{\sqrt{10}}\right)^2 + \left(\frac{1}{\sqrt{10}}\right)^2 = \frac{9}{10} + \frac{1}{10} = \frac{10}{10} = 1$$

which verifies.

As you progress, please attempt the exercises provided. This foundational knowledge will be instrumental as we advance to the theory of circular functions in our next section.

Thank you for engaging with this material, and I eagerly anticipate our continued journey through the fascinating world of trigonometry. Remember, practice is key to mastery, so take your time with the exercises, and I'll see you in the next section.

11.1.5 Practice Quiz

Question 1

Which of the following is the exact value of $\sin 30^\circ$?

(a) $\frac{1}{2}$
(b) $\frac{1}{3}$
(c) $\frac{1}{\sqrt{2}}$
(d) $\frac{\sqrt{3}}{2}$
(e) $\frac{2}{\sqrt{3}}$

Question 2

Which of the following is the exact value of $\cos 30^\circ$?

(a) $\frac{1}{2}$
(b) $\frac{1}{3}$
(c) $\frac{\sqrt{3}}{2}$
(d) $\frac{2}{\sqrt{3}}$
(e) $\frac{1}{\sqrt{2}}$

Question 3

Which of the following is the exact value of $\tan 45^\circ$?

(a) $\frac{1}{\sqrt{3}}$
(b) $\sqrt{3}$
(c) $\frac{1}{4}$
(d) 1
(e) $\frac{1}{2}$

Question 4

Which of the following is the exact value of $\tan 60^\circ$?

(a) $\sqrt{2}$
(b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$
(d) $\frac{1}{\sqrt{2}}$
(e) 1

Question 5

Which of the following is the exact value of $\sin^2 45^\circ$?

(a) $\frac{1}{\sqrt{2}}$
(b) $\frac{1}{2}$
(c) 1
(d) $\frac{3}{4}$
(e) $\frac{1}{4}$

Question 6

Which of the following is the exact value of $\cos^2 60^\circ$?

(a) $\frac{3}{4}$
(b) $\frac{1}{2}$
(c) 1
(d) $\frac{\sqrt{3}}{2}$
(e) $\frac{1}{4}$

Question 7

Which one of the following statements is false?

(a) $\sin^2 60^\circ = 3\cos^2 60^\circ$
(b) $\sin 45^\circ = \cos 45^\circ$
(c) $\sin 30^\circ = \cos 30^\circ$
(d) $\sin 20^\circ = \cos 70^\circ$
(e) $\sin^2 30^\circ = 3\cos^2 30^\circ$

Question 8

Suppose that θ is an acute angle such that $\tan\theta = 4$. Find $\sin\theta$.

(a) $\frac{4}{\sqrt{17}}$
(b) $\frac{1}{\sqrt{17}}$
(c) $\frac{\sqrt{17}}{4}$
(d) $\frac{1}{\sqrt{15}}$
(e) $\frac{4}{\sqrt{7}}$

Question 9

Suppose that θ is an acute angle such that $\tan\theta = 4$. Find $\cos\theta$.

(a) $\frac{1}{\sqrt{15}}$
(b) $\frac{1}{\sqrt{17}}$
(c) $\frac{4}{\sqrt{17}}$
(d) $\frac{4}{\sqrt{15}}$
(e) $\frac{7}{\sqrt{17}}$

Question 10

Suppose that θ is an acute angle such that $\cos\theta = \frac{2}{3}$. Find $\tan\theta$.

(a) $\frac{2}{\sqrt{13}}$
(b) $\frac{2}{\sqrt{3}}$
(c) 3
(d) $\sqrt{13}$
(e) $\frac{\sqrt{5}}{3}$

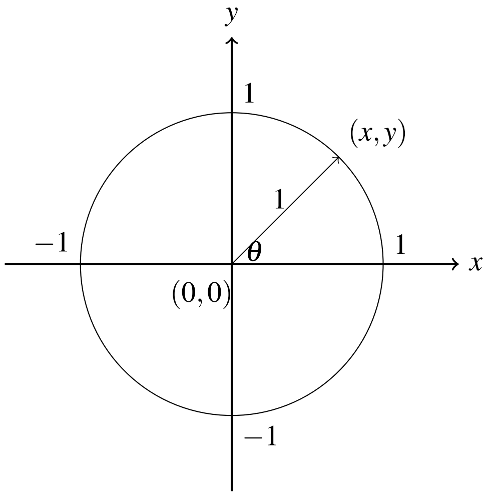
Answers

The answers will be revealed at the end of the module.

11.2 The unit circle and Trigonometry

Embarking on a journey through the realm of trigonometry, in this section, we explore the radian measure of angles, a pivotal concept that extends the definitions of sine, cosine, and tangent functions using the unit circle. This leads to the so called circular functions, y=sin(x) and y = cos(x) and their graphs.

11.2.1 Unit Circle and Radian Measure



We begin with the unit circle, a circle centered at the origin with a radius of 1. Any point x, y on this circle satisfies the equation $x^2 + y^2 = 1$. An angle θ, formed between the positive x-axis and the radius, is used to define the trigonometric functions for any angle.

But how do we measure angles?

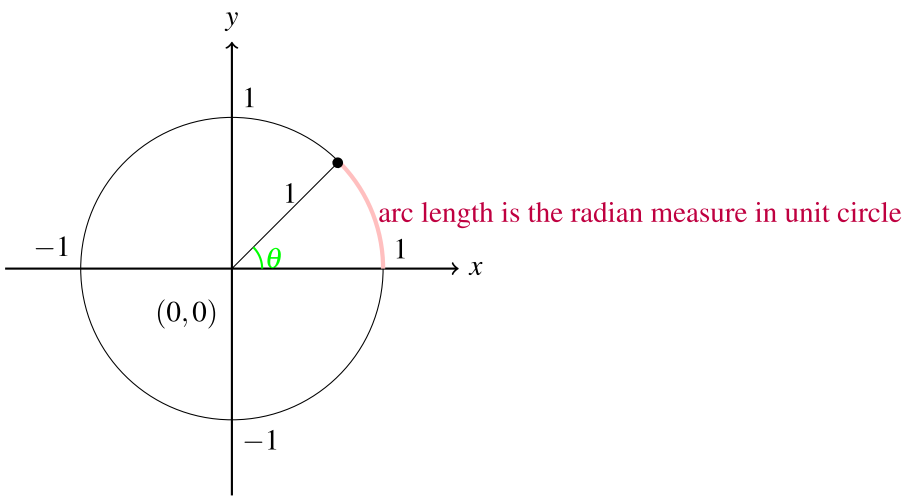
11.2.2 Degrees vs. Radians

Traditionally, angles have been measured in degrees. It's a familiar system where the angles of a triangle add up to 180 degrees. However, when it comes to calculus, radians are more suitable.
First, let us remind ourselves, there are 360 degrees in single full revolution of the circle. This convention was introduced by the ancient Babylonian Mathematicians. It's useful because 360 is a nice number with a lot of divisors:

$$360 = 2 \times 180 = 4 \times 90 = 8 \times 45 = 6 \times 60 = 2^3 \times 3^2 \times 5$$

And there are so many different divisors that capture lots of useful common angles.
The concept of measuring angles in radians comes from the arc length on the unit circle. **Radians measure the arc length on the unit circle**.

But what does that mean?



Here's the unit circle again. The point subtends an angle theta with positive X-axis. The red distance that we travel along the circle to get to our point is the arc length. And this is defined to be the radian measure of angle (for the case of unit circle).

A full revolution of the circle, which is 360 degrees, corresponds to an arc length of 2π radians, since the perimeter of the unit circle is $2\pi \times$ radius = $2\pi \times 1 = 2\pi$.
Hence, $360^\circ = 2\pi$ radians.
Dividing both sides by 2,

$$180^\circ = \pi \text{ radians}$$

Again, dividing both sides by 2,

$$90^\circ = \frac{\pi}{2} \text{ radians}$$

and so on.

11.2.3 Conversion Between Degrees and Radians

The conversion between degrees and radians is straightforward. Since there are 2π radians in a full revolution (360 degrees), one degree is equivalent to $\frac{\pi}{180}$ radians. Conversely, one radian equals $\frac{180}{\pi}$ degrees, which is approximately 57.3 degrees.

With radian measure, we drop the unit radian and just use number. So if an angle is quoted as a number, you know that radians are intended.
eg:

$$360^\circ = 2\pi \qquad 180^\circ = \pi \qquad 90^\circ = \frac{\pi}{2}$$

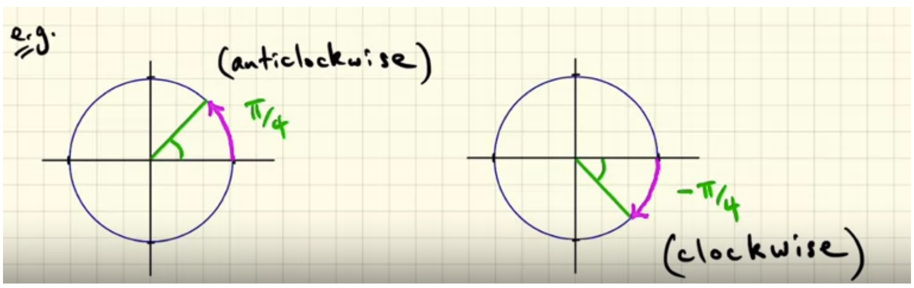
$$45^\circ = \frac{\pi}{4} \qquad 30^\circ = \frac{\pi}{6} \qquad 60^\circ = \frac{\pi}{3}$$

$$1^\circ = \frac{\pi}{180} \qquad 1 = (\frac{180}{\pi})^\circ = 57.3^\circ$$

11.2.4 Properties of Angles

11.2.4.1 Polarity

Just like numbers, angles can be positive, negative or zero.
Positive angles correspond to anti-clockwise rotation.
Negative angles correspond to clockwise rotation.



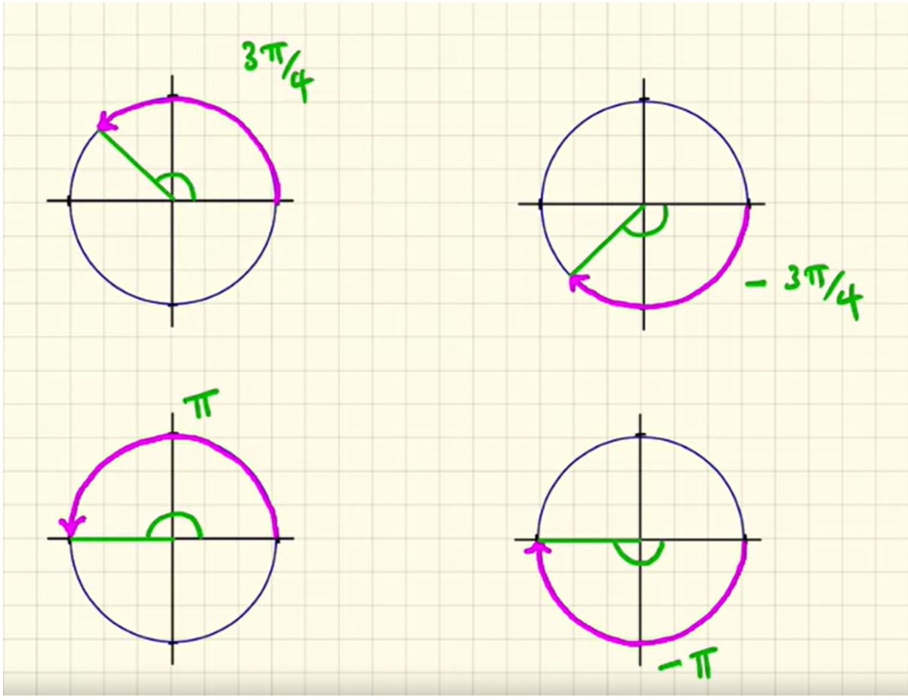


Figure 11.2: Depiction of Positive and Negative Angles

Note with this last pair, π and $-\pi$, we end up at the same point on the unit circle.

11.2.4.2 Equivalency of Angles

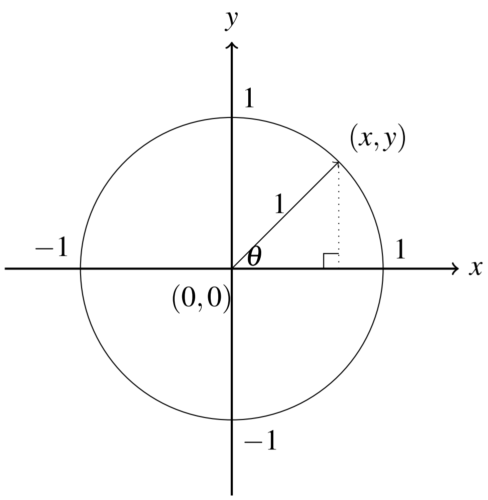
Angles are said to be equivalent if you end up with the same point on the unit circle. eg: *pi* and $-\pi$ are equivalent.
Angles that are large in magnitude can involve multiple revolutions of the unit circle. eg: 4 π uses 2 revolutions. 6 π uses 3 revolutions and so on.
The angles 2 π, 4 π and 6 π all end up back on the same starting point. so, they are equivalent.

11.2.5 Circular Functions

11.2.5.1 New definition of sin, cos, tan as circular functions

In the last section, we defined sin, cos and tan of an acute angle in terms of certain ratios of side lengths of right angled triangle. It turns out there is much more general way of defining these for any angle, exploiting coordinates of points that wind around the unit circle.

[0]Image 11.2 from MOOC Single Variable Calculus (University of Sydney)

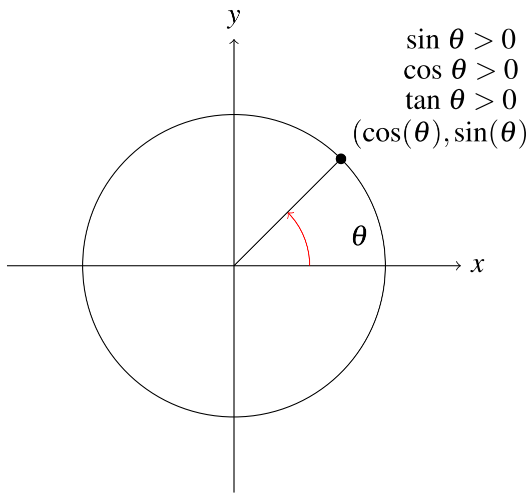


Here is again an unit circle with centre(0,0) and a point(x,y). We now define $\cos(\theta)$ to be the x-coordinate and $\sin(\theta)$ to be the y-coordinate. You can make sense of this because x is just horizontal component (adjacent side to θ) and y is vertical component (opposite side to θ). And since hypotenuse is 1, $\cos(\theta)$ would just be adjacent side and $\sin(\theta)$ would just be opposite side.

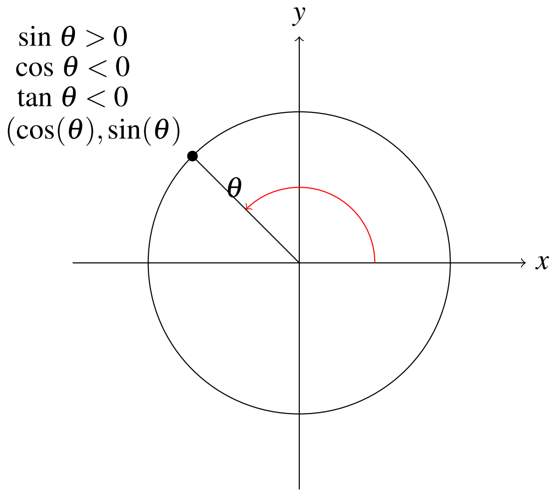
Since, the right angled triangle definition of trigonometric rations limited the θ to be less than 90 degree, this definition of trigonometric ratios will be helpful even when θ can be anything.

The sine and cosine functions are referred to as circular functions because they can be represented by coordinates on the unit circle. For any angle θ, $\cos(\theta)$ is the x-coordinate, and $\sin(\theta)$ is the y-coordinate of the point on the unit circle. The tangent function, $\tan(\theta)$, is defined as $\frac{\sin(\theta)}{\cos(\theta)}$, provided that $\cos(\theta) \neq 0$.

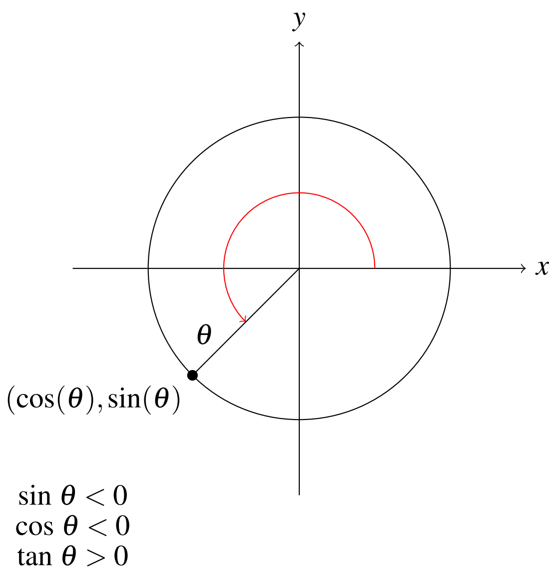
11.2.5.2 Sign of sin, cos and tan in various quadrants of the circle



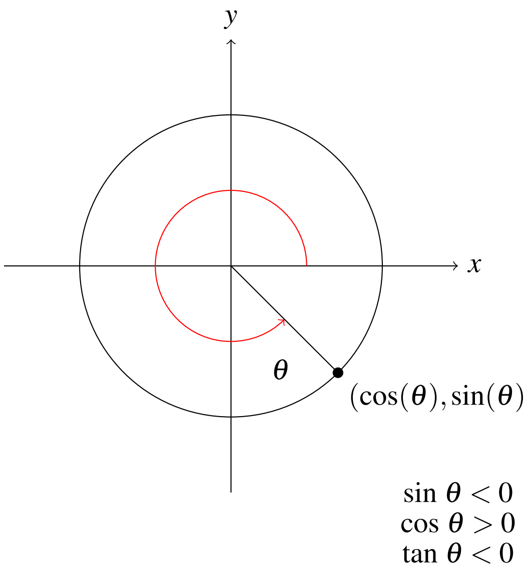
In the first quadrant of the circle, which is also referred to as the first quarter, both x and y coordinate are positive. Consequently, the trigonometric functions sine (sin) and cosine (cos) of an angle θ are positive, and so is the tangent (tan), because it is just the ratio of sine to cosine.



Moving to the second quadrant, where θ becomes an obtuse angle, the x coordinate is negative and y coordinate is positive. Here, $\sin(\theta)$ remains positive because the y-coordinate is positive, but $\cos(\theta)$ turns negative as the x-coordinate is negative. As a result, $\tan(\theta)$, being the ratio of sine to cosine, is negative.



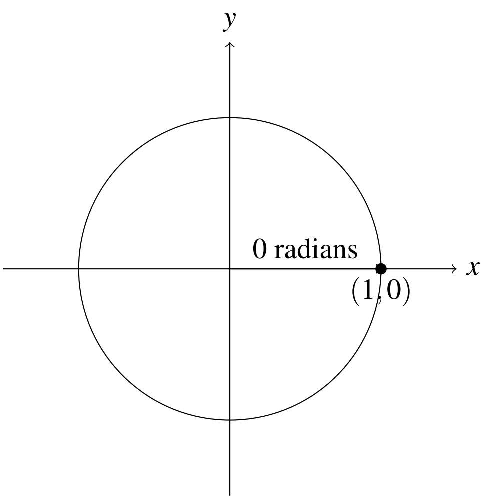
In the third quadrant, both coordinates are negative, which makes both $\cos(\theta)$ and $\sin(\theta)$ negative. However, since $\tan(\theta)$ is the ratio of two negative numbers, it becomes positive.



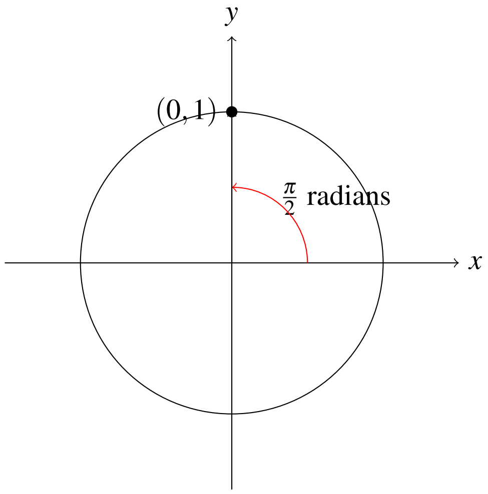
Finally, in the fourth quadrant, $\cos(\theta)$ is positive and $\sin(\theta)$ is negative, leading to a negative $\tan(\theta)$. This is because the x-coordinate is positive while the y-coordinate is negative.

11.2.6 Important Extreme Cases on the boundaries between the quadrants

In trigonometry, certain angles located at the boundaries between the quadrants of the Cartesian plane are considered special cases. These angles correspond to positions where one of the coordinates is zero and the other is either positive or negative one. Specifically, these angles are 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ radians.



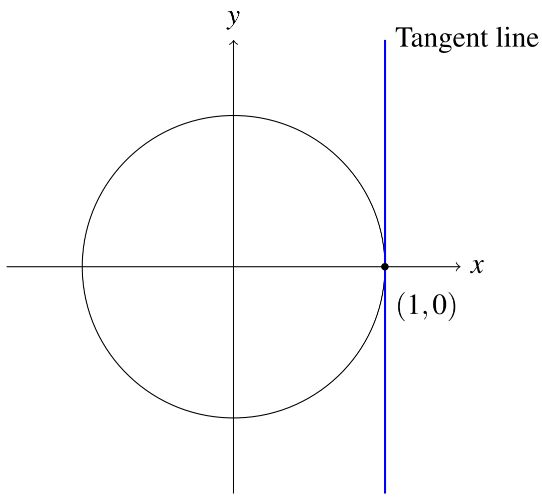
For instance, at an angle of 0 radians, which lies along the positive x-axis, the point on the unit circle is (1,0). Here, the cosine function, $\cos(0)$, is equal to 1, and the sine function, $\sin(0)$, is equal to 0. Consequently, the tangent function, $\tan(0)$, which is the ratio of sine to cosine, is $\frac{0}{1}$, resulting in a value of 0.



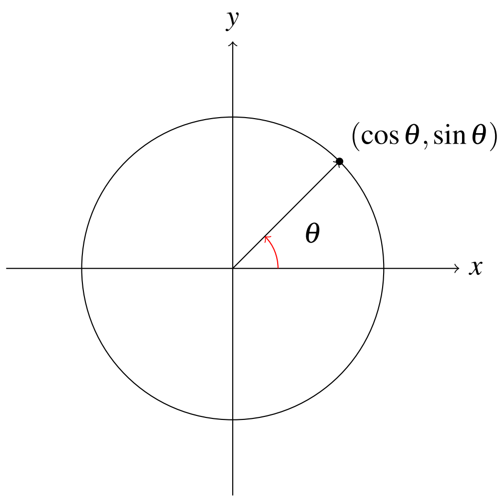
At an angle of $\frac{\pi}{2}$ radians, corresponding to 90 degrees and lying along the positive y-axis, the point on the unit circle is (0,1). In this position, $\sin\left(\frac{\pi}{2}\right)$ is 1, and $\cos\left(\frac{\pi}{2}\right)$ is 0. Attempting to calculate $\tan\left(\frac{\pi}{2}\right)$ becomes $\frac{1}{0}$ leading to division by zero, which is undefined in mathematics. This is why the tangent of $\frac{\pi}{2}$ radians is considered undefined.

The points for angles π and $\frac{3\pi}{2}$ radians, which lie on the negative x-axis and negative y-axis respectively, follow similar logic in terms of their sine, cosine, and tangent values. The geometric significance of the tangent being undefined at $\frac{\pi}{2}$ radians is related to the concept of the tangent line to the unit circle, which will be explored further in the upcoming subsection where we will try to understand the geometric significance of the fact that $\tan(\frac{\pi}{2})$ is undefined.

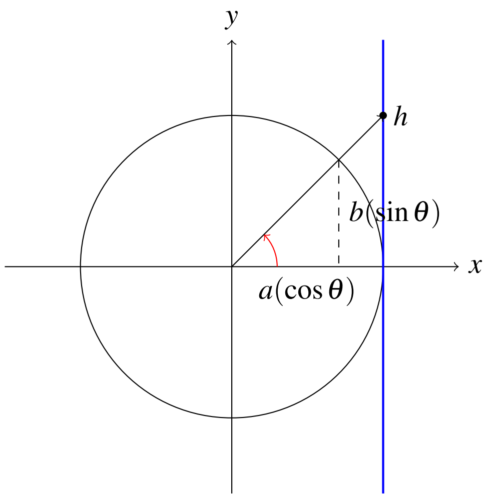
11.2.7 Tangent Function and Its Geometric Significance



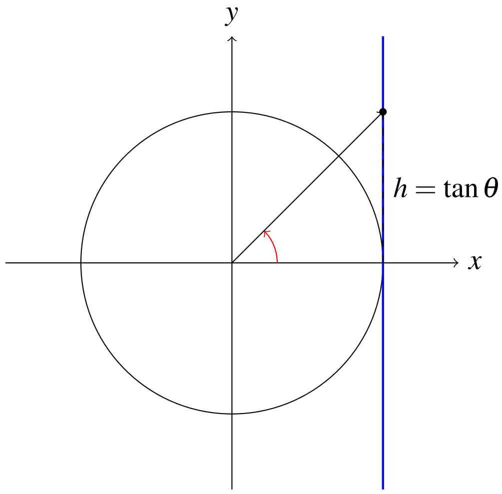
Let's revisit the unit circle again and delve into the concept of tangent lines and how they relate to trigonometric functions. Imagine the unit circle on a Cartesian plane, and draw a vertical line that just touches the circle at the point where the x-coordinate is one. This line is known as a tangent line to the unit circle because it touches the circle at exactly one point without cutting across it.



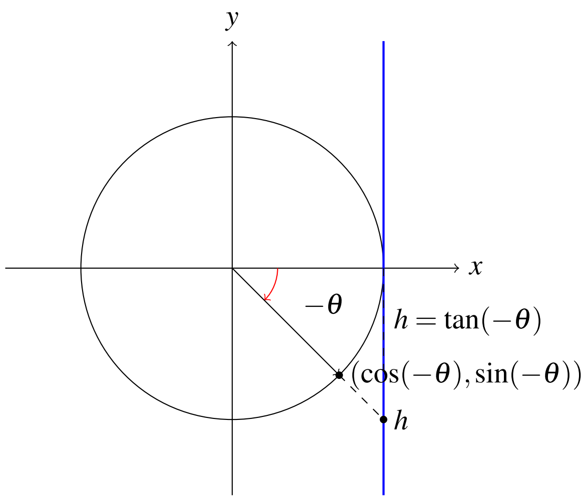
Consider a point on the unit circle that forms an angle θ with the positive x-axis. If we extend a line from the origin (the center of the circle) through this point, it will intersect the tangent line. This intersection creates a right-angled triangle within the circle, where the horizontal side length (along the x-axis) is denoted as a and the vertical side length (along the y-axis) is denoted as (b). In trigonometric terms, a represents $\cos(\theta)$, and b represents $\sin(\theta)$, which are the coordinates of the point on the unit circle.



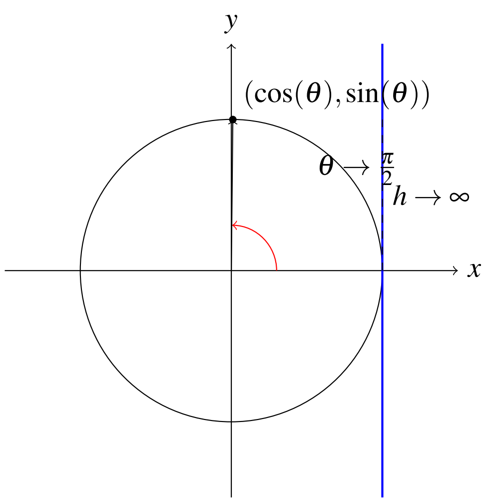
Now, let's focus on the point where the extended radius meets the tangent line. The vertical distance from this intersection point to the x-axis is labeled as h. This distance h can be understood by examining similar triangles formed by the radius and the tangent line. The ratio of b to a, which comes from the small right-angled triangle inside the unit circle, is equal to the ratio of h to one, derived from the larger right-angled triangle that includes the tangent line. Therefore, h is simply $\frac{b}{a}$, which is the definition of the tangent function, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.



This relationship between the tangent line and the tangent function gives rise to the term "tangent" in trigonometry. The length h represents $\tan(\theta)$ and is the distance you measure from the point where the extended radius intersects the tangent line to the x-axis.

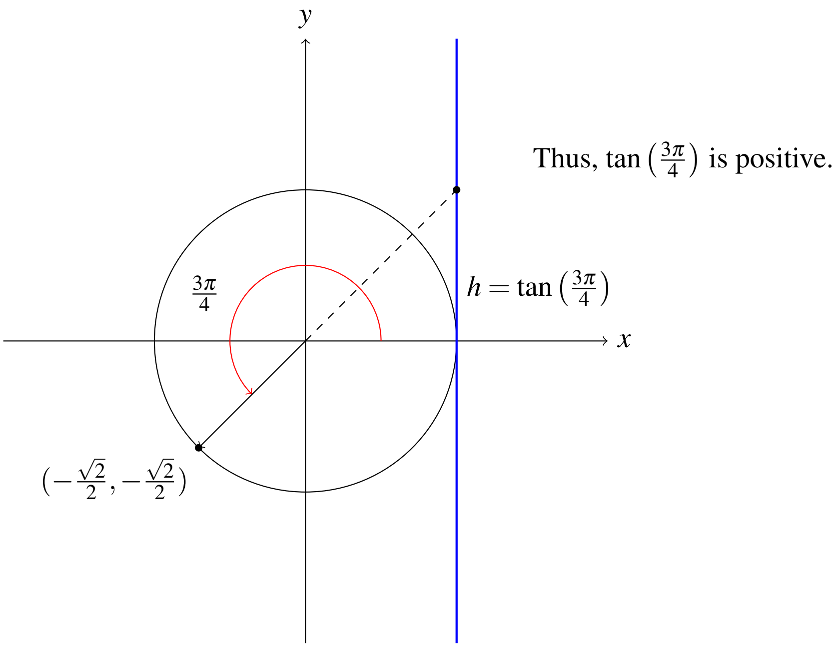


If θ is negative, indicating a clockwise rotation around the unit circle and staying within the fourth quadrant, the method still applies. However, the tangent line is intersected below the x-axis, resulting in a negative value for $\tan(\theta)$.



The tangent function becomes particularly interesting when θ approaches $\frac{\pi}{2}$ radians, or 90 degrees. As the radius rotates anticlockwise from the point (1,0) on the x-axis towards the point (0,1) on the y-axis and extends to touch the tangent line, the vertical distance h increases without bound. As θ gets closer to $\frac{\pi}{2}$, the value of $\tan(\theta)$ grows indefinitely, theoretically reaching infinity. This is why $\tan\left(\frac{\pi}{2}\right)$ is undefined—it represents a situation where the vertical distance h becomes infinitely large, a concept that cannot be represented by a finite number.

Now that we have seen two visual demostrations of tangent value based on tangent line, here is another exciting one. How is the tangent value positive when θ lies in third quadrant.

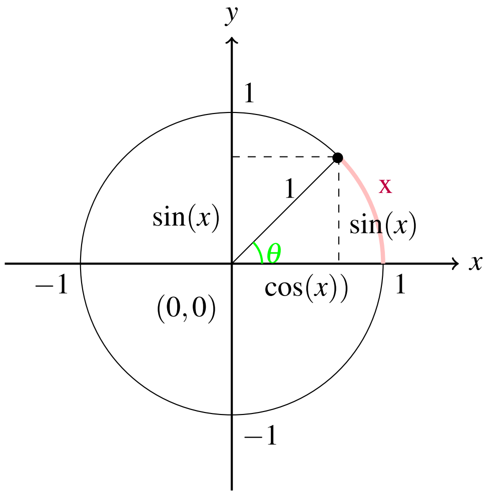


With this geometric intuititon, I think you can also see why tan value is negative in second quadrant.

11.2.8 Graphs of Sine, Cosine and Tangent

It's time to delve into the concepts of sine and cosine functions using the unit circle, and how they relate to angles and arc lengths.

When we talk about $\sin(\theta)$ and $\cos(\theta)$, we're considering θ (theta) as the input to a function. Traditionally, in mathematics, we often use x to represent the input variable. So, if we look at the unit circle, we can think of x as the arc length that corresponds to the angle θ. In this context, x is essentially the angle itself, measured in radians. Here's how you can visualize it on the unit circle:



Sin(x): To find the sine of an angle x, you can move directly across the vertical axis. The intersection on the vertical axis gives you the value of sin(x.)

Cos(x): Similarly, to find the cosine of an angle x, you can move straight down to the horizontal axis . The intersection on the horizontal axis gives you the value of cos(x).

11.2.8.1 Graph of Sin(x)

Now, focusing on sin(x), let's observe its behavior as x varies:

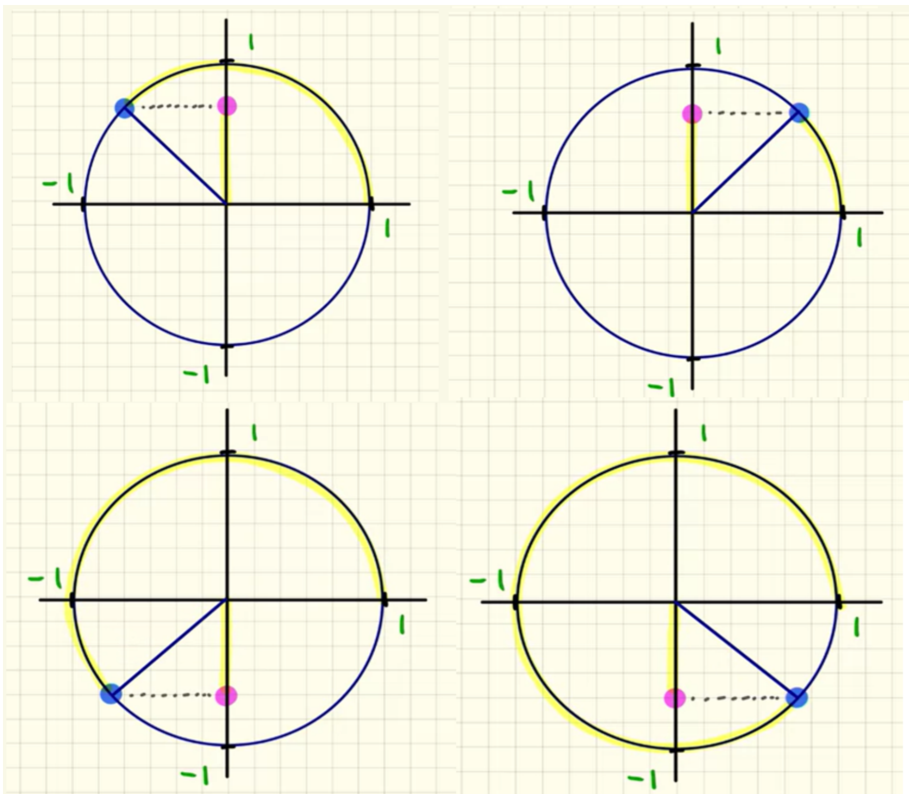
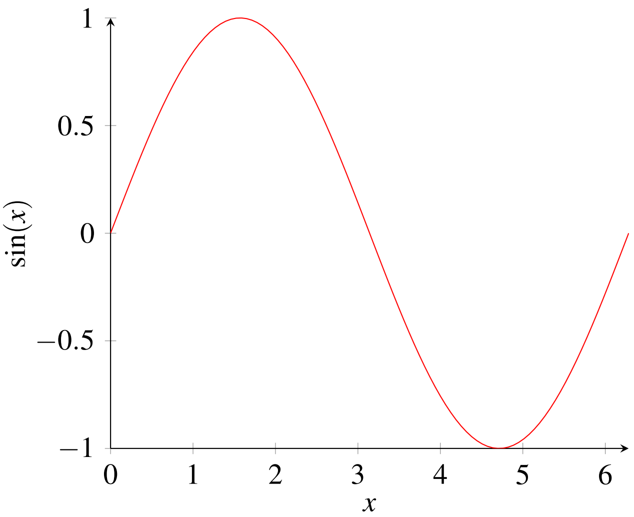


Figure 11.3: Variation of sin(x) with respect to x

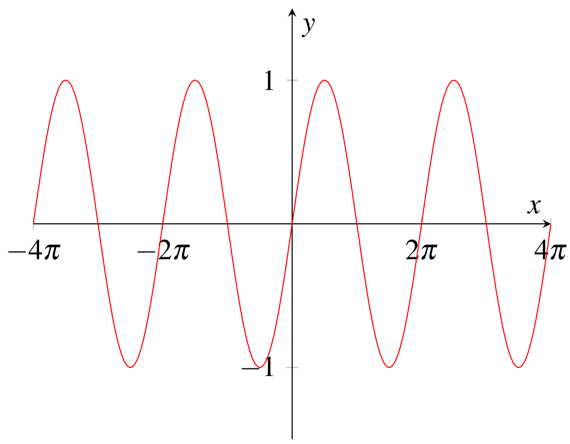
[0]Image 11.3 from MOOC Single Variable Calculus (University of Sydney)

As x increases from 0 to $\frac{\pi}{2}$, (a quarter of the way around the unit circle), sin(x) increases from 0 to 1. This is represented by a blue dot moving along the circle, carrying a pink dot up the vertical axis. Continuing from $\frac{\pi}{2}$ to π , sin(x) decreases back down from 1 to 0. From π to $\frac{3\pi}{2}$, sin(x) goes from 0 to -1, moving downwards on the vertical axis. Finally, as x goes from $\frac{3\pi}{2}$ to 2π , sin(x) returns from -1 back to 0.

As x keeps rotating around the unit circle, sin(x) oscillates between 0, 1, 0, -1, and back to 0. This oscillation is continuous and repeats indefinitely as x winds around the circle.



By defining the function with the rule y=sin(x) , we can then graph this oscillation. For x values ranging from 0 to 2π (6.28). , the graph captures this oscillation with a pink dot representing the y values. Y-values of the function is 0 at 3.14 (π) and is 1 at 1.57 ($\frac{\pi}{2}$). This creates a wave-like curve known as a sinusoidal curve.



The sinusoidal shape continues to repeat itself as x moves along the positive x-axis, reflecting the anticlockwise movement around the unit circle. Conversely, if x moves in the negative direction (clockwise movement), the sine function still oscillates, and the sinusoidal graph repeats endlessly in both directions.

Thus, the full sine curve extends along the entire real line, creating a wave that oscillates infinitely often. This is the essence of the sine function, a fundamental concept in trigonometry and mathematics.

11.2.8.2 Graph of cos(x)

Let's explore the behavior of the cosine function in a more detailed and comprehensive manner. The cosine function, denoted as cos(x), is one of the fundamental trigonometric functions and it's closely related to the unit circle.

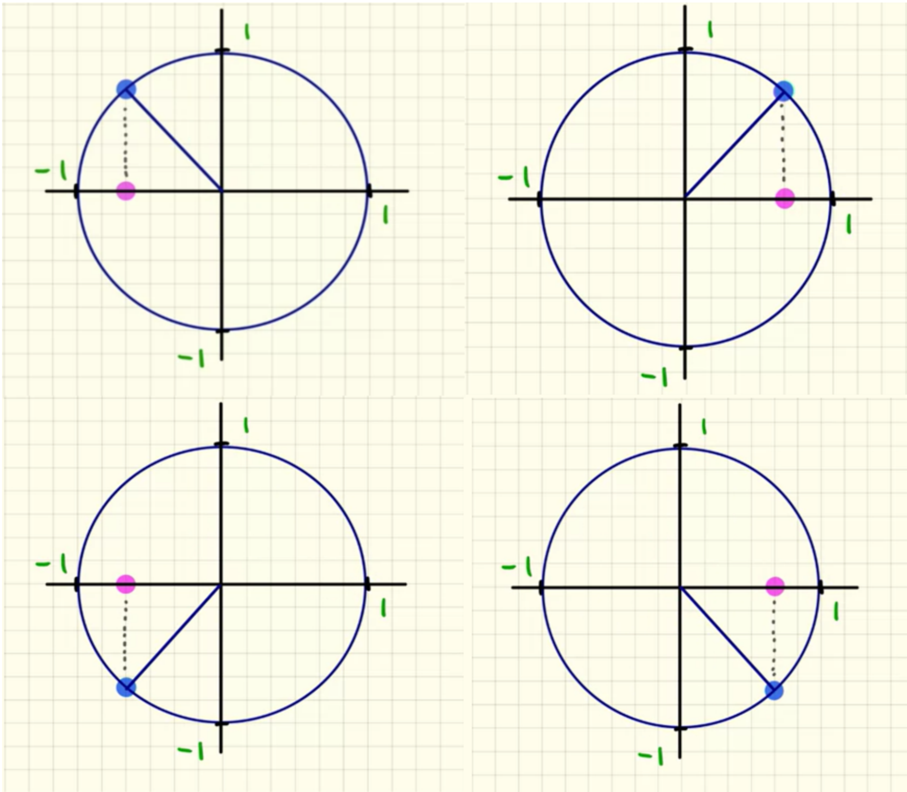


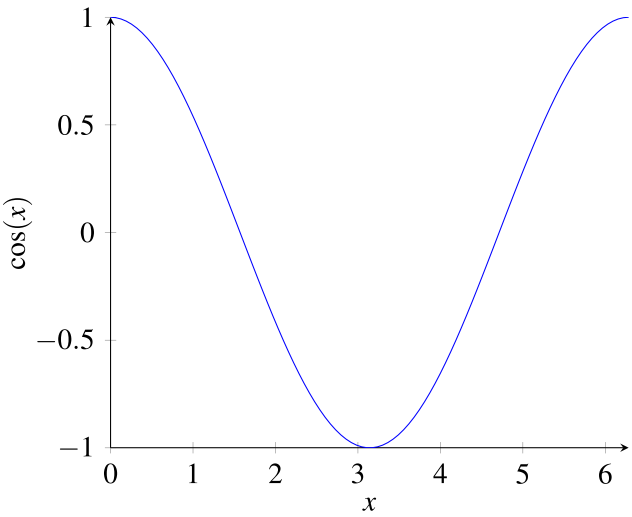
Figure 11.4: Variation of cos(x) with respect to x

Imagine the unit circle in front of you. The cosine of an angle x is represented by a point on the horizontal axis, which we'll mark with a pink dot for visualization. As the angle x varies from 0 to 2π (which is a full rotation around the circle), the position of the pink dot changes, reflecting the value of cos(x).

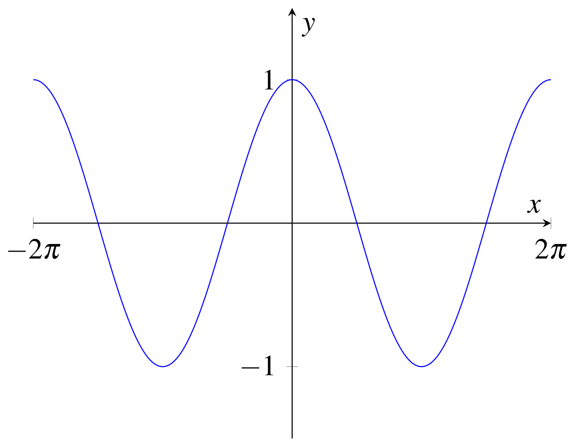
Here's a step-by-step breakdown of this movement:

Starting Point: $x = 0$**.** At $x = 0$, the pink dot is at 1 on the horizontal axis, so $\cos(0) = 1$.
Quarter Circle: $x = \frac{\pi}{2}$**.** As x increases to $\frac{\pi}{2}$, the pink dot moves leftward to 0, so $\cos\left(\frac{\pi}{2}\right) = 0$.
Half Circle: $x = \pi$**.** Continuing from $x = \frac{\pi}{2}$ to $x = \pi$, the pink dot moves further left to -1, so $\cos(\pi) = -1$.
Three-Quarter Circle: $x = \frac{3\pi}{2}$**.** While x goes from $x = \pi$ to $\frac{3\pi}{2}$, the pink dot comes back from -1 to 0, so $\cos\left(\frac{3\pi}{2}\right) = 0$.
Full Circle $x = 2\pi$: Completing the rotation at x=2π, the pink dot returns to 1 from 0, so cos($\frac{\pi}{2}$)=1, bringing us back to the starting point.

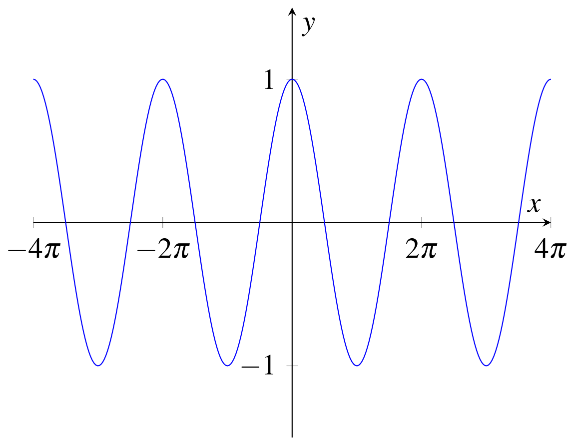
[0]Image 11.4 from MOOC Single Variable Calculus (University of Sydney)



As we continue to rotate around the unit circle, the value of cos(x) repeats this pattern indefinitely. This periodic nature of the cosine function is what creates its wave-like graph.



Now, let's define the function with the rule y=cos(x) . The graph of this function captures the oscillation of cos(x) as x varies from 0 to 2π . The resulting curve is a wave that starts at 1, dips down to -1, and returns to 1, completing one cycle of oscillation.

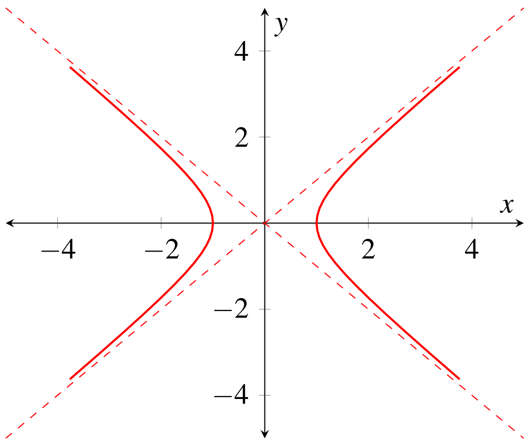


This pattern continues endlessly in both directions along the x-axis, forming the cosine curve. When you compare this curve to the sine curve, you'll notice they are identical in shape but shifted

horizontally.
This shift is known as a phase shift.

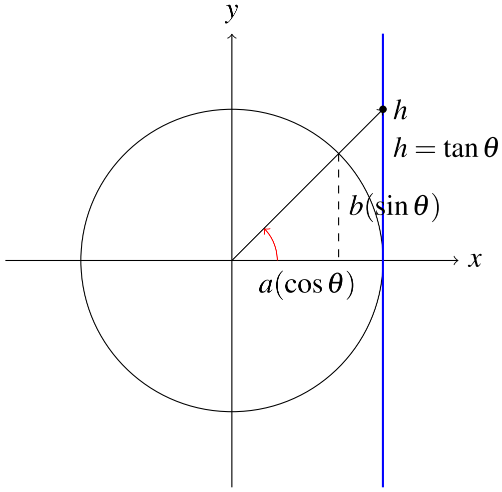
Both the sine and cosine functions are described as sinusoidal because of their wave-like patterns. They are also called circular functions because they can represent points on the unit circle when paired together, with cos(x) representing the x-coordinate and sin(x) representing the y-coordinate of a point on the circle.

While we've focused on circular functions, there are also hyperbolic functions (cosh(x),sinh(x)) which when plotted for different values of x produce a hyperbola, but that's a topic for another discussion.



11.2.8.3 Graph of tan(x)

Lastly, the function y=tan(x), which represents the tangent corresponding to an angle, has a very distinct and interesting graph.

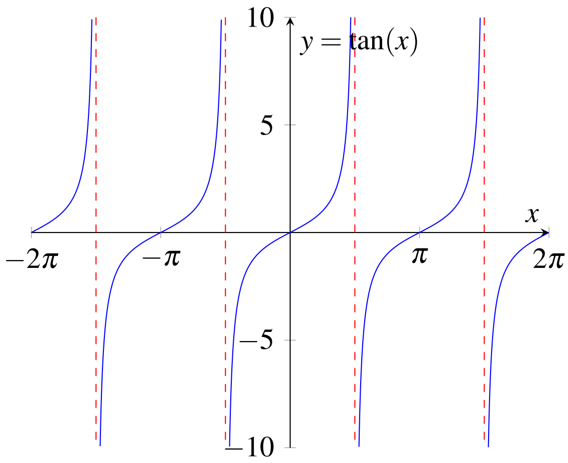


Here's a step-by-step breakdown of the change in the value of tan(θ) as θ goes from 0 to 2π:

Starting Point: $\theta = 0$**.** At $\theta = 0$, the length of tangent is 0.
Quarter Circle: $\theta = \frac{\pi}{2}$**.** As θ increases to $\frac{\pi}{2}$, the length of tangent increases to infinity.
Half Circle: $\theta = \pi$**.** Continuing from $\theta = \frac{\pi}{2}$ to $\theta = \pi$, we have to trace the tangent backward from the center. Thus, we get the negative value of tangent which progresses from -infinity to 0.

Three-Quarter Circle: $\theta = \frac{3\pi}{2}$**.** While θ goes from $\theta = \pi$ to $\frac{3\pi}{2}$, we have to trace the tangent backwards from the centre. Thus, we get postive value of tangent which progresses from 0 to infinity.
Full Circle $\theta = 2\pi$: While completing the rotation by going from $\theta = \frac{3\pi}{2}$ to θ=2π, the tangent now goes from -infinity to 0.

We can notice that at several points, curve suddenly breaks which can visualised in the graph of y=tan(x) below.



The red dashed lines which the graph of tan(x) seems to approach but never really touches are called asymptotes. In this case, since these lines are vertical, they are called vertical asymptotes. There are also other kind of asymptotes known as horizontal and oblique asymptotes that exist.

This series has covered the transition from degrees to radians, the definition of circular functions using the unit circle, and the geometric interpretation of the tangent function. The sinusoidal nature of the sine and cosine graphs has been illustrated, and readers are encouraged attempt the exercises provided.

Thank you for engaging with this section. We hope this comprehensive overview has enriched your understanding of trigonometry, and we look forward to your continued learning in our upcoming sections.

11.2.9 Practice Quiz

Question 1

Express the angle 300° in radians.
(a) $\frac{5\pi}{7}$ radians
(b) $\frac{5\pi}{6}$ radians
(c) $\frac{5\pi}{3}$ radians
(d) $\frac{7\pi}{4}$ radians
(e) $\frac{10\pi}{3}$ radians

Question 2

Express the angle $\frac{5\pi}{4}$ radians in degrees.
(a) 225°
(b) 135°

(c) 210°
(d) 245°
(e) 190°

Question 3

Which one of the following angles is equivalent to $\frac{3\pi}{4}$?
(a) $\frac{9\pi}{4}$ radians
(b) $\frac{-7\pi}{4}$ radians
(c) $\frac{7\pi}{4}$ radians
(d) $\frac{3\pi}{4}$ radians
(e) $\frac{-3\pi}{4}$ radians

Question 4

Which one of the following is the exact value of $\sin(\frac{5\pi}{2})$?
(a) $-\frac{1}{2}$
(b) 0
(c) -1
(d) 1
(e) $-\frac{\sqrt{3}}{2}$

Question 5

Which of the following is the exact value of $\cos(\pi)$?
(a) 0
(b) $\frac{1}{2}$
(c) $-\frac{1}{2}$
(d) 1
(e) -1

Question 6

Which of the following is the exact value of $\sin(-\frac{5\pi}{4})$?
(a) $-\frac{1}{2}$
(b) $\frac{1}{2}$
(c) $\frac{1}{\sqrt{2}}$
(d) $-\frac{1}{\sqrt{2}}$
(e) 0

Question 7

Which of the following is the exact value of $\cos(\frac{2\pi}{3})$?
(a) $\frac{\sqrt{3}}{2}$
(b) $-\frac{\sqrt{3}}{2}$
(c) $\frac{1}{2}$
(d) 0
(e) 1

Question 8

Which of the following is the exact value of $\tan(\frac{3\pi}{4})$?
(a) -1
(b) $-\frac{1}{\sqrt{3}}$
(c) $\frac{1}{4}$

(d) $\sqrt{3}$
(e) 1

Question 9

Which of the following is the exact value of $\tan(-\frac{5\pi}{6})$?
(a) $-\sqrt{3}$
(b) $\frac{1}{\sqrt{3}}$
(c) 0
(d) $\sqrt{3}$
(e) -1

Question 10

Which one of the following may be false for some real numbers z?
(a) $\sin z = \cos(z + \frac{3\pi}{2})$
(b) $\sin z = \cos(z - \frac{\pi}{2})$
(c) $\cos z = \sin(z + \frac{\pi}{2})$
(d) $\cos z = \sin(z - \frac{\pi}{2})$
(e) $\cos z = \sin(z - \frac{3\pi}{2})$

Answers

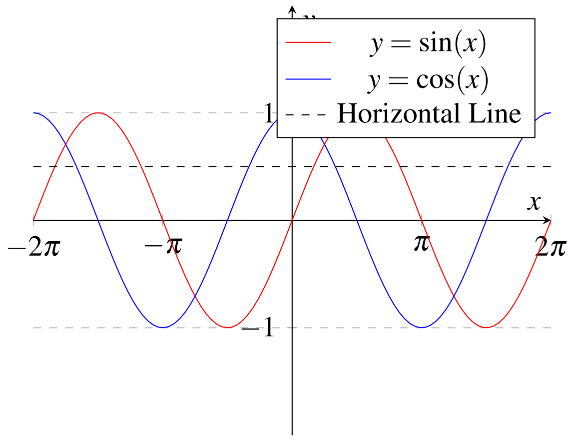
The answers will be revealed at the end of the module.

11.3 The inverse circular functions

11.3.1 Inverses of sine and cosine function

The sine and cosine functions, denoted as sin(x) and cos(x) respectively, are fundamental in trigonometry. They are defined for all real numbers, which we can interpret as angles in radians that rotate around a unit circle. The domain for both functions is the entire set of real numbers, denoted by R. As these functions trace points on the unit circle, their output values are limited between -1 and 1, inclusive. This range is represented by the interval [-1,1] .

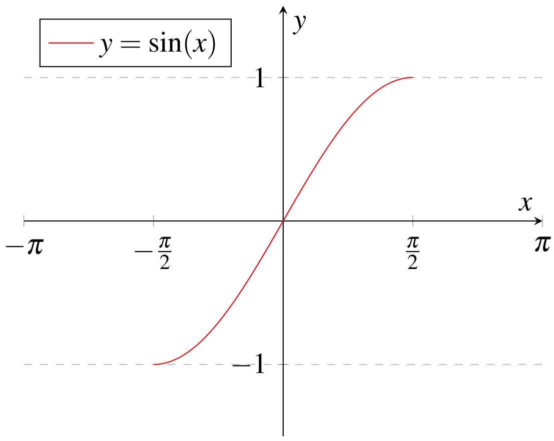
11.3.1.1 The Horizontal Line Test for sine and cosine function



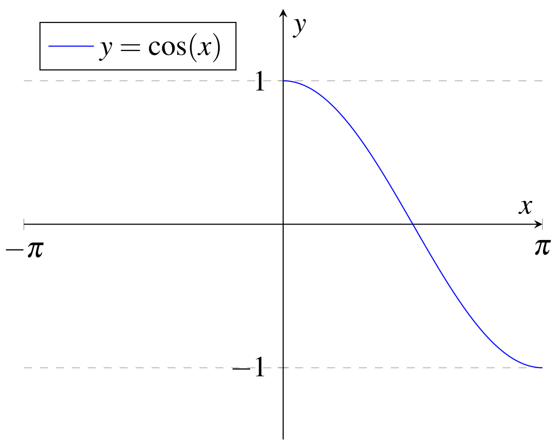
For a function to have an inverse, it must pass the horizontal line test. This means that any horizontal line should intersect the graph of the function at most once. However, the graphs of sine and cosine functions fail this test because a horizontal line intersects them at infinitely many points, indicating that they are not invertible in their current form.

11.3.1.2 Creating Invertible Fragments

To make these functions invertible, we restrict their domains to a portion where the horizontal line test is satisfied. For the sine function, we limit the domain to $[-\pi, \pi]$, capturing a segment of the curve that is symmetric about the origin and passes the horizontal line test. This symmetry is an example of an odd function, where rotating the graph 180 degrees around the origin leaves it unchanged.

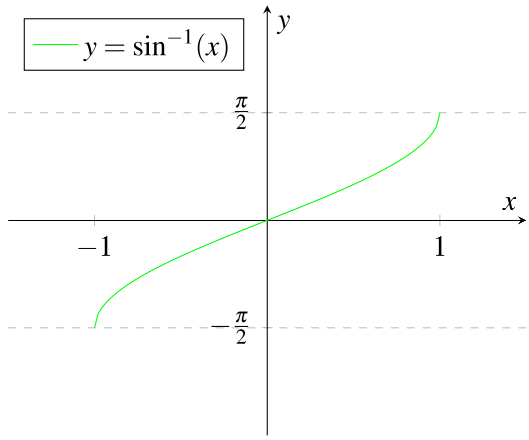


For the cosine function, the standard restriction is to the interval $[0, 2\pi]$, which also ensures that the horizontal line test is passed. The range for both restricted fragments remains $[1, 1]$.

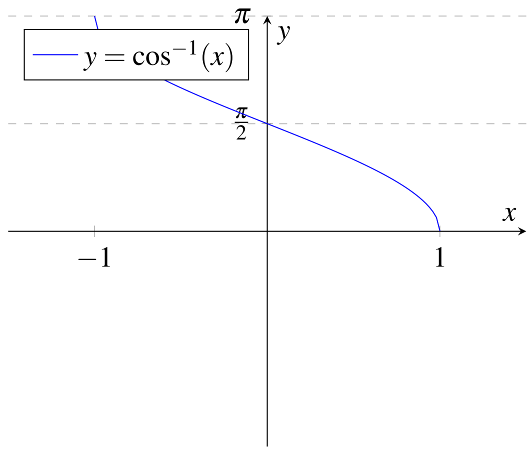


11.3.1.3 Visualizing the Inverse Functions of sine and cosine

When we reflect the restricted fragment of the sine curve over the line y=x, we obtain the graph of the inverse sine function, denoted as $sin^{-1}(x)$. or arcsin(x). This reflection swaps the roles of input and output, turning the original domain into the range and vice versa. Thus, the domain of the inverse sine function becomes [-1,1] , and its range becomes $[-\pi, \pi]$.



When we reflect the restricted fragment of the cosine curve over the line y=x , we obtain the graph of the inverse cosine function, denoted as $cos^{-1}(x)$. or arccos(x). This reflection swaps the roles of input and output, turning the original domain into the range and vice versa. Thus, the domain of the inverse sine function becomes [-1,1] , and its range becomes $[0, 2\pi]$.



11.3.2 Interpreting Inverse Sine and Cosine

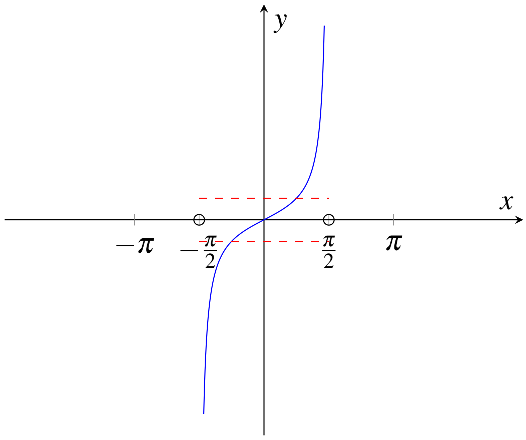
The inverse sine function essentially reverses the action of the sine function. It takes a number (representing the sine of an angle) and returns the angle itself. For instance, since $sin(\frac{\pi}{6})=\frac{1}{2}$, the inverse sine of $\frac{1}{2}$ is $\frac{\pi}{6}$ radians or 30 degrees. This helps us visualize the angle corresponding to a given sine value.

The inverse cosine function essentially reverses the action of the cosine function. It takes a number (representing the cosine of an angle) and returns the angle itself. For instance, since $cos(\frac{\pi}{3})=\frac{1}{2}$, the inverse cosine of $\frac{1}{2}$ is $\frac{\pi}{3}$ radians or 60 degrees. This helps us visualize the angle corresponding to a given sine value.

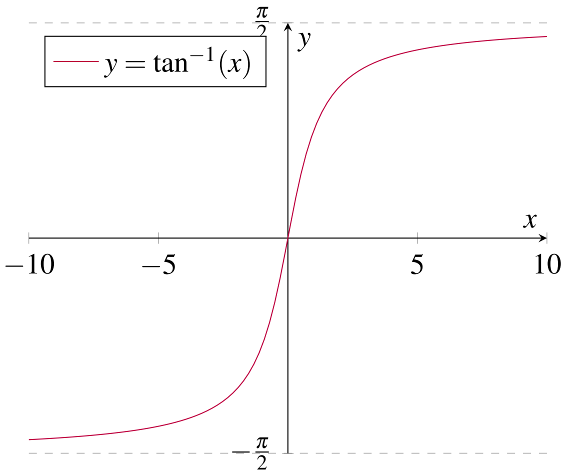
11.3.3 Inverse of tangent function

11.3.3.1 Inverse Tangent Function:

A similar process is applied to the tangent function, denoted as $\tan(x)$, to define its inverse, $tan^{-1}(x)$ or $\arctan(x)$.By restricting the domain of the tangent function appropriately i.e. from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we ensure that its graph also passes the horizontal line test, making it invertible. The domain of the restricted tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$ and the range of restricted tangent function is entire Real number $(-\infty, \infty)$.



Now, here is the graph of y=$\tan^{-1}(x)$ which is just the reflection of the graph of restricted y=tan(x) about the line y=x. Thus, its domain is entire Real number $(-\infty, \infty)$ and its range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.



The graph of $\tan^{-1}(x)$ is also commonly known as the signmoid curve. [1] This curve is bounded by two horizontal asymptotes and is significant in various fields, including computer science and neuroscience.

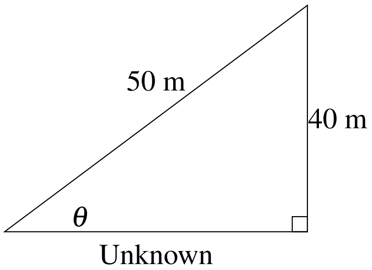
In summary, by restricting the domains of the sine, cosine and tangent functions, we can create segments of their graphs that are invertible. Reflecting these segments over the line y=x allows us to visualize and understand their inverse functions, which play a crucial role in trigonometry and calculus. The inverse functions take us back from a sine or tangent value to the corresponding angle, providing a bridge between numerical and angular representations.

11.3.4 Practical Application

11.3.4.1 Kite Flying

Consider a kite flying 40 meters above the ground, tethered by a 50-meter string. To find the angle of elevation (θ), we use the sine function:

[1] A sigmoid curve is a mathematical function that produces a characteristic "S"-shaped graph. This type of curve is common in many natural processes and is often used to describe phenomena that grow in a way that starts slowly, accelerates, and then slows down again as it approaches a maximum limit.



$\sin(\theta)=\frac{\text{opposite}}{\text{hypotenuse}} = \frac{40}{50} = 0.8$

Now, finding the inverse sine of 0.8, we get $\theta = 53°$ or 0.92 radian.

11.3.4.2 The Statue of Liberty

Imagine you're standing 250 meters away from the Statue of Liberty, which is 46 meters tall and θ stands on a pedestal of the same height. To find the angle that the statue subtends from your viewpoint, you can use trigonometry.

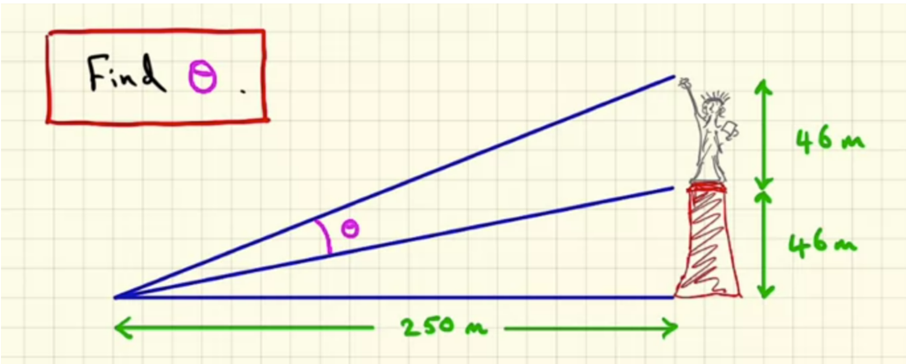


Figure 11.5: Statue of Liberty Problem

First thing you might want to do is to let the angle subtended by pedastal be ϕ.

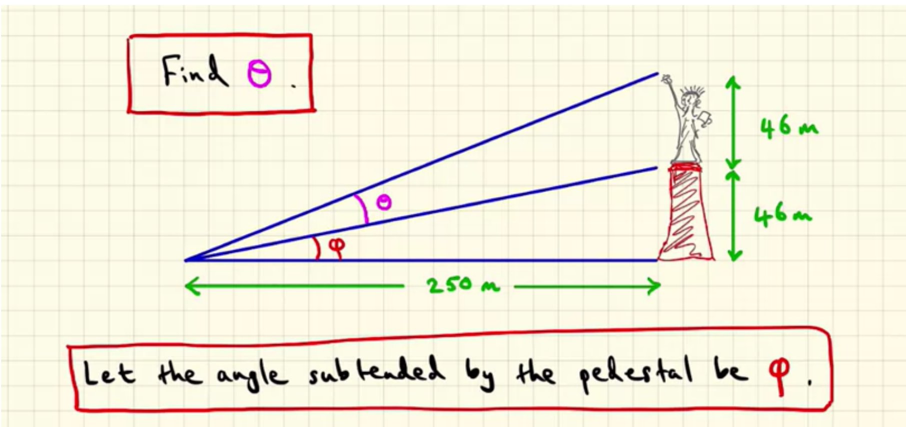


Figure 11.6: Supposing the angle

[1]Image 11.5 from MOOC Single Variable Calculus (University of Sydney)
[1]Image 11.6 from MOOC Single Variable Calculus (University of Sydney)

Consider the pedestal: there's a right-angled triangle formed by the pedestal, the ground, and the line of sight from your eye to the top of the pedestal.

This angle, which we called ϕ, has an opposite side of 46 meters (the height of the pedestal) and an adjacent side of 250 meters (your distance from the base of the pedestal). The tangent of ϕ is the ratio of these two lengths, $\frac{46}{250} = 0.184$.

Therefore, ϕ is the inverse tangent of this fraction.

$$\phi = \arctan(0.184) = 10.43^\circ$$

Next, consider the entire height of the Statue of Liberty, including the pedestal. This forms another right-angled triangle with an opposite side of 92 meters (the combined height of the statue and pedestal) and the same adjacent side of 250 meters.

The tangent of the angle $\phi + \theta$ for this larger triangle is $\frac{92}{250} = 0.368$.
Thus, $\phi + \theta$ is the inverse tangent of this fraction.

$$\phi + \theta = \arctan(0.368) = 20.2^\circ$$

To find θ, the angle subtended by just the statue, subtract ϕ from $\phi + \theta$. This gives us the value of $\theta = 20.2^\circ - 10.43^\circ = 9.77^\circ$. This means the Statue of Liberty appears to subtend nearly 10 degrees from a quarter of a kilometer away.

11.4 Recap of the chapter

Throughout this module, we've covered a wide array of mathematical concepts, all centered around the fundamental idea of a function. We've examined:

Linear Functions: Whose graphs are straight lines.
Quadratic Functions: Whose graphs are parabolas, and we've discussed solving them using methods like completing the square and the quadratic formula.
Polynomial Functions: Built from non-negative integer powers of x, which are straightforward to evaluate.
Fractional Powers: Which provide a way to understand square roots and other roots.
Function Composition and Inversion: Creating new functions by combining existing ones, and finding inverses by reflecting across the line $y = x$ or through algebraic manipulation.
Exponential Functions: Transitioning from power functions to exponential ones, particularly focusing on Euler's number e.
Logarithmic Functions: Exploring their properties and applications in scenarios like exponential growth and decay.
Trigonometric Functions: Introducing sine, cosine, and tangent functions and their roles in relating angles to ratios of sides in right-angled triangles.

We've also touched upon the inverse trigonometric functions, like inverse sine and inverse tangent, and demonstrated how they can convert numerical values back into angles, which is particularly useful in various practical applications.

As we conclude this chapter, it's important to practice the exercises, and fully digest the wealth of information presented. Thank you for engaging with this content, and I hope these explanations have been helpful. When you're ready, move on to the exercises to reinforce your understanding, and I look forward to continuing this mathematical journey with you.

11.4.1 Practice Quiz

Question 1

Recall that the conventional choice of range of the inverse sine function $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Which one of the following is equal to $\sin^{-1}\left(-\frac{1}{2}\right)$?

(a) $-\frac{\pi}{3}$
(b) $-\frac{\pi}{6}$
(c) $-\frac{7\pi}{6}$
(d) $-\frac{4\pi}{3}$
(e) $-\frac{5\pi}{6}$

Question 2

Recall that the conventional choice of range of the inverse sine function $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Which one of the following is equal to $\sin^{-1}(-1)$?

(a) $-\frac{\pi}{2}$
(b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$
(d) 0
(e) $-\pi$
(f) $\frac{3\pi}{2}$

Question 3

Recall that the conventional choice of range of the inverse cosine function $\cos^{-1}$ is $[0, \pi]$. Which one of the following is equal to $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$?

(a) $\frac{\pi}{6}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$
(d) $\frac{2\pi}{3}$
(e) $\frac{3\pi}{4}$
(f) $\frac{5\pi}{6}$

Question 4

Recall that the conventional choice of range of the inverse cosine function $\cos^{-1}$ is $[0, \pi]$. Which one of the following is equal to $\cos^{-1}(-1)$?

(a) $\frac{\pi}{2}$
(b) $\frac{\pi}{6}$
(c) $-\pi$
(d) π
(e) $\frac{\pi}{3}$

Question 5

Recall that the conventional choice of range of the inverse tan function $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which one of the following is equal to $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$?

(a) $\frac{\pi}{3}$
(b) $\frac{\pi}{6}$
(c) π
(d) $\frac{3\pi}{6}$

(e) $\frac{2\pi}{3}$

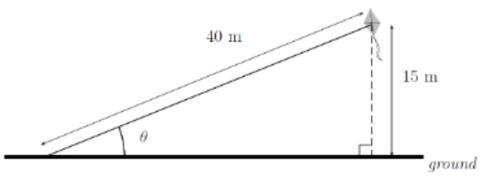
Question 6

Recall that the conventional choice of range of the inverse tan function $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which one of the following is equal to $\tan^{-1}(-\sqrt{3})$?

(a) $\frac{2\pi}{3}$
(b) $-\frac{\pi}{3}$
(c) $\frac{\pi}{3}$
(d) $\frac{\pi}{6}$
(e) $-\frac{\pi}{6}$

Question 7

A kite is attached to the ground by a piece of string of length 40 metres. The kite is flying 15 metres directly above the ground.

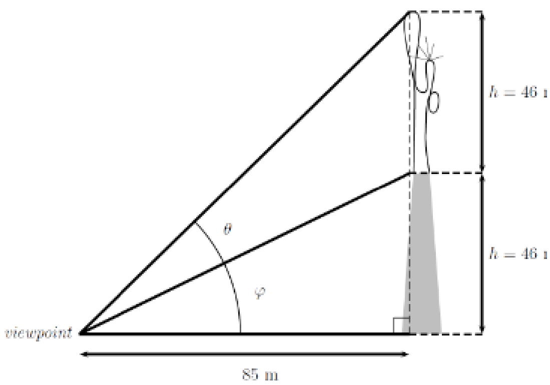


Which one of the following is a good estimate for the angle θ of inclination of the piece of string with the ground?

(a) $\theta = \cos^{-1}\left(\frac{3}{8}\right)$
(b) $\theta = \sin^{-1}\left(\frac{3}{8}\right)$
(c) $\theta = \sin^{-1}\left(\frac{5}{8}\right)$
(d) $\theta = \tan^{-1}\left(\frac{3}{8}\right)$
(e) $\theta = \cos^{-1}\left(\frac{3}{5}\right)$

Question 8

The Statue of Liberty and its pedestal are both 46 metres tall.

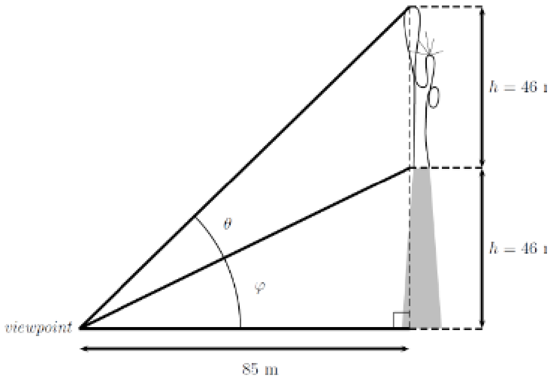


Which one of the following is a correct expression for the angle φ subtended by the pedestal, when viewed 85 metres from the base of the pedestal?

(a) $\varphi = \tan^{-1}\left(\frac{46}{85}\right)$
(b) $\varphi = \cos^{-1}\left(\frac{46}{85}\right)$
(c) $\varphi = \tan^{-1}\left(\frac{85}{46}\right)$
(d) $\varphi = \sin^{-1}\left(\frac{85}{46}\right)$
(e) $\varphi = \sin^{-1}\left(\frac{46}{85}\right)$

Question 9

The Statue of Liberty and its pedestal are both 46 metres tall.

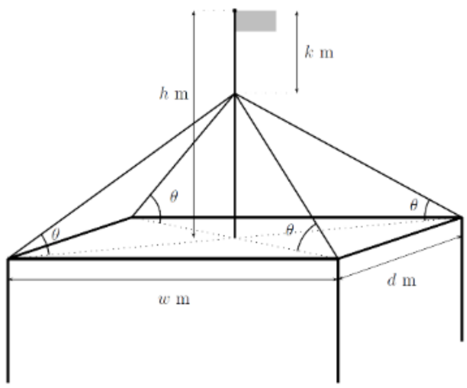


Which one of the following is a correct expression for the angle θ subtended by the statue, when viewed 85 metres from the base of the pedestal?

(a) $\theta = \sin^{-1}\left(\frac{92}{85}\right) - \sin^{-1}\left(\frac{46}{85}\right)$
(b) $\theta = \tan^{-1}\left(\frac{92}{85}\right)$
(c) $\theta = \tan^{-1}\left(\frac{92}{85}\right) - \tan^{-1}\left(\frac{46}{85}\right)$
(d) $\theta = \sin^{-1}\left(\frac{92}{85}\right)$
(e) $\theta = \tan^{-1}\left(\frac{46}{85}\right)$

Question 10

A vertical flag pole of height h metres is erected exactly in the middle of the flat roof of a building. The roof is rectangular of width w metres and depth d metres. The flag pole is stabilized by cables that join the corners of the roof top to the flag pole at a point k metres below the top of the flagpole.



Each piece of cable makes an angle θ with the horizontal. Which one of the following correctly describes θ in terms of w, d, h, and k?

(a) $\cos\theta = \sqrt{\frac{w^2+d^2}{w^2+d^2+(h-k)^2}}$
(b) $\cos\theta = \frac{w^2+d^2}{\sqrt{w^2+d^2+h^2-2hk}}$
(c) $\cos\theta = \frac{w+d^2}{\sqrt{w^2+4h^2-4hk}}$
(d) $\cos\theta = \sqrt{\frac{w^2+d^2}{w^2+d^2+h^2}}$
(e) $\cos\theta = \sqrt{\frac{w^2+d^2}{w^2+d^2+4(h-k)^2}}$

Answers

The answers will be revealed at the end of the module.

12. Assessment

12.1 Module Quiz

Question 1

Find $f(0)$ when f is the function given by the rule $f(x) = \frac{x-2}{x-1}$.

(a) 0
(b) 1
(c) -1
(d) 2
(e) -2

Question 2

Consider the following polynomial: $p(x) = x^2 + x - 12$.

Which one of the following is correct?

(a) $p(-4) = 0$
(b) $p(-6) = 0$
(c) $p(-1) = 0$
(d) $p(4) = 0$
(e) $p(-3) = 0$

Question 3

Suppose that f and g are functions with the rules $f(x) = 7x - 4$ and $g(x) = x^2$.

Find $(f \circ g)(x) = f(g(x))$.

(a) $7x^2 - 4$
(b) $7x^2 - 16$
(c) $7x^4 - 16$
(d) $(7x-4)^2$
(e) $49x^2 - 4$

Question 4

Suppose that f and g are functions with the rules $f(x) = 4x - 7$ and $g(x) = x^2$.

Find $(g \circ f)(x) = g(f(x))$.

(a) $16x^2 - 7$
(b) $4x^2 - 7$

(c) $4x^2+49$
(d) $(4x-7)^2$
(e) $4x^2-49$

Question 5

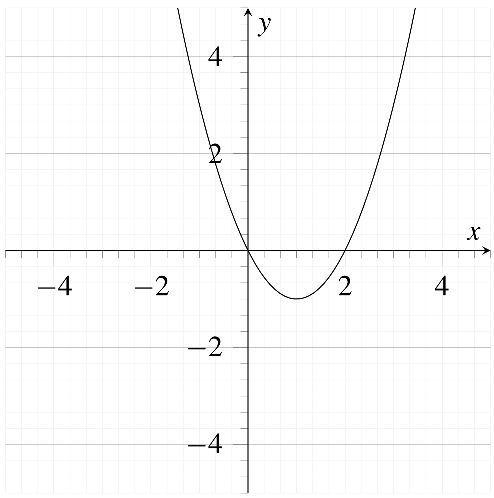
Solve for x given that $x^2+x-6=0$.
(a) $x=2,3$
(b) $x=-2,3$
(c) $x=-5,6$
(d) $x=2,-3$
(e) $x=-2,-3$

Question 6

Solve for x given that $x^2+x-1=0$.
(a) $x=\frac{3}{2},\frac{1}{2}$
(b) $x=\frac{3}{2},\frac{-1}{2}$
(c) $x=\frac{-1\pm\sqrt{5}}{2}$
(d) $x=\frac{1+\sqrt{3}}{2}$
(e) $x=\frac{1\pm\sqrt{5}}{2}$

Question 7

Which one of the functions below corresponds to the following parabola?



(a) $y=x^2+2x-1$
(b) $y=x(x-2)$
(c) $y=x^2-1$
(d) $y=x^2-2x+1$
(e) $y=x(x+2)$

Question 8

Find the range of the function f with the following rule:

$$f(x)=x^2+2x+3$$

(a) range $=[5,\infty)$
(b) range $=[4,\infty)$

(c) range $= [2, \infty)$
(d) range $= [1, \infty)$
(e) range $= [3, \infty)$

Question 9

Find the domain and range of the function f with the following rule:

$$f(x) = 1 + \sqrt{x-1}$$

(a) domain = $[1, \infty)$, range = $[0, \infty)$
(b) domain = $[1, \infty)$, range = $(-\infty, 1]$
(c) domain = $[1, \infty)$, range = $[1, \infty)$
(d) domain = $(-\infty, 1]$, range = $[1, \infty)$
(e) domain = $[-1, \infty)$, range = $[1, \infty)$

Question 10

Suppose that $f(x) = 7x - 2$ for all $x \in \mathbb{R}$. Find $f^{-1}(x)$.
(a) $\frac{2-x}{7}$
(b) $\frac{x-2}{7}$
(c) $\frac{1}{7x-2}$
(d) $\frac{x+2}{7}$
(e) $\frac{7}{x} + 2$

Question 11

Express the angle $\frac{3\pi}{2}$ radians in degrees.
(a) 210°
(b) 270°
(c) 350°
(d) 240°
(e) 300°

Question 12

Which one of the following is the exact value of $\sin\left(\frac{5\pi}{4}\right)$?
(a) $\frac{1}{\sqrt{2}}$
(b) 1
(c) -1
(d) $-\frac{1}{2}$
(e) $-\frac{1}{\sqrt{2}}$

Question 13

Express the angle 120° in radians.
(a) $\frac{3\pi}{4}$ radians
(b) π radians
(c) $\frac{7\pi}{6}$ radians
(d) $\frac{2\pi}{3}$ radians
(e) $\frac{5\pi}{3}$ radians

Question 14

Which one of the following is the exact value of $\tan(330^\circ)$?
(a) $\sqrt{3}$

(b) -1
(c) $-\frac{1}{\sqrt{2}}$
(d) 1
(e) $\frac{1}{\sqrt{3}}$

Question 15

Suppose that θ is an acute angle such that $\tan(\theta) = 2$. Find $\sin(\theta)$.
(a) $\frac{1}{\sqrt{3}}$
(b) $\frac{2}{\sqrt{3}}$
(c) $\frac{1}{3}$
(d) $\frac{3}{\sqrt{5}}$
(e) $\frac{2}{\sqrt{5}}$

Question 16

A kite is attached to the ground by a piece of string of length 35 meters. The kite is flying 14 meters directly above the ground.

Which one of the following is a good estimate for the angle θ of inclination of the piece of string with the ground?
(a) $\sin^{-1}\left(\frac{3}{5}\right)$
(b) $\cos^{-1}\left(\frac{2}{5}\right)$
(c) $\sin^{-1}\left(\frac{2}{5}\right)$
(d) $\cos^{-1}\left(\frac{3}{5}\right)$
(e) $\tan^{-1}\left(\frac{1}{6}\right)$

Question 17

Which one of the following expressions is equivalent to $\left(y^4x^{-3}\right)^{-2}$?
(a) $-\frac{y^8}{x^6}$
(b) $\frac{1}{y^8x^6}$
(c) $\frac{x^5}{y^2}$
(d) $\frac{y^2}{x^5}$
(e) $\frac{x^6}{y^8}$

Question 18

Which one of the following expressions is equivalent to $e^xe^{x^2}$?
(a) $e^{x(2-x)}$
(b) $e^{(x(1+x)}$
(c) e^{2x^2}
(d) $e^{x(2+x)}$
(e) e^{x^3}

Question 19

Solve for x when $3^x = 5$.
(a) $x = \frac{\ln 5}{\ln 3}$
(b) $x = \frac{\ln 3}{\ln 5}$
(c) $x = \frac{\ln 5}{3}$
(d) $x = \ln\left(\frac{5}{3}\right)$
(e) $x = \ln\left(\frac{3}{5}\right)$

Question 20

We have a sample of 100 g of cesium. The half-life of cesium is 30 years. How much of the sample will remain after 100 years (to the nearest tenth of a gram)?

(a) 9.8 g
(b) 9.7 g
(c) 9.6 g
(d) 9.5 g
(e) 9.9 g

13. In-text Element Examples

13.1 Referencing Publications

This statement requires citation [0]; this one is more specific [0, page 162].

13.2 Referencing Publications

This statement requires citation [0]; this one is more specific [0, page 162].

13.3 Link Examples

This is a URL link: LaTeX Templates. This is an email link: example@example.com. This is a monospaced URL link: `https://www.LaTeXTemplates.com`.

13.4 Lists

Lists are useful to present information in a concise and/or ordered way.

13.4.1 Numbered List

1. First numbered item
    a. First indented numbered item
    b. Second indented numbered item
        i. First second-level indented numbered item
2. Second numbered item
3. Third numbered item

13.4.2 Bullet Point List

- First bullet point item
    – First indented bullet point item
    – Second indented bullet point item
        ◦ First second-level indented bullet point item
- Second bullet point item
- Third bullet point item

13.4.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

13.5 International Support

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13.6 Ligatures

fi fj fl ffl ffi Ty

III Introducing the Differential Calculus

This module introduces techniques of differential calculus. We look at average rates of change which become instantaneous, as time intervals become vanishingly small, leading to the notion of a derivative. We then explore techniques involving differentials that exploit tangent lines. The module introduces Leibniz notation and shows how to use it to get information easily about the derivative of a function and how to apply it.

Learning Objectives

- connect seminal ideas including tangent lines, secants to curves, slopes and average rates of change
- understand and manipulate the definition of the derivative and use it to find derivatives from first principles
- use and apply Leibniz notation and differentials

14. Introduction

14.1 Introduction to Module 3

Welcome back! You've successfully navigated through the initial two modules, arming yourself with a robust set of precalculus tools. You've also mastered a diverse array of functions and the art of manipulating and graphing them on the x-y plane. As we venture into this next module, we embark on the first of two journeys into the realm of differential calculus. Our starting point is the exploration of average rates of change, which, as we narrow down time intervals to the slightest fraction, evolve into the concept of the derivative.

Consider velocity, our guiding example throughout this module—it's essentially the derivative of displacement over time. If you've ever monitored the speedometer in a moving vehicle, be it a car or a truck, you're already acquainted with the practical side of derivatives. The derivative represents the limit of specific ratios linked to curves, particularly the slopes of secant lines that inch closer to the slope of the tangent at a point of interest. These tangent lines serve as excellent proxies for the curve itself, and you'll discover methods that leverage differentials to harness this accuracy.

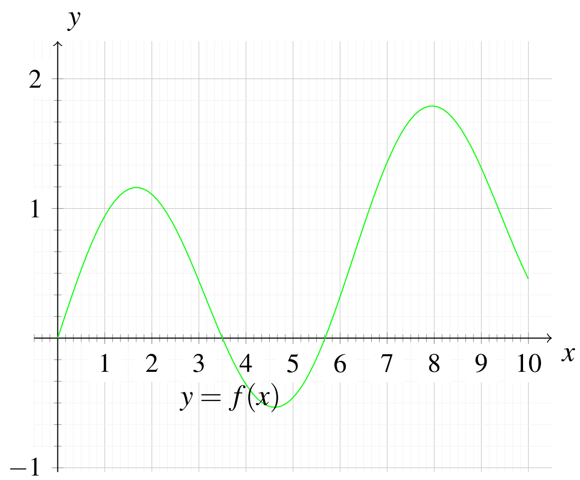
We delve into the mathematics with the timeless and sophisticated Leibniz notation, a legacy of Gottfried Leibniz, a pioneering figure in calculus alongside Newton in the 17th century. You'll swiftly become adept at using Leibniz notation, which will streamline your understanding and application of derivatives.

We trust that you'll find the content engaging and thought-provoking. The instructional journey, coupled with the extensive practice and challenges presented by numerous exercises, are designed to be of great benefit. I eagerly anticipate your continued engagement and participation.

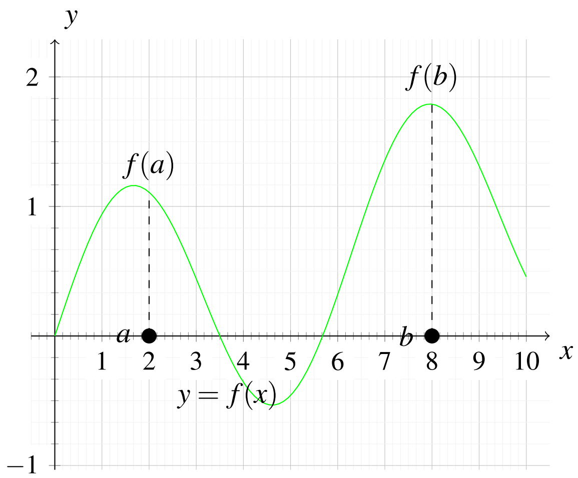
15. Rates of Change and Tangent Lines

15.1 Slope and Average Rates of Change

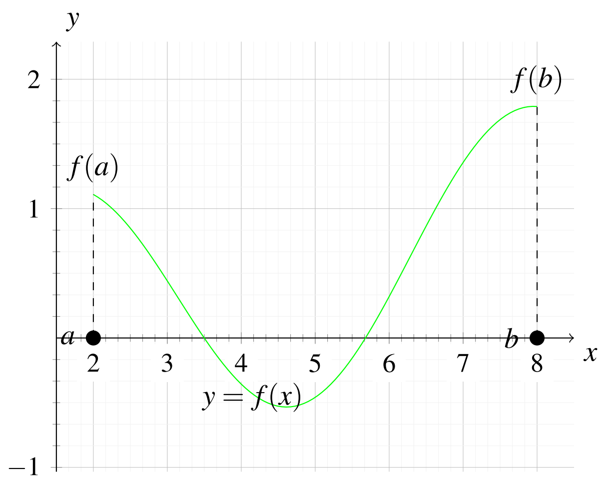
Calculus, at its core, is the study of change—how to understand it fully and deeply. It deals with change in functions that map inputs to outputs. In this section, we delve into how function outputs vary on average with changes in inputs. We limit our focus to a specific segment of the real line and determine the average change by calculating the slope of the line connecting two points on the function's graph. This concept is known as the average rate of change.



Function graphs illustrate the interplay between variables, such as x and y. Consider the graph of a volatile function, $y = f(x)$, and let's zoom in on the input range from a to b, where a is smaller than b.

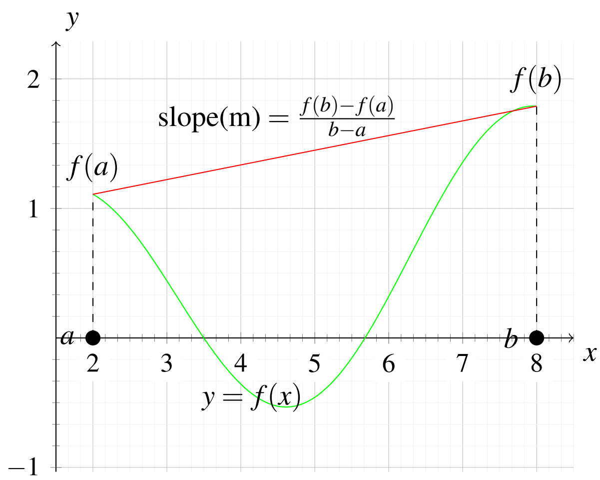


On this curve, the points corresponding to inputs a and b yield outputs $f(a)$ and $f(b)$, respectively. By narrowing down the domain to just this interval, we can clean up the graph and focus on the essential part.



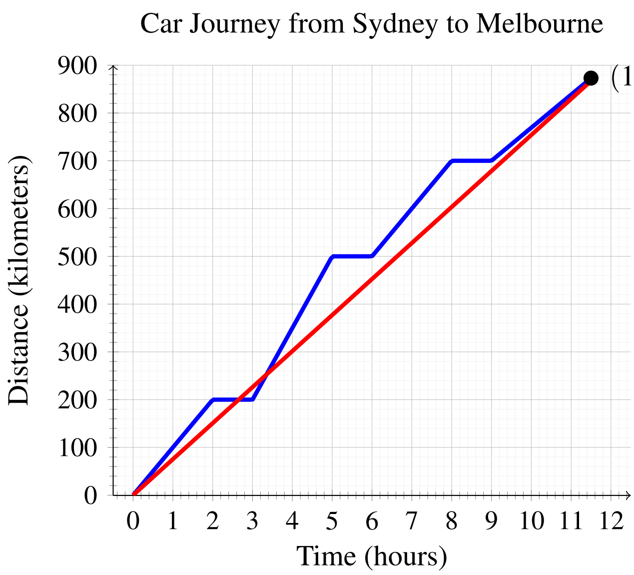
domain $= [a, b]$

Understanding how one variable influences or alters another is crucial, especially for making predictions. For instance, the central bank might analyze how adjustments to the base interest rate could impact economic indicators. Such analysis would likely involve complex functions with multiple variables. Here, we simplify to just two variables: an independent variable x that moves between a and b, and a dependent variable y, defined by the graph.



Functions can exhibit significant fluctuations. We're interested in the average extent of these fluctuations. If $f(b)$ is greater than $f(a)$, by how much? The total change in y is $f(b)-f(a)$. But to put this in perspective, we consider the rate of change relative to the interval length, $b-a$. A shorter interval suggests a quicker change, while a longer one indicates a slower pace. To quantify the average rate of change, we divide the difference in outputs by the difference in inputs, defining it as $\frac{f(b)-f(a)}{b-a}$.

Soon, you'll recognize this as something familiar. If we connect the endpoints on the graph with a straight line, this ratio is simply the slope of that line—the vertical rise over the horizontal run. This line smooths out the graph's minor undulations.



But why does this matter in the real world? Take road traffic authorities, for example, who are keenly interested in the average rate of change. Imagine a graph representing a car journey from Sydney to Melbourne. Starting from the origin, the vertical axis measures the distance from Sydney, and the horizontal axis tracks the time elapsed since departure. The journey spans 873 kilometers, ending in Melbourne, with the time axis starting at zero. The journey's duration, marked at the end of the x-axis, totals just over 11 and a half hours. This method of timekeeping, dividing hours and minutes into 60 parts, dates back to the Babylonians, who developed the base 60 system.

During the journey depicted by the flat sections of the graph, the vehicle remains stationary. One instances are where the driver took a break and the vehicle was not moving at all. Despite the graph's fluctuations and periods of inactivity, it's still possible to determine the overall average rate of change for the trip. By connecting the start and end points with a red line and calculating its slope—873 kilometers over roughly 11.5 hours—we arrive at an average speed of approximately 75.4 kilometers per hour, indicating my average velocity for the entire trip.

Focusing on a specific segment of the graph, let's say from t=3 to t=5, we see a more pronounced rate of change. This segment, aligns with the locations of speed cameras at 200kms and 500kms. These cameras calculate the average speed over that stretch. If the result exceeds the speed limit of 110 kilometers per hour, it constitutes a traffic violation, leading to a fine. In my case, I maintained a consistent speed across this distance. The average rate of change, determined by the slope—300 kilometers over 2 hours—suggests an average speed of 150 kilometers per hour, not within legal limits. Here, we see that even though the average speed over entire journey is 75.4 kms/hr which is within the speed limit, average speed over a short duration 150kms/hr is not.

A key issues arises: The graph's non-linear nature means that instantaneous speeds fluctuate. It's conceivable that, despite an average of 75.4 kilometers per hour, the speedometer might momentarily register speeds exceeding 75.4 kilometers per hour. Our goal is to grasp the concept of instantaneous speed, which will soon be defined using derivatives. Here's a preview: by sliding a line segment with the same slope as the average along the curve, we find points where it becomes tangent to the graph. On my drive between Coolac and North Gundagai, this occurred multiple times, reflecting the moments when my car's speedometer matched the average speed.

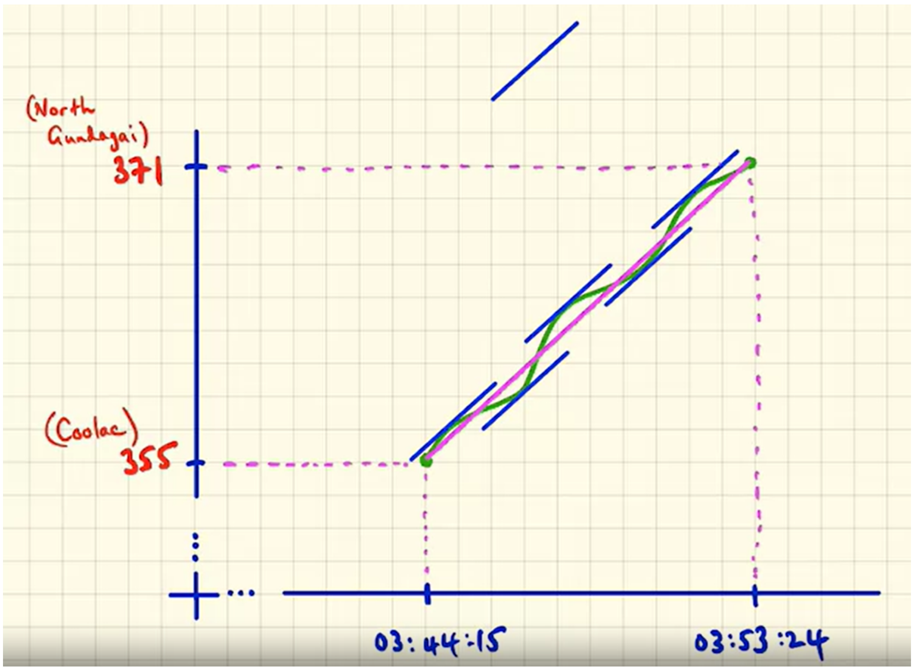


Figure 15.1: Instantaneous slope and average slope are equal at some point

During those instances, the speedometer in my car will reflect an instantaneous speed that aligns with the overall average of roughly 75.4 kilometers per hour.

[0]Image 15.1 from MOOC Single Variable Calculus (University of Sydney)

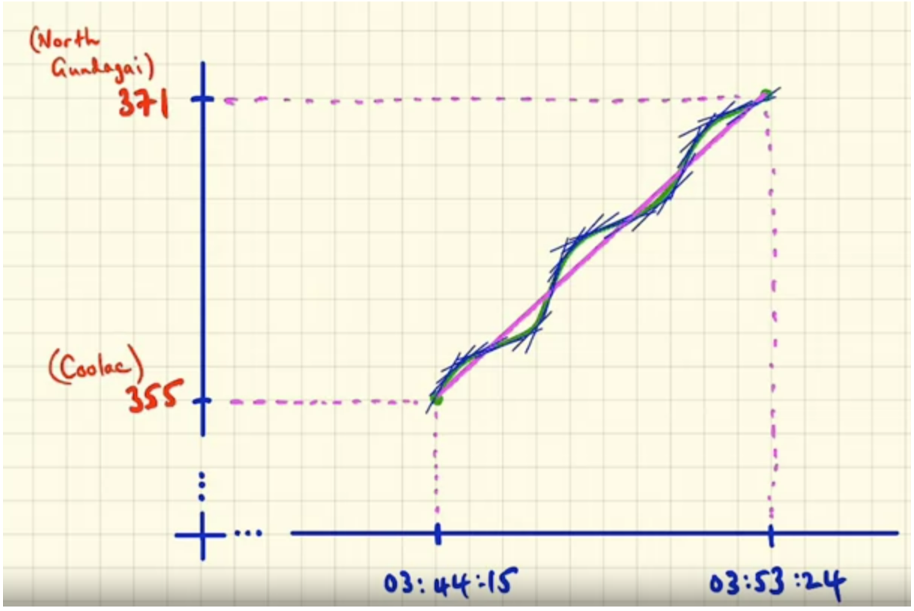


Figure 15.2: Instantaneous velocities at many points in the curve

However, if we were to capture a sequence of moments, we'd observe that the slopes of the tangent lines—representing instantaneous speeds—vary, sometimes being steeper or shallower than the average. These varying slopes of the tangents are what the speedometer displays moment by moment. If the average speed is over 75.4, then it's certain that at some point, the speedometer showed a speed exceeding 75.4, . On the other hand, if the average speed is below 75.4, it's not guaranteed that the instantaneous speed never surpassed 75.4.

For those accustomed to monitoring speedometers, whether as drivers or passengers, you've engaged with the concept of derivatives in a practical sense. This concept will be further developed and formalized in the upcoming parts of this module. Today, we've explored the average rate of change for a function over a specific interval and learned how to interpret this through the slope of the line connecting two points on a graph. This has provided us with insights into how quantities change on average. We delved into an example of a car journey, examining the average rate of change as average speeds across different segments of the trip. As we begin to consider instantaneous rates of change—like the readings on a car's speedometer—we're approaching a deeper understanding of derivatives. I encourage you to review the material and, when you feel prepared, to tackle the exercises. Thank you for tuning in, and I'm eager to connect with you again in the near future.

15.1.1 Practice Quiz

Question 1

A person drives in a car from Town A to Town B. The journey lasts 1 hour and 15 minutes. The distance between the towns is 74 kilometers. Estimate the average speed for the journey, correct to the nearest whole number.

(a) 63 kph
(b) 64 kph
(c) 60 kph
(d) 59 kph

[0]Image 15.3 from MOOC Single Variable Calculus (University of Sydney)

(e) 61 kph

Question 2

A person drives in a car from Town A to Town B, leaving at 2:15 pm and arriving at 3:10 pm. The distance between the towns is 74 kilometers. Estimate the average speed for the journey, correct to the nearest whole number.

(a) 82 kph
(b) 68 kph
(c) 81 kph
(d) 80 kph
(e) 67 kph

Question 3

A person drives in a car from Town A to Town B, leaving at 11:30 am. The average speed for the trip is 67 kph. The distance between the towns is 74 kilometers. Estimate the time of arrival to the nearest minute.

(a) 12:38 pm
(b) 12:40 pm
(c) 12:42 pm
(d) 12:36 pm
(e) 12:34 pm

Question 4

A person in a car takes 9 minutes and 34 seconds, to the nearest second, to drive between two average speed cameras located 17.84 km apart (to the nearest 10 meters). Estimate the average speed, correct to the nearest whole number, for the journey between the speed cameras.

(a) 112 kph
(b) 115 kph
(c) 111 kph
(d) 119 kph
(e) 121 kph

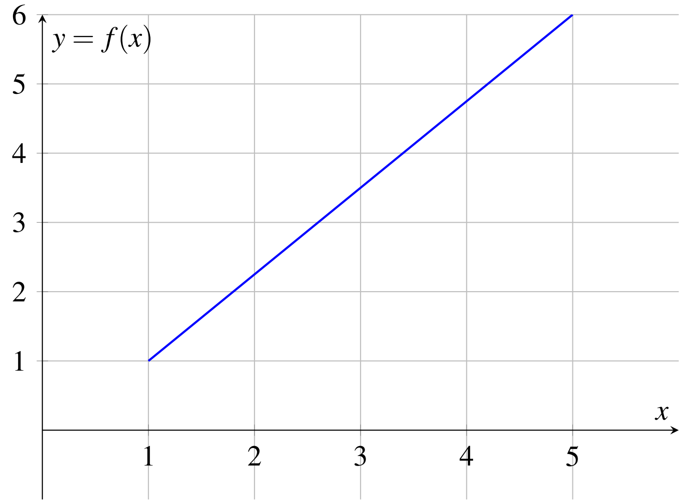
Question 5

A person in a car takes 17 minutes and 12 seconds, to the nearest second, to drive between two average speed cameras. The average speed over this distance is 109.5 kph, correct to one decimal place. Estimate the distance between the speed cameras, to the nearest 10 meters.

(a) 31.24 km
(b) 31.40 km
(c) 31.26 km
(d) 31.25 km
(e) 31.39 km

Question 6

Consider a function f with domain $[1,5]$ whose graph appears below. Find the average rate of change of f over the interval $[1,5]$.



(a) $\frac{5}{4}$
(b) $\frac{5}{6}$
(c) $\frac{6}{5}$
(d) $\frac{4}{5}$
(e) 1

Question 7

Let f be the function with rule $f(x) = x^2 - 6x + 3$ for $0 \leq x \leq 10$. Find the average rate of change of f over the interval $[0, 10]$.

(a) 4
(b) 0.4
(c) 40
(d) -0.4
(e) -4

Question 8

Let f be the function with rule $f(x) = x^2 - 6x + 3$ for $1 \leq x \leq 5$. Find the average rate of change of f over the interval $[1, 5]$.

(a) 0
(b) -0.4
(c) 0.4
(d) 2
(e) -2

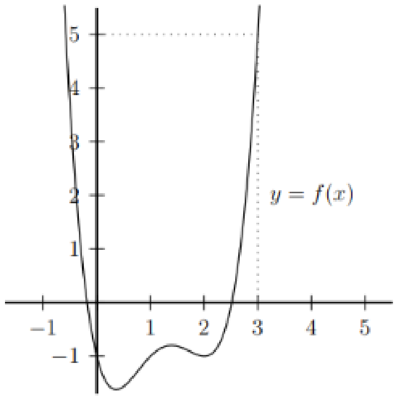
Question 9

Suppose that f is a function with domain $[-5, 5]$, such that $f(5) = 12$ and the average rate of change of f over the interval $[-5, 5]$ is 2. Find $f(-5)$.

(a) -12
(b) 32
(c) 10
(d) 8
(e) -8

Question 10

Consider a function f whose graph appears below. Find the average rate of change of f over the interval $[0, 3]$.



(a) $\frac{3}{5}$
(b) $\frac{5}{3}$
(c) $\frac{5}{2}$
(d) $\frac{4}{3}$
(e) 2

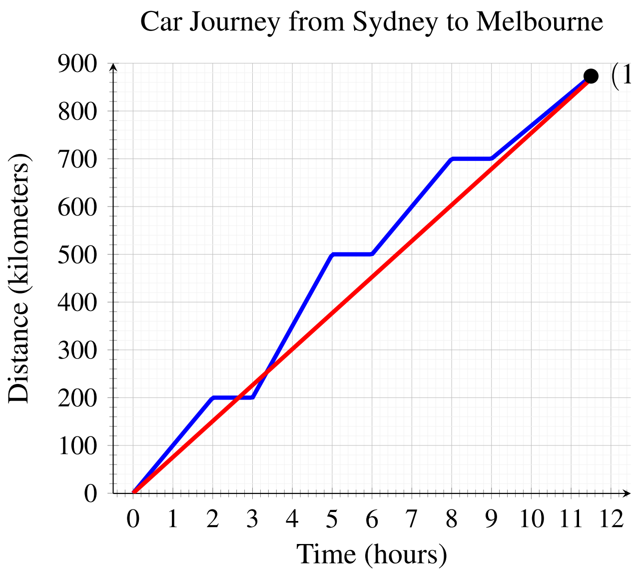
Answers

The answers will be revealed at the end of the module.

15.2 Displacement, Velocity and Acceleration

In this section, we delve into the concepts of displacement, velocity, and acceleration—key functions in physics that take time as their input and offer intuitive physical meanings. These concepts are foundational for understanding derivatives, which is the central topic of this module. We'll be building up to formal definitions of derivatives in upcoming videos.

Displacement functions track an object's location or the distance it has traveled at any given time.



Recall our previous example involving a displacement function that measured the distance between Sydney and Melbourne as a function of time over approximately 11.5 hours. We calculated the average velocity (average rate of change of displacement) for the trip by determining the slope of the line connecting the start and end points on our graph.

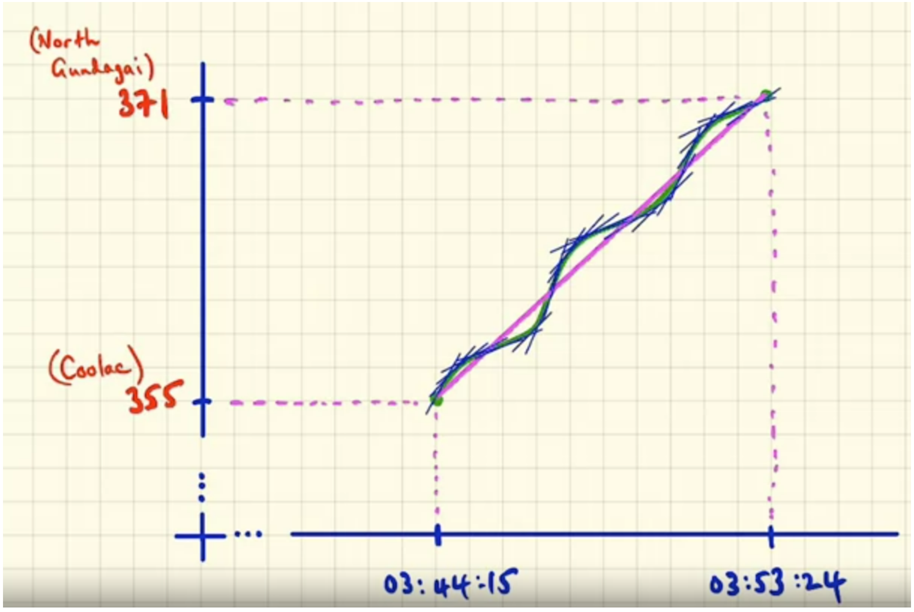


Figure 15.3: Instantaneous velocities at many points in the curve

We also touched on the concept of instantaneous speed—what you'd see on a car's speedometer—which corresponds to the slope of the tangent line at a specific point on our displacement graph.

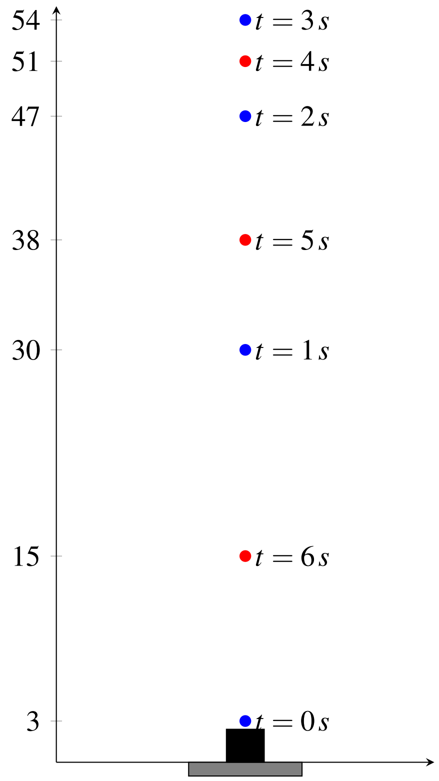
Typically, we use 'x' to represent the input and 'y' for the output. However, when dealing with displacement functions, it's common to use 't' for time as the input and 'x' = 'x(t)' for displacement as the output. In this context, 'x' becomes a dependent variable or an output variable.

15.2.1 Cannon - Ball Scenario

Imagine a cannon firing straight up. The red dots represent the scenario where the ball was rising up and blue dots represent the scenario where the ball was falling down.

[0]Image 15.3 from MOOC Single Variable Calculus (University of Sydney)

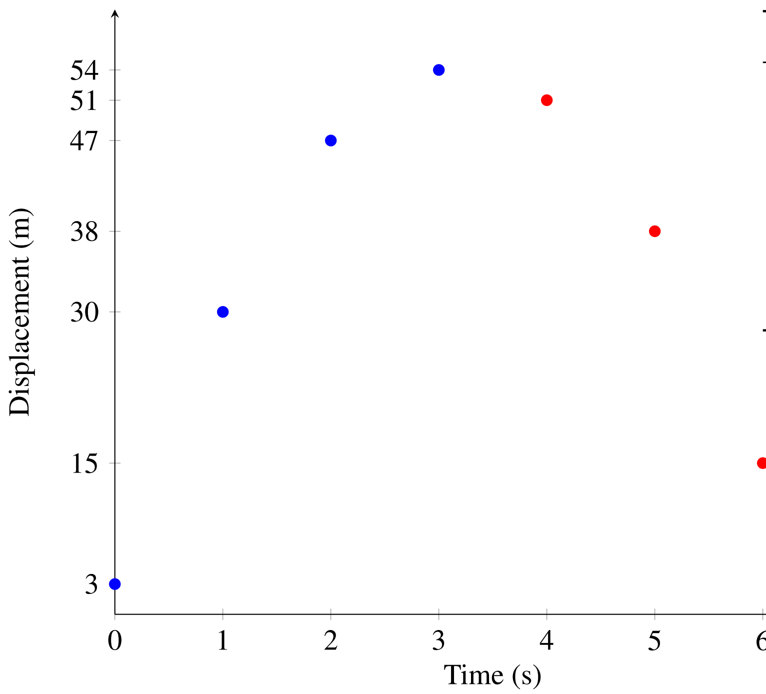
Cannonball Trajectory



15.2.1.1 Knowing the Trajectory

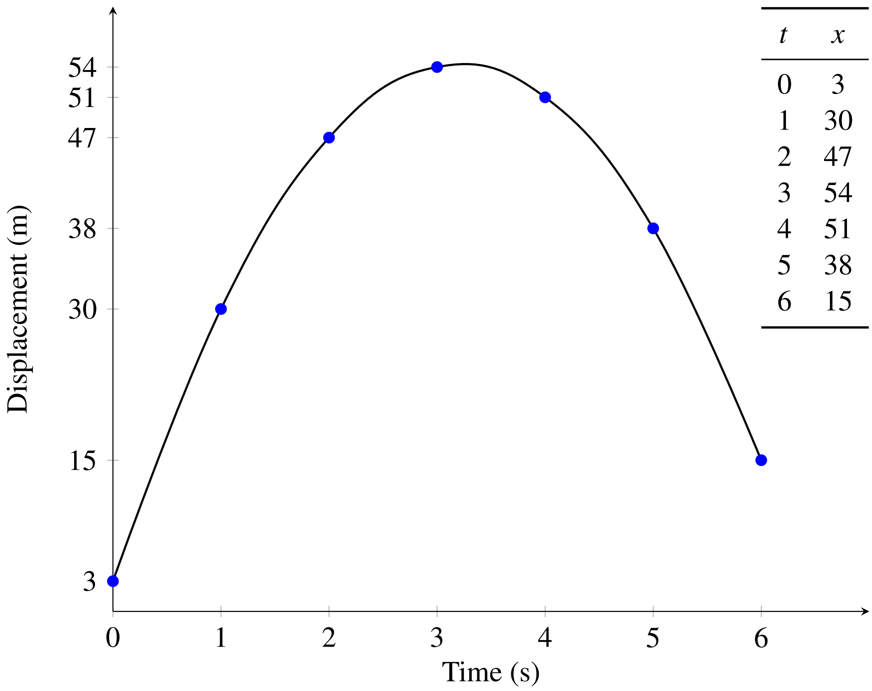
Now, if we try to represent the height of cannon ball as a function of time, the cannonball's path, influenced only by gravity (ignoring air resistance), forms an inverted parabola over six seconds.

Time vs. Displacement



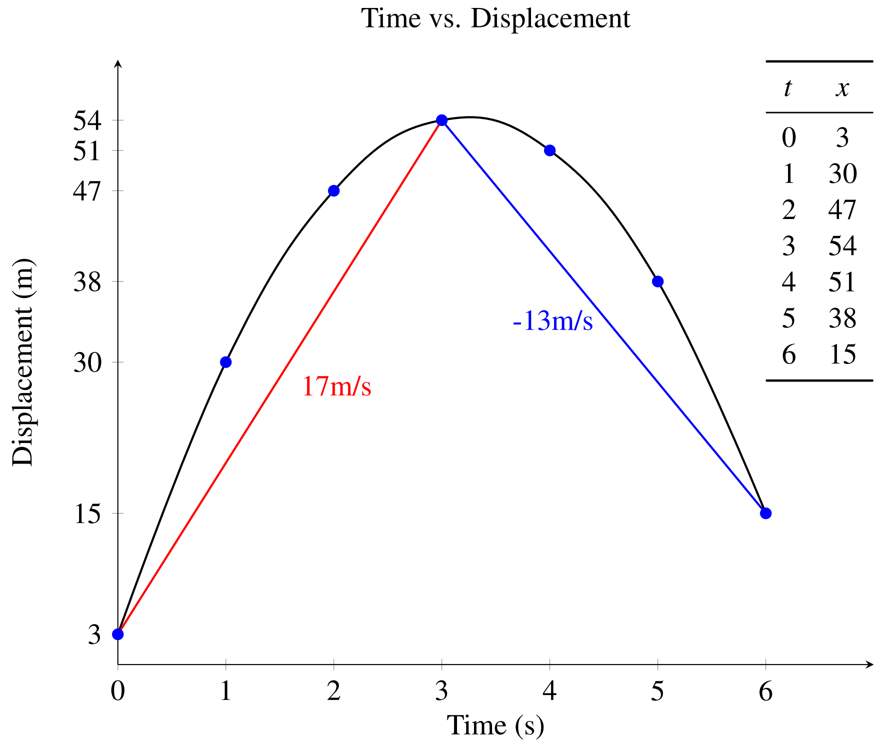
t	x
0	3
1	30
2	47
3	54
4	51
5	38
6	15

We consider the cannon's base at zero meters and the firing moment as time zero. A table lists the cannonball's height at various times, and connecting these points gives us the inverted parabola.



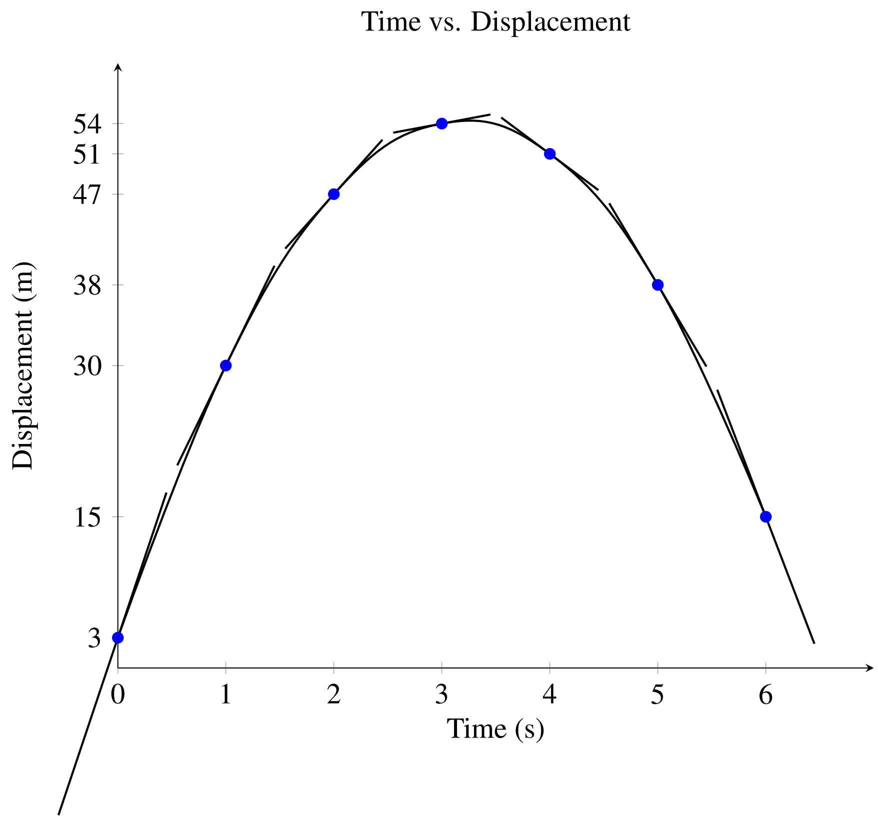
t	x
0	3
1	30
2	47
3	54
4	51
5	38
6	15

In the first three seconds, the cannonball ascends, slowing due to gravity. The average velocity during this period is the slope of the line between the points at t=0 and t=3, which is $\frac{54-3}{3-0} = 17$ meters per second. In the final three seconds, the cannonball reaches its peak and then descends. The average velocity here is $\frac{15-54}{6-3} =$ -13 meters per second, indicating downward movement since we've defined upward as positive.

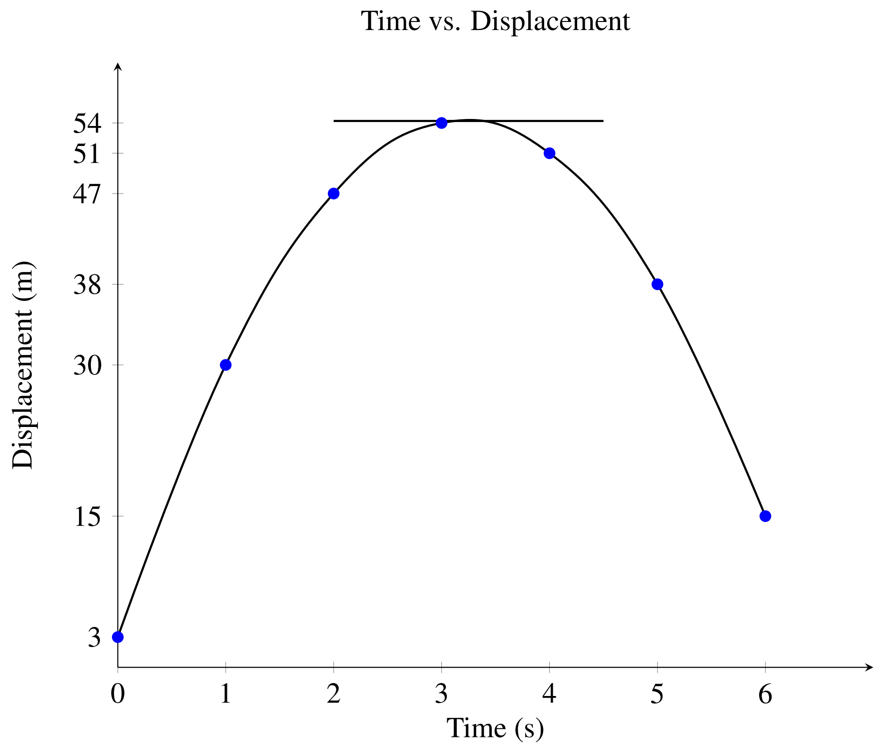


t	x
0	3
1	30
2	47
3	54
4	51
5	38
6	15

The underlying quadratic function for this motion is $x = -5t^2 + 32t + 3$, which models the scenario well. If we had a miniature speedometer attached to the cannonball, it would display the instantaneous velocity at any point, represented by the slope of the tangent line at that point.



At the parabola's apex, the tangent line is horizontal, indicating an instantaneous velocity of zero—the moment the cannonball stops ascending before descending.



By completing the square, we can determine the apex's exact location.

$$x = x(t) = -5t^2 + 32t + 3$$

$$x = -5(t^2 - \frac{32}{5}t) + 3$$

$$x = -5(t^2 - 2.t.\frac{32}{10} + (\frac{32}{10})^2 - (\frac{32}{10})^2) + 3$$

$$x = -5(t^2 - 2.t.\frac{32}{10} + (\frac{32}{10})^2) + 5.(\frac{32}{10})^2 + 3$$

$$x = -5(t^2 - 2.t.\frac{32}{10} + (\frac{32}{10})^2) + \frac{271}{5}$$

$$x = -5(t - \frac{32}{10})^2 + \frac{271}{5}$$

Now, this is a really useful form.

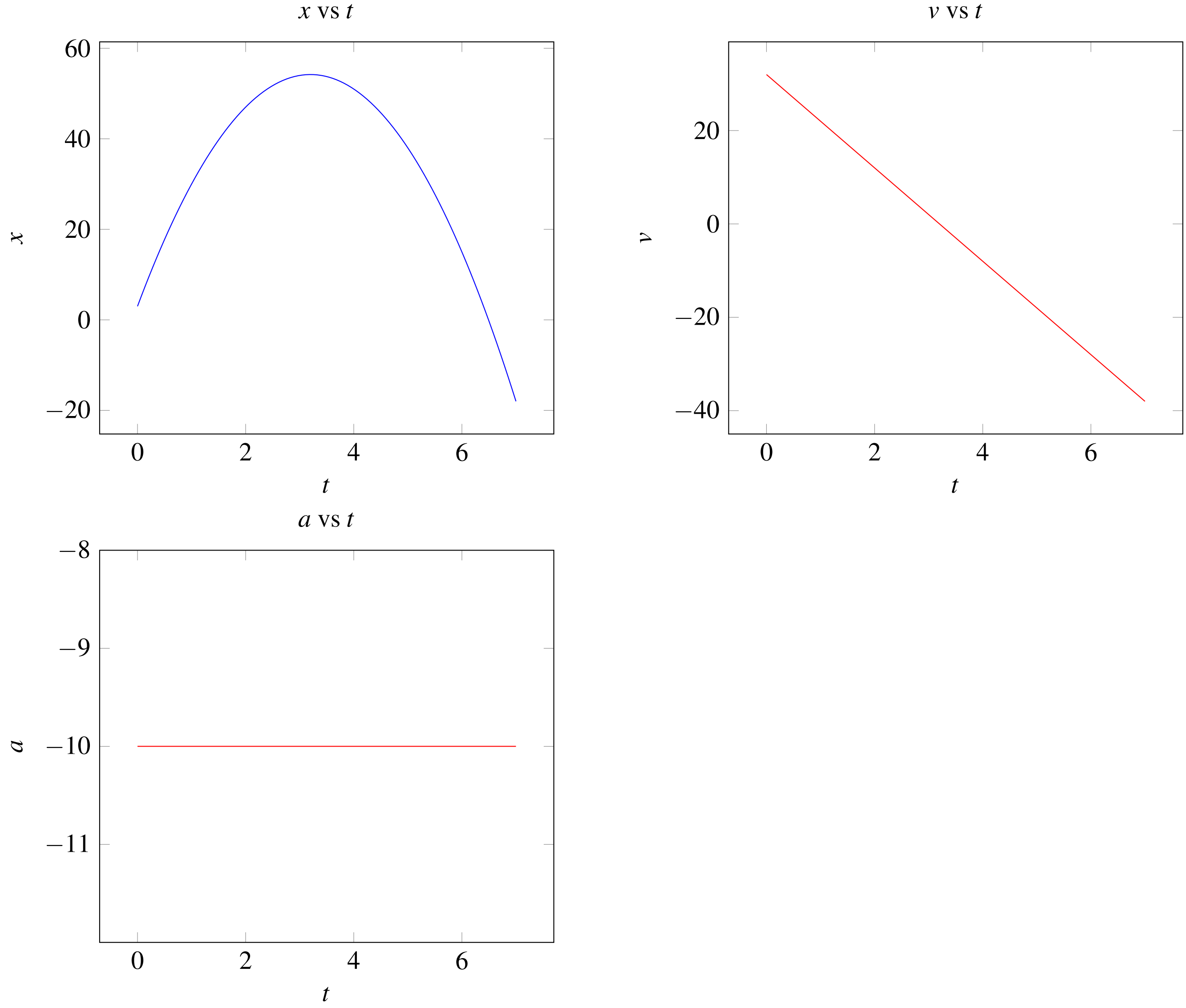
$$x = x(t) = -5(t - \frac{32}{10})^2 + \frac{271}{5}$$

We see that, for the value of x(t) to be maximum, the best we can do is to make $-5(t - \frac{32}{10})^2$ go to 0 by adjusting x to appropriate value i.e. $\frac{32}{10}$. This is because we see that whenever the expression is not 0, it's always negative. Thus, 0 is the maximum we can achieve.

Thus, the maximum value of x(t) is $0 + \frac{271}{5} = \frac{271}{5} = 54.2$ meters which occurs when t = $\frac{32}{10} = 3.2s$.

15.2.1.2 Knowing the velocity and acceleration

Now, let's consider the cannonball's velocity as a function of time, denoted by 'v'. We can extend our table to include 'v' and calculate the instantaneous velocities at different times. For instance, at t=0, the velocity is 32 meters per second, and it decreases uniformly over time.



$$x(t) = -5t^2 + 32t + 3$$

$$v(t) = -10t + 32$$

$$a(t) = -10$$

Acceleration, denoted by 'a', is the rate of change of velocity. In our example, acceleration is constant and negative, indicating deceleration. The function for 'a' is a constant a=-10.

With a quadratic function for displacement, a linear function for velocity, and a constant function for acceleration, we see a clear hierarchy. Once we master calculus tools, we'll be able to derive the functions for 'v' and 'a' from 'x' swiftly. Moreover, using integration techniques, we can work backward from 'a' to find 'v' and 'x'.

This example aligns with the physical world, where deceleration near Earth's surface is approximately 9.8 meters per second squared, similar to the 10 meters per second squared used here. Thus, our model is a good approximation of the cannonball's behavior.

This process of finding slopes of tangent lines to transition from displacement to velocity, and then to acceleration, is a general principle. Our example was unique in that the functions for velocity and acceleration were simpler than for displacement. But what happens beyond acceleration?

Today, we've explored the intricate relationships between displacement, velocity, and acceleration, and even ventured into the realms beyond. The connecting thread among these concepts is the derivative, which we're methodically approaching. Derivatives formalize the process of determining the slope of the tangent line at any point on a curve.
Please review the notes, and when you feel prepared, go ahead and tackle the exercises. Thank you for tuning in, and I eagerly anticipate our next session.

15.2.2 Practice Quiz

Question 1

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10$$

Estimate the height of the missile above the ground at the moment it was launched.
(a) 100 m
(b) 10 m
(c) 20 m
(d) 5 m
(e) 0 m

Question 2

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10$$

Estimate the height of the missile above the ground five seconds after launching.
(a) 510 m
(b) 385 m
(c) 395 m
(d) 490 m
(e) 390 m

Question 3

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10$$

Estimate the average velocity of the missile from $t = 0$ seconds to $t = 7$ seconds.
(a) 62 m/sec
(b) 64 m/sec

(c) 63 m/sec
(d) 66 m/sec
(e) 65 m/sec

Question 4

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10$$

Estimate the average velocity of the missile from $t = 7$ seconds to $t = 14$ seconds.
(a) -35 m/sec
(b) -5 m/sec
(c) -10 m/sec
(d) -65 m/sec
(e) -20 m/sec

Question 5

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10$$

Estimate the maximum height above the ground reached by the missile.
(a) 1010 m
(b) 490 m
(c) 500 m
(d) 510 m
(e) 650 m

Question 6

An arrow is shot upward from the surface of the moon. Its height $x = x(t)$ meters above the surface after t seconds is approximated by the formula

$$x(t) = 2 + 60t - 0.8t^2$$

Its velocity $v = v(t)$ m/sec after t seconds (and before it lands) is approximated by the formula

$$v = 60 - 1.6t.$$

Estimate the initial velocity of the arrow.
(a) -1.6 m/sec
(b) 58.4 m/sec
(c) 60 m/sec
(d) 2 m/sec
(e) -0.8 m/sec

Question 7

An arrow is shot upward from the surface of the moon. Its height $x = x(t)$ meters above the surface after t seconds is approximated by the formula

$$x(t) = 2 + 60t - 0.8t^2.$$

Its velocity $v = v(t)$ m/sec after t seconds (and before it lands) is approximated by the formula

$$v = 60 - 1.6t.$$

Estimate the time (to the nearest tenth of a second) at which the arrow is instantaneously at rest during its flight (before it lands).

(a) 37.5 sec
(b) 35.7 sec
(c) 37.2 sec
(d) 35.0 sec
(e) 32.7 sec

Question 8

An arrow is shot upward from the surface of the moon. Its height $x = x(t)$ meters above the surface after t seconds is approximated by the formula

$$x(t) = 2 + 60t - 0.8t^2.$$

Its velocity $v = v(t)$ m/sec after t seconds (and before it lands) is approximated by the formula

$$v = 60 - 1.6t.$$

Estimate the maximum height above the moon's surface reached by the arrow (to the nearest meter).

(a) 1127 m
(b) 1217 m
(c) 1109 m
(d) 1124 m
(e) 1192 m

Question 9

An arrow is shot upward from the surface of the moon. Its height $x = x(t)$ meters above the surface after t seconds is approximated by the formula

$$x(t) = 2 + 60t - 0.8t^2.$$

Its velocity $v = v(t)$ m/sec after t seconds (and before it lands) is approximated by the formula

$$v = 60 - 1.6t.$$

Estimate the constant acceleration of the arrow during its flight.

(a) 60 m/sec^2
(b) 1.6 m/sec^2
(c) -0.8 m/sec^2
(d) -3.2 m/sec^2
(e) -1.6 m/sec^2

Question 10

An arrow is shot upward from the surface of the moon. Its height $x = x(t)$ meters above the surface after t seconds is approximated by the formula

$$x(t) = 2 + 60t - 0.8t^2.$$

Its velocity $v = v(t)$ m/sec after t seconds (and before it lands) is approximated by the formula

$$v = 60 - 1.6t.$$

Estimate the average velocity of the arrow over the first ten seconds of its flight (to the nearest whole number.

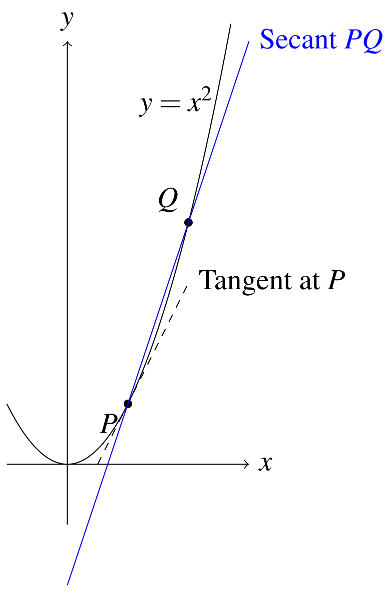
(a) 51 m/sec
(b) 54 m/sec
(c) 53 m/sec
(d) 50 m/sec
(e) 52 m/sec

Answers

The answers will be revealed at the end of the module.

15.3 Tangent lines and secants

In this section, we discuss how secants in geometry can be used to approximate slopes of tangent lines.



Slopes of lines are just fractions involving the vertical rise divided by the horizontal run.

$$\text{Slope} = \frac{\text{Vertical Rise}}{\text{Horizontal Run}}$$

The fractions associated with slopes of lines formed by secants happen to be linked to fractions associated with slopes of tangent lines by a natural limiting process. It's an amazing fact, that under certain conditions we can understand and control the behavior of approximations that occur, producing very precise outcomes. This leads us towards formal definitions of the derivative in terms of limits of fractions, which is the main theme of this section.

The underlying idea of calculus is to use tangent lines to approximate smooth curves, which arise as graphs of sophisticated functions that model interesting phenomena. Lines, of, course use simple arithmetic. So if we can achieve this then there's a substantial reduction in complexity. We have the possibility of creating very powerful and wide ranging tools.

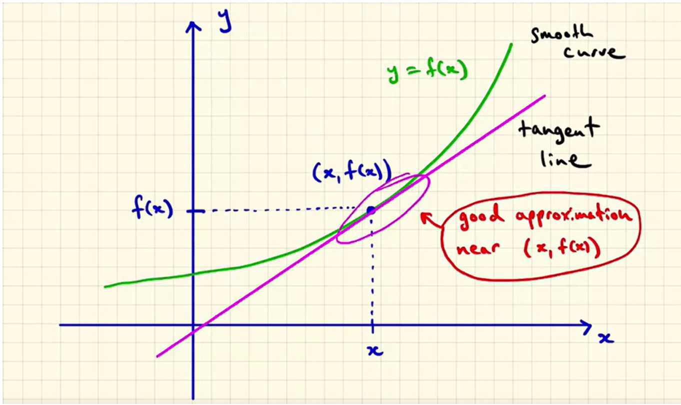
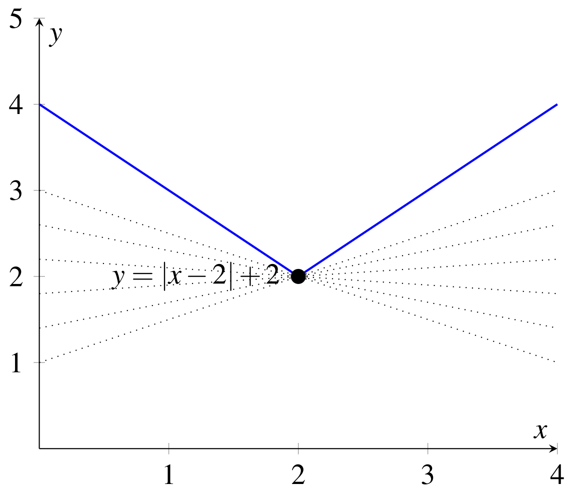


Figure 15.4: Approximate the value of a function at a point using tangent line

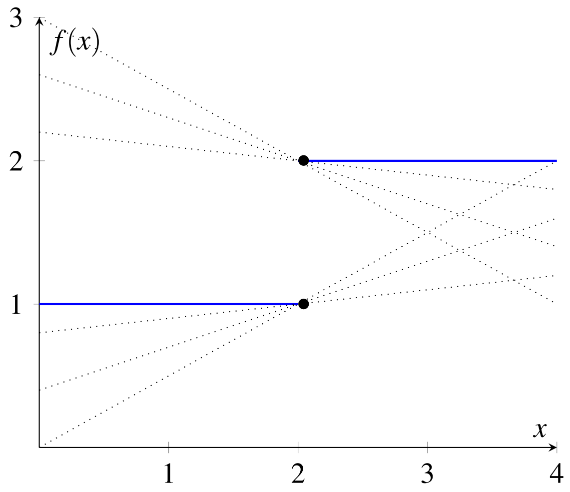
A tangent line, typically, just glances a curve at a single point of interest and acts as a good approximation to the curve near this point of intersection. There are cases where tangent lines actually cross the curve at the point of interest, but more about that later.

Warning, not all curves are smooth, some might need special treatment. For example, this curve has a sharp point (where literally infinite tangents are possible which touch curve at one point),

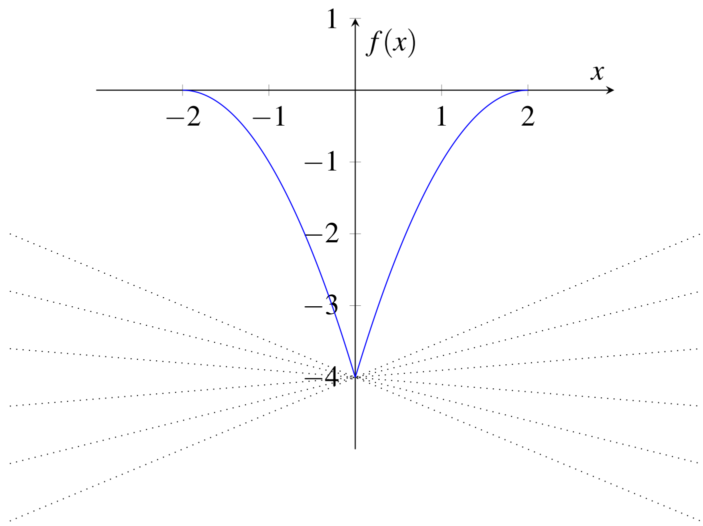


The curve below has a discontinuity where the curve breaks apart at some point (due to which infinite tangents are possible which touch the curve at only one point),

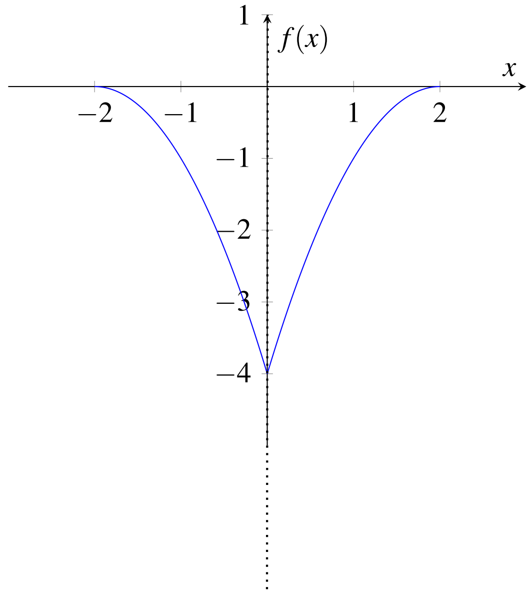
[0]Image 15.3 from MOOC Single Variable Calculus (University of Sydney)



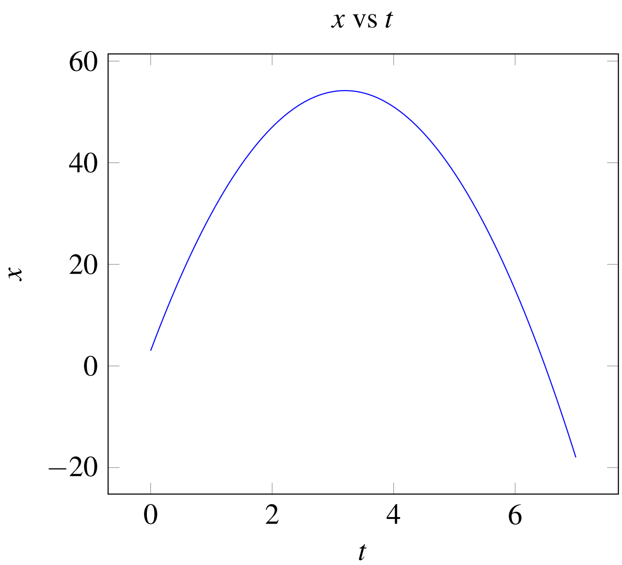
The curve below also has a sharp point (where we can literally draw infinite tangents at the sharp point).

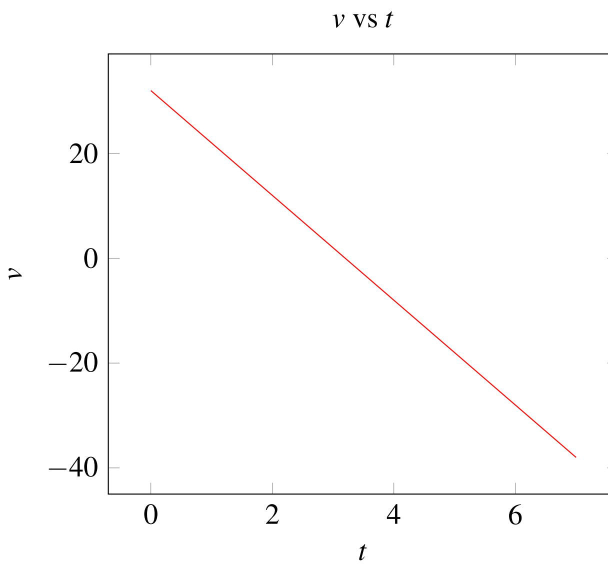


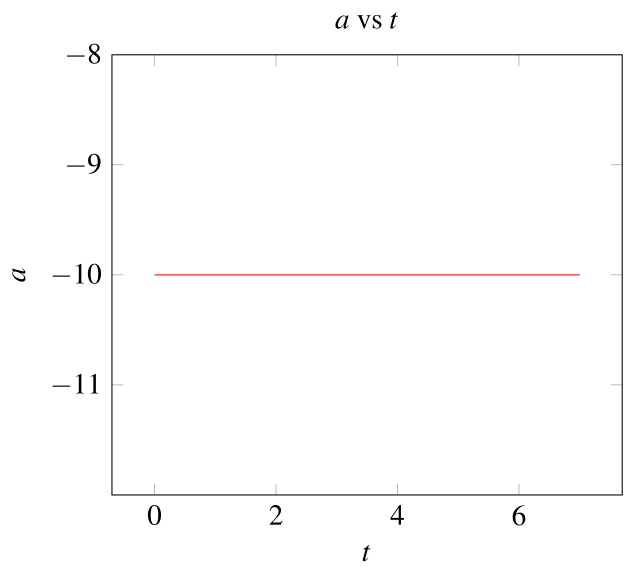
In each case there's some ambiguity about choosing an appropriate tangent line. The third curve is an interesting and important example and the sharp point is known as a **cusp**. There is, in fact, a useful and meaningful tangent line for this cusp example.



A vertical line that splits the cusp in the middle and even passes completely through the curve. Close to the cusp this vertical line is a very good approximation to the behavior of the curve on either side.







$$x(t) = -5t^2 + 32t + 3$$

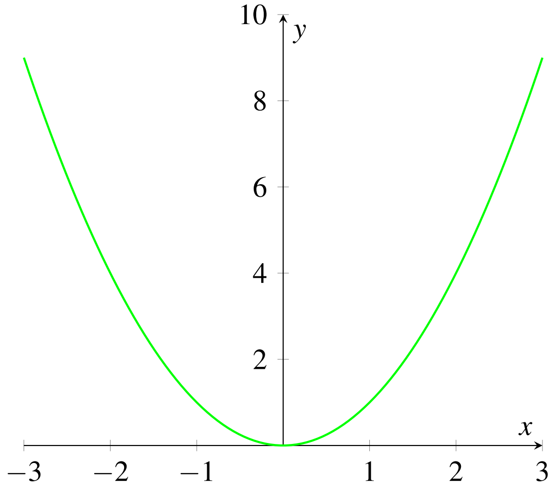
$$v(t) = -10t + 32$$

$$a(t) = -10$$

In the previous section, we looked at a particular displacement curve associated with a cannonball fired directly upwards and eventually coming back down to Earth under the influence of gravity. The curve was in fact described by a quadratic equation.

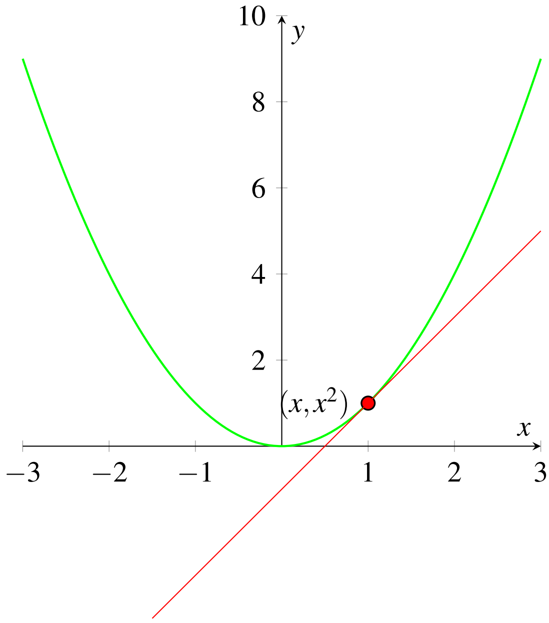
We looked at slopes of tangent lines in that example and a linear function was produced that represented the velocity. This looked rather special, but in fact turns out to be typical of behavior that occurs with all quadratic functions.

To try to understand what's going on let's carefully analyze the simplest quadratic of all, $y = x^2$. And see if we can extract some general principles.

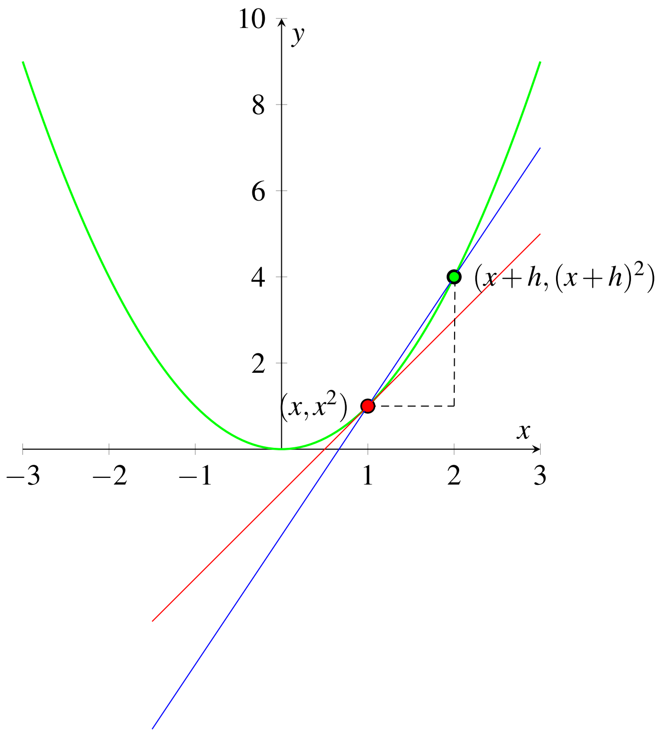


Note that we're reverting x again as the independent variable and y the dependent variable.

A typical point on the curve has coordinates x on the horizontal axis and x^2 on the vertical axis. And we can draw a tangent line to the curve passing through this point.



We can perturb the input x slightly simply by adding h to get a new input $x+h$. And then move up to the curve and across to the vertical axis to land at the value $(x+h)^2$. Hence the new point on the curve has coordinates $(x+h)$ and $(x+h)^2$.



We can join these two points on the green curve with a straight line segment, colored blue in the diagram. A line that joins two distinct points of a curve is called a secant in geometry. Because it's a line segment, this secant has a slope in the usual sense, that is a vertical raise divided by an horizontal run. You can see that the horizontal run in the diagram is h, there's no harm throughout this discussion in assuming that h is positive.

$$\text{Horizontal Run} = h$$

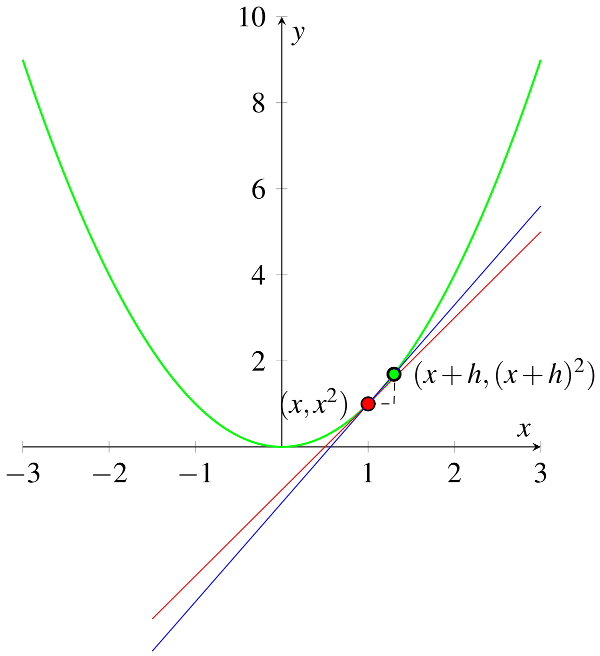
The vertical rise in this diagram is the difference in the y coordinates, namely, $(x+h)^2 - x^2$. We'll look at the quotient of these carefully in a moment.

$$\text{Vertical Rise} = (x+h)^2 - x^2$$

Note that as h is made smaller and smaller approaching zero without actually reaching zero, the slope of the secant approaches the slope of the tangent line.

$$\lim_{h \to 0} \text{ Slope of secant} = \text{ Slope of tangent}$$

And it's conventional to use these arrows ($\rightarrow$) as abbreviation for the word approaches. To see this visually imagine the second point slipping or sliding down the parabola towards the first point. Even though the secant gets shorter and shorter [1], the direction of the secant aligns more and more with the direction of the tangent line.



So that their slopes look like they're going to coincide and we think of them as actually coinciding in some kind of limit.

Note that even though h approaches zero, it doesn't actually reach zero as these diagrams require two distinct points in order to form a secant. If h were equal to zero then there would be only one point on the curve not two.

So returning to the earlier diagram with horizontal run h and vertical rise $(x+h)^2 - x^2$. Let's look at the quotient representing the slope of the secant.

$$\text{Slope of Secant} = \frac{(x+h)^2 - x^2}{(x+h) - x} = \frac{(x+h)^2 - x^2}{h}$$

[1] This is because Secant is technically defined as just the line segment between the two points on the curve.

The numerator, representing the vertical rise, can be expanded and then there's some cancellation.

$$\text{Slope of Secant} = \frac{x^2 + 2xh + h^2 - x^2}{h}$$

So the fraction becomes $2xh + h^2$ divided by h.

$$\text{Slope of Secant} = \frac{2xh + h^2}{h}$$

And the numerator factorizes and the h's cancel in the numerator and denominator simplifying further to $2x + h$.

$$\text{Slope of Secant} = \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h$$

As h gets closer and closer to zero this expression, $2x + h$, for the slope of the secant gets closer and closer to $2x$.

$$\lim_{h \to 0} \text{ slope of secant } = \lim_{h \to 0} 2x + h = 2x + 0 = 2x$$

We regard 2x as the limit of this process of taking slopes of secants. So 2x represents the slope of the tangent line.

$$\text{Slope of tangent line} = \lim_{h \to 0} \text{ slope of secant} = 2x$$

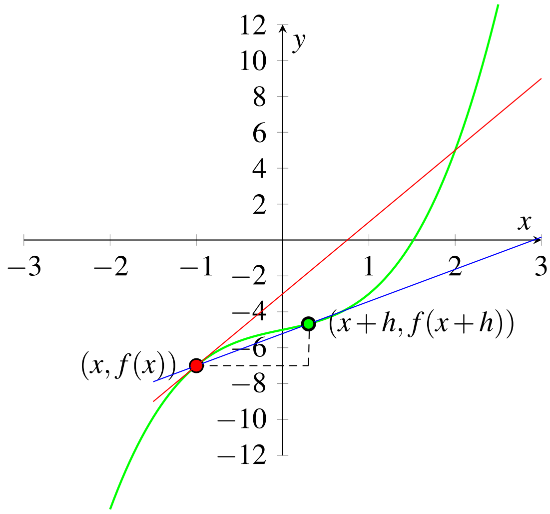
We've derived this very important result, that the slope of the tangent line to the parabola $y = x^2$ at the point of the coordinates $(x, y) = (x, x^2)$ is just 2x.

Now, if you go through the same process with $y = kx^2$, where k is some fixed constant then the previous calculation simply gets scaled by a factor of k. And we'll get the following result. The slope of the tangent line to the parabola $y = kx^2$ at the point with coordinates $(x, y) = (x, kx^2)$ is just $2kx$.

We can now discover a surprising connection with areas of circles. Consider a circle of radius r, it's area, capital A, is given by the formula $A = \pi r^2$. The perimeter, capital P, is given by the formula $P = 2\pi r$. That is, $2\pi r$, which is precisely the slope of the tangent line to the parabola $A = \pi r^2$ at the point with coordinates $(r, A) = (r, \pi r^2)$, where we interpret our previous result using r for x, capital A for y, and pi for k.

This isn't a coincidence, it's an instance of something quite profound in mathematics. Forming slopes of tangent lines is called differentiation. Finding areas of regions in the plane, like finding the area of a circle, is called integration. There's a remarkable connection between them described in the last section of this course called the "Fundamental Theorem of Calculus". Something as innocent looking as a simple relationship between slopes of tangent lines and areas of regions in the plane turns out to be one of the most transformative observations in the history of mathematics. And was first noticed by Newton and Leibniz in the 17th century.

The idea behind the calculation we just did for $y = x^2$ generalizes to any smooth curve $y = f(x)$.



Choose some point of interest (red dot in this figure) with coordinates x on the horizontal axis and $f(x)$ on the vertical axis. Now, attach a tangent line to the curve that passes through this point. Now, perturb the input x slightly by adding h to get a new input, $x+h$. The output corresponding to this new input is $f(x+h)$ marked on the vertical axis. We now join the two points on the curve to form this blue line segment, called a secant as before.

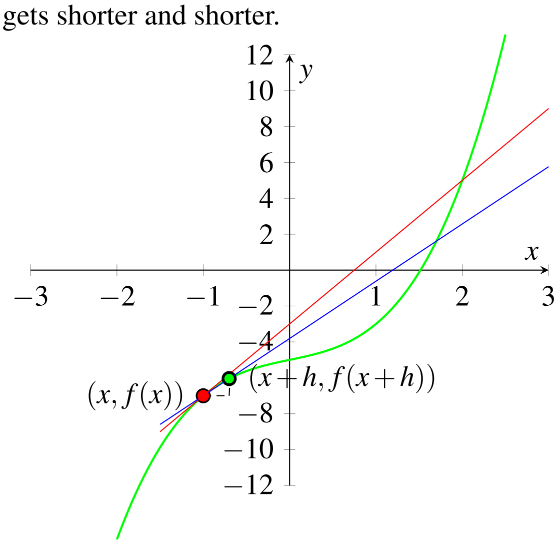
In moving from the first point to the second new point, the horizontal run is h and the vertical rise is $f(x+h) - f(x)$. So the slope of the secant is the quotient f(x+h)-f(x) divided by h.

$$\text{Slope of Secant} = \frac{\text{Vertical Rise}}{\text{Horizontal Run}} = \frac{f(x+h) - f(x)}{(x+h) - h} = \frac{f(x+h) - f(x)}{h}$$

The slope of the tangent line is the limiting slope of the secant as h tends towards zero.

$$\text{Slope of tangent} = \lim_{h \to 0} \text{ Slope of Secant}$$

You can see this visually as a new point slides down the curve towards the original point, the secant gets shorter and shorter.



But its slope gets more and more aligned with the slope of the tangent and we think of it as actually reaching the slope of the tangent in the limit.

$$\text{Slope of tangent} = \lim_{h \to 0} \text{ Slope of secant} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We capture this idea using this limit notation, lim is an abbreviation for limit. And together with $h \to 0$ beneath the full result reads, the slope of the tangent line is the limit as h approaches zero of $\frac{f(x+h)-f(x)}{h}$.

Whatever this limit evaluates to representing the slope of the tangent line is called the derivative of f at x. Now, this technical expression involving limits has several equivalent formulations that we'll explore in later videos. It's quite astonishing, as you'll see, that we're able to calculate and control these limits precisely with so many important functions f.

Today we considered in detail the process of taking the limiting slope of a secant to find the explicit formula for the slope of the tangent line to a parabola. The underlying method generalizes to any smooth function and is the basis for formal definitions of the derivative, which will be discussed and illustrated in detail in later sections.

The idea has two main ingredients, forming slopes of line segments, which are special kinds of fractions, and then seeing what happens in the limit. Because of the importance of the notion of limits we'll discuss them in more detail in the next two sections. Please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

15.3.1 Practice Quiz

Question 1

Consider the parabola $y = x^2$. Find the slope of the tangent line at the point $(1,1)$.

(a) 4
(b) 1
(c) 3
(d) 0
(e) 2

Question 2

Consider the parabola $y = x^2$. Find the slope of the tangent line at the point $(2,4)$.

(a) 8
(b) 2
(c) 0
(d) 4
(e) 1

Question 3

Consider the parabola $y = -x^2$. Find the slope of the tangent line at the point $(0,0)$.

(a) 0
(b) 1
(c) -1
(d) 2
(e) -2

Question 4

Consider the parabola $y = -x^2$. Find the slope of the tangent line at the point $(-2,-4)$.

(a) 0
(b) -2
(c) 4

(d) 2
(e) -4

Question 5

Find the coordinates of the point on the parabola $y = x^2$ where the slope of the tangent line is 10.
(a) $(20, 400)$
(b) $(25, 5)$
(c) $(5, 25)$
(d) $(10, 100)$
(e) $(5, 10)$

Question 6

Find the coordinates of the point on the parabola $y = -x^2$ where the slope of the tangent line is -6.
(a) $(3, -9)$
(b) $(-3, -9)$
(c) $(-6, -36)$
(d) $(-6, 36)$
(e) $(-3, 9)$

Question 7

Consider the parabola $y = 3x^2$. Find the slope of the tangent line at the point $(1, 3)$.
(a) 2
(b) 6
(c) 12
(d) 1
(e) 3

Question 8

Consider the parabola $y = 3x^2$. Find the slope of the tangent line at the point $(-2, 12)$.
(a) -1
(b) -6
(c) -12
(d) -4
(e) -2

Question 9

Find the coordinates of the point on the parabola $y = 3x^2$ where the slope of the tangent line is 4.
(a) $\left(\frac{2}{3}, \frac{4}{3}\right)$
(b) $\left(\frac{4}{3}, \frac{2}{3}\right)$
(c) $\left(\frac{3}{2}, \frac{9}{4}\right)$
(d) $\left(\frac{4}{3}, \frac{16}{3}\right)$
(e) $\left(\frac{3}{2}, \frac{27}{4}\right)$

Question 10

Find the coordinates of the point on the parabola $y = -3x^2$ where the slope of the tangent line is -12.
(a) $(2, -12)$
(b) $(-2, -12)$
(c) $(-1, 3)$
(d) $(-2, 12)$
(e) $(-1, -3)$

Answers

The answers will be revealed at the end of the module.

16. Limits

16.1 Different kind of Limits

16.1.1 Why did Limits come in the first place ?

The expression stretched to the limit is apt in calculus. We push functions all the way to the boundaries of possibilities. The concept of a limit captures precisely the notion of accessing values that at first sight appear to be forbidden or out of bounds.

In this section, we look at a range of phenomena that involve getting closer and closer to some kind of ideal point or value. We also make explicit some examples and notation involving infinities, which encapsulate the idea of being free to go on and on and on. How many times have we said ad infinitum in earlier sections?

Slopes of tangent lines to curves are a prototype for this idea of an idealized value. Imagine you inhabit a typical smooth curve, it's your universe, and you're unable to access points on the curve to create secants. You might never be able to jump off the curve onto the tangent line. But the way the secants behave near the point of interest may be enough to tell you what's happening out there on the tangent line.

In the last section, we demonstrated, by pure thought and manipulation of symbols, that the slopes of secants to the parabola $y = x^2$ approach the slope of the tangent line, which happens to be $2x$ at the point with input x, an amazing achievement. We realized that the underlying method works for any smooth curve, leading to the notion of a limit in this notation.

Lim is an abbreviation for limit and refers to some "limiting value" or behavior that is being "approached".

16.1.2 Two contrasting examples for exploration

Let's discuss two similar looking but contrasting examples,

$$f(x) = \frac{x^2 - 1}{x - 1} \qquad g(x) = \frac{x^2 - 2}{x - 1}$$

These are both examples of rational functions, ratios of polynomials. In both of these cases, x equals one creates a problem in the denominator because trying to evaluate them with x equals one means dividing by zero, which is forbidden.
g(1) = $\frac{1-2}{1-1}$ = Undefined

Let's do some exploration for x close to one.

f(1.1), you can check evaluates to 2.1.

f(1.01) evaluates to 2.01,

and f(0.9999) taking an input very close to it but just a tiny bit less than one, evaluates to 1.9999.

We seem to be getting closer and closer to two by taking our inputs closer and closer to one.

Now, let'stry the same thing using the rule for g, which at first sight appears to be only slightly different.

g(1.1), you can check evaluates to negative 7.9,

g(1.01) evaluates to nearly negative 98,a largish negative number, and

g(0.9999) evaluates to over 10,000, a very large positive number. The behavior of g for inputs near x equals one looks wild and unpredictable, compared with the apparently stable behavior of f.

16.1.3 Investigating the Reason

Let's try to understand why the values of f appear to be converging to two. In the rule for f,

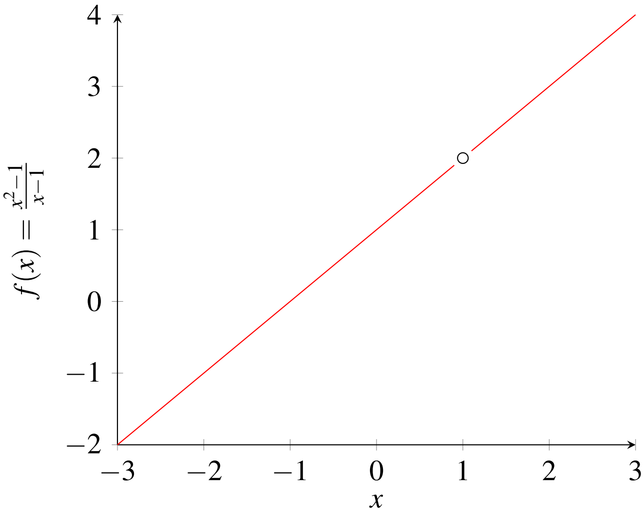
$$f(x) = \frac{x^2-1}{x-1}$$

the numerator factorizes as (x+1)(x-1), and we get cancellation,

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x+1$$

and the entire fraction simplifies to x+1.

Throughout, we're assuming that x doesn't equal one. So, the rule y = f(x) is almost exactly $y = f(x) = x+1$, the rule of a straight line. The only thing missing is a tiny hole with a rule for f that prohibits an input of x =1 .



Observe that as x approaches one from either side, we slide up and down the line approaching the hole, while at the same time, the values of y approach two on the y-axis. We use limit notation and say the $\lim_{x\to 1} f(x) = 2$. This confirms what we anticipated from our exploration.

So, what about this wild function g(x) = $\frac{x^2-1}{x-1}$?

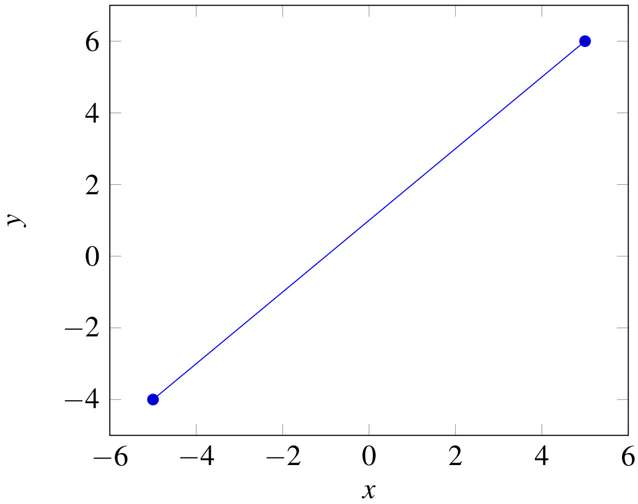
Let's try to simplify the fraction g(x) by doing a long division of polynomials, dividing $x^2 - 2$ by $x - 1$.

$$
\begin{array}{r|l}
 & \quad\quad x+1 \\
x-1 & \; x^2 \quad\quad -2 \\
 & -x^2+x \\
\hline
 & \quad\quad x-2 \\
 & \quad\;\; -x+1 \\
\hline
 & \quad\quad\;\; -1
\end{array}
$$

In a few short steps, we have the quotient of $x+1$, and a remainder of -1. Hence the fraction g(x) can be rewritten as the quotient $(x+1) - \frac{1}{x-1}$.

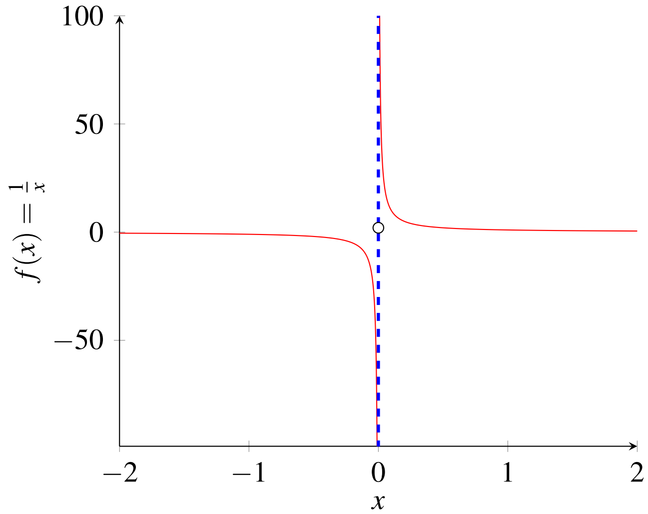
$$g(x) = (x+1) - \frac{1}{x-1}$$

Notice that the first part, x+1 is the rule for the line that we saw before in analyzing f.

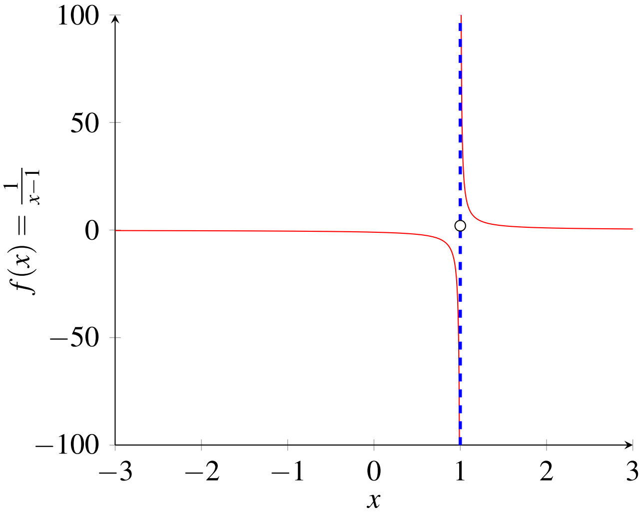


We're also taking away a piece $\frac{1}{x-1}$, which is the rule for hyperbola.

So, the rule for g involves a line and a hyperbola. Let's focus on $\frac{1}{x-1}$, which is closely related to the simple hyperbola, $\frac{1}{x}$.



To get the graph for $y = \frac{1}{x-1}$, you translate the hyperbola for $\frac{1}{x}$ to the right by 1 unit, and note that the line $x = 1$ becomes the vertical asymptote.



$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

As x approaches 1 from right using this arrow notation $(\to)$ with a little plus sign as a superscript, we see $\frac{1}{x-1}$ shoot off towards infinity, but which we mean, getting arbitrarily large and positive.

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

As x approaches one from left, but this time with a little minus sign as a superscript, we see $\frac{1}{x-1}$ shoot off towards $-\infty$, but which remain getting arbitrarily large and negative. So, we have these two contrasting behaviors of $\frac{1}{x-1}$ expressed using limit notation and the infinity and minus infinity symbols, which is notation for getting arbitrarily large and positive or negative.

Let's combine these with the overall rule for g.

$$g(x) = (x+1) - \frac{1}{x-1}$$

As x approaches 1 from left, x+1 approaches 1+1 = 2, which is straightforward, and $\frac{1}{x-1}$ becomes arbitrarily large and positive. So that when you take this away from something close to two, we get something large and negative. We capture this using limit notation.

$$\lim_{x \to 1^+} (x+1) - \frac{1}{x-1} = 1 + 1 - \infty = -\infty$$

By contrast, as x approaches one from right, again, x+1 approaches two. But this time, $\frac{1}{x-1}$ becomes arbitrarily large and negative. So that now, when we take this away, we get something large and positive, and we capture this using limit notation.

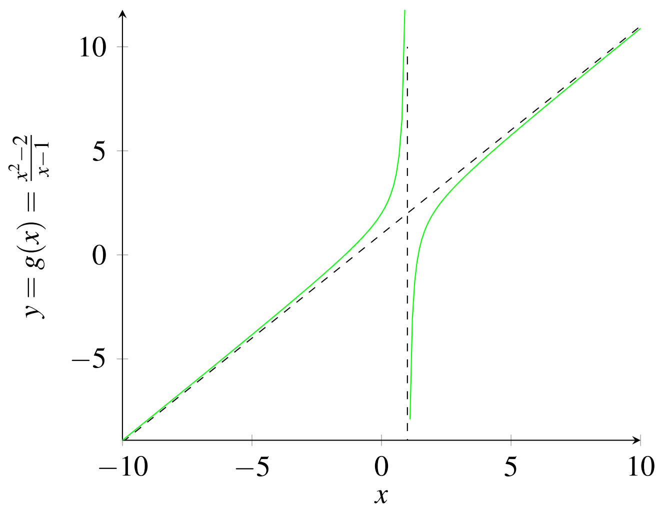
$$\lim_{x \to 1^-} (x+1) - \frac{1}{x-1} = 1 + 1 - (-\infty) = \infty$$

So, we have these two concise descriptions of the behavior of g(x) as x approaches one either from left or right.

But there's more information available from the hyperbola.

$$g(x) = \frac{x^2-2}{x-1} = (x+1) - \frac{1}{x-1}$$

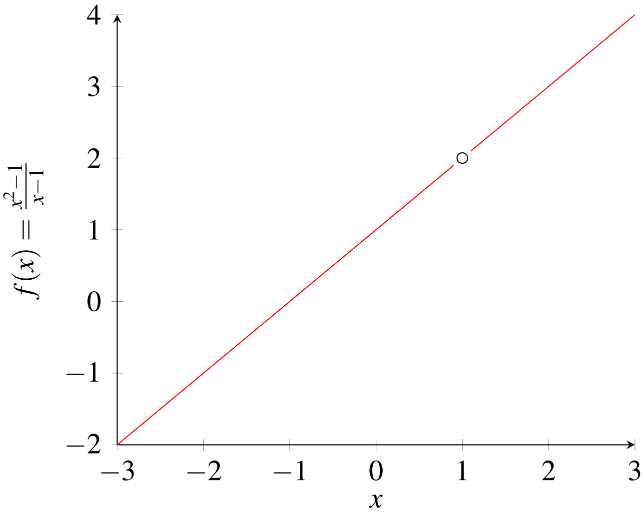
As x gets large, positively or negatively, $\frac{1}{x-1}$ approaches zero. We can see the overall effect on the rule for g(x). For large positive or negative x, g(x) becomes approximately x+1 because the piece, $-\frac{1}{x-1}$ is close to zero. The rule for g becomes more and more identical to the rule for the line $y = x+1$. The further you are away from the origin, geometrically, that line becomes an oblique asymptote for the curve. We can visualize what happens.



Here the axes, the vertical line $x = 1$ and the oblique line, $y = x+1$ dotted in. All of the asymptotic behavior is captured by this green curve in two pieces or branches which is the graph of $y = g(x) = \frac{x^2-2}{x-1}$. The line $x = 1$ is the vertical asymptote, and the line $y = x+1$ is the oblique asymptote, which becomes a better and better approximation to the curve, the further you are away from the origin. You might be curious how we knew to draw it exactly like this, and why, for example, there are no wrigley bits or strange things happening in between the asymptotic behavior.

In the next section, you'll be armed with techniques of curve sketching to determine very precisely the nature of curves like this one, by exploiting the derivative and the second derivative.

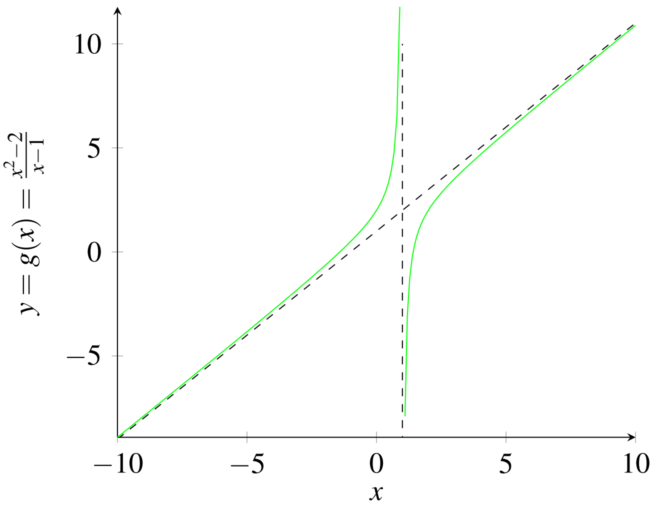
So, let's return to both our examples f and g. The graph of f we saw was in fact a line, but with one point missing.



$$\lim_{x\to 1^+} f(x) = 2 \qquad \lim_{x\to 1^-} f(x) = 2$$

As x approaches 1 from either side, f(x) approaches 2 captured by this limit statement, and this confirmed our exploration.

By contrast, now that we know about the graph of g, we can understand the wild fluctuations.



$$\lim_{x\to 1^+} \frac{x^2-2}{x-1} = \lim_{x\to 1^+} (x+1) - \frac{1}{x-1} = (1+1) - \infty = -\infty$$

$$\lim_{x\to 1^-} \frac{x^2-2}{x-1} = \lim_{x\to 1^-} (x+1) - \frac{1}{x-1} = (1+1) - (-\infty) = \infty$$

Two statements using limits describe what happens as we approach 1 from either side, explaining the large positive and negative values that we saw. Then we have this other interesting phenomenon that as x gets large, either positively or negatively, the curve is approximated by the straight line y=x+1, which forms an oblique asymptote.

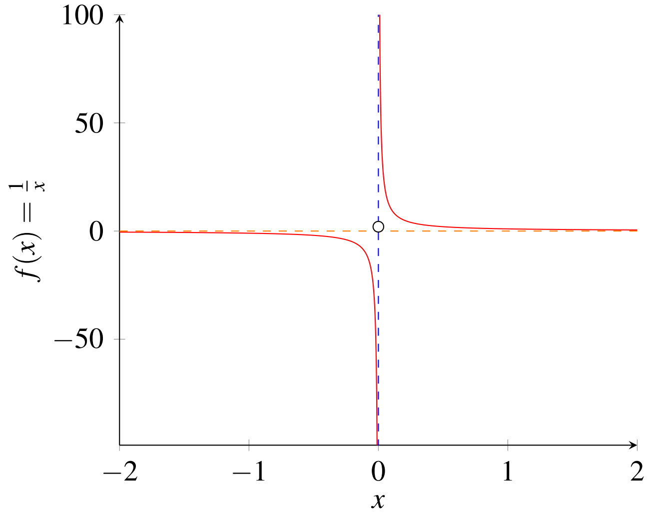
$$\lim_{x\to\infty} \frac{x^2-2}{x-1} = \lim_{x\to\infty} (x+1) - \frac{1}{x-1} = \lim_{x\to\infty} x+1$$

$$\lim_{x\to-\infty} \frac{x^2-2}{x-1} = \lim_{x\to-\infty} (x+1) - \frac{1}{x-1} = \lim_{x\to-\infty} x+1$$

16.1.4 Exploring some more nice limits

16.1.4.1 With Reciprocal Function

There are a number of important limits involving infinity symbols.



If you take the simplest hyperbola, the graph of $y = \frac{1}{x}$ that we used earlier, then $\frac{1}{x}$ approaches 0 as x gets arbitrarily large positively or negatively,

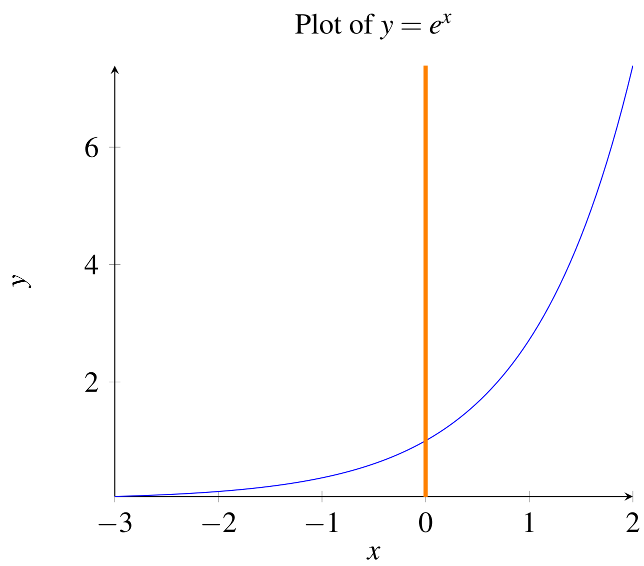
$$\lim_{x \to \infty} \frac{1}{x} = 0 \qquad \lim_{x \to -\infty} \frac{1}{x} = 0$$

captured by these two statements in limit notation, and the x-axis becomes a horizontal asymptote (denoted by orange color).

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

By contrast, if x approaches zero from above, $\frac{1}{x}$ gets arbitrarily large and positive, or if x approaches zero from below, $\frac{1}{x}$ gets arbitrarily large and negative, captured by these two statements in limit notation, and the y-axis becomes the vertical asymptote (denoted by blue color).

16.1.4.2 With exponential growth Function



$$\lim_{x \to \infty} e^x = \infty$$

If we look at the natural exponential function, we see that e^x gets arbitrarily large and positive as x does, captured by this statement in limit notation. And notice that we're using the infinity symbol as a value for the limit even though it's not a number. It's just notation and says that the values become arbitrarily large without bound.

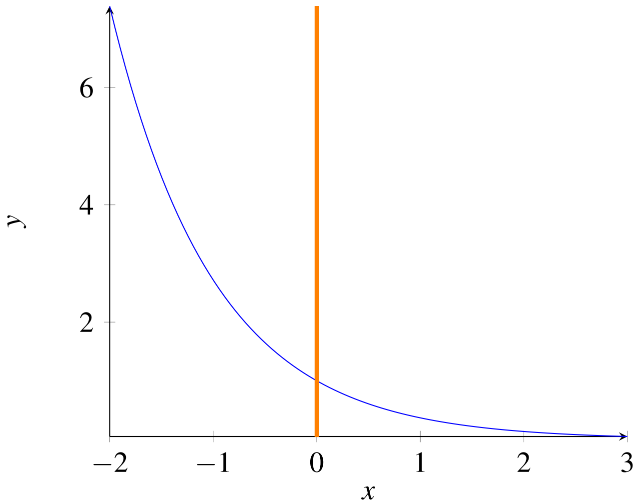
$$\lim_{x \to -\infty} e^x = 0$$

From the curve, you can see as x gets arbitrarily large and negative, the value of each to the x gets closer and closer to zero captured by this limit statement, and the x-axis becomes a horizontal asymptote. The contrasting behaviors of e^x as we move along y from the origin in each direction are captured concisely by the limit notation.

16.1.4.3 With exponential decay Function

If we reflect this curve in the y-axis, we get the curve $y = e^{-x}$,

Plot of $y = e^{-x}$



and we get the corresponding limit statements, which are mirror images of the first two.

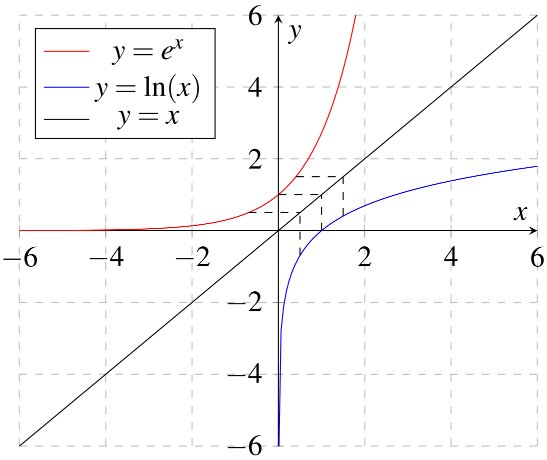
$$\lim_{x \to \infty} e^{-x} = 0$$

$$\lim_{x \to -\infty} e^{-x} = \infty$$

16.1.4.4 With logarithmic growth Function

If we reflect the curve $y = e^x$ in the line $y = x$, then we invert the function and get the natural logarithm.

Graphs of $y = e^x$, $y = \ln(x)$, and $y = x$



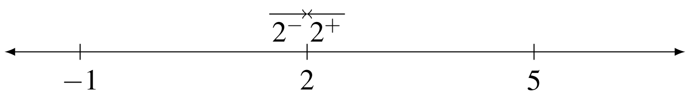
As x gets large and positive, so does ln(x), but, in fact, very slowly. This is called logarithmic growth in stark contrast to exponential growth. Instead of a rapid explosion, the natural logarithm grows at a snail's pace, and even then, a very, very almost unimaginably slow snail.

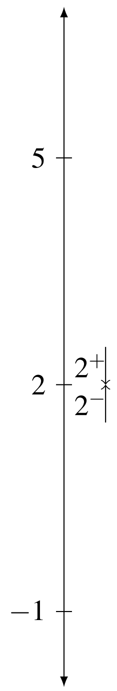
Finally, we note that as x approaches zero from above, ln(x) gets arbitrarily large and negative,

$$\lim_{x \to 0} \ln(x) = -\infty$$

expressed using this one-sided limit, and the negative half of the y-axis becomes a vertical asymptote.

We have introduced so many different ideas, all unified by the concept of a limit, which often captures or summarizes some kind of ideal value or behavior in the faraway distance, or evolving getting arbitrarily close to some prohibited input. All of those can be expressed concisely using limit notation, and various tricks using ∞ and $-\infty$ symbols to indicate getting arbitrarily large positively or negatively, and plus and minus symbols use the superscripts to indicate whether we're approaching something from the right or left, from above or below.





We saw examples of rational functions, where you can have, for example, a complicated looking rule reduced simply to a straight line with just one point missing.

By contrast, another function with almost the same rule but having much more complicated behavior involving an oblique asymptote.

Please re-read the material if you need , and when you're ready please attend the exercises. Thank you very much for reading even if you're also stretched to the limits by all of this new material. I look forward to seeing you again soon.

16.2 Limit Laws and identities

In today's section, we'll discuss and illustrate several limit laws that enable you to combine simple or known limits to calculate or understand the behavior of new limits that can arise and appear at first sight to be difficult or complicated. You'll become experts in no time.

16.2.1 First set of limit Laws

The first main collection of laws tells us that **limits respect arithmetic**. Arithmetic operations include addition, subtraction, multiplication, and division. These laws allow you to bring limits inside so to speak any of these operations provided all the components as simply limits actually exist.

1. The first law states that **"The limit of a sum is the sum of the limits."** Symbolically,

$$\lim(f(x)+g(x)) = \lim(f(x)) + \lim(g(x))$$

2. The second law says that **"The limit of a difference is the difference of the limits."** Expressed symbolically as before but with minus instead of plus.

$$\lim(f(x)-g(x)) = \lim(f(x)) - \lim(g(x))$$

3. The third law says that **"The limit of a product is the product of the limits."** Expressed symbolically this way now using multiplication.

$$\lim(f(x)\times g(x)) = \lim(f(x)) \times \lim(g(x))$$

   It follows as a special case that **the limit of a constant multiple is the constant multiple of the limits.** Expressed symbolically like this, where k is a constant. We often say constants come out the front.

$$\lim(k\times g(x)) = \lim(k)\times\lim(g(x)) = k\lim(g(x))$$

4. The fourth law says that **"The limit of a quotient is the quotient of the limits."** Expressed symbolically like this.

$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim(f(x))}{\lim(g(x))}$$

   Provided that at no stage are you dividing by zero. This last one's very important because we're often needing to deal with fractions in calculus with a possibility of zero as that denominator can become a serious and delicate issue.

Notice that in describing these laws I wasn't being explicit about what happens to x in order to be general to cover a range of possibilities. You can have x approaching some number a from either side,

$$\lim_{x\to a}$$

or approaching a number from the right,

$$\lim_{x\to a^+}$$

or from the left,

$$\lim_{x\to a^-}$$

the so-called one-sided limits.

We can also have x heading off towards infinity, by which we mean getting arbitrarily large and positive,

$$\lim_{x \to \infty}$$

or minus infinity getting arbitrarily large and negative.

$$\lim_{x \to -\infty}$$

16.2.2 Building Blocks of Limits

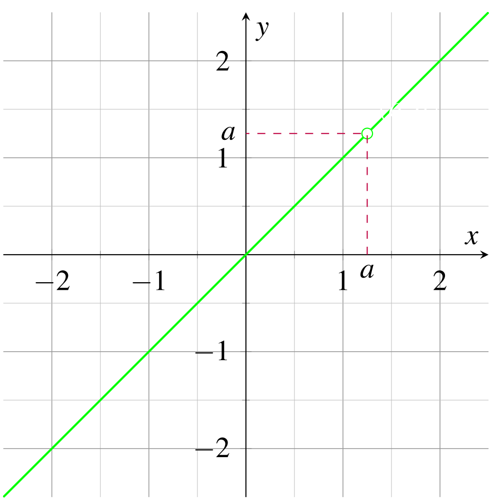
There are a few basic building blocks and you can combine them with the limit laws to progress very rapidly in many cases.

16.2.2.1 Limit involving Identity Function

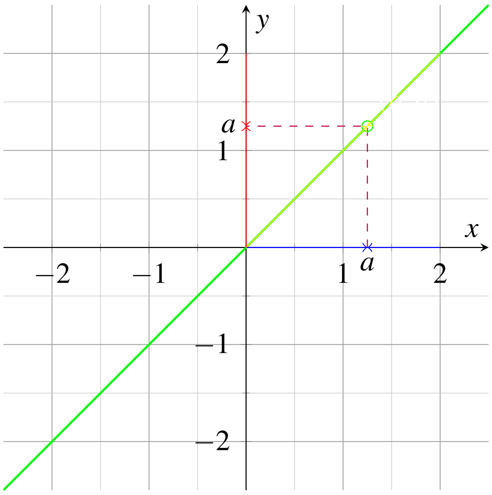
Firstly, the limit of x as x approaches a is clearly just a.

Theorem 16.1

$$\lim_{x \to a} x = a$$



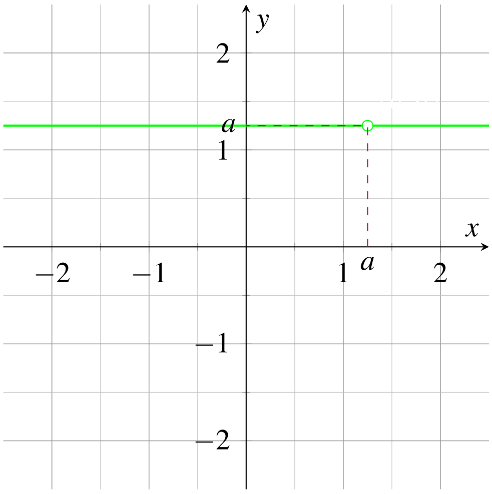
This is confirmed by looking at the graph of y=x which I've drawn but made a tiny hole at x=a because we're interested in limiting behavior as x approaches a but doesn't necessarily actually reach a. As x approaches a on the horizontal axis, y=x approaches a on the vertical axis.



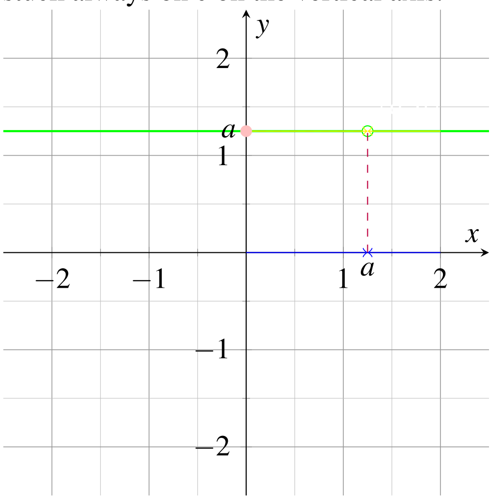
16.2.2.2 Limit involving Constant Function

Theorem 16.2

$$\lim_{x \to a} c = c$$



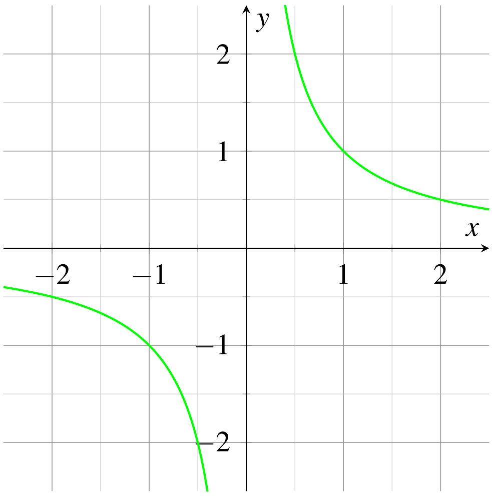
The limit of a constant c regardless of what x is doing is just the constant c. This is clear by looking at the graph of y = c. Again I've made a hole at x = a. As x approaches a on the horizontal axis, y is stuck always on c on the vertical axis.



16.2.2.3 Limit involving Reciprocal Function

Theorem 16.3

$$\lim_{x\to\infty}\frac{1}{x}=\lim_{x\to-\infty}\frac{1}{x}=0$$



If x gets very large and positive or large and negative then its reciprocal $\frac{1}{x}$ tends to zero, so that the limit of $\frac{1}{x}$ in both cases is equal to zero. This is clear from the graph which is a hyperbola. As x moves further and further away from the origin in either direction, the value of $y=\frac{1}{x}$ gets closer and closer to zero.

16.2.3 Examples

Let's look at some examples.

■ **Example 16.1** Let's find the limit as x approaches 2 of 2x-1.

$$\lim_{x\to 1} 2x-1$$

By the second law it becomes a difference of the limits of 2x and of one as x approaches two,

$$\lim_{x\to 1} 2x-\lim_{x\to 2} 1$$

which becomes the limit of 2 multiplied by the limit of x all take away 1,

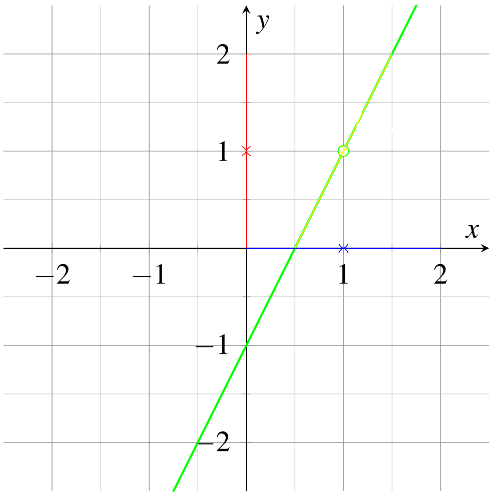
$$2\lim_{x\to 2} x-1$$

which quickly becomes 3.

$$2\times 1-1=2-1=1$$

You can see the effect of this visually. The graph of y = 2x -1 is just a straight line and we place a hole in the line corresponding to two on the horizontal axis which x is approaching.

■



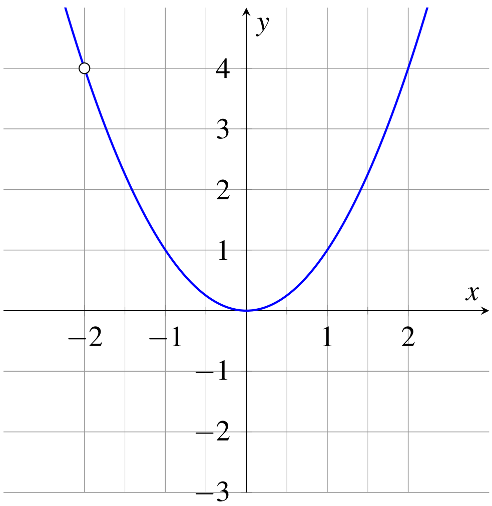
However, the hole lines up exactly with the value 1 on the y axis which is the limiting value of 2x - 1. In this case we can actually fill in the hole with x-coordinate two and y-coordinate being the limiting value as x approaches 1.

As another example, the limit of x^2 as x approaches -2 is by limit law of multiplication.

■ **Example 16.2**

$$\lim_{x \to -2} x^2 = \lim_{x \to -2} x . \lim_{x \to -2} x$$
$$= (-2).(-2) = (-2)^2 = 4$$

■



This also makes sense graphically, the graph is a parabola. If we put a hole in the graph at -2 on the horizontal axis which x is approaching, then the hole lines up exactly with 4 on the y-axis, the limiting value of x^2 .

Again, it's as though we can fill in the hole perfectly with x-coordinate -2 and y-coordinate being the limiting value as x approaches -2. The previous two examples are special cases of the general phenomenon.

To find the limiting value of the polynomial, simply evaluate it at the input that x is approaching. This relates to a property called continuity which we discuss in the next section.

As a more elaborate example, consider the limit of this cubic polynomial as x approaches one.

■ **Example 16.3**

$$\lim_{x\to 1} 3x^3 - 4x^2 + 2x + 1$$

$$3\lim_{x\to 1}(x^3) - 4\lim_{x\to 1}(x^2) + 2\lim_{x\to 1}(x) + \lim_{x\to 1} 1$$

$$3\lim_{x\to 1}(x.x.x) - 4\lim_{x\to 1}(x.x) + 2\lim_{x\to 1}(x) + \lim_{x\to 1} 1$$

$$3\lim_{x\to 1}(x).\lim_{x\to 1}(x).\lim_{x\to 1}(x) - 4\lim_{x\to 1}(x).\lim_{x\to 1}(x) + 2\lim_{x\to 1}(x) + \lim_{x\to 1} 1$$

$$3(\lim_{x\to 1}(x))^3 - 4(\lim_{x\to 1}(x))^2 + 2\lim_{x\to 1}(x) + \lim_{x\to 1} 1$$

$$= 3.1^3 - 4.1^2 + 2.1 + 1$$

$$= 3 - 4 + 2 + 1 = 2$$

■

Because the cubic is built up from x and constants by addition and multiplication, we can apply the limit laws to bring the limit inside the expression. This has precisely the effect of evaluating the polynomial at x equals one yielding in this case an answer of two.

We don't need to know what the graph actually looks like. In general, if p(x) is any polynomial then the limit of p(x) as x approaches a is just p(a).

If P(x) is a polynomial, $\lim_{x\to a} P(x) = P(a)$

The result of substituting a for x and evaluating the expression using simple arithmetic.

Let's look at a more elaborate example involving a rational function that is a quotient of two polynomials and try to deduce some asymptotic behavior by using the limit laws.

■ **Example 16.4**

$$\lim_{x\to\infty} \frac{3x^3 + 2x^2 - x + 7}{2x^3 - 7x^2 + 3x - 4}$$

■

Here's a very complicated expression. We want to see what happens as x goes to infinity.

The leading terms in the numerator and denominator involving x^3 are important as they will dominate the expression as x gets large. The remaining bits and pieces involving large powers of x start to look insignificant by comparison with the leading terms when x is large.

$$\lim_{x\to\infty} \frac{3x^3 + 2x^2 - x + 7}{2x^3 - 7x^2 + 3x - 4} = \lim_{x\to\infty} \frac{3x^3}{2x^3} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}$$

In the limit we expect the answer to be unchanged by simply throwing away these lower order insignificant pieces to get the limit of $3x^3$ divided by $2x^3$, but then this cancellation and we're just left with the limit of the constant $\frac{3}{2}$ which is just $\frac{3}{2}$.

So, for these heuristic reasons we expect the answer to be $\frac{3}{2}$.

For a more thorough or careful analysis we can do some algebraic manipulation first and then apply limit laws. The trick is to first divide through everything in the numerator and denominator by the highest power of x that appears, x^3.

$$\lim_{x\to\infty}\frac{3x^3+2x^2-x+7}{2x^3-7x^2+3x-4}=\lim_{x\to\infty}\frac{\frac{3x^3+2x^2-x+7}{x^3}}{\frac{2x^3-7x^2+3x-4}{x^3}}$$

This doesn't change the overall value of the expression. But by limit laws we can bring the limit inside the expression and see what happens to each of the simple pieces that we have created.

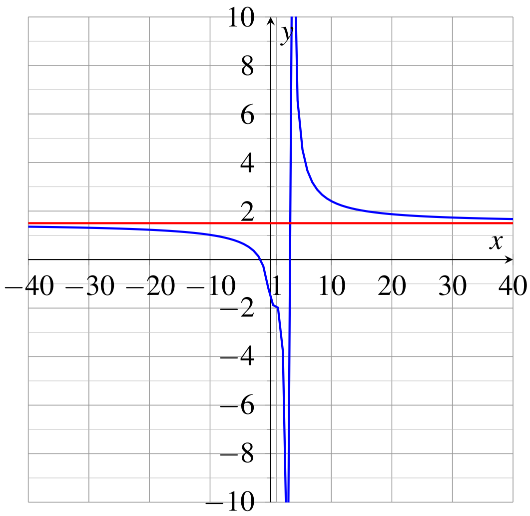
$$\frac{\lim_{x\to\infty}\frac{3x^3}{x^3}+\frac{2x^2}{x^3}-\frac{x}{x^3}+\frac{7}{x^3}}{\lim_{x\to\infty}\frac{2x^3}{x^3}-\frac{7x^2}{x^3}+\frac{3x}{x^3}-\frac{4}{x^3}}=\frac{\lim_{x\to\infty}3+\frac{2}{x}-\frac{1}{x^2}+\frac{7}{x^3}}{\lim_{x\to\infty}2-\frac{7}{x}+\frac{3}{x^2}-\frac{4}{x^3}}$$

The constants 3 and 2 are unaffected, but each of the fractions with x, x^2, and x^3 in the denominator have zero as their limit as x gets arbitrarily large

$$=\frac{3+0-0+0}{2-0+0-0}=\frac{3}{2}$$

and everything quickly becomes this expression which evaluates to $\frac{3}{2}$, which is what we were expecting earlier for heuristic reasons.

This tells us in fact, that the horizontal line $y=\frac{3}{2}$ will be an asymptote to the curve for this particular rational function.



16.2.4 The Squeeze Limit Law

We now discuss another important law that enables one to get information by making comparisons between known and unknown limits, aptly called the squeeze or sandwich law.

16.2.4.1 Statement of Squeeze theorem

If we have f(x) sandwiched in between g(x) and h(x), by which we mean g(x) is less than or equal to f(x), which is less than or equal to h (x) i.e. $g(x)\leq f(x)\leq h(x)$, for all x near x=a, and the limits

of g(x) and h(x) exists and are equal to L as x approaches a, then the limit of f of x also exists and equals this common limit L.
Mathematically,

$$\text{Say } g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then , taking limits in the original inequality,

$$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x)$$

$$L \leq \lim_{x \to a} f(x) \leq L$$

implies

$$\lim_{x \to a} f(x) = L$$

16.2.4.2 Intuition of Squeeze theorem

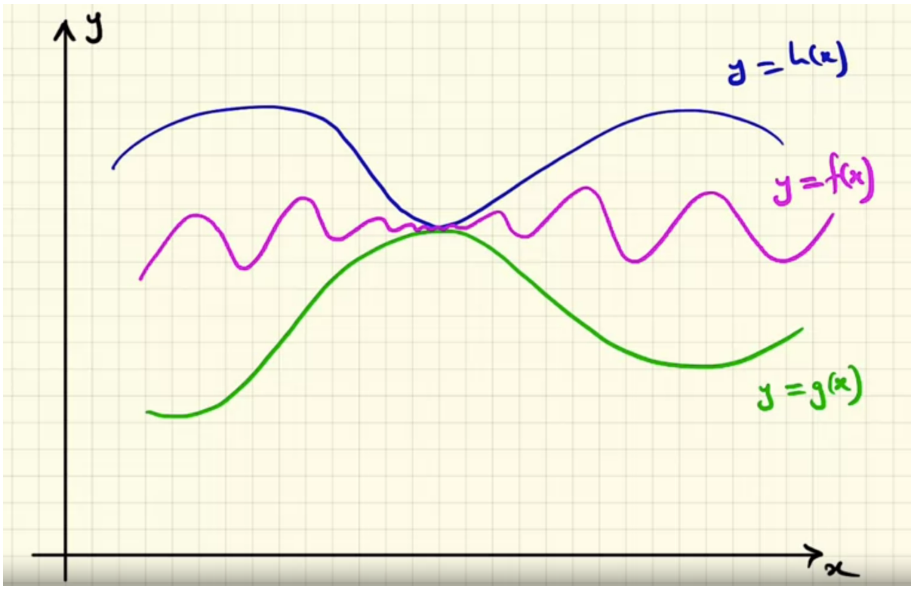


Figure 16.1: Demonstration of Squeeze/Sandwich Theorem

Imagine a curve for $y = g(x)$ sitting beneath a curve of $y = h(x)$ but they are pinched together at some point, and the curve for $y = f(x)$, even though possibly fluctuating wildly, is sandwiched in between. The point where they all pinch together corresponds to $x = a$ and $y = L$. Then all three limits exist and equal L as x approaches a.

For example, let's use this squeezing idea to investigate the important but difficult limit $\frac{\sin x}{x}$ as x goes to zero. This limit is crucial later for developing the derivative of the sine function. Let's do some exploration first to get a feel for what we might expect.

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

If you type in $\sin 1 \div 1$ in your calculator, you get about 0.84. $\sin 0.1 \div 0.1$ is about 0.998. $\sin 0.001 \div 0.001$ is really close to one. As x gets closer to zero, the calculations suggest $\frac{\sin x}{x}$ is getting close to one. So, I suspect the limiting value is one. Let's see if we can deduce this using general principles.

Here's part of the unit circle with some angle in the first quadrant, and we've extended the radius to meet the tangent line that touches the unit circle at the point one on the horizontal axis. Suppose that the angle, measured in radians, is x, so that the arc length along the unit circle subtended by the angle is x. The point where the line extending the radius meets the tangent is that height $\tan x$ above the horizontal axis. Moving across from the point on the unit circle to the vertical axis produces the value $\sin x$. Now we join the point on the unit circle to the point one on the horizontal axis, producing the short secant.

And now, extract this diagram involving two triangles and a wedge or sector from the unit circle sandwiched in between. If we shade in pink the smaller triangle, with base length one and height $\sin x$, then we get an area of $\frac{\sin x}{2}$. If we shade in green, the wedge or sector of the circle subtended by the angle x, then the area is the proportion, $\frac{x}{2\pi}$, of the area of the whole unit circle which is π, giving an area of $\frac{x}{2}$. If we shade in blue the area of the larger triangle that engulfs everything, with base length one and height $\tan x$, then we get an area of $\frac{\tan x}{2}$.

If we compare the three areas, we get that the pink area is less than or equal to the green area, which is less than or equal to the blue area. Thus we get a sequence of inequalities:

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}$$

So, take this inequality and multiply through by two and rewrite $\tan x$ as $\frac{\sin x}{\cos x}$, and divide everything through by $\sin x$ retaining the inequalities because $\sin x$ is positive in our diagram. Then reciprocate everything which reverses the inequalities, noting that everything is positive. Notice that $\frac{\sin x}{x}$ is squeezed or sandwiched in between one on the left and $\cos x$ on the right. These values on the left and right both tend towards the same limit one as x tends towards zero. This is obvious for one on the left which is constant. For $\cos x$ on the right, you can perhaps visualize in your mind's eye the curve for $\cos x$. And imagine as x moves towards zero, the value of $\cos x$ tends towards one, producing a limiting value of one.

But $\frac{\sin x}{x}$ is sandwiched in the middle. So, by the squeeze law, it has nowhere else to move but is also forced to go to one. This demonstrates that the limit of $\frac{\sin x}{x}$ as x approaches zero is indeed one as we suspected. Note that this argument used angles in the first quadrant of the unit circle, so that x was approaching zero from above. But the same argument can be adjusted to give the same result also if the angle is negative, that is if x is approaching zero from below.

Let's do another example that looks similar using $\frac{\tan x}{x}$. We don't need to go through an argument using the squeeze law again but can just utilize the result for $\frac{\sin x}{x}$.

First, write $\tan x$ as $\frac{\sin x}{\cos x}$ and then split the expression up into two factors; $\frac{\sin x}{x}$ multiplied by $\frac{1}{\cos x}$. And then use the product and quotient limit laws to express this as the limit of the first factor $\frac{\sin x}{x}$, by the limit of the second factor, which in turn is $\frac{1}{\cos x}$. Both of these limits are one so the expression quickly evaluates to one.

Let's do another example making things slightly harder still, asking for the limit of $\frac{\tan 2x}{x}$ as x tends to zero. Observe that we can rewrite the fraction by multiplying the top and the bottom by two and then bringing the constant two in the numerator out the front. So, this becomes $2 \times \lim_{x\to 0} \frac{\tan x}{2x}$, which we can rewrite as $\frac{\tan y}{y}$, by putting $y = 2x$, and notice that y tends to zero as x tends to zero. But the limit of $\frac{\tan y}{y}$ as y goes to zero is just one, our previous result using y instead of x. So we get $2 \times 1 = 2$.

We've made significant progress utilizing limit laws to deal with some quite difficult and tricky limits. We discussed and illustrated how limits respect arithmetic operations. So, the limits can be brought inside sums, differences, products, and quotients, and how we can force certain limits using the Squeeze Limit law when some possibly wild expression is sandwiched in between two expressions that we can control, and already know in advance converge to the same limit. We apply the Squeeze Limit law to deduce that $\frac{\sin x}{x}$ tends to one as x tends to zero, and use this to evaluate some limits involving the tan function. Please re-read the material if needed and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

16.2.5 Practice Quiz

Question 1

Evaluate the following limit:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

(a) -4
(b) 2
(c) 0
(d) 4
(e) -2

Question 2

Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x+1}$$

(a) 0
(b) 1
(c) -1
(d) 2
(e) -2

Question 3

Evaluate the following limit:

$$\lim_{x\to 3} \frac{x^2-9}{x-3}$$

(a) -6
(b) -3
(c) 6
(d) 0
(e) 3

Question 4

Evaluate the following limit:

$$\lim_{x\to 3} \frac{x^2-3x}{x-3}$$

(a) -6
(b) 6
(c) 0
(d) 3
(e) -3

Question 5

Evaluate the following limit:

$$\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$$

(a) 3
(b) -6
(c) 6
(d) 0
(e) -3

Question 6

Find the vertical asymptote for the following curve:

$$y = \frac{x^2-1}{x-\frac{1}{2}}$$

(a) $x=-1$
(b) $x=2$
(c) $x=-2$
(d) $x=1$
(e) $x=\frac{1}{2}$

Question 7

Evaluate the following limit:

$$\lim_{x\to \frac{1}{2}^+} \frac{x^2-1}{x-\frac{1}{2}}$$

(a) 0
(b) ∞
(c) -1
(d) $-\infty$
(e) 1

Question 8

Evaluate the following limit:

$$\lim_{x\to \frac{1}{2}^-} \frac{x^2-1}{x-\frac{1}{2}}$$

(a) ∞
(b) 1
(c) -1
(d) 0
(e) $-\infty$

Question 9

Find the oblique asymptote for the following curve:

$$y = \frac{x^2 - 1}{x - 2}$$

(a) $y = x$
(b) $y = x - 2$
(c) $y = x + \frac{1}{2}$
(d) $y = x + 1$
(e) $y = x - 1$

Question 10

Find the oblique asymptote for the following curve:

$$y = \frac{2x^2 + 5x + 2}{x + 1}$$

(a) $y = x - 1$
(b) $y = 2x + 1$
(c) $y = x + 1$
(d) $y = 2x + 2$
(e) $y = 2x + 3$

Answers

The answers will be revealed at the end of the module.

16.3 Limits and Continuity

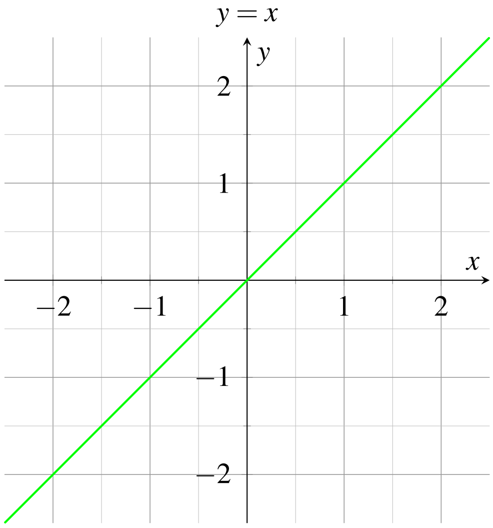
16.3.1 What is continuity?

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called continuous functions. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a discontinuity at a point where a break occurs. Another way of saying this intuitive concept is to say that a function whose graph does not have any break in it. **So, a function is said to be continuous on an interval if a graph can be drawn in the interval, at least in principle, without lifting your pen off the paper or a function whose graph does not have any break under the defined interval.**

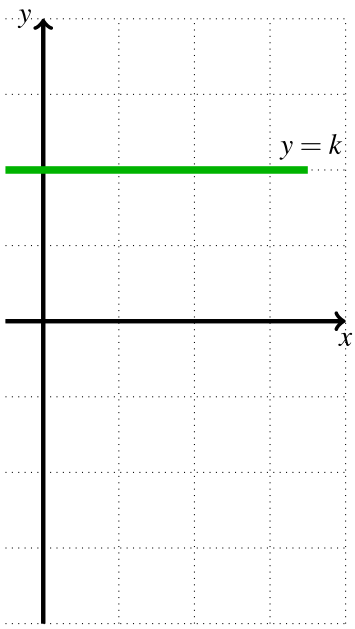
Continuous functions have many special properties. For this reason, many theorems in calculus assume that the function is continuous. So, when functions are used to model real-world problems, continuous functions are often used as possible. In this section, we look at and illustrate the important notion of continuity of a function, what can go wrong when discontinuities occur, and how we handle them. Including examples of when we can fix things using removable discontinuities.

16.3.2 A few examples for exploration

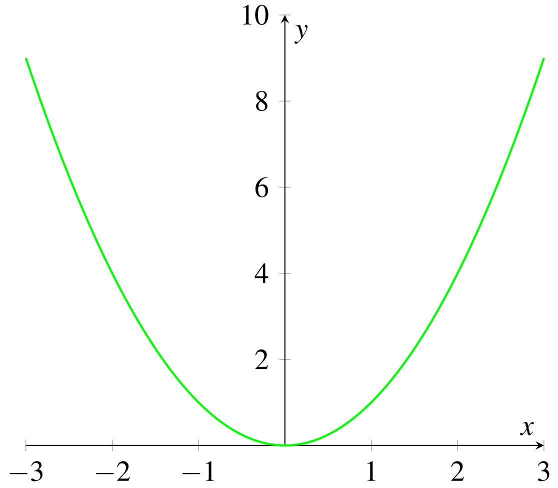
We will see a few examples to learn more about continuity.



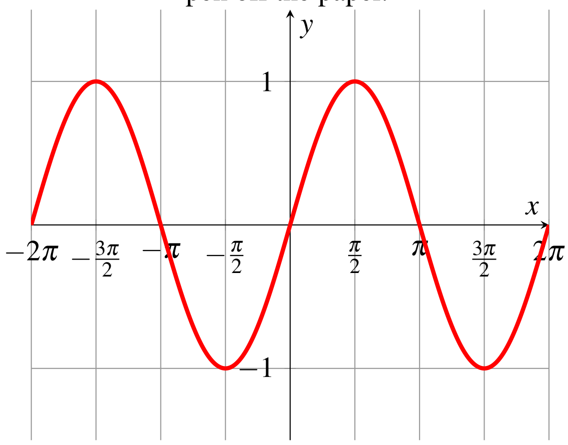
Here, the straight line y = x in this diagram was drawn without lifting the pen. The line, of course, goes on forever. So one has to imagine in principle being able to draw as much as we like, back and forth, as far as we like, without ever needing to lift the pen. In fact, this works for any graph of a linear function.



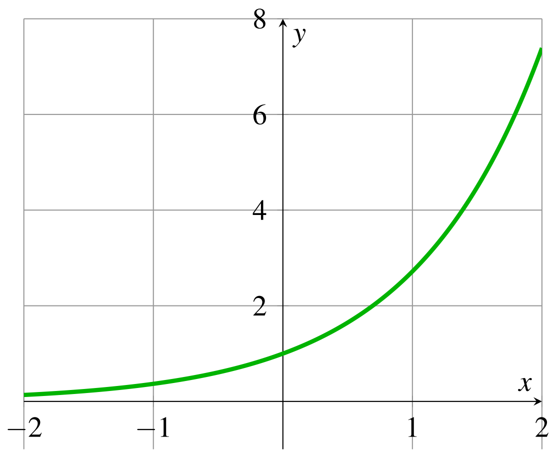
Here's a horizontal line which is the graph of y = k for some constant k. Which again, we can imagine drawing without lifting our pen off the paper.



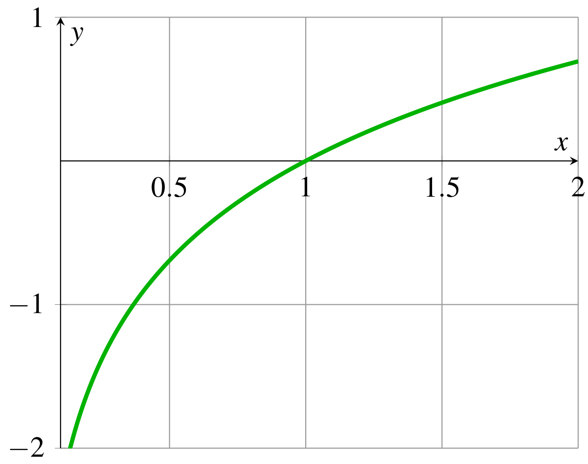
Here's the parabola $y = x^2$, which is also continuous, we can imagine drawing without lifting our pen off the paper.



Here's the function y = sin x, which wriggles back and forth, forever but continuously. This is also a continuous function. So, we can imagine drawing without lifting our pen off the paper.



The above function $y = e^x$ is also a continuous function. Which again, we can imagine drawing without lifting our pen off the paper.



The above function y = ln (x) is also a continuous function. So, all the functions we discussed above are continuous functions. This means all the above functions can be drawn on the graph without lifting our pen off the paper.

16.3.3 When is the function not continuous?

So, the question arises when the function is not continuous. All the examples of functions discussed above are continuous. So, now we will discuss the conditions when the function is not continuous at the specific point or at the given interval.

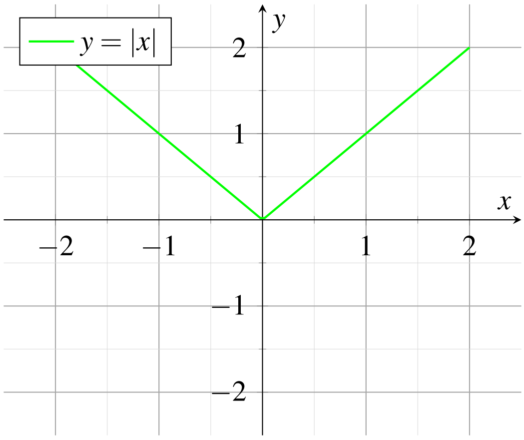
When we imagine drawing without lifting our pen, there are no holes or gaps. And the limiting values precisely fill in where your pen is supposed to move to, captured by the statement, the limits of

$$\lim_{x \to a} f(x) = f(a)$$

is the value of the function a, namely f(a). And this has to work for every input x equal to a in the domain of the function. There are no surprise jumps or leaps, and you can think of any limiting behavior of the function as coinciding with the actual value of the function. There are two main obstructions to continuity in practice. Firstly, we must consider denominators of fractions. We're not allowed to divide by zero, and so have to avoid values that cause the denominator of a fraction to be zero.

The second main problem is when a function is defined piecewise, which means that there are two more pieces in the description of the rule for the function. Leading to two or more pieces of the curve which have to match and join up with the endpoints, if we are to imagine drawing the combined curve without lifting a pen off the paper.

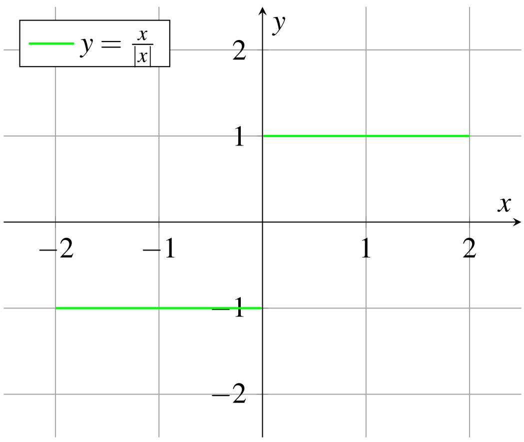
Here's an example of a piecewise function that happens to be continuous. Consider the usual magnitude function, f(x) = |x|. In fact defined in two pieces, as x, if x is greater than or equal to 0, and minus x if x is less than 0. The graph is in a V-shape, formed by taking the line y equals x in the first quadrant in the plane, and combining it with y equals -x in the second quadrant. The two pieces of the curve join up, matching perfectly at the origin. You can imagine drawing the V-shape without lifting your pen off the paper. The graph of f(x) = |x| is shown below.



By contrast, consider the following example, which turns out to be discontinuous. By which we mean that we'll be unable to draw the graph without lifting our pen off the paper. For this, we modify the previous example, and consider the function with rule f(x) = $\frac{x}{|x|}$ which again can be describe in pieces.

$$y = f(x) = \frac{x}{|x|} = \begin{cases} \frac{x}{x} = 1 & \text{if } x > 0 \\ \frac{x}{-x} = -1 & \text{if } x < 0 \\ x = undefined & \text{if } x = 0 \end{cases}$$

When x is positive, the magnitude of x is x itself, so dividing $\frac{x}{x}$ gives the value 1. When x is negative, the magnitude of x is -x, so dividing $\frac{x}{-x}$ gives the value -1. The rule is undefined for x equals 0. The graph of y = f(x) which consists of two horizontal lines is shown below. One at height y equals 1 for x greater than 0, and another at height y equals -1 for x less than 0. Note that these two lines get arbitrarily close to the y-axis, but don't actually meet it. You can think of there being two holes on the y-axis, and y equals 1 and y equals -1. It won't make the overall curve continuous by filling either of these holes.



For x approaching 0 from above, from the positive right-hand size, the y values are stuck at plus 1, so the limit from the right is plus 1. By contrast, for x approaching 0 from below, from the negative left-hand side, the y values are stuck at -1, so the limit from the left is -1. We have these contrasting one-sided limiting behaviors for $\frac{x}{|x|}$. We have a limit existing from the right, as x tend to 0 from above, namely + 1.

$$\lim_{x \to 0^+} \frac{x}{|x|} = 1$$

And a limit existing from the left as x tend to 0 from below, namely -1.

$$\lim_{x \to 0^-} \frac{x}{|x|} = -1$$

The overall two-sided limit does not exist if we allow x to approach 0 from both sides.

$$\lim_{x \to 0} \frac{x}{|x|} \text{does not exist}$$

There's no hope of filling in the holes to make the curve continuous so that you can draw it without lifting your pen off the paper.

From the above example, we can see that for a function to be continuous at a point, its limit has to exist and be defined at that point. Mathematically saying, **The function $f(x)$ is said to be continuous at the point $x = x_0$ if and only if**

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Recall that rational functions are ratios of polynomials, of the general form

$$y = f(x) = \frac{p(x)}{q(x)}, \textit{ where p(x) and q(x) are polynomials.}$$

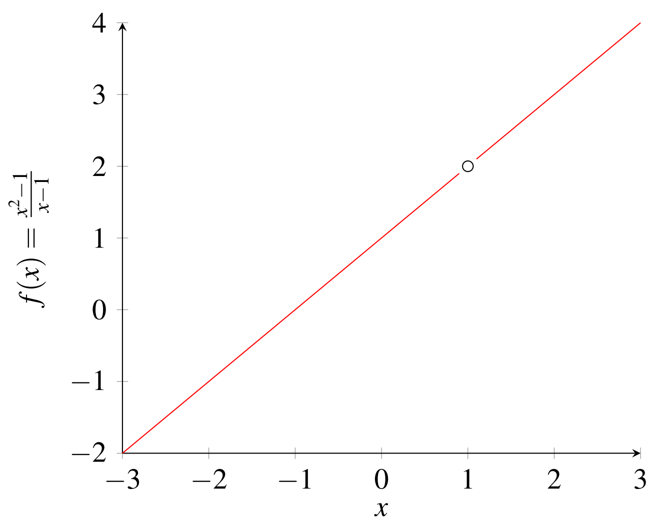
Problematic behavior occurs when denominators become zero, often but not always leading to asymptotes.

Again, Recall two contrasting examples from an earlier section,

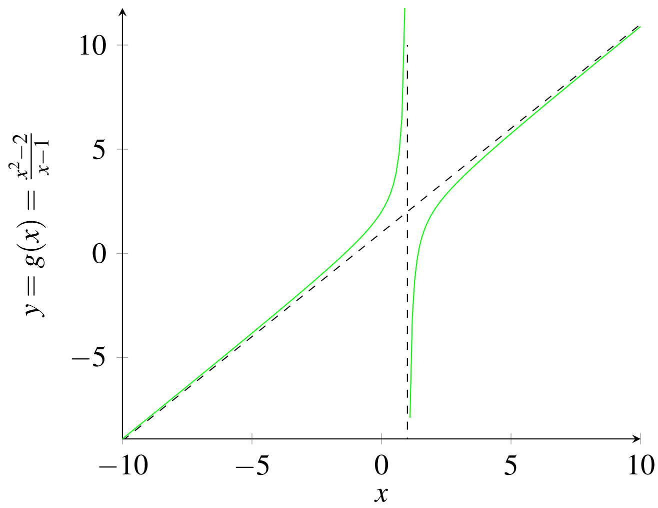
$$f(x) = \frac{x^2 - 1}{x - 1} \qquad\qquad g(x) = \frac{x^2 - 2}{x - 1}$$

Both of which would have 0 denominator if x = 1. So that the input x = 1 is problematic, but in quite different ways.

We produced the graph of f(x), which turns out to be a line with a hole at the point (1, 2). And the graph of g(x), which turns out to have an oblique asymptote and a vertical asymptote passing through the x-axis at x = 1.



In the case of the function f, the rule becomes f(x) = x + 1 for x not equal to 1, producing a line with just one point missing. If we fill in the hole, then we get the full, continuous curve. We say that the discontinuity is removable.



By contrast, if we look at the graph of g(x), no matter where we travel on the vertical asymptote corresponding to x = 1, we cannot find an input that will join up to two branches of the curve. They always remain separated. We say that the discontinuity is not removable.

16.3.4 Properties of the continuous functions

In this section, we'll discuss and illustrate several continuous functions properties that enable you to combine and understand the behavior of new functions that can arise and appear at first sight to be difficult or complicated. You'll become experts in no time.

The following are basic properties of continuous functions.

1. The first property is **"The function f(x) = c is a continuous function where c is any constant ."**
2. The second property is **"The function f(x) = x^n is a continuous function where n is any non-negative real number ."**
3. The third property is **"The function f(x) $\pm$ g(x) is a continuous function where both f(x) and g(x) are continuous functions."**
4. The fourth property is **"The function f(x) . g(x) is a continuous function where both f(x) and g(x) are continuous functions."**
5. The fifth property is **"The function $\frac{f(x)}{g(x)}$ is a continuous function where both f(x) and g(x) are continuous functions and $g(x) \neq 0$.**
6. The sixth property is **"The function $\sqrt[n]{f(x)}$ is a continuous function where f(x) is continuous function and x is restricted to those values that makes $\sqrt[n]{f(x)}$ real .**
7. The seventh property is **"The composite function fg(x) and gf(x) are continuous functions where f(x) and g(x) are continuous functions."**

16.3.5 Types of Discontinuity

In the above part, we have discussed the condition of a function to be discontinuous at a given interval or at a given specific point. Now we will have a look at the different types of discontinuous functions. A function $y = f(x)$ may have any one of the following types of discontinuity.

1. **Jump discontinuity:** If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$, then $f(x)$ is said to have a jump or an

ordinary discontinuity. In this case, $f(c)$ may or may not exist. The above discussed $\lim_{x\to 0}\frac{x}{|x|}$ is an example of jump discontinuity. As $\lim_{x\to 0}\frac{x}{|x|}$ does not exist as $\lim_{x\to 0^+}\frac{x}{|x|} \neq \lim_{x\to 0^-}\frac{x}{|x|}$

Here, h = $\left|\lim_{x\to c^-} f(x) - \lim_{x\to c^+} f(x)\right|$ is called the *height of the jump*.

2. **Removable discontinuity:** If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \neq f(c)$ or $f(c)$ is not defined, then $f(x)$ is said to have a removable discontinuity at $x = c$.

   For instance, a function f(x) is defined as:

   $$y = f(x) = \begin{cases} 2x+3 & \text{for } x < 1 \\ 4 & \text{for } x = 1 \\ 6x-1 & \text{for } x > 1 \end{cases}$$

   Let's check whether the function is continuous at x = 1:

   Left hand limit at $x = 1$ is

   $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+3) = 2\times 1+3 = 5$$

   Right hand limit at $x = 1$ is

   $$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(6x-1) = 6\times 1-1 = 5$$

   So,

   $$\lim_{x\to 1} f(x) \text{ exists and } \lim_{x\to 1} f(x) = 5$$

   But $f(1) = 4$

   $$\lim_{x\to 1} f(x) \neq f(1)$$

   Hence, $f(x)$ is not continuous at $x = 1$.
   This is a case of removable discontinuity.
   The given function will be continuous if $f(1)$ is also equal to 5. Thus the given function can be made continuous by defining the function in the following way:

   $$f(x) = \begin{cases} 6x-1 & \text{for } x > 1 \\ 2x+3 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \end{cases}$$

   So, this kind of discontinuity can be removed by redefining the function f(c) where c is the point.

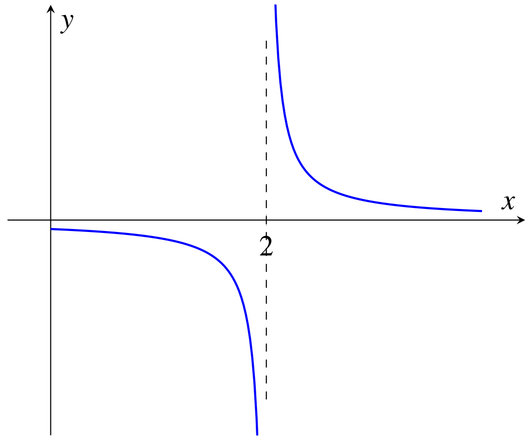
3. **Infinite discontinuity:** If one or both of $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ tend to $+\infty$ or $-\infty$, then $f(x)$ is said to have an infinite discontinuity.

The graph of $y = f(x) = \frac{1}{x-2}$ is given aside which has infinite discontinuity at x=2. Here

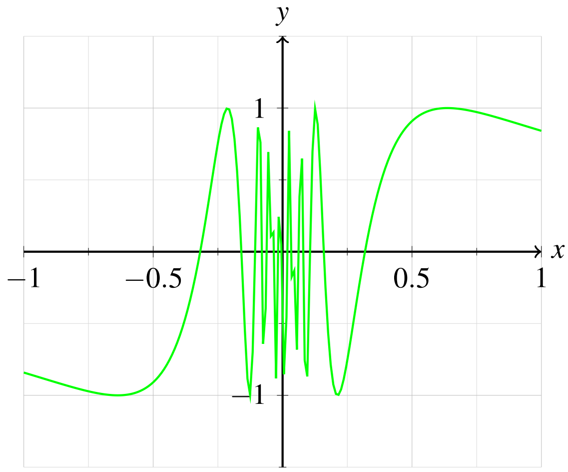
$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$

and

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty$$



4. **Oscillatory discontinuity:** A point of discontinuity that is neither a point of ordinary discontinuity, nor a removable discontinuity, nor an infinite discontinuity is called a point of oscillatory discontinuity. At such a point, the function may oscillate finitely or infinitely and does not tend to a finite limit or $+\infty$ or $-\infty$.
For example, $sin\frac{1}{x}$ oscillates finitely at x=0. So, $sin\frac{1}{x}$ has oscillatory discontinuity at point x=0.

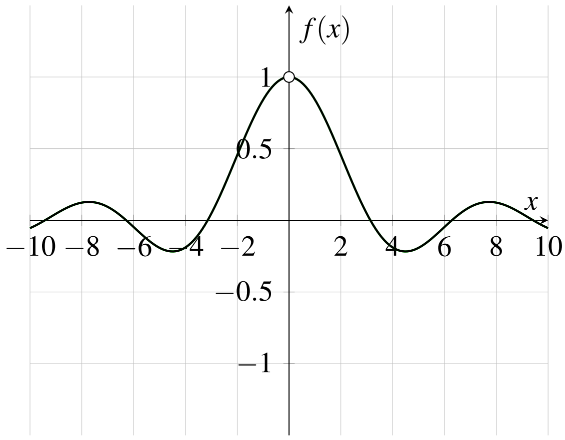


16.3.6 One great example to end this section

Before ending this section, we will describe one of the most famous and important examples of all that has a removable discontinuity. Recall that last time we carefully demonstrated using the Squeeze Limit law, that

$$\lim_{x\to 0} \frac{sinx}{x} = 1$$

. Let's consider the function f(x) = $\frac{sinx}{x}$, with domain the set of all nonzero real numbers. The rule is not defined for x = 0 because of x in the denominator, but it's defined for all other real numbers x.



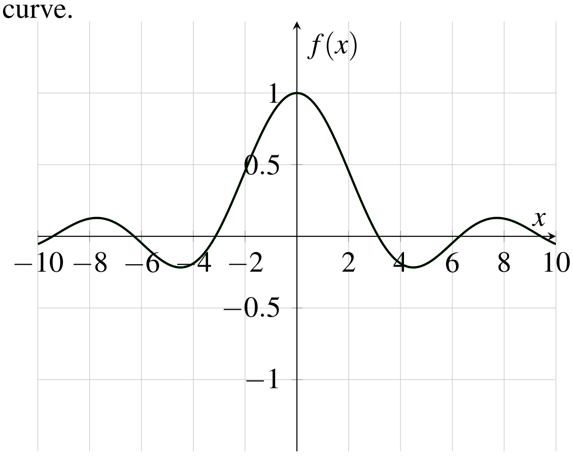
Here is the graph of f. Note that there is a hole for x = 0, because the rule is undefined there.

Because of the wavy periodic behavior of sine x, we expect the curve to undulate forever to the right and left. However, the amplitudes of the undulations decrease the further we're away from the origin, as the denominator x gets larger and larger. Multiplying by 1 on x has some kind of dampening effect on sin x. The graph also has perfect reflectional symmetry about the y-axis. It's an example of an even function, a concept that we'll talk about and use later in the course.

As x approaches 0 from either side, the limit of $\frac{\sin x}{x}$ is 1. Since the limit exists as x approaches 0, we can fill in the hole and adjust the function's rule. The adjustment is done below:

$$f(x) = \begin{cases} \frac{sinx}{x} & \text{if } x \neq 1 \\ 1 & \text{for } x = 0 \end{cases}$$

The discontinuity turns out to be removable, and we produce this beautiful, continuous, smooth curve.



It's a famous curve that has many application in areas of physics and engineering, such as signal analysis, acoustics, optics, and spectroscopy.

In this section, we included quite an advanced discussion about limits in the context now of continuity. We discussed the notion of a continuous function, whose graph we may think of as being drawn without lifting the pen off the paper. A property that can be defined in terms of limits, namely that the limiting behavior of the curve at any particular point actually takes the limiting value. We gave contrasting examples, some with a removable discontinuity at a particular point where the limit exists, including the famous sin x on x curve. In cases where it's not possible to remove the discontinuity, including examples such as where the left and right limits exist but are not equal, and where there's a vertical asymptote for a rational function.

16.3.7 Practice Quiz

Question 1

Which one of the following functions f is continuous over the entire real line?

(a) $f(x) = \sqrt{x}$
(b) $f(x) = \frac{1}{\cos x}$
(c) $f(x) = \tan x$
(d) $f(x) = \sin x$
(e) $f(x) = \frac{1}{x}$

Question 2

Which one of the following functions f is continuous over the entire real line?

(a) $f(x) = \frac{1}{x^2}$
(b) $f(x) = \ln x$
(c) $f(x) = \frac{x^2-1}{x-1}$
(d) $f(x) = \frac{1}{x}$
(e) $f(x) = \frac{x^2-1}{x+1}$

Question 3

Which one of the following functions f is not continuous over the entire real line?

(a) $f(x) = \cos x$
(b) $f(x) = x^2$
(c) $f(x) = \frac{1}{x+1}$
(d) $f(x) = e^x$
(e) $f(x) = |x|$

Question 4

Which one of the following functions f is not continuous over the entire real line?

(a) $f(x) = \frac{x^2-1}{x^2+1}$
(b) $f(x) = \frac{x^2-1}{x-1}$
(c) $f(x) = \frac{x^2-1}{x+1}$
(d) $f(x) = x^2 - 1$
(e) $f(x) = \frac{1}{x^2+1}$
(f) $f(x) = x^2 + 1$

Question 5

Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ k & \text{if } x = 1 \end{cases}$$

(a) $k = 1$
(b) $k = -1$
(c) $k = -2$
(d) $k = 2$
(e) $k = 0$

Question 6

Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x \neq -1 \\ k & \text{if } x = -1 \end{cases}$$

(a) $k = -1$
(b) $k = 1$
(c) $k = 0$
(d) $k = 2$
(e) $k = -2$

Question 7

Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} k & \text{if } x \geq 1 \\ \frac{x-1}{x-1} & \text{if } x < 1 \end{cases}$$

(a) $k = 0$
(b) $k = 1$
(c) $k = 4$
(d) $k = 3$
(e) $k = 2$

Question 8

Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 1 \\ k-x & \text{if } x < 1 \end{cases}$$

(a) $k = 1$
(b) $k = 2$
(c) $k = 4$
(d) $k = 0$
(e) $k = 3$

Question 9

Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} kx^2 - 1 & \text{if } x \geq 2 \\ x^3 + k & \text{if } x < 2 \end{cases}$$

(a) $k = 2$
(b) $k = 3$
(c) $k = 1$
(d) $k = 0$
(e) $k = 4$

Question 10

Question 10 Which one of the following functions f does not have a removable discontinuity at x = 1?

(a) $f(x) = \frac{x^2-x}{x-1}$
(b) $f(x) = \frac{|x-1|}{x-1}$
(c) $f(x) = \frac{x-1}{sin(x-1)}$
(d) $f(x) = \frac{sin(x-1)}{x-1}$
(e) $f(x) = \frac{x^2-1}{x-1}$

Answers

The answers will be revealed at the end of the module.

17. The derivative

17.1 The derivative as a limit

In this section, we formally define the derivative as a limit in two different but equivalent ways and practice these definitions on the cubing function that takes x to x^3. In this lesson, we'll distinguish between the interpretation and the definition of a derivative. And by the end of the lesson, you'll be seeing derivatives everywhere. Recall from earlier section, that we've looked at the notion of an average rate of change of a function over some interval of inputs, which is just the slope of the line joining the end points of the curve. Recall for displacement function, as a function of time. This produces the average velocity for the trip. If we imagine the time interval becoming shorter and shorter, in fact as vanishingly small, recall that we get the instantaneous velocity, which is the idea behind what you see on your speedometer. This is represented by the slope of the tangent line to the curve at the point of interest.

17.1.1 Limits in derivatives

Recall that we've seen this idea informally in an earlier video by constructing slopes of secants to curves and watching what happens as they get shorter and shorter. As the secants vanish away, the slopes of the secants get closer and closer to the slope of the tangent line. We think of them as reaching the slope of the tangent line in the limit and we express this in limit notation.

$$\text{slope of tangent line} = \text{limiting slope of secant} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

We take this now as the definition of the derivative of the function f given by the rule y equals f(x) at the input x. The derivative of f(x) denoted by $f'(x)$(read as f dash of x) is

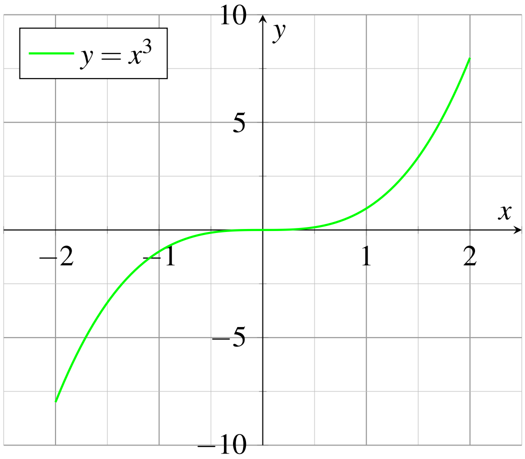
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This represents the slope of the tangent line to the curve y equals f(x) at the point with coordinates x, f(x).

17.1.2 Exploring through examples

Let's see an example of a squaring function. In the case of f(x) = x^2, the squaring function, recall that we calculated this limit in an earlier section, but the derivative is 2x. So,$f'(x)$ is 2x.

Let's move from squares to cubes and consider the function y equals f (x) that takes an input x to execute and think about what the derivative might be.



The graph of this function looks like this, with a 180-degree rotational symmetry about the origin, it turns out to be an odd function, an important concept we'll discuss in detail later.

Apart from being a lot steeper, this curve has an important distinguishing feature from the parabola that describes the squaring function. If you draw miniature tangent lines to the curve throughout, you'll discover that the tangent to the curve at the origin, which is just the x-axis in this case, crosses the curve. The origin becomes an inflection point for this curve. Another important concept we'll discuss in detail in a later chapter. Already you can see for that particular tangent line, the slope is zero since it's just the x-axis in fact which is horizontal. But what about slopes of tangent lines in general for f (x) = x^3 ? Well, let's work through the definition of the derivative and say what happens. Start off with the definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{(3x^(2) + 3xh + h^2)h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

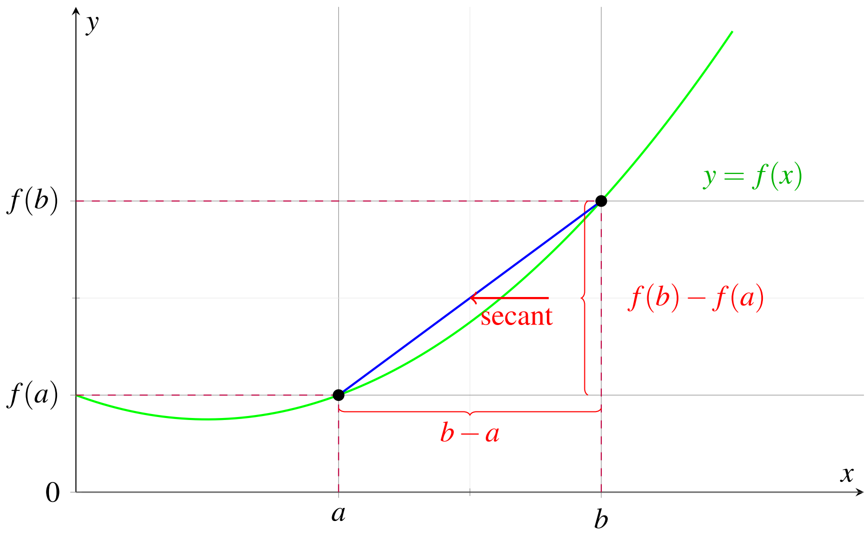
$$= \lim_{h \to 0} (3x^2 + 3x(0) + (0)^2)$$

$$= 3x^2$$

This solves our original problem. The derivative of f(x) = x^3 is $3x^2$.
Recapping from earlier sections with this new notation, we've already seen that if f(x) = x^2 and the derivative is 2x. And just now we saw that if f(x) is x^3, then the derivative is $3x^2$. You can easily work through the details for f (x) equals x^4, though it takes a bit longer and find that the derivative is $4x^3$.

This all instances of the general pattern. For f(x) = x^n, for any exponent n, the derivative f'(x) = nx^{n-1}. That is, the exponent falls down to the front and you get a new exponent by subtracting one. If you know about something called a binomial expansion, you can verify this directly already for any positive integer n from the definition of the derivative. When we've developed some general techniques with finding derivatives, this result will follow quite easily and works for any real expanded n, which is very nice indeed.

There's an alternative limit definition of the derivatives that come about naturally by taking the average rate of change of the function over an interval and seeing what happens in the limit as the interval shrinks to nothing at the left-hand end point.



Here's our typical curve y = f(x). To find at least an interval from a to b, producing outputs f(a) and f(b) for the respective endpoints, and then we draw the secant joining the end points which has slope $\frac{f(b)-f(a)}{b-a}$, and this is just the average rate of change. If we allow $b \to a$, then the secant lines up more and more in the direction of the tangent line to the curve at x equals a. So, that its slope the, average rate of change approaches the slope of the tangent line, which is just the derivative of the f(x) = a, denoted by f'(a).

In simpler form: As $b \to a$, the average rate of change should approach the derivative $f'(a)$.

So, we expect an alternative definition of the derivative:

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Now, this looks quite different to our original definition. So, let's see if we can recover the original definition by rearranging this formula somehow.

Put h = b - a, because we want to get h in the denominator, and put x = a because we want x in the game.

Then, $b = a + h = x + h$, and $h \to 0$ as $b \to a$.

Since, x=a and b-a= h:

$$f'(x) = f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Above, we recover our original definition of the derivative that we introduced early in this section. So, everything is working nicely as expected.

17.1.3 Revisiting a previous example

Let's revisit the derivative of the cubing function x^3 but using this alternative definition. The definition is in terms of

$$f'(a) = \lim_{b\to a} \frac{f(b)-f(a)}{b-a}$$

So,let's find the derivative of x^3 using alternative definition.

$$f'(a) = \lim_{b\to a} \frac{f(b)-f(a)}{b-a} = \lim_{b\to a} \frac{b^3-a^3}{b-a}$$

$$= \lim_{b\to a} \frac{(b-a)(b^2+ab+a^2)}{b-a} = \lim_{b\to a}(b^2+ab+a^2)$$

$$= \lim_{b\to a}(a^2+a^2+a^2) = 3a^2$$

This shows f'(a)= $3a^2$. Now replacing a=x:

$$f'(x) = 3x^2$$

This is exactly what we obtained using the original definition which is very pleasing. Today, we see that many ideas we've encountered previously are all coming together using the unifying concept of a **limit**. We define the derivative to be the slope of the tangent line to a curve which has two quite natural and equivalent formulae in terms of limits. We use both of these formulae to find the derivative of the function that takes x to x^3 and the answer turns out to be $3x^2$. In later sections, we'll apply this limit formulae for the derivative to other important functions. Then in the next module, develop some general techniques and principles that enable one to find swaths of derivatives easily and quickly in a variety of different settings. Please read the notes and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

17.1.4 Practice Quiz

Question 1

Find $f'(2)$ when $f(x) = x^2$.

(a) 1
(b) 3
(c) 2
(d) $2x$
(e) 4

Question 2

Find $f'(-2)$ when $f(x) = x^2$.

(a) -4
(b) 2
(c) 4
(d) $2x$
(e) -2

Question 3

Find $f'(2)$ when $f(x) = x^3$.

(a) 8
(b) 12
(c) 3
(d) 6
(e) 9

Question 4

Find $f'(-2)$ when $f(x) = x^3$.

(a) -6
(b) 12
(c) 6
(d) -8
(e) -12

Question 5

Find $f'(1)$ when $f(x) = x^4$.

(a) 1
(b) 2
(c) 8
(d) 4
(e) 3

Question 6

Find $f'\left(-\frac{1}{2}\right)$ when $f(x) = x^4$.

(a) -$\frac{1}{8}$
(b) $\frac{1}{2}$
(c) 32
(d) -$\frac{1}{4}$
(e) -$\frac{1}{2}$

Question 7

Find $f'(10)$ when $f(x) = x^{10}$.

(a) $10x^9$
(b) 10^9
(c) 10^{10}
(d) 10^{11}
(e) $9x10^{10}$

Question 8

Find $f'(1)$ when $f(x) = \frac{1}{x} = x^{-1}$.

(a) $\frac{1}{2}$
(b) -1
(c) $-\frac{1}{2}$
(d) 0
(e) 1

Question 9

Find $f'(2)$ when $f(x) = \frac{1}{x^2} = x^{-2}$.

(a) -1
(b) $-\frac{1}{8}$
(c) -4
(d) $-\frac{1}{4}$
(e) $-\frac{1}{2}$

Question 10

Find $f'\left(\frac{1}{8}\right)$ when $f(x) = x^{2/3}$.

(a) $\frac{4}{3}$
(b) $\frac{3}{4}$
(c) $\frac{1}{12}$
(d) $-\frac{3}{4}$
(e) $-\frac{4}{3}$

Answers

The answers will be revealed at the end of the module.

17.2 Finding derivatives from first principles

In this section, we use the limit definition to rapidly expand our repertoire of derivatives. The process of taking derivatives of functions is called differentiation. We discussed the additivity of the derivative and how that applies in particular to provide simple rules for differentiating polynomials, explain why the natural exponential function replicates itself under differentiation, and sketch a proof that the derivative of the sine function is the cosine function.

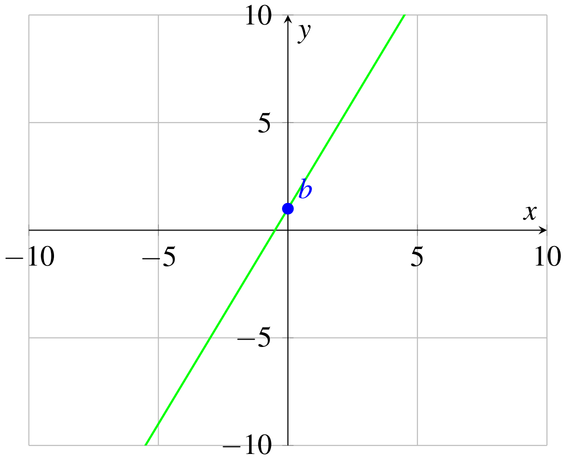
17.2.1 The derivative of a linear function

We will start from where we left off in the previous section. We gave a definition of the derivative in terms of limits last time. Let y = f (x) be the rule for a function f. Remember, we defined the derivative of f (x) denoted by f'(x) to be the

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Recall, this limit formula captures precisely the value of the slope of the tangent line to the curve at the point of interest. By now, you're probably starting to picture this phenomenon in your mind's eye and the idea of the limiting slope of secants. We've already applied this formula to discover that if f (x) = x^2, then the derivative is 2x, and if f (x) = x^3, then the derivative is $3x^2$. More generally, if f(x)= x^n, then the derivative is nx^{n-1}.

The simplest functions that differentiate quickly are the linear functions with rules of the form f(x) = ax + b, where a and b are constants.



These graphs are just straight lines. If you reflect for a moment, the only reasonable way for a given straight line to be approximated by a tangent line is to take the line itself, and the approximation is so good they match perfectly. In other words, a straight line coincides with this tangent line at every point and has slope a so that the derivative must be just a. You can verify for yourself that this also quickly follows from the definition of the limit.

(The derivative of a linear function: Let $f(x) = ax+b$ where a and b are constants be a linear function, whose graph is a line. Then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a(x+h)+b-(ax+b)}{h}$$

$$= \lim_{h\to 0}\frac{ax+ah+b-ax-b}{h} = \lim_{h\to 0}\frac{ah}{h} = \lim_{h\to 0} m = a,$$

which is just the slope a of the associated line. This is to be expected, because a line coincides with its tangent line at every point.

So, f(x) = ax + b $\implies f'(x) = a.$

In particular, if a is equal to zero, so that f(x) = b is a constant function, then the derivative is zero.

$$f(x) = b(constant) \implies f'(x) = 0.$$

If f(x) = x, then the derivative is 1.

So, $f(x) = x \implies f'(x) = 1.$

17.2.2 The derivative is additive

What does it mean for the derivative to be additive? This question came up when you moved ahead in this course. The derivatives additive by which we mean that if f(x) is the sum of two rules, u(x) and v(x), then the derivative of f is the sum of the derivatives of u and v.

f(x) = u(x) + v(x) $\implies f'(x) = u'(x)+v'(x)$

So, above equation also implies:

$$f(x) = u(x)-v(x) \implies f'(x) = u'(x)-v'(x)$$

Constants come out the front in the sense that if f (x) is k times g (x), where k is a constant, then the derivative of f is just k times the derivative of g.

$$f(x) = k.g(x) \implies f'(x) = k.g'(x) \qquad \text{if k is a constant}$$

This follows quickly from either of the limit definitions and the fact that limits respect constant multiples. For example,

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{b \to a} \frac{kg(b) - kg(a)}{b - a} = k\left(\lim_{b \to a} \frac{g(b) - g(a)}{b - a}\right) = kg'(a),$$

and the result follows, taking $x = a$.

For example, if $f(x) = 10x^3$ then $f'(x) = (10)(3x^2) = 30x^2$.

17.2.3 Derivative of a polynomial

It's now straightforward to differentiate any polynomial. You differentiate any power of x you see, multiply through by any constant you see, and add everything up. For example, let f(x) be $f(x) = x^4 - 2x^3 + 5x^2 - 3x + 8$ having polynomial degree four. Then, the derivative is formed, first by differentiating x^4 , which is $4x^3$, then move on to the next term involving x cubed. So, you subtract two times $3x^2$, then move on to the next term involving x^2. So, you add five times 2x, then move on to the next term involving x, so subtract three times 1. Finally, move on to the constant term whose derivative is zero, and add everything up to get the cubic. With some practice, you'll go immediately to the following last step.

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 5x^2 - 3x + 8 \\ \Rightarrow f'(x) &= 4x^3 - 2(3x^2) + 5(2x) - 3(1) + 0 \\ &= 4x^3 - 6x^2 + 10x - 3 \end{aligned}$$

The derivative of a polynomial: From the derivatives of power functions (discussed previously) and the fact that derivatives respect addition and constant multiples, we are able to quickly obtain the derivative of any polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1} + a_nx^n$$

of degree $n \geq 1$ where $a_0, \ldots, a_n$ are constants, namely,

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1},$$

which is a polynomial of degree $n - 1$, that is, of degree one less than that of $p(x)$.
Notice that differentiating a polynomial decreases its degree by one. So, you can always make a polynomial disappear if we differentiate it often enough. For example, suppose f(x)=$3x^2 - 5x + 7$, a quadratic function, then a derivative is linear function. If we differentiate again, we get what's called the second derivative, which in this case, the constant 6. If we differentiate it again, we get what is called the third derivative which in this case is zero.

Let $f(x) = 3x^2 + 5x - 7$.
Then $f'(x) = 6x + 5$,
$f''(x) = 6$ (second derivative),

$f'''(x) = 0$ (third derivative).

Differentiating three times makes this quadratic disappear and that's true for any quadratic. If you take a polynomial of degree n and differentiate it n+1 times, then it disappears. Next, we'll look at a function that turns out to be indestructible with respect to differentiation.

Let $y = f(x)$. The derivative of f is denoted by $y' = f'(x)$, and is also called the *first derivative*. The *second derivative* is the derivative $(y')'$ of the (first) derivative and denoted more simply just by $y'' = f''(x)$.

The *third derivative* is the derivative $(y'')'$ of the second derivative and denoted by $y''' = f'''(x)$.

We can continue differentiating to get *higher-order derivatives*. Because of the proliferation of dashes, it is conventional to denote the result of differentiating n times by $y^{(n)} = f^{(n)}(x)$, which is an abbreviation for using n dashes. For example, $y^{(3)} = y'''$ and $y^{(4)} = y'''' = y^{(4)}$, and so on.

If $p(x)$ is a polynomial of degree $n \geq 1$ then, as noted above, $p'(x)$ is a polynomial of degree $n-1$. It follows that the higher order derivative $p^{(k)}$, the result of differentiating the polynomial $k \leq n$ times, is a polynomial of degree $n-k$. In particular $p^{(n)}$ is a constant polynomial (of degree 0), so one further differentiation will produce zero. This shows that, by differentiating a polynomial sufficiently often, eventually the higher-order derivatives can be made to "disappear" in the sense of becoming zero.

17.2.4 Derivative of the exponential function is itself

Now, we'll look at a function that turns out to be indestructible with respect to differentiation. Consider the function $y = e^x$. Recall from an earlier section, that the tangent line to it's curve at the y intercept has slope 1. The base is Euler's number e and is chosen deliberately so that this property holds. We want to transform this fact into a very specific limit. Let's have a look at the curve for x close to zero.

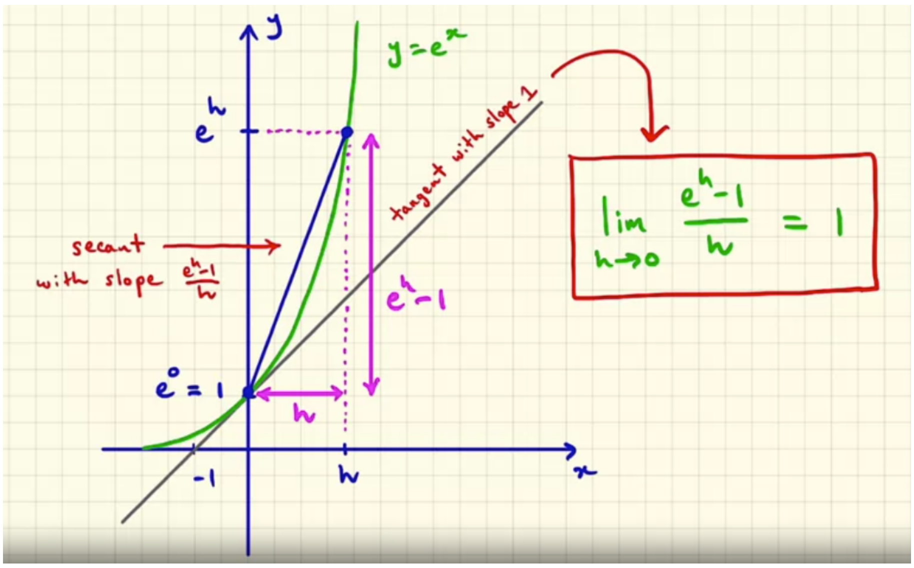


Figure 17.1: Demonstration of $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

If we input x equals h some small positive real number and move up to the curve, then we have the value e^h on the y axis. Now, draw a secant that joins the curve's y intercept to this point on the curve. The slope of the secant is the vertical rise $e^h - 1$ over the horizontal run h which is $\frac{e^h-1}{h}$. As h tends to zero, this slope tends to the slope of the tangent line which is 1.

$$\text{Thus, } \lim_{h\to 0}\frac{e^h-1}{h} = 1$$

. This innocuous-looking limit is very special, and in fact, the foundation for the central role Euler's number e plays in mathematics.

Consider the natural exponential function $y = f(x) = e^x$. We claim that

$$y' = f'(x) = e^x.$$

But, with our derivative notation, this slope is just $f'(0)$. Hence, we have that

$$1 = f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{0+h}-e^0}{h} = \lim_{h\to 0}\frac{e^h-1}{h}.$$

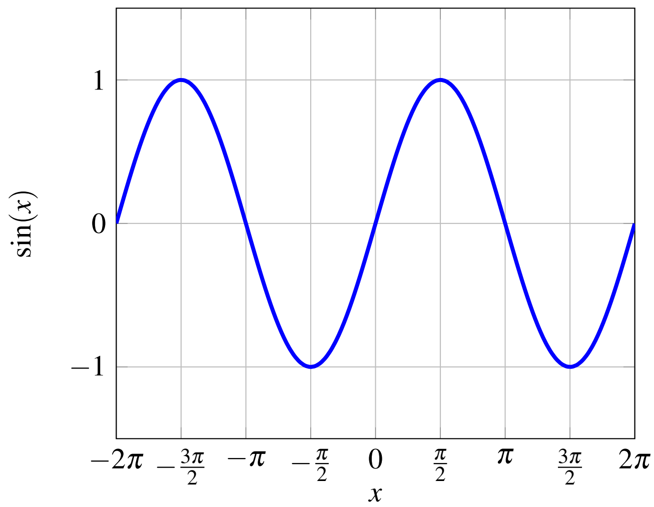
Remarkably, we can use this to prove that $f'(x) = e^x$ for all x. Using an exponential law at the third step, and the above limit at the second last step, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$
$$= e^x\left(\lim_{h\to 0}\frac{e^h-1}{h}\right) = e^x(1) = e^x,$$

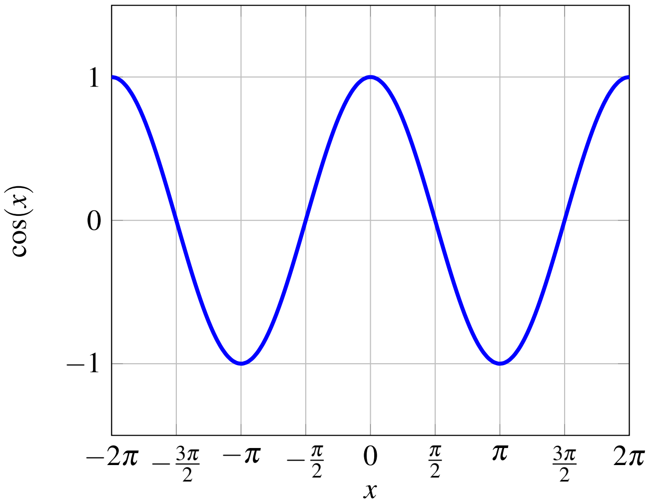
as required. Thus the natural exponential function reproduces itself by differentiation, and coincides with all of its higher derivatives, by contrast with polynomial functions, which eventually become zero by repeated differentiation.

17.2.5 The derivative of the sine function is the cosine function

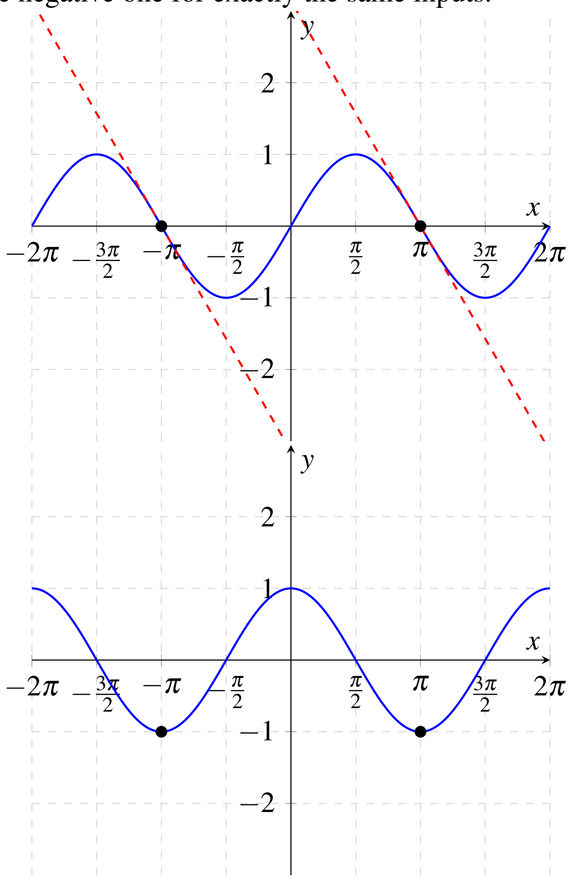
The circular functions y = sin x and y = cos x form a pair of indestructible functions from the point of view of differentiation in an interesting roundabout way as you'll soon see. We claim that the derivative of sine x is cos x. We are going to sketch a proof. This will be quiet an advanced mathematical argument. If you don't follow all the detail, don't worry. I'll first try to convince you visually that the claim seems plausible. Here's a graph of y equals sine x, and directly beneath it we've drawn the graph of y equals cos x.



[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)



Here are points on the sine curve, where the tangent lines are horizontal, having slope equal to zero. Notice, the cos x takes the value zero for exactly the same inputs. Let's also focus on points where the tangent lines of y equals sine x, have slope +1. Notice, the cos x takes the value +1 for exactly the same inputs. Finally, focus on points where the tangent lines y = sine x, have slope negative one, and notice that cos x takes the value -1 for exactly the same inputs. Finally, focus on points where the tangent lines y equals sine x, have slope negative one, and notice that cos x takes the value negative one for exactly the same inputs.



So, our claim is looking plausible based on these samples of points.

Consider $y = f(x) = \sin x$. We claim that

$$y' = f'(x) = \cos x.$$

Before proving this, first observe that the slope of the tangent line to the sine curve at the origin is 1 (which coincides with $\cos(0)$). This follows because of the fact

$$\lim_{h \to 0} \frac{\sin h}{h} = 1,$$

which we carefully proved using the squeeze law, but stated here using h instead of x, and because the ratio $\frac{\sin h}{h}$ is the slope of the secant joining the origin to the point on the curve for $x = h$.

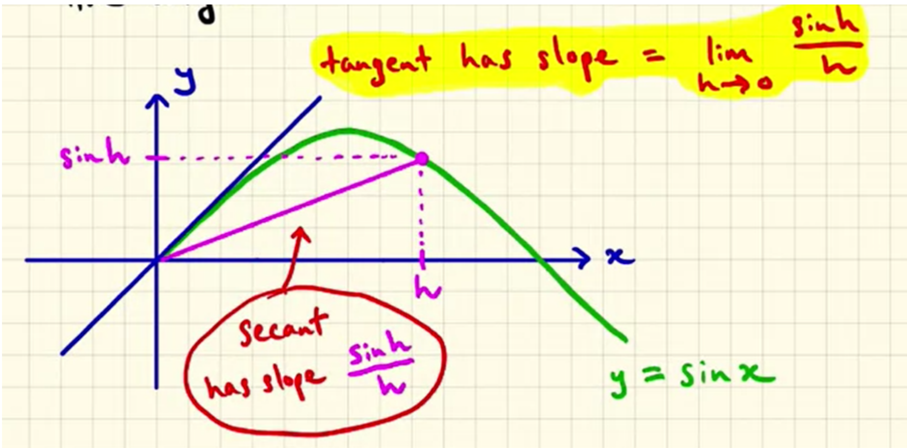


Figure 17.2: Demonstration of sin x with tangent and secant line

Let's also look at the tangent to the cosine curve at x equals zero. Because the cosine curve turns around at that point (0,1), the tangent is horizontal. So, must have slope zero at $x = 0$.

If one joins the point (0, 1) to some nearby point (h, cos h) to form a secant, then it has(negative) slope $cos\frac{h}{h-1}$, which must therefore approach 0 as h approaches 0.

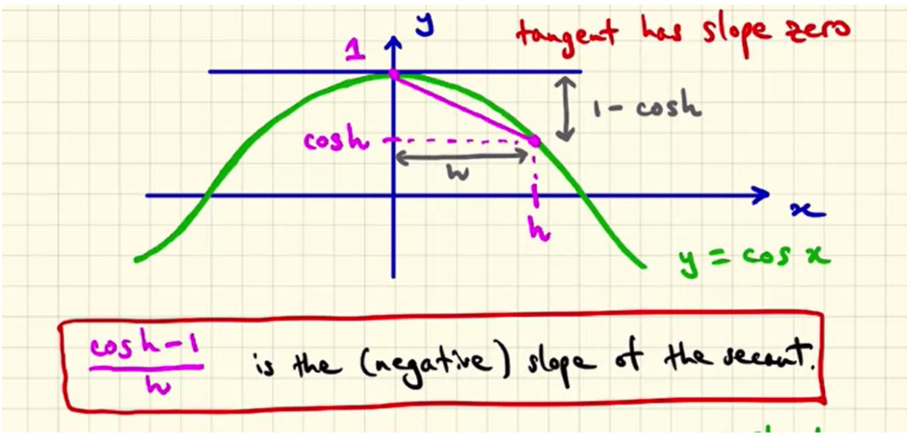


Figure 17.3: Demonstration of cos x with tangent and secant line

We therefore get the following limit:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

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[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

We can now find the derivative of $\sin x$ as follows, using these two special limits at the second last step, after expanding the numerator at the third step using the formula for rearranging sums and angles (explained later in these notes), and our usual tricks for sine of a fraction and manipulating limits:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$= \sin x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)+\cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$
$$= \sin x\cdot 0+\cos x\cdot 1 = \cos x,$$

as required.

17.2.6 The derivative of the cosine function is the negative sine function:

we have already seen the derivative of sine function is cosine function graphically and using first principle. You can also prove it graphically like we did above.

Consider $y = f(x) = \cos x$.
We claim that

$$y' = f'(x) = -\sin x.$$

We again use the two special limits that were used in the previous proof and the same tricks for manipulating fractions and limits, but now, in the third step, use the formula from the advanced trigonometry for the cosine of a sum of angles which is $cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} = \lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$= \cos x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)-\sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$

$$= \cos x\cdot(0)-\sin x\cdot(1) = -\sin x, \qquad \text{as required.}$$

You're probably curious now about the derivative of tan x, you're not likely to guess the answer. It is in fact, the reciprocal of the square of cos x. You'll have to wait till you turn the next pages to study new section to see why. We showed before, that the derivative of e^x is e^x. We've just shown that the derivative of sine x is cos x, and mentioned that the derivative of cos x is -sine x. This means that the circular functions are indestructible in the following sense, if f(x) is sine x and the derivative is cos x, so each derivative, the second derivative, is negative sine x. So, each derivative, the third derivative, is negative cos x. So, its derivative, the fourth derivative, quickly becomes sine x, and we're back to where we started. Differentiating, add infinitum, just reproduces the same cycle of functions over and over again forever. Recall in an earlier section, we mentioned simple harmonic motion as an example of a displacement function, which happens to be the sine curve. Its derivative, the velocity is the cosine curve, its second derivative, the acceleration, is the negative sine curve, its third derivative, the jerk, is a negative cosine curve, it's fourth derivative, the snap, is the original sine curve, and the process goes on forever reproducing these four curves by differentiation. Today, we've really come a long way using the limit definition to rapidly expand our repertoire. We discussed and illustrated simple rules for differentiating polynomials, and noted, that by differentiating often enough we can make a polynomial disappear in the sense of going to zero. We explained why the natural exponential function reproduces a perfect copy of itself under differentiation. So, is in a certain sense indestructible. Sketched a proof that the derivative of sine is cosine, and mentioned that the derivative of cosine is negative sine, leading to an indestructible sequence of functions, sine, cosine, negative sine, and negative cosine, that reproduces itself forever under differentiation. Please read properly, and when you're ready please attempt the exercises. Thank you very much for reading, and I look forward to seeing you again soon.

17.2.7 Practice Quiz

Question 1

Find $f'(x)$ when $f(x) = 7x - 6$.

(a) 7
(b) 6
(c) -6
(d) -7
(e) 1

Question 2

Find $f'(x)$ when $f(x) = x^2 + 3x + 1$.

(a) 2x + 2
(b) x + 3
(c) 2x + 3
(d) x + 2
(e) 2x + 1

Question 3

Find $f''(x)$ when $f(x) = 4x^3 - 7x^2 + 2x - 5$.

(a) 12x - 14
(b) 24x - 14
(c) 12x - 7
(d) 24x - 7
(e) 9x - 10

Question 4

Find $f'(x)$ when $f(x) = 3e^x$.

(a) $3xe^{x-1}$
(b) $3xe^x$
(c) $3e^x$
(d) $9e^x$
(d) e^{3x}

Question 5

Find $f''(x)$ when $f(x) = 3e^x$.

(a) 0
(b) $9e^x$
(c) $3e^x$
(d) $3xe^x$
(e) $3(x-1)e^{x-2}$

Question 6

Find $f'(x)$ when $f(x) = \sin x + \cos x$.

(a) $\frac{\cos x - \sin x}{2}$
(b) $\sin x - \cos x$
(c) $\cos x - \sin x$
(d) $-\cos x - \sin x$
(e) $\sin x + \cos x$

Question 7

Find $f''(x)$ when $f(x) = \cos x$.

(a) $\sin x$
(b) $\cos x$
(c) $-\sin x + \cos x$
(d) $-\cos x$
(e) $-\sin x$

Question 8

Evaluate $f'(0)$ when $f(x) = x^3 - 4x^2 + 6$.

(a) -8
(b) 3
(c) 6
(d) 0
(e) -6

Question 9

Evaluate $f'(0)$ when $f(x) = e^x + x$.

(a) e
(b) $e+1$
(c) 3
(d) 2
(e) 1

Question 10

Evaluate $f''(\pi)$ when $f(x) = \sin x - \cos x$.

(a) 0
(b) -1
(c) -2
(d) 1
(e) 2

Answers

The answers will be revealed at the end of the module

18. Leibniz Notation and Differentials

18.1 Leibniz Notations

In earlier sections, we've introduced and developed the limit definition of the derivative formally expressed in the notation of functions, so that if f is a function, then the derivative of f(x) with respect to x is denoted by f'(x), with a dash as a superscript. In today's video, we introduce and apply a very commonly used notation for the derivative called $\frac{dy}{dx}$, that looks like a fraction and is known as Leibniz's notation. In fact, $\frac{dy}{dx}$ is still formally speaking a limit, but the idea of thinking of the derivative as a fraction and deliberately using notation as a fraction goes back to the pioneering work of Gottfried Leibniz, one of the founders in parallel to Isaac Newton of calculus in the 17th century, and after whom this notation is named. Leibniz thought in terms of very powerful underlying heuristics. He considered the derivative to be a fraction involving infinitesimals, which idealized infinitely small numbers that interact using an arithmetic that extends that of the real number system. His notation was so carefully chosen that it has survived intact for hundreds of years. It's interesting that Leibniz's number system wasn't placed on a formal rigorous footing until the 1960s, which is not so long ago. If you want to read more about this fascinating topic, you can look up terms on the Internet such as nonstandard analysis or the hyperreal number system.

18.1.1 Connecting the Dots

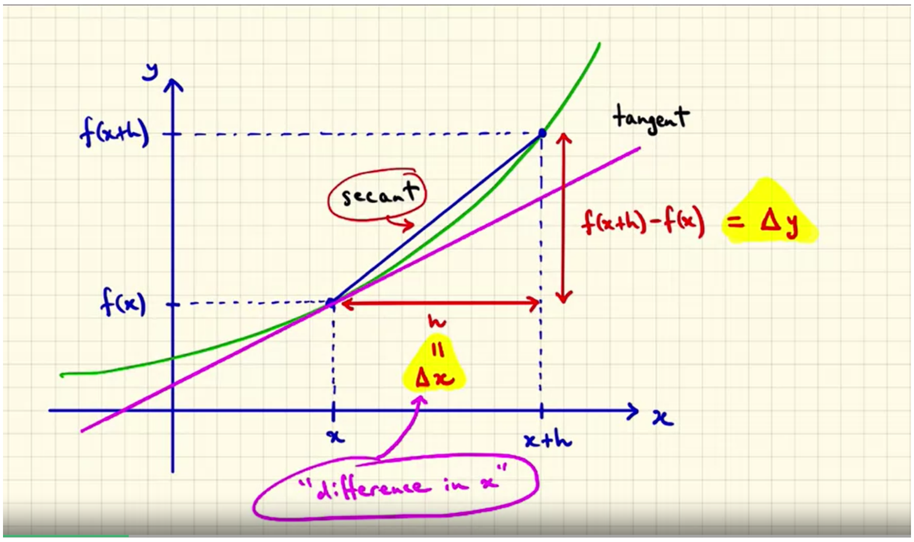

Figure 18.1: Demonstration of a function using tangent lines

We begin by revisiting a familiar diagram intended to represent a general curve, y = f(x), and a point of interest with coordinates (x, f(x)), and a tangent line to the curve at that point, and a nearby point with coordinates (x + h, f(x) + h). We connect the two points with a straight line segment called a secant, the horizontal runners h and the vertical rise is f(x + h) - f(x). We think of adding h to x as a slight variation or perturbation of input, and we introduce some new notation and give h the name Δx. Delta being the Greek capital letter corresponding to the Latin d, and think of Δ for difference. So, Δx equal to h is the difference in the x-coordinate, and we may simply say difference in x or change in x. As we perturb the x-coordinate, this induces a perturbation in the y-coordinate, and the change in effect in the vertical direction f(x + h) - f(x), we may now rename Δy, and simply say difference in y or change in y. The slope of the secant that we recognized before as f(x + h) - f(x), now becomes, using this notation, $\frac{\Delta y}{\Delta x}$. The change in y over the change in x. The slope of the tangent line that we recognized as limiting slope of the secant can now be expressed using this new notation as

$$\lim_{h\to 0}\frac{\Delta y}{\Delta x}$$

. We can now rewrite the limit definition of the derivative of y= f(x), or equivalently as the $\lim_{h\to 0}\frac{\Delta y}{\Delta x}$. This has a very beautiful notational abbreviation $\frac{dy}{dx}$, just spoken dy dx and called Leibniz notation in honor of Gottfried Leibniz. Informally, in the limit, you can think of the greek Δx turning into the Latin lowercase dx, and the greek Δy turning into the Latin lowercase dy. Leibniz thought of dx and dy as idealized mathematical objects, representing some kind of infinitely small numbers that had their own arithmetic, that paralleled the arithmetic of the real numbers. In modern terminology, dx and dy are called differentials. They become very useful heuristic devices,both in applications and also later in the final module when we manipulate integrals.

18.1.2 Revisiting the earlier examples using Leibniz notation

Let's revisit some earlier examples, but interpreted or expressed using Leibniz notation. First, if y = k is a constant function, then the graph is a horizontal line with no change in the y-value, giving slope zero everywhere. So, $\frac{dy}{dx}$ is zero.

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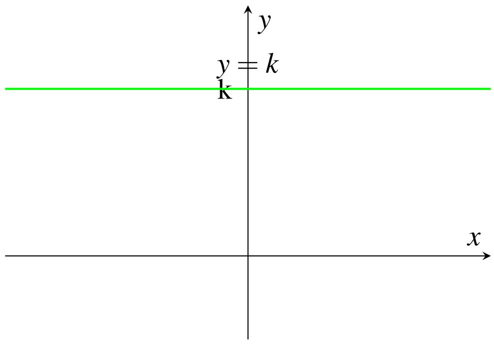
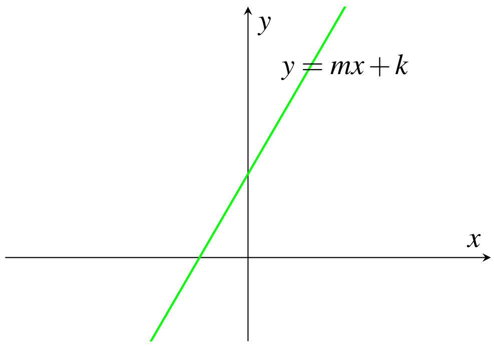


Next, if $y = mx + k$ is a linear function, where m is the slope and k is y-intercept, then y changes m units uniformly for each unit that exchanges. So, $\frac{dy}{dx}$ is the constant m.



We know that previously that the derivative is additive, which means that the derivative of a sum is the sum of the derivatives. For example, if y is a quadratic function at x, say $x^2 + bx + c$, where a, b, and c are constants, then $\frac{dy}{dx}$ is the sum of the derivatives of each of the pieces, which can be expressed in this way.

$$\begin{aligned} \text{e.g.} \quad & y = ax^2 + bx + c \\ \Rightarrow \quad & \frac{dy}{dx} = \frac{d}{dx}\left(ax^2\right) + \frac{d}{dx}(bx) + \frac{d}{dx}(c) \end{aligned}$$

So, the derivative of ax^2, regarded as the root for the function that takes x to x^2, can be rewritten as $\frac{d}{dx}(ax^2)$, which is $2ax$. The derivative of bx can be written as $\frac{d}{dx}(bx)$, which is b, and the derivative of the constant function c can be rewritten as $\frac{d}{dx}(c)$, which is zero. The entire derivative simplifies to $2ax + b$.

We mentioned in general that the derivative of

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ for any fixed exponent } n.$$

We carefully proved that the derivative of e^x is e^x, reproducing itself exactly under differentiation.

$$\frac{d}{dx}(e^x) = e^x \text{ (note } x \text{ is the exponent)}$$

Notice that the variable x is the exponent. It is a common error by students to confuse the exponential rule for differentiation with the previous rule and guess that the answer for the derivative of e^x should be the result of bringing the exponent x down to the front, and then creating a new exponent by subtracting one, which is completely wrong and off track. This is a subtle error and the result of confusing the contrasting roles of the variable x used in building power functions, where x is a base, and exponential functions, where x is an exponent.

common error: $\frac{d}{dx}(e^x) = xe^{x-1}$

We've sketched a proof that the derivative of sine x is cos x, and mentioned that the derivative of cos x is -sine x.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Good notation can be amazingly transformative, You might recall we developed the quadratic formula really from scratch.Another derivation at the time to quite a few lines and some tricks, it was relatively painless using algebraic notation. Imagine as a medieval mathematician, without our notation, trying to express everything in mixtures of words and hybrids of symbols and trying to avoid any heresies associated with using negative numbers. Leibniz notation, $\frac{dy}{dx}$, is truly a miracle of inventiveness. So, simple, yet so powerful. It places emphasis on the roles of the variables x and y, where the differential associated with x, appears in the denominator of some kind of fraction and the differential dy, in the numerator. These distinctions are invisible or opaque, in the function notation, f'(x), for the derivative. If one wants to reverse the roles of the variables x and y, it makes perfectly, good sense using Leibniz notation, to tip $\frac{dy}{dx}$ upside down, that is form $\frac{dx}{dy}$, interchanging the differentials in the numerator and the denominator. If these were ordinary fractions, the effect would be to form the reciprocal.

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

This equation $\frac{dx}{dy}$ equals the reciprocal of $\frac{dy}{dx}$, is an irresistible deduction from the notation and intuition of Leibniz and turns out to be a theorem about derivatives of inverse functions. Interchanging the roles of x and y of input and output, has the effect of inverting the original function. So, this formula tells us how to differentiate the inverse function. It becomes a nontrivial and useful theorem about derivatives.

18.1.2.1 Why is it useful?

Here, we will see how Leibniz notation is useful while operating on inverse functions. Let's use this idea to find the derivative of the natural logarithm function, knowing that it's the inverse of the natural exponential function. Start off with $y = e^x$, for which we already know the derivative is e^x. We want information about the inverse function. So, tip the derivative upside down, which both reverses the roles of x and y and also reciprocates the derivative $\frac{dy}{dx}$ of the original exponential function. This is just $\frac{dx}{dy} = \frac{1}{e^x} = \frac{1}{y}$. But $y = e^x$, so undoing this gives $x = \ln(y)$. So, saying that $\frac{dx}{dy} = \frac{1}{y}$, is the same as saying $\frac{d}{dy}(\ln y) = \frac{1}{y}$. We now revert back using x as the symbol for the input. We get that the derivative of $\ln(x)$ with respect to x is the reciprocal of x.

Derivative of the natural logarithm:

$$\text{Put} y = e^x, \text{ so } \frac{dy}{dx} = e^x,$$

So,

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{e^x} = \frac{1}{y}.$$

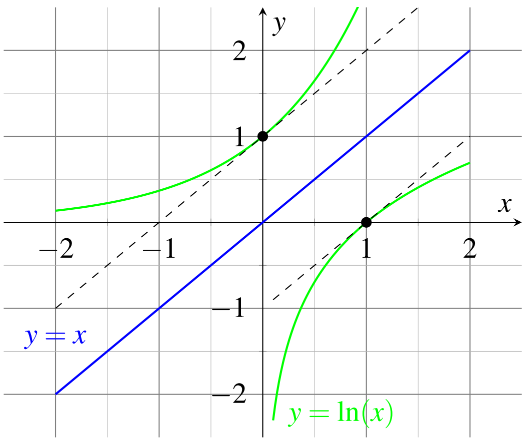
But $x = \ln y$, so:

$$\frac{d}{dy}(\ln y) = \frac{1}{y}.$$

Reverting to x as an input:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Amazing, such a simple and elegant answer! The formula makes sense visually. Here are the familiar natural logarithm and exponential functions on the same diagram and we get from one to the other by reflecting in the line y = x. Euler's number e was chosen so that the slope of the tangent to the curve, $y = e^x$ at the y intercept, is 1. So, this tangent is parallel to the line y = x. When you do the reflection, the reflected tangent line, now touching the curve y = log (x) at the x intercept, retains the same slope. Both of these tangent lines, have slope one. Of course one is $\frac{1}{1}$. So, this matches the formula for the derivative of log (x). You can visualize the tangent to the curve at $x = 2$ and if your diagram is accurate, you will see, that it has a slope of a $\frac{1}{2}$, also matching the formula for the derivative. We can do the same thing for x =3 and get a slope of a $\frac{1}{3}$ and anywhere on the x axis you'll find the slope of the tangent to the curve is the reciprocal of x or $\frac{1}{x}$.



18.1.2.2 Let's see another example

Let's try this idea, tipping the Leibniz derivative upside down, to differentiate the square root function. Start off with the squaring function, $y = x^2$, and focus on $x \geq 0$, so that x is just the square root of y. We know the derivative of y with respect to x is just $2x$, and this is two times the square root of y. We now form $\frac{dx}{dy}$, which is the reciprocal of $\frac{dy}{dx}$, which becomes $\frac{1}{2\sqrt{y}}$. But $\frac{dx}{dy}$ is just $\frac{d}{dy}$ of the square root of y, so we can express everything in terms of y. We now revert to using the symbol x as a typical input. So, $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Thus, we discover that to differentiate $\sqrt{x}$, you just take the reciprocal of twice the square root. This is not something you're ever likely to guess.

We can relate this to fractional powers because the square root of x is just $x^{1/2}$. This rule for differentiation then becomes $\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$. Note that this can be rewritten as $\frac{1}{2}x^{-1/2}$, and so, we get this nice formula."So, let's simplify and clean up the text below, without all the language complexities."lexities."

Put $\quad y = x^2, \quad$ for $\quad x \geq 0,$

$x = \sqrt{y}$ and $\dfrac{dy}{dx} = 2x = 2\sqrt{y}$

Hence

$$\frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

Reverting to x as input,

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

We have:

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

Using fractional power notation:

$$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$$

i.e.$\dfrac{d}{dx}\left(x^{\frac{1}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}}$

consistent with $\dfrac{d}{dx}(x^n) = nx^{n-1}$ when $n = \dfrac{1}{2}$

18.1.3 Recapping this section

Before finishing this section let's recap what we have studied in this lesson and solve some derivatives using Leibniz notation.

Some common derivatives and properties using Leibniz notation:

(a) $\frac{d}{dx}(k) = 0$ for any constant k, $\frac{d}{dx}(x) = 1$, $\frac{d}{dx}(x^2) = 2x$ $\frac{d}{dx}(x^3) = 3x^2$.
(b) $\frac{d}{dx}(x^n) = nx^{n-1}$ for any exponent n (though possibly with restrictions on x).
(c) $\frac{d}{dx}(ky) = k\frac{dy}{dx}$ (constants come out the front).
(d) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ (the derivative is additive).
(e) $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$.
(f) $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Inverting the derivative using Leibniz notation:

If the function $y = f(x)$ is invertible then the rule for the derivative of the inverse function f^{-1} can be deduced by 'inverting' $\frac{dy}{dx}$, as though it were an ordinary fraction, to get

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}},$$

and expressing the right-hand side in terms of y. If we revert to using x as an input then we produce a rule for the derivative of the inverse function.

Examples and derivations:

1. Find the derivative $\frac{dy}{dx}$ when $y = 4x^3 - x^2 + 6x - 9$.
Solution: We have

$$\frac{dy}{dx} = \frac{d}{dx}(4x^3) - \frac{d}{dx}(x^2) + \frac{d}{dx}(6x) - \frac{d}{dx}(9) = 4(3x^2) - 2x + 6 = 12x^2 - 2x + 6.$$

2. Find the derivative $\frac{dy}{dx}$ when $y = e^x - 3\ln x + 4x^2$.
Solution: We have

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) - \frac{d}{dx}(3\ln x) + \frac{d}{dx}(4x^2) = e^x - \frac{3}{x} + 8x.$$

3. Find the derivative $\frac{dy}{dx}$ when $y = 2\sin x - 3\cos x$.
Solution: We have

$$\frac{dy}{dx} = \frac{d}{dx}(2\sin x) - \frac{d}{dx}(3\cos x) = 2\cos x - 3(-\sin x) = 2\cos x + 3\sin x.$$

4. Find the derivative $\frac{dy}{dx}$ when $y = \theta^4 + 7\cos\theta - 5\sin\theta$.
Solution: We have

$$\frac{dy}{dx} = \frac{d}{d\theta}(\theta^4) + \frac{d}{d\theta}(7\cos\theta) - \frac{d}{d\theta}(5\sin\theta) = 4\theta^3 - 7\sin\theta - 5\cos\theta.$$

5. We use Leibniz notation to explain why $\frac{d}{dx}(\ln y) = \frac{1}{y}$.

Proof: Put $y = e^x$ so that $x = \ln y$ and, from earlier work,

$$\frac{dy}{dx} = e^x = y.$$

Hence, reciprocating, we get

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{y}.$$

Reverting to using x as an input, we get

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

as required.

6. We use Leibniz notation to explain why $\frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3x^{2/3}}$.

Proof: Put $y = x^3$, so that $x = \sqrt[3]{y} = y^{1/3}$ and, from earlier work,

$$\frac{dy}{dx} = 3x^2 = 3y^{2/3}.$$

Hence, reciprocating, we get

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3y^{2/3}}.$$

Reverting to using x as an input, we get

$$\frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3x^{2/3}},$$

as required.

Note that, using fractional power notation, this becomes

$$\frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3},$$

which is a special case of the formula $\frac{d}{dx}(x^n) = nx^{n-1}$, when $n = \frac{1}{3}$.

In this section, we introduced and practiced the Leibniz notation for the derivative called $\frac{dy}{dx}$, which appears as a fraction involving the differentials dx and dy. Even though the derivatives formally defined as a limit of fractions, thinking of it as behaving like a fraction in its own right, turns out to be very useful especially for applications and for making heuristic leaps of the imagination. Once such leap, is to invert $\frac{dy}{dx}$ to get $\frac{dx}{dy}$ which then can be used to get information about the derivative of the associated inverse function. We exploited this trick, to discover that the derivative of the natural logarithm function, is the reciprocal function and also to differentiate the square root and cube root functions. Please read the notes, and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

18.1.4 Practice Quiz

Question 1

Find $\frac{dy}{dx}$ when $y = 2x^3 - 4x + 8$.

(a) $5x^2 - 4x$
(b) $6x^2 - 4$
(c) $6x^2 - 4x$
(d) $4x - 4$
(e) $5x^2 - 4$

Question 2

Find $\frac{dy}{dx}$ when $y = 2\ln x$.
(a) 1
(b) $\frac{2}{x}$
(c) $\frac{2}{x^2}$
(d) $1 - \frac{2}{x}$
(e) $\frac{2}{x} - 2$

Question 3

Find $\frac{dz}{dx}$ when $z = e^x$.
(a) e^x
(b) e^x
(c) xe^{x-1}
(d) e^{-x}
(e) te^{t-1}

Question 4

Find $\frac{dy}{du}$ when $y = u^2 - 1$.
(a) 0
(b) $2u$
(c) $2u - 1$
(d) $2u - 1$
(e) $2u$

Question 5

Find $\frac{du}{d\theta}$ when $u = -3\sin\theta$.
(a) $-\frac{\cos\theta}{3}$
(b) $3\cos\theta$
(c) $-3\cos\theta$
(d) $\cos\theta$
(e) $\frac{3}{\cos\theta}$

Question 6

Find $\frac{du}{d\theta}$ when $u = -3\cos\theta$.
(a) $-3\sin\theta$
(b) $\frac{\sin\theta}{3}$
(c) $3\sin\theta$
(d) $3\sin\theta$
(e) $-\frac{\sin\theta}{3}$

Question 7

Find $\frac{dx}{dy}$ when $y = e^x$.
(a) $\frac{1}{y}$
(b) e^x
(c) xe^{x-1}
(d) e^y
(e) $\frac{1}{x}$

Question 8

Find $\frac{dx}{dy}$ when $y = 2x^2$ (assuming $x \geq 0$).

(a) $4x$
(b) $\frac{1}{2\sqrt{2y}}$
(c) $\frac{1}{2\sqrt{y}}$
(d) $2\sqrt{y}$
(e) $\frac{\sqrt{2}}{\sqrt{y}}$

Question 9

Find $\frac{dx}{dy}$ when $y = -x^3$.

(a) $-3x^2$
(b) $-\frac{3y^{2/3}}{3}$
(c) $y^{-2/3}$
(d) $\frac{y^{-2/3}}{3}$
(e) $\frac{y^{2/3}}{3}$

Question 10

Find $\frac{dx}{dy}$ when $y = 8x^3$.

(a) $24x^2$
(b) $\frac{y^{1/3}}{24}$
(c) $\frac{8y^{-2/3}}{3}$
(d) $\frac{y^{-2/3}}{6}$
(e) $\frac{y^{2/3}}{8}$

Answers

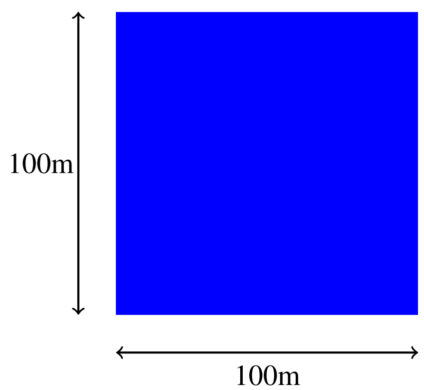
The answers will be revealed at the end of the module.

18.2 Differentials and Applications

In the previous section, we introduced Leibniz's notation which expresses the derivative in the form of a ratio of differentials. In this video, we'll discuss some contrasting applications by interpreting differentials as small changes in the respective variables. This is closely related to the tangent line approximation of the curve near any particular point of interest.

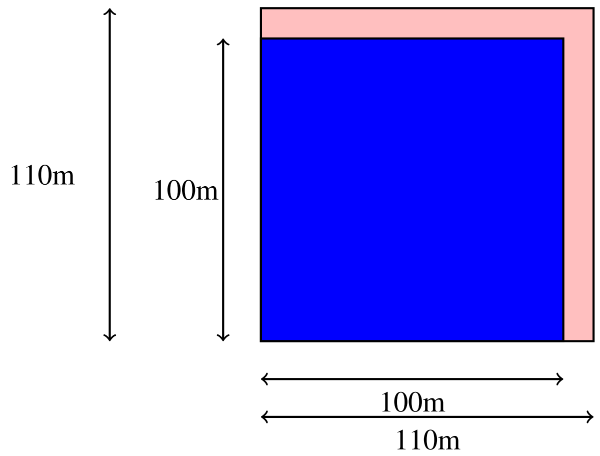
18.2.1 Interpreting the equation linking differentials

In our first example, we take a square and consider the increase in area as we change the width slightly.



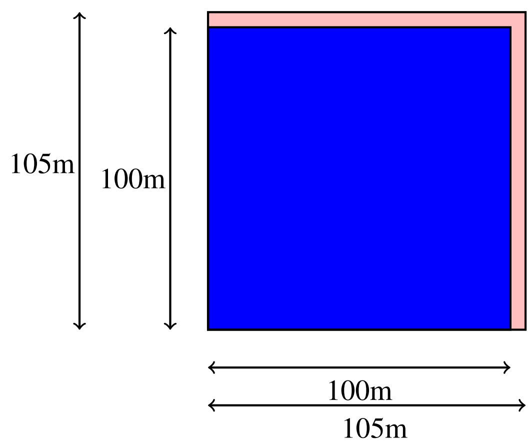
Here we've drawn a square of width a 100 meters and colored in the area in blue. Let A denote the the area of a square of side length x, so A equals x^2. The value of A when x equals a 100 is 10,000. So, this square has area 10,000 square meters. How does the area change if we increase the width by say 10 meters, or 5 meters, or 1 meter, or 1 centimeter?

In the first case, we expand the width by 10 meters to become 110 meters.

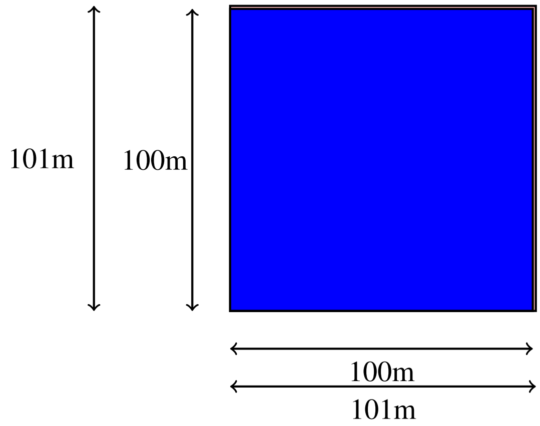


We've shaded in pink, the new extra area, which is the area of the larger square minus the area of the original square, and you can check this is exactly 2,100 square meters.

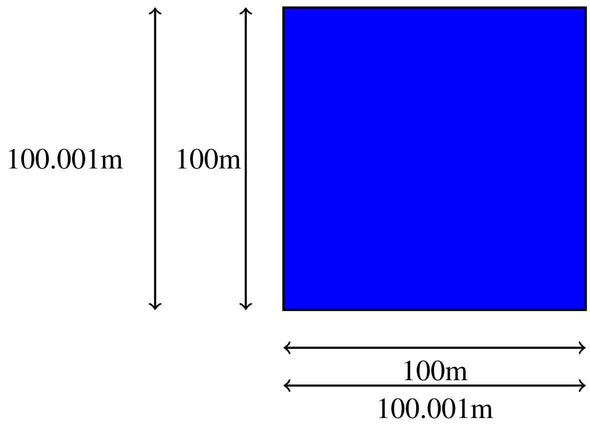
In the second case, we expand the width now only by 5 meters to become 105 meters, and we can calculate this extra pink area to be exactly 1,025 square meters.



In the third case, we expand the width by just 1 meter to become 101 meters, and the extra area becomes exactly 201 square meters.



In the fourth case, we expand the width by miniscule of one centimeter that is 0.01 meters, and the extra area turns out to be 2.0001 square meters.



Though the exact calculations are not difficult in this example, I want to show you how you can exploit Leibniz's notation of the derivative to get fast, and perhaps surprisingly accurate estimates for these extra areas. We have this function A equalsx^2. The derivative of A with respect to x is just 2x. The trick is to treat the differentials dA and dx like numbers, and $\frac{dA}{dx}$ like an ordinary fraction. So, the multiplying through by dx, we get dA equals $2xdx$ which we think of as an equation involving differentials. This is an idealized form of an approximation namely, ΔA is approximately equal to $2x\Delta x$. This gives us a good approximation formula for the change in area ΔA in terms of x andΔx, the change in the width x.

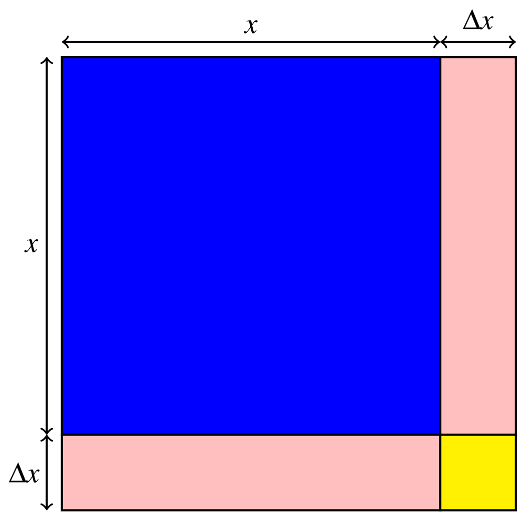
$$A = A(x) = x^2$$

$$\frac{dA}{dx} = 2x$$

$$dA = 2x\,dx$$

$$\Delta A \approx 2x\Delta x$$

This says that the change in the area of a square is approximately two times the width times the change in the width. In our example with the blue square x equals 100 meters, in case one, the change in x is 10 meters, so the formula for the change in area gives approximately 2,000 square meters. In case two, the change in x is 5 meters and the formula gives the change in area approximately 1,000 square meters. In case three, Δx is 1, and ΔA is approximately 200 square meters. In case four, Δx is 0.01 and ΔA is approximately 2 square meters.
We can compare those estimates with a true increases in area in each of the four cases. One can see that the estimates are quite close to the true increases, and in fact the quality of the estimates improves as the change in x gets smaller and smaller. The changes in area themselves get smaller but the proportional error in the estimates gets vanishingly small as $\Delta x \to 0$. This is related to the underlying idea of tangent lines used to approximate curves which motivates the definition of the derivative.
In this particular example, there's also a clear visual explanation of why the estimate should be so good. Take a general square of side length x and add a small change in x to the width. To form a slightly larger square of width $x + \Delta x$, color of the original square is blue and the oblong areas are pink. There's a new much small square colored beige in the diagram that makes up the corner.



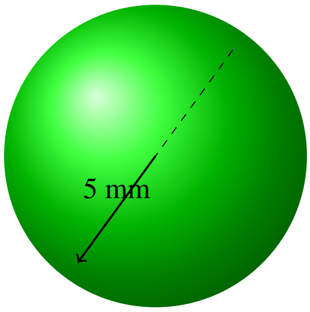
The area of the largest square $(x+\Delta x)^2$, which we can expand out.

$$\text{Area } = (x+\Delta x)^2$$

$$= x^2 + 2x\Delta x + (\Delta x)^2$$

The part of this expansion $2x$ times Δx is in fact the formula for the estimate of increase that we obtained by considering differentials before, which is the result of combining the two pink colored areas. The last part of the expansion $(\Delta x)^2$ is the area of the tiny beige square and it's exactly the error in our estimates. As $\Delta x \to 0$, this beige square vanishes out of sight becoming insignificant compared to the pink areas, and the quality of the approximation improves towards perfect agreement.

The next example is an application of differentials to estimate changes in the volume of a sphere. Suppose you're manufacturing metal ball bearings in the shape of hopefully close to perfect spheres of radius five millimeters. However, your manufacturing process is not perfect, and you expect up to about 2% error in the true radius for any particular ball bearing. Our problem is to estimate the percentage error in the volume of metal required to manufacture these ball bearings. Let V be the volume of a sphere of radius r.



It’s a fact and also follows from methods we learned in the final module on integral calculus, that v is $\frac{4}{3}\pi r^3$. Let’s differentiate regarded as a function of r. The four-thirds Pi is a constant that can come out the front of the derivative of r^3. Remember, the derivative of x^3 with respect to x is $3x^2$ so using r instead of x, the derivative of r^3 with respect to r is $3r^2$.

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{4}{3}\pi \frac{d}{dr}(r^3) = \frac{4}{3}\pi(3r^2)$$

$$\Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$$

$$\Rightarrow \quad dV = 4\pi r^2\, dr$$

$$\Rightarrow \quad \boxed{\Delta V \approx 4\pi r^2\, \Delta r}$$

Hence $\frac{dV}{dr}$ simplifies quickly to $4\pi r^2$. Multiplying through by the differential dr gives dV equals $4\pi r^2 dx$ which we can interpret as an equation involving differentials, which is really an idealization of an approximation involving changes in v and r. ΔV is approximately $4\pi r^2 \Delta r$, where ΔV is the change in v propagated by Δr, a small change in r. In our application, v represents the volume of a ball bearing of radius r equals five millimeters, and we’re told to expect up to two percent error in the radius. Here’s all the information so far. Δr may be positive or negative depending on whether the actual radius is a bit more or a bit less than five millimeters. We can ignore the sign by taking the absolute value and then the information about the percentage error is really saying that the proportion, the magnitude of $\frac{\Delta r}{r} \leq 0.02 \quad (= 2\%)$. We’re working towards estimating the percentage error in the volume. So, we really want the proportion of the volume V represented by the magnitude of the change in volume ΔV. Thus, we want to estimate the fraction the magnitude of $\frac{\Delta V}{V}$.
Here’s a summary of what we now have, and what we’re looking for. We just have to put the pieces together. The magnitude of

$$\frac{|\Delta V|}{V} \approx \frac{|4\pi r^2 \Delta r|}{\frac{4}{3}\pi r^3} = \frac{4\pi r^2 |\Delta r|}{\frac{4}{3}\pi r^3}$$

$$= \frac{3|\Delta r|}{r} \leq 3(0.02) = 0.06$$

Thus, we have the fraction, the magnitude of $\frac{\Delta V}{V}$ lg0.06, and we therefore expect up to about 6%t variation in the volume of the ball bearings. It’s worth noting that the mathematics shows that this estimate is independent of the size of the ball bearings. This method only makes an approximate prediction when translating differentials into actual changes in the variables. Because the percentage error 2% was small, we expect the approximation to be quite good.

In the next example, we apply differentials to estimate cube roots, and also interpret our answers in terms of a tangent line to the cube root curve. Let's estimate some cube roots close to the cube root of 64 which is 4. Say the cube roots of 70, 65, and 63. These numbers are all in the vicinity of the 64 whose cube root is exactly 4. So, expect to get cube roots in the vicinity of four. Note that cube roots are fractional powers with exponents one-third. So, we consider the function $y = f(x) = x^{\frac{1}{3}}$. Then the derivative we saw last time is $\frac{1}{3}x^{\frac{-2}{3}}$. Multiplying through by dx this becomes an equation involving differentials which may be interpreted as the following approximation involving changes in y and x. Δy is approximately equal to $\frac{1}{3}x^{\frac{-2}{3}}\Delta x$.

Put $\quad y = f(x) = x^{\frac{2}{3}},$

So, $\quad \dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}},$

So $\quad dy = \dfrac{1}{3}x^{-\frac{2}{3}}\,dx,$

which is an idealised version of the approximation

$$\Delta y \approx \frac{1}{3}x^{-\frac{2}{3}}\Delta x$$

In our particular problem, all the values of x are in the vicinity of 64. So, in forming this approximation, we take x equal to 64 so that y equals its cube root which is 4, and the approximation for Δy simplifies in a few steps to $\frac{\Delta x}{48}$.

Put $\quad x = 64, \quad$ so $\quad y = 64^{\frac{1}{3}} = 4,$

and

$$\Delta y \approx \frac{1}{3}(64^{-\frac{2}{3}})\Delta x = \frac{1}{3}(4^{-2})\Delta x = \frac{\Delta x}{48}$$

Now, 48 is close to 50, so we can approximate Δy even more coursely by $\frac{\Delta x}{50}$, if we want to. Of course, $\frac{\Delta x}{48}$ is more accurate, but $\frac{\Delta x}{50}$ may produce simpler rounded numbers if we're only looking for rough approximations. We have all the information we need to start making approximations. Let's estimate the cube root of 70. To move from 64 to 70, Δx, the change in x is 6. So, Δy should be approximately $\frac{6}{50}$ which is equal to 0.12. The cube root of 70 which is y plus the change in y, Δy, will be approximately $4 + 0.12$ which is 4.12.

Estimating $\quad \sqrt[3]{70} = 70^{\frac{1}{3}}$

$$\Delta x = 70 - 64 = 6, \quad \Delta y \approx \frac{6}{50} = 0.12,$$

so

$$\sqrt[3]{70} = y + \Delta y \approx 4 + 0.12 = 4.12.$$

(In fact $\quad \sqrt[3]{70} = 4.1212853\ldots$)

The cube root of 70 in fact has this decimal expansion. So, a rough and ready approximation is correct to two decimal places.

We will estimate the cube root of 65 using the same manipulations, but Δx now becomes one, and Δy is approximately $\frac{1}{50}$ which is equal to 0.02. So, the cube root of 65 which is $y+\Delta y$ now becomes $4+0.02$ which is 4.02. The true value again agrees with the approximation to two decimal places.

Estimating $\sqrt[3]{65} = 65^{\frac{1}{3}}$

$$\Delta x = 65 - 64 = 1, \quad \Delta y \approx \frac{1}{50} = 0.02,$$

so

$$\sqrt[3]{65} = y + \Delta y \approx 4 + 0.02 = 4.02.$$

(In fact $\sqrt[3]{70} = 4.020725759\ldots$)

If we use the more accurate estimate for Δy, that is, $\frac{\Delta x}{48}$, then our approximation for the cube root of 65 becomes more refined, which astonishingly agrees to the true value to almost four decimal places.

Finally, estimating the cube root of 63 this time Δx is -1 because the change from 64 to 63, we must take away one unit, and now Δy is approximately $\frac{-1}{50}$ which is negative 0.02. So, the cube root of 63 is approximately 4 minus 0.02 which is 3.98. Again, agreeing to the true value to two decimal places.

Estimating $\sqrt[3]{63} = 63^{\frac{1}{3}}$

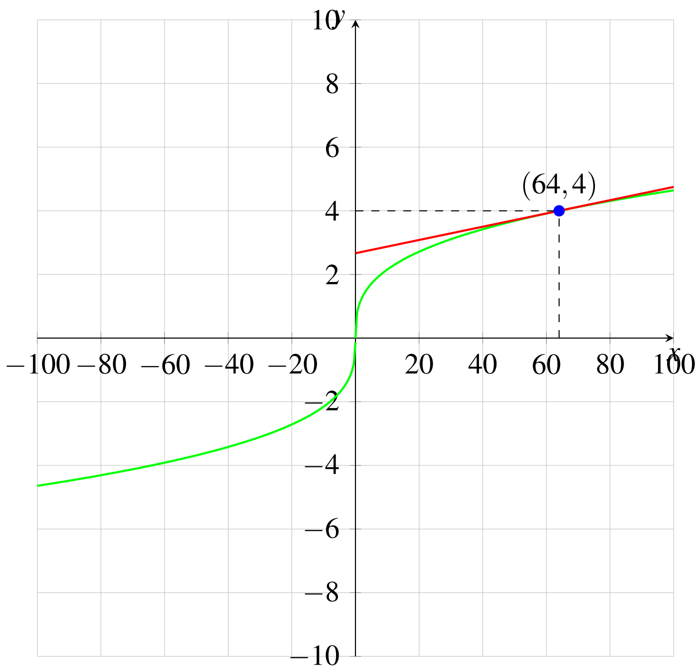
$$\Delta x = 63 - 64 = -1, \quad \Delta y \approx \frac{-1}{50} = -0.02,$$

so

$$\sqrt[3]{63} = y + \Delta y \approx 4 - 0.02 = 3.98.$$

(In fact $\sqrt[3]{70} = 3.975092\ldots$)

Using the more accurate approximation with 48 instead of 50 in the denominator, we get a more refined estimate which again agrees to the true value to four decimal places. To interpret what's happening visually, here's the graph of $y = x^{\frac{1}{3}}$.



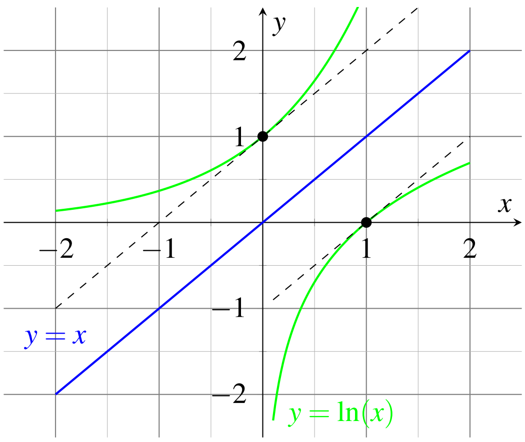
We're focusing on behavior near x equals 64 giving this point on the graph with y equals 4. The underlying idea is to take the tangent line to the curve at this point which is a very good approximation to the curve nearby. The value of the derivative at x equals 4 is in fact one of the 48 following from our earlier calculations, and it's not hard to find the equation of the tangent line which is$y = \frac{x}{48} + \frac{8}{3}$, and we can use this equation to estimate y values on the actual curve. If you plug in x equals 65 and x equals 63, the equation of the line produces the same approximations we found earlier correct to four decimal places implied by the differentials. In fact, using differentials and tangent lines to approximate points on the curve are entirely equivalent processes, just expressed with different notation.

18.2.2 Rule of Seventy

In this final example, we'll look at a rule of thumb used by bankers and investors to estimate how many years it takes for a sum of money, the principal, to double in value when invested with a given compounding annual interest rate. This is called the Rule of Seventy.

It estimates the number of years for the principle to double is 70 divided by the interest rate. For example, if the interest rate is 1%, then you expect to take 70 divided by 1, which is 70 years for the principle to double. If the interest rate is 2%, then you expect to take 70 divided by 2 which is 35 years to double. If the interest rate is 7%, then you expect to take 70 over 7 which is 10 years to double and so on. The rule of 70 is only a good estimate for small interest rates. Now why does it work? Though the rule itself is very simple, the underlying reason why it works is profound and relies ultimately on a property of Euler's number e.

Recall that the importance of e stems from the fact that the slope of the tangent line to the curve, $y = e^x$ at the y intercept is 1. So that when we reflect in the line y=x to form the inverse function, this tangent then becomes a tangent line to the curve $y = log(x)$ and its x intercept also of slope one.



The equation of this tangent line to the curve of the natural logarithm, you can check quickly is y= x-1. And it's a very good approximation for the curve near x=1. Thus lnx is approximately equal to x-1 for x close to 1. Recall that the derivative of $lnx$is$\frac{1}{x}$, so when x equals 1, the value of the derivative is exactly 1, the slope of the tangent line.

$\frac{d}{dx}(\ln x) = \frac{1}{x}$
so, for $x = 1$,
$\frac{d}{dx}(\ln x) = \frac{1}{1} = 1,$
so
$d(\ln x) = 1\,dx = dx.$

$\boxed{d(\ln x) = dx}$ $\boxed{\Delta(\ln x) \approx \Delta x}$

But
$\Delta(\ln x) = \ln(1+\Delta x) - \ln(1)$
$= \ln(1+\Delta x) - 0 = \ln(1+\Delta x)$
so

$\boxed{\ln(1+\Delta x) \approx \Delta x}$

or

$\boxed{\ln(1+\Delta) \approx \Delta}$

In terms of differentials, if we multiply through by dx, we get d (ln(x)) = dx. This becomes an approximation, $\Delta(\ln(x)$ is approximately equal to Δx, which we expect to be a good estimate when x is close to 1. Making room to continue, d$\Delta(\ln(x)$, the change of ln(x) is log(1) plus the change in x take way the log(1), which is just $\ln(1+\Delta x)$, because log(1) is 0. Combined with the above, this gives $\ln(1+\Delta x)$ is approximately equal to Δx. As a simple abstract statement, this just says that ln $\ln(1+\Delta x)$ is approximately equal to Δ for any small numbers Delta. If we want to estimate $\ln(1+\Delta)$ when Δ is small, we can just replace this expression by Δ itself!

What on Earth has this got to do with the rule of 70? Let P denote the initial principle being invested and i denote the interest rate. Let y=y(x) be the value of the investment after x years. So P is the value of the investment when x = 0 so that y(0) = P, and the general formula is $y = P(1+\frac{i}{100})^x$, which uses an exponential growth model as the investment uses compound interest.
We want the number of years x, after which the original principal, P, has doubled to 2P. So by the formula for y(x) we get this $y = P(1+\frac{i}{100})^x$ expression for P.

We can divide through by P and then take natural logs of both sides.But remember, taking logs brings the exponent down to the front. So the right hand side becomes x times ln (1) + i divided by 100, and then dividing through, x becomes this complicated looking fraction.

$$2P = y(x) = P\left(1+\frac{i}{100}\right)^x,$$
$$\text{i.e.}\quad 2 = \left(1+\frac{i}{100}\right)^x$$

Then

$$\ln 2 = \ln\left(1+\frac{i}{100}\right)^x = x\ln\left(1+\frac{i}{100}\right)$$

So

$$x = \frac{\ln 2}{\ln\left(1+\frac{i}{100}\right)}.$$

Now comes the magic. Recall, that $\ln(1+\Delta x)$ is approximately equal to Δ for small Δ. Here, take Δ to be $\frac{i}{100}$, which is a small number. And then we can replace $\ln(1+\frac{i}{100})$, up to an approximation.Thus x becomes approximately the simple fraction:

$$x = \frac{\ln 2}{fraci100)} \qquad = \frac{100\ln 2}{i}$$

The upshot is that to estimate the number of years, we just divide 100 ln(2) by the interest rate.But 100 ln(2) is 69.3 to one decimal place, which is close to 70. So the number of years is approximately 70 divided by the interest rate, which explains the rule of 70.
Now 100 ln(2) , in fact, rounds down to 69. So the number of years is also approximately 69 divided by the interest rate, known as the rule of 69, which is slightly more accurate then the rule of 70.And 69 is also not too far off 72, which is a nice number, and we get the so called rule of 72, which is less accurate, but often used as a rule of thumb if the interest rate is an integer that divides 72 exactly.
For example, if the interest rate is 6%, then 72 divided by 6 is 12, so expect about 12 years for the principle to double. It's slightly awkward to divide 6 into 70 or into 69, and in any case both answers round up to 12.

18.2.3 Examples and derivations

Let's see some examples to see how exactly the rule of 70, 72 and 69 works.

1. **Use the Rule of Seventy to estimate how long it takes for the principal to double when invested at $1\%, 5\%$, and 10% compound annual interest rates respectively.**

   Solution: We estimate that it takes $\frac{70}{1} = 70$ years, $\frac{70}{5} = 14$ years, and $\frac{70}{10} = 7$ years respectively for the principal to double.

2. **Estimate the annual interest rate $i\%$ (to the nearest tenth of one percent) required that a given principal should double within about 25 years.**

   Solution: We want i such that $25 \approx \frac{70}{i}$, that is,

   $$i \approx \frac{70}{25} \approx 2.8.$$

   Thus, we estimate that using an interest rate of about 2.8%, the principal should double in about 25 years.

3. **We explain where the Rule of Seventy comes from.**

Let P be the principal that has been invested initially at the compound interest rate of $i\%$ per annum. Let $y = y(t)$ be the value of the investment after t years, so $P = y(0)$ and

$$y = y(t) = P\left(1+\frac{i}{100}\right)^t.$$

We want to estimate the number of years t such that $y(t) = 2P$, that is,

$$2P = P\left(1+\frac{i}{100}\right)^t,$$

which becomes, after cancelling P from both sides,

$$2 = \left(1+\frac{i}{100}\right)^t.$$

Taking natural logarithms of both sides, we get

$$\ln 2 = t\ln\left(1+\frac{i}{100}\right).$$

Rearranging, and using the approximation $\ln(1+\Delta) \approx \Delta$ noted earlier, taking $\Delta = \frac{i}{100}$ we get

$$t = \frac{\ln 2}{\ln\left(1+\frac{i}{100}\right)} \approx \frac{\ln 2}{\frac{i}{100}} = \frac{100\ln 2}{i},$$

which we expect to be a good approximation for small i. But the numerator can be approximated by 69, 70, and 72:

$$t = \frac{100\ln 2}{i} \approx \frac{69.3}{i} \approx \frac{69}{i} \approx \frac{70}{i} \approx \frac{72}{i}.$$

This gives rise to the Rule of Seventy, by dividing it into 70.

It also leads to the Rule of Sixty-nine (slightly more accurate), by dividing it into 69, and the Rule of Seventy-two (the least accurate), by dividing it into 72.

The Rule of Seventy-two is a convenient rule of thumb if the interest rate happens to exactly divide 72. For example, if the principal is invested at 6% per annum, then the rule estimates that it takes about $\frac{72}{6} = 12$ years to double. It is slightly awkward to divide 69 into 6.9 or 70, and in both cases, the answer rounds up to 12.

4. **Find an equation relating the differentials dy and dx, and an associated approximation relating small changes Δy and Δx, when $y = x^2$.**

Solution: We have $\frac{dy}{dx} = y' = 2x$, so that

$$dy = 2x\,dx \quad \text{and} \quad \Delta y \approx 2x\,\Delta x.$$

Let y (square metres) denote the area of the paddock of side length x (metres), so $y = x^2$. We are considering making small changes to $x = 100$ and $y = 100^2 = 10{,}000$.

In the first case, $\Delta x = 10$, and by our formula,

$$\Delta y \approx 2x\Delta x = 2 \times 100 \times 10 = 2,000,$$

so we expect an increase of about 2,000 square metres. But

$$y(110) - y(100) = 110^2 - 100^2 = 2,100,$$

so the true change in area is 2,100 square metres, which agrees with our estimate to within 100 square metres.

In the second case, $\Delta x = 5$, and by our formula,

$$\Delta y \approx 2x\Delta x = 2 \times 100 \times 5 = 1,000,$$

so we expect an increase of about 1,000 square metres. But

$$y(105) - y(100) = 105^2 - 100^2 = 1,025,$$

so the true change in area is 1,025 square metres, which agrees with our estimate to within 25 square metres.

In the third case, $\Delta x = 1$, and by our formula,

$$\Delta y \approx 2x\Delta x = 2 \times 100 \times 1 = 200,$$

so we expect an increase of about 200 square metres. But

$$y(101) - y(100) = 101^2 - 100^2 = 201,$$

so the true change in area is 201 square metres, which agrees with our estimate to within 1 square metre.

In the fourth case, $\Delta x = 0.01$ (since 1 cm equals 0.01 m), and by our formula,

$$\Delta y \approx 2x\Delta x = 2 \times 100 \times 0.01 = 2,$$

so we expect an increase of about 2 square metres. But

$$y(100.01) - y(100) = 100.01^2 - 100^2 = 2.0001.$$

so the true change in area is 2.0001 square metres, which agrees with our estimate to within 0.0001 square metres.

5. **Find the equation of the tangent line to the curve $y = x^3$ at the point (3,9) and use it to estimate 3.01^3, and compare the estimate with the true value.**

   Solution: We have $y' = 3x^2$, so the slope of the tangent line is $2 \times 3 = 6$. Hence the tangent line must have equation

   $$y = 6x + k,$$

   for some constant k. But the point (3,9) lies on this line, so 9 = 6(3) + k = 18 + k, so that $k = -9$. Hence the equation of the tangent line is

   $$y = 6x - 9.$$

Using the tangent line, with input $x = 3.01$, gives the estimate

$$3.01^3 \approx 6(3.01) - 9 = 9.06.$$

In fact, $3.01^3 \approx 9.0601$, which agrees with the estimate to three decimal places.

6. **Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point (9,3) and use it to estimate $\sqrt{9.5}$ and $\sqrt{8.5}$, and compare with the true values.**

Solution: We have $y' = \frac{1}{2\sqrt{x}}$ so the slope of the tangent line is $\frac{1}{6}$. Hence the tangent line must have equation

$$y = \frac{x}{6} + k,$$

for some constant k. But the point (9,3) lies on this line, so $3 = \frac{9}{6} + k = \frac{3}{2} + k$, so that $k = \frac{3}{2}$. Hence the equation of the tangent line is

$$y = \frac{x}{6} + \frac{3}{2}.$$

Using the tangent line, with input $x = 9.5$, gives the estimate

$$\sqrt{9.5} \approx \frac{9.5+9}{6} = \frac{18.5}{6} = 3.0833.$$

In fact, $\sqrt{9.5} \approx 3.0822$, to four decimal places, which agrees with the estimate to two decimal places.

7. **Find the equation of the tangent line to the curve $y = x^{1/3}$ at the point (64,4) and use it to estimate $\sqrt[3]{70}$, $\sqrt[3]{65}$ and $\sqrt[3]{63}$, and compare with the true values.**

Solution: We have $y' = \frac{1}{3}x^{-2/3}$, so the slope of the tangent line is $\frac{1}{3} \times 64^{-2/3} = \frac{1}{48}$. Hence the tangent line must have equation

$$y = \frac{x}{48} + k,$$

for some constant k. But the point (64,8) lies on this line, so

$$4 = \frac{64}{48} + k = \frac{4}{3} + k, \text{ so that } k = \frac{8}{3}.$$

Hence the equation of the tangent line is

$$y = \frac{x+128}{48}.$$

Using the tangent line, with input $x = 70$, gives the estimate

$$\sqrt[3]{70} \approx \frac{70+128}{48} = 4.125.$$

In fact, $\sqrt[3]{70} \approx 4.1213$, to four decimal places, which agrees with the estimate to almost two decimal places.

Using the tangent line, with input $x = 65$, gives the estimate

$$\sqrt[3]{65} \approx \frac{65+128}{48} = 4.02083.$$

In fact, $\sqrt[3]{65} \approx 4.0207$, to four decimal places, which agrees with the estimate to almost four decimal places.

Using the tangent line, with input $x = 63$, gives the estimate

$$\sqrt[3]{63} \approx \frac{63+128}{48} = 3.97916.$$

In fact, $\sqrt[3]{63} \approx 3.9791$, to four decimal places, which again agrees with the estimate to almost four decimal places.

8. **Imagine that you are manufacturing metal ball bearings in the shape of spheres with radius 5 mm, with up to 2% error in the radius. The problem is to use differentials to estimate the percentage error in volume.**

   Solution: The volume $V = V(r)$ of a sphere of radius r is given by the formula

   $$V = V(r) = \frac{4\pi r^3}{3}.$$

   Hence $dV/dr = 4\pi r^2$, so that $dV = 4\pi r^2 dr$, becoming the following approximation, relating small changes ΔV in the volume to small changes Δr in the radius:

   $$\Delta V \approx 4\pi r^2 \Delta r.$$

   In our application, $r = 5$. We are not sure whether the fluctuations in the radius cause it to be smaller or greater than $r = 5$. To capture all possibilities, we use the magnitude of Δr, and express the given information in terms of a bound on the true proportion of Δr that this error in the radius represents:

   $$\left|\frac{\Delta r}{r}\right| \leq 0.02.$$

   Working now with the proportion of V that the magnitude of ΔV represents, we have

   $$\left|\frac{\Delta V}{V}\right| \approx \frac{4\pi r^2 \Delta r}{\frac{4\pi r^3}{3}} = \frac{3\Delta r}{r} \leq 3 \times 0.02 = 0.06.$$

   Hence an estimate for the volume of the ball-bearings is up to about 6%. Notice that this final answer is independent of the value of the radius r.

This is the final section for module three, and we've come so far so quickly in transition from pre-calculus. We're well set up now to launch into the main techniques and applications of differential calculus in module four, and integral calculus in module five.

We began this module by discussing average rates of change. And especially focused on interpretations as average speed or velocity in the context of distance or displacement functions. What you see on the speedometer of a car is a physical representation of what we think of as instantaneous velocity, which in mathematical terms is the limiting behavior of average velocity when the time intervals become vanishingly small. All of this has a general interpretation in terms of slopes of tangent lines to curves, which we saw were described as limits of slopes of secants. The slope of the tangent line is called the derivative. his leads naturally into limit definitions of the derivative and several natural equivalent formulations, including Leibniz notation, which expresses the derivative in terms of differentials.

We discussed a variety of possible behaviors of curves of functions, captured succinctly using variations of limit notation, including examples of asymptotic behavior. We also discussed the important notion of continuity. Now today we exploited the notation of Leibnitz for the derivative to see how to translate idealized equations involving differentials, into approximations involving small changes in the variables expressed by Δ notation. This has some surprising applications such as explaining the rule of Seventy. Please read the notes, and when you're ready, please attempt the exercises.

18.2.4 Practice Quiz

Question 1

Use the Rule of Seventy to estimate how long it takes (to the nearest year) for principal to double when invested at 3 per cent annual compound interest.

(a) 23 years
(b) 20 years
(c) 22 years
(d) 21 years
(e) 24 years

Question 2

Use the Rule of Seventy to estimate the annual compound interest rate (to one decimal place) required for invested principal to double in value after 50 years have elapsed.

(a) 1.2
(b) 1.4
(c) 1.3
(d) 1.5
(e) 1.6

Question 3

Which one of the following equations involving differentials comes from the rule $y = 3x^2$?

(a) $dy = 6dx$
(b) $dz = 6dy$
(c) $dx = 6dy$
(d) $dy = 6xdx$
(e) $dy = 3dx$

Question 4

Which one of the following equations involving differentials comes from the rule $x = \sqrt{t}$?

(a) $dt = \frac{dx}{2\sqrt{t}}$
(b) $dx = \frac{dt}{\sqrt{t}}$
(c) $dt = \frac{2dx}{\sqrt{t}}$
(d) $dx = \frac{dt}{2\sqrt{t}}$
(e) $dt = \frac{2dx}{\sqrt{x}}$

Question 5

Which one of the following approximations involving ΔA and Δr comes from the formula $A = \pi r^2$ for the area A of a circle of radius r?

(a) $\Delta A \approx \pi \Delta r$
(b) $\Delta A \approx \pi r \Delta r$
(c) $\Delta r \approx \frac{\pi \Delta A}{r}$
(d) $\Delta A \approx 2\pi r \Delta r$
(e) $\Delta A \approx 2r\Delta r$

Question 6

Which one of the following approximations involving ΔV and Δx comes from the formula $V = x^3$ for the volume V of a cube of side length x?

(a) $\Delta V \approx 6\Delta x$
(b) $\Delta V \approx 3x^2 \Delta x$
(c) $\Delta V \approx 6x\Delta x$
(d) $\Delta V \approx 3x\Delta x^2$
(e) $\Delta V \approx 3x^2 \Delta x$

Question 7

Cubes of volume $V = x^3$ of side length x (in some appropriate units) are manufactured using a material that produces up to 1.5% error in the volume. Estimate an upper bound for the error in the side lengths of the cubes that come out of this process.

(a) 3%
(b) 4.5%
(c) 0.5%
(d) 5%
(e) 1.5%

Question 8

Find the equation of the tangent line to the curve $y = e^x$ at the point $(0,1)$.

(a) $y = ex + 1$
(b) $y = 1 - x$
(c) $y = e^{-x} - 1$
(d) $y = x + 1$
(e) $y = -x + 1$

Question 9

Find the equation of the tangent line to the curve $y = x^2$ at the point $(10,100)$.

(a) $y = 20x - 100$
(b) $y = 20x + 100$
(c) $y = x + 90$
(d) $y = 10x$
(e) $y = 20x$

Question 10

Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(36,6)$.

(a) $y = 12x - 36$
(b) $y = \frac{x-36}{12}$
(c) $y = \frac{x+36}{12}$
(d) $y = 12x + 36$
(e) $y = \frac{x+72}{18}$

Answers

The answers will be revealed at the end of the module.

19. Assessment

19.1 Module Quiz

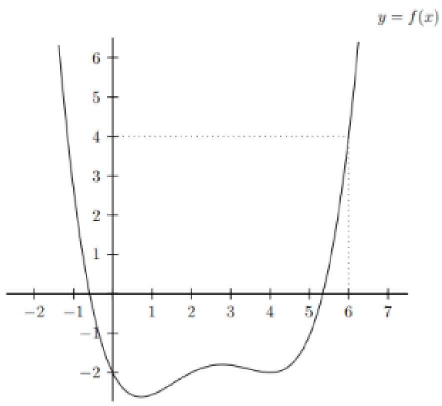
Question 1

A person drives in a car from Town A to Town B. The journey lasts 2 hours and 20 minutes. The distance between the towns is 126 kilometres. Find the average speed for the journey in kilometres per hour (kph).

(a) 57 kph
(b) 58 kph
(c) 56 kph
(d) 54 kph
(e) 55 kph

Question 2

Consider a function f whose graph appears below. Find the average rate of change of f over the interval $[0, 6]$.



(a) $\frac{3}{2}$
(b) 1
(c) 0
(d) -1
(e) $\frac{2}{3}$

Question 3

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10.$$

Estimate the height of the missile above the ground six seconds after launching.

(a) 430 m
(b) 400 m
(c) 420 m
(d) 390 m
(e) 410 m

Question 4

A missile is launched vertically in the air from a platform with displacement function $z = z(t)$ meters above the ground, at time t seconds after launching, approximated by the formula

$$z(t) = -5t^2 + 100t + 10.$$

Estimate the average velocity of the missile from $t = 0$ seconds to $t = 5$ seconds.

(a) 75 m/sec
(b) 90 m/sec
(c) 80 m/sec
(d) 96 m/sec
(e) 85 m/sec

Question 5

Evaluate the following limit:

$$\lim_{x\to 2} 2x - 3.$$

(a) -2
(b) -1
(c) 0
(d) 1
(e) 2

Question 6

Evaluate the following limit:

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5}.$$

(a) 0
(b) 10
(c) -10
(d) 5
(e) -5

Question 7

Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} \frac{x^2-16}{x-4} & \text{if } x \neq 4 \\ k & \text{if } x = 4 \end{cases}.$$

(a) $k = -16$
(b) $k = 16$
(c) $k = 8$
(d) $k = -12$
(e) $k = 12$

Question 8

Find the vertical asymptote for the following curve:

$$y = \frac{x^2 - 1}{x - 4}.$$

(a) $x = -1$
(b) $x = 4$
(c) $x = 1$
(d) $x = -4$
(e) $x = 0$

Question 9

Find the oblique asymptote for the following curve:

$$y = \frac{x^2 - 4}{x - 5}.$$

(a) $y = x - 2$
(b) $y = x + 2$
(c) $y = x - 5$
(d) $y = x + 5$
(e) $y = x$

Question 10

Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sin(2x)}{4x}.$$

(a) $\frac{1}{2\sqrt{2}}$
(b) $\frac{1}{2}$
(c) $\frac{1}{4}$
(d) $\frac{1}{2}$
(e) 0

Question 11

Evaluate the following limit:

$$\lim_{x \to -\infty} \frac{6x^3 - 5x^2 + 2x - 1}{2x^3 + x^2 + x + 1}.$$

(a) 0
(b) 6
(c) -1
(d) 3
(e) ∞

Question 12

Consider the parabola $y = 4x^2$. Find the slope of the tangent line at the point $(3, 36)$.

(a) 72
(b) 24
(c) 12
(d) 6
(e) 36

Question 13

Find $f'(3)$ when $f(x) = x^3$.

(a) 27
(b) 9
(c) 3
(d) 6
(e) 12

Question 14

Find $f'(x)$ when $f(x) = 4e^x$.

(a) 4e
(b) $4e^{x-1}$
(c) $4e^{-1}$
(d) $4e^x$
(e) $4xe^x$

Question 15

Find $f''(x)$ when $f(x) = 5x^3 - 3x^2 + 7x - 2$.

(a) $15x - 6$
(b) $15x - 3$
(c) $30x - 3$
(d) $30x - 12$
(e) $30x - 6$

Question 16

Find $\frac{dx}{dt}$ when $x = 7\ln t$.

(a) $\frac{1}{7t}$
(b) 1
(c) $\frac{t}{7}$
(d) $\frac{7}{t}$
(e) $\frac{7}{t}$

Question 17

Find $\frac{du}{d\theta}$ when $u = -4\cos\theta$.

(a) $\frac{-\sin\theta}{4}$
(b) $4\sin\theta$
(c) $\sin\theta$
(d) $-4\sin\theta$
(e) $\frac{4}{\sin\theta}$

Question 18

Find $\frac{dx}{dy}$ when $y = 9x^2$ (assuming $x \geq 0$).

(a) $\frac{1}{3\sqrt{y}}$
(b) $\frac{9}{2\sqrt{y}}$
(c) $\frac{3}{2\sqrt{y}}$
(d) $\frac{1}{6\sqrt{y}}$
(e) $18x$

Question 19

Which one of the following equations involving differentials comes from the rule $y = 4x^2$?
(a) $dx = 8ydy$
(b) $dy = 4xdx$
(c) $dy = 8xdx$
(d) $dx = 8dy$
(e) $dy = 8dx$

Question 20

Find the equation of the tangent line to the curve $y = 2x^2$ at the point $(5,50)$.
(a) $y = 20x - 50$
(b) $y = 11x$
(c) $y = 20x + 50$
(d) $y = 11x - 5$
(e) $y = 9x + 5$

Answers

The answers will be revealed at the end of the module.

20. Answer Key

20.1 Parabolas and Quadratics

Answers

1 (a)
2 (c)
3 (d)
4 (a)
5 (c)
6 (c)
7 (e)
8 (c)
9 (d)
10 (e)

20.2 The Quadratic Formula

Answers

1 (e)
2 (a)
3 (c)
4 (d)
5 (c)
6 (d)
7 (e)
8 (c)
9 (b)
10 (b)

20.3 Functions as Rules, with Domain, Range, and Graph

Answers

1 (c)
2 (a)
3 (d)
4 (a)
5 (b)
6 (d)
7 (e)
8 (a)
9 (d)
10 (b)

20.4 Polynomial and Power Functions

Answers

1 (b)	5 (a)	9 (c)
2 (e)	6 (e)	10 (e)
3 (d)	7 (e)	
4 (a)	8 (a)	

20.5 Composite Functions

Answers

1 (c)	5 (c)	9 (d)
2 (d)	6 (a)	10 (c)
3 (c)	7 (b)	
4 (e)	8 (b)	

20.6 Inverse Functions

Answers

1 (b)	5 (b)	9 (d)
2 (a)	6 (a)	10 (b)
3 (d)	7 (b)	
4 (c)	8 (d)	

20.7 The Exponential Function

Answers

1 (c)	5 (d)	9 (c)
2 (d)	6 (e)	10 (c)
3 (a)	7 (b)	
4 (b)	8 (a)	

20.8 The Logarithmic Function

Answers

1 (e)	5 (a)	9 (c)
2 (d)	6 (a)	10 (c)
3 (a)	7 (a)	
4 (c)	8 (b)	

20.9 Exponential Growth and Decay

Answers

1 (b)	5 (d)	9 (b)
2 (d)	6 (b)	10 (c)
3 (b)	7 (c)	
4 (a)	8 (b)	

20.10 Sine, Cosine, and Tangent

Answers

1 (a)
2 (c)
3 (d)
4 (c)
5 (b)
6 (e)
7 (c)
8 (a)
9 (b)
10 (e)

20.11 The Unit Circle and Trigonometry

Answers

1 (c)
2 (a)
3 (b)
4 (d)
5 (d)
6 (e)
7 (b)
8 (c)
9 (c)
10 (a)

20.12 Inverse Circular Functions

Answers

1 (b)
2 (a)
3 (e)
4 (d)
5 (b)
6 (e)
7 (b)
8 (a)
9 (c)
10 (e)

20.13 Assessment

Answers

1 (d)
2 (a)
3 (a)
4 (d)
5 (d)
6 (c)
7 (b)
8 (c)
9 (c)
10 (d)
11 (b)
12 (e)
13 (d)
14 (e)
15 (e)
16 (c)
17 (b)
18 (b)
19 (a)
20 (e)

20.14 Slope and Average Rate of Change

Answers

1 (d)
2 (c)
3 (d)
4 (a)
5 (d)
6 (a)
7 (a)
8 (a)
9 (e)
10 (b)

20.15 Displacement, Velocity, and Acceleration

Answers

1 (b)
2 (b)
3 (e)
4 (b)
5 (d)
6 (c)
7 (a)
8 (a)
9 (e)
10 (b)

20.16 Tangent Lines and Secants

Answers

1 (e)
2 (d)
3 (a)
4 (c)
5 (c)
6 (a)
7 (b)
8 (c)
9 (a)
10 (a)

20.17 Different Kinds of Limits

Answers

1 (d)
2 (a)
3 (c)
4 (d)
5 (d)
6 (e)
7 (e)
8 (b)
9 (a)
10 (e)

20.18 Limits and Continuity

Answers

1 (d)
2 (b)
3 (c)
4 (b)
5 (d)
6 (e)
7 (b)
8 (e)
9 (a)
10 (b)

20.19 The Derivative as a Limit

Answers

1 (e)
2 (a)
3 (b)
4 (b)
5 (e)
6 (e)
7 (c)
8 (b)
9 (d)
10 (a)

20.20 Finding Derivatives from First Principles

Answers

1 (a)	5 (c)	9 (d)
2 (c)	6 (c)	10 (b)
3 (b)	7 (d)	
4 (c)	8 (d)	

20.21 Leibniz Notation

Answers

1 (b)	5 (c)	9 (d)
2 (b)	6 (d)	10 (d)
3 (a)	7 (a)	
4 (b)	8 (b)	

20.22 Differentials and Applications

Answers

1 (e)	5 (d)	9 (a)
2 (b)	6 (b)	10 (c)
3 (d)	7 (a)	
4 (d)	8 (d)	

20.23 Assessment

Answers

1 (d)	8 (b)	15 (e)
2 (e)	9 (c)	16 (d)
3 (a)	10 (e)	17 (b)
4 (a)	11 (d)	18 (d)
5 (d)	12 (b)	19 (c)
6 (b)	13 (a)	20 (a)
7 (c)	14 (d)	

IV

erties and Applications of Derivative

This module continues the development of differential calculus by introducing the first and second derivatives of a function. We use sign diagrams of the first and second derivatives and from this, develop a systematic protocol for curve sketching. The module also introduces rules for finding derivatives of complicated functions built from simpler functions, using the Chain Rule, the Product Rule, and the Quotient Rule, and how to exploit information about the derivative to solve difficult optimisation problems.

Learning Objectives

- connect properties of curves with their derivatives and second derivatives
- use sign diagrams to find extreme values, local maxima and minima and inflections of curves
- use and apply the Product, Quotient and Chain rules for differentiation
- solve maximisation and minimisation problems

21. Introduction

21.1 Introduction to Module 4

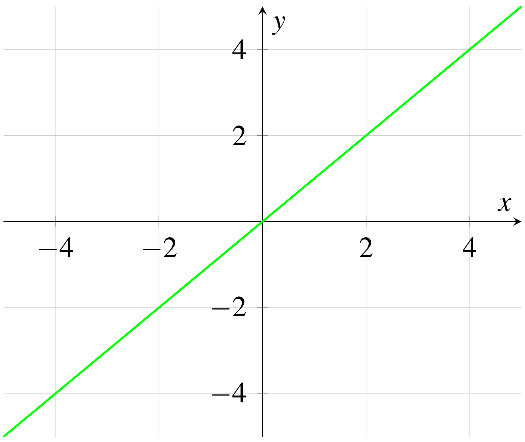
You've just completed the third module introducing the derivative and Leibniz notation. This next module continues the development of differential calculus by introducing you to a myriad of techniques and ways of exploiting the first and second derivatives of the function. You'll learn how to recognize when a function is increasing or decreasing by looking at the sign diagram of the first derivative, indicating turning points associated with local and global maxima and minima and how to recognize when a curve is concave up or concave down by looking at the sign diagram of the second derivative, indicating points of inflection. You'll develop a systematic protocol for curve sketching including information gleaned from the sign diagrams for the first and second derivatives, as well as information about x and y intercepts and any asymptotic behavior, which tells us what happens when quantities get very large in magnitude. You'll learn rules for finding derivatives of complicated functions built from simple functions using the Chain Rule, the Product Rule, and the Quotient Rule, and you'll be up to exploit information about the derivative to solve difficult optimization problems. Again, we hope that you find the material interesting and stimulating, that you find the videos helpful, and that the practice and challenges provided by the many exercises are beneficial. I look forward very much to your continued attention and participation.

22. First Derivatives and Turning Point

22.1 Increasing and Decreasing Functions

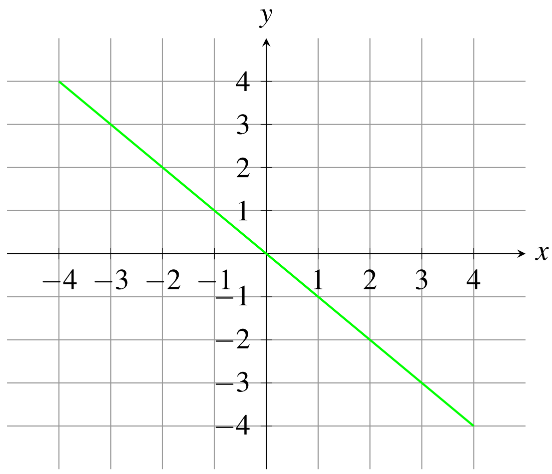
In this section, we discuss what it means for a function to be increasing or decreasing, provide some contrasting examples and relate these ideas to the sign of the derivative. That is, whether the derivative is positive or negative over a given interval. We say that a function f given by rule y equals f(x) is increasing if the outputs y gets bigger as the inputs x get bigger. By getting bigger, we mean moving in the positive direction, to the right along the x-axis and upwards along the y-axis.

The simplest example would be the function y = x, whose curve is a straight line sloping upwards.

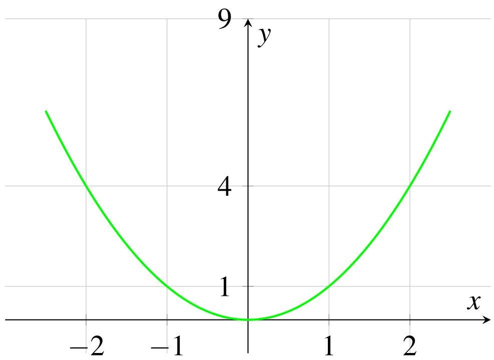


Naturally, as x gets bigger, y = x also gets bigger. A more precise definition involves, the less than relation. If a and b are two numbers and a < b, then b > a. Thus, y = f(x) is the rule for an increasing function if a < b implies f(a) < f(b) for inputs a and b. This makes precise the idea that as x gets bigger, going from a to b, then f(x) also gets bigger going from f(a) to f(b). Often, the contexts in which the rule of the function is being used restrict the inputs. Typically, a and b will come from some interval on the real line that is of particular relevance for a given problem. This will become important later when we interpret sign diagrams and sketch curves.

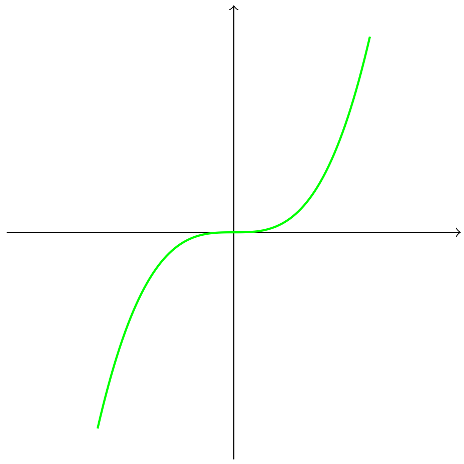
We say that a function f given by the rule y = f(x) is decreasing if the outputs y gets "smaller" as inputs x get "bigger". By "getting smaller", we mean moving in the negative direction to the left along the x axis and downwards along the y axis. A simple example would be the function y = -x, whose curve is a straight line sloping downwards.



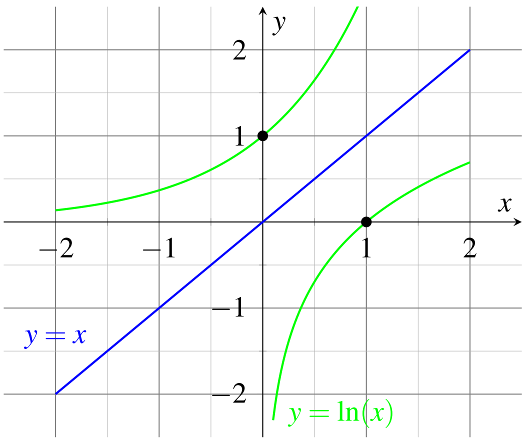
As x gets "bigger" in the sense of moving in the positive direction y equal to minus x gets "smaller" in the sense of moving in the negative direction. More precisely, y = f(x) is the rule for a decreasing function if a < b implies f(a)> f(b) for inputs a and b. Note how the inequality less than for inputs turns around into greater than for outputs. Functions whose graphs are lines fall into three classes. They're increasing if the slope of the line is positive, decreasing if the slope is negative or neither increasing nor decreasing if the slope is zero, that is, if the function is constant. Consider the simplest quadratic function $y = x^2$, whose graph is a parabola with lowest point at the origin. The function is increasing, if inputs are restricted to being greater than or equal to zero, but decreasing if inputs are restricted to being less than or equal to zero. If inputs are allowed to be taken from anywhere on the real line, then the function is neither increasing nor decreasing.



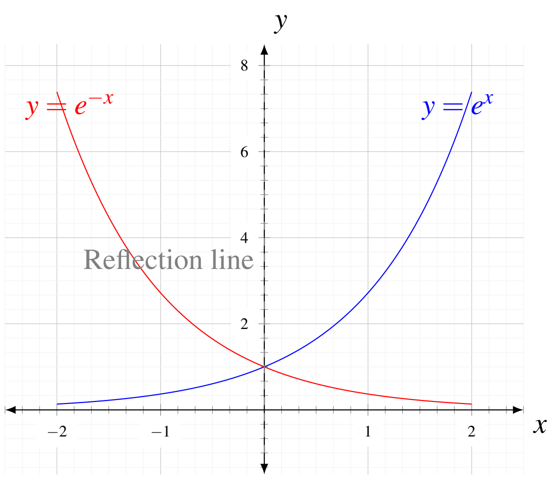
Consider the simplest cubic polynomial function $y = x^3$, whose graph looks like this with an inflection at the origin where the tangent line passes through the curve. This function is increasing on all of the real line even though it seems to flatten out momentarily at the origin. It's always the case that if a < b, then $a^3 < b^3$.



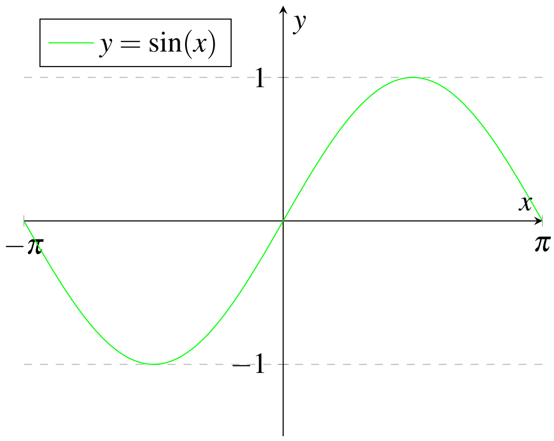
Here are the graphs, the natural logarithm and exponential functions on the same diagram, they form a natural pair of increasing functions obtained from each other by reflecting on the line y = x.

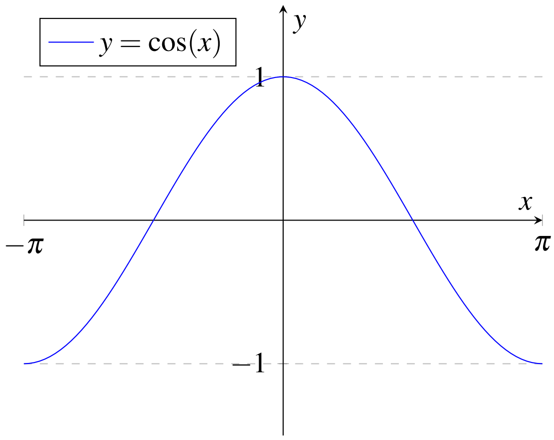


The exponential function is increasing on all of the real line without any restriction on inputs. The natural logarithm function, however, is increasing only over the positive reals. Remember, you aren't allowed to take the logarithm of zero or of a negative number. The fact that both are increasing on their respective domains is not an accident, it's a nice fact, elaborated on a bit later, that if f is increasing then the inverse function f^{-1} is also increasing. The function associated with exponential decay y = e^{-x} is by contrast decreasing on all of the real line. This is to be expected because it's the result of reflecting the graph of the increasing function y = e^x in the y-axis. Reflecting graphs in the y-axis interchanges increasing and decreasing functions.

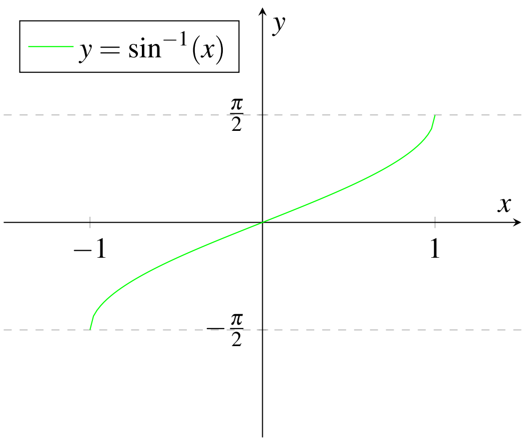


The circular functions $y = sinx$ and $y = cosx$ undulate forever backwards and forwards. So, neither increasing nor decreasing over the real line.

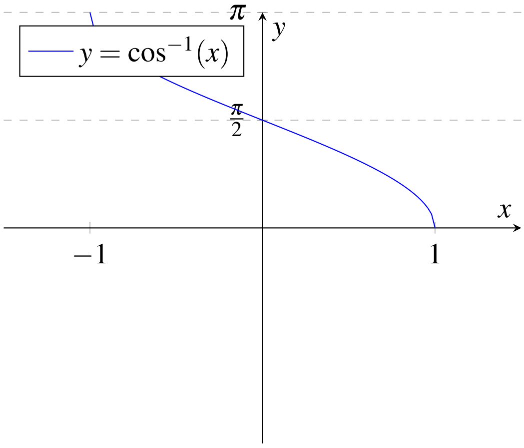




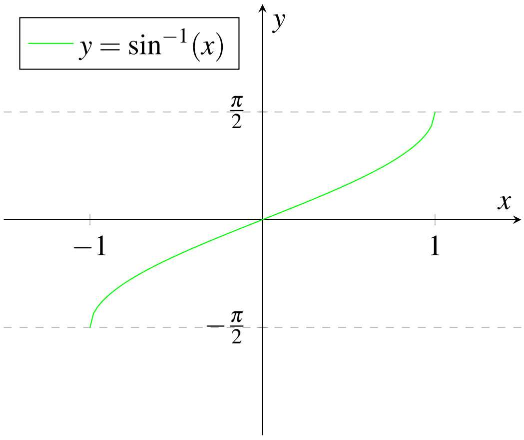
If we restrict the domain of the sine curve to the interval from $-\frac{\Pi}{2}$ to $\frac{\Pi}{2}$, we get a fragment of the curve which now represents an increasing function. This is the standard restriction that's used to invert the sine function.



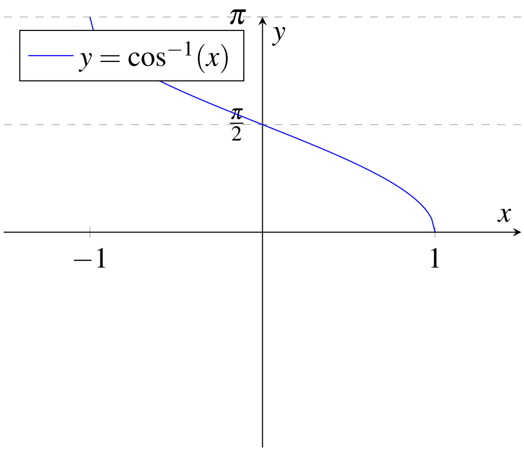
If we restrict the domain of the cosine curve to the integral from 0 to Π we get a fragment of the curve which now represents a decreasing function. This is the standard restriction used to invert the cosine function.



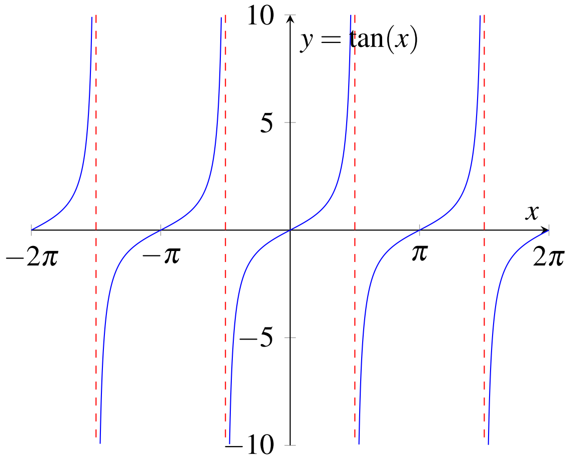
It's a general fact that increasing and decreasing functions are invertible because they satisfy the horizontal line test and these restrictions of the sine and cosine functions are two such examples. The natural exponential and logarithm functions are also examples that we've remarked upon earlier. It's a general fact explaining the notes that the inverse of an increasing function is increasing and the inverse of a decreasing function is decreasing. You can see how this works geometrically with these restrictions of sine and cosine. For example, take the restriction of the sine curve which is increasing and the line y = x reflect and you produce the inverse sine function which is also increasing.



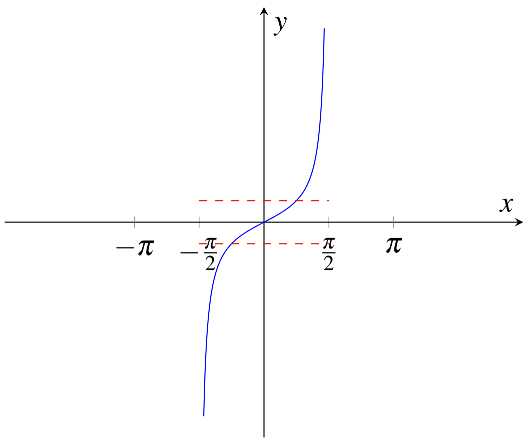
You should take the restriction of the cosine curve which is decreasing and the line y= x, make be a bit more room in the vertical direction, reflect and you produce the inverse cosine function, which is also decreasing.



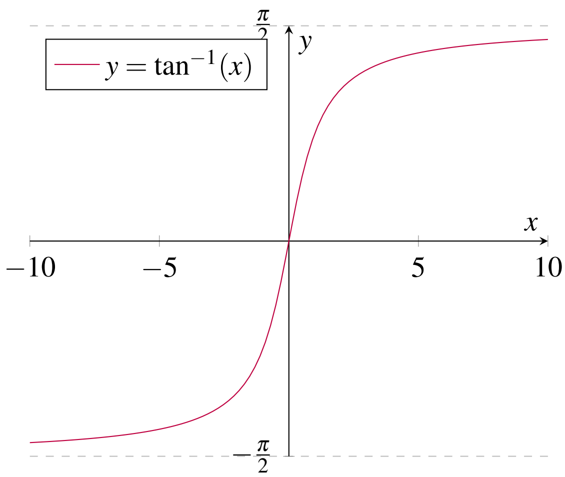
Here's the graph of the tan function which is neither increasing nor decreasing over its entire domain. Though it's comprised of these infinitely repeating pieces sandwiched between and separated by vertical asymptotes and each of these pieces represents an increasing function.



We will take one of the asymptotes here, the result of restricting the tan function to the interval strictly between $-\frac{-\Pi}{2}$ and $\frac{-\Pi}{2}$. This is the standard restriction to make the tan function invertible and the result is an increasing function.



Here's the graph of the inverse tan function which is also increasing.



22.1.1 Physical examples

What about a physical example drawn from real life? Here's a curve we looked at in an earlier section that describes the distance function of a car trip from Sydney to Melbourne.

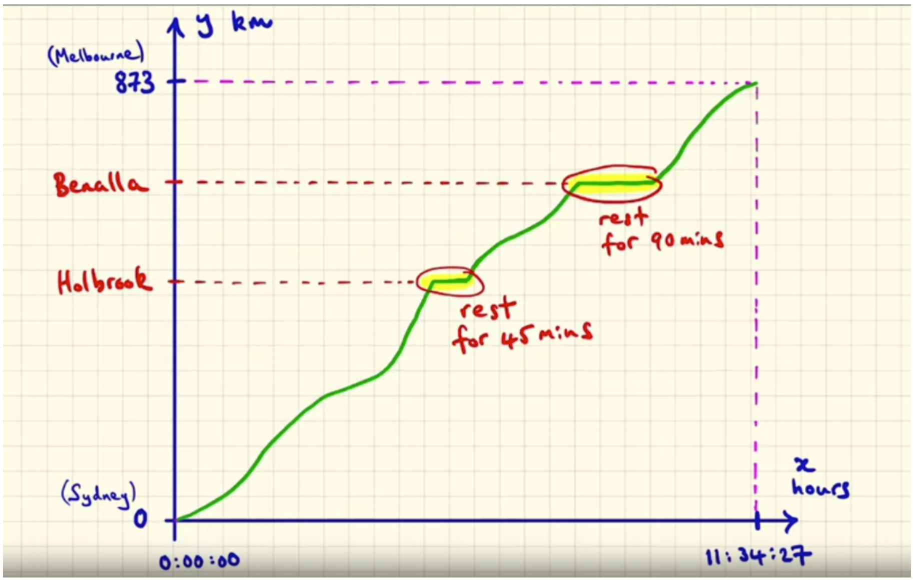


Figure 22.1: Function of car trip

While it's generally sloped upwards as you move from left to right, there are two places where the curve is flat corresponding to rest periods in the trip with the car isn't moving.

22.1.2 Connecting the Dots

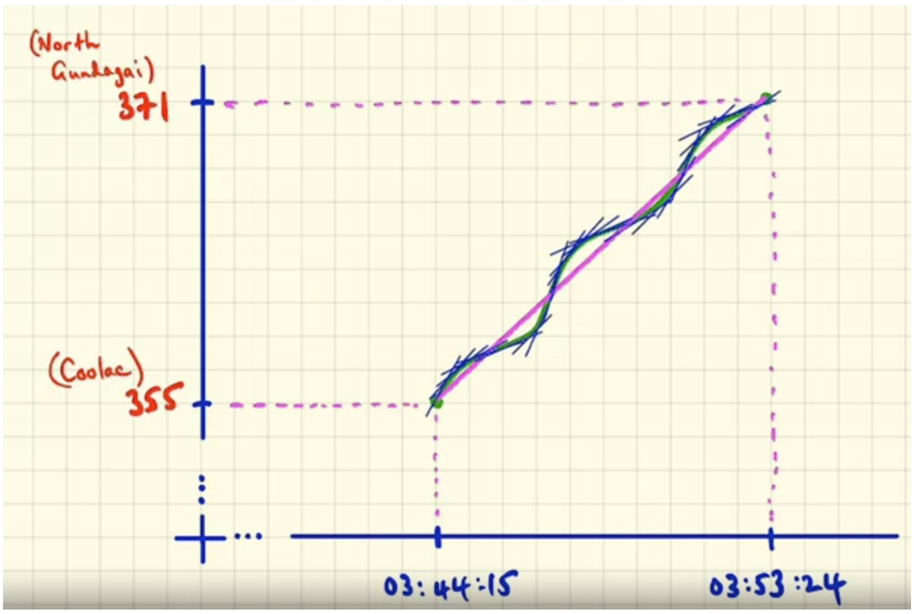


Figure 22.2: Demonstration of where curve is flat

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

So, overall the function is not increasing over the entire time interval. But there are many intervals of time on that trip when the car is moving all the time such as this section from Coolac to North Gundagai and over that time period, the function is increasing. If you're inside the car, but not looking outside, you could be sure the car is actually moving because you could always say a positive velocity on the speedometer, even though it might be fluctuating from moment to moment. These positive velocities correspond to slopes of tangent lines to the curve which are the derivatives of the function. It's clear then that if all the derivatives are positive, then the car is moving at positive velocities, moving closer to North Gundagai from Coolac, which just means, in our mathematics parlance that the associated function is increasing. This is a physical demonstration of the following mathematical fact. **If the derivative of a function $f(x)$ is positive on an interval, then the function is increasing. We have correspondingly, if the derivative of a function $f(x)$ is negative on an interval then the function is decreasing.** You can think of this physically in terms of a car moving backwards in the reverse direction with negative velocities.

Consider the function with rule $f(x) = x^3 - x$.

Let $f(x) = x^3 - x$.

Then

$$f'(x) = 3x^2 - 1$$

If $3x^2 > 1$ then $f'(x) > 0$, and this occurs if $x^2 > \frac{1}{3}$, i.e.,

$$x > \frac{1}{\sqrt{3}} \text{ or } x < -\frac{1}{\sqrt{3}}$$

Thus the curve $y = x^3 - x$ will be increasing on the intervals

$$\left(\frac{1}{\sqrt{3}}, \infty\right) \text{ and } \left(-\infty, -\frac{1}{\sqrt{3}}\right)$$

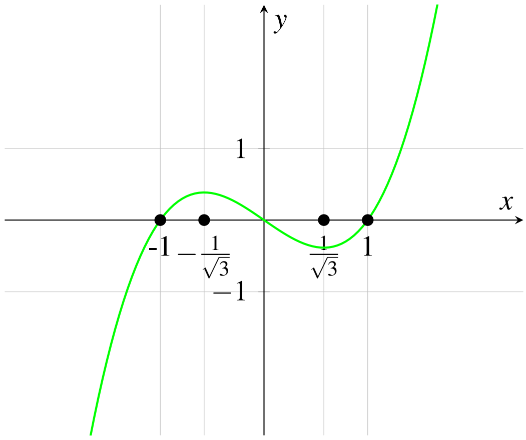
If $3x^2 < 1$ then $f'(x) < 0$, and this occurs if $x^2 < \frac{1}{3}$, i.e.,

$$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Thus the curve $y = x^3 - x$ will be decreasing.

on the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
Notice that the rule of f factorizes as $x(x+1)(x-1)$, which tells us that the curve for this rule must cross the x axis at zero, -1 and 1. We're also aware of the importance of$\pm\frac{1}{\sqrt{3}}$.

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)



So, here's the curve crossing the x-axis at those points we noted before, increasing for x < $-\frac{1}{\sqrt{3}}$ and for x > $\frac{1}{\sqrt{3}}$ and decreasing for x in between. What we've just completed in fact, are some of the main steps in the technique of curve sketching which is the topic of a future lesson. Notice the importance of the sign of the derivative, whether it's positive or negative and that can be captured in something called a sign diagram for the derivative in the form of a table, with a horizontal line representing the real line and we note the important points $\pm\frac{1}{\sqrt{3}}$ where the derivative is zero and the changing sides of the derivatives from positive to negative to positive which correspond to the curve is increasing, decreasing and then increasing again.

22.1.3 Examples and derivatives

Let's see some examples and how to solve them and derive them.

1. **We explain why reflecting an increasing function in the y-axis produces a decreasing function.**

   Suppose $y = f(x)$ is increasing over some domain. Let $y = g(x)$ be the function corresponding to reflecting the graph of $y = f(x)$ in the y-axis, so that

   $$g(x) = f(-x)$$

   for any input x for g. Suppose that a and b are inputs for g and $a < b$. Then $-a$ and $-b$ are inputs for f and $-b < -a$, since multiplying by -1 reverses the inequality. Hence, we have

   $$g(b) = f(-b) < f(-a) = g(a),$$

   since f is increasing. Thus $g(a) > g(b)$, completing the verification that g is decreasing.

   A similar argument shows that reflecting a decreasing function in the y-axis produces an increasing function.

2. **We explain why an increasing function is invertible and why its inverse function must also be increasing. Suppose that $y = f(x)$ is increasing. If the horizontal line test fails, then there would be two inputs, say a and b such that $a < b$ and $f(a) = f(b)$, which contradicts that $f(x)$ is increasing. Hence the horizontal line test is satisfied and f is invertible.**

We show that the inverse function f^{-1} is also increasing. Suppose c and d are inputs for f^{-1} such that $c < d$. Our task is to show that $f^{-1}(c) < f^{-1}(d)$. Put

$$a = f^{-1}(c) \quad \text{and} \quad b = f^{-1}(d),$$

so that $f(a) = c$ and $f(b) = d$ by definition of the inverse function. If $a = b$ then $c = f(a) = f(b) = d$, contradicting that $c < d$. Hence $a \neq b$. If $a < b$ then $f(a) < f(b)$, since f is increasing, so that $d = f(b) < f(a) = c$, again contradicting that $c < d$. The only alternative is that $a < b$, so that

$$f^{-1}(c) = a < b = f^{-1}(d),$$

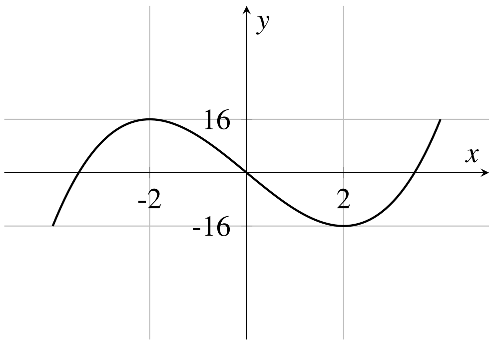
completing the verification that f^{-1} is increasing.

A similar argument shows that a decreasing function is invertible and its inverse is also decreasing. This also follows from the result about increasing functions, by reflecting a given decreasing function in the y-axis, using the fact that the reflected function is now increasing with an increasing inverse function; this new inverse function, in turn, reflects in the y-axis, which it is easy to check produces a decreasing inverse function corresponding to the original decreasing function.

3. **Consider the function $y = f(x) = x^3 - 3x$.**

   Then, $y' = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$,

   which is positive if $x > 1$ and if $x < -1$, and negative if $-1 < x < 1$. Thus f is increasing for $x > 1$, becomes decreasing for $-1 < x < 1$ and then becomes increasing again for $x > 1$. This increasing-decreasing-increasing behavior can be seen on the graph of the function $y = f(x)$. Soon you will develop techniques to understand fully how to obtain such a graph and all of its important features.



That's a good point to close the discussion for today, leading into a more general discussion about sign diagrams in the next section. In today's section, we discussed what it means for a function to be increasing or decreasing and saw many familiar examples interpreted with this new terminology and noted that if the derivative is positive on an interval then the function is increasing. Or, if the derivative is negative, then the function is decreasing. Please read the notes and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

22.1.4 Practice Quiz

Question 1

Which one of the following rules describes a function f that is increasing over all of its natural domain?

(a) $f(x) = \sin x$
(b) $f(x) = \cos x$
(c) $f(x) = x^3$
(d) $f(x) = 2 - x$
(e) $f(x) = x^2$

Question 2

Which one of the following rules describes a function f that is not increasing over all of its natural domain?
(a) $f(x) = e^x$
(b) $f(x) = 2x^2 + 6$
(c) $f(x) = \sqrt{x}$
(d) $f(x) = 2x + 6$
(e) $f(x) = \ln x$

Question 3

Which one of the following rules describes a function f that is decreasing over all of its natural domain?
(a) $f(x) = \tan x$
(b) $f(x) = -x^2$
(c) $f(x) = -\sin x$
(d) $f(x) = e^{-x}$
(e) $f(x) = x^{1/3}$

Question 4

Which one of the following rules describes a function f that is not decreasing over all of its natural domain?
(a) $f(x) = -\ln x$
(b) $f(x) = -x$
(c) $f(x) = -\sqrt{x}$
(d) $f(x) = \sqrt{x} - 2$
(e) $f(x) = -1$

Question 5

Over which one of the following intervals is the function $y = x^2$ increasing?
(a) $[-1, 1]$
(b) $(0, \infty)$
(c) $(-\infty, 1]$
(d) $[-1, \infty)$
(e) $(-\infty, 0]$

Question 6

Over which one of the following intervals is the function $y = \sin x$ increasing?
(a) $[-\pi, 0]$
(b) $[0, \pi]$
(c) $[\pi/2, \pi]$
(d) $[-\pi, \pi]$
(e) $[-\pi/2, \pi/2]$

Question 7

Over which one of the following intervals is the function $y = x - x^2$ decreasing?

(a) $(-\infty, 1/2]$
(b) $[0, 1]$
(c) $[1/2, \infty)$
(d) $(0, \infty)$
(e) $(-\infty, 0]$

Question 8

Over which one of the following intervals is the function $y = x^3 - 3x$ decreasing?
(a) $[1, \infty)$
(b) $[-1, 1]$
(c) $[0, 3]$
(d) $(-\infty, -1]$
(e) $[-\sqrt{3}, \sqrt{3}]$

Question 9

Which one of the following implications holds where a and b are inputs for an increasing function $y = f(x)$?
(a) $b < a \Rightarrow f(b) > f(a)$
(b) $a < b \Rightarrow f(a) \geq f(b)$
(c) $b < a \Rightarrow f(b) < f(a)$
(d) $a < b \Rightarrow f(a) = f(b)$
(e) $a < b \Rightarrow f(b) \leq f(a)$

Question 10

Let $y = f(a)$ be a function with domain $\mathbb{R}$.
Which one of the following statements may be false?
(a) If f is increasing then f is invertible and f^{-1} is increasing.
(b) If f is decreasing then f is invertible and f^{-1} is decreasing.
(c) If f is increasing or f is decreasing then f is invertible and f^{-1} has domain $\mathbb{R}$.
(d) If f is invertible and f^{-1} is increasing then f is increasing.
(e) If f is invertible and f^{-1} is decreasing then f is decreasing.

Answers

The answers will be revealed at the end of the module.

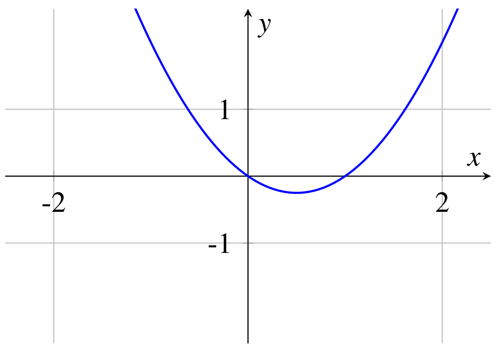
22.2 Sign Diagrams

In this section, we introduce general sign diagrams, for rules of functions, and provide contrasting examples, illustrating how they clarify and inform our understanding of graphs of functions. A sign diagram is a special type of table with two rows. The first row for indicating values of a variable x, say, and the second row for some expression involving the variable. For example, $x^2 - x$, which we factorize if possible. The line separating the rows is a copy of the real number line. On that line, we indicate important points relevant to the behavior of the expression, such as when it becomes zero or undefined.

For example, $x^2 - x$ x becomes zero if x equals zero and one. So, we mark off two points on the line and label them with zero and one. Below this line, we indicate the way the expression behaves with a zero or undefined at the indicated points, and whether positive or negative between these points and on neither side to the right and left. So, for this particular expression, the value is positive for x < 0 or x > 1, and negative only for x between 0 and 1. Later, I will give an example with expression is undefined at a particular point. This completes the sign diagram.

x		0		1	
x(x-1)	$+$	0	$-$	0	$+$

This particular expression is a quadratic, so the associated curve is a parabola. Bowl-shaped up because the coefficient of x^2 is positive and passing through the x-axis at 0 and 1. The parabola sits above the x-axis for x < 0 and x >1, corresponding to the plus signs in the sign diagram, and below the x-axis for x between 0 and 1, corresponding to the minus sign.



The sign of the derivative gives us information about where a function might be increasing or decreasing. So, let's investigate the derivative of $y = x^2 - x$, denoted by y', which in this case is $2x-1$. Observe, that this derivative is 0 precisely when $x = \frac{1}{2}$. So, we can draw a sign diagram marking off the point $\frac{1}{2}$, zero directly beneath, and then plus to the right, where $2x-1$ will be positive, and minus to the left where it will be negative.

x		$\frac{1}{2}$	
y'	$-$	0	$+$

The signs tells us that the curve is increasing for $x > \frac{1}{2}$ and decreasing for $x > \frac{1}{2}$. Both of these, exactly match what we see on the curve above. This also confirms that the point with coordinates of half negative a quarter is the lowest point on the parabola. We informally draw a small line sloping downwards beneath any minus sign to indicate decreasing, and informally, a line sloping upwards beneath any plus sign to indicate increasing.

Now, what about the derivative of y' called the second derivative denoted by y" double dash? The derivative of $2x-1$ is just the constant 2 and the sign diagram is particularly simple. There are no x values where the second derivative is zero or undefined. The second derivative is positive everywhere indicated by a plus sign.

x		0		1	
y"			$+$		

We have an informal notation which is a **U** (bowl-shaped up symbol), which has a technical term, concave up. When we see a plus sign for the second derivative, then we know the curve has some kind of bowl-shaped up behavior over the relevant interval. We'll say more about the technical meaning of concavity in a later section. In the case of above parabola, you can imagine the tangent line to the curve as you pass from left to right having increasing slope in the sense of becoming more and more positive as you move from left to right. This is exactly what the sign diagram for the second derivative is telling us, causing the curve to have the shape of a ball facing upwards. This upright parabola is concave up. But if we tipped it over, it will become what we call concave down. Then, there will be a negative sign instead of a plus sign and an inverted bowl-shaped down or sad face symbol($\bigcap$) in the sign diagram. We'll illustrate that in a moment.

22.2.1 Revisiting an example

We revisit an example discussed towards the end of the previous section. Consider y = $x^3 - x$, which factorizes as $x(x+1)(x-1)$. So, that y equals 0 when x equals -1, 0, and 1. We can draw a sign diagram for y, noting these important values x that cause y to be zero. Then nothing an alternating pattern of plus and minus symbols as you move right to left past the points that we've marked on the x-axis in sign diagram below.

x		-1		0		1	
y	$-$	0	$+$	0	$-$	0	$+$

This tells us when the curve sits above and below the x-axis. To find out where the curve is increasing or decreasing, we form a sign diagram for the derivative, which in this case, is $3x^2 - 1$. So, that y' to zero, when $x = \pm\frac{1}{\sqrt{3}}$. We get the sign diagram that we mentioned at the end of the last section.

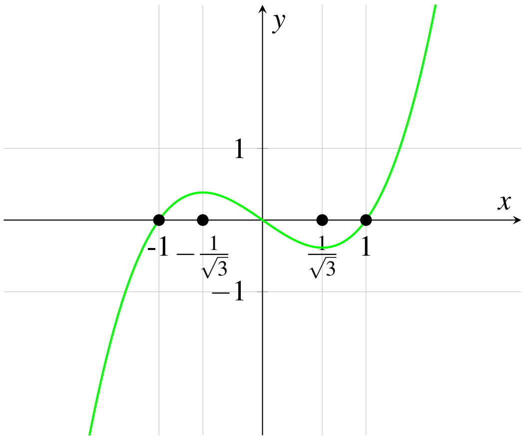
x		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
y'	$+$	0	$-$	0	$+$

We now add our up or down notation to indicate visually when the curve is increasing or decreasing according to whether the sign of the derivative is positive or negative. When the derivative is positive function is increasing and vice-versa. Now, we can be sure how the curve should undulate. In this case, up, down, and then up again as we move from left to right.

What about the second derivative? Can this give us any useful information? The first derivative is $3x^2 - 1$. So, the second derivative, y" is 6x. This is zero when x is zero. We can draw the sign diagram for y", indicating that it's zero when x equals zero, and then it has the same sign as x, positive, when x is positive, and negative when x is negative.

x		0	
y"	$-$	0	$+$

You might remember in the last example, we used a bowl-shaped happy face symbol(U) when the second derivative was positive. So, we'll use that again here for x < 0. But when the second derivative is negative, we use a bowl-shaped down or sad face symbol($\bigcap$). We now have three sign diagrams that give a fairly complete description of how the curve should behave. Where it sits above or below the x-axis, where it's increasing or decreasing, and where it's bowl-shaped up or down. The important points marked off on the sign diagrams, then correspond to important features of the graph so we note them on the x-axis on the graph below.



Then, fitting the curve matches the information about the signs of y, y', and y". Of note, are other points where the curve is turning around. A peak at $-\frac{1}{\sqrt{3}}$ and a trough at $-\frac{1}{\sqrt{3}}$, then sad face shape for negative x and this happy face shape for positive x. The point where the curve changes from sad to happy is called an inflection. The tangent line actually passes through the curve at a point of inflection. It's worth noticing and relates to something we call the second derivative test, which we'll talk about in a future lesson, that the point on the curve for $x = -\frac{1}{\sqrt{3}}$, that looks like the top of a hill, correlates with the imagery suggested by the sign diagrams for both the first, and second derivatives. Similarly, for the point on the curve, $x = \frac{1}{\sqrt{3}}$ that looks like the bottom of the valley.

The next example also was discussed in an earlier section and it's quite difficult. Consider the rational function g with rule $g(x) = \frac{x^2-2}{x-1}$, which can be rewritten in different ways, for example,

$$g(x) = \frac{x^2-2}{x-1} = \frac{(x+\sqrt{2})(x-\sqrt{2})}{x-1} = x+1-\frac{1}{x-1} = x+1+\frac{1}{1-x}.$$

The third and fourth expressions make clear the asymptotic behaviour of the curve, pictured below, which has been discussed previously. Clearly $g(x)$ is undefined at $x = 1$, and $g(x) = 0$ when $x = \pm\sqrt{2}$. By considering signs of the factors in the second expression, we get the following sign diagram for $g(x)$:

x		$-\sqrt{2}$		1		$\sqrt{2}$	
y	$-$	0	$+$	u	$-$	0	$+$

This matches the behaviour of the curve, which sits above the x-axis when $x > \sqrt{2}$ and when $-\sqrt{2} < x < 1$, and sits below the x-axis when $x < -\sqrt{2}$ and when $1 < x < \sqrt{2}$. To get the sign diagram for the derivative, we need to differentiate g and get

$$g'(x) = \frac{d}{dx}(x+1) + \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

As practice with one of the limit definitions of the derivative, put $h(x) = \frac{1}{1-x}$. Then, for any $a \neq 1$, we have

$$h'(a) = \lim_{b\to a}\frac{h(b)-h(a)}{b-a} = \lim_{b\to a}\frac{\frac{1}{1-b}-\frac{1}{1-a}}{b-a} = \lim_{b\to a}\frac{1-a-(1-b)}{(b-a)(1-b)(1-a)}$$

$$= \lim_{b\to a}\frac{b-a}{(b-a)(1-b)(1-a)} = \lim_{b\to a}\frac{1}{(1-b)(1-a)} = \frac{1}{(1-a)^2}.$$

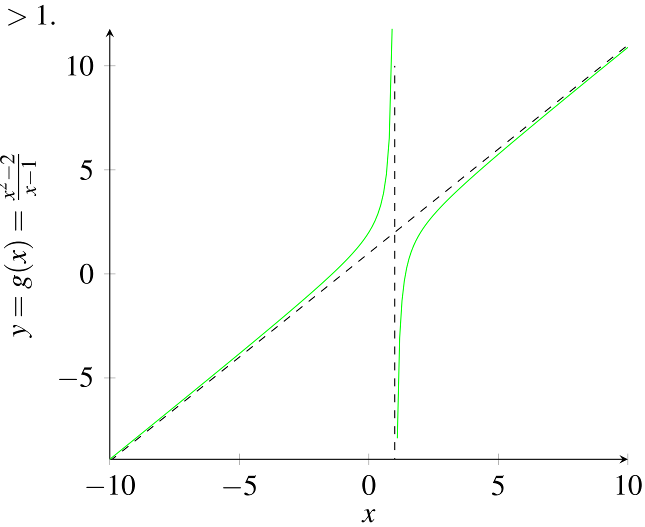
This shows that $h'(x) = \frac{1}{(1-x)^2}$, so that,

$$g'(x) = 1 + h'(x) = 1 + \frac{1}{(1-x)^2}.$$

Observe that $g'(1)$ is not defined and that $g'(x)$ is positive for all $x \neq 1$, so we get the following sign diagram:

x		1	
$g(x)$	$+$	u	$+$

This also matches the behaviour of the curve below, which is increasing both for $x < 1$ and for $x > 1$.



This behavior predicted by the sign diagram is exactly matched on the curve, but for the branch of the curve to the left to the vertical asymptote and to the branch to the right. There's more that we can say by considering the sign diagram for the second derivative. But we'll save that up for later section. Today, we discussed sign diagrams in general for rules of functions, and interpreted a positive or a negative sign for the derivative as indicating that the function is increasing or decreasing respectively. A positive sign for the second derivative as indicating that the graph of the function has a bowl-shaped up or smiley face appearance. A negative sign for the second derivative indicates that the graph has a bowl-shaped down or sad face appearance. When you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

22.2.2 Practice Quiz

Question 1

Which one of the following is the correct sign diagram for the function $y = x^2 - 2x$?

(a)

x		-2		0	
y	$-$	0	$+$	0	$-$

(b)

x		-2		0	
y	$+$	0	$-$	0	$+$

(c)

x		$-\sqrt{2}$		$\sqrt{2}$	
y	$+$	0	$-$	0	$+$

(d)

x		0		2	
y	−	0	+	0	−

Question 2

Which one of the following is the correct sign diagram for the function $y = -x^2 + 4x - 3$?

(a)

x		1		3	
y	+	0	−	0	+

(b)

x		−3		1	
y	+	0	−	0	+

(c)

x		−3		1	
y	−	0	+	0	−

(d)

x		1		3	
y	−	0	+	0	−

(e)

x		−3		3	
y	+	0	−	0	+

Question 3

Which one of the following is the correct sign diagram for the function $y = 4x - x^3$?

(a)

x		−2		0		2	
y	+	0	−	0	+	0	−

(b)

x		−2		0		2	
y	+	0	+	0	−	0	−

(c)

x		−2		0		2	
y	−	0	+	0	+	0	−

(d)

x		−2		0		2	
y	+	0	−	0	−	0	+

(e)

x		−2		0		2	
y	−	0	+	0	−	0	+

Question 4

Which one of the following is the correct sign diagram for the function $y = (x+1)(x-1)(x-3)$?

(a)

x		−3		−1		1	
y	+	0	−	0	−	0	+

(b)

x		−1		1		3	
y	+	0	−	0	+	0	−

(c)

x		−1		1		3	
y	−	0	+	0	+	0	−

(d)

x		−3		−1		1	
y	−	0	+	0	−	0	+

(e)

x		−1		1		3	
y	−	0	+	0	−	0	+

Question 5

Which one of the following is the correct sign diagram for the derivative y of the function $y = x^2 - 2x$?

(a)
$$\begin{array}{c|ccc} x & & 1 & \\ \hline y' & + & 0 & - \end{array}$$

(b)
$$\begin{array}{c|ccc} x & & -1 & \\ \hline y' & + & 0 & - \end{array}$$

(c)
$$\begin{array}{c|ccc} x & & -1 & \\ \hline y' & - & 0 & + \end{array}$$

(d)
$$\begin{array}{c|ccc} x & & 2 & \\ \hline y' & - & 0 & + \end{array}$$

(e)
$$\begin{array}{c|ccc} x & & 1 & \\ \hline y' & - & 0 & + \end{array}$$

Question 6

Which one of the following is the correct sign diagram for the derivative y' of the function $y = 4x - x^3$?

(a)
$$\begin{array}{c|ccccc} x & & -\frac{2}{\sqrt{3}} & & \frac{2}{\sqrt{3}} & \\ \hline y' & + & 0 & - & 0 & + \end{array}$$

(b)
$$\begin{array}{c|ccccc} x & & -\frac{2}{\sqrt{3}} & & \frac{2}{\sqrt{3}} & \\ \hline y & - & 0 & + & 0 & - \end{array}$$

(c)
$$\begin{array}{c|ccccc} x & & -\frac{2}{3} & & \frac{2}{3} & \\ \hline y & - & 0 & + & 0 & - \end{array}$$

(d)
$$\begin{array}{c|ccccc} x & & -\frac{\sqrt{3}}{2} & & \frac{\sqrt{3}}{2} & \\ \hline y' & + & 0 & - & 0 & + \end{array}$$

(e)
$$\begin{array}{c|ccccc} x & & -\frac{\sqrt{3}}{2} & & \frac{\sqrt{3}}{2} & \\ \hline y' & - & 0 & + & 0 & - \end{array}$$

Question 7

Which one of the following is the correct sign diagram for the derivative y” of the function $y = 4x - x^3$?

(a)
$$\begin{array}{c|ccc} x & & \frac{2}{3} & \\ \hline y'' & + & 0 & - \end{array}$$

(b)
$$\begin{array}{c|ccc} x & & \frac{2}{3} & \\ \hline y'' & - & 0 & + \end{array}$$

(c)
$$\begin{array}{c|ccc} x & & 0 & \\ \hline y'' & - & 0 & + \end{array}$$

(d)
$$\begin{array}{c|ccc} x & & \frac{3}{2} & \\ \hline y'' & + & 0 & - \end{array}$$

(e)

x		0	
y”	+	0	−

Question 8

Which one of the following is the correct sign diagram for the derivative y’ of the function $y = x + \frac{1}{x}$?

(a)

x		−1		0		1	
y’	+	0	−	u	+	0	−

(b)

x		−1		0		1	
y’	+	0	+	u	−	0	−

(c)

x		−1		0		1	
y’	+	0	−	u	−	0	+

(d)

x		−1		0		1	
y’	−	0	+	u	+	0	−

(e)

x		−1		0		1	
y’	−	0	+	u	−	0	+

Question 9

Which one of the following is the correct sign diagram for the derivative y’ of the function $y = x - \frac{1}{x}$?

(a)

x		−1		0		1	
y’	+	0	−	u	+	0	−

(b)

x		0	
y’	+	u	+

(c)

x		−1		0		1	
y’	+	0	−	u	−	0	+

(d)

x		−1		0		1	
y’	−	0	+	u	+	0	−

(e)

x		0	
y’	−	u	−

Question 10

Suppose that the following is the sign diagram for y' for some function $y = f(x)$ that is differentiable on the entire real line:

x		1		2		3	
y’	−	0	+	0	+	0	−

Which one of the following statements must be true?

(a) $f(0.5) > f(1.5)$
(b) $f(1.5) < f(2.5)$
(c) $f(2.5) < f(3.5)$
(d) $f(2) = 0$
(e) $f(1.5) > 0$

Answers

The answers will be revealed at the end of the module.

22.3 Maxima and Minima

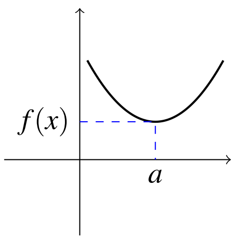
In today's lesson, we discuss local and global maxima and minima, collectively called extrema for rules of functions, which are related to turning points of curves. We provide some contrasting examples and illustrate how to search for extrema by looking for the changing sign of the derivative to see where a function changes from being increasing to decreasing or from decreasing to increasing. We finish with the discussion of the closed interval method.

22.3.1 Extrema

Let $y = f(x)$ be the rule for a function f.

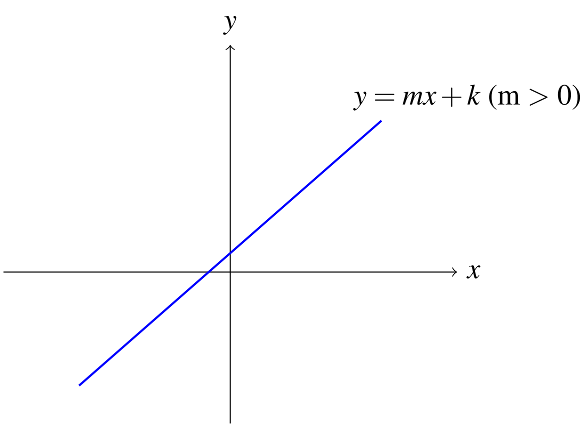
We say that the value $f(a)$, corresponding to the input $x = a$, is a local minimum, if

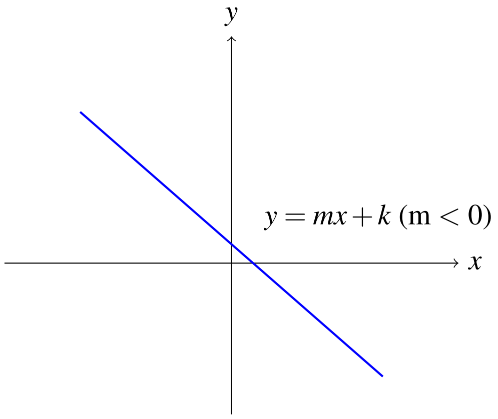
$f(a) \leq f(x)$ for all x near a.



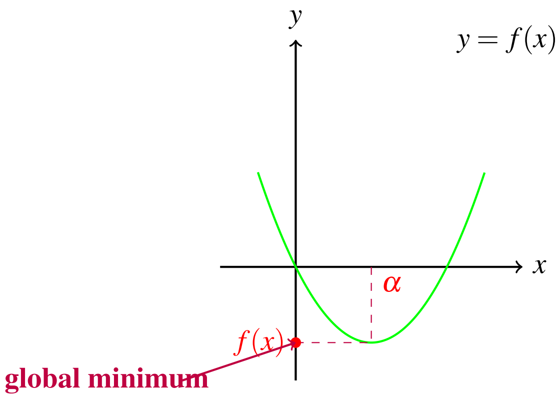
We may refer to either a maximum or a minimum as an extremum, and if the previous properties hold for all x in the domain of the function, then the extremum is said to be global. Just think of a global property is one that holds everywhere. To be "near" a on the real number line means ranging over some interval surrounding a. That interval which could be quite small is often called a neighborhood of a by mathematicians.

Extrema may not exist even for the simplest functions. For example, any linear function, where the slab is non-zero increases and decreases smoothly and uniformly without bound. There's no opportunity for any value of such a function to be the greatest or the least in any neighborhood of its input. However, by contrast, any linear function with zero slope that is a constant function y = k has only one value. So, that constant k becomes simultaneously both a global maximum and minimum value of the function.

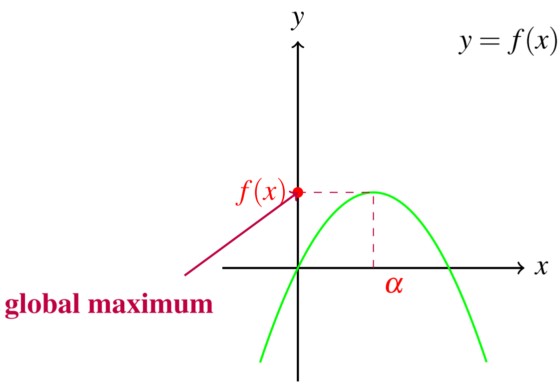




Consider now a function f, whose rule $f(x)$ is a quadratic. The graph is a parabola. Opening upwards if the coefficient of x^2 is positive or downwards if it's negative. In the upwards facing case, the apex, also called the vertex of the parabola is the lowest point. When the parabola is situated somewhere in the xy plane, and one moves across to the vertical axis, we produce a y value which is less than or equal to all y values produced by the parabola. So, it becomes the global minimum.



In above diagram, the apex is in the fourth quadrant, but it could be located in any quadrant. Wherever it is, when we move across to the y-axis, we always produce the global minimum. In the downward-facing case, the apex now is at the highest point. So, wherever the parabola is situated in the plane when we move across to the y-axis, we produce the global maximum.



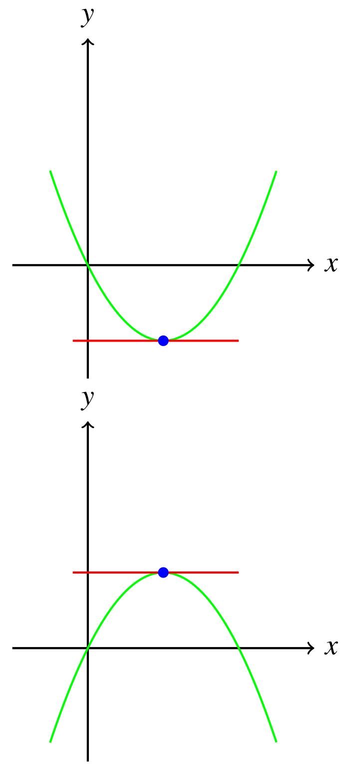
Let $f(x) = ax^2 + bx + c$ where a, b, c are constants $(a \neq 0)$.

The graph of f is a parabola

facing upwards if $a > 0$, in which case
the apex corresponds to the global minimum,

facing downwards if $a < 0$, in which case
the apex corresponds to the global maximum.

So, we have complete information in the two cases of quadratic functions.
The apex, in either case, is also called a turning point. In both cases, the tangent line to the curve at the apex is horizontal, so has zero slope.



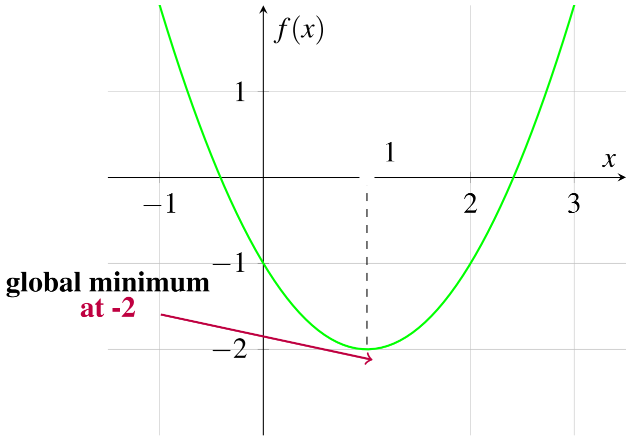
These are special cases of a general phenomenon. **A turning point for a general curve is a place where the curve changes from increasing to decreasing or from decreasing to increasing.** If any given turning point has a tangent line, which is usually the case as most of the time our curves are smooth, then the slope is zero. But the slope of the tangent line is the derivative of the function. So, if we want to find turning points, the natural place to look would be where the derivative is zero. Let's apply the derivative to quickly find the turning points and global minimum of quadratic equation:$x^2 - 2x - 1$.

Consider the function $y = f(x) = x^2 - 2x - 1$. Then $y' = 2x - 2 = 2(x-1)$, so that $y' = 0$ when $x = 1$, and we get the following sign diagram:

$$
\begin{array}{c|ccc}
x & & 1 & \\
\hline
g(x) & - & 0 & + \\
 & \searrow & & \nearrow
\end{array}
$$

The graph is a parabola and the turning point is $(1, -2)$. The value $y = -2$ is the global minimum.

All of this is confirmed by drawing the associated parabola.



Notice, when we draw the sign diagram for the derivative, we see that the derivative is negative to the left of $x = 1$ and positive to the right, confirming that the function is decreasing from the left and increasing to the right of the apex.

The next example involves a cubic polynomial, and the problem is to find all local and global extrema, firstly, over the whole real line and secondly, over the restricted interval between 0 and 5. Find all local and global extrema for the cubic function

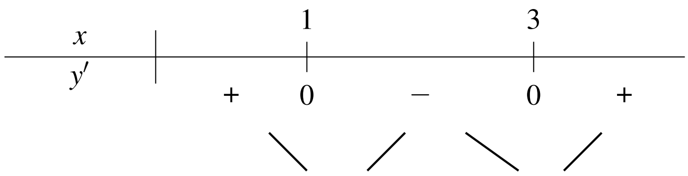
$$y = f(x) = x^3 - 6x^2 + 9x + 1$$

over (a) the entire real line $\mathbb{R}$, and (b) over the interval $[0,5]$.

Solution: For part (a), observe that

$$y' = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3),$$

so that $y' = 0$ when $x = 1, 3$ and we get the following sign diagram:



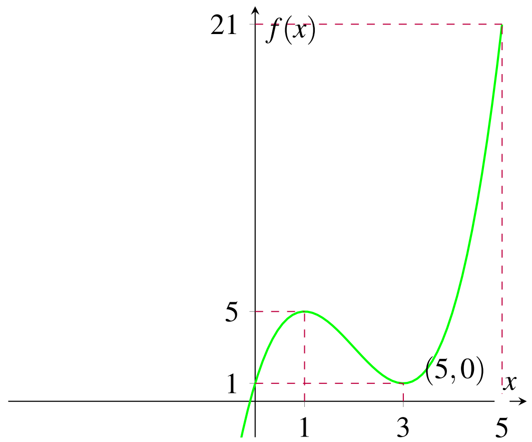
The turning points occur at $x = 1$ and $x = 3$ producing a local maximum $f(1) = 5$ and a local minimum $f(3) = 1$. Neither of these local extrema can be global, because the cubic takes arbitrarily large positive and negative values over the entire real line (as the rule for the function is dominated by x^3 for large inputs x).

For part (b), we apply the closed interval method and only need to check values of the function at the endpoints of the interval $[0,5]$ and where the derivative is zero. Thus, utilising values obtained above for $x = 1$ and $x = 3$, we only need to compare the following:

$$f(0) = 1, \quad f(1) = 5, \quad f(3) = 1, \quad f(5) = 21.$$

The smallest value is 1, which is therefore the global minimum. The largest value is 21, which is therefore the global maximum.

Note that we can draw these conclusions without knowing what the graph looks like. In fact, the curve has the following appearance, consistent with the information in the sign diagram for the derivative:



By drawing the sign diagram for the derivative, we should be able to tell what kind of extrema occur and where. So, we get an up-down-up pattern of behavior of the curve with turning points at $x = 1$, where there is a local maximum, and $x = 3$ where there is a local minimum. The local maximum is the value of the function at $x = 1$, which is 5, and the local minimum is the value at $x = 3$, which is 1. Neither of these local extrema can be global because with non-restricted domain, the cubic can take arbitrarily large positive and negative values. The question is really how far the curve falls away to the left and how much it climbs to the right. The global maximum in interval [0,5] is just the largest of these which is 21, and the global minimum is the smallest which is 1. This solves the second part of our problem. The graph is actually quite steep at $x = 5$. So, to see what's going on, we need to adjust the scale of the axes, and then you can see how easily the local maximum of 5 is surpassed by the global maximum of 21. The global minimum, in fact, is achieved exactly twice, both at the left-hand endpoint of the interval and at $x = 3$ inside the interval. The technique that we just used applies generally to curves and is called the closed interval method.

Closed interval method: To find the global maximum and minimum values of a continuous function f defined for $a \le x \le b$ $(a, b$ fixed), it suffices to consider the largest and smallest values of $f(c)$ where $c = a$, $c = b$ and c such that $a < c < b$ and $f'(c) = 0$ or $f'(c)$ is undefined.

In the last example, the derivative was defined everywhere. This is a very powerful result because for most functions, there are an infinite number of possible inputs, and this is saying that you only have to restrict attention typically to just a finite handful of points. In the last example, which is quite a sophisticated cubic, we only needed to evaluate the function at four points. We finished with an example, where the derivative is not defined everywhere.

Now, we are going to find the global maximum and minimum of the function f where

$$f(x) = |x-1| + 3 \qquad \textbf{for } -2 \le x \le 5.$$

Observe first that, if $x \ge 1$, then

$$f(x) = |x-1| + 3 = x - 1 + 3 = x + 2,$$

and if $x < 1$ then

$$f(x) = |x-1| + 3 = 1 - x + 3 = 4 - x.$$

Hence the rule for f can be expressed as follows:

$$f(x) = \begin{cases} x+2 & \text{if } x \ge 1 \\ 4-x & \text{if } x < 1. \end{cases}$$

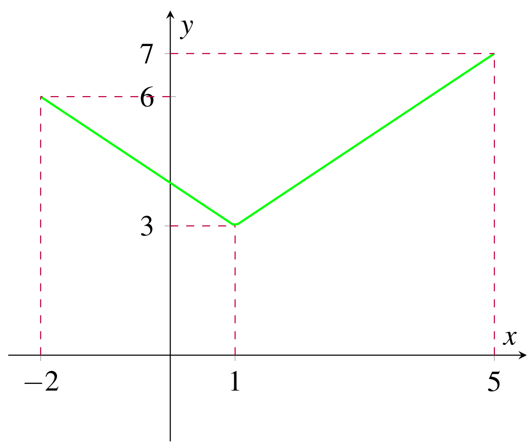
The graph is the result of joining two line segments of slopes 1 and -1 together at the point (1,3). No unique tangent line to the curve exists at this point, and the derivative does not exist for $x = 1$. Nevertheless, the derivative exists for $x \neq 1$ and can be expressed as follows:

$$f'(x) = \begin{cases} 1 & \text{if } x > 1 \\ -1 & \text{if } x < 1 \,. \end{cases}$$

Note that the derivative is nonzero for all $x \neq 1$. Hence, applying the closed interval method, it suffices to compare the following values:

$$f(-2) = |-2-1| + 3 = 6\,, \quad f(1) = |1-1| + 3 = 3\,, \quad f(5) = |5-1| + 3 = 7\,.$$

The smallest value is 3, which is therefore the global minimum. The largest value is 7, which is therefore the global maximum. Note that these values could be gleaned directly from the graph, which is shown below.



Let's see one last similar example before ending this chapter. We will find the global maximum and minimum of the function f where

$$f(x) = |x-2| + |x+1|\,, \qquad \textbf{for } -3 \leq x \leq 3\textbf{.}$$

Observe first that, if $x \geq 2$, then

$$f(x) = |x-2| + |x+1| = x-2+x+1 = 2x-1\,.$$

If $-1 \leq x < 2$ then

$$f(x) = |x-2| + |x+1| = 2-x+x+1 = 3\,.$$

If $x < -1$ then

$$f(x) = |x-2| + |x+1| = 2-x-x-1 = 1-2x\,.$$

Hence the rule for f can be expressed as follows:

$$f(x) = \begin{cases} 2x-1 & \text{if } x \geq 2 \\ 3 & \text{if } -1 \leq x < 2 \\ 1-2x & \text{if } x < -1\,. \end{cases}$$

The graph is the result of joining together three line segments of slopes 2, 0, and -2 at the points (2,3) and (-1,3). No unique tangent line to the curve exists at these points, and the derivative does not exist for $x = -1$ or for $x = 2$. Nevertheless, the derivative exists for $x \neq -1, 2$ and can be expressed as follows:

$$f'(x) = \begin{cases} 2 & \text{if } x > 2 \\ 0 & \text{if } -1 < x < 2 \\ -2 & \text{if } x < -1\,. \end{cases}$$

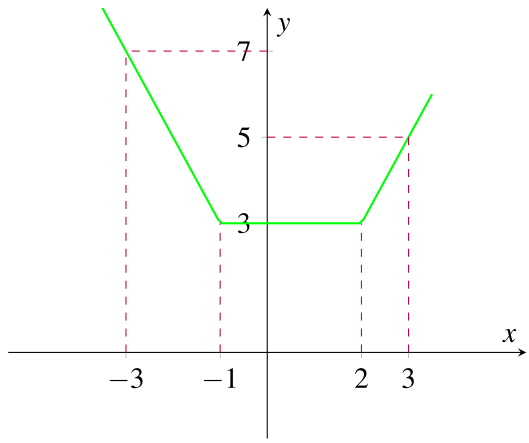
The derivative is zero or undefined therefore for all x in the interval $[-1,2]$. Applying the closed interval method, it suffices to look at the value $f(x)$ for such x and also for the endpoints, when $x = -3$ and $x = 3$. For all $x \in [-1,2]$ we have $f(x) = 3$. At the endpoints, we have

$$f(-3) = |-3-2| + |-3+1| = 5+2 = 7$$

and

$$f(3) = |3-2| + |3+1| = 1+4 = 5\,.$$

The smallest value is 3, which is therefore the global minimum. The largest value is 7, which is therefore the global maximum. Note, again, that these values could be gleaned directly from the graph, which is shown below.



Note that though we looked at the graph, the closed interval method can be followed blindly and mechanically without any visual assistance. It's important to have such methods to deal with functions whose graphs are so complicated that it may be too difficult to draw them or even imagine what they look like. In this, we discussed local and global maxima and minima, collectively called extrema, for rules of functions, considered associated turning points, where the curve changes from increasing to decreasing or decreasing to increasing and found them by exploiting the derivative and the fact that the tangent line to a curve at a turning point must be horizontal and have zero slope. We finished with discussion and illustrations of the closed interval method, which locates global extrema by narrowing attention to endpoints of the domain and points where the derivative of zero is undefined. Please read properly and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

22.3.2 Practice Quiz

Question 1

Find the global minimum for the function f with rule $f(x) = x^2 - 2x + 1$.

(a) 1

(b) 0
(c) -2
(d) 2
(e) -1

Question 2

Find the global maximum for the function f with rule $f(x) = 4 + 6x - 3x^2$.

(a) 4
(b) 1
(c) -5
(d) 5
(e) 7

Question 3

Find the global maximum and minimum for the function f with rule $f(x) = 2x^2 - 8x + 7$ for $x \in [-3, 3]$.

(a) 49, 1
(b) 13, -1
(c) 49, -1
(d) 1, -1
(e) 1, 31

Question 4

Which one of the following statements is true for the function f with rule $f(x) = x^3$?

(a) f has a turning point at (-1, -1) and -1 is a local minimum.
(b) f has a turning point at (0, 0) and 0 is a local minimum.
(c) f has a turning point at (0, 0) and 0 is a local maximum.
(d) f has a turning point at (1, 1) and 1 is a local maximum.
(e) f has no turning points.

Question 5

Which one of the following statements is true for the function f with rule $f(x) = x^4$?

(a) f has a turning point at (0, 0) and 0 is a local maximum.
(b) f has a turning point at (-1, -1) and -1 is a local minimum.
(c) f has a turning point at (0, 0) and 0 is a local minimum.
(d) f has a turning point at (1, 1) and 1 is a local maximum.
(e) f has no turning points.

Question 6

Consider the function f with the following rule:

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

Which one of the following statements is true?

(a) f has no turning points.
(b) f has a turning point at (0, 1) and 1 is a local minimum.
(c) f has a turning point at (-1, 8) and 8 is a local minimum.
(d) f has a turning point at (2, -19) and -19 is a local minimum.
(e) f has a turning point at (-1, 8) and 8 is a local maximum.

Question 7

Find the global maximum and minimum for the function f with rule

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

for $x \in [0, 3]$.
(a) 8, -19
(b) -8, -19
(c) 8, 1
(d) 1, -19
(e) 1, -8

Question 8

Consider the function f with the following rule:

$$f(x) = x^3 - 5x + 3$$

Which one of the following statements is true?
(a) f has a turning point at (-1, 7) and 7 is a local maximum.
(b) f has a turning point at (1, -1) and -1 is a local maximum.
(c) f has a turning point at (-1, 1) and 1 is a local minimum.
(d) f has no turning points.
(e) f has a turning point at (0, 3) and 3 is a local minimum.

Question 9

Find the global maximum and minimum for the function f with rule

$$f(x) = |x + 2| - 4$$

for $x \in [-3, 3]$.
(a) 1, -4
(b) 1, -3
(c) 1, -2
(d) -3, -4
(e) -2, -4

Question 10

Find the global maximum and minimum for the function f with rule

$$f(x) = |x - 1| + |x - 2|$$

for $x \in [0, 4]$.
(a) 5, 0
(b) 5, 3
(c) 3, 2
(d) 3, 1
(e) 5, 1

Answers

The answers will be revealed at the end of the module.

23. Second Derivatives and curve sketching

23.1 Concavity and Inflections

In this section, we formally discuss concavity, which is a property of curves with the slopes of tangent lines are either increasing or decreasing, leading to notions of concave up and concave down, and points of inflection where concavity changes either from concave up to down or from concave down to up, which will also point to the property that the tangent lines pass through the curve.

23.1.1 Concavity

We begin by discussing concavity. Think of the property concave up of a curve as corresponding to the curve being in a "bowl-shaped up" configuration.

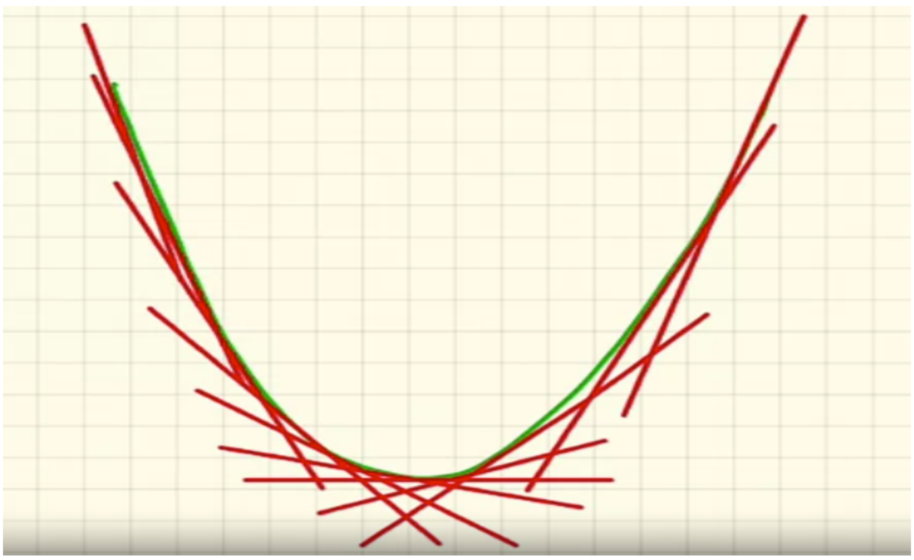


Figure 23.1: Parabola and Tangent lines

If you look at miniature tangent lines to the curve as you pass from left to right, you'll notice

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

that the slopes change from being very steep and negative, to shallow and negative, to being zero as you move through the turning point, to being shallow and positive, to being steep and positive. The slopes of the tangent lines are increasing. This corresponds to the derivative of the rule of the original function being itself an increasing function. So, its derivative, the second derivative of the original function denoted by y", should be positive(y" > 0). Now, consider the opposite property concave down, which corresponds to the curve being in a bowl-shaped down configuration.

Figure 23.2: Parabola and Tangent lines

Now, if you look at miniature tangent lines to the curve as you pass from left to right, you'll notice that the slopes change from being very steep and positive, to shallow and positive, to being zero as you move through the turning points, to being shallow and negative, to being steep and negative. The slopes of the tangent lines are decreasing. This corresponds to the derivative of the rule of the original function being itself a decreasing function. So, the second derivative of the original function, y", should be negative(y" < 0).

23.1.2 Inflection

A point of inflection or we simply say an inflection, is a point where the concavity changes either from being concave up to being concave down, or from concave down to concave up. We can picture the first case in a couple of ways transitioning from bowl-shaped up to bowl-shaped down, and the moments of transition are called inflections.

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

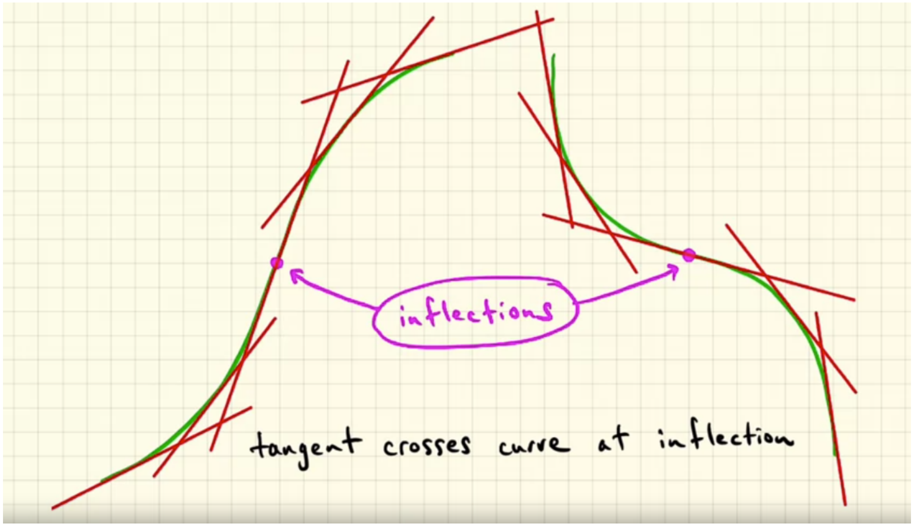


Figure 23.3: Curve and Tangent lines

You can watch the way miniature tangent lines to the curve behaves as you pass from left to right for the first curve displayed here, and for the second curve. You might notice that the tangent line crosses the curve at the point of inflection. The slope of the tangent line which is the derivative denoted by y', reaches a maximum at that moment of transition at the inflection point. So, its derivative denoted y" must become zero. In summary, for the case where concavity changes from concave up to down, the derivative reaches a maximum and the second derivative is zero. Consider the second type of transition from concave down to concave up. We can picture that in a couple of ways transitioning from bowl-shaped down to bowl-shape up. In above picture, note the moments of transition, the inflections. Now watch the way the miniature tangent lines behave as you pass from left to right. For the first curve and for the second curve, and again, the tangent line crosses the curve at the point of inflection. This time the derivative y dashed reaches a minimum at that moment of transition at the inflection, and again, the second derivative y" must become zero. In summary, for the case where concavity changes from concave down to up, the derivative reaches a minimum and the second derivative is zero. So, we've considered variations in the way concavity can change and the effects they have on the derivative and second derivative. The upshot of this is that changes in concavity may be detected by looking at the changes in the sign of the second derivative, and all of this information will be captured in its sign diagram. To see how this works, let's sketch the curve $y = 9x - x^3$.
First observe that

$$y = 9x - x^3 = x(9 - x^2) = x(3 + x)(3 - x)$$

so that $y = 0$ when $x = 0, \pm 3$. We have

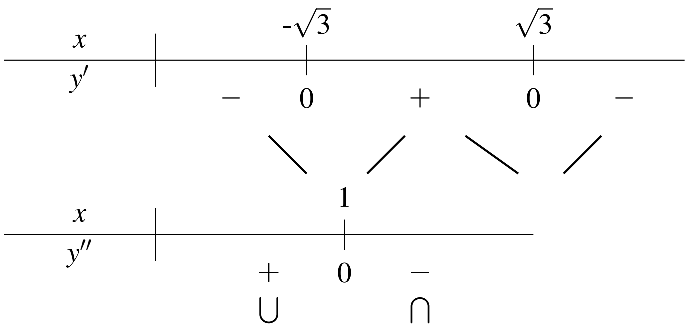
$$y' = 9 - 3x^2 = 3(3 - x^2) = 3(\sqrt{3} + x)(\sqrt{3} - x)$$

so that $y' = 0$ when $x = \pm\sqrt{3}$. Further

$$y'' = -6x$$

so $y'' = 0$ when $x = 0$. We get the following sign diagrams for the first and second derivatives:

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

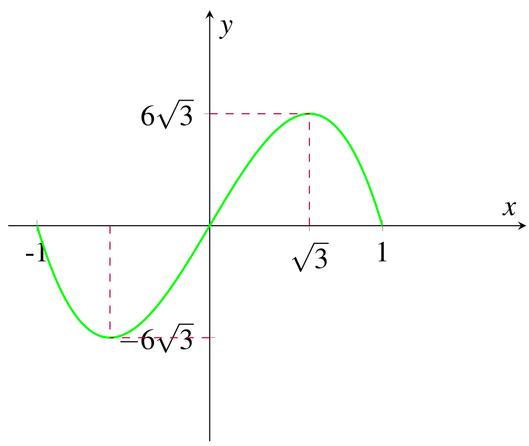


These indicate that there is a local minimum at $x = -\sqrt{3}$, a local maximum at $x = \sqrt{3}$ and a point of inflection when $x = 0$, at which point the curve changes from concave up to concave down. The original factorisation of y tells us also that the curve crosses the x-axis at $x = -3, 0$ and 3. The local maximum is

$$y(\sqrt{3}) = 9\sqrt{3} - 3\sqrt{3} = 6\sqrt{3},$$

and the local minimum is

$$y(-\sqrt{3}) = -9\sqrt{3} + 3\sqrt{3} = -6\sqrt{3}.$$



Here are the important points on the x-axis and we have fitted on the curve which is concave up to the left of zero, concave down to the right, with a local minimum at $x = -\sqrt{3}$ and a local maximum at $x = \sqrt{3}$. The local maximum and minimum values, you can quickly work out from the rule for the function. And they become $\pm 6\sqrt{(3)}$, and the origin is the points of inflection where the concavity changes.

We'll finish with an example of a cusp and sketch the curve $y = x^{\frac{2}{3}}$.
First observe, using the rule for differentiating a power of x (by bringing the exponent to the front and creating a new exponent by subtracting one), that

$$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

Differentiating again, we get

$$y'' = \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)x^{-4/3} = \frac{2}{-9x^{4/3}} = \frac{-2}{9(x^{2/3})^2},$$

written this way just to emphasise that the denominator is always positive for $x \neq 0$. We get the following sign diagrams, noting that both y' and y'' are undefined for $x = 0$:

x		0	
y'	$-$	u	$+$
	$\cup$		$\cap$

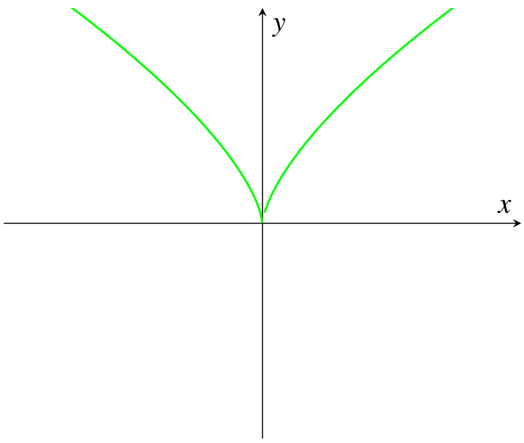
x		1	
y''	$-$	0	$+$
	$\searrow$		$\nearrow$

Note that $y = 0$ when $x = 0$, and the curve, which includes the origin, is continuous, that is, can be drawn without lifting the pen off the paper. The sign diagram for the derivative tells us that there must be a global minimum at $x = 0$. The sign diagram for the second derivative tells us also that the curve is concave down on both the negative and positive halves of the real line.

To understand the slope near the origin, we can observe the following one-sided limits:

$$\lim_{x \to 0^+} y' = \lim_{x \to 0^+} \frac{2}{3x^{1/3}} = \infty \quad \text{and} \quad \lim_{x \to 0^-} y' = \lim_{x \to 0^-} \frac{2}{3x^{1/3}} = -\infty\,,$$

noting that as x gets close to zero from the positive side, then y' gets arbitrarily large and positive, whilst if x get close to zero from the negative side, then y' gets arbitrarily large and negative. The first limit tells us that the tangent lines to the curve, as we move from the right towards zero, become closer and closer to being vertical, using steep positive slopes. The second limit tells us that, as we move from the left towards zero, the tangent lines again become closer and closer to being vertical, but using steep negative slopes. We can now draw the graph that matches all of this information producing a continuous curve with a sharp point, a cusp at the origin, which also coincides with the global minimum value of y = 0. This completes the sketch of the curve.



In this section, we discussed notions of concavity of a curve, including the properties of being concave up and concave down, and points of inflection where concavity changes either from concave up to down or from concave down to up. We illustrated the ideas by working through a couple of contrasting examples, including a cusp and exploited information provided by the sign diagrams to the first and second derivatives. Please re-read if you didn't understood and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

23.1.3 Practice Quiz

Question 1

Which one of the following statements is true for the curve $y = 2x - 4$?

(a) The curve is concave up everywhere.
(b) The curve has a point of inflection at $(0, -4)$.
(c) The curve is not concave up or concave down anywhere.
(d) The curve has a point of inflection at $(2, 0)$.
(e) The curve is concave down everywhere.

Question 2

Which one of the following statements is true for the curve $y = x^2 - 4x + 4$?

(a) The curve is concave up everywhere.
(b) The curve has a point of inflection at $(0, 4)$.
(c) The curve is concave down everywhere.
(d) The curve has a point of inflection at $(2, 0)$.
(e) The curve is not concave up or concave down anywhere.

Question 3

Which one of the following statements is true for the curve $y = 2 + 6x - x^2$?

(a) The curve is concave down everywhere.
(b) The curve is concave up everywhere.
(c) The curve is not concave up or concave down anywhere.
(d) The curve has a point of inflection at $(-3, -25)$.
(e) The curve has a point of inflection at $(3, 11)$.

Question 4

Which one of the following statements is true for the curve $y = x^3 - 1$?

(a) The curve is concave down everywhere.
(b) The curve is concave up everywhere.
(c) The curve has a point of inflection at $(1, 0)$.
(d) The curve is not concave up or concave down anywhere.
(e) The curve has a point of inflection at $(0, -1)$.

Question 5

Which one of the following statements is true for the curve $y = x^2 - x^3$?

(a) The curve has exactly three turning points and two points of inflection.
(b) The curve is concave up everywhere.
(c) The curve has exactly two turning points and one point of inflection.
(d) The curve has exactly one turning point and no points of inflection.
(e) The curve is concave down everywhere.

Question 6

Which one of the following statements is true for the curve $y = x^3 - 3x^2 + 3x$?

(a) The curve is concave up everywhere.
(b) The curve has no turning points and one point of inflection.
(c) The curve is concave down everywhere.
(d) The curve has no turning points and no points of inflection.
(e) The curve has exactly two turning points and one point of inflection.

Question 7

Which one of the following statements is true for the curve $y = |x| + 6$?
(a) The curve is concave down for $x < 0$ and concave up for $x > 0$.
(b) The curve has exactly one turning point and no points of inflection.
(c) The curve has no turning points and no points of inflection.
(d) The curve has exactly one turning point and one point of inflection.
(e) The curve is concave up for $x < 0$ and concave down for $x > 0$.

Question 8

Which one of the following statements is true for the curve $y = 2 - x^2/2$?
(a) The curve is concave down for $x < 0$ and concave up for $x > 0$.
(b) The curve is concave up for $x < 0$ and concave down for $x > 0$.
(c) The curve is concave up for $x < 0$ and concave down for $x > 0$.
(d) The curve is concave down for $x < 0$ and concave up for $x > 0$.
(e) The curve is not concave up or concave down anywhere.

Question 9

Consider the function f with the following rule:

$$f(x) = x^5 - 5x + 3$$

Which one of the following statements is true?
(a) f has exactly three points of inflection.
(b) The curve $y = f(x)$ is concave down for $x < 0$ and concave up for $x > 0$.
(c) The curve $y = f(x)$ is concave up for $x < 0$ and concave down for $x > 0$.
(d) f has exactly two points of inflection.
(e) f has no points of inflection.

Question 10

Consider the function f with the following rule:

$$f(x) = (x^2 - 1)^3$$

Then the following hold:

$$f'(x) = 6x(x^2 - 1)^2 \quad \text{and} \quad f''(x) = (6x^2 - 1)(3x^2 - 1)$$

Use these facts to decide which one of the following statements is true?
(a) f has exactly one turning point and two points of inflection.
(b) f has exactly one turning point and four points of inflection.
(c) f has exactly two turning points and one points of inflection.
(d) f has exactly one turning point and no points of inflection.
(e) f has exactly five turning points and four points of inflection.

Answers

The answers will be revealed at the end of the module.

23.2 Curve Sketching

In this section, we bring together a range of ideas and techniques that we've been discussing over several videos that lead to an ordered and systematic checklist for sketching a curve in the plane. Of central importance are the sign diagram for the derivative, which tells us where the curve is increasing or decreasing, and the sign diagram for the second derivative, which tells us where the curve is concave up or down.

We begin by making some general remarks about curve sketching, by which we mean, more specifically, sketching or drawing the graph of the function y=f(x) in the xy-plane. The scales of the x and y axes don't need to be the same, and the axes are positioned on the page depending on where the important features of the graph will appear. Always aim for simplicity and clarity, and focus on the main qualitative features. Extra details, such as coordinates of points, can be added depending on the requirements of any particular problem.
It's a fairly natural list of typical features to look for. Firstly, look for the y-intercept, where the curve crosses the vertical axis, in the case that $x = 0$ is in the domain of the function, which is simply found by evaluating f(0). And then look for the x-intercepts, where the curve crosses the horizontal axis, if they're likely to be straightforward to find or simple to describe. But they may be difficult, as they're the solutions to the equation $f(x) = 0$. Indeed, a given problem might be to attempt to solve the equation $f(x) = 0$, typically by using some approximation method from advanced calculus, and the main purpose of the sketch of the curve may be just to get started, with the final aim of knowing roughly where the x-intercepts might be. Then look for asymptotic behaviour, which roughly speaking means, seeing what happens when things get large, such as when x gets arbitrarily large and positive, or arbitrarily large and negative, leading to possible horizontal or oblique asymptotes, or whether there are any vertical asymptotes near which y can get arbitrarily large and positive or negative. Then we look for where the curve might be increasing, where the derivative could be positive, or decreasing, where the derivative could be negative, which helps us find turning points, where the derivative might be zero, and these could be local or global maxima or minima. This information typically is stored in the sign diagram for the derivative. Then we look for where the curve might be concave up, where the second derivative might be positive, or concave down, where the second derivative might be negative, which helps us find possible points of inflection, where the second derivative might be zero. This information typically is stored in the sign diagram for the second derivative.

23.2.1 Contrasting examples

Let's work through the above list to sketch the curve of different functions which look like similar but are very different. Let's start with the cubic function. Let $y = f(x) = 2x^3 - 9x^2 + 12x - 1$.
The y-intercept is $y(0) = -1$. (Finding the x-intercepts looks difficult, but we may obtain information about them as a consequence of the rest of this curve sketching exercise.)

The asymptotic behaviour is not very interesting. The cubic is dominated by the $2x^3$ term, so we have

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty,$$

which are abbreviations for saying that $f(x)$ gets arbitrarily large and positive as x does, and $f(x)$ gets arbitrarily large (in magnitude) and negative as x does. In particular, there are no horizontal asymptotes.
You may be curious to know how we are sure there is no oblique asymptotic behaviour. Suppose, by way of contradiction that there exists some oblique asymptote with equation

$$y = mx + c$$

for some constants m and c. Certainly m can't be zero, for otherwise we would have a horizontal asymptote, and we have already seen that this is not the case. This hypothetical asymptote would have to have the property that the curve $y = f(x)$ gets arbitrarily close to it as x gets large, which would imply that the ratio of $mx+c$ with $f(x)$ should tend to 1. But, using our usual tricks for evaluating limits,

$$\lim_{x\to\infty}\frac{mx+c}{f(x)} = \lim_{x\to\infty}\frac{mx+c}{2x^3-9x^2+12x-1} = \lim_{x\to\infty}\frac{\frac{m}{2}+\frac{c}{x}}{1-\frac{9}{2x}+\frac{12}{2x^2}-\frac{1}{2x^3}} = 0,$$

which of course does not equal 1. A similar contradiction is reached by taking the limit as $x \to -\infty$, so there is no chance of there being an oblique asymptote.

We have

$$y' = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2),$$

so that $y' = 0$ when $x = 1, 2$. Further

$$y'' = 12x - 18 = 6(2x-3),$$

so $y'' = 0$ when $x = \frac{3}{2}$. We get the following sign diagrams for the first and second derivatives:

x		1		2	
y'	$+$	0	$-$	0	$+$
	↗		↘		↗

x		1	
y''	$-$	$\frac{3}{2}$	$+$
	↘		↗

The diagram for y' indicates that there is a turning point at $x = 1$, with local maximum

$$f(1) = 2 - 9 + 12 - 1 = 4,$$

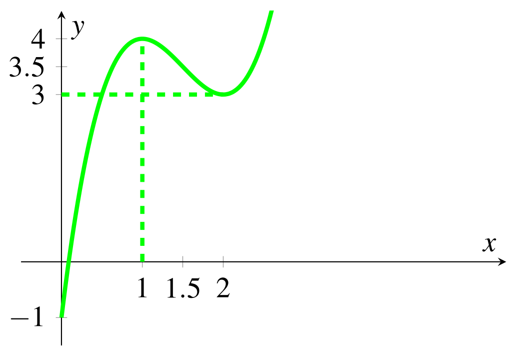
and a turning point at $x = 2$, with local minimum

$$f(2) = 16 - 36 + 24 - 1 = 3.$$

The diagram for y'' indicates that there is a point of inflection at $x = \frac{3}{2}$ and the corresponding y-value is

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 12\left(\frac{3}{2}\right) - 1 = \frac{7}{8}.$$

All of these important features are highlighted in the sketch below. Notice that, in fact, there is only one x-intercept, something that was not at all evident from the rule of the function, and it is forced to lie somewhere between 0 and 1, and the sketch suggests quite close to 0.



Notice that a sketch tells us that there's only one x-intercept, and the curve crosses the x-axis somewhere between 0 and 1, and quite close to 0. If one were trying to solve the equation for x where this cubic is set equal to 0, then this sketch would tell you, firstly, that there's only one solution, and, secondly, where to start looking to find an approximation to this unique solution.

Now we'll sketch the graph of a by now familiar rational function g with the rule $g(x) = \frac{x^2-2}{x-1}$. Many of its main features have been explored in earlier sections. Let's go through the checklist systematically. Firstly, the y-intercept is 2, the result of evaluating the rule when x equals zero. The x-intercepts turn out to be straightforward, simply when the numerator x^2 is 0, which occurs when x is $\pm\sqrt{2}$. The asymptotic behavior has been explored thoroughly in an earlier section. The limit as x approaches one from above of g(x) is minus infinity, which means g(x) gets arbitrarily large and negative, and the limit as x approaches one from below becomes positive infinity, which means g(x) gets arbitrarily large and positive, and x=1 is a vertical asymptote.

The y-intercept is $g(0) = 2$. We have considered this curve before and exploited the fact that the rule can be expressed in different ways:

$$g(x) = \frac{x^2-2}{x-1} = x + \frac{(x+1)}{x-1} = x+1+\frac{1}{x-1}.$$

The x-intercepts are clearly $\pm\sqrt{2}$. The line $x = 1$ is a vertical asymptote, and the line $y = x+1$ is an oblique asymptote. Put $h(x) = \frac{1}{x-1}$. We have also seen before that the derivative is

$$g'(x) = 1 + h'(x) = 1 + \frac{1}{(x-1)^2}$$

with the following sign diagram:

x		0	
$g''(x)$	$+$	u	$-$
	$\cup$		$\cap$

The derivative can also be put in the following form:

$$g'(x) = 1 + h'(x) = 1 + \frac{1}{(1-x)^2} = 1 + (1-x)^{-2}.$$

The second derivative therefore becomes

$$g''(x) = \frac{d}{dx}\left(1 + (1-x)^{-2}\right) = -2\frac{d}{dx}\left((1-x)^{-3}\right).$$

We have been told that $\frac{d}{dx}x^n = nx^{n-1}$ for any exponent n, and it follows that

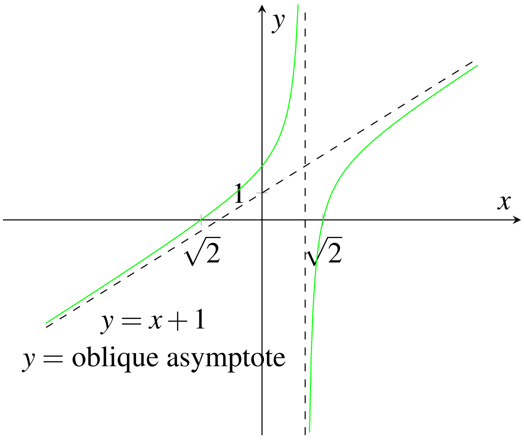
$$\frac{d}{dx}(1-x)^n = n(1-x)^{n-1},$$

since the curve $y = (1-x)^n$ is obtained from the curve $y = x^n$ by a horizontal shift one unit to the right, so the slope of tangent lines is unaffected. We can apply this fact in our case, when $n = -2$, to conclude that

$$g''(x) = \frac{d}{dx}\left((1-x)^{-2}\right) = -2(1-x)^{-3} = -\frac{2}{(1-x)^3}.$$

It follows that $g''(1)$ is undefined and the sign of $g''(x)$ is the opposite of the sign of $x-1$.

Now we can gather all of this information, and set up the axes for the sketch of the curve, noting the important points two on the y-axis and plus and minus root two on the x-axis, and the oblique and vertical asymptotes. This pattern of concave up, followed by concave down, is matched by the concavity behaviour in the sketch below.



Then we see this familiar curve in two branches, consistent with all of this information. The feature that we didn't discuss previously was concavity. And we see the concave up branch to the left of the vertical asymptote and the concave down branch to the right. Notice that these are not full smiley or sad faces, so to speak. The slope is increasing in the left-hand branch of the graph and decreasing in the right-hand branch, which is consistent with the pattern of plus and minus in the sign diagram for the second derivative. The reason for the oblique asymptote is that the rule for the function splits up into a linear piece, $x+1$, take away this extra piece, $\frac{1}{1-x}$. This means that to the right, the curve is approaching the oblique asymptote from below because we are taking a small positive piece away from the linear piece. But, to the left, the curve is approaching the oblique asymptote from above, because we are taking away a small negative piece , which is the same as adding a small positive piece. It's interesting to ask, what happens if we add $\frac{1}{x-1}$ in the rule for the function instead of taking it away?

So let's work through the checklist for sketching the curve with this variation of the previous example, where we call the function f,

$$y = f(x) = x+1+\frac{1}{x-1} = \frac{x^2}{x-1}.$$

The y-intercept is $f(0) = 0$ and $y = 0$ when $x = 0$, so there is only one x-intercept and the curve passes through the origin. As in the previous example, the line $x = 1$ is a vertical asymptote and the line $y = x+1$ is an oblique asymptote. The derivative becomes, similar to the previous example in the first three steps,

$$f'(x) = \frac{d}{dx}(x+1)+\frac{d}{dx}\left(\frac{1}{x-1}\right) = 1+\frac{d}{dx}\left((x-1)^{-1}\right) = 1-(x-1)^{-2}$$

$$= 1-\frac{1}{(x-1)^2} = \frac{(x-1)^2-1}{(x-1)^2} = \frac{x^2-2x+1-1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Differentiating again, we get

$$f''(x) = \frac{d}{dx}\left(\frac{x(x-2)}{(x-1)^2}\right) = \frac{d}{dx}\left((x-1)^{-2}\right) = -2\left((x-1)^{-3}\right) = \frac{2}{(x-1)^3}.$$

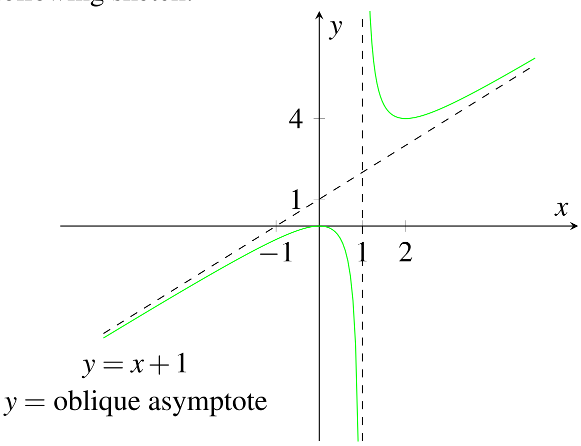
x		0	
$f'(x)$	$+$	0	$-$
	$\nearrow$		$\searrow$

From these formulae for $f'(x)$ and $f''(x)$ we get the following sign diagrams:

x		0	
y''	$-$	u	$+$
	$\cap$		$\cup$

And so we had quickly build the sign diagram with the pattern minus, plus, sad face, followed by smiley face, so that the curve is concave down for x < 1, and concave up for x >n 1. Putting this information together, drawing the axes and two asymptotes, we get the curve in two branches, where the first branch concave down, achieving a local maximum corresponding to the origin, and the second branch concave up, achieving a local minimum corresponding to the point (2,4).

Thus there is a turning point at $x = 0$, with local maximum $y = 0$, and a turning point at $x = 2$, with local minimum $y = 4$. The curve is concave down for $x < 1$ and concave up for $x > 1$, and we get the following sketch:



Notice how by contrast with the previous example, the curve gets closer and closer to the oblique asymptote from above as we move to the right and from below as we move to the left. In this section, we produced a checklist for sketching curves in the plane and worked through the details in three contrasting examples. The first involved a cubic function, where there are no asymptotes, but there were two turning points and an inflection. Interestingly, the sketch shows there's exactly one x-intercept, and in this case roughly where to find it, which will be useful if one wanted to go further and solve an associated cubic equation. The second and third examples involved rational functions with similar rules but contrasting behavior. Both of them had two branches and identical vertical and oblique asymptotes but approached differently. One curve had no turning points, whilst the other curve had two. They both had shifts in concavity as one passes across the vertical asymptote, so to speak, which we may think of as creating some kind of inflection of plus and minus infinity. Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading, and I look forward to seeing you again soon.

23.2.2 Practice Quiz

Question 1

Find the y-intercept for the curve $y = \frac{16-(x-2)^2}{x^2-1}$.

(a) -8
(b) -12
(c) 0
(d) -4
(e) -16

Question 2

Find the x-intercepts for the curve $y = \frac{16-(x-2)^2}{x^2-1}$.

(a) -6, 2
(b) -4, 4
(c) 6, 8
(d) -2, 6
(e) -2, 0

Question 3

Which one of the following statements is true for the curve $y = \frac{1}{x}$?

(a) There is exactly one x-intercept and exactly one y-intercept, both equal to 1.
(b) There is only one asymptote to the curve and it is the x-axis.
(c) The curve does not have any asymptotes.
(d) There is only one asymptote to the curve and it is the y-axis.
(e) Both the x and y axes are asymptotes to the curve.

Question 4

Which one of the following statements is true for the curve $y = \frac{2}{x-1}$?

(a) The asymptotes are the x-axis and the line $y = x$.
(b) The asymptotes are the y-axis and the line $y = x$.
(c) The asymptotes are the y-axis and the line $y = x+1$.
(d) The asymptotes are the y-axis and the line $y = x-1$.
(e) The asymptotes are the x-axis and the line $y = x-1$.

Question 5

Which one of the following statements is true for the curve $y = \frac{x}{x^2-1}$?

(a) There is only one asymptote to the curve and it is the x-axis.
(b) There is exactly one x-intercept and exactly one y-intercept, both equal to 0.
(c) There is only one asymptote to the curve and it is the y-axis.
(d) The curve does not have any asymptotes.
(e) Both the x and y-axes are asymptotes to the curve.

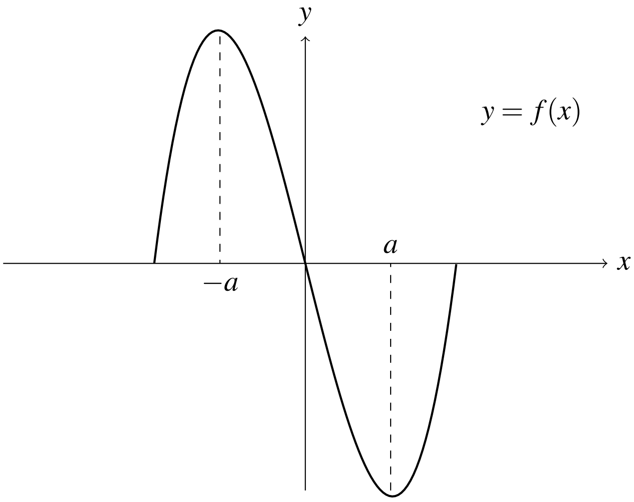
Question 6

Use long division of polynomials to find the oblique asymptote to the curve $y = \frac{x^3-2x^2+x-1}{x^2+1}$.

(a) $y = x-2$
(b) $y = x+2$
(c) $y = x-1$
(d) $y = x$
(e) $y = x+1$

Question 7

Consider the following curve $y = f(x)$ where a is some positive real number.



Which one of the following pairs of sign diagrams for y' and y'' best matches this curve?

(a)

x		$-a$		a	
y'	$-$	0	$+$	0	$-$

x		0	
y”	$+$	0	$-$

(b)

x		$-a$		a	
y'	$-$	0	$+$	0	$-$

x		0	
y”	$-$	0	$+$

(c)

x		$-a$		a	
y'	$+$	0	$-$	0	$+$

x		0	
y”	$+$	0	$-$

(d)

x		$-a$		a	
y'	$+$	0	$+$	0	$+$

x		2	
y”	$-$	0	$-$

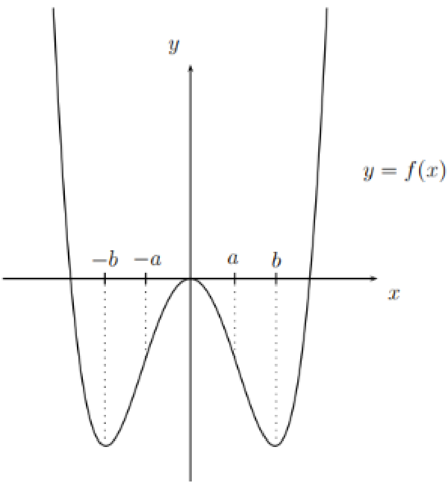
(e)

x		-3		3	
y'	$+$	0	$-$	0	$+$

x		0	
y”	$-$	0	$+$

Question 8

Consider the following curve $y = f(x)$ where a and b are some positive real number.



Which one of the following pairs of sign diagrams for y' and y'' best matches this curve?

(a)

x		$-b$		0		b	
y'	$-$	0	$+$	0	$-$	0	$+$

x		$-a$		a	
y”	$-$	0	$+$	0	$-$

(b)

x		$-b$		0		b	
y'	$+$	0	$-$	0	$+$	0	$-$

x		$-a$		a	
y''	$-$	0	$+$	0	$-$

(c)

x		$-b$		0		b	
y'	$-$	0	$+$	0	$-$	0	$+$

x		$-a$		a	
y''	$+$	0	$-$	0	$+$

(d)

x		$-b$		0		b	
y'	$+$	0	$-$	0	$-$	0	$+$

x		$-a$		a	
y''	$-$	0	$+$	0	$-$

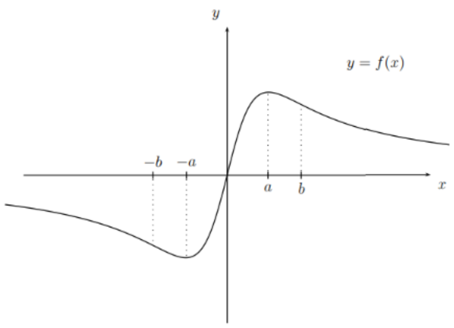
(e)

x		$-b$		0		b	
y'	$+$	0	$-$	0	$+$	0	$-$

x		$-a$		a	
y''	$+$	0	$-$	0	$+$

Question 9

Consider the following curve $y = f(x)$ where a and b are some positive real number.



Which one of the following pairs of sign diagrams for y' and y'' best matches this curve?

(a)

x		$-a$		a	
y'	$-$	0	$+$	0	$-$

x		$-b$		0		b	
y''	$-$	0	$+$	0	$-$	0	$+$

(b)

x		$-a$		a	
y'	$+$	0	$-$	0	$+$

x		$-b$		0		b	
y''	$+$	0	$-$	0	$+$	0	$-$

(c)

x		$-a$		a	
y'	$-$	0	$+$	0	$-$

x		$-b$		0		b	
y''	$+$	0	$-$	0	$+$	0	$-$

(d)

x		$-a$		a	
y'	$-$	0	$+$	0	$-$

x		$-b$		0		b	
y''	$-$	0	$+$	0	$+$	0	$-$

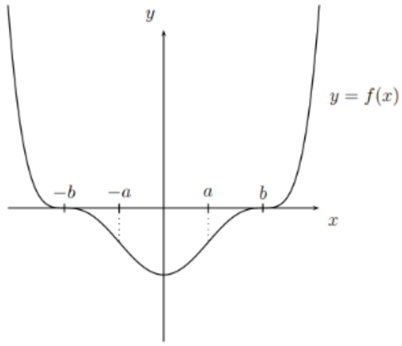
(e)

x		$-a$		a	
y'	$+$	0	$-$	0	$+$

x		$-b$		0		b	
y''	$-$	0	$+$	0	$-$	0	$+$

Question 10

Consider the following curve $y = f(x)$ where a and b are some positive real number.



Which one of the following pairs of sign diagrams for y' and y'' best matches this curve?

(a)

x		$-b$		0		b	
y’	$-$	0	$+$	0	$+$	0	$-$

x		$-b$		$-a$		a		b	
y”	$+$	0	$-$	0	$+$	0	$-$	0	$+$

(b)

x		$-b$		0		b	
y’	$+$	0	$-$	0	$+$	0	$-$

x		$-b$		$-a$		a		b	
y”	$-$	0	$+$	0	$-$	0	$+$	0	$-$

(c)

x		$-b$		0		b	
y’	$-$	0	$-$	0	$+$	0	$+$

x		$-b$		$-a$		a		b	
y”	$+$	0	$-$	0	$+$	0	$-$	0	$+$

(d)

x		$-b$		0		b	
y’	$-$	0	$-$	0	$+$	0	$+$

x		$-b$		$-a$		a		b	
y”	$-$	0	$+$	0	$-$	0	$+$	0	$-$

(e)

x		$-b$		0		b	
y’	$-$	0	$+$	0	$-$	0	$+$

x		$-b$		$-a$		a		b	
y”	$+$	0	$-$	0	$+$	0	$-$	0	$+$

Answers

The answers will be revealed at the end of the module.

24. The Chain Rule

24.1 The Chain Rule

There are three important techniques or formulae for differentiating more complicated functions in terms of simpler functions, known as the Chain Rule, the Product Rule, and the Quotient Rule. In this section, we introduce and explain the first of these, the Chain Rule, which is considered by many to be the simplest to remember, explain, and apply.
Let y = f(u), be a function of the variable u, while u itself is a function of another variable x, say u = g(x). Using Leibniz's notation, the Chain Rule says simply; $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. You can think of that differential du in the numerator, canceling with du in the denominator as though these were ordinary fractions. There's an equivalent statement using the function dash notation for the derivative.$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. The two statements look quite different but in fact carry exactly the same information, and I'll come back to the function dash notation version later. For now, let's focus on the version of the chain rule using Leibniz's notation and work through some examples.

Here, $y = (-3x+6)^2$, and we want to find the derivative $\frac{dy}{dx}$. We can solve this directly by expanding the brackets to get the quadratic $9x^2 - 36x + 36$, and then differentiating in the usual way to get $18x - 36$, which factorizes as $18(x-2)$.
We can also solve this indirectly using the chain rule. To do this, we put $y = u^2$, where $u = -3x+6$, so that y becomes a particularly simple function of u. The chain rule says

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

But the derivative of y with respect to u is just $2u$ since $y = u^2$, and the derivative of u with respect to x is just -3. So, this product becomes $2u \cdot (-3)$, which is $-6u$. Converting back into an expression involving x, this becomes $-6(-3x+6)$, which quickly simplifies to $18(x-2)$, which is the same answer that we obtained by the direct method.
It's interesting that we get to the same answer by quite different routes. In this example, there's about the same amount of work using either method. In more elaborate examples, the chain rule can save a lot of time and work. With little bit of practice, you will develop fluency quickly and find the chain rule effective, powerful and easy to use.
In the next example, we want to find the derivative of $y = e^{-x}$. To set up the chain rule, we put $y = e^u$, where $u = -x$. Then again, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. But $y = e^u$, so $\frac{dy}{du}$ is just e^u, and $\frac{du}{dx} = -1$. So,

this becomes $-e^u$, which becomes $-e^{-x}$, getting an expression in terms of x. This shows that the derivative of e^{-x} is $-e^{-x}$.

More generally, we can consider $y = e^{kx}$, where k is a constant, and imitate the previous calculation when k was equal to -1. Put $u = kx$ so that $y = e^u$. Again, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, which, similar to before, becomes $e^u \cdot k$, which is ke^u, which becomes ke^{kx}. This shows that the derivative of e^{kx} is ke^{kx}.

You may be curious to know why the chain rule works, and we now sketch a proof. We take the heuristic idea of canceling differentials in the numerator and denominator, and relate this to the formal definition of the derivative in terms of limits. Using Leibniz's notation, the derivative is defined as the limit as Δx tends to zero of the quotient $\frac{\Delta y}{\Delta x}$, where Δx is a small change in x and Δy is the corresponding small change in y. We also have the variable u somewhere in-between acting like a stepping stone in the composition of functions, and the change in x causes a change in u denoted by Δu.
We can suppose in this sketch that Δu is non-zero, and then insert Δu both in the numerator and denominator of the fraction without affecting the overall value of the expression of which we're taking the limit. But the limit of a product is the product of the limits. So, we can rewrite this as

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right),$$

and we can adjust the way we've written the first limit. Because we expect Δu to get really, really small as Δx does, that is $\Delta u \to 0$ as $\Delta x \to 0$.
Then recognize the first limit as a definition of $\frac{dy}{du}$ and the second limit as a definition of $\frac{du}{dx}$. This completes the sketch of the proof that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Here's the statement of the chain rule again with both versions. Let's focus now on the version using function dash notation. The explicit connection is that y is a function of u with rule $f(u)$, and u is a function of x with rule $g(x)$. So, we're feeding the rule for g into the rule for f, which creates the composite function $(f \circ g)(x)$. Then $(f \circ g)'(x)$ is the derivative of y with respect to x, not with respect to u, which we had previously just called y', also called $\frac{dy}{dx}$ in Leibniz's notation, which is $\frac{dy}{du} \cdot \frac{du}{dx}$ by the first form of the chain rule that we have discussed already in detail.

But $\frac{dy}{du}$ becomes $f'(u)$ and $\frac{du}{dx}$ becomes $g'(x)$ in function notation, which is $f'(g(x)) \cdot g'(x)$ because $u = g(x)$, which gives us the required statement of the chain rule using function notation.

Let's practice this version of the chain rule on an example. Let f and g be functions with rules $f(x) = x^3$ and $g(x) = x^2 - 1$. We wish to find the derivatives of the composite functions $f \circ g$ and $g \circ f$. Certainly, the derivative of f is $3x^2$ and the derivative of g is $2x$. So, applying the chain rule using function notation, we'll get

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

which is

$$f'(x^2 - 1) \cdot g'(x).$$

This becomes

$$3(x^2-1)^2 \cdot 2x,$$

which simplifies to

$$6x(x^2-1)^2.$$

On the other hand, applying the chain rule to these functions but composing in the other order, we have

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x),$$

which becomes

$$g'(x^3) \cdot f'(x),$$

which simplifies to

$$2x^3 \cdot 3x^2 = 6x^5.$$

Notice that we didn't need to know the rules for $f \circ g$ or $g \circ f$ and can follow the formula for the chain rule quite mechanically.

Of course, we can solve this problem also without the chain rule. In the case of $f \circ g$, the rule becomes

$$f(g(x)) = f(x^2-1) = (x^2-1)^3.$$

Expanding this out gives

$$(x^2-1)^3 = x^6 - 3x^4 + 3x^2 - 1,$$

with derivative

$$6x^5 - 12x^3 + 6x,$$

which factorizes as

$$6x(x^4 - 2x^2 + 1),$$

and further factorizes to

$$6x(x^2-1)^2,$$

which agrees with the answer we found quickly and indirectly using the chain rule.
In the case of $g \circ f$, the rule becomes $g(f(x))$, which can be rewritten as $g(x^3)$. This is $(x^3)^2 - 1 = x^6 - 1$. The derivative of this is $6x^5$, which agrees with the answer we found using the chain rule.

24.1.1 Examples

Let's see some solved example and end this section too.
1. Find $\frac{dy}{dx}$ when $y = (3x-7)^4$.

Solution: Put $u = 3x-7$, so that $\frac{du}{dx} = 3$, and $y = u^4$, so that $\frac{dy}{du} = 4u^3$. Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3(3) = 12u^3 = 12(3x-7)^3.$$

2. Find $\frac{dy}{dx}$ when $y = (2x^2+3x+1)^3$.

Solution: Put $u = 2x^2+3x+1$, so that $\frac{du}{dx} = 4x+3$, and $y = u^3$, so that $\frac{dy}{du} = 3u^2$. Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(4x+3) = 3(2x^2+3x+1)^2(4x+3).$$

3. Find $\frac{dy}{dt}$ when $y = e^{-5t}$.

Solution: Put $u = -5t$, so that $\frac{du}{dt} = -5$, and $y = e^u$, so that $\frac{dy}{du} = e^u$. Hence

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = e^u(-5) = e^{-5t}(-5) = -5e^{-5t}.$$

4. Find $\frac{dy}{dx}$ when $y = e^{x^2}$.

Solution: Put $u = x^2$, so that $\frac{du}{dx} = 2x$, and $y = e^u$, so that $\frac{dy}{du} = e^u$. Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^{x^2}(2x) = 2xe^{x^2}.$$

5. Find $\frac{dy}{dx}$ when $y = \sin(6\theta)$.

Solution: Put $u = 6\theta$, so that $\frac{du}{d\theta} = 6$, and $u = \sin u$, so that $\frac{dy}{du} = \cos u$. Hence

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = (\cos u)(6) = 6\cos(6\theta).$$

6. Find $\frac{dy}{dx}$ when $y = \cos(\sin x)$.

Solution: Put $u = \sin x$, so that $\frac{du}{dx} = \cos x$, and $y = \cos u$, so that $\frac{dy}{du} = -\sin u$. Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\sin u)(\cos x) = -\cos x \sin(\sin x).$$

7. Find $\frac{dy}{dt}$ when $x = e^{\cos(5t)}$.

Solution: Put $u = \cos(5t)$ and $v = 5t$, so that $\frac{du}{dt} = -5\sin(5t)$, and $u = \cos x$, so that $\frac{dy}{du} = -\sin u$.
Also $x = e^u$, so that $\frac{dx}{du} = e^u$. Hence

$$\frac{dx}{dt} = \frac{dx}{du}\frac{du}{dt} = (e^u)(-5\sin(v)) = -5e^{\cos(5t)}\sin(5t).$$

8. Let f and g be functions such that $f(x) = x^2$ and $g(x) = x^2+1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution: Observe that $f(x) = 3x^2$ and $g(x) = 2x$, so that

$$(f \circ g)(x) = f(g(y)) = x^2 + (y^2) = 6x(x^2+1)^2,$$

and

$$(g \circ f)(x) = g(f(x)) = 2x(x+1) = 2x(3x^2+1).$$

9. Let f and g be functions such that $f(x) = x^2 + x + 1$ and $g(x) = (x-1)^3$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution: Observe that $f(x) = 2x+1$ and $g(y) = 3(x-1)^3$, so that

$$(f \circ g)(x) = f(g(y)) = 2(x-1)^3(y+3) - 1/3(2x-1)^2,$$

and

$$(g \circ f)(x) = g(f(x)) = 3(x+1)^2(x+1) = 3x^2(x+1).$$

In this section, we introduced the Chain Rule, using both Leibniz's notation and function notation, and applied it to differentiate composite functions in several contrasting examples. Please re-read it, if you didn't get it, and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

24.1.2 Practice Quiz

Question 1

Find $\frac{dy}{dx}$ when $y = (2x-1)^2$.

(a) $\frac{(2x-1)^3}{3}$
(b) $\frac{(2x-1)^3}{6}$
(c) $2x$
(d) $2(2x-1)$
(e) $4(2x-1)$

Question 2

Find $\frac{dy}{dx}$ when $y = (4x+7)^3$.

(a) $3(4x+7)^2$
(b) $3x^2$
(c) $12(4x+7)^2$
(d) $84(4x+7)^2$
(e) $\frac{(4x+7)^4}{16}$

Question 3

Find $\frac{dy}{dx}$ when $y = (2-3x)^6$.

(a) $-18(2-3x)^5$
(b) $6(2-3x)^5$
(c) $18(2-3x)^5$
(d) $-36(2-3x)^6$
(e) $36(2-3x)^6$

Question 4

Find $\frac{dy}{dx}$ when $y = (x^2 - x + 1)^4$.
(a) $4(x^2 - x + 1)^3$
(b) $8x(x^2 - x + 1)^3$
(c) $4(2x - 1)(x^2 - x + 1)^3$
(d) $4(x^2 - x + 1)^3$
(e) $4(x^2 - 1)^3$

Question 5

Find $\frac{dy}{dx}$ when $y = e^{7x}$.
(a) $7e^{7x}$
(b) e^7
(c) e^{7x}
(d) $\frac{e^{7x}}{7}$
(e) $7xe^{7x-1}$

Question 6

Find $\frac{dx}{dt}$ when $x = e^{-t}$.
(a) $-te^{-t-1}$
(b) $-e^{-t}$
(c) e^{-t}
(d) $-e^t$
(e) $-te^{-t}$

Question 7

Find $\frac{dy}{dx}$ when $y = \sin(3x)$.
(a) $-\cos(3x)$
(b) $3\cos(3x)$
(c) $\cos(3x)$
(d) $3\cos x$
(e) $-3\cos x$

Question 8

Find $\frac{dx}{dt}$ when $x = \cos(3\theta)$.
(a) $-3\theta\sin(3\theta)$
(b) $-\sin(3\theta)$
(c) $\sin(3\theta)$
(d) $3\sin(3\theta)$
(e) $-3\sin(3\theta)$

Question 9

Let f and g be functions with rules $f(x) = 5 - x^3$ and $g(x) = x^2$. Find $(f \circ g)'(x)$.
(a) $5 - 6x^5$
(b) $2x(5 - x^6)$
(c) $-6x^5$
(d) $2x(5 - 8x^3)$
(e) $5 - x^6$

Question 10

Let f and g be functions with rules $f(x) = 5 - x^3$ and $g(x) = x^2$. Find $(g \circ f)'(x)$.

(a) $6x^2(5-x^3)$
(b) $18x^5$
(c) $-6x^2(5-x^3)$
(d) $-27x^6$
(e) $-3x^2(5-x^3)^2$

Answers

The answers will be revealed at the end of the module.

24.2 Applications of the Chain Rule

In today's video, we demonstrate two contrasting applications of the Chain Rule, to begin to understand the behavior of the Gaussian curve, $y = e^{-x^2}$, related to the normal probability distribution used in statistics, and to predict how long it takes for an ice cube to melt away completely.

Our first example is a Gaussian curve, $y = e^{-x^2}$, which is often described as bell-shaped. It's actually the graph of something called a probability density function for the normal distribution used in statistics. So, it is of fundamental importance in applications to science and many other disciplines. The curve is named after Carl Friedrich Gauss, another giant in the development of modern mathematics who lived and worked from the end of the 18th century into the middle of the 19th century. It's a common debating theme, who was a greater mathematician, Euler or Gauss. The description 'bell-shaped' is possibly slightly misleading. There are many curves you could say are in the shape of a bell, but this is the one most people have in mind when they refer to a bell-shaped curve. We'll meet another important curve in the shape of a bell in a later section known as the Witch of Maria Agnesi, after the famous female 18th-century Italian mathematician Maria Agnesi. But more about that later.

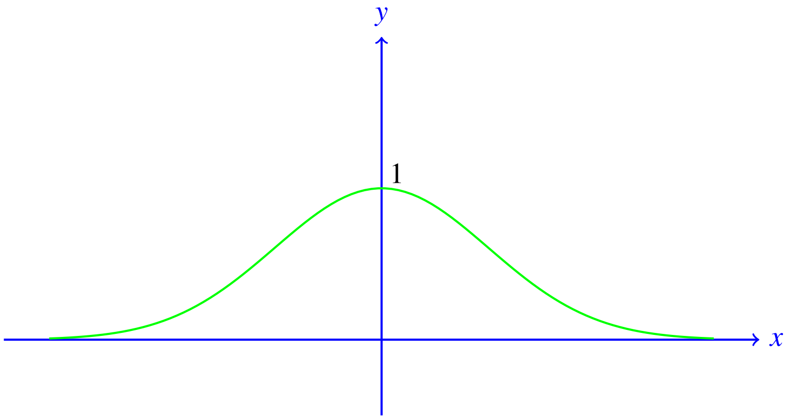
I'd like to apply the methods of curve sketching that we've discussed in earlier sections to the function $y = e^{-x^2}$. In the course of doing this, you will see an application of the chain rule. We first find the y-intercept, which is one. There are, in fact, no x-intercepts because any power of e, in particular, e^{-x^2} for any x, is positive. As to asymptotic behavior, observe that e^{-x^2}, which is the reciprocal of e^{x^2}, tends to zero as x gets arbitrarily large and positive or negative. So, the x-axis is a horizontal asymptote.

We next look at properties of the derivative. This is where the chain rule comes in handy. Put $y = e^u$ where $u = -x^2$, so $\frac{dy}{du} = e^u$, and $\frac{du}{dx} = -2x$. Hence, the derivative y', which is $\frac{dy}{dx}$ in Leibniz notation becomes $\frac{dy}{du} \times \frac{du}{dx}$ by the chain rule, which now becomes $e^u \times (-2x)$. Which is $-2x \times e^{-x^2}$. It's important to know when the derivative is zero. In this case, this only happens when $x = 0$, noting that the factor e^{-x^2} is never zero. We can then build a simple sign diagram by noting that y' is zero when x is zero.

x		1	
y'	$+$	0	$-$
	$\diagdown$		$\diagup$

We have a pattern of plus-minus, indicating that the curve is increasing for $x < 0$, decreasing for $x > 0$ and achieving a maximum at $x = 0$.

We can put this all together so far drawing the axes and noting the y-intercept. The asymptotic behavior of the x-axis, both to the far left and far right, and the fact that there is a turning point at the y-intercept when $x = 0$. Because the curve is increasing from the left and decreasing to the right, it's natural to fill in the rest of the curve to obtain below bell shape.



To create this shape, there appear to be two natural inflections where the curve changes concavity both to the right and to the left. The curve has natural symmetry above the vertical axis because e^{-x^2} doesn't change if you replace x by $-x$, an instance of an even function. To be sure about the way the curve behaves, and in particular, to find these inflections, we need to go further and investigate the second derivative y''. But y' is quite complicated. We don't yet have the tools to find y'' easily. To do so, we employ the so-called product rule, which tells us how to differentiate a product of expressions. But we defer the complete analysis until after next time when the product rule will be explained and illustrated.

The next example involves some quite intricate mathematics, so you might not be able to follow all the details at first. I hope you'll nevertheless persevere, maybe re-reading the text time to time because the mathematics describes a very beautiful physical phenomenon with surprising accuracy and involves an elegant and powerful application of the chain rule.

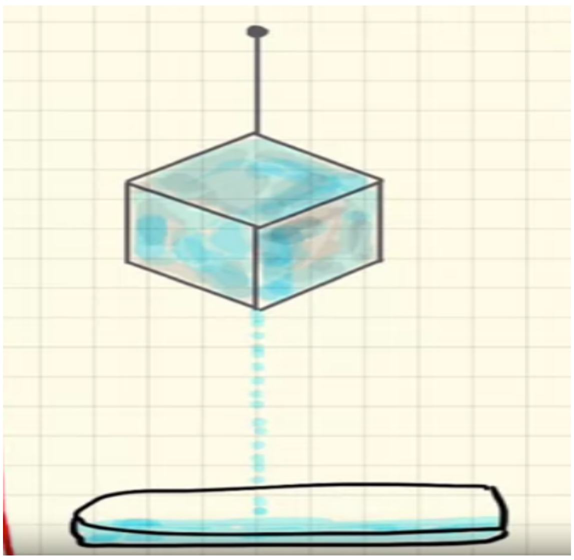


Figure 24.1: Figure of Suspended Ice

In this next example, we've created an ornamental sculpture in the form of a perfect ice cube, which we've suspended from the ceiling in a warm room. One hundred liters of water were used to create the original ice cube. We notice after three hours, that 10 liters of water have accumulated in a tray below the dripping ice cube. Now the problem is, to use mathematics to estimate the number of

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

hours it takes the ice cube to melt away to nothing. We begin by denoting the side length of the cube by x units. What the units turn out to be is unimportant. We don't even need to know what the original value of x is or even to know what exact volume of frozen ice corresponds to, say, one liter of water. It's very important, however, to think of the value of x as a function that changes with time t,$x = x(t)$. The units of time will be hours since the question asks "How many hours it takes for the cube to melt away to nothing?" You might notice in this example, we're using t as the independent variable and x as the dependent variable, similar to the conventions used in describing displacement functions.

There are two quantities that turn out to be important in the modeling that precedes the analysis. These are the volume of the cube, denoted by V, which is also a function of time t and takes the value x^3 (cubic units) or $V = V(t) = x^3$, for whatever value x happens to be at that particular time. The surface area of the cube, denoted by A, is another function of time t and takes the value $6x^2$ since there are six faces to the cube, each of area x^2 (square units). So,$A = A(t) = 6x^2$

So, we have these three quantities associated with the melting ice cube: x, V, and A, all functions of time t. In fact, we want to know when the ice cube disappears, so we want the time t at which all of these will be zero. The key physical fact, that starts our mathematical modeling, is that the rate of melting of the volume of the cube is proportional to the surface area of the cube. This makes good intuitive sense since the surface of the cube is where the cube interfaces directly with the warm air in the room. So, the more area there is on the surface of the cube, the more the associated substance of the cube, measured by volume, should disintegrate from solid into liquid and drip into the tray below.

The rate at which the volume changes with time is the derivative $\frac{dV}{dt}$, and this fact states that $\frac{dV}{dt}$ is a constant multiple of the surface area A, say k times A for some constant k. But $A = 6x^2$, so $\frac{dV}{dt} = 6kx^2$.

We're developing a list of important facts. Note also that $\frac{dV}{dx}$, the derivative of x^3 with respect to x, is $3x^2$. We'd like information about the width x of the cube and how it changes with time t. So, it's natural to ask what the derivative $\frac{dx}{dt}$ might be. The link between this and the other derivatives is the chain rule, which in this case says that $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$.

But $\frac{dV}{dt} = 6kx^2$ and $\frac{dV}{dx} = 3x^2$. So, this says that $6kx^2 = 3x^2 \cdot \frac{dx}{dt}$. Dividing through gives $\frac{dx}{dt} = \frac{6kx^2}{3x^2}$, which simplifies to $2k$. Hence, $\frac{dx}{dt}$ is simply $2k$, which is another constant. If the rate of change of x with respect to time t is constant, then the function x must be a linear function of t with slope $2k$, so that x must have the form $x = 2kt + C$ for some constant C.

This is an important step going backwards from a derivative to a function. It's a fact, and part of a more general theory that we'll explore in the final module, that if the derivative is constant then the function that it comes from must be a linear function. If you think about this geometrically, if the slopes of all the tangent lines of a curve are constant, then the curve must be a straight line. That's what's happening here. Our variables are t and x, rather than the usual x and y. This is significant progress to see that $V = x^3$ and $x = 2kt + C$ where k and C are some constants.

We haven't yet used the given information that 10 liters of water dripped off the cube in the first three hours, which represents 10 percent of the volume, leaving intact 90 percent or $\frac{9}{10}$ of the original volume. So, $V(3) = \frac{9}{10}V(0)$.
We can use the fact that $V = x^3$ to rewrite this as x^3 evaluated at $t = 3$ is $\frac{9}{10}$ of x^3 evaluated at $t = 0$.

But we know that x evaluated at $t = 0$ is just the constant C and x evaluated at $t = 3$ becomes $6k + C$.

So, we get a nice equation linking the constants k and C. We can rewrite this as $(6k+C)^3 = 0.9C^3$. Now, solve for C, first by taking cube roots to get $6k+C = \sqrt[3]{0.9}C$. Rearranging gives finally $C = \frac{6k}{\sqrt[3]{0.1}-1}$.

We have an elegant relationship between the constants C and k and can now ask the time t at which the cube disappears by melting away, which is when the width x becomes zero. That is, when $2kt+C=0$. We can substitute in the formula for C in terms of k. We therefore want t such that this equation holds.

Rearranging the information and noticing that k cancels out, we get $t = \frac{3}{1-\sqrt[3]{0.9}}$, which our calculator tells us is approximately 86.9 to one decimal place. Thus, the answer to our original problem is that it takes almost 87 hours for the cube to melt away completely.

If you look carefully at the mathematical analysis, you'll see that the final answer depends only on the proportion of the cube that had melted after three hours. Clearly, the warmer the temperature, the larger would be this proportion. One could generalize the problem to include this proportion as a variable parameter and deduce a general formula. It's interesting that the model predicts the cube disappears completely. There is a time at which the width and the volume actually become zero. This is in contrast to exponential decay models where the amount of the given substance tends towards zero very rapidly but never actually reaches zero. In the case of our melting ice cube, the width turns out to be a decreasing linear function of time.

In this section, we discussed two contrasting examples that relied on the chain rule in order to progress towards a solution. The first example looked at the Gaussian bell-shaped curve related to the normal probability distribution used in statistics. The Chain Rule assisted us in finding the derivative of the associated function. In a future section, I will go further with this example to find and analyze the second derivative. But to achieve this, we'll use the product rule which is introduced in the next section. In the second example, we performed a detailed analysis of a melting ice cube. We used the chain rule to connect together different aspects of the problem expressed in terms of derivatives. We obtained a model in which the main measure, the width of the cube, becomes a decreasing linear function so that the cubed disappears completely. This is in contrast to exponential decay models where the value of the function decreases and gets arbitrarily close to zero, but that's not actually reach zero. Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

25. The Product Rule

25.1 The Product Rule

In this section, we introduce, explain, and illustrate the Product Rule, which enables you to differentiate a product of expressions in terms of the derivatives of the factors. Consider a function y of x which is a product of u and v, where u and v are themselves functions of x. The product rule states that the derivative of y with respect to x is u times the derivative of v plus v times the derivative of u, expressed this way using Leibniz notation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Equivalently, using function notation and the dash to denote the derivative, the product rule says $y' = uv' + vu'$.

For example, let y be the product of a quadratic and a linear polynomial, and we want to find the derivative y'. We'll give two solutions. The first solution is direct without using the product rule. We expand the brackets and gather together the terms to form this cubic polynomial:

$$y = (x^2 + 3x - 4)(2x - 5)$$

Expanding this gives:

$$y = 2x^3 - 5x^2 + 6x^2 - 15x - 8x + 20$$

$$y = 2x^3 + x^2 - 23x + 20$$

Then, differentiate immediately to get the quadratic:

$$y' = 6x^2 + 2x - 23$$

The second solution uses the product rule. We write y as the product uv where $u = x^2 + 3x - 4$ and $v = 2x - 5$, so that the derivative of u is $2x + 3$ and the derivative of v is 2. By the product rule, $y' = uv' + vu'$, and we just put all the pieces together:

$$u = x^2 + 3x - 4$$
$$u' = 2x + 3$$
$$v = 2x - 5$$
$$v' = 2$$

Applying the product rule:

$$y' = uv' + vu'$$
$$y' = (x^2 + 3x - 4)(2) + (2x - 5)(2x + 3)$$

Expand and simplify:

$$y' = 2x^2 + 6x - 8 + 4x^2 + 6x - 10x - 15$$
$$y' = 6x^2 + 2x - 23$$

which, of course, agrees with our first solution. The solutions are the same, but the pathways are quite different. Which pathway you take depends on the problem at hand.

Here is another example. Again, y is a product of two expressions involving x and we want to find the derivative y'. We can solve this directly by expanding the brackets, simplifying one of the expressions using an exponential law, and then differentiating directly piece by piece, using the fact repeatedly that the derivative of e^{kx} is ke^{kx} where k is a constant, which is a result we established in an earlier video using the chain rule. Then tidying this up a little to get $-3e^{3x} - 2e^{-x} - 2e^{2x}$.

An alternative solution is to apply the product rule recognizing that y is the product uv, where u is the first factor, $1 + e^{-x}$, and v is the second factor, $2 - e^{3x}$. So, that u' quickly becomes $-e^{-x}$ and v' becomes $-3e^{3x}$. By the product rule, $y' = uv' + vu'$. We can put all the pieces together, expand, simplify, and gather terms to get finally $-3e^{3x} - 2e^{2x} - 2e^{-x}$, which agrees with our first solution. Using the product rule is often the only natural way to proceed, and of course, one tries to avoid going back to the original limit definition of the derivative.

Here's such an example. We want to find the derivative y' when y is the product of $x \sin x$. We'll present the solution using the product rule and Leibniz's notation. The derivative is $\frac{dy}{dx}$ which is $\frac{d}{dx}(x \sin x)$, which by the product rule, is the first factor x times the derivative of the second factor plus the second factor, $\sin x$, times the derivative of the first factor. The derivative of $\sin x$ is $\cos x$ and the derivative of x is one, so we get $x \cos x + \sin x \times 1$, which is simply $x \cos x + \sin x$. Notice how just in a few short steps, we are able to painlessly differentiate a sophisticated function. You're very unlikely to guess this answer and extremely unlikely to navigate through the limit definition of the derivative, which would quickly become a quagmire in an example like this. You can appreciate the power of results like the product rule that provides such succinct and elegant solutions.

We've only stated the product rule and used it, and you might be curious to know why it works. Here's a sketch of proof why it works, and what follows now is quite advanced, tricky, and intricate. So, you shouldn't worry if you find it difficult to follow all of the details.

Let y be a product of u and v where y, u, and v are functions of x. A small change in x called Δx propagates a small change in y called Δy. Well, Δy is the difference between y evaluated at x and y evaluated at $x+\Delta x$. That is,

$$\Delta y = y(x+\Delta x) - y(x)$$

. But y is $u \times v$, so this becomes

$$u(x+\Delta x) \times v(x+\Delta x) - u(x) \times v(x)$$

.

Now comes the magic sleight of hand. We insert

$$-u(x+\Delta x) \times v(x) + u(x+\Delta x) \times v(x)$$

which altogether is zero, so it doesn't change the overall value of the expression. Now, why would we do such a thing making it look so much more complicated? Well, it's common in proofs in mathematics to introduce terms to make things look more complex or elaborate, but to facilitate something that gets you past an obstruction or leads to something ultimately more simple than what you started with. The trick here, using an expansion of zero expressed as an addition and subtraction of the same thing, has lots of applications in algebra. You might remember we employed a similar trick to complete the square when manipulating quadratics in earlier sections.

$$\begin{aligned}
&= u(x+\Delta x)\left(v(x+\Delta x) - v(x)\right) + \left(u(x+\Delta x) - u(x)\right)v(x) \\
&= u(x+\Delta x)\left(v(x+\Delta x) - v(x)\right) + v(x)\left(u(x+\Delta x) - u(x)\right) \\
&= u(x+\Delta x)\Delta v + v(x)\Delta u.
\end{aligned}$$

Observe above that in the first two terms of this expression, there is a common factor of $u(x+\Delta x)$, which we can bring outside of $v(x+\Delta x) - v(x)$. In the second half, there is a common factor of $v(x)$ which we can bring outside on the right of $u(x+\Delta x) - u(x)$. Which is significant progress because $v(x+\Delta x) - v(x)$ is just Δv, the change in v, and $u(x+\Delta x) - u(x)$ is just Δu, the change in u. Now, you start to see how everything is starting to simplify. We have Δy equals $= u(x+\Delta x)\Delta v + v(x)\Delta u.$, and we can divide everything through by Δx, setting up something that looks close to the derivative $\frac{dy}{dx}$.

Hence, by the limit definition of the derivative, using Leibniz notation, and making free use of limit laws,

$$\begin{aligned}
&\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)\Delta v + v(x)\Delta u}{\Delta x} \\
&= \left(\lim_{\Delta x \to 0} u(x+\Delta x)\right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} v(x)\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right) \\
&= \left(\lim_{\Delta x \to 0} u(x+\Delta x)\right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) + v(x)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right) \\
&= u(x)\frac{dv}{dx} + v(x)\frac{du}{dx},
\end{aligned}$$

(using continuity of u at the last step). This completes the sketch of the proof of the product rule.

25.1.1 Solved Examples

Before ending this section, let's see some solved examples.

Examples

1. Differentiate $y = (3x^2+6x-7)(4x+1)$.

   Solution (direct): Expanding the brackets, we have

   $$y = 12x^3+3x^2+24x^2+6x-28x-7 = 12x^3+27x^2-22x-7,$$

   so that

   $$y' = 36x^2+54x-22.$$

   Solution (using the Product Rule): We have $y = uv$ where $u = 3x^2+6x-7$, so that $u' = 6x+6$, and $v = 4x+1$, so that $v' = 4$. By the Product Rule,

   $$y' = uv'+vu' = (3x^2+6x-7)(4)+(4x+1)(6x+6)$$

   $$= 12x^2+24x-28+24x^2+24x+6x+6 = 36x^2+54x-22,$$

   which, of course, agrees with the direct solution.

2. Differentiate $y = (3-e^{2x})(2+e^{-x}-e^{5x})$.
   Solution (direct): Expanding the brackets, and using an exponential law, we have

   $$y = 6+3e^{-x}-3e^{5x}-2e^{2x}-e^{2x}e^{-x}+2e^{2x}e^{5x}$$

   $$= 6+3e^{-x}-3e^{5x}-2e^{2x}-e^{x}+2e^{7x},$$

   so that, using the fact that $\frac{d}{dx}(e^{kx}) = ke^{kx}$ for any constant k,

   $$y' = -3e^{-x}-15e^{5x}-4e^{2x}-e^{x}+7e^{7x}.$$

   Solution (using the Product Rule): We have $y = uv$ where $u = 3-e^{2x}$, so that $u' = -2e^{2x}$, and $v = 2+e^{-x}-e^{5x}$, so that $v' = -e^{-x}-5e^{5x}$. By the Product Rule,

   $$y' = uv'+vu' = (3-e^{2x})(-e^{-x}-5e^{5x})+(2+e^{-x}-e^{5x})(-2e^{2x})$$

   $$= -3e^{-x}+e^{2x}e^{-x}-15e^{5x}+5e^{2x}e^{5x}-4e^{2x}-2e^{2x}e^{-x}+2e^{2x}e^{5x}$$

   $$= -3e^{-x}-15e^{5x}+e^{x}+5e^{7x}-4e^{2x}-2e^{x}+2e^{7x}$$

   $$= -3e^{-x}-15e^{5x}-4e^{2x}-e^{x}+7e^{7x},$$

   which, of course, agrees with the direct solution.

3. Find $\frac{dy}{dt}$ when $y = t^2e^{-3t}$.
   Solution: We have $y = uv$ where $u = t^2$ and $v = e^{-3t}$, so that $\frac{du}{dt} = 2t$ and $\frac{dv}{dt} = -3e^{-3t}$.
   By the Product Rule,

   $$\frac{dy}{dt} = \frac{d}{dt}(t^2e^{-3t}) = t^2\left(-3e^{-3t}\right)+e^{-3t}(2t) = t^2(-3)e^{-3t}+2te^{-3t} = e^{-3t}(2t-3t^2).$$

4. Find $\frac{dy}{dt}$ when $y = x^2 \cos(2x)$.
Solution: We have $y = uv$ where $u = x^2$, so that $u' = 2x$, and $v = \cos(2x)$, so that $v' = -2\sin(2x)$, by the Chain Rule. By the Product Rule,
$$y' = u'v + uv' = x^2(-2\sin(2x)) + \cos(2x)(2x) = 2x(\cos(2x) - x\sin(2x)).$$
5. Find $\frac{dy}{dt}$ when $t = e^{\ln(2t)}$.
Solution: We have $u = \ln(2t)$, then $u' = 2t$, and, applying the Chain Rule, noting a common natural variation in the notation, just to avoid introducing a new symbol in making the substitution.
$$\frac{d}{dt}(e^{\ln(2t)}) = \frac{d}{dt}(t) = \frac{1}{t}.$$
By the Product Rule,
$$\frac{du}{dx} = \frac{du}{dt} = e^{\left(\frac{1}{t}\right)}(\ln(2t))(2t) = (1 + \ln(2t)).$$
6. Find $\frac{dx}{dt}$ when $t = \sin^3(\theta)\cos^4(\theta)$.
Solution: By the Product and Chain Rules, we have
$$\frac{dx}{dt} = \left(\sin^3(\theta)\cos^4(\theta)\right)' = \left(\sin^3(\theta)\right)'\cos^4(\theta) + \cos^4(\theta)\left(\sin^3(\theta)\right)'.$$
7. Find y when $y = e^x \sin x \cos x$.
Solution: By the Product Rule,
$$y' = (e^x)' \sin x \cos x + e^x (\sin x \cos x)'.$$
$$y' = e^x(\sin x \cos x) + e^x\left(\cos^2(x) - \sin^2(x)\right).$$
$$y' = e^x(\sin x \cos x + \cos^2(x) - \sin^2(x)).$$

In this section, we introduced and illustrated the product rule which enables you to differentiate a product of expressions in terms of the derivatives of the factors. We then sketched a proof which involved some subtle algebraic manipulation and the limit definition of the derivative using Leibniz notation. Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

25.1.2 Practice Quiz

Question 1

Find y' when $y = (x^2 + 1)(3x^2 - 6)$.
(a) $x(5x^2 + 3x^2 - 12)$
(b) $x(5x^2 + 3x^2 - 12)$
(c) $6x^3$
(d) $(2x + 1)(3x^2 - 6)$
(e) $5x^4 + 3x^2 - 12$

Question 2

Find y' when $y = (2 + e^{3x})(1 - e^{-x})$.
(a) $3e^x$
(b) $3(e^x + e^{-x})$
(c) $2e^{-x} - 2e^{2x} + 3e^{3x}$
(d) $2e^{-x} + 2e^x - 3e^x$
(e) $2e^{-x} - 2e^x + 3e^x$

Question 3

Find $\frac{dy}{dt}$ when $y = te^{-2t}$.
(a) $-2e^{-2t}$
(b) $e^{-2t}(1-2t)$
(c) $-2te^{-2t}$
(d) $e^{-2t}(1+2t)$
(e) $e^{-2}(1+t)$

Question 4

Find $\frac{du}{dt}$ when $u = t\ln t$.
(a) $1+t$
(b) $1+\frac{1}{t}$
(c) $\ln t$
(d) $t\ln t+\frac{1}{t}$
(e) $1+\ln t$

Question 5

Find y' when $y = x^2 \sin x$.
(a) $2x\sin x+\cos x$
(b) $x(\sin x+2\cos x)$
(c) $2x\cos x$
(d) $x(2\sin x-2\cos x)$
(e) $x(\cos x+2\sin x)$

Question 6

Find $\frac{dy}{d\theta}$ when $y = \theta\cos\theta$.
(a) $\theta(\cos\theta+\sin\theta)$
(b) $\cos\theta-\theta\sin\theta$
(c) $\cos\theta+\sin\theta$
(d) $\cos\theta-\sin\theta$
(e) $\theta(\cos\theta-\sin\theta)$

Question 7

Find $\frac{dy}{d\theta}$ when $x = \sin\theta\cos\theta$.
(a) $\cos^2\theta-\sin^2\theta$
(b) $\cos^2\theta-\sin\theta$
(c) $\cos^2\theta+\sin^2\theta$
(d) $\cos\theta\sin\theta$
(e) $-\cos\theta\sin\theta$

Question 8

Find $\frac{dx}{d\theta}$ when $x = \sin^2\theta\cos^3\theta$.
(a) $\sin\theta\cos^2\theta(2\cos^2\theta+3\sin^2\theta)$
(b) $-6\sin^2\theta\cos^3\theta$
(c) $\sin\theta\cos^3\theta(2\cos^2\theta-3\sin^2\theta)$
(d) $2\sin\theta\cos^4\theta-3\sin^3\theta\cos^2\theta$
(e) $6\sin^2\theta\cos^3\theta$

Question 9

Find $\frac{dy}{du}$ when $y = (u^2+1)\sqrt{u^2-1}$.

(a) $\frac{3u^2-1}{\sqrt{u^2-1}}$
(b) $\frac{u(3u^2-1)}{\sqrt{u^2-1}}$
(c) $\frac{u(3u^2+1)}{\sqrt{u^2-1}}$
(d) $\frac{u(2u^2-1)}{\sqrt{u^2-1}}$
(e) $\frac{2u^2-1}{\sqrt{u^2-1}}$

Question 10

Find y' when $y = (x+1)e^x \sin x$.
(a) $e^x(1+x)\sin x + e^x(2+x)\cos x$
(b) $e^x(\cos x + e^x \sin x)$
(c) $e^x(1+x)(\sin x + \cos x)$
(d) $e^x \cos x$
(e) $e^x(2+x)\sin x + e^x(1+x)\cos x$

Answers

The answers will be revealed at the end of the module.

25.2 Applications of the Product Rule

In this section, we apply the Product Rule to further explore the behavior of the Gaussian curve $y = e^{-x^2}$ using the second derivative, and to give the proof by mathematical induction of the formula for differentiating a power of x with the exponent as a positive integer.

In an earlier section, we used methods of curve sketching to go a long way towards describing the graph of the function $y = e^{-x^2}$. We observed that the y-intercept is 1, that there are no x-intercepts, and the x-axis is a horizontal asymptote. We used the Chain Rule to find the derivative y' and built its associated sign diagram. We put this all together, noting the y-intercept, the asymptotic behavior of the x-axis both to the far left and far right, and the fact that there is a turning point when $x = 0$. Because the curve is increasing to the left and decreasing to the right, it's natural to fill in the rest of the curve to obtain this bell shape. To create this shape, there appear to be true natural inflection points where the curve changes concavity. To be sure about the way the concavity changes, we need to go further and investigate the second derivative y''. Because the first derivative is quite complicated, we'll use the product rule to work out its derivative.

We have $y' = -2x \cdot e^{-x^2}$, which we can write as $u \cdot v$, where $u = -2x$ and $v = e^{-x^2}$. Then $u' = -2$, and v is our original y, so v' is, in fact, the same as y'. By the Product Rule, y'', which is the derivative of y', is $uv' + vu'$. This expression becomes, after a couple of steps, $2(2x^2 - 1)e^{-x^2}$. Notice that the factor e^{-x^2} is always positive. Hence, the second derivative is zero precisely when $2x^2 - 1 = 0$. That is, when $x = \pm\frac{1}{\sqrt{2}}$. We can build the sign diagram, noting the important points for x where $y'' = 0$, and the pattern of $+-+$, producing a pattern of concave up, concave down, concave up, as we pass from left to right.

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
y''	$+$	0	$-$	0	$+$
	$\cup$		$\cap$		$\cup$
	Concave up		Concave down		Concave up

This tells us that there are inflections when $x = \pm\frac{1}{\sqrt{2}}$. We can update our previous information in sketching the Gaussian curve, confirming what we expected the concavity might be, and locating the points of inflection. This now completes a thorough investigation of the curve.

Our second application of the Product Rule is to provide a proof of the formula for the derivative of x^n. You will recall that we carefully proved, from the limit definition, that the derivative of x^2 is $2x$, and the derivative of x^3 is $3x^2$. This is a special case of a general pattern that the derivative of x^n is $n \cdot x^{n-1}$, a result that we've used on several occasions without explaining why it's true. The aim of this next application is to carefully prove this formula for all positive integers n, that is, for $n = 1, 2, 3, 4$, and so on. The argument that I'm about to give is called more formally a proof by mathematical induction, which is a very common and powerful proof technique in mathematics. You can read more about it if you wish, though I think you'll be able to follow the main ideas. For many of you, this might be your very first proof by mathematical induction and will serve you well as a prototype example of the technique, in case you go on to do more advanced mathematics.
The claim that we are proving is, in fact, infinitely many statements as n passes through the set of positive integers. We've already carefully checked this in the cases $n = 1$, 2, and 3. The case $n = 1$ matches up nicely with the fact that the derivative of x is 1. The cases $n = 2$ and 3 match up with facts we already know. The problem for us now is to somehow prove the cases $n = 4$, 5, 6, and so on. We don't want to spend a lot of time and effort on each case, and we certainly don't have an infinite amount of time to go through each and every positive integer separately.

This is where you might start to see the power of thinking abstractly. What follows is like a thought experiment. Suppose that by whatever means, we've been able to prove the formula for some particular positive integer n, and let's refer to this formula, only for this particular n, by $\star$. In formal proofs, the statement $\star$ is referred to as the inductive hypothesis. Our aim is to prove statement $\star$ with n replaced by the next positive integer $n+1$. This new statement would be that the derivative of x^{n+1} is $(n+1) \cdot x^n$, and we can refer to this as $\star\star$.

If we can prove that $\star$ implies $\star\star$, then we get an instantaneous infinite chain reaction from the case $n = 1$, and then $\star$ must hold for all positive integers. Why are we able to say that? Well, we've verified $\star$ for $n = 1$, and also for $n = 2$ and 3, but that's not, in fact, important. It's enough that we've verified it for $n = 1$. $\star$ implying $\star\star$ means the formula must be true for $1+1 = 2$, and then feeding 2 back into n for $2+1 = 3$, and then feeding 3 back into n for $3+1 = 4$, and repeating for $4+1 = 5$, for $5+1 = 6$, and so on, forever racing through all the positive integers instantaneously.

So, let's prove $\star\star$ after supposing that $\star$ is true for some particular positive integer n. Notice that x^{n+1} is the product of x^n with x. So, we can apply the product rule to conclude the derivative is x^n times the derivative of the second factor, plus x times the derivative of the first factor. Which becomes $x^n \cdot 1 + x\cdot$ (and at this point we invoke statement $\star$) $n \cdot x^{n-1}$. Then this tidies up in a few steps to become $(n+1) \cdot x^n$. This establishes the statement $\star\star$, hence, $\star$ implies $\star\star$. This implication is called the inductive step in formal proofs by mathematical induction. It shows, by the chain reaction effect that I mentioned earlier, that $\star$ holds for all positive integers n.

Let us see the above things mathematically and formally.
We use the Product Rule to verify that, for all x and for all positive integers n, we have

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof: We use a method known, more formally as *proof by mathematical induction*, which you may see explained in further courses in mathematics. You should be able to follow the main idea, even without a formal introduction to the general method.

The statement above is a claim about all positive integers, $n = 1, 2, 3, 4$, and so on, forever. We first check that it holds in the simplest case $n = 1$:

$$\frac{d}{dx}(x^1) = \frac{d}{dx}(x) = 1 = 1 \cdot x^0 = 1 \cdot x^{1-1},$$

which indeed is the case. Now suppose that the claim holds for some particular positive integer n, that is,

$$\frac{d}{dx}(x^n) = nx^{n-1}. \tag{1}$$

Now this may look like we are cheating, assuming what we want to prove, but that is not the case. We are only assuming the claim holds for one particular value of n, and then seeing if we can use this assumption to leverage up to other values of n.

Observe that $x^{n+1} = x^n \cdot x = uv$ where $u = x^n$ and $v = x$. Then $u' = nx^{n-1}$, by our assumption (*), and $v' = 1$. Hence, by the Product Rule,

$$\frac{d}{dx}(x^{n+1}) = uv' + vu' = x^n(1) + x(nx^{n-1}) = x^n + nx^n = (n+1)x^n,$$

that is,

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n. \tag{**}$$

We have carefully verified the following implication:

$$(*) \implies (**).$$

We checked earlier that (*) holds when $n = 1$, so this implication implies that (**) holds with $n = 1$ by using $n = 1 + 1 = 2$.

But then the implication implies that (**) holds when $n = 2$, which becomes statement (*) again, but now with $2 + 1 = 3$.

Feeding back into the implication, we get (**) holding when $n = 3$, which is (*) again, but now with $n = 3 + 1 = 4$.

Feeding back into the implication, we get (**) holding when $n = 4$, which is (*) again, but now with $n = 4 + 1 = 5$.

We repeat this process of climbing up from one positive integer to the next, forever. By an "instantaneous" infinite chain-reaction, we therefore get that (*) holds for all positive integers n. This completes the proof.

This completes this particular application of the product rule for all positive integers n and for all real numbers x. If we restrict attention to positive real numbers x only, then we can go much further and prove that to differentiate a general power, say x^α for any real number α, you again bring the exponent to the front and make a new exponent by subtracting one. This very general fact follows from the chain rule and properties of logs and exponentials. To see why, express x^α as $e^{\alpha \ln x}$, and then the derivative of x^α becomes the derivative of e^u, where $u = \alpha \ln x$. Which becomes $\frac{d}{du}e^u \cdot \frac{du}{dx}$ by the chain rule and each piece is then straightforward. The derivative of e^u with respect to u is just e^u, and the derivative of u with respect to x is just $\frac{\alpha}{x}$ because the derivative of $\ln x$ is $\frac{1}{x}$. Thus the derivative becomes $e^u \cdot \frac{\alpha}{x}$, which can be rewritten as $x^\alpha \cdot \alpha \cdot x^{-1}$. Which simplifies to $\alpha \cdot x^{\alpha-1}$, and this completes the proof of our formula.

Again let' see the above bluffing into a formal and concise form.
We can go much further if we assume throughout that x is positive. We claim that, for any real number α,

$$\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}.$$

The restriction that $x > 0$ ensures that the power x^α is always sensibly defined.
Proof: We make free use of properties of logs and exponentials throughout. Observe first that

$$x^\alpha = \left(e^{\ln x}\right)^\alpha = e^{\alpha \ln x} = e^u,$$

where $u = \alpha \ln x$, so that

$$\frac{du}{dx} = \alpha \frac{d}{dx}(\ln x) = \frac{\alpha}{x}.$$

By the Chain Rule

$$\frac{d}{dx}(x^\alpha) = \frac{d}{dx}(e^u) = \frac{d}{du}(e^u)\frac{du}{dx} = e^u\left(\frac{\alpha}{x}\right) = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1},$$

completing the proof.

In this section, we applied the product rule first to complete our curve sketching analysis of the Gaussian curve from an earlier section, in particular to understand the curve's behavior with respect to concavity and to locate the inflection points. Secondly, to prove that the derivative of x^n is $n \cdot x^{n-1}$ for all real numbers x and all positive integers n. This proof, in fact, is an example of the technique of proof by mathematical induction. We then used the chain rule and properties of logs and exponentials to see how to generalize this formula for arbitrary real exponents. Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

25.2.1 Practice Quiz

Question 1

Find $\frac{dy}{dx}$ when $y = e^{x^2}$.
(a) $2xe^{x^2}$
(b) e^x
(c) e^{x^2}
(d) $2e^{x^2}$
(e) $x^2e^{x^2-1}$

Question 2

Find $\frac{dy}{dx}$ when $y = e^{-t^2}$.

(a) $-te^{-t^2}$
(b) $-t^2e^{-1}$
(c) e^{-t^2}
(d) $-e^{-t^2}$
(e) $-2te^{-t^2}$

Question 3

Find $\frac{dy}{dx}$ when $y = e^{x^2+bx+c}$, where a, b and c are constants.

(a) $(ax^2+bx+c)e^{x^2+bx+c-1}$
(b) $(2ax+b)e^{x^2+bx+c}$
(c) e^{x^2}
(d) $2axe^{x^2}$
(e) e^{x^2+bx+c}

Question 4

Find y' when $y = \sqrt{x^2+1}$.

(a) $\frac{1}{2\sqrt{x^2+1}}$
(b) $\frac{1}{x\sqrt{x^2+1}}$
(c) $\frac{1}{\sqrt{x^2+1}}$
(d) $\frac{x}{\sqrt{x^2+1}}$
(e) $\frac{2x}{\sqrt{x^2+1}}$

Question 5

Find y' when $y = \sin(\cos x)$.

(a) $\sin x \cos(\cos x)$
(b) $\cos(\cos x)$
(c) $-\sin x \cos(\cos x)$
(d) $-\cos(\sin x)$
(e) $-\cos x \cos(\cos x)$

Question 6

Find $\frac{du}{d\theta}$ when $u = \cos(\sin\theta)$.

(a) $-\sin(\cos\theta)$
(b) $\sin(\sin\theta)$
(c) $\cos\theta\cos(\cos\theta)$
(d) $\cos\theta\sin(\sin\theta)$
(e) $-\cos\theta\sin(\sin\theta)$

Question 7

Find $\frac{dy}{dx}$ when $y = \sin^2(\pi t)$.

(a) $-2\sin(\pi t)\cos(\pi t)$
(b) $2\sin(\pi t)\cos(t)$
(c) $2\pi\sin(t)\cos(\pi t)$
(d) $2\pi\sin(\pi t)\cos(\pi t)$
(e) $2\pi\sin(t)$

Question 8

The volume V of a sphere of radius r is given by the formula $V = \frac{4\pi r^3}{3}$.

Suppose that a snowball initially of radius $r = 100$ cm is shrinking uniformly at the rate of 1 cm per hour, so that $r = r(t) = 100 - t$.

Use the Chain Rule to find a formula for $\frac{dV}{dt}$, the rate at which the volume V of the snowball is shrinking (in cubic centimeters per hour) after t hours have elapsed.

(a) $\frac{dV}{dt} = 4\pi(100-t)^2$
(b) $\frac{dV}{dt} = -12\pi(100-t)^2$
(c) $\frac{dV}{dt} = 8\pi(100-t)$
(d) $\frac{dV}{dt} = 8\pi(100-t)$
(e) $\frac{dV}{dt} = 4\pi(100-t)^2$

Question 9

A balloon, in the shape of a sphere of volume $V = \frac{4}{3}\pi r^3$ cubic cm, where r cm is its radius, is being inflated at the constant rate of 150 cubic cm per second. How fast is the radius of the balloon increasing at the moment $r = 30$?

(a) $\frac{\pi}{25}$ cm/sec
(b) $\frac{1}{25\pi}$ cm/sec
(c) $\frac{1}{24\pi}$ cm/sec
(d) $\frac{24}{\pi}$ cm/sec
(e) $\frac{\pi}{24}$ cm/sec

Question 10

A circle of radius r centered at the origin is described, in the first and second quadrants by the curve

$$y = \sqrt{r^2 - x^2}.$$

Use the Chain Rule to find the slope of the tangent line to the circle at any given point (x, y) on this part of the circle.

(Your answer should also agree with the fact that any tangent to a circle is perpendicular to the radius.)

(a) $\frac{2x}{\sqrt{r^2-x^2}}$
(b) $-\frac{2x}{\sqrt{r^2-x^2}}$
(c) $-\frac{x}{2\sqrt{r^2-x^2}}$
(d) $\frac{x}{\sqrt{r^2-x^2}}$
(e) $-\frac{x}{\sqrt{r^2-x^2}}$

Answers

The answers will be revealed at the end of the module.

26. The Quotient Rule

26.1 The Quotient Rule

In today's video, we introduce, explain, and illustrate the Quotient Rule which enables us to differentiate a quotient of expressions in terms of the derivatives of the numerator and denominator.

Consider a function y of x which is a quotient u divided by v where u and v are themselves functions of x. The quotient rule states that the derivative of y with respect to x is v times the derivative of u, minus u times the derivative of v all over v^2, expressed this way using Leibniz notation:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Equivalently, using function notation and a dash to denote the derivative, the quotient rule says y' is $vu' - uv'$ all over v^2:

$$y' = \frac{vu' - uv'}{v^2}.$$

It might look complicated at first, but it's remarkably elegant considering the task at hand, which is to differentiate a possibly highly sophisticated function expressed as a quotient. That we can do it at all is surely something of a miracle.

The quotient rule, in fact, follows from the product and the chain rules. The reason is worth investigating, as it consolidates your understanding of these rules and the way they fit together and makes your technique more robust and stronger.

Put y equal to $\frac{u}{v}$, which we can write as a product u times $\frac{1}{v}$. Which we can further rewrite as u times v^{-1}. Because this expression is a product, naturally we can apply the product rule. So that

$$\frac{dy}{dx} = \text{first factor } u \times \text{the derivative of the second factor} + \text{second factor } v^{-1} \times \text{the derivative of the first factor.}$$

But the derivative of v^{-1} with respect to x may be expanded to become

$$\frac{d}{dv}(v^{-1}) \times \frac{dv}{dx} \quad \text{by the chain rule, which becomes} \quad -1 \times v^{-2} \times \frac{dv}{dx}.$$

Which can be rewritten as

$$-\frac{1}{v^2}\frac{dv}{dx}.$$

Hence

$$\frac{dy}{dx} = u \times \left(-\frac{1}{v^2}\frac{dv}{dx}\right) + v^{-1} \times \frac{du}{dx}.$$

And the first and second pieces can be rewritten so the whole thing becomes

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Which is exactly the form of the derivative in the quotient rule. This produces the Leibniz form which immediately converts into the rule using dash notation.

For example, let y equal the quotient $\frac{x^2+1}{x}$. And our task is to find the derivative, y'. We can solve this directly by splitting y up into two fractions, which becomes $x + x^{-1}$, and then differentiate each piece. The derivative of the first piece is 1 and the derivative of the second piece is $-1 \times x^{-2}$. Which becomes $1 - \frac{1}{x^2}$, which we can leave as that or rewrite as a single fraction, $\frac{x^2-1}{x^2}$.

An alternative solution is to apply the quotient rule because y is the quotient of u divided by v where u is the numerator $x^2 + 1$, and v is the denominator x. So that u' is $2x$, and v' is 1. So by the quotient rule, y' is $\frac{vu'-uv'}{v^2}$, and we can put all the pieces together. Expand, simplifying a couple steps to get $\frac{x^2-1}{x^2}$ which agrees with our first direct solution.

Here's a more complicated expression for y and our task again is to find the derivative y'. Observe that y is the quotient, $\frac{u}{v}$, where u is the numerator and v is the denominator with derivatives u' and v' respectively. We apply the quotient rule, and put all the pieces together. The derivative y' is complicated, and I'm not even going to try to simplify it. I just made up this example for the purposes of illustration, and have no idea what the graph of the function looks like, or even whether the function corresponds to some interesting or important problem, or physical phenomenon.

The point I want to emphasize in an example like this is that we can find the derivative quickly by means of the rules that we developed. And the process becomes quite mechanical and procedural. Something quite remarkable has happened. Just in the space of a few videos, we have moved from basic definitions of the derivative to being able to differentiate such sophisticated functions fluently and easily. Even though we don't know what the curve looks like, this expression for the derivative, in principle, allows us to find slopes of tangent lines at any given point in which the expression is defined. And therefore, to find the equation of the tangent line at a given point and thereby approximate values of the function. In my opinion, this is an extraordinary achievement. And if you follow the development up to this point, then you should be congratulated on coming so far so quickly.

In an earlier section, I told you what the derivative of $\tan x$ turns out to be. Let's see if we can discover it ourselves. Observe that $\tan x$ is $\frac{\sin x}{\cos x}$, which becomes a fraction $\frac{u}{v}$ where $u = \sin x$ and $v = \cos x$. So that u' is $\cos x$, and v' is $-\sin x$. By the quotient rule, the derivative of $\tan x$ with respect to x becomes $\frac{vu'-uv'}{v^2}$. And we put the pieces together. And the denominator becomes $\cos^2 x$. Whilst, the numerator becomes $\cos^2 x + \sin^2 x$, which you might recognize as simplifying to 1 by the Pythagorean identity. And the answer becomes $\frac{1}{\cos^2 x}$.

This agrees with the answer that I told you some time ago without giving any reason. Now, there's some common notation and terminology that you should become aware of involving another trig function called the secant. We put $\sec x = \frac{1}{\cos x}$, the reciprocal of $\cos x$, where sec is an abbreviation for secant. This word is related to lines, drawing certain points in diagrams involving the unit circle, in the context of more advanced trigonometry, but I won't go into the details. And you can look it up yourselves, if you wish, out of interest.

This new terminology, the derivative of $\tan x$ becomes the square of $\sec x$, which is written as $\sec^2 x$. You can continue to think of the derivative as $\frac{1}{\cos^2 x}$ if you like. But you'll almost certainly see the sec notation in your mathematical travels. The secant function simply reciprocates the value of the cosine function. There's special terminology also for reciprocating the value of the sine function. Put $\csc x$ equal to x equal to $\frac{1}{\sin x}$, where csc and cosec are both common abbreviations for cosecant.

There's one more, put $\cot x$ equal to x equal $\frac{\cos x}{\sin x}$, which is the reciprocal of $\tan x$. And here cot and cotan are common abbreviations for cotangent.

Thus cosecant and cotangent, again have geometric interpretations in more advanced trigonometry which I won't go into here. And we will not need either of them in this course.

We have from before, the derivative $\tan x$ is $\frac{1}{\cos^2 x}$ expressed also as $\sec^2 x$. For completeness, though we will not need it, and you can check it yourself if you wish, the derivative $\cot x$ or x is $\frac{-1}{\sin^2 x}$. Which you can express as $-\csc^2 x$ using either abbreviation.

26.1.1 Examples

Before wrapping this section, let us see some solved examples.

1. Find y' when $y = \frac{x^3+2}{x^2-5}$.

   Solution: We have $y = \frac{u}{v}$ where $u = x^3+2$, so that $u' = 3x^2$, and $v = x^2-5$, so that $v' = 2x$. Then, by the Quotient Rule,

$$y' = \frac{vu'-uv'}{v^2} = \frac{(x^2-5)(3x^2)-(x^3+2)(2x)}{(x^2-5)^2} = \frac{3x^4-15x^2-2x^4-4x}{(x^2-5)^2}$$

$$= \frac{x^4-15x^2-4x}{(x^2-5)^2} = \frac{x(x^3-15x-4)}{(x^2-5)^2}$$

2. Find y' when $y = \frac{1+e^x}{2+e^{2x}}$.

   Solution: We have $y = \frac{u}{v}$ where $u = 1+e^x$, so that $u' = e^x$, and $v = 2+e^{2x}$, so that $v' = 2e^{2x}$. Then, by the Quotient Rule,

$$y' = \frac{vu'-uv'}{v^2} = \frac{(2+e^{2x})(e^x)-(1+e^x)(2e^{2x})}{(2+e^{2x})^2} = \frac{2e^x+e^{3x}-2e^{2x}-2e^{3x}}{(2+e^{2x})^2}$$

$$= \frac{2e^x-2e^{2x}-e^{3x}}{(2+e^{2x})^2} = \frac{2e^x(1-e^x)-e^{3x}}{(2+e^{2x})^2}$$

3. Find $\frac{dy}{dx}$ when $y = \frac{x^3}{t^4}$.
Solution: We have $y = \frac{u}{v}$ where $u = x^3$ and $v = t^4$, so that $u' = 3x^2$ and $v' = 4t^3$. By the Quotient Rule,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(t^4)(3x^2) - (x^3)(4t^3)}{(t^4)^2} = \frac{3t^4x^2 - 4x^3t^3}{t^8}$$

$$= \frac{3x^2t - 4x^3}{t^8}$$

4. Find $\frac{dw}{dx}$ when $y = \frac{\ln t}{u^2}$.

Solution: We have $y = \frac{u}{v}$ where $u = \ln t$ and $v = u^2$, so that $u' = \frac{1}{t}$ and $v' = 3u^2$. By the Quotient Rule, regardless of the symbols chosen, we multiply the bottom by the derivative of the top, take away the top times the derivative of the bottom, and put all over the bottom squared to get:

$$y' = \frac{vu' - uv'}{v^2} = \frac{u^2\left(\frac{1}{t}\right) - (\ln t)(2u)}{u^4} = \frac{1 - 3\ln u}{u^4}$$

5. Find y' when $y = \frac{xe^x}{\cos^2 x}$.

Solution: We have $y = \frac{u}{v}$ where $u = xe^x$, so that $u' = e^x + xe^x$, and $v = \cos x$, so that $v' = -\sin x$. By the Quotient Rule,

$$y' = \frac{vu' - uv'}{v^2} = \frac{\cos^2 x(e^x + xe^x) - (xe^x)(-2\sin x)}{\cos^4 x}$$

$$= \frac{e^x\cos^2 x + xe^x\cos^2 x + 2xe^x\sin x}{\cos^4 x}$$

6. Find $\frac{du}{d\theta}$ when $u = \frac{\theta^2 - 1}{\sqrt{\theta^2 + 1}}$.

Solution: We have $u = \frac{u}{v}$ where $u = \theta^2 - 1$, so that $u' = 2\theta$, and $v = \sqrt{\theta^2 + 1} = (\theta^2 + 1)^{1/2}$, so that, by the Chain Rule,

$$\frac{dv}{d\theta} = \frac{1}{2}(\theta^2 + 1)^{-1/2}(2\theta) = \frac{\theta}{\sqrt{\theta^2 + 1}}.$$

By the Quotient Rule, regardless of the symbols chosen, we multiply the bottom by the derivative of the top, take away the top times the derivative of the bottom, and put all over the bottom squared, to get

$$\frac{du}{d\theta} = \frac{vu' - uv'}{v^2} = \frac{(\sqrt{\theta^2 + 1})(2\theta) - (\theta^2 - 1)\left(\frac{\theta}{\sqrt{\theta^2+1}}\right)}{(\sqrt{\theta^2 + 1})^2}$$

$$= \frac{(\theta^2 + 1)(2\theta) - (\theta^2 - 1)(\theta)}{(\theta^2 + 1)^{3/2}} = \frac{2\theta(\theta^2 + 1) - \theta(\theta^2 - 1)}{(\theta^2 + 1)^{3/2}}$$

$$= \frac{\theta(2\theta^2 + 2 - \theta^2 + 1)}{(\theta^2 + 1)^{3/2}} = \frac{\theta(\theta^2 + 3)}{(\theta^2 + 1)^{3/2}}.$$

In this section, we introduced and illustrated the quotient rule, which enables you to differentiate a quotient of expressions in terms of the derivatives of the numerator and denominator. We explained how it comes about as a result of applying the product and chain rules by expressing the quotient as a product, the numerator with the reciprocal of the denominator. We gave some contrasting examples and importantly, found a formula for the derivative of $\tan x$. Namely, the reciprocal of $\cos^2 x$, which can also be expressed as $\sec^2 x$, where sec is an abbreviation for secant. The values of the secant function are the reciprocals of the values of the cosine function. Please re-read if you didn't get it, and when you're ready, please attempt the exercises. Thank you very much for reading, and I look forward to seeing you again soon.

26.1.2 Practice Quiz

Question 1

Find y' when $y = \frac{x^2+1}{x^3-6}$.

(a) $\frac{x^4+3x^2+12}{(x^3-6)^2}$
(b) $-\frac{x(x^3+3x+12)}{(x^3-6)^2}$
(c) $\frac{x^3+3x+12}{(x^3-6)^2}$
(d) $-\frac{x^4-3x^2-12}{(x^3-6)^2}$
(e) $\frac{x(x^3+3x+12)}{(x^3-6)^2}$

Question 2

Find y' when $y = \frac{2+e^{3x}}{1-e^{-x}}$.

(a) $3e^{4x}$
(b) $\frac{2e^{-x}-4e^{2x}+3e^{3x}}{(1-e^{-x})^2}$
(c) $-\frac{2e^{-x}-4e^{2x}+3e^{3x}}{(1-e^{-x})^2}$
(d) $\frac{2e^{-x}-4e^{2x}+3e^{3x}}{(1-e^{-x})^2}$
(e) $\frac{2e^{-x}+4e^{2x}-3e^{3x}}{(1-e^{-x})^2}$

Question 3

Find $\frac{dy}{dt}$ when $y = \frac{t}{e^{2t}}$.

(a) $\frac{2t-1}{e^{2t}}$
(b) $\frac{1+2t}{e^{2t}}$
(c) $\frac{1}{2e^{2t}}$
(d) $\frac{1-2t}{e^{4t}}$
(e) $\frac{1-2t}{e^{2t}}$

Question 4

Find $\frac{du}{dt}$ when $u = \frac{t}{(\ln t)}$.

(a) $\frac{\ln t-1}{(\ln t)^2}$
(b) $\frac{1-t\ln t}{t(\ln t)^2}$
(c) $\frac{1-\ln t}{(\ln t)^2}$
(d) $\frac{1}{(\ln t)^2}$
(e) $\frac{t\ln t-1}{t(\ln t)^2}$

Question 5

Find y' when $y = \frac{x^2}{\sin x}$.

(a) $\frac{x(x\cos x+2\sin x)}{\sin^2 x}$
(b) $\frac{x(x\cos x-2\sin x)}{\sin^2 x}$
(c) $\frac{2x}{\cos x}$
(d) $-\frac{x(x\sin x+2\cos x)}{\sin^2 x}$
(e) $\frac{2(x\sin x-x\cos x)}{\sin^2 x}$

Question 6

Find $\frac{dy}{d\theta}$ when $y = \frac{\theta}{\cos\theta}$.

(a) $\frac{\theta\sin\theta-\cos\theta}{\cos^2\theta}$
(b) $-\frac{1}{\sin\theta}$
(c) $\frac{\cos\theta+\theta\sin\theta}{\cos^2\theta}$
(d) $\frac{\cos\theta-\theta\sin\theta}{\cos^2\theta}$
(e) $\frac{\cos\theta-\theta\sin\theta}{\cos^2\theta}$

Question 7

Find $\frac{dx}{d\theta}$ when $x = \frac{\cos\theta}{\sin\theta}$.

(a) $\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta}$
(b) $-\frac{1}{\sin^2\theta}$
(c) $\frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta}$
(d) $\tan\theta$
(e) $-\frac{1}{\sin^2\theta}$

Question 8

Find $\frac{dx}{d\theta}$ when $x = \frac{\sin^2\theta}{\cos^3\theta}$.

(a) $\frac{\sin\theta(2\cos^2\theta+3\sin^2\theta)}{\cos^4\theta}$
(b) $-\frac{2}{3\cos\theta}$
(c) $\frac{\sin\theta(3\sin^2\theta-2\cos^2\theta)}{\cos^4\theta}$
(d) $-\frac{\sin\theta(2\cos^2\theta+3\sin^2\theta)}{\cos^4\theta}$
(e) $\frac{\sin\theta(2\cos^2\theta-3\sin^2\theta)}{\cos^4\theta}$

Question 9

Find $\frac{dy}{du}$ when $y = \frac{u^2+1}{\sqrt{u^2-1}}$.

(a) $\frac{u(1-3u^2)}{(u^2-1)^{3/2}}$
(b) $\frac{u(3u^2-1)}{(u^2-1)^{3/2}}$
(c) $\frac{u}{\sqrt{u^2-1}}$
(d) $\frac{u(3-u^2)}{(u^2-1)^{3/2}}$
(e) $\frac{u(u^2-3)}{(u^2-1)^{3/2}}$

Question 10

Find y' when $y = \frac{x+1}{e^x\sin x}$.

(a) $\frac{x\sin x+(x+1)\cos x}{e^x\sin^2 x}$

(b) $\frac{1}{e^x(\sin x+\cos x)}$
(c) $\frac{-x\sin x-(x+1)\cos x}{e^x\sin^2 x}$
(d) $\frac{x\sin x-(x+1)\cos x}{e^x\sin^2 x}$
(e) $\frac{(x+1)\cos x-x\sin x}{e^x\sin^2 x}$

Answers

The answers will be revealed at the end of the module.

26.2 Applications of the Quotient Rule

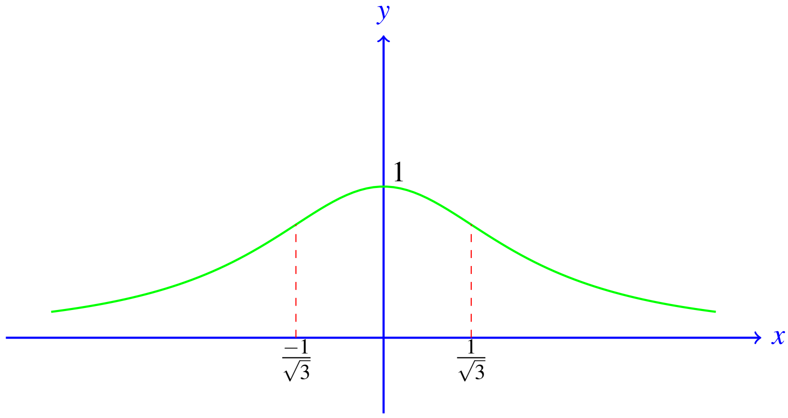
In this section, we apply the Quotient Rule to explore the behavior of the curve $y=\frac{1}{x^2+1}$, known as the Witch of Maria Agnesi. Our main example is a curve of fundamental importance in the theory of rational functions, that is, ratios of polynomials, and involves reciprocating the quadratic x^2+1, which has a special place in mathematics. Being the simplest polynomial you can think of that does not have any roots over the real number system since it's always greater than or equal to one for any real number x, it cannot possibly be zero. In fact, this innocent and unassuming quadratic leads to the construction of the arithmetic of complex numbers $\mathbb{C}$, with quite remarkable properties which I hope you will see if you take more advanced courses in mathematics that follow from this course. Consider the function with the rule $y=\frac{1}{x^2+1}$. Its domain is the entire real number line, since x^2+1 is always nonzero. One can ask what its graph looks like. The resulting curve is known as the Witch of Maria Agnesi, named after an Italian mathematician, Maria Agnesi, who lived and worked in the 18th century, and wrote one of the first treatises on calculus. The word "witch" in fact comes about by a linguistic accident in translating texts from Latin, though happily the name of the curve survives and is evocative of a witch's hat, as you'll see in a moment.

Let's work through our usual checklist for sketching a curve. In this case, for $y=\frac{1}{x^2+1}$. The y-intercept is the value of y when $x=0$, which is 1. There are no x-intercepts as the reciprocal of x^2+1 is always positive. For asymptotic behavior, it's clear that the limit of $\frac{1}{x^2+1}$ is 0 when x gets large, and positive or large and negative so that the x-axis becomes a horizontal asymptote. We then investigate the derivative which gives us information about when the curve is increasing or decreasing, and for finding any turning points.

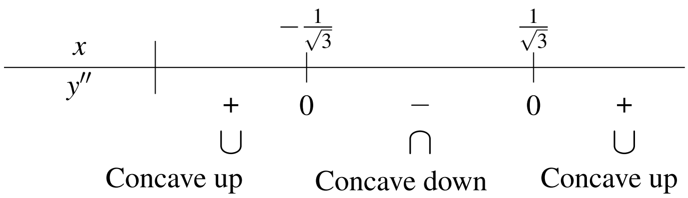
To set up the chain rule, we write y as $(x^2+1)^{-1}$, which becomes u^{-1} where $u=x^2+1$. So, the derivative y' which is $\frac{dy}{dx}$ becomes $\frac{dy}{du}\times\frac{du}{dx}$ by the chain rule, which is $-u^{-2}\times 2x$, which we can rewrite as $\frac{-2x}{u^2}$, and then express everything in terms of x to get $y'=\frac{-2x}{(x^2+1)^2}$. Notice that the denominator is always positive and $y'=0$ precisely when the numerator is 0, that is when $x=0$. We can now easily build the sign diagram. We have $y'=0$ when $x=0$, and either side of 0 has the opposite sign of x, because the denominator is positive, so the sign is determined by the numerator giving the pattern positive-negative corresponding to increasing, decreasing with a global maximum occurring when $x=0$.

x		0	
y'	$+$	0	$-$
	$\searrow$		$\nearrow$

We can put all of this information together locating the y-intercept, the fact that the x-axis is an asymptote and the turning point of $x=0$. The qualitative behavior parallels what we saw when we analyzed the Gaussian curve, $y=e^{-x^2}$ in an earlier section. One expects to be able to join the pieces together to create a smooth curve that also looks bell-shaped. There appear to be two points of inflection where concavity changes. To be sure about the behavior with regards to concavity and to locate the inflections, we need to investigate the second derivative y''. Because the first derivative y' is a rational function, we should expect to apply the quotient rule.



To investigate the second derivative, we write y' as $\frac{u}{v}$ where u is the numerator $-2x$, and v is the denominator $(x^2+1)^2$. So, u' is -2, and v' is $\frac{dv}{dx}$ in Leibniz's notation, which is $\frac{dv}{dw} \times \frac{dw}{dx}$ by the chain rule, if we make the substitution $w = x^2+1$, which becomes $2w \times 2x$ which is $4x \times (x^2+1)$. We now have all the ingredients for the quotient rule, and inserting all the pieces gives this complicated looking expression which simplifies after some effort to the rational function $\frac{2(3x^2-1)}{(x^2+1)^3}$ as you can check yourself, and please do it. It's good exercise in algebraic manipulation. Note that the denominator ends up being the third power of x^2+1, because of cancellation with a copy of x^2+1 in the numerator, thus we end up with this very nice expression for y''. Note that the denominator is always positive being the cube of x^2+1 which is positive. Also, y'' is zero precisely when the numerator is zero, which occurs when $x = \pm\frac{1}{\sqrt{3}}$. We can now build the sign diagram with y'' being zero for $x = \pm\frac{1}{\sqrt{3}}$, the pattern of positive, negative, positive determined by the sign of $3x^2-1$, corresponding to concave up, concave down, concave up with inflections occurring when $x = \pm\frac{1}{\sqrt{3}}$.



We can add this extra information to the previous compilation when we worked through the curve sketching checklist. This completes the sketch of the curve $y = \frac{1}{x^2+1}$ affectionately known as the witch of Maria Agnesi. The curve is also in the shape of a bell though it certainly is not the same as a Gaussian curve $y = e^{-x^2}$ even though they share many similar qualitative features.

I would like to finish by establishing a remarkable connection between the witch of Maria Agnesi and trigonometry.Recall the single branch of the tan curve that results by restricting the domain to the interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, sandwiched between two vertical asymptotes, and satisfies the horizontal line test, certified as suitable for inversion with the inverse function obtained by reflecting in the line $y = x$ to get the arctan function which is increasing and sandwiched in between two horizontal asymptotes.

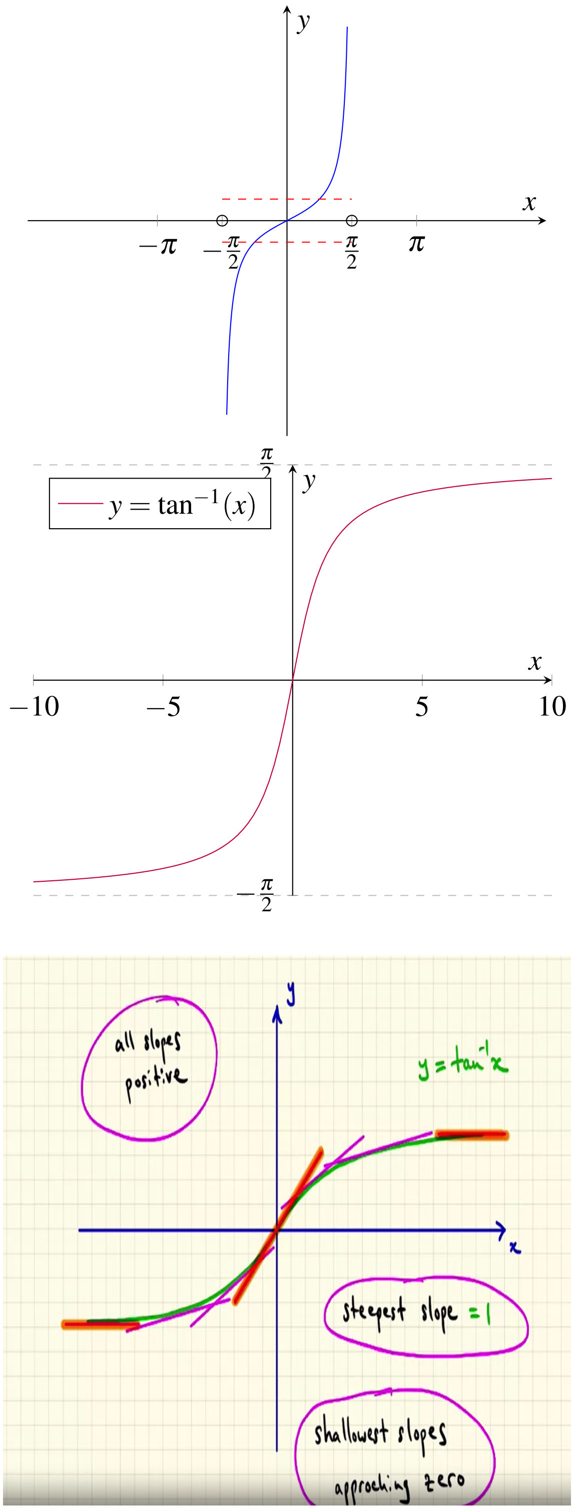


Figure 26.1: Slopes of Tangent line

We can see how the slopes of miniature tangent lines behave as we move from left to right along the arctan curve. Notice how the slopes are all positive and the steepest slope appears to be at the origin and takes the value one, and the shallowest slopes appear to the far left and far right and appear to be approaching zero. The slopes of tangent lines are just the derivatives and this behav-

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

ior suggests that the derivative of the arctan function behaves qualitatively like the function that gives rise to the Witch of Maria Agnesi. Maybe, if $y = \arctan x$, then $y' = \frac{1}{x^2+1}$. It turns out to be true.

The proof emerges from the magic of Leibniz's notation. Watch. Start with a circular identity, $\sin^2 x + \cos^2 x = 1$. Divide through by $\cos^2 x$ eventually to get $\tan x$ somehow into the game. This becomes $\left(\frac{\sin x}{\cos x}\right)^2 + 1 = \frac{1}{\cos^2 x}$, which can be rewritten as $\tan^2 x + 1 = \sec^2 x$. Since $\tan x = \frac{\sin x}{\cos x}$ and from last time $\sec x$ is the new name for $\frac{1}{\cos x}$. So, we have this very nice trig identity, closely related to the circular identity which says

$$\tan^2 x + 1 = \sec^2 x$$

. Put $y = \tan x$ so that $x = \arctan y$, and $\frac{dy}{dx} = \sec^2 x$ from the previous section, which is $\tan^2 x + 1$ from the identity we have just established, and this is just $y^2 + 1$. Now, we tip the derivative upside down using Leibniz's notation, the trick we used in an earlier section to get $\frac{dx}{dy}$ equal to the reciprocal of $\frac{dy}{dx}$ which is $\frac{1}{y^2+1}$. But $x = \arctan y$, so this says that $\frac{d}{dy}(\arctan y) = \frac{1}{y^2+1}$. We usually prefer to use the symbol x as a typical input. So, replacing y by x gives $\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$, which is the rule for the Witch of Maria Agnesi, confirming our guess before based on visualizing slopes of tangent lines. This result is one of the main ingredients in the theory of integration of rational functions, some aspects of which we will touch on in the final module of this course.

In this section, we applied the quotient rule to help us understand the second derivative of the rule for the function whose graph is the so-called Witch of Maria Agnesi. The rule for this curve is $y = \frac{1}{x^2+1}$ which turns out to be the derivative of the arctan function. Please re-read if you didn't get it, and when you're ready, please attempt the exercises. Thank you very much for reading, and I look forward to seeing you again soon.

26.2.1 Practice Quiz

Question 1

Find y' when $y = \frac{1}{x^2+4}$.

(a) $\frac{-1}{(x^2+4)^2}$
(b) $\frac{-2x}{(x^2+4)^2}$
(c) $\frac{1}{(x^2+4)^2}$
(d) $\frac{2x}{(x^2+4)^2}$
(e) $\frac{-2x}{x^2+4}$

Question 2

Find y'' when $y = \frac{1}{x^2+4}$.

(a) $\frac{2(4-3x^2)}{(x^2+4)^3}$
(b) $\frac{2(4-3x^2)}{(x^2+4)^2}$
(c) $\frac{2(4-3x^2)}{(x^2+4)}$
(d) $\frac{2(3x^2-4)}{(x^2+4)^3}$
(e) $\frac{-2}{(x^2+4)^3}$
(f) $\frac{2(3x^2-4)}{(x^2+4)^5}$

Question 3

Which one of the following statements about the curve $y = \frac{1}{x^2+4}$ is true?

(a) The curve is concave down everywhere with a global maximum of $\frac{1}{4}$.
(b) The line $y = \frac{1}{4}$ is a horizontal asymptote for the curve.
(c) The curve has a global minimum of $\frac{1}{4}$ with inflections at $x = \pm\frac{2}{\sqrt{3}}$.
(d) The curve is concave up everywhere with a global minimum of $\frac{1}{4}$.
(e) The curve has a global maximum of $\frac{1}{4}$ with inflections at $x = \pm\frac{2}{\sqrt{3}}$.

Question 4

Find y' when $y = \frac{x}{x^2+1}$.

(a) $\frac{1-3x^2}{(x^2+1)^2}$
(b) $\frac{-1}{(x^2+1)^2}$
(c) $\frac{1-x^2}{(x^2+1)^2}$
(d) $\frac{3x^2-1}{(x^2+1)^2}$
(e) $\frac{x^2-1}{(x^2+1)^2}$

Question 5

Find y'' when $y = \frac{x}{x^2+1}$.

(a) $\frac{2x(x^2-3)}{(x^2+1)^4}$
(b) $\frac{2(x^2-3)}{(x^2+1)^3}$
(c) $\frac{2(x^2-3)}{(x^2+1)^3}$
(d) $\frac{2}{(x^2+1)^3}$
(e) $\frac{2x(3-x^2)}{(x^2+1)^4}$

Question 6

Which one of the following statements about the curve $y = \frac{x}{x^2+1}$ is true?

(a) The curve is concave up for $x \geq 0$ and concave down for $x < 0$.
(b) The curve is concave up for $x < 0$ and concave down for $x \geq 0$.
(c) The curve does not have any asymptotes.
(d) The curve has a global maximum of $\frac{1}{2}$ and a global minimum of $-\frac{1}{2}$.
(e) The curve is decreasing for $-1 \leq x \leq 1$ and increasing for $x < -1$ and $x > 1$.

Question 7

Find y' when $y = \tan^{-1}(3x)$.

(a) $\frac{1}{3x^2+1}$
(b) $\frac{3}{9x^2+1}$
(c) $\frac{3}{x^2+1}$
(d) $\frac{3x}{3x^2+1}$
(e) $\frac{1}{9x^2+1}$

Question 8

Find y' when $y = \tan^{-1}\left(\frac{x}{3}\right)$.

(a) $\frac{9}{3x^2+1}$
(b) $\frac{9}{x^2+9}$
(c) $\frac{3}{x^2+9}$

(d) $\frac{3}{x^2+3}$
(e) $\frac{1}{x^2+9}$

Question 9

Which one of the following functions has the property that $y' = \frac{1}{x^2+4}$?

(a) $y = \frac{1}{2}\tan^{-1}(2x)$
(b) $y = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)$
(c) $y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$
(d) $y = 2\tan^{-1}(2x)$
(e) $y = 2\tan^{-1}\left(\frac{x}{2}\right)$

Question 10

Which one of the following functions has the property that $y' = \frac{x}{x^2+1}$?

(a) $y = \frac{1}{2}\ln(x^2+1)$
(b) $y = x\ln(x^2+1)$
(c) $y = \frac{x}{2}\ln(x^2+1)$
(d) $y = \frac{1}{4}\ln(x^2+1)$
(e) $y = \ln(x^2+1)$

Answers

The answers will be revealed at the end of the module.

27. Optimisation

27.1 Optimisation

In this section, we use calculus to solve several contrasting problems in optimization, which means finding the smallest or largest values of some quantity of interest.
In the first problem, we're asked to minimize the sum of a real number and its reciprocal. You can experiment with a few numbers to get a feeling for what the answer might be before we actually do the calculus.

$$2+\frac{1}{2}=\frac{5}{2}, \quad 10+\frac{1}{10}=\frac{101}{10}, \quad 1+\frac{1}{1}=2,$$

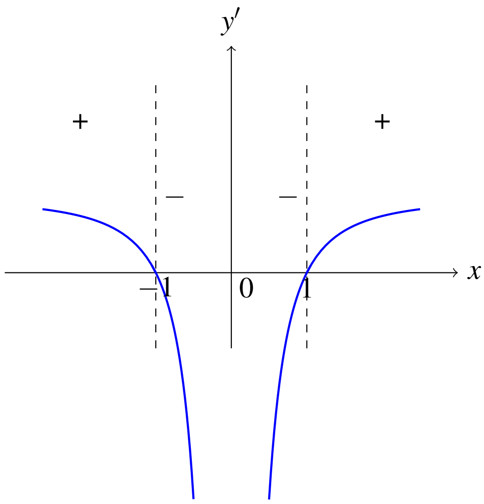
$$0.5+\frac{1}{0.5}=1+2=\frac{5}{2}, \quad 0.1+\frac{1}{0.1}=\frac{1}{10}+10=\frac{101}{10}.$$

Just by experimentation, one can quickly become convinced that 2 is the smallest such sum (when one adds 1 to its reciprocal, which is also 1), but how can one be sure?

To turn this into mathematics, we put $y=f(x)=x+\frac{1}{x}$, which can be rewritten as $x+x^{-1}$. The strategy is to use the derivative to find the turning point, which should tell us what the minimum might be. The derivative is $y'=1-x^{-2}$, which we can rewrite in a couple of steps as $\frac{x^2-1}{x^2}$. Thus, $y'=0$ when $x=\pm 1$. Note that y' is undefined when $x=0$.

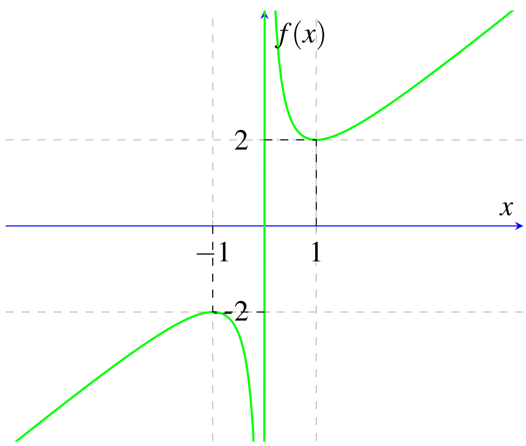
So, we get the following sign diagram, with the pattern positive, negative, negative again, positive, which indicates the pattern increasing, decreasing, decreasing again, increasing, with a maximum when $x=-1$ and a minimum when $x=1$.

x		-1		0		1	
y'	$+$	0	$-$	u	$-$	0	$+$
	↗		↘		↘		↗



In fact, we only need to focus on positive x and the corresponding part of the sign diagram highlighted here, producing a global minimum for $x > 0$ occurring when $x = 1$, with value $f(1) = 2$. This solves the original problem. The answer is that the minimum sum of the positive real number with its reciprocal is 2 and this occurs by adding 1 to its reciprocal. The mathematics tells us it's impossible to do any better than this.

Though we don't need to, we can, if we like, graph the curve $y = x + \frac{1}{x}$, which splits into two branches, with the y-axis being a vertical asymptote and the line $y = x$ being an oblique asymptote. The branch of the curve in the first quadrant is directly relevant for our original problem and the turning point with coordinates $(1, 2)$ corresponds to the solution.

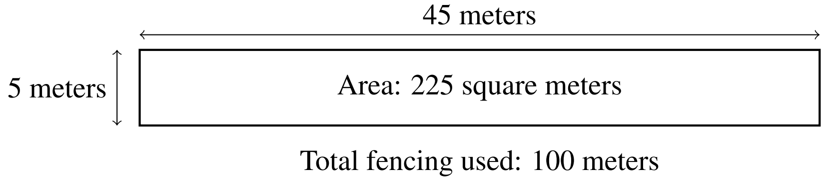


This indicates that the curve is increasing for $x < -1$, decreasing for $-1 < x < 0$, decreasing again for $0 < x < 1$ and increasing for $x > 1$, with a maximum at $x = -1$ and a minimum at $x = 1$.

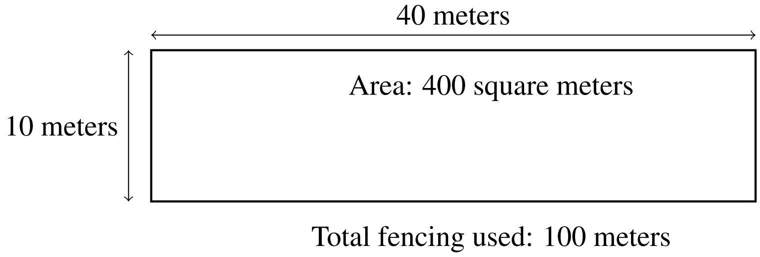
However, the application to our original problem only uses $x > 0$, and so we get the minimum sum of a positive number with its reciprocal to be

$$f(1) = 1 + \frac{1}{1} = 2.$$

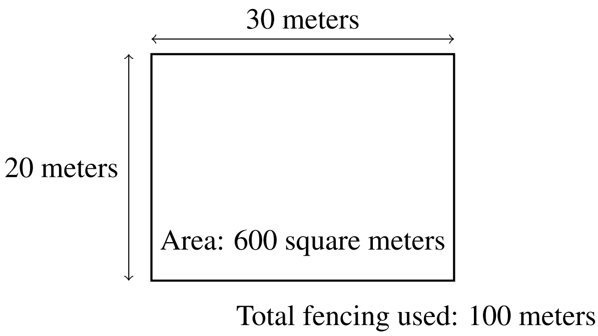
In the next optimization problem, we're asked to build a rectangular enclosure using 100 meters of fencing in order to maximize the area. What should we do? For example, if we make a thin rectangle, only five meters deep and 45 meters wide, then we use up 100 meters of fencing and the area becomes 225 square meters.



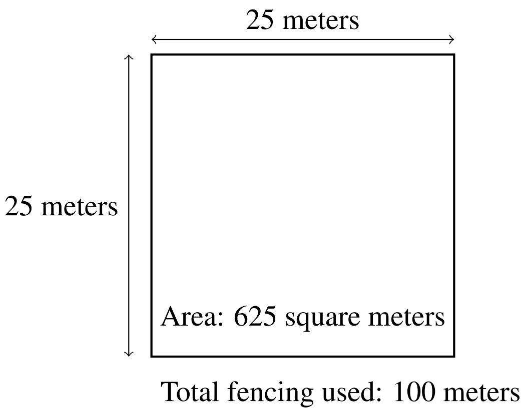
If we try making it 10 meters deep and now 40 meters wide, we get a larger area of 400 square meters, but we can do better.



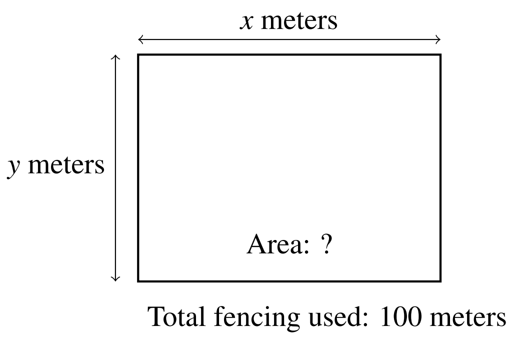
If it's 20 meters deep and 30 meters wide, we increase the area to 600 square meters. We can still do better.



If it's 25 meters deep and 25 meters wide, then the area becomes 625 square meters.



Notice now that we've produced a perfect square, and it looks like there's no way to make the area any larger. It seems reasonable to guess intuitively that a perfect square should solve the problem. Let's do the math to see if it confirms our intuition.



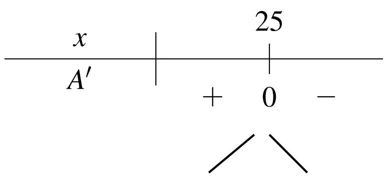
Here's a diagram representing a general enclosure with side lengths x and y meters. Note that the diagram is just a guide to assist us in creating the mathematics. In fact, we're expecting to get a perfect square in the end, but we're not sure. So, just try to imagine any typical rectangle, but realize this is only to help set up the mathematics. Denote the area of the rectangle by A square meters so that $A = xy$. The constraint is that we only have 100 meters of fencing. So, we add up all the sides of the rectangle, that is $x+y+x+y$ to get 100.We can now regard A as a function of x only, giving

$$A = xy = x(50-x) = 50x - x^2.$$

Its derivative with respect to x is

$$A' = 50 - 2x,$$

which is zero precisely when $x = 25$. We can now build its sign diagram with the pattern: positive, negative, and increasing then decreasing, with a maximum occurring when $x = 25$. It follows that the area A is maximized when x and y are both equal to 25. So, indeed the enclosure turns out to be a perfect square as we suspected all along, with maximum area 625 square meters. The math guarantees that there's no chance of doing better than this.



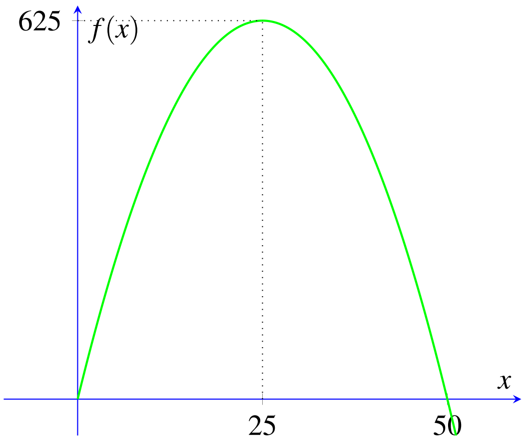
This indicates that the area function A is increasing for $x < 25$ and decreasing for $x > 25$, attaining a global maximum

$$A = 25(50-25) = 625$$

when $x = 25$. Thus the maximum area that can be achieved by the rectangular enclosure is 625 square meters, and this occurs when $x = y = 25$, so that the enclosure is indeed square.

(Note that x is nonnegative by the physical constraint of the problem, though this maximum also holds when A is regarded as a quadratic function for all real x. The global maximum corresponds to the apex of an inverted parabola.)

The area function A turned out to be a quadratic in x. So of course, we can also visualize the solution in terms of the associated parabola. Here's the information we used about the derivative.



It matches perfectly with the actual parabola, which passes through the x-axis at 0 and 50 with apex exactly halfway between, corresponding to the solution. The problem is constrained by having non-negative side lengths with $x = 0$ or $x = 50$ at the boundaries of the actual physical domain, the rectangle becomes infinitely thin with zero area. Note also that the second derivative is -2 which is less than zero, so the curve should be concave down, which indeed corresponds to the fact that the parabola is facing downwards.

Our final optimization problem is quite difficult and uses all of the techniques that we've been building up over several sections. We're unlikely to be able to guess the answer, so we rely on producing a carefully developed mathematical solution.

We revisit the Statue of Liberty which you might remember is 46 meters high, standing on a pedestal which is also 46 meters high. The problem is to find the horizontal distance x meters from the base of the pedestal that maximizes the angle θ subtended by the statue, and also, to find this largest angle.

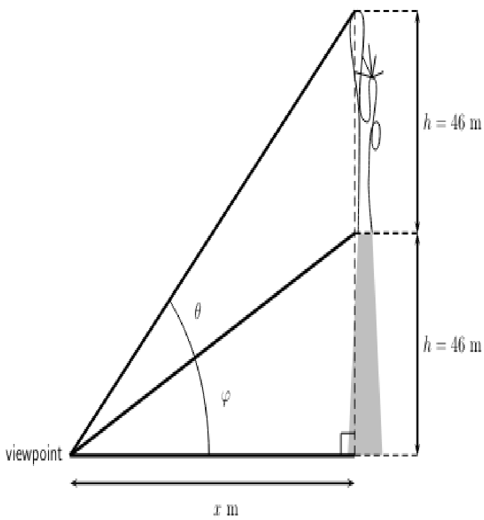


Figure 27.1: Statue of Liberty

It's useful, especially in developing formulae, to refer to the common height of the statue and pedestal using the symbol h. Though it hasn't been asked for in the problem, we give a name ϕ to

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

the angle subtended by the pedestal.

By inspecting right-angled triangles, we have

$$\tan\varphi = \frac{h}{x} \quad \text{and} \quad \tan(\theta+\varphi) = \frac{2h}{x}.$$

Hence

$$\varphi = \tan^{-1}\left(\frac{h}{x}\right) \quad \text{and} \quad \theta+\varphi = \tan^{-1}\left(\frac{2h}{x}\right),$$

so that

$$\theta = \tan^{-1}\left(\frac{2h}{x}\right) - \varphi = \tan^{-1}\left(\frac{2h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right).$$

Looking at the derivative of θ with respect to x is likely to lead to progress in determining the maximum possible value of θ. But the derivative of a difference is the difference of the derivatives, so that, by the Chain Rule,

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\tan^{-1} u\right)\frac{du}{dx} - \frac{d}{dx}\left(\tan^{-1} v\right)\frac{dv}{dx}$$

where $u = \frac{2h}{x}$ and $v = \frac{h}{x}$, so that

$$\frac{du}{dx} = -\frac{2h}{x^2} = -\frac{2h}{x^2} \quad \text{and} \quad \frac{dv}{dx} = -\frac{h}{x^2} = -\frac{h}{x^2}.$$

But we know that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

(from an earlier section where we introduced the witch of Maria Agnesi). Putting all of these ingredients together, we get

$$\begin{aligned}
\frac{d\theta}{dx} &= \left(\frac{1}{1+u^2}\right)\left(-\frac{2h}{x^2}\right) - \left(\frac{1}{1+v^2}\right)\left(-\frac{h}{x^2}\right) \\
&= \left(\frac{1}{1+\left(\frac{2h}{x}\right)^2}\right)\left(-\frac{2h}{x^2}\right) - \left(\frac{1}{1+\left(\frac{h}{x}\right)^2}\right)\left(-\frac{h}{x^2}\right) \\
&= \left(\frac{1}{1+\frac{4h^2}{x^2}}\right)\left(-\frac{2h}{x^2}\right) + \left(\frac{1}{1+\frac{h^2}{x^2}}\right)\left(\frac{h}{x^2}\right) \\
&= \left(\frac{x^2}{4h^2+x^2}\right)\left(-\frac{2h}{x^2}\right) + \left(\frac{x^2}{h^2+x^2}\right)\left(\frac{h}{x^2}\right) \\
&= -\frac{2h}{4h^2+x^2} + \frac{h}{h^2+x^2} = \frac{-2h(h^2+x^2)+h(4h^2+x^2)}{(4h^2+x^2)(h^2+x^2)} \\
&= \frac{-2h(h^2+x^2)+h(4h^2+x^2)}{(4h^2+x^2)(h^2+x^2)} \\
&= \frac{-2h^3-2hx^2+4h^3+hx^2}{(4h^2+x^2)(h^2+x^2)} = \frac{2h^3-hx^2}{(4h^2+x^2)(h^2+x^2)}
\end{aligned}$$

$$= \frac{h(2h^2 - x^2)}{(4h^2 + x^2)(h^2 + x^2)}.$$

Thus we get

$$\frac{d\theta}{dx} = \frac{h(2h^2 - x^2)}{(4h^2 + x^2)(h^2 + x^2)},$$

and notice, on the right-hand side, that h and the denominator are always positive.

In what follows we make the underlying assumption throughout that x is positive (which accords with the physical constraints of the original problem involving the Statue). Hence

$$\frac{d\theta}{dx} = 0 \text{ precisely when } 2h = x^2, \text{ that is, } x^2 = 2h^2, \text{ so that}$$

$$x = \sqrt{2}h.$$

We can build the following sign diagram for $\frac{d\theta}{dx}$ (only considering $x > 0$), noting that the sign is completely determined by the expression $2h^2 - x^2$:

x		$\sqrt{2}h$	
$\frac{d\theta}{dx}$	$+$	0	$-$
	$\nearrow$		$\searrow$

Hence θ is increasing for $x < \sqrt{2}h$ and decreasing for $x > \sqrt{2}h$, achieving a global maximum when the horizontal distance to the Statue is

$$x = \sqrt{2}h = 46\sqrt{2} \approx 65\text{m}.$$

The maximum angle then becomes

$$\theta = \tan^{-1}\left(\frac{2h}{\sqrt{2}h}\right) - \tan^{-1}\left(\frac{h}{\sqrt{2}h}\right) = \tan^{-1}(\sqrt{2}) - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 19.5°.$$

Notice that the final answer for θ is independent of the height h of the Statue and its pedestal. We reach the same answer for the maximum angle subtended by any statue that is mounted on a pedestal of the same height.

The distance $\sqrt{2}h$ that achieves this maximum angle of course varies with h. In our problem we achieve a maximum angle that is slightly less than 20 degrees and at a horizontal distance from the Statue of about 65 meters.

In this section, we discussed and solved three contrasting optimization problems. All the solutions relied on determining and understanding the behavior of the sign of the derivative. The first two problems have solutions that one could reasonably guess in advance, either by experimenting or thinking about the problem intuitively. The third problem however, was very difficult with the final answer that I think will be impossible to guess. The derivation relied on the derivative of the inverse tan function, "The witch of Maria Agnesi," together with quite intricate algebraic manipulations. I hope you'll be able to follow through and check all of the details. By doing so, you'll develop very strong technique and a thorough understanding of the underlying principles. Please re-read if you didn't get it and, when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

27.1.1 Practice Quiz

Question 1

Which one of the following expressions represents the derivative $\frac{dy}{dx}$, where

$$y = x + \frac{1}{2x}$$

(a) $2x^2 - 1$
(b) $\frac{x^2-1}{2x^2}$
(c) $\frac{2x^2-1}{2x^2}$
(d) $\frac{x^2-1}{2x}$
(e) $\frac{2x^2+1}{2x^2}$

Question 2

Find the minimum value obtained by adding a positive real number to the reciprocal of its double, that is, find the minimum value of

$$y = x + \frac{1}{2x}$$

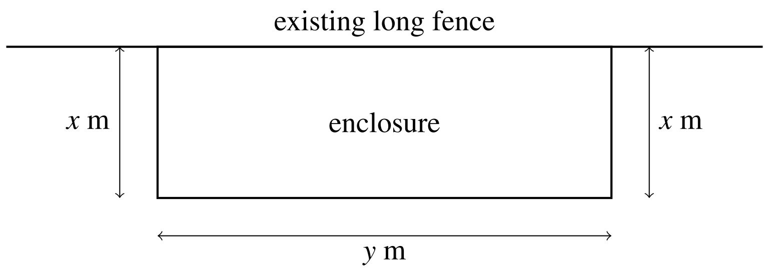
for $x \geq 0$.
(a) $\frac{4}{3}$
(b) $\frac{3}{2}$
(c) $\sqrt{2}$
(d) $\frac{\sqrt{2}+1}{2\sqrt{2}}$
(e) $\frac{1}{\sqrt{2}}$

Question 3

Find the largest possible area of a rectangular enclosure closure that uses a total of 80 metres of fencing.
(a) 600 m^2
(b) 800 m^2
(c) 1200 m^2
(d) 400 m^2
(e) 325 m^2

Question 4

A rectangular enclosure is built using a long existing fence and three new sides that use a total of 80 metres of fencing.

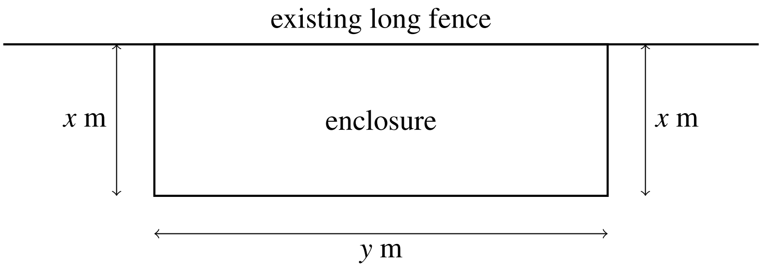


Denote the length of the two new sides perpendicular to the existing fence by x metres, and the length of the new side parallel to the existing fence by y metres. Let A m^2 be the area of the enclosure. Which one of the following expressions describes A as a function of x?

(a) $A = 4x(20 - x)$
(b) $A = x(40 - x)$
(c) $A = \frac{x(80-x)}{2}$
(d) $A = 2x(20 - x)$
(e) $A = 2x(40 - x)$

Question 5

A rectangular enclosure is built using a long existing fence and three new sides that use a total of 80 metres of fencing.



Denote the length of the two new sides perpendicular to the existing fence by x metres, and the length of the new side parallel to the existing fence by y metres. Find the largest possible area of the enclosure.
(a) 325 m^2
(b) 800 m^2
(c) 600 m^2
(d) 400 m^2
(e) 1200 m^2

Question 6

Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{1}{x}\right)$.
(a) $\frac{1}{x^2+1}$
(b) $\frac{-1}{x^2-1}$
(c) $\frac{1}{x^2-1}$
(d) $\frac{-1}{x^2+1}$
(e) $\frac{x^2}{x^2+1}$

Question 7

Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{h}{x}\right)$, where h is a constant.
(a) $\frac{-h}{x^2+h^2}$
(b) $\frac{h}{x^2+h^2}$
(c) $\frac{-h}{x^2+1}$
(d) $\frac{1}{x^2+h^2}$
(e) $\frac{-h}{x^2+1}$

Question 8

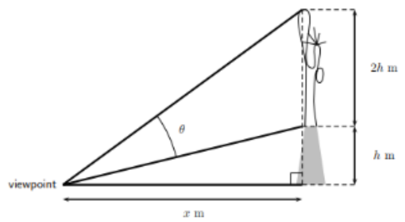
Find $\frac{dy}{dx}$ when

$$y = \tan^{-1}\left(\frac{3}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right).$$

(a) $\frac{2(x^2-3)}{(x^2+9)(x^2+1)}$
(b) $\frac{2(3-x^2)}{(2-x)(x^2-1)}$
(c) $\frac{(2x-9)(2x-1)}{(x^2+9)(2x-1)}$
(d) $\frac{3(4x^2-2)}{(x^2+9)(x^2+1)}$
(e) $\frac{4(2x-3)(2x-1)}{(x^2+9)(x^2+1)}$

Question 9

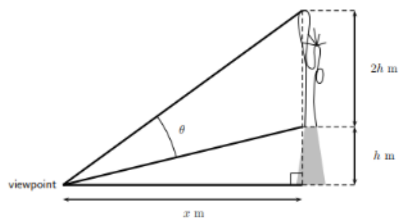
A statue of height $2h$ metres is mounted on a pedestal of height h metres, where h metres is some positive height. Thus the statue is twice as high as its pedestal.



Which one of the following is a correct expression for the angle θ subtended by the statue, when viewed z metres from the base of the pedestal?
(a) $\theta = \tan^{-1}\left(\frac{2h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$
(b) $\theta = \tan^{-1}\left(\frac{3h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$
(c) $\theta = \tan^{-1}\left(\frac{3h}{x}\right) - \tan^{-1}\left(\frac{2h}{x}\right)$
(d) $\theta = \tan^{-1}\left(\frac{2h}{x}\right) + \tan^{-1}\left(\frac{h}{x}\right)$

Question 10

A statue of height $2h$ metres is mounted on a pedestal of height h metres, where h metres is some positive height. Thus the statue is twice as high as its pedestal.



Find the largest possible angle θ subtended by the statue.
(a) $\frac{\pi}{3}$
(b) $\frac{\pi}{6}$
(c) $\tan^{-1}\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)$
(d) $\frac{\pi}{4}$
(e) $\tan^{-1}\left(\frac{\sqrt{3}}{1+\sqrt{3}}\right)$

Answers

The answers will be revealed at the end of the module.

27.2 The second derivative Test

In this section, we state the second derivative test which makes explicit the technique we've already been using in developing our skills in curve sketching. This test enables us to infer from the sign of the second derivative whether certain points on a curve correspond to a maximum or minimum. Suppose we have a function f with a rule $y = f(x)$, such that everything is well-behaved and in some particular input $x = c$. By well-behaved we mean, generally, that the curve is at least continuous, that is, can be drawn without lifting the pen off the page. We assume that the curve is smooth enough that the first and second derivatives exist at all points of interest.

The test comes in two parts. The first part asserts that if the derivative of f evaluated at $x = c$ is zero, and the second derivative is positive at that point, then the value $f(c)$ is a local minimum and it might even be global, although we can't be sure. The second part asserts that if, again, the derivative is zero, but now the second derivative is negative, then the value $f(c)$ is a local maximum and might possibly be global. Both of these correspond to facts about curves that are probably now becoming quite familiar to you. In the first case, the curve is concave up or bowl-shaped up. Now I'm used to using a smiley face symbol and the apex clearly suggests a minimum. In the second case, the curve is concave down or bowl-shaped down, and again, we're used to using a sad face symbol and the apex clearly suggests a maximum.

We have a simple test in two parts:
(a) If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a local minimum.
(b) If $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a local maximum.
For example, the shape and behavior of parabolas fit nicely with the second derivative test. Consider the function f whose rule is the quadratic

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are constants such that $a \neq 0$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. The turning point occurs when $f'(x) = 0$, that is, when $x = -\frac{b}{2a}$.

If $a > 0$ then $f''(x) = 2a > 0$, for all x and, in particular, when $x = -\frac{b}{2a}$. In this case, part (a) of the Second Derivative correctly predicts that the turning point corresponds to a minimum, coinciding with the fact that the associated parabola is facing upwards.

By contrast, if $a < 0$ then $f''(x) = 2a < 0$ for all x and, in particular, when $x = -\frac{b}{2a}$. In this case, part (b) of the Second Derivative Test correctly predicts that the turning point corresponds to a maximum, coinciding with the fact that the associated parabola is facing downwards.

So that the associated parabola must be facing downwards which, again, matches exactly what we expect from a quadratic with a coefficient of x^2 now is negative. Now, let's use this test to determine extrema, if any, for two curves. Firstly, $y = xe^x$, and secondly $y = xe^{-x}$.

Consider $y = xe^x$. By the product rule, its derivative is easily seen to be $e^x + xe^x$, which factorizes as $(1+x)e^x$.

Consider the functions f and g whose rules are

$$f(x) = xe^x \quad \text{and} \quad g(x) = xe^{-x}.$$

From the Product Rule, it follows that

$$f'(x) = e^x + xe^x = (1+x)e^x \quad \text{and} \quad f''(x) = e^x + (1+x)e^x = (2+x)e^x.$$

Then $f'(x) = 0$ if and only if $1 + x = 0$ (since e^x is always positive), that is, $x = -1$. Observe that

$$f''(-1) = (2-1)e^{-1} = e^{-1} = \frac{1}{e} > 0.$$

By part (a) of the Second Derivative Test, $f(-1) = -e^{-1} = -\frac{1}{e}$ must be a local minimum. In fact, we know more: this is a global minimum. One can see this from the sign diagram for $f'(x)$, noting that its sign coincides with the sign of the factor $(1+x)$:

x		-1	
$f'(x)$	$-$	0	$+$
	$\searrow$		$\nearrow$

From the Product Rule, and the fact that $\frac{d}{dx}(e^{-x}) = -e^{-x}$, we see that

$$g'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} \quad \text{and} \quad g''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}.$$

Then $g'(x) = 0$ if and only if $1 - x = 0$ (since e^{-x} is always positive), that is, $x = 1$. Observe that

$$g''(1) = (1-2)e^{-1} = -e^{-1} = -\frac{1}{e} < 0.$$

By part (b) of the Second Derivative Test, $g(1) = e^{-1} = \frac{1}{e}$ must be a local maximum. In fact, we know more: this is a global maximum. One can see this from the sign diagram for $g'(x)$, noting that its sign coincides with the sign of the factor $(1-x)$:

x		1	
$g'(x)$	$+$	0	$-$
	$\nearrow$		$\searrow$

Note that if we had looked at the sign diagrams for the first derivatives, we would have known immediately about the global minimum for f and the global maximum for g, so the Second Derivative Test becomes superfluous. However, the Second Derivative Test is valuable as an extra check or as part of exploration. Mathematicians are always on the look out for alternative ways of reaching any given conclusion, to help safeguard against errors in our discovering errors.
Then, for y", noting that x = 1 indeed falls within the region where the curve is concave down and we get the extra information about the concavity and the existence of an inflection when x = 2. These are both natural steps in the process of curve sketching. Notice that the y and x intercepts are above zero so the curve passes through the origin.

x		2	
y''	$-$	0	$+$
	$\cap$		$\cup$
	Concave down		Concave up

The other usual ingredient is to look for asymptotic behavior. It's clear that as x gets large and negative, then $x \cdot e^{-x}$ explodes negatively. What's not clear is the behavior of $x \cdot e^{-x}$ as x gets large and positive. The expression can be rewritten as the fraction $\frac{x}{e^x}$, and both the numerator and denominator are getting arbitrarily large and positive, and it's not obvious what happens to the fraction of one large number divided by another large number. There's a sophisticated tool from advanced calculus, like a supercool Maserati of limit laws, that enables us to transform this limit into something simpler. Its advanced trick is called L'Hôpital's Rule. Often called the hospital rule by students because it's so good at fixing things, it is named after the 17th-century French mathematician Guillaume de l'Hôpital. If you take more advanced courses in calculus, you'll become quite familiar with this rule, but it takes some care. Under certain conditions, it says you can differentiate the top and bottom of a fraction and get the same limit. It turns out to be applicable in other situations though I don't want to get into the details of why. We can differentiate x in the numerator to get one, and e^x in the denominator to get e^x back again, and we now get a much simpler limit. In fact, the usual limit associated with exponential decay, which is zero. The upshot of all this is the$\lim_{x\to\infty} xe^{-x}$ is zero, so that the positive x-axis becomes a horizontal asymptote for the curve.

$$\lim_{x\to\infty} xe^{-x}$$

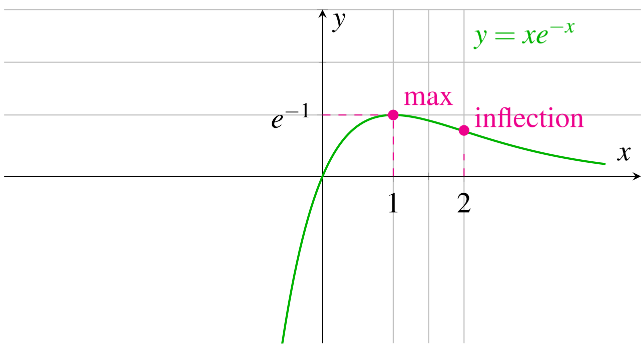
$$= \lim_{x\to\infty} \frac{x}{e^x}$$

Using L'Hôpital's Rule

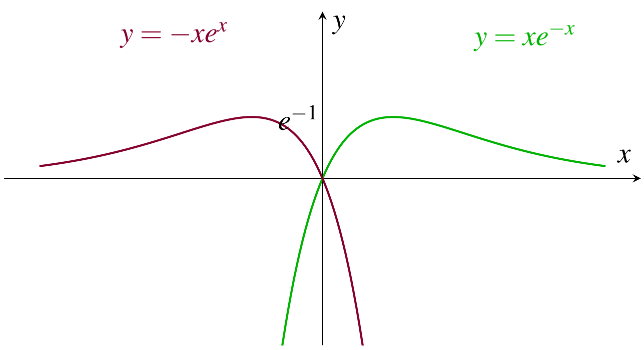
$$\lim_{x\to\infty} \frac{1}{e^x}$$

$$\lim_{x\to\infty} e^{-x}$$

Putting all this together produces a sketch of the curve, with all the important features that we've noted before.



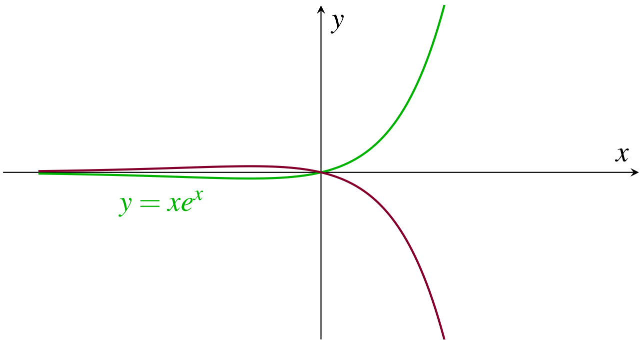
What about the curve from part (a) of our earlier problem? We can in fact get the sketch by playing around with reflections in the plane applied to the curve we've just looked at in detail. First, we're reflecting the vertical y-axis to get the following diagram.



Algebraically, this corresponds to replacing x with $-x$ in the rule for the function $y = x \cdot e^{-x}$., which quickly simplifies to $y = -x \cdot e^{x}$.

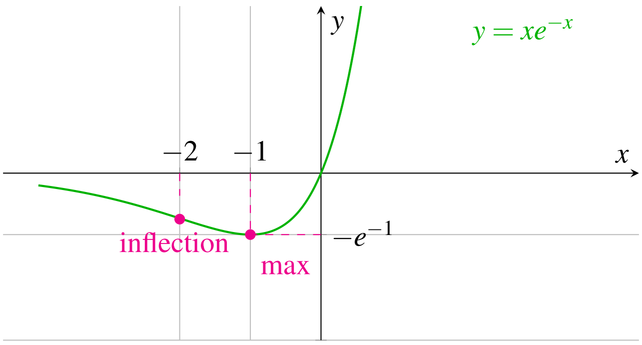
Thus the curve $y = xe^{-x}$ is transformed into the curve $y = -xe^{x}$.

Now, we take this transformed curve and reflect it in the x-axis to obtain the following diagram.



Algebraically, this corresponds to taking the negative of the y-values, which quickly becomes $y = xe^{x}$.

This, in fact, produces the rule for the curve for part (a) of our original problem. Then all of the corresponding features such as the turning point, the inflection, and also the negative x-axis as a horizontal asymptote become apparent.



Notice also how the two curves can be obtained from one another by rotating 180 degrees about the origin. It's a general fact important in geometry and algebra, that if you follow any reflection by another refection, the overall effect is a rotation. If we create a transparency of the original curve and flip it over vertically, and then horizontally, we see the overall effect of a rotation.

But we can perform any other pair of reflections, and the overall effect is always a rotation. You can also see this effect with a coin. You flip it over twice, in any odd manner, you get a rotation from the original position. I'd like to finish by remarking about relationships between pairs of functions, as this leads naturally into the theory of differential equations, which you're likely to see in any further courses on calculus. The simplest examples that give rise to exponential growth and decay, $y = e^{x}$ and e^{-x} form a natural pair. They turn out to be what are called fundamental solutions of the differential equation:

$$y'' - y = 0$$

You start with either of them, differentiate twice and take away what you started with, you get zero. This equation is called differential because it involves some kind of derivative, in this case, y''. The two functions are called fundamental solutions because they're basic building blocks and turn out to be fundamentally different in a sense that can be made precise in advanced mathematics. You could think of them as analogous to hydrogen and helium atoms in chemistry.

Another natural pair are the circular functions $y = \sin x$ and $y = \cos x$. If you start with either of them, differentiate twice and add what you started with, you get zero. This is the solution to the equation:

$$y'' + y = 0$$

Even though you might think of circular functions as completely different from exponential functions, there's a strong connection from the point of view of differential equations. To get from one pair to the other, you just alter the associated differential equation by turning a minus into a plus.

We can also pair $y = e^x$ with $y = xe^x$, one of the curves we started in this section, which you can think of as a modified form of exponential growth. They turn out to be solutions to the equation

$$y'' - 2y' + y = 0$$

, as you can check easily if you wish. If you pair $y = e^{-x}$ with $y = xe^{-x}$, some kind of modified form of exponential decay, you get solutions of the same equation as the previous pair but with minus replaced by plus. What if we put a pair of pairs together? Say e^x, e^{-x}, $\sin x$, and $\cos x$. These turn out to be fundamental solutions to the equation $y^{(4)} - y = 0$ ($y^{(4)} = y''''$) What if we go the whole hog and add to this list the two curves we studied today, $y = xe^x$ and xe^{-x}. These turn out to be fundamental solutions to the equation $y^{(6)} - y^{(4)} - y'' + y = 0$. You can easily check these are indeed solutions, but you might wonder how on earth one could come up with a single unifying equation like this. There are indeed methods a bit like magic tricks that you'll learn if you go on to study more advanced courses on calculus and linear algebra. There are so many intriguing possibilities and surprising connections. Many people devote their lives to trying to discover and understand solutions to differential equations, working out how they fit together and relate to one another. They're like organic chemists, discovering new and revolutionary organic compounds or genetic engineers seeking ways to splice together genes to cure or prevent diseases. I hope this very brief introduction might stimulate your interest to learn more about this fascinating topic, drawing upon connections between many different branches of mathematics with myriad applications to science and other disciplines.

This module is finally drawing to a close. The key idea or focus throughout has been the derivative and the careful development of techniques of differentiation. We started with increasing and decreasing functions, saw how to recognize this behavior using the first derivative where being positive corresponds to increasing and negative to decreasing. So, the transition from one to the other indicates the presence of a turning point, where the value of the function achieves a maximum or minimum. All of this is captured concisely using the sine diagram for the derivative. We studied concavity of curves, where concave up or bowl-shaped up behavior is indicated by a positive second derivative, and concave down or bowl-shaped down behavior is indicated by a negative second derivative, and transition from one to the other indicates presence of the point of inflection. All of this is captured concisely using the sign diagram for the second derivative. We adopted useful mnemonic devices or symbols involving a smiley face for concave up and a sad face for concave down. We developed thorough and systematic protocols for curve sketching including intercepts of the axes, horizontal, vertical, and oblique asymptotes and all of the information captured by the sign diagrams for the derivative and second derivative. We introduced and applied several rules for differentiation including the Chain Rule associated with composition functions, the Product Rule associated with multiplication, and the Quotient Rule associated with division. We looked at some contrasting examples of optimization, where the task is to find out where values of a function might be minimized or maximized, and just now looked at the useful criteria involving the second derivative called the second derivative test.

Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon in the final module, where we launch into integral calculus.

27.2.1 Practice Quiz

Question 1

Consider the function f given by the following rule:

$$f(x) = 3x^2 - 6x + 7.$$

Which one of the following statements about f is true?
(a) f has a local extremum corresponding to some point $(x, f(x))$ for which $f''(x) = 0$.
(b) $f''(x)$ is positive for all x and f has a global minimum.
(c) $f''(x)$ is negative for all x and f has a global maximum.
(d) $f''(x)$ is negative for all x and f has a global minimum.
(e) $f''(x)$ is positive for all x and f has a global maximum.

Question 2

Consider the function f given by the following rule:

$$f(x) = 5 + x - 4x^2.$$

Which one of the following statements about f is true?
(a) $f''(x)$ is negative for all x and f has a global maximum.
(b) f has a local extremum corresponding to some point $(x, f(x))$ for which $f''(x) = 0$.
(c) $f''(x)$ is positive for all x and f has a global maximum.
(d) $f''(x)$ is negative for all x and f has a global minimum.
(e) $f''(x)$ is positive for all x and f has a global minimum.

Question 3

Consider the function f given by the following rule:

$$f(x) = e^x.$$

Which one of the following statements about f is true?
(a) f has a local extremum corresponding to some point $(x, f(x))$ for which $f''(x) = 0$.
(b) $f''(x)$ is negative for all x and f has a global maximum.
(c) $f''(x)$ is positive for all x and f has a global minimum.
(d) $f''(x)$ is negative for all x and f has a global minimum.
(e) $f''(x)$ is positive for all x and f has a global maximum.

Question 4

Consider the function f given by the following rule:

$$f(x) = x^3 - 12x.$$

Which one of the following statements about f is true?

(a) $f'(2) = f''(2) = 0$ and f has a local maximum at $x = 2$.
(b) $f'(2) = 0, f''(2) \neq 0$ and f does not have local minimum or maximum at $x = 2$.
(c) $f'(2) = 0, f''(2) < 0$ and f has a local maximum at $x = 2$.
(d) $f'(2) = 0, f''(2) > 0$ and f has a local minimum at $x = 2$.
(e) $f'(2) = 0, f''(2) = 0$ and f has a local minimum at $x = 2$.

Question 5

Consider the function f given by the following rule:

$$f(x) = x^3 - 12x.$$

Which one of the following statements about f is true?
(a) $f'(-2) = 0, f''(-2) < 0$ and f has a local maximum at $x = -2$.
(b) $f'(-2) = 0, f''(-2) = 0$ and f has a local minimum at $x = -2$.
(c) $f'(-2) = 0, f''(-2) \neq 0$ and f does not have local minimum or maximum at $x = -2$.
(d) $f'(-2) = f''(-2) = 0$ and f has a local maximum at $x = -2$.
(e) $f'(-2) = f''(-2) = 0$ and f has a local minimum at $x = -2$.

Question 6

Consider the function f given by the following rule:

$$f(x) = x^3 - 3x^2 + 3x - 1.$$

Which one of the following statements about f is true?
(a) $f'(1) = 0, f''(1) < 0$ and f has a local maximum at $x = 1$.
(b) $f'(1) = f''(1) = 0$ and f does not have local minimum or maximum at $x = 1$.
(c) $f'(1) = 0, f''(1) \neq 0$ and f has a local maximum at $x = 1$.
(d) $f'(1) = 0, f''(1) > 0$ and f has a local minimum at $x = 1$.
(e) $f'(1) = f''(1) = 0$ and f has a local maximum at $x = 1$.

Question 7

Consider the function f given by the following rule:

$$f(x) = xe^{x/2}.$$

Which one of the following statements about f is true?
(a) $f'(-4) = 0, f''(-4) < 0$ and f has a local maximum at $x = -4$.
(b) $f'(-4) = 0, f''(-4) > 0$ and f has a local maximum at $x = -4$.
(c) $f'(0) = f''(0) = 0$ and f does not have local minimum or maximum at $x = 0$.
(d) $f'(-2) = 0, f''(-2) < 0$ and f has a local minimum at $x = -2$.
(e) $f'(-2) = 0, f''(-2) > 0$ and f has a local minimum at $x = -2$.

Question 8

Consider the function f given by the following rule:

$$f(x) = xe^{-x/2}.$$

Which one of the following statements about f is true?
(a) $f'(4) = 0, f''(4) > 0$ and f has a local minimum at $x = 4$.
(b) $f'(0) = f''(0) = 0$ and f does not have local minimum or maximum at $x = 0$.
(c) $f'(-2) = 0, f''(-2) < 0$ and f has a local maximum at $x = -2$.
(d) $f'(2) = 0, f''(2) < 0$ and f has a local minimum at $x = 2$.
(e) $f'(4) = 0, f''(4) < 0$ and f has a local maximum at $x = 4$.

Question 9

Consider the function f given by the following rule:

$$f(x) = x + \tan^{-1} x.$$

Which one of the following statements about f is true?
(a) $f'(0) = 0$ and f has an inflection at $x = 0$.
(b) $f'(0) > 0$ and f has a local minimum at $x = 0$.
(c) $f'(0) < 0$ and f has a local maximum at $x = 0$.
(d) $f'(0) = 0$ and f has an inflection at $x = 0$.
(e) $f'(0) < 0$ and f has an inflection at $x = 0$.

Question 10

Consider the function f given by the following rule:

$$f(x) = x - \tan^{-1} x.$$

Which one of the following statements about f is true?
(a) $f''(0) > 0$ and f has a local minimum at $x = 0$.
(b) $f''(0) < 0$ and f has a local minimum at $x = 0$.
(c) $f''(0) > 0$ and f has a local maximum at $x = 0$.
(d) $f''(0) = 0$ and f has an inflection at $x = 0$.
(e) $f''(0) < 0$ and f has an inflection at $x = 0$.

Answers

The answers will be revealed at the end of the module.

28. Assessment

28.1 Module Quiz

Question 1

Find $\frac{dy}{dx}$ when $y = (2x-1)(3x+5)$.

(a) $12x+3$
(b) $6x+7$
(c) $12x+7$
(d) $6x+9$
(e) $12x+5$

Question 2

Find $\frac{dy}{dx}$ when $y = (3x-1)^3$.

(a) $9(3x-1)^2$
(b) $9x(3x-1)^2$
(c) 27
(d) $3x(3x-1)^2$
(e) $3(3x-1)^2$

Question 3

Find $\frac{dy}{dx}$ when $y = e^{5t}$.

(a) e^{5t}
(b) $5e^{5t}$
(c) e^5
(d) $\frac{e^{5t}}{5}$
(e) $5te^{5t-1}$

Question 4

Find $\frac{du}{d\theta}$ when $u = \cos(2\theta)$.

(a) $2\sin(\theta)$
(b) $-\sin(2\theta)$
(c) $\sin(2\theta)$
(d) $2\sin(2\theta)$
(e) $-2\sin(2\theta)$

Question 5

Find y' when $y = e^{3x^2}$.
(a) e^{3x^2}
(b) $6e^{6x}$
(c) $18x^3e^{3x^2-1}$
(d) $3x^2e^{3x^2-1}$
(e) $6xe^{3x^2}$

Question 6

Find y'' when $y = e^{-2x^2}$.
(a) $16x^2e^{-2x^2}$
(b) $4(2x^2-1)e^{-2x^2}$
(c) $(4x^2-1)e^{-2x^2}$
(d) $4(2x^2-1)e^{-2x^2}$
(e) $2(4x^2-1)e^{-2x^2}$

Question 7

Find $\frac{dy}{dx}$ when $y = \frac{3x+5}{2x-1}$.
(a) $\frac{3}{x} - 13(2x-1)^2$
(b) $-\frac{13}{(2x-1)}$
(c) $\frac{13-3x}{(2x-1)^3}$
(d) $\frac{3x-8}{(2x-1)^2}$
(e) $\frac{13}{(2x-1)}$

Question 8

Find $\frac{du}{dt}$ when $u = \frac{\ln t}{t^3}$.
(a) $\frac{1-3\ln t}{t^4}$
(b) $\frac{t^2-3\ln t}{t^6}$
(c) $\frac{3\ln t-1}{t^6}$
(d) $\frac{3\ln t-1}{t^5}$
(e) $\frac{1-3\ln t}{t^6}$

Question 9

Find y' when $y = \tan^{-1}\left(\frac{x}{2}\right)$.
(a) $\frac{2}{x^2+4}$
(b) $\frac{2}{4x^2+1}$
(c) $\frac{1}{x^2+4}$
(d) $\frac{4}{x^2+4}$
(e) $\frac{2}{2x^2+1}$

Question 10

Over which one of the following intervals is the function $y = x^2 - 1$ increasing?
(a) $[-1,1]$
(b) $(-\infty,1]$
(c) $[0,\infty)$
(d) $[-1,\infty)$
(e) $(-\infty,0]$

Question 11

Which one of the following is the correct sign diagram for the derivative y' of the function $y = x^3 - 6x$?

(a)

x		-1		1	
y'	$+$	0	$-$	0	$+$

(b)

x		-1		1	
y'	$-$	0	$+$	0	$-$

(c)

x		$-\sqrt{2}$		$\sqrt{2}$	
y'	$+$	0	$-$	0	$+$

(d)

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
y'	$-$	0	$+$	0	$-$

(e)

x		$-\sqrt{2}$		$\sqrt{2}$	
y'	$-$	0	$+$	0	$-$

Question 12

Which one of the following is the correct sign diagram for the derivative y'' of the function $y = x^3 - 6x$?

(a)

x		0	
y''	$-$	0	$+$

(b)

x		1	
y''	$-$	0	$+$

(c)

x		0	
y''	$+$	0	$+$

(d)

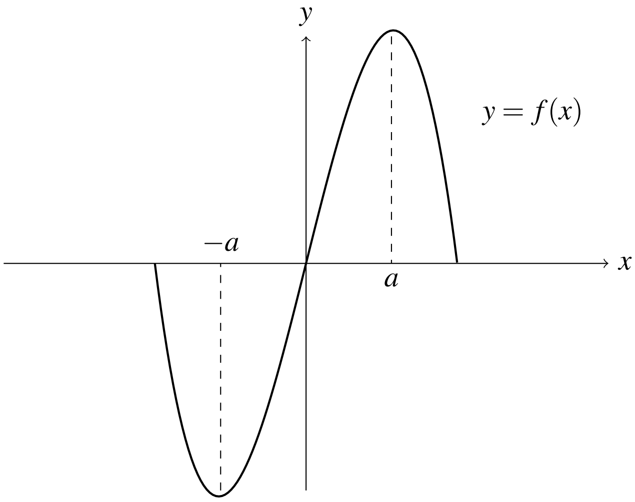
x		$\frac{3}{2}$	
y''	$+$	0	$-$

(e)

x		1	
y''	$+$	0	$-$

Question 13

Consider the following curve $y = f(x)$ where a is some positive real number.



Which one of the following pairs of sign diagrams for y' and y'' best matches this curve?

(a)

x		$-a$		a	
y’	$-$	0	$+$	0	$-$

x		0	
y”	$+$	0	$-$

(b)

x		$-a$		a	
y’	$-$	0	$+$	0	$-$

x		0	
y”	$-$	0	$+$

(c)

x		$-a$		a	
y’	$+$	0	$-$	0	$+$

x		0	
y”	$+$	0	$-$

(d)

x		$-a$		a	
y’	$+$	0	$-$	0	$+$

x		2	
y”	$-$	0	$+$

(e)

x		-3		3	
y’	$+$	0	$-$	0	$+$

x		0	
y”	$-$	0	$-$

Question 14

Which one of the following statements is true for the curve $y = x^3 + 1$?
(a) The curve is concave up everywhere.
(b) The curve has a turning point at (0, 1).
(c) The curve has a point of inflection at (0, 1).
(d) The curve is concave down everywhere.
(e) The curve has a point of inflection at (-1, 0).

Question 15

Which one of the following statements is true for the curve $y = x^4 + 2x^2$?
(a) The curve has exactly three turning points and two points of inflection.
(b) The curve is concave up everywhere.
(c) The curve has exactly two turning points and one point of inflection.
(d) The curve has exactly one turning point and no points of inflection.
(e) The curve is concave down everywhere.

Question 16

Find the global minimum for the function f with rule $f(x) = x^2 - 4x + 5$.
(a) -3
(b) 5
(c) 0
(d) 1

(e) -1

Question 17

Consider the function f with the following rule:

$$f(x) = 2x^3 - 3x^2 - 12x - 2$$

Which one of the following statements is true?
(a) f has a turning point at (0, -2) and -2 is a local minimum.
(b) f has a turning point at (2, -22) and -22 is a local minimum.
(c) f has a turning point at (2, -22) and -22 is a local maximum.
(d) f has a turning point at (-1, -5) and -5 is a local minimum.
(e) f has no turning points.

Question 18

Find the global maximum and minimum for the function f with the rule

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

for $x \in [0, 3]$.
(a) $4, -5$
(b) $11, 4$
(c) $4, -16$
(d) $-5, -16$
(e) $11, -16$

Question 19

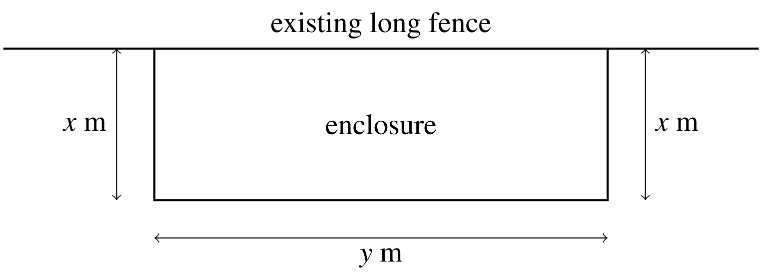
Find the minimum possible value of

$$y = x + \frac{1}{9x}$$

for $x > 0$.
(a) $\frac{3}{10}$
(b) $\frac{13}{18}$
(c) $\frac{9}{16}$
(d) $\frac{9}{10}$

Question 20

A rectangular enclosure is built using a long existing fence and three new sides that use a total of 40 metres of fencing.



Denote the length of the two new sides perpendicular to the existing fence by x metres, and the length of the new side parallel to the existing fence by y metres, so that $2x + y = 40$. Find the largest possible area of the enclosure.

(a) 160 m^2
(b) 220 m^2
(c) 200 m^2
(d) 150 m^2
(e) 192 m^2

Answers

The answers will be revealed at the end of the module.

29. Answer Key

Increasing and Decreasing Functions

1 (c)
2 (b)
3 (d)
4 (d)
5 (b)
6 (c)
7 (c)
8 (b)
9 (c)
10 (c)

Sign Diagrams

1 (d)
2 (d)
3 (a)
4 (b)
5 (e)
6 (c)
7 (c)
8 (b)
9 (b)
10 (b)

Maxima and Minima

1 (a)
2 (b)
3 (c)
4 (e)
5 (c)
6 (d)
7 (a)
8 (b)
9 (a)
10 (a)

Concavity and Inflections

1 (c)
2 (d)
3 (e)
4 (e)
5 (a)
6 (e)
7 (b)
8 (b)
9 (d)
10 (e)

Curve Sketching

1 (d)
2 (b)
3 (e)
4 (c)
5 (b)
6 (a)
7 (d)
8 (c)
9 (d)
10 (d)

The Chain Rule

1 (e)
2 (c)
3 (a)
4 (c)
5 (a)
6 (b)
7 (b)
8 (e)
9 (c)
10 (c)

Applications of the Chain Rule

1 (a)
2 (e)
3 (b)
4 (e)
5 (e)
6 (e)
7 (d)
8 (b)
9 (c)
10 (b)

The Product Rule

1 (a)
2 (c)
3 (b)
4 (e)
5 (e)
6 (b)
7 (a)
8 (a)
9 (b)
10 (e)

Applications of the Product Rule

1 (a)
2 (d)
3 (a)
4 (a)
5 (a)
6 (b)
7 (b)
8 (c)
9 (b)
10 (e)

The Quotient Rule

1 (e)
2 (b)
3 (e)
4 (c)
5 (a)
6 (d)
7 (b)
8 (a)
9 (b)
10 (d)

Applications of the Quotient Rule

1 (b)
2 (d)
3 (e)
4 (c)
5 (e)
6 (c)
7 (b)
8 (b)
9 (c)
10 (a)

Optimization

1 (c)
2 (c)
3 (d)
4 (d)
5 (b)
6 (d)
7 (b)
8 (a)
9 (b)
10 (e)

The Second Derivative Test

1 (b)
2 (a)
3 (c)
4 (b)
5 (c)
6 (e)
7 (e)
8 (d)
9 (d)
10 (d)

Assessment

1 (c)
2 (a)
3 (b)
4 (e)
5 (e)
6 (b)
7 (d)
8 (a)
9 (a)
10 (c)
11 (d)
12 (a)
13 (a)
14 (b)
15 (c)
16 (d)
17 (b)
18 (c)
19 (a)
20 (c)

V Introducing the Integral Calculus

This fifth and final module introduces integral calculus, looking at the slopes of tangent lines and areas under curves. This leads to the Fundamental Theorem of Calculus. We explore the use of areas under velocity curves to estimate displacement, using averages of lower and upper rectangular approximations. We then look at limits of approximations, to discover the formula for the area of a circle and the area under a parabola. We then develop methods for capturing precisely areas under curves, using Riemann sums and the definite integral. The module then introduces indefinite integrals and the method of integration by substitution. Finally, we discuss properties of odd and even functions, related to rotational and reflectional symmetry, and the logistic function, which modifies exponential growth.

Learning Objectives

- antidifferentiate simple functions and recognise important connections, such as between displacement, velocity and acceleration
- relate areas under curves to Riemann sums and rectangular and trapezoidal approximations
- understand, use and apply definite and indefinite integrals and the Fundamental Theorem of Calculus
- use and apply simple integration techniques, including integration by substitution and the methods of partial fractions and integration by parts

30. Introduction

30.1 Introduction to Module 5

Welcome back. We've just completed the third and fourth modules introducing and discussing differential calculus, which is really all about studying slopes of tangent lines to curves. This final module introduces and discusses integral calculus, which is really all about the study of areas under curves. Remarkably, slopes of tangent lines and areas under curves turn out to be intrinsically linked leading to the Fundamental Theorem of Calculus. This final module begins by illustrating how one can use areas under velocity curves to estimate displacement using averages of lower and upper rectangular approximations, and replicating thought experiments, originally due to the ancient Greeks involving limits of approximations to discover the formula for the area of a circle and the area under a parabola. We then formalize the method of Riemann sums using rectangular approximations to areas under curves over a given interval, leading to the definite integral, which is defined to be the limit of the Riemann sums, and captures precisely areas under a curve. We can calculate the definite integral exactly under certain conditions using the Fundamental Theorem of Calculus, providing a simple and elegant formula involving taking anti-derivatives, which reverses the process of forming derivatives. We introduce indefinite integrals and illustrate the method of integration by substitution, closely related to the Chain Rule that you learned about in the previous module. We discuss properties of odd and even functions related to rotational and reflectional symmetry, and the logistic function, which modifies exponential growth by introducing an inhibition factor with important applications to population dynamics. Finally, we come full circle and provide you with some context and insight into how calculus was originally conceived by Newton in the 17th century through his remarkable estimate, of the escape velocity of a rocket. Again, we hope that you'll find the material interesting and stimulating, that you find the videos helpful, and that the practice and challenges provided by the many exercises are beneficial. I look forward very much to your continued attention and participation.

31. Velocity and Displacement

31.1 Inferring Displacement from Velocity

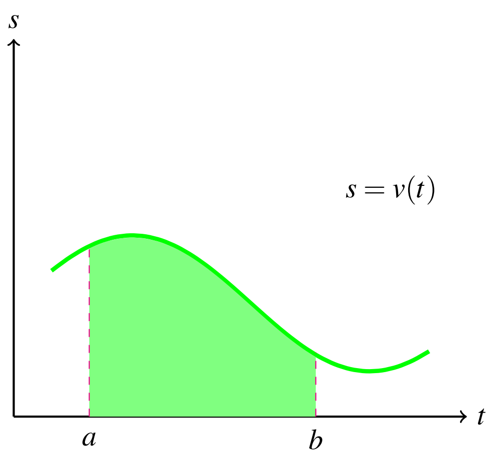
In today's lesson, we go through an extended example involving data points from the velocity curve of an accelerating vehicle to estimate the displacement or distance traveled. This exemplifies the idea of using the derivative to get information about the original function. As you'll see, the calculation or estimation of areas under curves is the key to moving backwards so to speak from the derivative. The underlying reason is that derivatives are alleged to instantaneous rates, which are limits of fractions. To undo a fraction, you expect multiplication to be involved, and of course, multiplication of real numbers is related to and may be defined in terms of areas of rectangles.

31.1.1 Introduction to integral calculus

This is our first foray into integral calculus. Two key words will become familiar to you are integration and antidifferentiation. In common language, to integrate, means to put things together into a coherent whole. The word anti-differentiate is not part of the common language and is only used by mathematicians, and it means to undo differentiation in the sense that we can make precise. That is to go backward from derivatives to the functions from which they came. For us, putting things together will have a technical meaning of finding areas, which involves multiplication and converting rates into quantities, which is some kind of process by which we unpack the derivative. The central theme is to use information about the derivative to gain information or insight concerning the original function. Think of all that hard work in the previous two modules where we created derivatives. Now, we going in the reverse direction, developing techniques for undoing the derivative. This leads naturally to setting up and solving differential equations, which are equations in which at least one derivative appears somewhere. The study of differential equations will be for you a natural continuation following on from this introductory course in calculus. Working backwards from the derivative it includes for example, knowing the velocity, the instantaneous rate of change of displacement. To get back to or find displacement, or knowing acceleration the instantaneous rate of change of velocity to get back to or find velocity, or knowing the rate of absorption of a drug, to determine the amount of the drug that's been absorbed, or knowing the rate of growth of the colony of bacteria to determine or predict size of the population, or knowing the rate of spread of a disease to estimate or predict the number of people that will become infected.

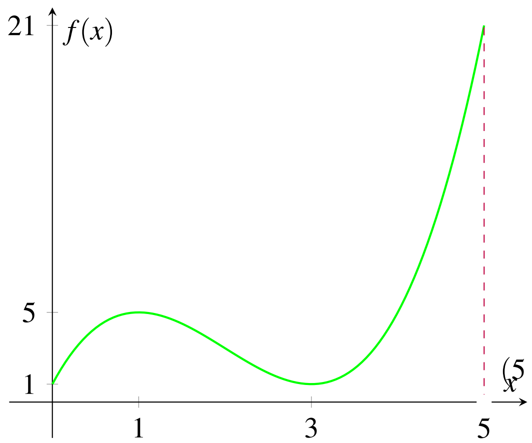
31.1.2 Motivating example

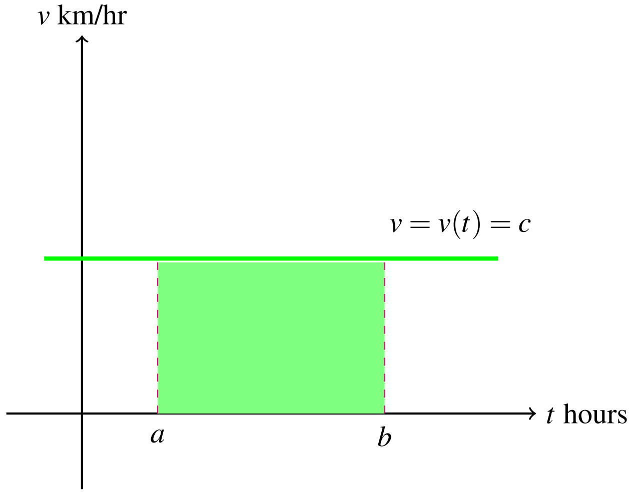
Let's focus now on perhaps the most important motivating example. Velocity denoted by v. The derivative of displacement denoted by x, both of which are functions of time t. We can keep in our minds that v is $\frac{dx}{dt}$. If we have enough information about v and even without a formula for $\frac{dx}{dt}$, then we can find or estimate displacement x by finding areas under the velocity curve.



Here's a typical velocity curve, and we're interested in the displacement over a particular interval of time as t passes say from a to b. We move up the curve from t = a and t = b to create a region in the plane colored here in green, bounded by the curve the horizontal axis and the vertical lines through a and b. It turns out that this green area represents the change in position, the displacement x of the object of this particular time interval.

To try to understand this phenomenon, we first look at the simplest case when the velocity v is constant, say c. So, that the velocity curve is a horizontal line passing through c on the vertical axis.





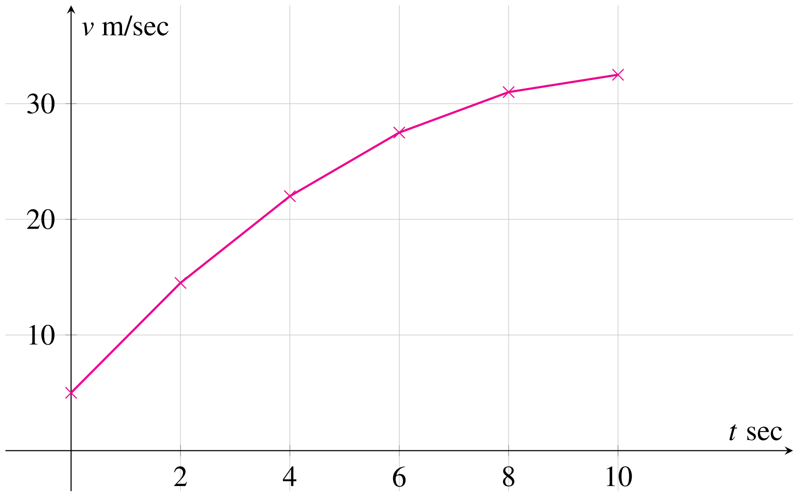
We move up to the curve from t = a to t = b and obtain a rectangular region colored green in this diagram. To make it more concrete, let's measure the velocity in say kilometers per hour and time t in hours, and you could imagine that we're traveling in a car alongside a straight road. Note that in the diagram, the scales on the horizontal and vertical axes are typically different, and can change according to the numbers involved. For example, if c = 60, a = 5 and b = 9, then we'd be traveling at 60 kilometers per hour for four hours. So, the distance traveled will be 4 times 60 equal to 240 kilometers, represented exactly by the area of the green rectangle of width 4 units and height 60 units. In general, the width of the rectangle is b - a and the height is the constant c. So, the green area is c(b-a). Exactly representing the number of kilometers traveled by multiplying the constant velocity by the number of hours spent traveling.

Of course in practice, the velocity fluctuates. Let's go through a typical example. To estimate the distance traveled by a car that is accelerating over an interval of 10 seconds so that its speed keeps increasing. We will assume the car is traveling along a straight road in one direction. So, we can use the words velocity and speed interchangeably. The velocities were measured every two seconds, and recorded here in meters per second to keep the units consistent and to simplify the calculation that follows.

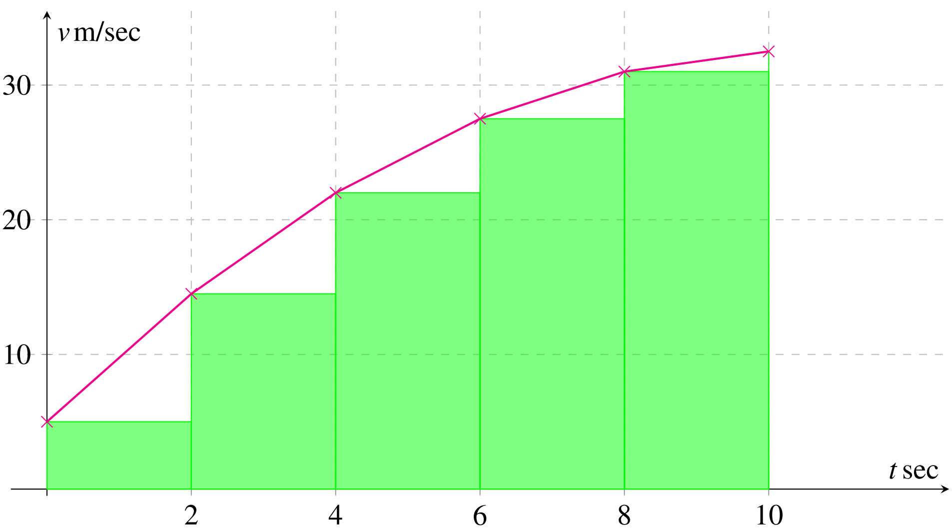
t sec	**0**	**2**	**4**	**6**	**8**	**10**
v m/sec	**5**	**14.5**	**22**	**27.5**	**31**	**32.5**

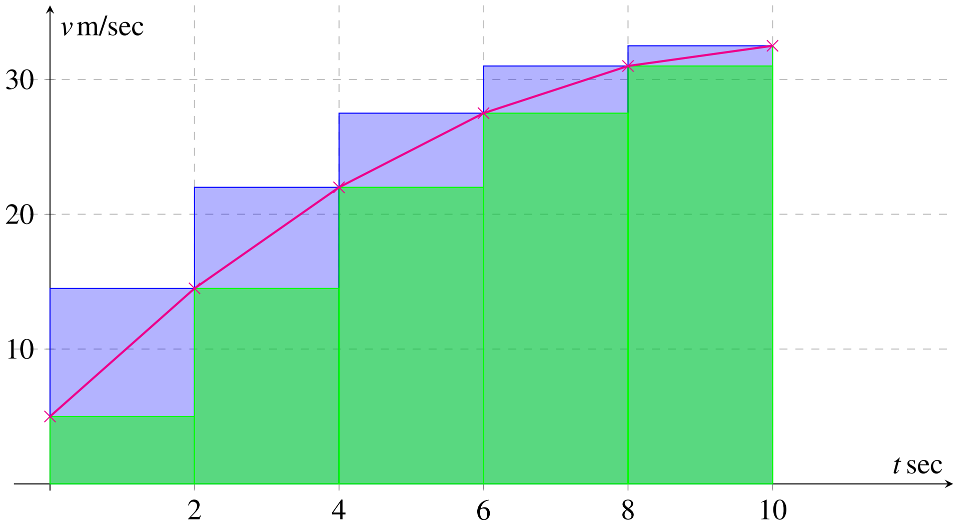
Note, to put this in context, in case you're used to measuring speeds in kilometers per hour, that 90 kilometers per hour converts to 25 meters per second. You can see from the table that the car must have been going really fast after about five or six seconds.

We can introduce a horizontal axis for time t and the vertical axis for velocity v and plot the points from the table. We can join the points to get what we expect will be a reasonable approximation to the true velocity curve of the car.



Now, in the first two seconds, the car accelerates from 5 to 14.5 meters per second. So, it's always traveling at a speed greater than or equal to 5 meters per second much of the time considerably faster than that. Any car moving at a constant speed of five meters per second would travel a shorter distance, which is represented by the green rectangular area in the diagram. This green area sitting beneath the curve then becomes a lower bound for the distance traveled by the accelerating car over the first two seconds. Similarly, we can draw a rectangle for the period from two to four seconds. This area represents the distance traveled by a car moving at the constant speed of 14.5 meters per second. This again is a lower bound for the distance traveled by the accelerating car, which is always moving at a speed greater than or equal to 14.5 meters per second over this interval. We can continue drawing in rectangles beneath the curve for time t equals four to six seconds, t equals six to eight seconds and t equals eight to 10 seconds. We get a lower bound on the actual total displacement of the accelerating car by adding up the areas of all these green rectangles or columns sitting beneath the curve. All of the columns have width two units. So to get each successive shaded area, we just multiply the heights by two.





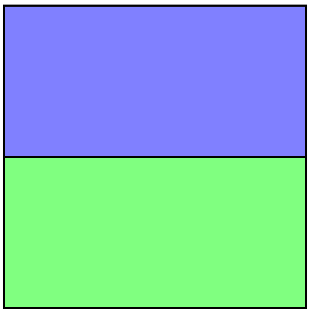
As the car is accelerating, its speed keeps increasing throughout. So we just take the velocities in the table as heights of the rectangles in succession from five all the way up to 31 and this sum quickly evaluates to 200. Thus, a lower bound for the distance traveled by the car becomes 200 meters. The previous estimate used rectangles sitting beneath the curve. If we use upper bounds for the velocities on each of the sub intervals, then we build rectangles that sit above the curve. In this diagram, we enlarge the green area in each case to the blue area, so that the combined blue and green areas form columns that encompass all of the area under the curve. The area of each larger rectangle or column now represents the distance traveled by moving at a constant speed greater than or equal to the speed of the accelerating car on any particular interval in two seconds. By adding up all of the areas, we should get an upper bound for the total distance traveled. Again the width of each column is two units and heights now moved from 14.5 and upper bound for the first interval successively through the heights in the table finishing at 32.5 and upper bound for the last sub interval. These areas all add up to give 255. Hence, we estimate an upper bound for the true distance travelled by our car to be 255 meters.
Upper bound for displacement (distance travelled)
for these 10 seconds would be

$$U = \text{sum of areas of combined shaded columns}$$

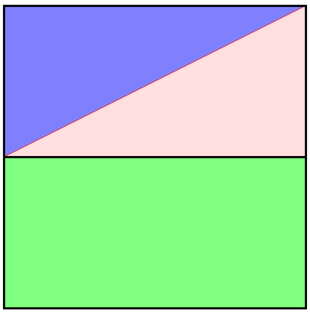
$$U = (2 \times 14.5) + (2 \times 22) + (2 \times 27.5) + (2 \times 31) + (2 \times 32.5)$$

Upper bound for displacement (distance travelled)
] Putting this together, we found a lower bound and an upper bound and note that the true distance traveled is somewhere in between. The number halfway between these two bounds seems likely to be a reasonable estimate and that number is just the average, which turns out to be 227.5. Hence finally, we estimate that the car traveled about 227.5 meters over those 10 seconds. In fact, we expect this average to be a slight underestimate. As you can see from the sketch that the curve appears to be concave down.

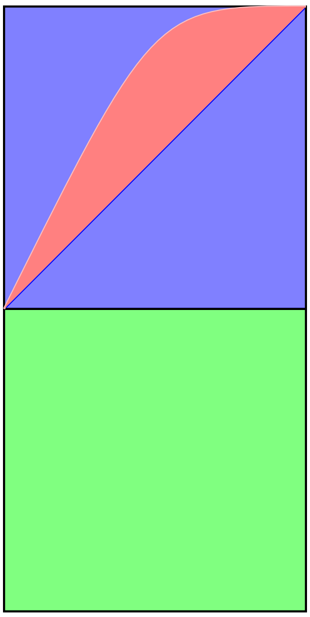
I'll try to explain the significance of the concavity. Consider any rectangle, split it into two pieces and color the low part of the rectangle green and the upper part blue.



Now, draw a diagonal across the upper part upwards from left to right and coloring pink everything below this diagonal.



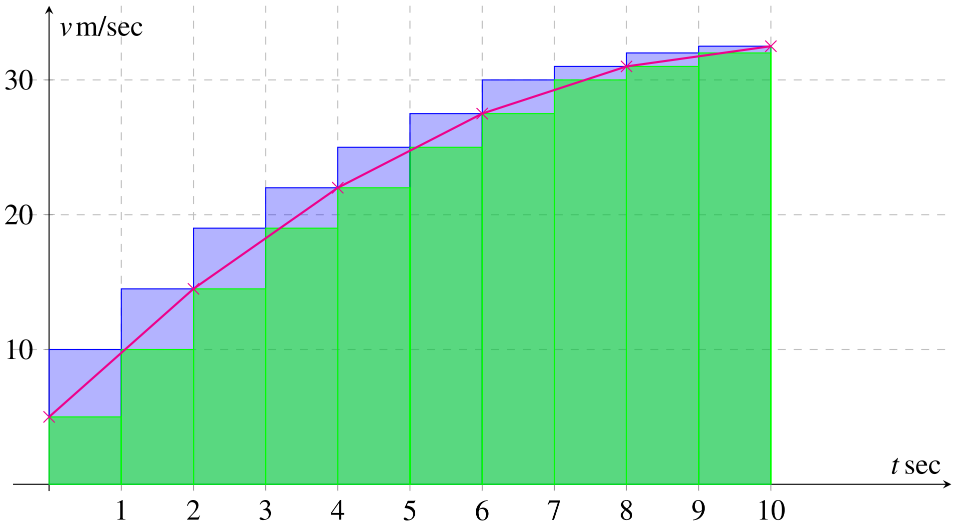
This pink area, is the green area plus half of the blue area, which can be expressed as one half of the green area plus the combined green and blue areas. In other words, the pink area is the average of the small and large areas from the previous two diagrams. The shape colored in pink is an example of a trapezium in geometry. In our example with an accelerating car, in a particular interval of two seconds, we had a green rectangle sitting below a fragment to the velocity curve; the combined green and blue rectangles sitting above the curve. And there's this trapezium shape, which we have colored in pink and as we've just explained, the pink area is the average of the areas of the upper and lower rectangles.
Let's add to this diagram, a curved fragment joining the points of the diagonal that creates the trapezium and draw it so that it's concave down as in our physical example.



Because of the concavity, there's a tiny piece of area not included by the pink shading, which we've colored beige. The pink area, the area with trapezium therefore becomes a slight underestimate to the area under this curve fragment. Nevertheless, the trapezium becomes an excellent approximation for the area if the curved fragment is almost a straight line. This leads to a technique known as the trapezoidal rule for approximating areas under curves using trapeziums, which is: **If curve is almost a straight line, trapeziums become excellent approximations for areas under curve.** In the case of our velocity curve, you can see visually that though concave down overall, the individual fragments of the curve are close to being straight lines. So, we expect this average, the upper and lower bounds, which is the aggregate of the averages of the areas of the lower and upper rectangles to be very good if not excellent approximation to the true distance traveled by the car. The previous estimate was based on a table with six data points. Clearly, if we had more data, then we expect to be able to improve our estimate. Suppose that the velocity was measured every second instead of every two seconds.

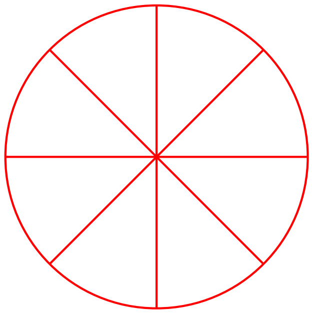
Here's a new table with the original data, including measurements taken for each second in between.

t sec	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
v m/sec	**5**	**10**	**14.5**	**19**	**22**	**25**	**27.5**	**30**	**31**	**32**	**32.5**



As before, we can plot all of the points and join them with a smooth curve. Again, we build rectangles to sit beneath the curve and then shade in the areas in green. We can build rectangles that sit above the curve by extending the lower rectangles and shade in the additional areas in blue. As before, we can calculate lower and upper bounds to the area under the true velocity curve. Note that the width of each rectangle or column is one unit, since the time intervals are now only one second each. Adding up the green areas gives a lower bound of 216. Whilst combining the green and blue areas gives an upper bound of 243.5. We can again take the average to produce a revised estimate of 229.75 meters for the distance traveled by the car. Because the curve appears to be concave down, we still expect this to be a very slight underestimate. The curve fragments appear to be very close to a straight line segments on each of these, now very short time intervals. So we expect to have produced an excellent estimate of the distance traveled.

In this section, we began with some general remarks concerning moving backwards, so to speak, from rates to original quantities or, in mathematical language, using the derivative to gain information about the original function. Moving backwards from velocity to displacement becomes a prototype for this phenomenon. We saw that the connection involves finding areas under curves. We discussed in detail how we could use a table of values for the velocity to estimate the distance traveled by an accelerating car over a given time interval. We use rectangular approximations that sit both below and above the velocity curve to get lower and upper bounds for the distance traveled. By taking the average of these bounds, we appear to get a very good if not excellent approximation to the true displacements. Taking averages is in fact related to using trapezoids to estimate the area under curve. Please re-read if you didn't get it and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.


(a) Eight sector circle

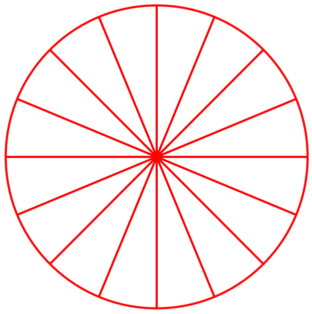

(b) Sixteen sector circle

Figure 31.1: Comparison of circle divisions

$$P(r)\,\Delta r \le \Delta A \le P(r+\Delta r)\,\Delta r$$

31.1.3 Practice Quiz

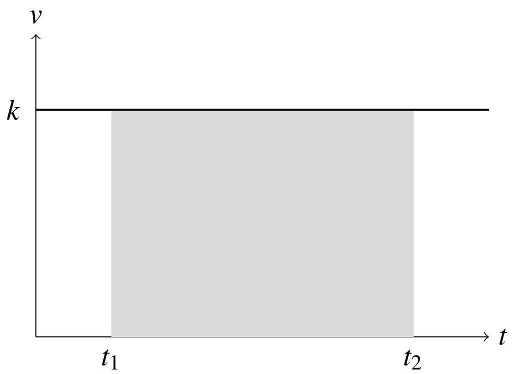
Question 1

A car travels along a straight road at 50 kilometres per hour for 5 seconds. How far did the car travel?

(a) $\frac{125}{9}\,m$
(b) $\frac{625}{9}\,m$
(c) $250\,m$
(d) $\frac{250}{9}\,m$
(e) $625\,m$

Question 2

The velocity curve of a particle is given below, in compatible units.



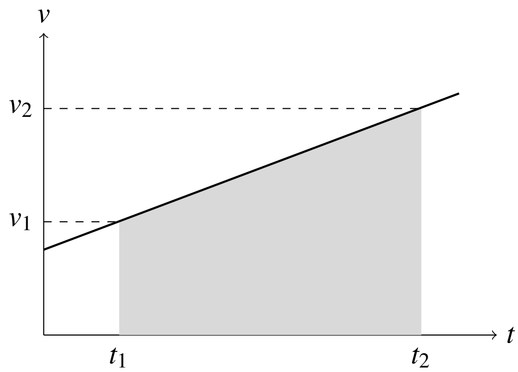
Find an expression for the shaded area under the curve, representing the displacement of the particle over the time interval $t_1 \le t \le t_2$.

(a) $\frac{k(t_2-t_1)}{2}$
(b) $k(t_2-t_1)$

(c) $k(t_1+t_2)$
(d) $k(t_1-t_2)$
(e) $\frac{k(t_1+t_2)}{2}$

Question 3

The velocity curve of a particle is given below, in compatible units.

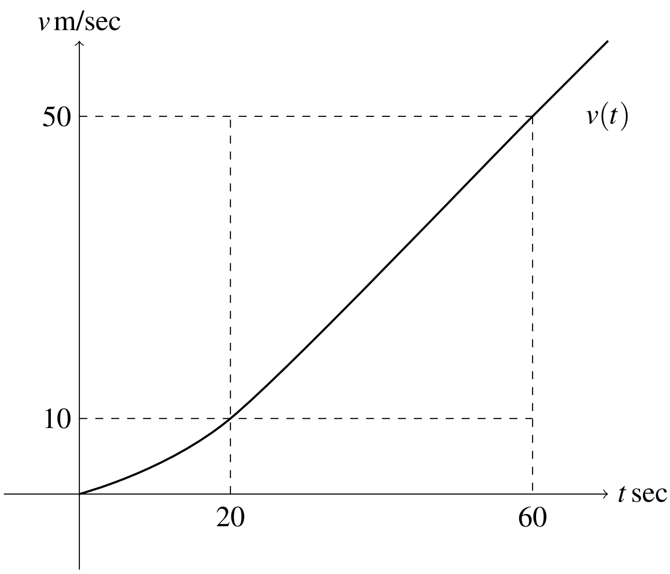


Find an expression for the shaded area under the curve, representing the displacement of the particle over the time interval $t_1 \leq t \leq t_2$.

(a) $(v_1+v_2)(t_2-t_1)$
(b) $\frac{(v_2-v_1)(t_2+t_1)}{2}$
(c) $\frac{(v_2+v_1)(t_2+t_1)}{2}$
(d) $\frac{(v_2-v_1)(t_2-t_1)}{2}$
(e) $\frac{(v_2+v_1)(t_2-t_1)}{2}$

Question 4

The velocity curve $v = v(t)$ m/sec of an accelerating object moving along a straight line is shown below, as a function of t sec, and is shaped slightly concave upwards.



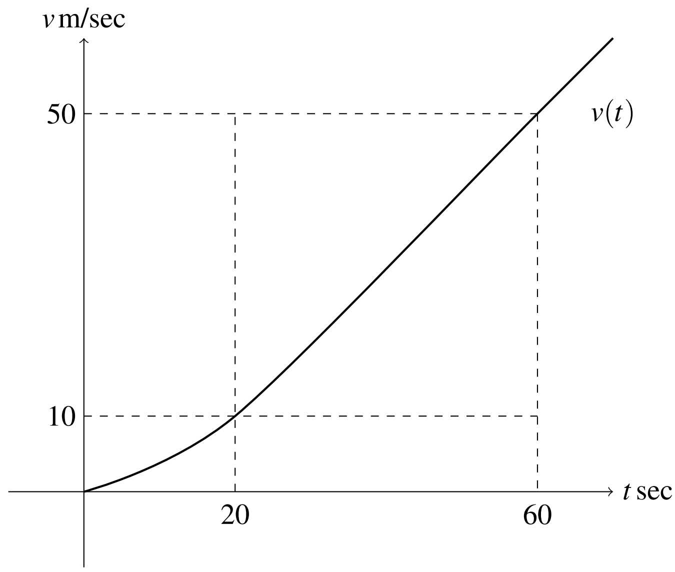
Use a lower rectangle that sits beneath the curve, but just reaches the curve at the left-hand endpoint, to find a lower bound for the distance traveled for $20 \leq t \leq 60$.

(a) 600 m

(b) 1,000 m
(c) 400 m
(d) 500 m
(e) 200 m

Question 5

The velocity curve $v = v(t)$ m/sec of an accelerating object moving along a straight line is shown below, as a function of t sec, and is shaped slightly concave upwards.

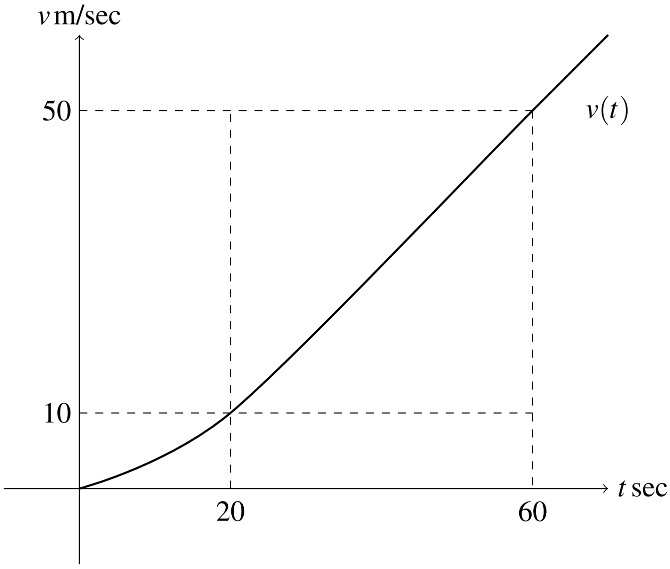


Use an upper rectangle that sits above the curve, but just touches the curve at the right-hand endpoint, to find an upper bound for the distance traveled for $20 \leq t \leq 60$.

(a) 3,000 m
(b) 600 m
(c) 1,000 m
(d) 2,000 m
(e) 1,600 m

Question 6

The velocity curve $v = v(t)$ m/sec of an accelerating object moving along a straight line is shown below, as a function of t sec, and is shaped slightly concave upwards.

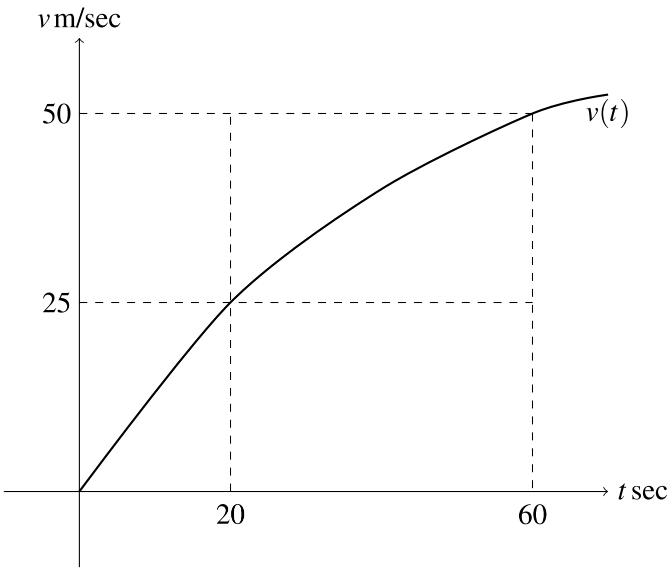


Estimate the distance traveled by the object, for $20 \leq t \leq 60$, by taking the average of the areas of the lower and upper rectangles that just touch the curve, and decide whether this provides an overestimate or underestimate of the true distance traveled.

(a) 1,000 m, overestimate
(b) 1,200 m, overestimate
(c) 1,500 m, overestimate
(d) 1,500 m, underestimate
(e) 1,200 m, underestimate

Question 7

The velocity curve $v = v(t)$ m/sec of an accelerating object moving along a straight line is shown below, as a function of t sec, and is shaped slightly concave downwards.



Estimate the distance traveled by the object, for $20 \leq t \leq 60$, by taking the average of the areas of the lower and upper rectangles that just touch the curve, and decide whether this provides an overestimate or underestimate of the true distance traveled.

(a) 1,500 m, underestimate

(b) 1,600 m, overestimate
(c) 1,200 m, underestimate
(d) 1,500 m, overestimate
(e) 1,600 m, underestimate

Question 8

A car accelerates from rest reaching a speed of 22 m/sec after five seconds. The following velocities are recorded at the end of each second:

Time from rest (sec)	0	1	2	3	4	5
Velocity (m/sec)	0	8	14	18	21	22

Use the average of lower and upper rectangular approximations to estimate the distance travelled after the five seconds have elapsed. (You may assume the velocity curve is increasing throughout.)
(a) 76 m
(b) 73 m
(c) 75 m
(d) 72 m
(e) 74 m

Question 9

A car comes to a stop five seconds after the driver slams on the brakes. While the brakes are on, the following velocities are recorded:

Time since brakes applied (sec)	0	1	2	3	4	5
Velocity (m/sec)	26	18	12	7	3	0

Use the average of lower and upper rectangular approximations to estimate the distance travelled over those five seconds as the car comes to a halt. (You may assume the velocity curve is decreasing throughout.)
(a) 52 m
(b) 53 m
(c) 54 m
(d) 50 m
(e) 51 m

Question 10

A car accelerates smoothly from rest reaching a speed of 30 m/sec after fifteen seconds. The following velocities are recorded at the end of each three second period:

Time from rest (sec)	0	3	6	9	12	15
Velocity (m/sec)	0	10	18	24	28	30

Use the average of lower and upper rectangular approximations to estimate the distance travelled after the fifteen seconds have elapsed. (You may assume the velocity curve is increasing throughout.)
(a) 245 m
(b) 275 m
(c) 285 m
(d) 265 m
(e) 255 m

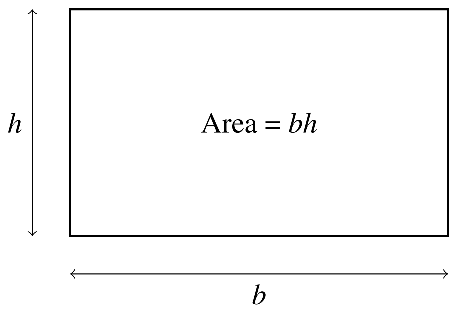
Answers

The answers will be revealed at the end of the module.

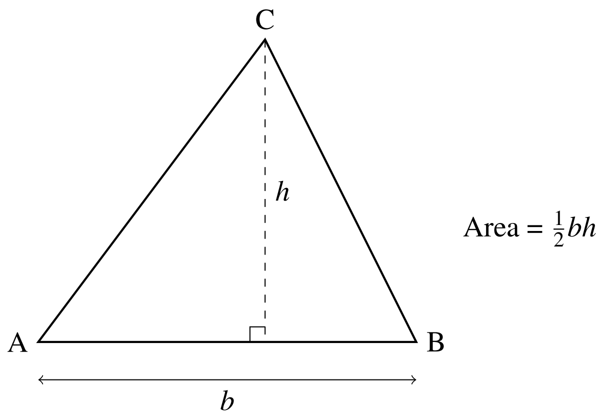
32. Areas under Curves, Riemann Sums and definite integrals

32.1 Areas bounded by curves

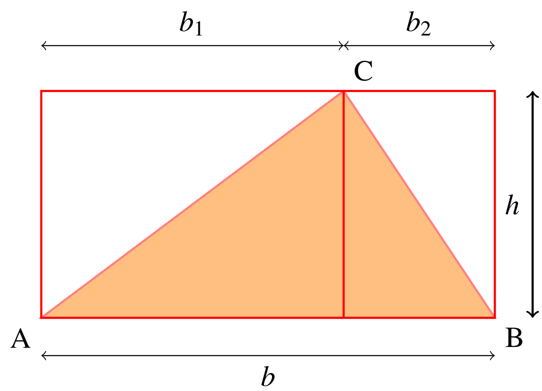
In this section, we go through two important and quite subtle calculations, both of which were known to the ancient Greeks. We use the perimeter to find a formula for the area of a circle, and calculate precisely a certain area bounded by a parabola in the x, y plane. We begin with two very familiar areas.



The area of a rectangle, of course is the base times the height, and in fact this is how one defines multiplication of two positive real numbers in the first place. The area of the triangle, I expect you know, is half the base times the height.



Already something interesting is happening as this enables one in principle to find areas of any geometric figure bounded by straight lines, since such figures can be split up into triangular pieces. To see why this formula holds, one has to relate a typical triangle in some way to rectangles.

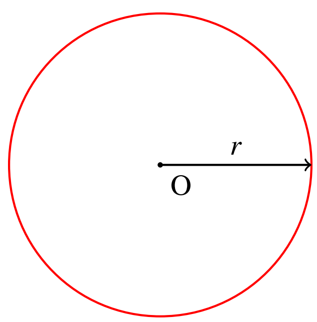


In this particular figure, one can form altitude of height h from the apex of the triangle to make the horizontal base of length b, and then use it to form two rectangles that enclose the triangle. One of width b1, and the other one width b2 say. Notice that the area of the triangle, colored beige, splits up into two pieces. The first of which is exactly half the area of the rectangle with width b1, and the second of which is exactly half the area of the rectangle with width b2. So, putting together the areas of these rectangles, and using the fact that b1 + b2 = b, we get to the beige area is 0.5 of b times h.

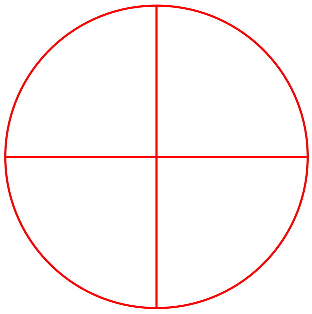
$$\text{beige area} = \frac{1}{2}(\text{area of rectangle with width } b_1)$$
$$+\frac{1}{2}(\text{area of rectangle with width } b_2)$$
$$= \frac{1}{2}b_1 h + \frac{1}{2}b_2 h = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}bh$$

This isn't the most general diagram for a triangle, but you can easily adapt this argument to all triangles.

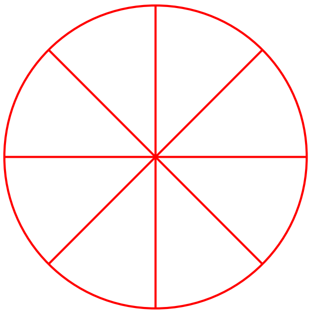
What about areas bounded by curves that are not straight lines? The most classic example would be a circle say of radius r. I expect you know the formula for the area of a circle, but do you know why it works? We discussed the perimeter in a very early section which we can call p, and note the fact that $p = \pi d$ or $2\pi r$.

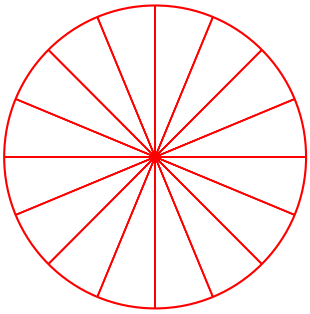


We are going to engage in a thought experiment used by the ancient Greeks. Imagine dividing a circle into N pieces called sectors, which are like pieces of pizza. Here for example, N equals four.

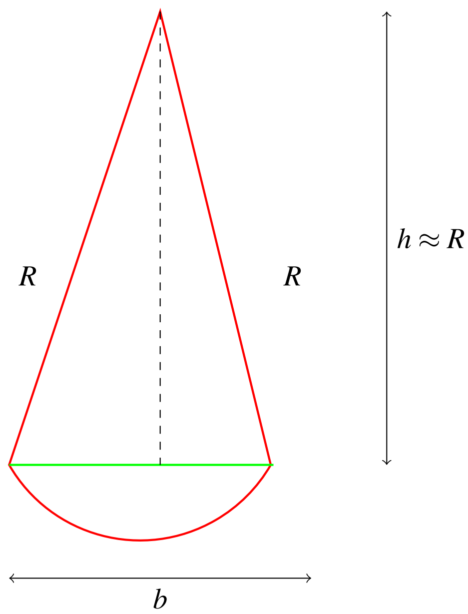


Dividing all the pieces in half we can get N equals eight, and in half again we can get N equals 16.



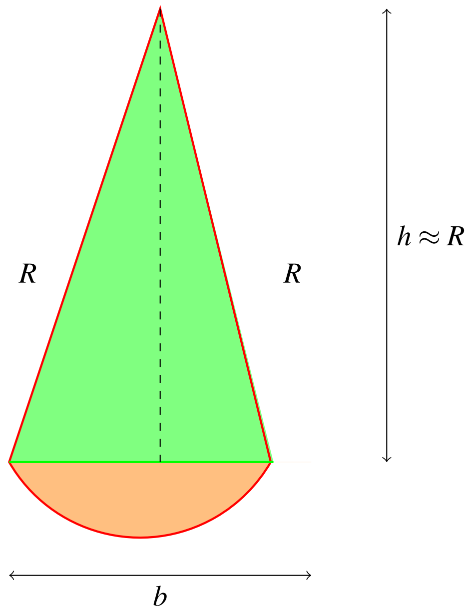


We can imagine going on dividing the sectors in half indefinitely and they get thinner and thinner.Suppose we stop halving the segments at some stage, and join up the points on the circle using successive small line segments where all the radii emanating from the centre meet the circle, to form this grain polygonal shape that appears to be hugging the perimeter. Suppose we do this so that all of the sectors have exactly the same shape, and here's a picture of one enlarged, including a small green line segment joining up the two points on the perimeter.

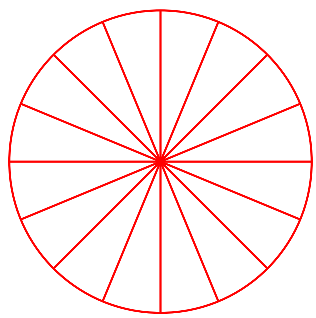


The drawing is not too thin to make it easier to follow the mathematics associated with the diagram. We have the two radii of length R emanating from the center, which form a triangle together with the green line segment of length b say. Denote the height or altitude of the triangle by h say. Notice that h is approximately equal to the radius r, because the altitude emanating from the center of the circle almost reaches the perimeter. If you imagine making more and more thinner sectors, by allowing the number N of sectors to become arbitrarily large, then you can see h getting closer and closer to the radius. We can express this precisely using limit notation.

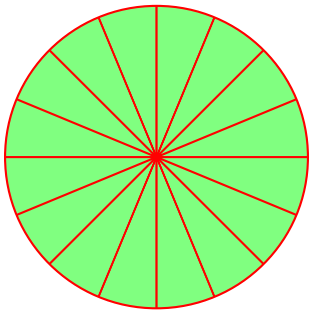
$$\lim_{N\to\infty} h = R$$



What about the area of the sector? It falls naturally into two pieces, the area of the triangle colored green bounded by the two radii and the line segments, and a little bit left over between the line segment and the perimeter colored beige. The area of the sector is approximately the area of the triangle, which is $\frac{1}{2}bh$.



Here's our circle again, split up into N sectors with the same size and shape, each with area $\frac{1}{2}bh$, and the limit as N goes to infinity of h is the radius R. Observe that if we circumnavigate the circle using the grain line segments of length b, we get N lots of b approximating the perimeter P. The more we subdivide the circle into thinner and thinner sectors, the closer N times b will be to the actual perimeter P. In fact the limit as $\lim_{N\to\infty} Nb = P$. How does all of this information relate to the area of the circle?



Here, all the triangles approximating the sectors have been colored in green. There is only a tiny bit of area of the circle not included, which you can hardly say between the green line segments and the perimeter. So, the area of the circle is approximately the sum of the areas of the triangles, which is N times a half bh, which we can rewrite as $\frac{Nb}{2}h$. The approximation gets better and better as N gets larger, and in fact the area is precisely the limit of this expression as N tends to infinity. But we have enough information to evaluate this limit.

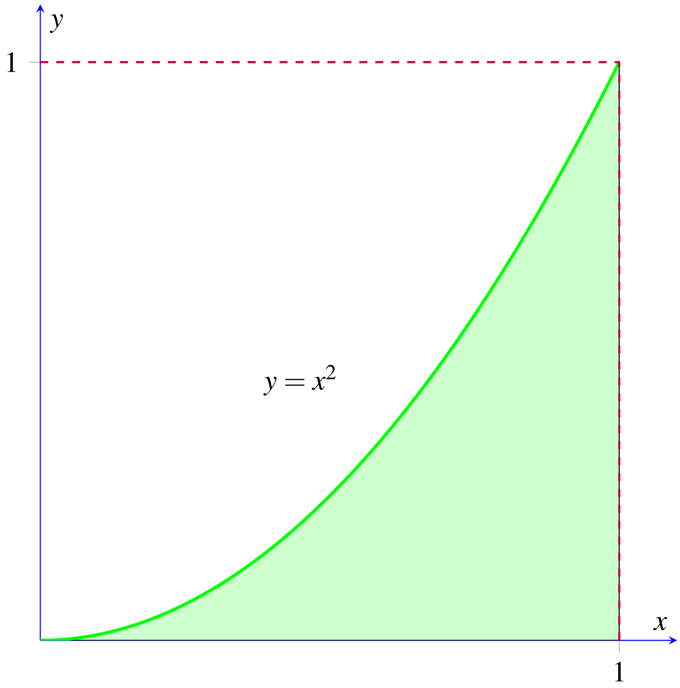
$$\begin{aligned}\text{Area of circle} &= \lim_{N\to\infty}\frac{1}{2}(Nb)h \\ &= \frac{1}{2}\left(\lim_{N\to\infty} Nb\right)\left(\lim_{N\to\infty} h\right) = \frac{1}{2}PR\end{aligned}$$

The limit of Nb is just the perimeter P, and the limit of h is the radius r. But the perimeter is $2\pi r$. So, the expression a $\frac{1}{2}Pr$ becomes $\frac{1}{2}2\pi r\cdot r$. Which simplifies quickly to πr^2.

Thus, we answer our earlier question: the formula for the area of a circle is $A = \pi R^2$.

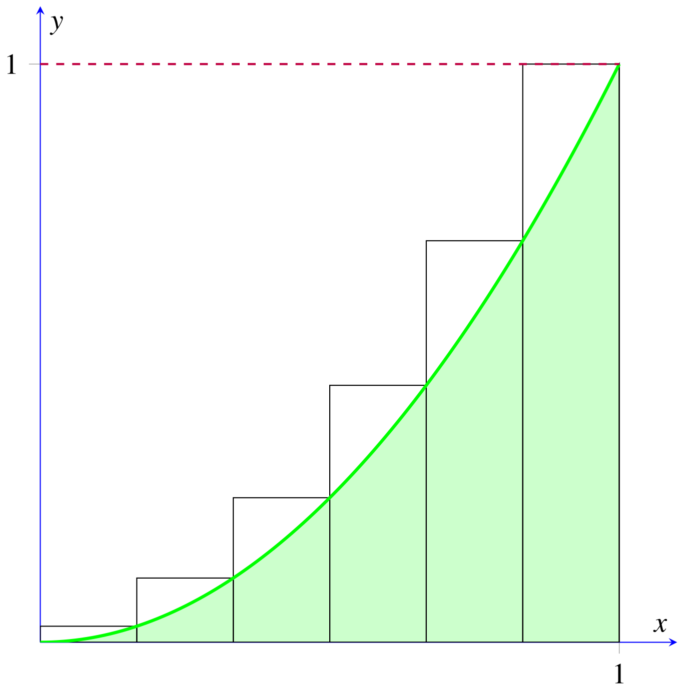
Look carefully at this in light of our knowledge of the derivative. The area A is πR^2, and the perimeter P is $2\pi R$. The derivative $\frac{dA}{dR}$ is also $2\pi R$. This isn't a coincidence. We call the area an antiderivative of the perimeter, and soon you'll see this fact in a wider context. Functions associated with areas bounded by curves are intrinsically related to the functions describing the curves by going backwards from differentiation.

The Greeks also thought carefully about areas associated with another classical curve, the parabola. However, a parabola extends infinitely and doesn't close up like a circle. Nevertheless, we can position a parabola in the x,y plane and consider the area between the curve and the x-axis over a specific interval. Let's simplify everything as much as possible and focus on the area under the parabola $y = x^2$ for x between 0 and 1, shaded here in green.



If you use graph paper, as shown in this diagram, you can get a very good estimate by adding up tiny sub-squares and fragments of sub-squares. By doing this, you can probably make an educated guess about the answer. But we'll try to emulate the Greeks and find the exact answer without guessing.

Remember in the previous section, we estimated displacement from the velocity curve by using rectangular approximations. We do something similar here. First of all, divide the interval from zero to one on the x-axis into n equal sub-intervals of width $\frac{1}{n}$ where n is a positive integer. You can think of n as large, though in this particular diagram n =6. Then move up to the curve and build rectangles that sit above the curve.



It's enough for us to consider upper rectangles only though you can perform a similar calculation using lower rectangles if you wish. Then we shade in the areas inside the rectangles which includes a green area under the curve and extra areas not colored in beige which sit above the curve but still inside the rectangles. Note that all the rectangles have a width of $\frac{1}{n}$ units, adding up the areas of these upper rectangles, we get:

$$\frac{1}{n} \times \left(\frac{1}{n}\right)^2 \quad \text{for the first rectangle since the height is the value} \quad x^2 \quad \text{when} \quad x = \frac{1}{n}$$

$$+\frac{1}{n} \times \left(\frac{2}{n}\right)^2 \quad \text{for the second rectangle since the height is the value} \quad x^2 \quad \text{when} \quad x = \frac{1}{n}$$

$$+\frac{1}{n} \times \left(\frac{3}{n}\right)^2 \quad \text{for the third rectangle and so on all the way up to the final rectangle}$$

$$+\frac{1}{n} \times \left(\frac{n}{n}\right)^2 \quad \text{for the final rectangle since the height is the value} \quad x^2 \quad \text{when} \quad x = \frac{1}{n}$$

Now we get:

$$\text{Area} = \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

Denote the green area under the curve by A. Then A is approximated by this expression for the areas of the rectangles. As n gets larger and larger, the rectangles get thinner and thinner and the extra beige area becomes vanishingly small so that A becomes the limit of this expression as n goes to infinity.

$$\text{A} = \lim_{n\to\infty} \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

We have a chance of evaluating this limit if we can understand or control the sum of all those squares inside the brackets. A very succinct way of writing a sum like that is to use Sigma notation, using the Greek letter capital sigma as an abbreviation for repeated addition. Here we're adding up i-squared as I

$$\sum_{i=1}^{n} i^2$$

If you're not sure about Sigma notation, we will see a little bit here. We're going to use it a lot in the rest of the section. I should also warn you that what follows is quite advanced material and we move very quickly. Don't worry if you don't follow every step and see frequently if you want to check each of the manipulations slowly and carefully.

Sigma notation: A sum of numbers, typically with many terms, is often abbreviated using *sigma notation*, employing the Greek capital Σ for 'sum':

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n = \sum_{i=m}^{n} a_i.$$

The general term or expression is indicated by a_i, say, where the subscript i ranges from m to n, say, in steps of one at a time, where $m \leq n$. The smallest value of the subscript m appears below the Σ symbol and the largest value n appears above the symbol.

For example, if $a_i = i$, $m = 1$ and $n = 100$, then

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 100 \quad \text{and} \quad \sum_{i=1}^{100} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 100^2,$$

the sums of the first 100 consecutive integers, and squares of consecutive integers, respectively. We have the following simple but useful properties, which allow one to break a sum into pieces and to bring constants outside the sum, where $b_m, \ldots, b_n$ form any other list of numbers and c is a constant:

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \quad \text{and} \quad \sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i.$$

Now, let's return to where we left off.

It is far from obvious how to evaluate this limit. There is a trick involving looking at n^3 in a clever way, called a telescoping sum. For example, a 'telescope' for 5^3 looks like the following:

$$\begin{aligned} 5^3 &= 5^3 - 4^3 + 4^3 - 3^3 + 3^3 - 2^3 + 2^3 - 1^3 + 1^3 - 0^3 \\ &= (5^3 - 4^3) + (4^3 - 3^3) + (3^3 - 2^3) + (2^3 - 1^3) + (1^3 - 0^3). \end{aligned}$$

Now, we do something that looks totally bizarre and unexpected, known as the telescoping sum method. Though we want to sum squares, we look at n^3 and open it up like a telescope by successively adding together $i^3 - (i-1)^3$, climbing all the way down to $1^3 - 0^3$.

$$n^3 = n^3 - (n-1)^3 + (n-1)^3 - (n-2)^3 + \ldots - 2^3 + 2^3 - 1^3 + 1^3 - 0^3$$

Now, this looks absurdly complicated, but we're just playing around with zero. You might remember how we used expansions of zero in the trick of completing the square and also in proving the product rule for differentiation. Here, we're using the same trick repeated many times to produce this huge telescope. You can see how terms cancel out producing zeros all the way to the end where we get to 0^3 which is also zero. So, this huge long telescope really is just n^3 expressed in a complicated way for the purpose of discovering something quite remarkable and surprising.

Now, we use Sigma notation to express all of this very concisely. Again, if you're not used to Sigma notation, please read about it in the above part. Here, we can rewrite n^3 as a sum of $i^3 - (i-1)^3$ as i ranges from 1 to n:

$$n^3 = \sum_{i=1}^{n} \left(i^3 - (i-1)^3\right)$$

Let's play with this expression that uses Sigma notation. What follows is quite intricate and you should give a lot of time to see and understand. We first expand the difference of the two cubes and see that it simplifies to:

$$n^3 = n^3 - (n-1)^3 + (n-1)^3 - (n-2)^3 + \ldots - 2^3 + 2^3 - 1^3 + 1^3 - 0^3$$

$$= \sum_{i=1}^{n} (i^3 - (i-1)^3) = \sum_{i=1}^{n} (i^3 - 3i^2 + 3i - 1) = \sum_{i=1}^{n} (3i^2 - 3i + 1)$$

But the sigma notation means a gigantic sum, so we can split this into three separate pieces. The first piece becomes three times the sum of i^2. The second piece becomes minus three times the sum of i, and the third piece is the sum of one n times, which is just n.

$$= 3\left(\sum_{i=1}^{n} i^2\right) - 3\left(\sum_{i=1}^{n} i\right) + \left(\sum_{i=1}^{n} 1\right)$$

So, now we get this even more complicated way of expressing n^3. Where is all this leading? There's another surprise often referred to as the trick of Gauss.

There's a story about Gauss when he was a child. The teacher asked the class to add up all the numbers from one to 100. Almost immediately, the pupil Gauss answered 5,050, astonishing everyone. How could a child add up 100 numbers so quickly in his head? Well, let's think more generally about adding up all the numbers from one to a very large number n and call the total sum S.

$$1 + 2 + 3 + \cdots + n - 1 + n = S$$

Gauss's trick is to write the same sum S but backwards and then add it to itself.

$$1 + 2 + 3 + \cdots + n - 1 + n = S$$
$$n + n - 1 + n - 2 + \cdots + 2 + 1 = S$$
$$n + 1 + n + 1 + n + 1 + \cdots + n + 1 + n + 1 = 2S$$

Notice how the numbers line up and you get n copies of $n+1$ on the left and twice S on the right. So, the two times S is $n \cdot (n+1)$. So, dividing by two, the sum S becomes this nice formula $\frac{n(n+1)}{2}$. So, that's how Gauss did it. He multiplied 100 by 101 and divided by two, which is 5,050.

We therefore get $2S = n(n+1)$, so that, dividing by 2, and also using $\sum$ notation,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

In sigma notation,

$$\sum_{i=1}^{100} i = 1 + 2 + \cdots + 100 = \frac{100 \times 101}{2} = 5050.$$

What's the relevance of this? You can see in our complicated expression for n^3, we have $= \sum_{i=1}^{n} i$. Which is just the sum S that we evaluated using the trick of Gauss. So, let's substitute that expression in and see what happens.

$$n^3 = 3\left(\sum_{i=1}^{n} i^2\right) - \frac{3n(n+1)}{2} + n.$$

We get yet another complicated expression for n^3. But before throwing up our hands in despair, notice that we've made remarkable progress. The thing we're really interested in finding related to our area problem for the parabola is the sum of squares and you can see it trying to hide inside this equation. But that's not a problem for us because we just rearrange the furniture using algebraic manipulation and isolate the sum of the squares and then just carefully simplify the expression which involves powers of n. You can stop reading perhaps and check each step if you like. The upshot is, that we get this very elegant formula for the sum of squares. Rearranging this, we get the following elegant expression for the sum of squares of consecutive integers:

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}\left(n^3 + \frac{3n(n+1)}{2} - n\right) = \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2n\right)$$

$$= \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{n(n+1)(2n+1)}{6}.$$

We can now return to finding the area A under the parabola, by evaluating the limit we found previously, using this formula and our usual tricks for manipulating limits:

$$A = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{1\cdot 2}{6} = \frac{1}{3}.$$

Seeing that the answer quickly simplifies to $\frac{1}{3}$, we finally get the answer. In fact, it was discovered by the Greeks thousands of years ago that the area under the parabola between zero and one is exactly $\frac{1}{3}$. No approximations. You can say that this was a really difficult calculation and much more difficult than finding the area of a circle.

Later in this module, we'll learn a method that exploits the antiderivative of x^2, and the answer of $\frac{1}{3}$ will drop out in a couple of steps. However, I wanted you to see this extended, bare-handed calculation as it involves so many different ideas and techniques that you can apply in all sorts of other mathematical settings.

In this section, we discussed two examples of finding areas bounded by curves. Firstly, we used the perimeter to find a formula for the area of a circle, and secondly, we calculated the area under the parabola $y = x^2$ in the xy-plane over the interval from zero to one, which turns out to be exactly $\frac{1}{3}$. It's remarkable that by using fairly abstract ideas involving approximations and taking limits, we get exact answers in both cases.

In the second example, we used sigma notation, which is an abbreviation for adding lots of things together; telescoping sums, which involve elaborate ways of playing around with zero; the Gauss trick of adding successive numbers together; lots of algebraic manipulation; and our techniques of evaluating limits.

Please re-read if you didn't get it and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

32.1.1 Practice Quiz

Question 1

Use the trick of Gauss to add up consecutive integers from 1 to 200, that is, find the sum

$$1+2+3+\cdots+199+200.$$

(a) 20,200
(b) 20,000
(c) 20,050
(d) 20,100
(e) 20,150

Question 2

Use Sigma notation to find an expression for the sum of consecutive integers from 1 to 200, that is, the sum

$$1+2+3+\cdots+199+200.$$

(a) $\sum_{i=1}^{199} i$
(b) $\sum_{i=1}^{200}(i+1)$
(c) $\sum_{i=1}^{200}(i-1)$
(d) $\sum_{i=0}^{199} i$
(e) $\sum_{i=1}^{200} i$

Question 3

Use Sigma notation to find an expression for the sum of squares of consecutive integers from 1 to 200, that is, the sum

$$1^2+2^2+3^2+\cdots+199^2+200^2.$$

(a) $\left(\sum_{i=1}^{200} i\right)^2$
(b) $\sum_{i=0}^{199} i^2$
(c) $\sum_{i=1}^{199}(i^2+(i+1)^2)$
(d) $\sum_{i=1}^{200}(i+1)^2$
(e) $\sum_{i=1}^{200} i^2$

Question 4

Use the trick of Gauss to add up consecutive even integers from 2 to 200, that is, find the sum

$$2+4+6+\cdots+198+200.$$

(a) 10,200
(b) 10,100
(c) 10,000
(d) 10,150
(e) 10,250

Question 5

Find an expression that is equivalent to the following sum of even integers, where n is a fixed large positive integer:

$$2+4+6+8+\cdots+2(n-1)+2n.$$

(a) $2\left(\sum_{i=1}^{2n} i\right)$
(b) $2\left(\sum_{i=1}^{n} i\right)$
(c) $\sum_{i=1}^{2n}(2i)$
(d) $\sum_{i=1}^{2n}(2n)$
(e) $2\left(\sum_{i=1}^{n} n\right)$

Question 6

Evaluate

$$\sum_{i=2}^{6}(2i+3).$$

(a) 52
(b) 54
(c) 56
(d) 53
(e) 55

Question 7

Evaluate

$$\sum_{i=2}^{101} 3.$$

(a) 302
(b) 300
(c) 306
(d) 297
(e) 303

Question 8

Evaluate

$$\sum_{i=1}^{100}(2i-3).$$

(a) 10,100
(b) 9,900
(c) 9,800
(d) 9,700
(e) 10,000

Question 9

Find n such that

$$\sum_{i=1}^{n} i = 210.$$

(a) 20
(b) 19
(c) 18
(d) 21
(e) 22

Question 10

Evaluate

$$\sum_{i=1}^{100}\left(\frac{1}{i}-\frac{1}{i+1}\right).$$

(a) $\frac{99}{100}$
(b) $\frac{100}{101}$
(c) $\frac{100}{99}$
(d) $\frac{101}{100}$
(e) 1

Answers

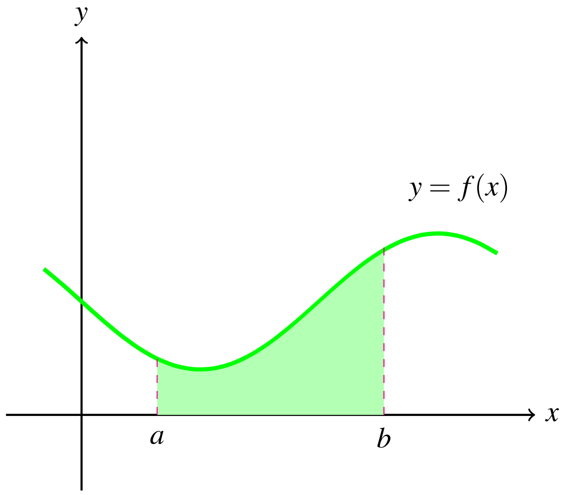
The answers will be revealed at the end of the module.

32.2 Riemann Sums and definite integrals

In this section, we formalize the methods that we have been using in earlier section where we estimated areas using rectangular approximations. This leads to the method of Riemann sums, which involves partitioning the interval of interest into tiny subintervals over which rectangles are formed reaching up to the curve. As the partitions get finer and finer, as the distances along the x-axis between successive points tend to zero, the Riemann sums approach a limit which we think of as the area under the curve. This area is called the definite integral, denoted by a stylized S called an integration symbol, which you can think of as some kind of continuous sum. The symbolism is all due to Leibnitz, the great visionary from the 17th century. It wasn't until the 19th century that the mathematics had evolved sufficiently to be regarded as having secure and rigorous foundations.

We describe the method of Riemann sums attributed to the famous 19th-century mathematician Bernhard Riemann. Though in fact, the method and proofs were the combination of combined efforts in mathematical evolution by many mathematicians spanning the 17th, 18th, and 19th centuries.

We consider a curve $y = f(x)$, drawn here for x between a and b, where $a < b$. We find the area, denoted by A, between the curve and the x-axis over this interval by forming approximations and then taking some kind of limit.



The method proceeds in four steps. In the first step, we break the interval up into smaller pieces called subintervals by choosing numbers $t_1, t_2, \ldots, t_{n-1}$ in between a and b, where n is some positive integer, which you can think of as being large. We also have $t_0 = a$ and $t_n = b$.
Divide this interval into n subintervals (not necessarily of equal width) by choosing

$$t_0 = a < t_1 < t_2 < t_3 < \cdots < t_{n-1} < t_n = b,$$

We call the sequence of numbers a partition of the interval from a to b. In the second step, we use the partition to form rectangles that reach up to the curve, and then sum all of the areas together with the aim of approximating the area under the curve. In this drawing, the rectangles all sit beneath the curve, but in fact, one has complete freedom to create rectangles that use any points at all within the subintervals.

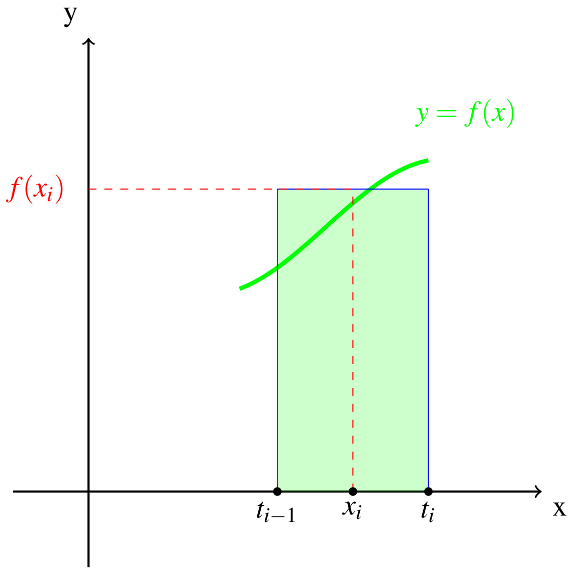
so that $[t_{i-1}, t_i]$ denotes the i-th subinterval for $i = 1$ to n. Choose points

$$x_1, x_2, \ldots, x_n$$

within successive subintervals, that is, $x_i \in [t_{i-1}, t_i]$ for $i = 1$ to n. Denote by Δx_i the width of the i-th subinterval, that is

$$\Delta x_i = t_i - t_{i-1},$$

for $i = 1$ to n.



If we focus on the i-th subinterval, for a typical i between 1 and n, we can choose any x_i between t_{i-1} and t_i. Move up to the curve and form a rectangle of height $f(x_i)$, which has an area that we can calculate easily. Its width, which we call Δx_i, is just the difference $t_i - t_{i-1}$. The area becomes the height $f(x_i)$ multiplied by this width Δx_i.

The third step is to add up all of these areas. Our area A under the curve will be:

$$A \approx f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdot f(x_n)\Delta x_n$$

This large sum can be written concisely using sigma notation with the Greek letter capital sigma Σ for sum, which is an abbreviation for adding up all of these areas, with the typical expression

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

. This expression, which will have many terms if n is large, is called a Riemann sum.

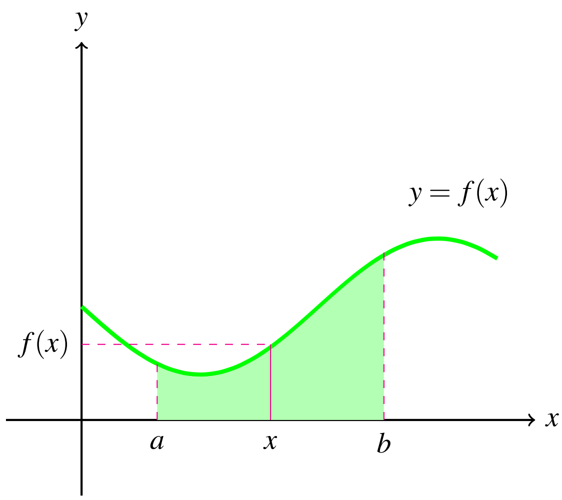
Now, we can stop at this step if we wish to get a very good approximation to the area A if we've chosen n large enough, and we'll come back in a moment to illustrate this with an example. But there's an important fourth step, which is to see what happens to the Riemann sum in the limit as n gets arbitrarily large. If all the conditions are right, such as the curve being continuous and we followed the previous steps carefully, then the approximation to the area A turns into an equality. We reach the actual area A by taking the limit of the Riemann sums.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

We have a special notation for this using a stylized S called an integral symbol, with a few features I'll mention in a moment.

$$\int_a^b f(x)\,dx$$

The overall expression is called a definite integral. There's also something called an indefinite integral which we'll come to in a later section. Notice how a differential dx appears, replacing the Δx_i. $f(x)$ replaces $f(x_i)$. The jagged sigma symbol is replaced by the nice smooth integral symbol, and we have Leibnitz to thank for that, and the limit notation has disappeared. The i ranging from 1 to n is replaced with a and b, the endpoints of our interval, with a as a subscript and b as a superscript. To aid our intuition, it can be helpful to think of the expression $f(x)\,dx$ as the area of an infinitely thin rectangle of width dx and height $f(x)$.



Here's the curve again with x ranging between a and b. You can imagine forming this infinitely thin rectangle from x up to the curve, almost like a single line segment with height f(x) and width, the infinitesimal differential dx.

$$\int_a^b f(x)\,dx$$

So, that when we form the definite integral, it's as though we're taking a continuous sum of all these infinitesimal areas f (x) times dx as x ranges from a to b. That's certainly how Leibnitz thought of it.

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

The Riemann sum is a discrete quantity involving finitely many terms representing an approximation which you can think of when we take the limit as becoming a continuous process that finds the actual exact area under the curve, the definite integral also called the Riemann integral. It's a very important theorem proved carefully in the 19th century, that if y = f(x) is a continuous function $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

exist provided the $\Delta x_i \to 0$.

To guarantee existence to this limit, we have to make sure the partition gets finer and finer, so that the width of the rectangles goes towards zero as n gets arbitrarily large. This result is very subtle and difficult to prove. The value of the limit is independent of the choices you make in the first step when forming the partition of the interval, and you don't even need the widths of the subintervals to be the same size. Also in the second step, when you have complete freedom to choose your x_i from anywhere within the $i^t h$ subinterval. There are doubly infinitely many different choices as you perform these steps and incredibly they all produce the same limit as n gets arbitrarily large.

32.2.1 Definite Integral

$$\int_a^b f(x)\,dx$$

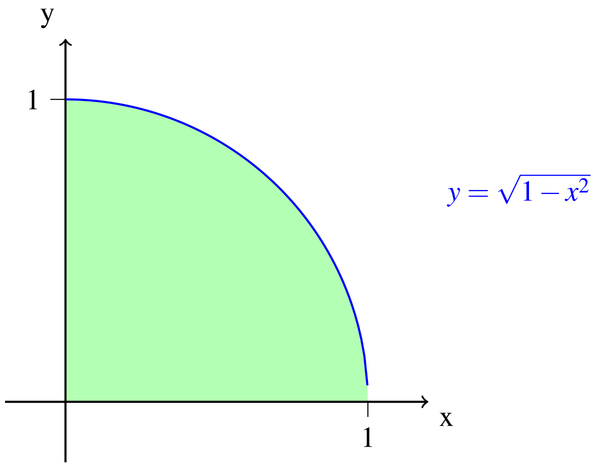
Here are the main features of the definite integral. We have the differential dx at the right, the terminals a and b reminding you of the endpoints of the interval, and the rule for the function f(x) sandwiched in the middle referred to as the integrand.

Let's use a Riemann sum to estimate the area under the curve representing the unit circle as x ranges from zero to one. The equation of the circle is $x^2 + y^2 = 1$. So, rearranging this, y is the positive square root of $1 - x^2$:

$$y = \sqrt{1 - x^2}$$

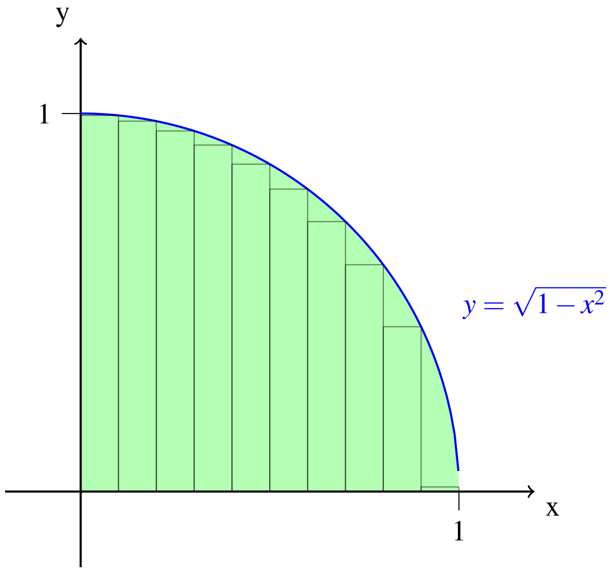
The area is the definite integral using $\sqrt{1-x^2}$ as the integrand and 0 and 1 as the limits of integration.

$$\int_0^1 \sqrt{1-x^2}\,dx$$



But, this represents the area of one quarter of the unit circle, which you can see is just $\frac{\pi}{4}$.

So, in fact, we're going to use the Riemann sums to approximate $\frac{\pi}{4}$. In this illustration, we'll use 10 sub-intervals, all of width 0.1. We're going to produce two approximations using rectangles that sit beneath the curve to get a lower bound for the true area, and rectangles that sit above the curve to get an upper bound. We first create what is called a lowest sum by moving up to the curve, making sure all the rectangle sit beneath the curve. Then, take the areas colored green, which we add up to form what we call capital L for lower sum.

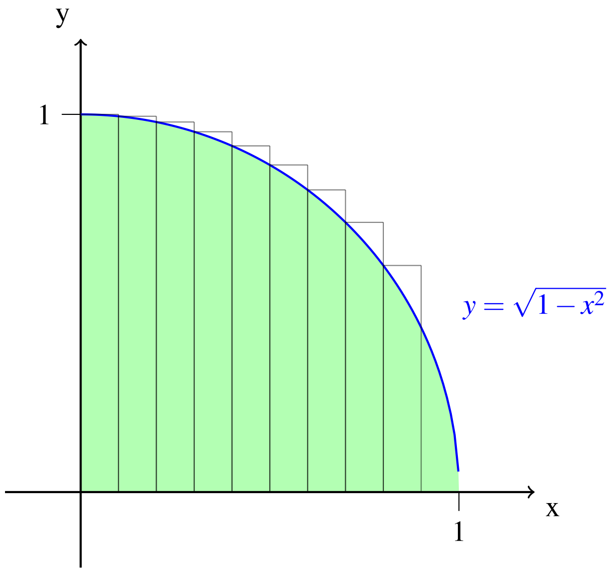


Each of the rectangles has a common width of 0.1 units, which comes out as a common factor, and heights the y values. The square root of $\sqrt{1-x^2}$ for the discrete inputs x equal to 0.1, 0.2, 0.3, all the way up to 1.

$$L = 0.1\left(\sqrt{1-0.1^2}+\sqrt{1-0.2^2}+\sqrt{1-0.3^2}+\sqrt{1-0.4^2}+\sqrt{1-0.5^2}\right.$$

$$\left.+\sqrt{1-0.6^2}+\sqrt{1-0.7^2}+\sqrt{1-0.8^2}+\sqrt{1-0.9^2}+\sqrt{1-1^2}\right)\approx 0.72613$$

You can check this evaluates to approximately 0.72613, and this becomes our lower estimate for the area $\frac{\pi}{4}$. We can repeat this process, but now forming what we call an upper sum by moving up to the curve, and using rectangles that sit above the curve, and obtain the areas by topping up the green areas already in the diagram with this extra white areas.



We can then add up all the areas to get what we call U for upper sum. Noting that again, the interval widths are all 0.1 coming out as common factor. Now, the heights of the rectangles are the y values as x goes from 0, 0.1, 0.2, all the way up to 0.9, which you can evaluate to get approximately 0.82613.

$$U = 0.1\Big(\sqrt{1-0.0^2}+\sqrt{1-0.1^2}+\sqrt{1-0.2^2}+\sqrt{1-0.3^2}+\sqrt{1-0.4^2}$$
$$+\sqrt{1-0.5^2}+\sqrt{1-0.6^2}+\sqrt{1-0.7^2}+\sqrt{1-0.8^2}+\sqrt{1-0.9^2}\Big) \approx 0.82613$$

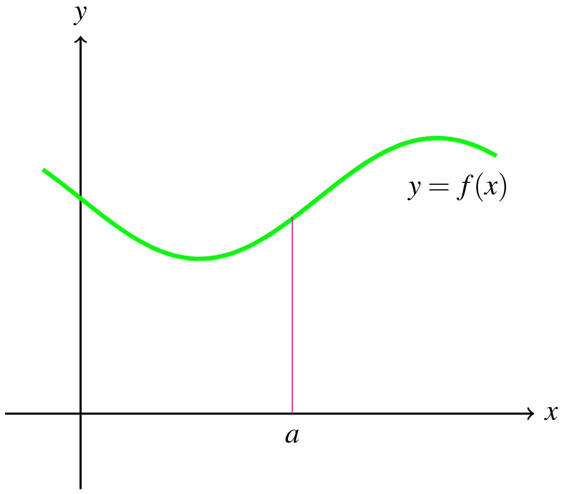
If you're astute, you'll notice from the diagram that in this case, the upper sum is just the lowest sum shifted to the right 0.1 units plus the area of the very first rectangle, which is 0.1.

$$U = 0.1(\sqrt{1-0.0^2}) + L$$
$$U \approx 0.82613$$

Then, we can get u by adding 0.1 to L. In any case, we have a lower bound L and an upper boundary for the area of one-quarter of the circle. The true area is somewhere in between, so we take the average, which turns out to be about 0.776.

$$= \frac{L+U}{2} \approx \frac{0.7263+0.8263}{2}$$
$$\approx 0.776$$

You can compare this with the true value of $\frac{\pi}{4}$ which is 0.785, and see that they almost agree to two decimal places. Now our approximation is an underestimate. This is to be expected as the curve is concave down, and taking averages in this way corresponds to approximating the curve using trapezoids as explained in an earlier section. Trapezoids sit just slightly below the curve when it's concave down. If we use more subdivisions of the interval, then we'd expect getting a more accurate estimates of $\frac{\pi}{4}$. We have some further technical definitions involving definite integrals. Preceding discussion assumed throughout, we were finding areas over an interval a b, where $a < b$. What if the terminals a and b are the same?



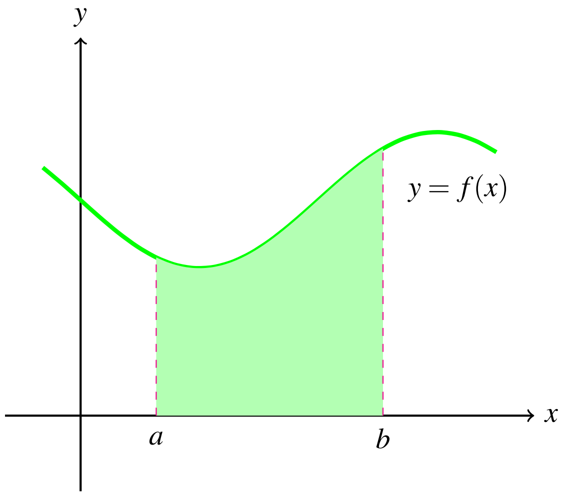
It's as though we're integrating over a degenerate, infinitely thin interval with zero area. So, it makes sense to define the definite integral from a to a to be zero:

$$\int_a^a f(x)\,dx = 0$$

There's also a rule that says that we can swap the positions of the limits of integration. But then, we have to multiply the definite integral by -1.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

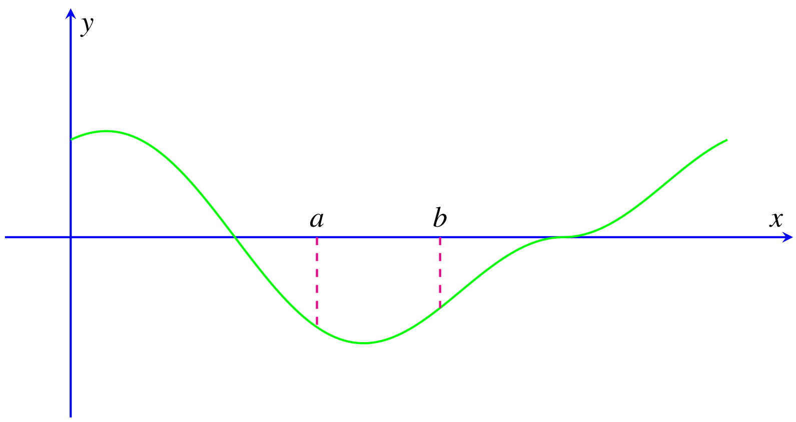
This makes perfectly good sense. Consider again the interval from a to b, where a as usual is less than b or $a < b$.



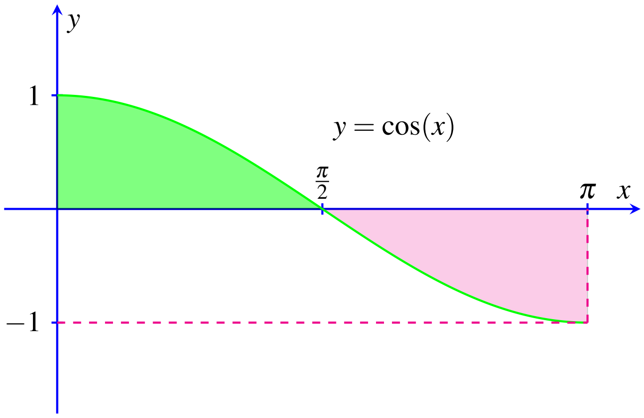
If we decide to change direction by putting the car into reverse and move instead from b backwards towards a, then we'd be reversing the direction that we're traveling along that same interval, which has the effect of creating negative bases, and hence negative areas of rectangles.

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

If you look carefully inside the expression for the Riemann sum, the term Δx_i, the difference in the x values, will become negative instead of positive as you move from right to left instead of from left to right. The signed area of the rectangle inside the sum gets multiplied by the negative base Δx_i. Now, in all the diagrams we've used so far, the curve $y = f(x)$ has been sitting above the x-axis, so all the y values used to build the rectangles in the Riemann sum have been positive. If the curve goes below the x-axis, then the y values become negative. Used as heights of rectangles in the Riemann sum formula, that will contribute a negative sign to the areas. Thus, areas under curves as we go from a to b, where $a < b$, are negative.



For example, consider the curve, y = cos x, x moving from 0 to π.



For x from 0 to $\frac{\pi}{2}$, the curve is above the x-axis, and the area is positive, colored green. However, for x from $\frac{\pi}{2}$ to π, the curve is below the x-axis, and the area is negative, colored pink. Due to symmetry, the magnitudes of the areas, ignoring the signs, are the same. Therefore, the indefinite integral from 0 to π must evaluate to zero as the positive and negative areas exactly cancel out.

$$\int_0^{\pi} cos(x)\,dx = 0$$

We can actually split the integral into two pieces. The integral from 0 to $\frac{\pi}{2}$, giving the green area, plus the integral from $\frac{\pi}{2}$ to π, adding on the pink area, gives a combined total of zero.

$$\int_0^{\pi} cos(x)\,dx = \int_0^{\frac{\pi}{2}} cos(x)\,dx + \int_{\frac{\pi}{2}}^{\pi} cos(x)\,dx = 0$$

This is an example of a general fact that definite integrals might be broken up into pieces. If we want to integrate over a long interval, say from a to c with b somewhere in the middle, we can split the definite integral into two pieces: the integral from a to b, and then from b to c, and combine them by addition.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx$$

This makes good sense as we are creating one large area by adding together the small areas in succession.

There are other important theorems or let say properties we will be discussing here.

Theorem 32.1

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

Firstly, the definite integral is additive, a way of splitting an integral into pieces, which means the integral of a sum is the sum of the integrals and difference between integral is the difference of integrals. This means that you can split the integrand up into pieces, integrate them separately, and then combine the answers using addition and subtraction.

Theorem 32.2

$$\int_a^b c\,dx = c(b-a) \quad \text{and} \quad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{for any constant } c.$$

As usual, constants come out the front. Both of these properties follow from corresponding properties about limits of sums and limits of constant multiples and the fact that definite integrals are themselves limits of Riemann sums.

Theorem 32.3

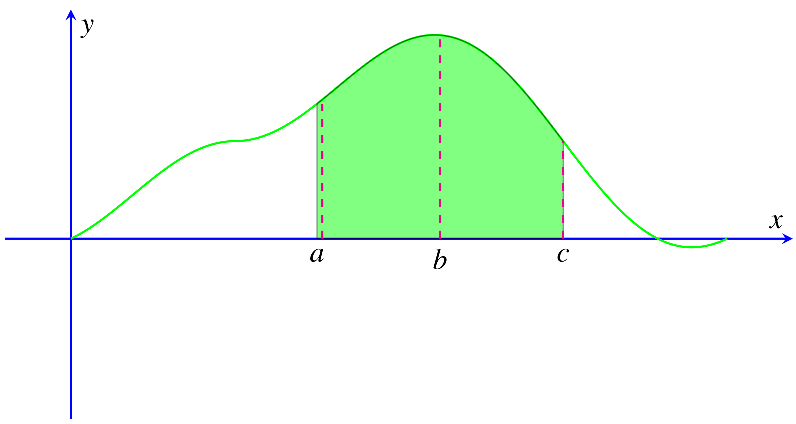
$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad (\text{defined also when } a \geq b).$$

As we have discussed above, integral of function at same bound point is zero and when bound points are swapped in same integral sign of the integral is changed.

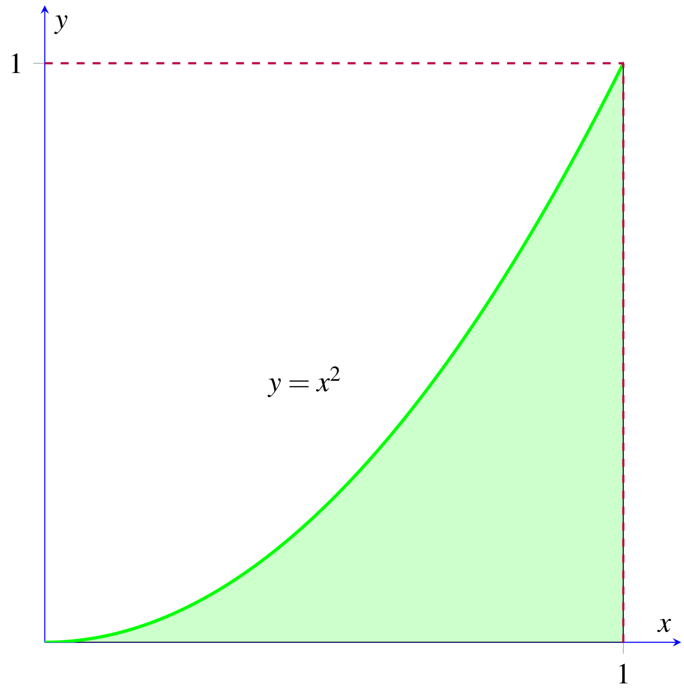
Theorem 32.4

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

This theorem is also discussed above and how we can split the integral into two pieces.



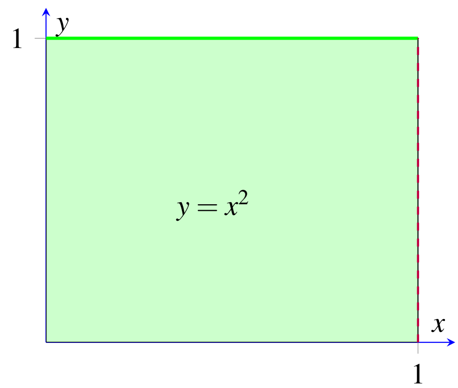
When integrating over a long interval, from a to c with b somewhere in the middle, we can split the definite integral into two pieces: the integral from a to b, and then from b to c, and combine them by addition.



Let's apply these principles to an example. Recall from last time that the area under the parabola $y = x^2$ as x goes from 0 to 1 is $\frac{1}{3}$, which says that the definite integral evaluates to exactly $\frac{1}{3}$.

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

Even easier is the area under the constant function $y = 1$ as x goes from 0 to 1, which of course is one square unit.



Now, it gives us another definite integral.

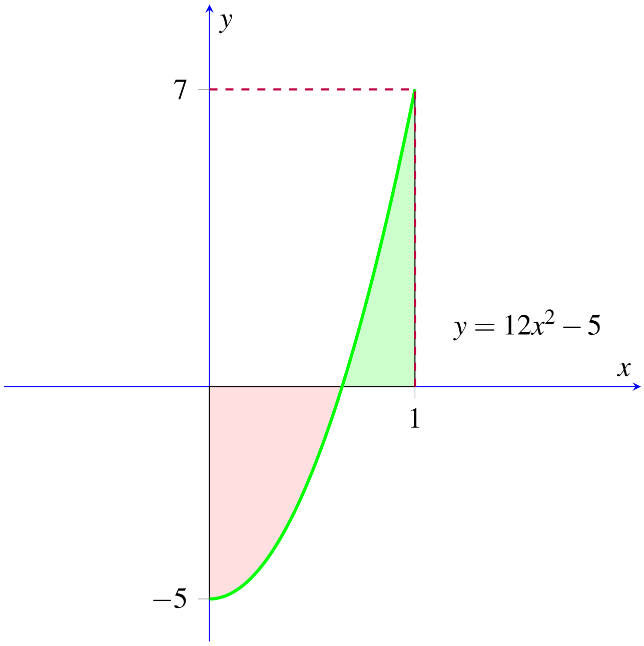
$$\int_0^1 1\,dx = 1$$

So, we have these two definite integrals and can use them to find the integral of, say $12x^2 - 5$ over the same interval. First, split the integrand into two pieces, which we integrate separately, then bring constants out front, and evaluate using the known values above.

$$\int_0^1 12x^2 - 5\,dx$$

$$= \int_0^1 12x^2\,dx + \int_0^1 (-5)\,dx$$
$$= 12\int_0^1 x^2\,dx + (-5)\int_0^1 1\,dx$$
$$= 12\left(\frac{1}{3}\right) - 5(1) = 4 - 5 = -1$$

The answer turns out to be negative one. By following these rules, we can predict that the net signed area turns out to be negative. You can see the effect by drawing the portion of the parabola for this interval.



It doesn't matter that my scales are different for the x- and y-axes. Notice that the pink area under the x-axis doesn't appear to exceed the green area above the x-axis in magnitude.

32.2.2 Examples

Let's see some solved examples to have a better understanding of this section.

1. Evaluate $\int_{-2}^{5} 3\,dx$, the area under the constant function 3 over the interval $[-2,5]$.

   Solution: The definite integral is the area of a rectangle of height 3 units and width 7, so

   $$\int_{-2}^{5} 3\,dx = 3\times 7 = 21.$$

2. Evaluate and interpret the definite integral $\int_5^{-2} 3\,dx$.

   Solution: Moving backwards along the interval $[-2,5]$ has the effect of measuring the base of a rectangle (appearing in the Riemann sum) as negative. Since the integrand is positive, one expects to produce a negative area. This is confirmed by the following formal manipulation, using the answer to the previous exercise, where we multiply the definite integral by negative one when we swap the terminals:

$$\int_5^{-2} 3\,dx = -\int_{-2}^{5} 3\,dx = -21.$$

3. Evaluate and interpret the definite integral

$$\int_2^2 (f(x)+3g(x)-7h(x))\,dx,$$

   where f, g, and h are any continuous functions defined on an interval containing $x = 2$.

   Solution: The definite integral evaluates to zero, as the interval $[2,2] = \{2\}$ over which we are integrating is degenerate, consisting of a single point, so the area under the curve vanishes.

4. Evaluate

$$\int_0^1 (2+6x^2)\,dx,$$

   given that $\int_0^1 x^2\,dx = \frac{1}{3}$.

   Solution: We have

$$\int_0^1 (2+6x^2)\,dx = \int_0^1 2\,dx + 6\int_0^1 x^2\,dx = (2\times 1) + 6\left(\frac{1}{3}\right) = 2+2 = 4.$$

5. Evaluate

$$\int_1^5 (6-f(x)+2g(x))\,dx,$$

   given that $\int_1^5 f(x)\,dx = 14$ and $\int_1^5 g(x)\,dx = -5$.
   Solution: We have

$$\int_1^5 (6-f(x)+2g(x))\,dx = \int_1^5 6\,dx - \int_1^5 f(x)\,dx + 2\int_1^5 g(x)\,dx$$
$$= (6\times(5-1)) - 14 + 2(-5) = 24-14-10 = 0.$$

6. Find

$$\int_1^7 f(x)\,dx,$$

   given that $\int_1^3 f(x)\,dx = 2$ and $\int_3^5 f(x)\,dx = 5$ and $\int_5^7 f(x)\,dx = -2$.

Solution: We have

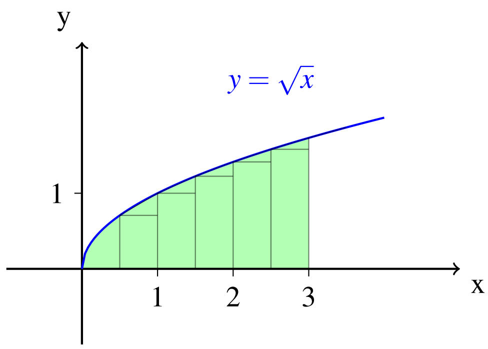
$$\int_1^7 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$$

$$= \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx - \int_5^7 f(x)\,dx = 5 + 2 - (-2) = 9.$$

7. Use a lower Riemann sum with 6 subintervals, by finding the areas of the lower rectangles beneath the curve, as indicated in the following diagram, to find a lower bound for the definite integral

$$\int_0^3 \sqrt{x}\,dx,$$

quoting your answer to two decimal places.



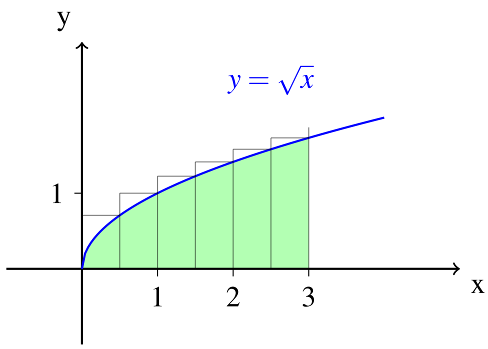
Solution: Each rectangle has width 0.5 and the heights of the lower rectangles are, in succession from left to right,

$$0, \sqrt{0.5}, 1, \sqrt{1.5}, \sqrt{2}, \sqrt{2.5}.$$

Hence the lower Riemann sum is

$$0.5(0 + \sqrt{0.5} + 1 + \sqrt{1.5} + \sqrt{2} + \sqrt{2.5}) \approx 2.96.$$

8. Use an upper Riemann sum with 6 subintervals, by finding the areas of the upper rectangles, as indicated in the following diagram, to find an upper bound for the definite integral $\int_0^3 \sqrt{x}\,dx$, quoting your answer to two decimal places.



Solution: Each rectangle has width 0.5 and the heights of the rectangles are, in succession from left to right,

$$\sqrt{0.5}, \quad 1, \quad \sqrt{1.5}, \quad \sqrt{2}, \quad \sqrt{2.5}, \quad \sqrt{3}.$$

Hence the upper Riemann sum is

$$0.5(\sqrt{0.5}+1+\sqrt{1.5}+\sqrt{2}+\sqrt{2.5}+\sqrt{3}) \approx 3.83.$$

9. Use the average of the lower and upper bounds found in the previous two exercises to estimate $\int_0^3 \sqrt{x}\,dx$, quoting your answer to one decimal place.
Comment on whether you think this estimate will be an underestimate or an overestimate of the true area under the curve.
Solution: The average of the lower and upper bounds from the previous exercise is

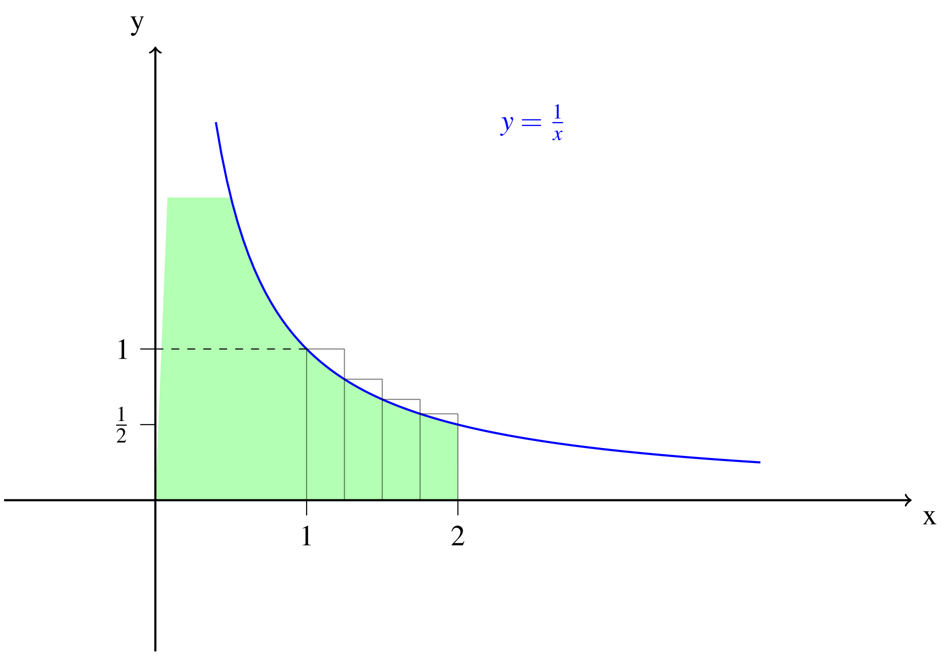
$$\frac{2.96+3.83}{2} \approx 3.4,$$

to one decimal place.
The average of the lower and upper Riemann sums for an increasing function, such as this one, uses areas of trapeziums, so should provide an underestimate of the true area under the curve, since the curve is concave down so sits just above the lines joining endpoints of the subintervals.
Soon you will be able to evaluate this area exactly using the Fundamental Theorem of Calculus. In fact,

$$\int_0^3 \sqrt{x}\,dx = \frac{2}{3}(3)^{3/2} \approx 3.5,$$

to one decimal place, which is slightly more than the estimate, confirming our expectations.

10. Use the average of the lower and upper Riemann sums, using 4 subintervals,



to estimate $\int_1^2 \frac{1}{x}\,dx$, rounding off your final answer to three decimal places. Comment on whether you expect your estimate to be an underestimate or overestimate of the true area under the curve.
Solution: The partition of the interval uses the points

$$1, 1.25, 1.5, 1.75, 2.$$

Each subinterval has width 0.25. To form the lower Riemann sum L, we use the right-hand endpoints of the subintervals, and we get the lower bound

$$L = 0.25 \times \left(\frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75} + \frac{1}{2}\right) \approx 0.6345,$$

to four decimal places. To form the upper Riemann sum U, we use the left-hand endpoints of the subintervals, and we get the upper bound

$$U = 0.25 \times \left(\frac{1}{1} + \frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75}\right) \approx 0.7595,$$

to four decimal places. The average of L and U is (keeping everything exact, this time),

$$\frac{L+U}{2} = 0.125 \times \left(\frac{1}{1} + \frac{2}{1.25} + \frac{2}{1.5} + \frac{1}{1.75} + \frac{1}{2}\right) \approx 0.697.$$

We estimate the area under the curve to be about 0.697, and expect this to be a slight overestimate, since the curve is concave up, so the curve sits slightly below lines joining endpoints of subintervals.
Soon you will be able to evaluate this area exactly using the Fundamental Theorem of Calculus. In fact,

$$\int_1^2 \frac{1}{x}\,dx = \ln 2 \approx 0.693,$$

a true decimal place, which is slightly less than the shaded region, confirming our expectation that the estimate is an overestimate.

In this section, we discussed in detail the method of Riemann sums, which uses rectangles to approximate areas under curves. There are four steps: first, partition the interval forming N sub-intervals; then, build rectangles that reach up to the curve; next, add together the areas to form the Riemann sum; and finally, take the limit of these expressions as N gets arbitrarily large. It's a major theorem that if everything is well-behaved, and the length of sub-intervals tends to zero, then the limit exists. This limit of the Riemann sums is called the definite integral, also known as the Riemann integral, and represents the exact area under the curve.

We express this symbolically using an integral sign, which is like a stylized S for sum. Important features of this notation include the integrand, which is the value of the function multiplied by the differential dx, and the use of limits of integration, which are just the endpoints of the interval over which we're working. We used the Riemann sum to approximate the area under one-quarter of the unit circle by taking the average of lower and upper bounds.

We also mentioned important properties of the definite integral: that areas are multiplied by negative one if one swaps the limits of integration; that areas beneath the x-axis have a negative sign; that we can split an integral into pieces by integrating over subintervals and summing the results; that we can split the integrand into pieces and integrate each part separately; and finally, that we can factor constants out of the integral. Please re-read if you didn't get it and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

32.2.3 Practice Quiz

Question 1

Evaluate $\int_{-1}^{1} 4\,dx$, the area under the constant function 4 over the interval $[-1,1]$.
(a) 8
(b) 4
(c) -8
(d) 0
(e) -4

Question 2

Evaluate $\int_{-1}^{1} 4\,dx$.
(a) -4
(b) 8
(c) 0
(d) -8
(e) 4

Question 3

Evaluate:

$$\int_{1}^{1} (2f(x) - 3g(x) + 4)\,dx$$

where f and g are defined on an interval containing 1.
(a) 0
(b) 1
(c) 4
(d) 2
(e) -3

Question 4

Evaluate:

$$\int_{0}^{1} (2f(x) - 3g(x) + 4)\,dx$$

given that $\int_0^1 f(x)\,dx = 2$ and $\int_0^1 g(x)\,dx = -3$.
(a) 17
(b) 0
(c) 3
(d) 13
(e) -1

Question 5

Evaluate

$$\int_0^1 (1-3x^2)\,dx$$

given that $\int_0^1 x^2\,dx = \frac{1}{3}$.

(a) $\frac{2}{3}$
(b) 1
(c) 0
(d) $-\frac{2}{3}$
(e) -3

Question 6

Find

$$\int_2^6 f(x)\,dx$$

given that $\int_3^6 f(x)\,dx = -1$ and $\int_2^3 f(x)\,dx = 5$.

(a) -4
(b) 4
(c) -5
(d) 5
(e) -6

Question 7

Find

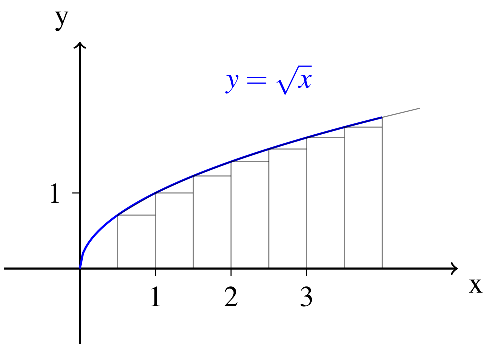
$$\int_2^3 g(x)\,dx$$

given that $\int_2^4 g(x)\,dx = 3$ and $\int_3^4 g(x)\,dx = 7$.

(a) -3
(b) 5
(c) -4
(d) 4
(e) 10

Question 8

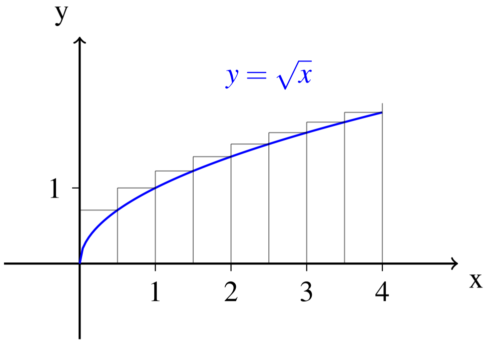
Use a lower Riemann sum with 8 subintervals, by finding the areas of the lower rectangles beneath the curve, as indicated in the following diagram,



to estimate the definite integral $\int_0^4 \sqrt{x}\,dx$, rounding off the final answer to one decimal place.
(a) 4.9
(b) 5.1
(c) 4.8
(d) 5.0
(e) 4.7

Question 9

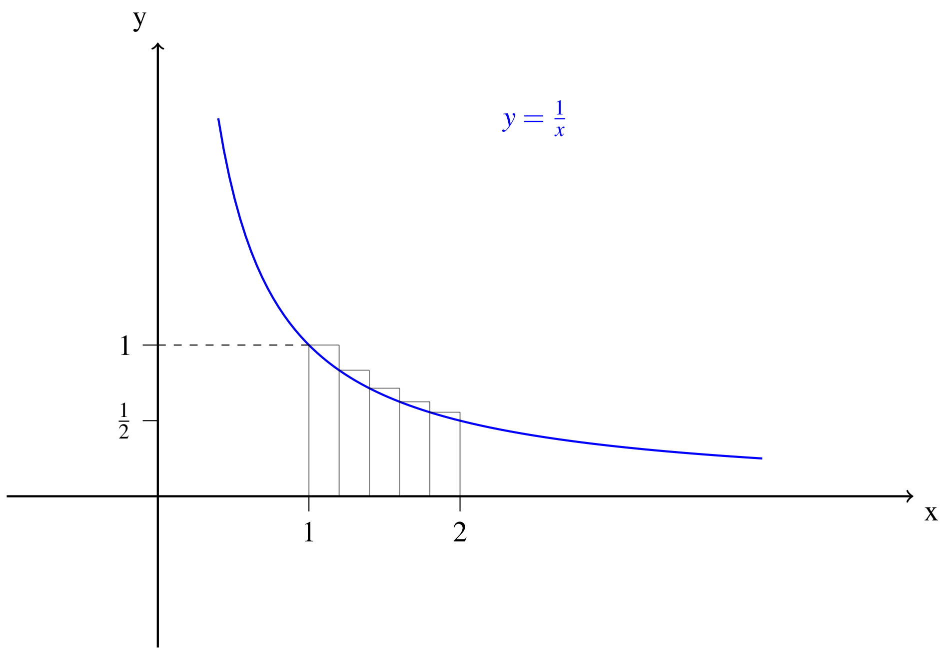
Use an *upper* Riemann sum with 8 subintervals, by finding the areas of the upper rectangles, as indicated in the following diagram,



to estimate the definite integral $\int_0^4 \sqrt{x}\,dx$, rounding off the final answer to one decimal place.
(a) 5.6
(b) 5.9
(c) 5.8
(d) 6.0
(e) 5.7

Question 10

Use the average of the lower and upper Riemann sums, using 5 subintervals,



to estimate the definite integral $\int_1^2 \frac{dx}{x}$, rounding off the final answer to two decimal places.
(a) 0.68

(b) 0.70
(c) 0.67
(d) 0.69
(e) 0.71

Answers

The answers will be revealed at the end of the module.

33. The fundamental Theorem of Calculus and Indefinite Integrals

33.1 The fundamental theorem of calculus and indefinite integrals

In this section, we introduce and apply the Fundamental Theorem of Calculus, which provides an elegant formula that uses antiderivatives to find exact values for definite integrals, that is, areas under curves. We discuss common antiderivatives and how they are represented using indefinite integrals, which use the same symbolism as definite integrals except the terminals are missing.

33.1.1 How do we find definite integrals?

The definite integral is a number representing the area under a curve over some given interval.

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

It's represented symbolically by putting together an integration symbol, an integrand $f(x)$ representing the rule of the function, a differential dx telling us what input variable is being used throughout, and terminals a and b that are endpoints of the interval. In principle, we can find the value of that definite integral by taking a certain limit with an expression known as a Riemann sum, discussed and illustrated last time. This looks like a very involved and complicated way to find areas under curves. Under certain circumstances, it seems that a miracle occurs and all this complexity can be avoided. The Fundamental Theorem of Calculus states that the definite integral can be evaluated as a difference of two expressions, $F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

The terminology needs explaining, which we'll do in a moment. This theorem holds that whenever the function f, whose rule appears in the integrand of the integral, is continuous over the interval from $[a\,,\,b]$ — that is, you can imagine drawing the curve without lifting your pen off the paper. Before illustrating applications to this theorem, we need first to spend time clarifying and exploring carefully the notion of antiderivative. To say $F(x)$ is an antiderivative of $f(x)$ means that the derivative with respect to x is $f(x)$. In previous sections, you're used to taking the derivative in all sorts of settings, but here, $f(x)$ is not being differentiated; instead, it appears as a derivative itself. To say that the derivative of $F(x)$ is $f(x)$ is entirely equivalent to saying that $F(x)$ is an antiderivative of $f(x)$. These two statements convey the same information. There is a subtlety, however. Notice how $f(x)$ is the derivative of $F(x)$, which is an antiderivative. It turns out that antiderivatives are not unique; in fact, all antiderivatives of $f(x)$ differ by addition of a constant. I'll explain why later in the section. You'll see constants appearing a lot in integration formulas, typically represented by a capital C called a constant of integration, but more about that later. The whole point of the Fundamental Theorem is to find areas under curves as simply as possible, and the technique relies on finding antiderivatives of functions. If we're able to do this, then we get exact answers, with no need for approximations. There's a twist to all of this: the process of differentiation is relatively straightforward once you've learned the basic rules, like the Chain, Product, and Quotient rules, which turn differentiation essentially into a mechanical procedure. Going the other way, however, forming antiderivatives is far from mechanical and requires great skill and ingenuity.

Let's start to build a table with some common useful functions, their derivatives and antiderivatives. For example, here are some simple decreasing powers of x. You're familiar with their derivatives. The derivative of x is 1, so the x becomes an antiderivative of 1. Antiderivatives can differ by a constant. So, let's add capital C to cover all possibilities. This is the C mentioned before, the constant of integration. Let's see all other function in the below table. In general, the derivative of x^n is nx^{n-1} and an antiderivative of x^n is the result of adding one to the exponent, which becomes $n+1$ and placing $n+1$ in the denominator, and don't forget to add C. Now, there's an important exception: we can only divide by $n+1$ if it's non-zero. So, we have to stipulate that $n \neq -1$.

Function	Derivative	Antiderivative
x^3	$3x^2$	$\frac{x^4}{4}+C$
x^2	$2x$	$\frac{x^3}{3}+C$
x	1	$\frac{x^2}{2}+C$
1	0	$x+C$
x^n	nx^{n-1}	$\frac{x^{n+1}}{n+1}+C$

Let's focus then on some negative parts of x, including the problematic case when $x = x^{-1}$. Again, the derivatives are straightforward. The antiderivatives of x^{-2} and x^{-3} are taken care of by the pattern we observed earlier for antidifferentiating x^n. What about an antiderivative of x^{-1}? It's a non-trivial fact that the derivative of $\ln x$ is $\frac{1}{x}$. So, $\ln x$ is an antiderivative and as usual, we add C. But we're not yet out of the woods; this only makes sense if x is positive. You can't take the log of a negative number, yet x^{-1} makes sense for negative x. There's a method for handling the negative case which I'll mention later.

Now, let's consider the two circular functions. You know their derivatives, but be careful to include the minus sign when differentiating $\cos x$. Since the derivative of $\sin x$ is $\cos x$, $\sin x$ is an antiderivative of $\cos x$, and don't forget the $+C$. The derivative of $\cos x$ is $-\sin x$. To antidifferentiate $\sin x$, we adjust by adding a minus sign to $\cos x$, and, as always, don't forget the $+C$.

Function	**Derivative**	**Antiderivative**
$\frac{1}{x} = x^{-1}$	$-x^{-2} = -\frac{1}{x^2}$	$\ln x + C \quad (x > 0)$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3} = -\frac{2}{x^3}$	$-x^{-1} + C = -\frac{1}{x} + C$
$\frac{1}{x^3} = x^{-3}$	$-3x^{-4} = -\frac{3}{x^4}$	$-\frac{x^{-2}}{2} + C = -\frac{1}{2x^2} + C$
$\sin x$	$\cos x$	$-\cos x + C$
$\cos x$	$-\sin x$	$\sin x + C$

Let's also address the natural exponential and logarithm functions whose derivatives we know well. Antidifferentiating e^x is straightforward. Now, for a challenging one: an antiderivative of $\ln x$ is $x \ln x - x$. This result requires insight; there's a systematic method using integration by parts, which you'll learn in advanced calculus courses. It's easy to verify that it works. If you differentiate the expression using the product rule and simplifying, you'll confirm that it simplifies back to $\ln x$, confirming that we have indeed found an antiderivative.

Function	**Derivative**	**Antiderivative**
e^x	e^x	$e^x + C$
$\ln x$	$\frac{1}{x}$	$x \ln x - x + C$

Check:

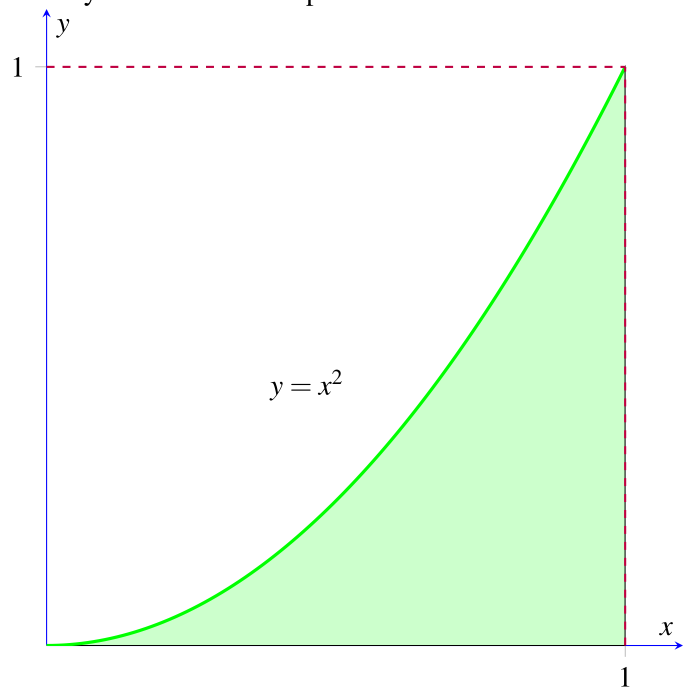
$$\frac{d}{dx}(x \ln x - x) = x\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x) - 1$$

$$= x\left(\frac{1}{x}\right) + (\ln x)(1) - 1 = 1 + \ln x - 1 = \ln x$$

We've spent time discussing antiderivatives, but we haven't used the fundamental theorem yet to find any areas under curves. The theorem tells us that the area is $F(b) - F(a)$. Because this expression is so useful, it gets its own square bracket notation. Write $[F(x)]_a^b$ as an abbreviation for $F(b) - F(a)$. When people see this, they often say "evaluate F between a and b", which technically means $F(b) - F(a)$. The theorem now has this compact form, and we say the area under the curve $f(x)$ for x between a and b is $F(x)$ evaluated between a and b, where $F(x)$ is any antiderivative of $f(x)$.

$$\int_a^b f(x)\, dx = [F(x)]_a^b$$

Let's try it out on an example. Let's find the area under the parabola $y = x^2$ for x between 0 and 1.



We already know the answer is $\frac{1}{3}$ from an earlier section, so this is a good test case for this new method. The area is an antiderivative of x^2 evaluated between 0 and 1. We know an antiderivative, namely $\frac{x^3}{3}$. So, we just evaluate this by substituting 1 for x and 0 for x and taking the difference, which quickly becomes $\frac{1}{3}$ as expected.

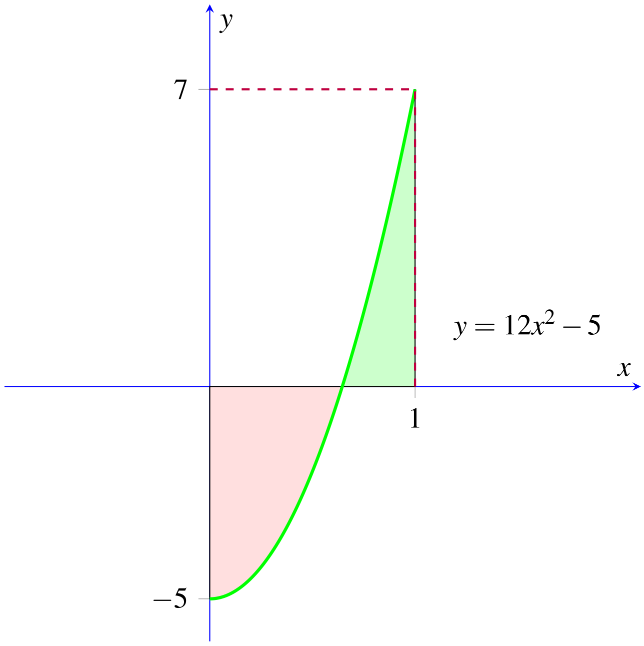
$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1^3}{3} - \frac{0^3}{3}$$

$$= \frac{1}{3} - 0 = \quad \frac{1}{3}$$

Notice how effortless this becomes; no complicated telescoping sums, Gauss trick, or any of the other paraphernalia we've used in an earlier video.

Let's look at another example from an earlier section and evaluate it by direct methods: the area under the curve $y = 12x^2 - 5$, again for x between 0 and 1.



The area is just an antiderivative of $12x^2 - 5$ evaluated at 0 and 1.

$$\int_0^1 (12x^2 - 5)\,dx = \left[4x^3 - 5x\right]_0^1$$

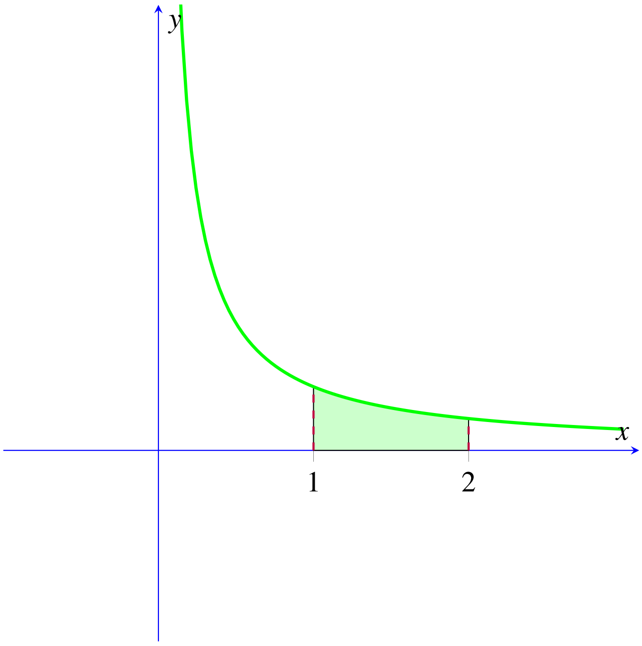
$$= \left(4(1)^3 - 5(1)\right) - \left(4(0)^3 - 5(0)\right)$$

$$= (4-5) - (0-0)$$

$$= -1$$

How do we know how to put $4x^3 - 5x$ inside the square brackets? Just as we can differentiate piece by piece and bring constants out in front, we can perform similar manipulations to find antiderivatives. We discussed before that an antiderivative of x^2 is $\frac{x^3}{3}$, so an antiderivative of $12x^2$ should be 12 times $\frac{x^3}{3}$, which is $4x^3$. An antiderivative of 1 is x, so an antiderivative of -5 should be -5 times x. Adding these two antiderivatives together gives $4x^3 - 5x$. Then substituting in 1 for x and 0 for x and taking the difference, the answer quickly simplifies to -1, which happily coincides with the answer we found in an earlier section.

Let's find the area under the branch of the hyperbola $y = \frac{1}{x}$ in the first quadrant for x between 1 and 2.



The area is given by this definite integral.

$$\int_1^2 \frac{1}{x}\,dx = \int_1^2 \frac{dx}{x}$$

$$= [\ln x]_1^2$$

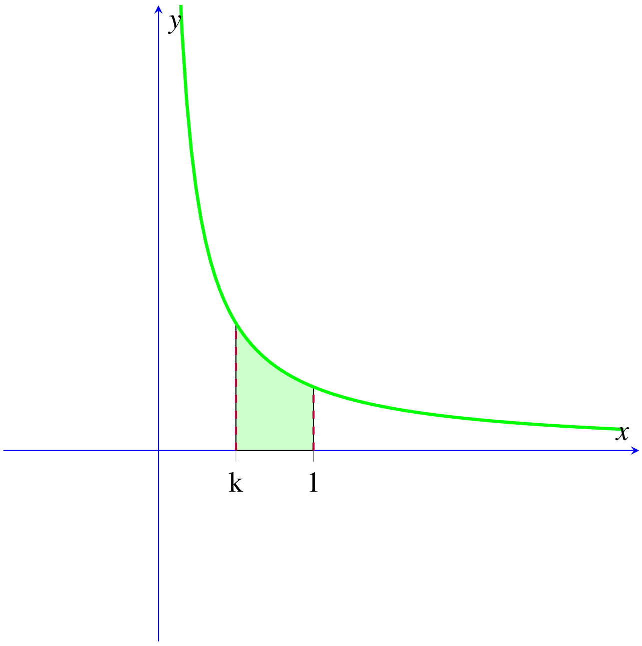
$$= \ln 2 - \ln 1$$

$$= \ln 2 - 0$$

$$= \ln 2$$

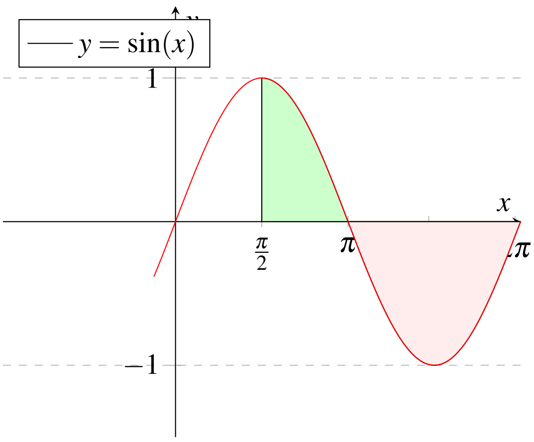
It's a common and harmless abuse of notation to put the integrand and the differential together as a single fraction, making everything more compact. By the fundamental theorem, all we have to do is evaluate some antiderivative of $\frac{1}{x}$ between 1 and 2. But $\ln x$ is an antiderivative, so evaluating between 1 and 2 gives $\ln 2 - \ln 1$, which becomes just $\ln 2$. Hence, the area we're looking for is the natural logarithm of 2.

Notice that there's nothing special about 2 in this calculation. We can replace 2 with any positive real number k and the area turns out to be $\ln k$. This diagram implicitly assumes k is greater than 1. Suppose instead that $0 < k < 1$.



The definite integral is still $\ln k$ as before, but now integrating from 1 to k in fact moves from right to left, instead of left to right, so that the green area drawn here, calculated in a backward direction, becomes negative. This matches nicely the fact that $\ln k$ is negative. If we really want to regard the green area as positive, we have to integrate forward from k to 1, and we can swap terminals by multiplying by -1 to get $-\ln k$, which is indeed positive and all is well.

Let's find the area under the sine curve between $\frac{\pi}{2}$ and π. An antiderivative is $-cosx$.



Evaluating between $\frac{\pi}{2}$ and π quickly produces the answer 1.

$$\int_{\pi/2}^{\pi} \sin x \, dx = [-\cos x]_{\pi/2}^{\pi}$$

$$= -\cos\pi - \left(-\cos\left(\frac{\pi}{2}\right)\right)$$

$$= -(-1) - (-0)$$

$$= 1$$

$$\int_{\pi/2}^{\pi} \sin x \, dx = [-\cos x]_{\pi/2}^{\pi}$$

$$= -\cos\pi - \left(-\cos\left(\frac{\pi}{2}\right)\right)$$

$$= -(-1) - (-0)$$

$$= 1$$

Amazing. Who would have expected the area to be such a nice number? If we integrate from $\frac{\pi}{2}$ all the way to 2π, then the curve slips below the x-axis and you expect by the symmetry to add two lots of the green area, which is called pink in the diagram, to be counted negatively, so the total answer should be -1. Let's check.

$$\int_{\pi/2}^{2\pi} \sin x \, dx = [-\cos x]_{\pi/2}^{2\pi}$$

$$= -\cos 2\pi - \left(-\cos\left(\frac{\pi}{2}\right)\right)$$

$$= -(1) - (-0)$$

$$= -1$$

We evaluate the same antiderivative, but now, between $\frac{\pi}{2}$ and 2π and quickly see that the answer simplifies to -1 as expected.

Here's the fundamental theorem again.

The Fundamental Theorem of Calculus:

$\int_a^b f(x)\,dx = F(b) - F(a)$

where $F(x)$ is any antiderivative of $f(x)$.

We also write

$F(x) = \int f(x)\,dx$

without terminals

We have a natural notation to describe the antiderivative. We write $F(x)$ using the same expression as the definite integral but without the terminals. This is called an indefinite integral. The indefinite integral describes an antiderivative of the integrand $f(x)$, so is a function of x. By contrast, the definite integral is an area under the curve, so it's a real number.

We mentioned earlier the fact that antiderivatives of $f(x)$ differ by a constant. This assumes all functions are continuous on a particular interval of interest. I'll briefly explain the reason.

Suppose that $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, so that their derivatives are both equal to $f(x)$. We want to show that their rules are the same, but differ by a constant. The trick is to differentiate the difference $F(x) - G(x)$, which is the difference in the derivatives, which is $f(x) - f(x)$, which is zero.

Reason: Suppose $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, so $F'(x) = G'(x) = f(x)$.

$$\text{Then} \quad \frac{d}{dx}(F(x) - G(x)) = F'(x) - G'(x) = f(x) - f(x) = 0.$$

$$\text{so} \quad F(x) - G(x) = C \quad \text{for some constant } C,$$

$$\text{so} \quad F(x) = G(x) + C.$$

If the derivative is zero, then the tangent lines to the curve with rule $F(x) - G(x)$ must be horizontal everywhere on the interval, which means, the curve must be itself just a horizontal line. So, $F(x) - G(x) =$ some constant C. So, that $F(x) = G(x) + C$. Indeed, we've shown the two antiderivatives differ by a constant. This explains the origin of the constant of integration C that you will see all the time in so-called integration formulas, which are just equations linking functions to their antiderivatives and use the indefinite integral notation.

For example, this one just says that the general antiderivative of x is $\frac{x^2}{2} + C$. This one says that the general antiderivative of x^2 is $\frac{x^3}{3} + C$. These are special cases of the formula that says the general antiderivative of x^n is $\frac{x^{n+1}}{n+1} + C$, provided $n \neq -1$. It's common to use the same capital C for the constant of integration.

The exceptional case to integrate x^{-1}, also written as the integral of $\frac{dx}{x}$, is $\ln|x| + C$, provided $x \neq 0$. This includes the use of absolute value signs to cover both positive and negative x.

We also have the integral of e^x which is $e^x + C$, and two integration formulas for the circular functions. Since the derivative of $\tan x$ is $\sec^2 x$, we get that the integral of $\sec^2 x$ is $\tan x + C$.

There are many more formulas which you'll see if you open the covers of almost any calculus textbook. There are also some general principles for manipulating indefinite integrals which are made explicit in the accompanying notes.

In today's section, we introduced and applied the fundamental theorem of calculus, which provides a simple and elegant formula using antiderivatives to find exact values for definite integrals. We found antiderivatives for a number of common functions and then used some of them to try out this new method to calculate areas under a variety of curves. You also expressed antiderivatives using indefinite integrals, which use the same symbolism as definite integrals except the terminals are missing. Then listed a number of common integration formulas. These all employ a constant of integration typically denoted by adding capital C, which arises because of the fact that all antiderivatives of a given function differ by a constant.

33.1.2 Examples and derivation

1. Find the definite integral $\int_0^1 (6x^2+4x-1)\,dx$.
   Solution: Using the antiderivative from the previous exercise (without the "plus C"), and the Fundamental Theorem of Calculus, we get

$$\int_0^1 (6x^2+4x-1)\,dx = \left[2x^3+2x^2-x\right]_0^1 = 16+8-2-(2+2-1) = 19.$$

2. Find the indefinite integral $\int (\cos x - \sin x)\,dx$.
   Solution: Again we can integrate each piece separately, and combine the constants as a single "plus C" at the right, to get

$$\int (\cos x - \sin x)\,dx = \int \cos x\,dx - \int \sin x\,dx = \sin x + \cos x + C.$$

3. Find the definite integral $\int_{\pi/4}^{\pi/3} (\cos x - \sin x)\,dx$.
   Solution: Using the result of the previous exercise, we have

$$\int_{\pi/4}^{\pi/3} (\cos x - \sin x)\,dx = [\sin x + \cos x]_{\pi/4}^{\pi/3} = \sin\frac{\pi}{3} + \cos\frac{\pi}{3} - \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = 1+0-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) = 1-\sqrt{2}.$$

4. Find the indefinite integral $\int 2x\sqrt{x}\,dx$.
   Solution: We rewrite the integrand as a single fractional power of x, and as usual rewrite any constant to the right as a "plus C", to get

$$\int 2x\sqrt{x}\,dx = 2\int x^{3/2}\,dx = \frac{4x^{5/2}}{5} + C = \frac{4\sqrt{x^5}}{5} + C.$$

5. Find the definite integral $\int_1^4 2x\sqrt{x}\,dx$.
   Solution: Using the result of the previous exercise, we have

$$\int_1^4 2x\sqrt{x}\,dx = \left[\frac{4x^{5/2}}{5}\right]_1^4 = \frac{4}{5}\left[x^{5/2}\right]_1^4 = \frac{4}{5}(32-1) = \frac{124}{5}.$$

6. Find the definite integral $\int_2^1 (x^2-2x^3)\,dx$.
   Solution: By the formula for integrating powers of x for each of the two pieces, we get

$$\int_2^1 (x^2-2x^3)\,dx = \left[-\frac{x^3}{3}+x^2\right]_2^1 = \left(-\frac{1}{2}+1\right) - \left(-\frac{8}{3}+4\right) = -\frac{1}{6}.$$

7. Find the definite integral $\int_{-1}^1 \frac{e^x-e^{-x}}{2}\,dx$.
   Solution: An obvious antiderivative of e^x is e^x. For e^{-x}, it is not so hard to guess an antiderivative, because its derivative is $-e^{-x}$, so multiplying through by negative one, gives

$-e^{-x}$. Hence, considering the indefinite integral first, and playing with minus signs for e^{-x}, we get

$$\int \frac{e^x - e^{-x}}{2}\,dx = \frac{1}{2}\left(\int e^x\,dx - \int e^{-x}\,dx\right) = \frac{1}{2}\left(e^x + e^{-x}\right) + C.$$

Hence, we get

$$\int_{-1}^{1} \frac{e^x - e^{-x}}{2}\,dx = \frac{1}{2}\left[e^x + e^{-x}\right]_{-1}^{1} = \frac{1}{2}\left(e + e^{-1} - e^{-1} - e\right) = 0.$$

8. We explain why there is a magnitude sign appearing in the standard integration formula $\int \frac{dx}{x} = \ln|x| + C$.
For $x > 0$, certainly $\ln x$ is an antiderivative of $\frac{1}{x}$, so there is some constant C_1 such that

$$\int \frac{dx}{x} = \ln x + C_1. \tag{1}$$

However, let us now consider $x < 0$. Then $-x > 0$ and the expression $\ln(-x)$ makes perfectly good sense. Put $u = -x$, so that $\frac{du}{dx} = -1$. By the Chain Rule,

$$\frac{d}{dx}(\ln(-x)) = \frac{d}{du}(\ln u)\frac{du}{dx} = \left(\frac{1}{u}\right)(-1) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}.$$

So, in this case, $\ln(-x)$ is an antiderivative of $\frac{1}{x}$, so there must be some constant C_2 such that

$$\int \frac{dx}{x} = \ln(-x) + C_2. \tag{2}$$

If $C_1 = C_2 = C$ then (1) and (2) can be combined in a single equation,

$$\int \frac{dx}{x} = \ln|x| + C, \tag{3}$$

exploiting the fact that $|x| = x$ for $x > 0$, and $|x| = -x$ for $x < 0$. This explains the usual formula.
In fact, (1) and (2) are more general than (3), because one could create a single antiderivative of $y = \frac{1}{x}$ using different constants in (1) and (2) for the different branches of the hyperbola, for example

$$F(x) = \begin{cases} \ln x + 3 & \text{if } x > 0 \\ \ln(-x) + 7 & \text{if } x < 0 \end{cases}$$

which would not be covered by (3). However, in practice, one is usually only concerned with one branch of the hyperbola, where the domain is a connected interval, so (3) suffices. Note that, because of this flexibility on the different branches of the hyperbola, the result guaranteeing uniqueness of the antiderivative up to constants holds true for a single constant C. This is not a contradiction, because the function $y = \frac{1}{x}$ is not continuous on the whole real line.

Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

Practice Quiz

Question 1

Find the indefinite integral $\int \left(3x^2+2x+1\right) dx$.

(a) $6x+2+C$
(b) $\frac{x^3}{3}+\frac{x^2}{2}+x+C$
(c) x^3+x^2+x+C
(d) $x^3+x^2+\frac{x}{2}+C$
(e) x^3+x^2+C

Question 2

Find the definite integral $\int_1^3 \left(3x^2+2x+1\right) dx$.

(a) 39
(b) 36
(c) 30
(d) 27
(e) 33

Question 3

Find the indefinite integral $\int (\sin x+\cos x)\, dx$.

(a) $\sin x-\cos x+C$
(b) $\sin x+\cos x+C$
(c) $\frac{\cos x-\sin x}{2}+C$
(d) $\cos x-\sin x+C$
(e) $\frac{\sin x-\cos x}{2}+C$

Question 4

Find the definite integral $\int_0^{\pi/2} (\sin x+\cos x)\, dx$.

(a) $\frac{3}{2}$
(b) 0
(c) $\frac{\pi}{2}$
(d) 1
(e) 2

Question 5

Find the indefinite integral $\int 3\sqrt{x}\, dx$.

(a) $\frac{2x\sqrt{x}}{3}+C$
(b) $\frac{3}{2\sqrt{x}}+C$
(c) $\frac{3}{2x\sqrt{x}}+C$
(d) $\frac{3x\sqrt{x}}{2}+C$
(e) $2x\sqrt{x}+C$

Question 6

Find the definite integral $\int_4^9 3\sqrt{x}\, dx$.

(a) 44
(b) 42
(c) 38
(d) 40
(e) 36

Question 7

Find the indefinite integral $\int \left(\frac{e^x+1}{2}\right) dx$.

(a) $\frac{e^x+x}{2}+C$
(b) $\frac{e^{x+1}}{2(x+1)}+\frac{x}{2}+C$
(c) $\frac{e^x+x^2}{4}+C$
(d) $\frac{2e^x+x^2}{2}+C$
(e) $\frac{e^x+x}{2x}+C$

Question 8

Find the definite integral $\int_{-1}^{1} \left(\frac{e^x+1}{2}\right) dx$.

(a) $\frac{e^2-2e+1}{2e}$
(b) $\frac{e^2+2e-1}{2e}$
(c) $\frac{e^2+e+1}{e}$
(d) $\frac{e^2+e+2}{2e}$
(e) $\frac{e^2+2e-1}{e}$

Question 9

Find the indefinite integral $\int \left(x^{-3}-x^{-4}\right) dx$.

(a) $\frac{3x-2}{6x^3}+C$
(b) $\frac{2x-3}{6x^3}+C$
(c) $\frac{3-2x}{6x^3}-\frac{x^4}{4}+C$
(d) $\frac{2-3x}{6x^3}+C$
(e) $\frac{3+2x}{6x^3}+C$

Question 10

Find the indefinite integral $\int_{-2}^{-1} \left(x^{-3}-x^{-4}\right) dx$.

(a) -1
(b) 0
(c) $-\frac{2}{3}$
(d) -2
(e) $-\frac{5}{6}$

Answers

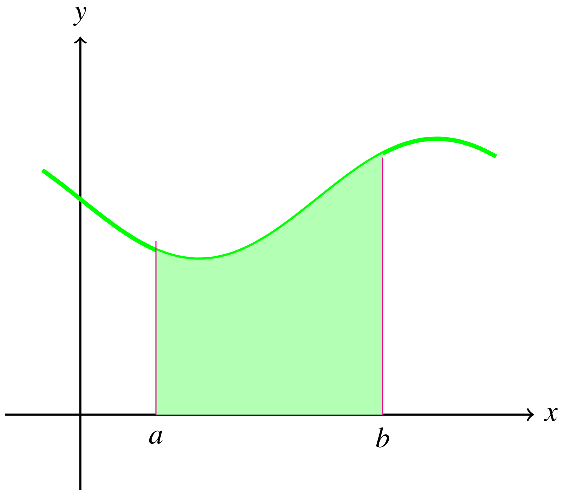
The answers will be revealed at the end of the module.

33.2 Connection between areas and derivatives

33.2.1 Part 1

In the previous section, we introduced and applied the Fundamental Theorem of Calculus, which provides an elegant formula, under certain conditions, for finding exact values for definite integrals, that is, areas under curves. In this section, we explain the connection between areas and derivatives, which is the main idea that leads to a proof of this remarkable theorem. We also apply the main idea to use the derivative to relate areas of circles to perimeters, and volumes of spheres to surface areas. Recall that the definite integral is the area under the curve $y = f(x)$ as x goes from a, the left-hand endpoint of the interval, to b, the right-hand endpoint.

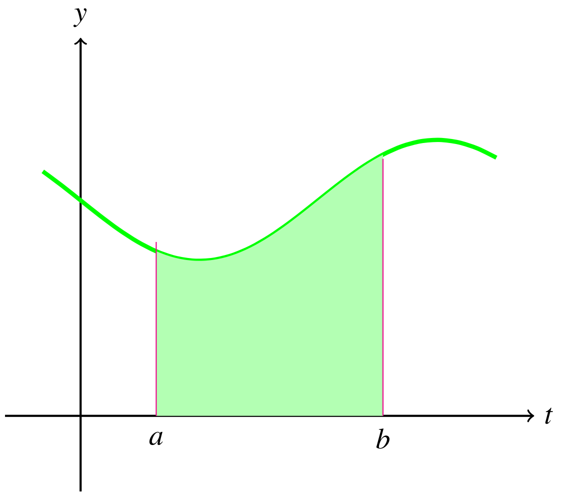
$$\int_a^b f(x)\,dx$$



Assuming the curve is continuous, then the area is described succinctly by the Fundamental Theorem of Calculus, as $F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$, by which we mean that its derivative is $f(x)$. The main purpose of today's section is to explain why this works. Before starting, it should be remarked that this is one of the most important theorems in mathematics. Though the Ancient Greeks employed thought experiments with approximations and limits, the basic ingredients of calculus, it took a couple of thousand years before mathematicians were able to recognize and articulate the main ideas that lead to the statement and proof of the theorem. So you shouldn't expect the ideas to be straightforward. We have to be radical in our thinking, flexible and prepared to experiment and abruptly change our point of view.

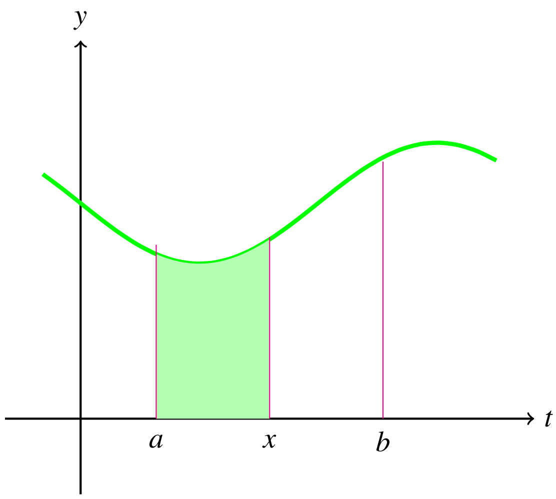
The first step is to change the name of the input variable that is used to form the curve with respect to which we're taking the area. Let's use t instead of x, because we want to reserve x for something else in a moment. Wherever we see an x, we replace it with a t.

$$\int_a^b f(t)\,dt$$



So the horizontal axis is labeled by t. The curve is a function of t, and the integrand and differential use t. We're still finding the same area under the same curve, just using a different name for the input variable. What's so radical about that? We do something completely unexpected. We still allow x to move from a to b, but now use x instead of b as the right-hand endpoint of an interval, with respect to which we take areas under the curve.

$$A(x) = \int_a^x f(t)\,dt$$



We create what we call an area function, $A(x)$, using the same curve, but now representing the area under the curve from a up to x, wherever x happens to be in the interval. The value of $A(x)$ clearly is able to vary as x varies. Being an area under the curve, the value $A(x)$ is still represented by a definite integral, but now x appears as the upper terminal, instead of having a role in the specification of the integrand and differential. On the face of it, this looks like a ridiculous and unnecessary complication, as we're only interested in the area under the curve from a to b. But you've already seen many examples in earlier sections, where we've made things more complicated in order to get past a barrier or obstruction, to solve a problem or make things simpler later. The phenomenon you're about to witness is one of the most spectacular examples of problem solving in the history of mathematics. Calculus is a study of how things change, especially about instantaneous rates of change.

Let's think about the way the area function changes. First of all, make note of what happens at the two extremes, when $x = a$ and $x = b$, the endpoints of the interval. When $x = a$, both terminals in the definite integral are the same and remember this gives an area of zero.

$$A(x) = \int_a^a f(t)\,dt = 0$$

In the diagram above, when $x = a$, the green area vanishes. However, if $x = b$, then $A(b)$ becomes our original definite integral, and the green area under the curve goes all the way from a to b. What about the instantaneous rate of change of the area function? This is the derivative, defined as

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

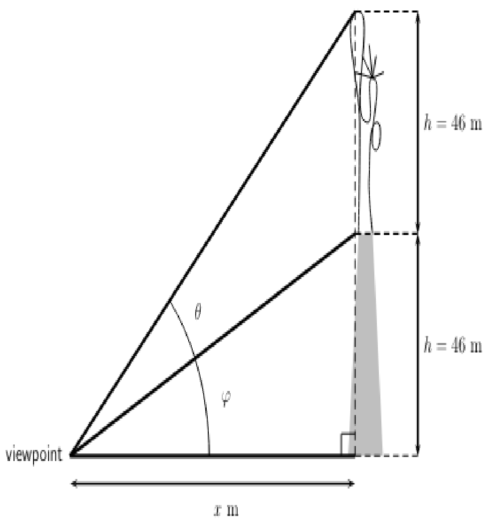


Figure 33.1: Statue of Liberty

We relate the ingredients of this limit to the diagram representing the area function. We assume throughout the following discussion that h is small and positive, and that the curve $y = f(t)$ is increasing as in this diagram. These assumptions greatly simplify the argument, but nevertheless suffice to illustrate all of the important key ideas. Because h is positive, $x+h$ is slightly to the right of x. We move up to the curve, and note $f(x)$ and $f(x+h)$ on the y-axis. There's a small piece of the curve over the interval from x to $x+h$, which we enclose with lower and upper rectangles. The height of the lower rectangle is $f(x)$, and the height of the upper rectangle is $f(x+h)$. The width of both rectangles is h. We color in the area of the lower rectangle blue and the area under the curve between the lower and upper rectangles beige. The value of $A(x+h)$ is the area all the way from a to $x+h$, whilst the value of $A(x)$ is the area from a to x. So when you take $A(x)$ away from $A(x+h)$, you just get the area under the curve between x and $x+h$, which is just the blue area plus the beige area.

$$A(x+h) - A(x) = \text{(blue area)} + \text{(beige area)}$$

We color, in pink, the area in the upper rectangle that's not under the curve. So, if we add this to the blue and beige areas, then we get an upper bound for $A(x+h) - A(x)$. On the other hand, if we only consider the blue area, then we also get a lower bound. But the blue area is just the area of the lower rectangle, which is $h \cdot f(x)$, and the blue, beige, and pink areas combine to give the area of the upper rectangle which is $h \cdot f(x+h)$. So we get this nice cascade of inequalities, with $A(x+h) - A(x)$ sandwiched in the middle. Let's divide through by the positive number h, retaining the same inequalities throughout, so that the h's cancel on the left and right, and we conclude that

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h)$$

So let's see what happens as h goes to zero. $f(x)$ just stays the same, and $f(x+h)$ becomes $f(x)$, because we're assuming throughout that the function f is continuous. The expression in the middle goes to the derivative $A'(x)$. But this must become $f(x)$, by the Squeeze Theorem, since both the expressions on the left and right have $f(x)$ as the same limit. There's nowhere else to move, but for all three limits to coincide with $f(x)$.

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

$$f(x) \quad = \quad A'(x) \quad = \quad f(x)$$

Reflect on what we've just done. We defined an area function by allowing the upper terminal of the definite integral, which we now call x, to vary from a to b, and discovered that its derivative is simply the value of f at x. How remarkable! We've learned so many rules for manipulating derivatives, and we're used to analyzing and creating complicated functions, and the area function looks like it probably is some very weird and exotic function. But when we differentiate it with respect to x, we just reproduce the value of f at x. This beautiful, simple, and elegant relationship between areas and curves evaded detection for thousands of years.

What's this got to do with the Fundamental Theorem of Calculus? Well, it tells us that the area function is an antiderivative of the function that gives rise to the original curve. The Fundamental Theorem gives us a formula in terms of $F(x)$, which is an antiderivative of $f(x)$. But remember, antiderivatives of a given function differ by a constant. So $F(x)$ must equal $A(x)+C$ for some constant C.

To find C, we can substitute $x=a$, so $F(a)$ becomes $A(a)+C$, which is the definite integral with a as both upper and lower terminals, plus C, which is zero plus C, which of course is just C.

$$F(a) = A(a) + C = \int_a^a f(t)\,dt + C$$

So the constant is $F(a)$. So $F(x)$ becomes $A(x)+F(a)$. In particular, $F(b)$ becomes $A(b)+F(a)$. And rearranging gives $A(b) = F(b) - F(a)$.

But $A(b)$ is the original area under the curve over the entire interval from a to b, which shows that our original definite integral is indeed given by the formula provided by the Fundamental Theorem of Calculus, finally completing the proof.

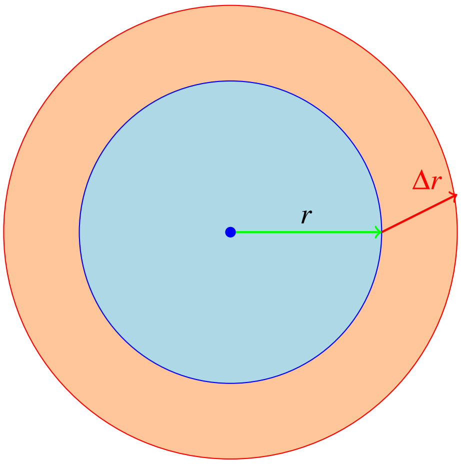
$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Now, there were certain simplifications in the way we set up this proof in the accompanying diagrams, so it's really only a sketch. But all of the main ideas behind the Fundamental Theorem have been captured by what we've done.

33.2.2 Part 2

Now that we've introduced the idea of an area function, we can apply this idea to find some perhaps surprising and unexpected connections. The first is to explain the relationship that we noted in earlier sections between the formulas for the perimeter and area of a circle.

Suppose we have a circle of radius r. Consider the effects on area and perimeter that occur by making a small change in the radius, say Δr, which is positive in the diagram below. Color the area of the original circle in blue, and color the pathway around the circle with Δr in beige as shown in the diagram below.



So the blue area we can call A, which is a function of r, $A(r)$, given by the formula πr^2. The beige area is the change in A, ΔA, which is the difference between A evaluated at $(r+\Delta r)$ and A evaluated at r.

Let P denote the perimeter of the circle. This area is shaded in blue and is also a function of r, given by the formula $2\pi r$. Till now we get the following equations.

$$A = A(r) = \pi r^2 = \text{blue area}$$

$$\Delta A = A(r+\Delta r) - A(r) = \text{beige area}$$

$$P = P(r) = \text{perimeter} = 2\pi r$$

The area of the beige pathway is represented symbolically by ΔA. The resultant change in the radius is Δr, and the pathway has two sides: a smaller side length, which is the perimeter of the circle with radius r, and a larger side length, which is the perimeter of the circle with radius $r+\Delta r$.

Such an area must be bounded below by the small perimeter multiplied by the width of the path and bounded above by the larger perimeter multiplied by the same path width.

And so we get this chain of inequalities.

$$P(r)\,\Delta r \leq \Delta A \leq P(r+\Delta r)\,\Delta r$$

We can divide through by Δr, and get $\frac{\Delta A}{\Delta r}$ sandwiched between $P(r)$ and $P(r+\Delta r)$.

$$P(r) \leq \frac{\Delta A}{\Delta r} \leq P(r+\Delta r)$$

As Δr approaches 0, $P(r)$ just stays the same.

$$\lim_{\Delta r \to 0} P(r) = P(r)$$

And $P(r+\Delta r)$ becomes $P(r)$, because the perimeter function is continuous. And $\frac{\Delta A}{\Delta r}$ becomes the derivative $\frac{dA}{dr}$.

$$\lim_{\Delta r \to 0} P(r+\Delta r) = P(r)$$

$$\lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = \frac{dA}{dr}$$

All three limits must coincide by the squeeze theorem because of the chain of inequalities, and the fact that the limits on the left and right are equal.

$$P(r) = \frac{dA}{dr} = P(r)$$

We conclude that the derivative $\frac{dA}{dr}$ becomes the perimeter function P, explaining this fact that we've noted before directly from the respective formulas.

$$\frac{dA}{dr} = P(r)$$

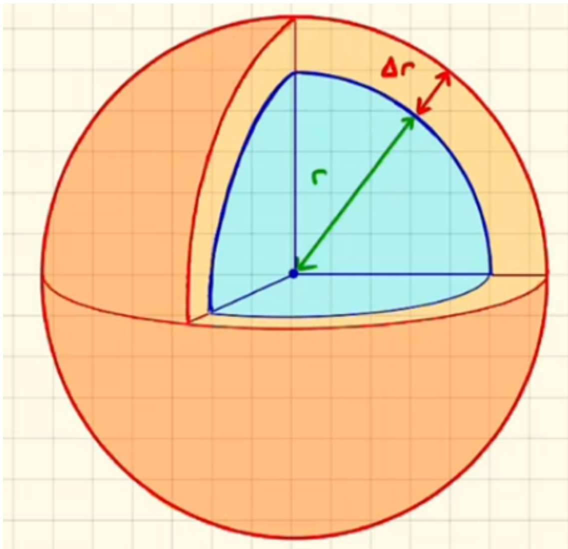


Figure 33.2: Sphere

It's natural then to ask if there's a similar connection between the volume of a sphere and its surface area. Consider a sphere of radius r. Denote its volume by V and its surface area by S, both of which are functions of r. Consider again, now in the context of the sphere, the effects of a small positive change, Δr, in the radius. This will propagate small changes in both the volume and surface area of the sphere. The change in volume ΔV, will be the volume of the outer layer, which we will refer to as the crust, by analogy with the crust of the earth. This crust, so to speak, is colored light beige, and the outer surface of the expanded sphere is colored slightly darker beige. The inner sphere, the core, with the original radius r, is colored blue.

The crust has a uniform thickness of Δr throughout. Where it interfaces with the blue inner sphere, we have a smaller blue surface area. So the volume of the crust must be bounded below by the blue

[0] Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

surface area, whatever that might be, multiplied by Δr. On the other hand, the crust interfaces with the exterior outside world through the largest surface area of the expanded sphere. So the volume of the crust must be bounded above by the darker beige surface area multiplied by Δr. And so we get a chain of inequalities.

(blue surface area) $\times \Delta r \quad \leq \quad$ **(beige crust)** $\quad \leq \quad$ **(dark beige surface area)** $\times \Delta r$

But the surface area of the inner blue core is represented symbolically by $S(r)$. And the darker beige, surface area of the larger sphere by $S(r+\Delta r)$. So we can write our chain of inequalities more concisely.

$$S(r) \times \Delta r \leq \Delta V \leq S(r+\Delta r) \times \Delta r$$

Dividing through by Δr, we get:

$$S(r) \leq \frac{\Delta V}{\Delta r} \leq S(r+\Delta r)$$

As $\Delta r \to 0$, $S(r)$ of course stays the same. $S(r+\Delta r)$ goes to $S(r)$ by continuity of the surface area function, and $\frac{\Delta V}{\Delta r}$ becomes the derivative $\frac{dV}{dr}$.

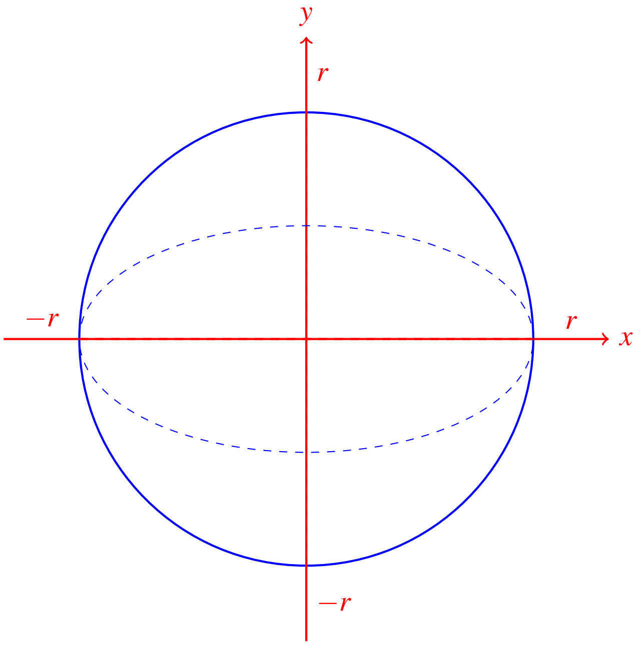
$$\frac{dV}{dr} = S(r)$$

By the squeeze law, again, all three limits are equal.

$$S(r) = \frac{\Delta V}{\Delta r} = S(r+\Delta r)$$

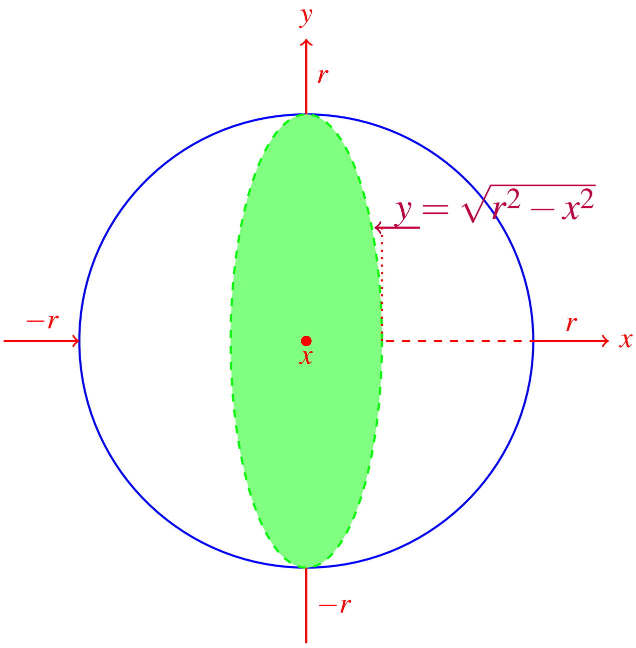
This proves the that the derivative of the volume of the sphere, must be the surface area.

Whatever the formula is for the volume, we can get a formula for the surface area by differentiating. We've established this amazing connection, simply by thought experiments and a few diagrams. To apply the fruit of our labors, we'd like to come up with some actual formulae. With some lateral thinking, we can figure out a formula for the volume fairly quickly. And then use this indirectly to get the surface area by differentiating. It's actually very difficult to find the surface area formula directly.



Here's a sphere of radius r, again, but imagine a copy of the xy plane forming a vertical cross-section. So that we get this circular profile that crosses the axis at $\pm r$.

The circle that we see in this profile has the equation $x^2 + y^2 = r^2$. So in the xy plane, y takes the value $\pm\sqrt{r^2 - x^2}$. Now comes the lateral thinking: we imagine slicing through the sphere vertically, but in a direction perpendicular to the xy plane. This produces a green circular vertical cross-section with the x-axis passing exactly through its center.



If this green circle intersects the x-axis at a particular value of x, then its radius is the positive $\sqrt{r^2 - x^2}$. We want the total volume of the sphere, and the green cross-section is like an infinitely thin slice. The idea, intuitively, is to imagine the area of all these infinitely thin vertical cross-sections being added together, so to speak, as x moves from $-r$ to r along the x-axis. It's like the continuous sums that we've been discussing for the definite integral. But now, we use the cross-sectional area of these circles as the integrand. You just imagine of this cross-section as x moves from left to right. But in fact, we're thinking of a continuous movement from $x = -r$ through to $x = r$. To set up the integration, we need to know the area of the circular cross-section for any particular x.

But the radius is $y = \sqrt{r^2 - x^2}$, and the area of the circle is πy^2, which equals $\pi(r^2 - x^2)$.

So we're really looking at the definite integral with integrand $\pi(r^2 - x^2)$, with limits $\pm r$, as we're thinking of x as moving all along the interval from $-r$ to r.

Volume of sphere = "continuous sum of cross-sections"

$$= \int_{-r}^{r} \pi(r^2 - x^2)\, dx$$

In setting this up, we're appealing to your intuition. We're in fact using something called the disk method taught in more advanced courses in calculus, which formally relates the definite integral to the limit of Riemann sums where the pieces being added up are volumes of so-called disks with certain thicknesses. These become, in the limit, as things get thinner and thinner, infinitely thin circular cross-sections. The volume we're finding is also called a volume of revolution, which you can look up and read about if you wish. So the volume of this sphere is this particular definite integral, and the constant π can come out the front. We wish to form an antiderivative of $r^2 - x^2$, and evaluate it for x between $-r$ and r.

Volume of sphere = $\pi \int_{-r}^{r} (r^2 - x^2)\, dx$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

To find this antiderivative, observe that r^2 is a constant, so that contributes $r^2 x$. Then we take away an antiderivative of x^2, which is $\frac{x^3}{3}$.

$$
\begin{aligned}
\text{Volume of sphere} &= \pi \int_{-r}^{r} \left(r^2 - x^2\right) dx \\
&= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\
&= \pi \left(\left[r^2(r) - \frac{r^3}{3}\right] - \left[r^2(-r) - \frac{(-r)^3}{3}\right]\right) \\
&= \pi \left(r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right) \\
&= \pi \left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right) \\
&= \pi \left(\frac{2r^3}{3} + \frac{2r^3}{3}\right) \\
&= \frac{4}{3}\pi r^3
\end{aligned}
$$

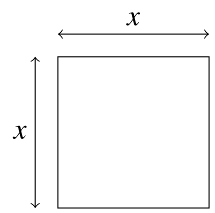
These pieces combine to give us our full antiderivative. Evaluating it for x between $-r$ and r, simplifies to $\frac{4}{3}\pi r^3$. Thus we get the well-known formula for the volume of a sphere. We now get an explicit formula for the surface area by taking the derivative with respect to r, which quickly simplifies to $4\pi r^2$. Thus finally, we reproduce the well-known formula for the surface area of a sphere.

In this section, we explained in detail the connection between areas and derivatives, which is the main idea that leads to a proof of the Fundamental Theorem of Calculus. To do this, we changed that point of view, and considered an area function built out of a definite integral, where the upper terminal is a variable. The derivative of this function turns out to be the value of the integrand. Using properties of anti-derivatives, the statement of the fundamental theorem quickly followed. We then applied the main idea involving an area function to explain why the derivative of the area with respect to the radius produces the perimeter. We adapted that idea to explain why the derivative of the volume of a sphere with respect to the radius produces the surface area. We then worked out a formula for the volume of a sphere directly, by integrating the circular cross-sectional area. By differentiating this answer, we finally obtained an explicit formula for the surface area of the sphere. Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

33.2.3 Practice Quiz

Question 1

Consider a square of side-length $x > 0$.

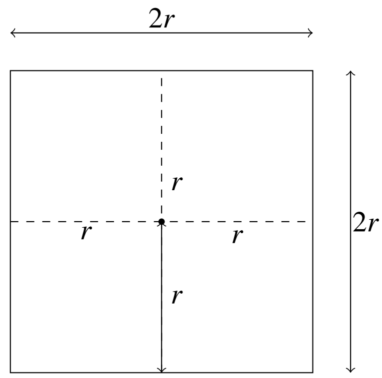


Let $A = x^2$ denote the area and $P = 4x$ denote the perimeter of the square. Which one of the following statements is true?

(a) $\frac{dA}{dx} = 4$
(b) $\frac{dA}{dx} = P$
(c) $\frac{dA}{dx} = 2P$
(d) $\frac{dA}{dx} = 4P$
(e) $\frac{dA}{dx} = \frac{P}{2}$

Question 2

Define the *radius* r of a square to be the shortest distance from the point in the middle of the square (its *centre*) to any of the four sides. Thus the side length of the square is $2r$.

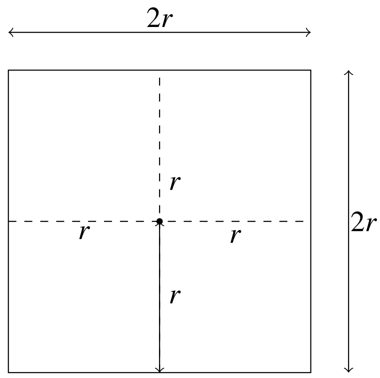


Let $P = P(r)$ denote the perimeter of a square with radius r. Which one of the following is the correct formula for P?

(a) $P = 4r$
(b) $P = 8r$
(c) $P = 2\pi r$
(d) $P = 16r$
(e) $P = 6r$

Question 3

Define the *radius* r of a square to be the shortest distance from the point in the middle of the square (its *centre*) to any of the four sides. Thus the side length of the square is $2r$.

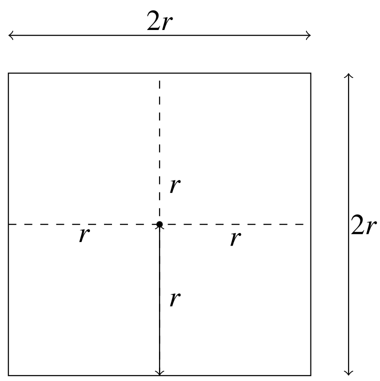


Let $A = A(r)$ denote the area of a square with radius r. Which one of the following is the correct formula for A?

(a) $A = r^2$
(b) $A = \pi r^2$
(c) $A = 2r^2$
(d) $A = 4r^2$
(e) $A = 3r^2$

Question 4

Define the *radius* r of a square to be the shortest distance from the point in the middle of the square (its *centre*) to any of the four sides. Thus the side length of the square is $2r$.

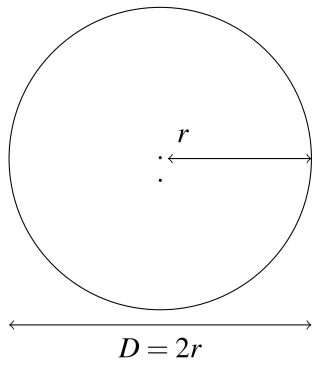


Let $A = A(r)$ denote the area and $P = P(r)$ denote the perimeter of a square with radius r. Which one of the following statements is true ?

(a) $\frac{dA}{dr} = P$
(b) $\frac{dA}{dr} = \frac{P}{2}$
(c) $\frac{dA}{dr} = 6P$
(d) $\frac{dA}{dr} = 2P$
(e) $\frac{dA}{dr} = 4P$

Question 5

Consider a circle of radius r.

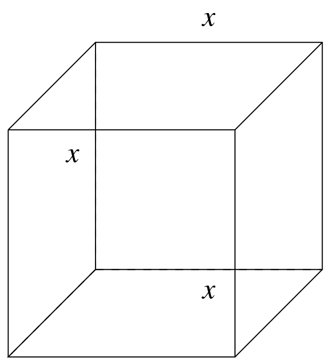


Recall that the area A of the circle is given by the formula $A = \pi r^2$, and its perimeter P by the formula $P = 2\pi r$. Recall that the *diameter* of the circle is $D = 2r$. Which one of the following statements is true?

(a) $\frac{dA}{dD} = 2P$
(b) $\frac{dA}{dD} = \frac{P}{2}$
(c) $\frac{dA}{dD} = 4P$
(d) $\frac{dA}{dD} = P$
(e) $\frac{dA}{dD} = \frac{P}{4}$

Question 6

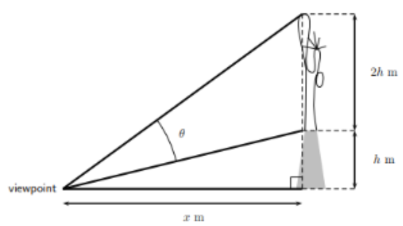
Consider a cube of side-length $x > 0$.



(a) $\frac{dV}{dx} = 2S$
(b) $\frac{dV}{dx} = 4S$
(c) $\frac{dV}{dx} = S$
(d) $\frac{dV}{dx} = \frac{S}{2}$
(e) $\frac{dV}{dx} = 3S$

Question 7

Define the *radius* r of a cube to be the shortest distance from the point in the middle of the cube (its *centre*) to any of the six faces.

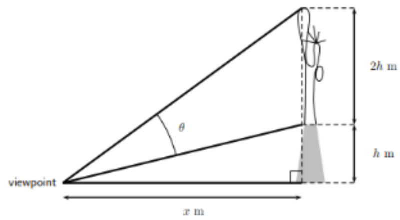


Thus the side length of the cube is $2r$. Let $S = S(r)$ denote the surface area of a cube with radius r. Which one of the following is the correct formula for S?

(a) $S = 4\pi r^2$
(b) $S = 12r^2$
(c) $S = 6r^2$
(d) $S = 8r^2$
(e) $S = 24r^2$

Question 8

Define the *radius* r of a cube to be the shortest distance from the point in the middle of the cube (its *centre*) to any of the six faces. Thus the side length of the cube is $2r$.

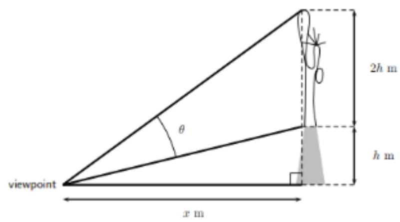


Let $V = V(r)$ denote the surface area of a cube with radius r. Which one of the following is the correct formula for V?

(a) $V = \frac{4\pi r^3}{3}$
(b) $V = 4r^2$
(c) $V = r^3$
(d) $V = 8r^3$
(e) $V = 2r^2$

Question 9

Define the *radius* r of a cube to be the shortest distance from the point in the middle of the cube (its *centre*) to any of the six faces.

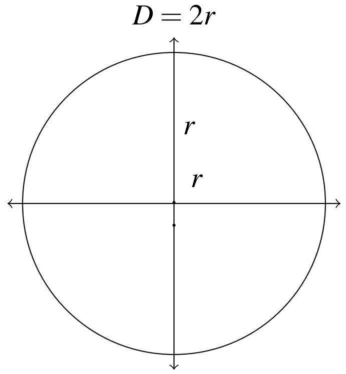


Thus the side length of the cube is $2r$. Let $S = S(r)$ denote the surface area of a cube with radius r. Which one of the following is the correct formula for S?

(a) $\frac{dV}{dr} = 2S$
(b) $\frac{dV}{dr} = 4S$
(c) $\frac{dV}{dr} = 16P$
(d) $\frac{dV}{dr} = \frac{S}{2}$
(e) $\frac{dV}{dr} = S$

Question 10

Consider a sphere of radius r and recall that its volume V is given by the formula $V = \frac{4\pi r^3}{3}$ and its surface area S by the formula $S = 4\pi r^2$. Recall also that the *diameter* of the sphere is $D = 2r$.



Which one of the following statements is true?

(a) $\frac{dV}{dD} = 2S$
(b) $\frac{dV}{dD} = \frac{S}{4}$
(c) $\frac{dV}{dD} = 4S$
(d) $\frac{dV}{dD} = \frac{S}{2}$
(e) $\frac{dV}{dD} = S$

Answers

The answers will be revealed at the end of the module.

34. Integration by Substitution

34.1 Integration by Substitution

34.1.1 Part 1

In this section, we introduce and illustrate an important and useful method for anti-differentiation, known as integration by substitution. This method is closely related to the Chain Rule, one of the differentiation rules that we discussed in the previous module. This method provides us with a tool for reducing the complexity of a given integral by making an intelligent or judicious choice of substitution of variables, followed by algebraic manipulation of the integrand and differential.

The word integration has several meanings, usually clear from context. It may refer to the process of finding an indefinite integral, which is an anti-derivative of the integrand, producing a function. It can also refer to the process of finding a definite integral, which is the area under a given curve over a given interval, producing a real number. The expression integration by substitution typically refers to a rule or method for anti-differentiation closely related to the Chain Rule for differentiation. As you'll see, it makes free use of differentials and takes full advantage of the utility of Leibniz's notation. There's also a version of this method involving definite integrals, and special care needs to be taken to keep track of the correct limits.

Before formalizing the method, let's look at the following example and follow our instincts for manipulating expressions, in particular those involving differentials. Our aim is to anti-differentiate $4x \cdot (x^2+1)$.

$$\int 4x(x^2+1)\,dx = \int (4x^3+4x)\,dx$$

$$= x^4 + 2x^2 + C$$

A direct method is to expand the integrated, exactly as above, and then anti-differentiate each piece, noting that $4x^3$ is the derivative of x^4 and $4x$ is the derivative of $2x^2$. Putting the pieces together with a constant of integration.

The second method is indirect and avoids the initial expansion.

The complicated part of the integrand appears to be x^2+1, so let's see what happens if we put u equal to this.

Put $u = x^2 + 1$,

The derivative of u with respect to x is $2x$, and we get the equation of differentials $du = 2x\,dx$.
Then,

$$\frac{du}{dx} = 2x \quad , \quad \text{so} \quad du = 2x\,dx.$$

Hence,

$$\int 4x(x^2+1)\,dx = \int 2(x^2+1)\,2x\,dx$$

The aim is to express everything, if possible, in terms of u, in order to simplify the integral. So, we move a factor of $2x$ in the integrand to sit next to the differential dx, and notice that the expression becomes $\int 2u\,du$, replacing x^2+1 by u and $2x\,dx$ by du. The problem simplifies to anti-differentiating $2u$ with respect to u, which is just $u^2 + C'$, where C' is a constant of integration.

$$\begin{aligned} &= \int 2u\,du \\ &= u^2 + C' \\ &= (x^2+1)^2 + C'. \end{aligned}$$

Expressing everything in terms of x, we get the final solution: $(x^2+1)^2 + C'$. So, let's compare our two solutions. On the face of it, they appear to be different. In fact, the differences in the way things are expressed create an illusion. If we expand out the second solution, we reproduce the first solution by gathering together the constants and setting $C = 1 + C'$.

$$(x^2+1)^2 + C' = x^4 + 2x^2 + 1 + C' = x^4 + 2x^2 + C$$

$$\text{where} \quad C = 1 + C'$$

When one anti-differentiates using different pathways, it's quite common to end up with expressions that look superficially different but which, in fact, represent the same function as you carefully examine the constants that appear.

Let's look at an example involving circular functions. As you know, an anti-derivative of $\cos x$ is $\sin x$. What about an anti-derivative of $\cos 2x$?

We can guess that it should look something like $\sin 2x$. If we differentiate $\sin 2x$ with respect to x using the chain rule and a substitution of $u = 2x$, we quickly get $2\cos 2x$, which is two times more than the integrand $\cos 2x$ above.

$$\begin{aligned} &\frac{d}{dx}(\sin 2x) = \frac{d}{du}(\sin u)\frac{du}{dx} \quad \text{where} \quad u = 2x \\ &= (\cos u)(2) = 2\cos u = 2\cos(2x) \end{aligned}$$

So, we compensate by dividing $\sin 2x$ by two, yielding finally the anti-derivative $\frac{\sin 2x}{2} + C$. This is called the guess and check method, which usually works well when you have to handle constant multiples of variables. For a more thorough systematic approach, we make a substitution upfront by putting $u = 2x$. So that $\frac{du}{dx} = 2$, so that $du = 2dx$, which we can rearrange as $dx = \frac{1}{2}du$. The aim is to convert everything involving x to something simpler involving u. So $\cos 2x$ becomes $\cos u$, and we can replace dx by $\frac{1}{2}du$.

Hence,

$$\int \cos(2x)\,dx = \int \cos u \left(\frac{1}{2}du\right)$$
$$= \frac{1}{2}\int \cos u\,du = \frac{1}{2}(\sin u + C')$$
$$= \frac{\sin u}{2} + \frac{C'}{2} = \frac{\sin(2x)}{2} + C$$

as before (putting $C = \frac{1}{2}C'$)

We can bring the constant $\frac{1}{2}$ out the front of a simpler integral involving u. So that we get $\frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C'$. Which becomes $\frac{\sin 2x}{2} + C$, where $C = \frac{1}{2}C'$, giving the same answer that we obtained using the guess and check method. With practice, we usually skip the two steps involving a constant C', which I included just so you can see exactly how it all fits together. Let's develop a general framework for examples like these, called the substitution rule for integration.

Recall the chain rule for differentiation, which says

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

where we think of the differentials du canceling in the numerator and denominator as though these expressions were ordinary fractions.

Suppose y is an antiderivative of $f(u)$ with respect to u,
i.e.

$$\frac{dy}{du} = f(u) \quad \text{and} \quad \int f(u)\,du = y + C \quad \text{for some constant } C,$$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f(u)\frac{du}{dx},$$

so that y is an antiderivative of $f(u)\frac{du}{dx}$ with respect to x,
so

$$\int \left(f(u)\frac{du}{dx}\right) dx = y + C$$

This can be expressed also as an indefinite integral, namely,

$$\int f(u)\frac{du}{dx}\,dx$$

all brackets together, as the integrand with respect to x, which becomes again $y + C$, and there's no harm in using the same general C as before.

So, $y + C$ appears in two places, linking the two indefinite integrals together. Thus, we get equality of these two integrals, known as the integration by substitution formula.

$$\int f(u)\,du = \int \left(f(u)\frac{du}{dx}\right) dx$$

$$\int f(u)\,du = \int f(u)\frac{du}{dx}\,dx$$

Brackets are usually deleted, which makes the formula very easy to remember because you can think of the differentials as canceling. Treating the derivative as an ordinary fraction, just as we do in remembering the chain rule. The formula can also be written using function notation for the derivative. If $u = g(x)$, the rule for some function g with input x, then we can rewrite the formula as

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx$$

often written in the other order:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

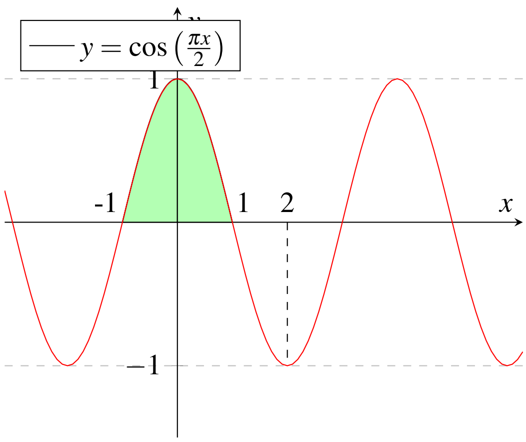
You often see the expressions written down in the opposite order, the reason being that one tries, if possible, to recognize some complicated integral as the left-hand side, which dissolves by the formula into the much simpler right-hand side.

There's also a version using definite integrals, where we add terminals, say a and b for the variable x on the left-hand side, and $g(a)$ and $g(b)$ for the corresponding variable u on the right-hand side.

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

The use of the terminals is quite delicate and takes some care, and one needs to accurately match the terminals to the variable that appears in the differential used to form the definite integral. It's common to make errors in manipulating terminals, or to overlook the need to change them as one switches between variables.

Consider the following example, which is to find the area under the curve $y = \cos\left(\frac{\pi x}{2}\right)$ for x going from -1 to 1. We'll approach the solution in two different ways.



In the first solution, we start off by working with an indefinite integral and only later introduce the terminals. We remove the terminals, and try to find the integral of $\cos\left(\frac{\pi x}{2}\right)$ with respect to x.

$$\int \cos\left(\frac{\pi x}{2}\right)\,dx = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) + C$$

In fact, we quickly get $\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)+C$, using a guess and check method by an appropriate adjustment of the constant out the front.

Guess & check:

$$\frac{d}{dx}\left(\sin\left(\frac{\pi x}{2}\right)\right)=\frac{d}{du}(\sin u)\frac{du}{dx}\quad\text{where}\quad u=\frac{\pi x}{2}$$
$$=(\cos u)\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$$

The derivative of $\sin\left(\frac{\pi x}{2}\right)$, one can quickly say, is $\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$. So, the adjustment required to get the anti-derivative above is to multiply by $\frac{2}{\pi}$.

Because the anti-derivative is a function of x, we just put the terminals back in to get the original definite integral and evaluate it in the usual way using the fundamental theorem of calculus. We substitute 1 for x, -1 for x, and take the difference.

$$\int_{-1}^{1}\cos\left(\frac{\pi x}{2}\right)\,dx=\left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1}$$
$$=\frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)\right)$$
$$=\frac{2}{\pi}\left(1-(-1)\right)=\frac{2}{\pi}(2)=\frac{4}{\pi}$$

And see that it quickly evaluates to $\frac{4}{\pi}$.

Alternatively, we can solve this using definite integrals all the way through. We use a substitution $u=\frac{\pi x}{2}$, so that $\frac{du}{dx}=\frac{\pi}{2}$, and $du=\frac{\pi}{2}dx$. Which we can rearrange to get $dx=\frac{2}{\pi}du$.

Put $u=\frac{\pi x}{2}$, so $dx=\frac{2}{\pi}du$.

We'll express everything in terms of u, including the terminals. So $x=1$, the original upper terminal, converts to $u=\frac{\pi}{2}$, and $x=-1$, the original lower terminal, converts to $u=-\frac{\pi}{2}$.

Conversion of terminals:

$$\begin{cases} x=1 & \Rightarrow u=\frac{\pi}{2} \\ x=-1 & \Rightarrow u=-\frac{\pi}{2} \end{cases}$$

We now rewrite the original definite integral, by replacing $\frac{\pi x}{2}$ in the integrand by u. The differential dx by $\frac{2}{\pi}du$. The upper terminal by $\frac{\pi}{2}$ and the lower terminal by $-\frac{\pi}{2}$.

Hence

$$\int_{-1}^{1}\cos\left(\frac{\pi x}{2}\right)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos u\left(\frac{2}{\pi}du\right)$$

The constant term $\frac{2}{\pi}$ can come out the front, and then we evaluate in the usual way by the fundamental theorem of calculus. Using $\sin u$ as an anti-derivative of $\cos u$ with respect to u. And replacing u by $\frac{\pi}{2}$, u by $-\frac{\pi}{2}$ and taking the difference. Which quickly simplifies to $\frac{4}{\pi}$, agreeing with the answer we found by the first solution. Using either method, we find the green shaded area under the curve turns out to be $\frac{4}{\pi}$.

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos u\, du$$

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos u\, du = \frac{2}{\pi}[\sin u]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{2}{\pi}(1-(-1))$$

$$= \frac{2}{\pi}(2) = \frac{4}{\pi}$$

34.1.2 Part 2

The remaining examples in this section increase substantially in terms of difficulty, and you should re-read the text often to check the details and your understanding. You might greatly benefit from consulting the accompanying examples section at the last of this chapter and ensure that you feel comfortable with the easier substitution examples.

Next, we illustrate a tricky antiderivative using a mixture of circular functions in the integrated $\int \sin^3 x \cos^2 x\, dx$. It's not obvious what substitution could simplify the calculation. With experience, you realize that dealing with the even power $\cos^2 x$ turns out to be the most problematic. Shortly, you'll see that the odd power of $\sin^3 x$ can be advantageous. So, we attempt a substitution using $u = \cos x$.

Thus, $\frac{du}{dx} = -\sin x$, and $du = -\sin x\, dx$. Now, we rewrite the integrand to exploit this du. We extract a factor of $\sin x$ from $\sin^3 x$, leaving $\sin^2 x$, which can be rewritten as $1-\cos^2 x$, using the circular identity.

$$\int \sin^3 x \cos^2 x\, dx = \int \sin x(\sin^2 x)\cos^2 x\, dx$$

$$= \int \sin x(1-\cos^2 x)\cos^2 x\, dx$$

Rearranging the integrand places $-\sin x$ next to dx, balanced with an additional minus sign at the front.

$$= -\int (1-\cos^2 x)\cos^2 x(\sin x)\, dx$$

Now, the substitution works its magic. We replace $\cos x$ wherever it appears in the integrand with u, and $-\sin x\, dx$ is replaced by du. Thus, our original integral transforms into

$$= -\int (1-u^2)u^2\, du$$

Expanding the integrand and bringing the minus sign inside yields a simple polynomial in u, which is straightforward to ant differentiate step by step. Finally, everything is expressed in terms of x, and the original problem is solved.

$$= -\int u^2 - u^4\,du = \int u^4 - u^2\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

The next example is a difficult integration of a rational function where the integrand is a fraction with a linear polynomial in the numerator and a quadratic in the denominator as shown as below:

$$\int \frac{4x+7}{x^2+2x+2}\,dx$$

Denominators usually create problems. So, let's try to rewrite the quadratic in a simpler form. Using the completing the square technique, we can express it as $(x+1)^2+1$. This simplification motivates us to make a substitution $u = x+1$, so that $\frac{du}{dx} = 1$ and $du = dx$.

The numerator can also be transformed in terms of u by adding and subtracting 1 inside brackets, rearranging to get $4u+3$. We steadily work towards expressing everything in terms of u. The integrand thus becomes $\frac{4u+3}{u^2+1}$, and the differential dx is simply replaced by du.

$$\int \frac{4x+7}{x^2+2x+2}\,dx = \int \frac{4u+3}{u^2+1}\,du$$

This represents substantial progress, allowing us to further simplify by decomposing the integrand into components and integrating them separately.

$$= \int \frac{4u}{u^2+1}\,du + 3\int \frac{du}{u^2+1}$$

$$= \int \frac{4u}{u^2+1}\,du + 3\tan^{-1} u + C$$

The first piece we'll come back to in a moment, but the second piece, with a constant 3 at the front of the integral of $\frac{du}{u^2+1}$, becomes $3\tan^{-1}(u)+C$. From an earlier section, we know that the derivative of $\tan^{-1}(x)$ is $\frac{1}{x^2+1}$, describing the curve known as the witch of Maria Agnesi. We express this as an indefinite integral involving x, which we're applying here with u instead of x.

$$= \int \frac{4u}{u^2+1}\,du + 3\tan^{-1}(x+1) + C$$

Notice that the constant of integration, when multiplied by 3, remains another constant, which we can simply rename as C.

We can go a step further and rewrite $\tan^{-1}(u)$ as $\tan^{-1}(x+1)$. We still need to figure out the first piece. Again, the denominator is the problem, so let's make a new substitution: say $v = u^2+1$, so

that $\frac{dv}{du} = 2u$. Hence, $dv = 2u\,du$. To set things up, we bring the constant 2 to the front, replace the denominator with v, and the new numerator (including the differential) with dv. This produces the much simpler expression $2\int \frac{dv}{v}$.
Hence

$$\int \frac{4u}{u^2+1}\,du = 2\int \frac{2u\,du}{u^2+1} = 2\int \frac{dv}{v}$$

Recall that the integral of $\frac{dx}{x}$ is $\ln|x| + C$, which we can apply here.

$$= 2\ln v + C$$

Notice that the constant of integration is multiplied by 2, which we again rename C. We splice this into our earlier answer.

$$= 2\ln v + 3\tan^{-1}(x+1) + C$$

There are actually two constants of integration, both of which are called C, and when added together, they form another constant, which we again call C. This kind of notation, calling everything C, is universally used and relatively harmless, but you should be aware of what you're doing when manipulating constants.

We want the antiderivative in terms of the original variable x, so we convert v back to u^2+1, and the remaining u back to $x+1$.

$$= 2\ln(u^2+1) + 3\tan^{-1}(x+1) + C$$

$$= 2\ln\left((x+1)^2+1\right) + 3\tan^{-1}(x+1) + C$$

Thus, we've solved the original problem.

We'll finish with a very difficult problem, which is to anti-differentiate the fraction

$$\frac{\sqrt{x}}{1+\sqrt{x}}$$

. There are a few different substitutions we could try, but we'll go all the way and set u equal to the entire denominator, which can be rewritten as $1+x^{1/2}$. Then, $\frac{du}{dx}$ becomes $\frac{1}{2}x^{-1/2}$, which is $\frac{1}{2\sqrt{x}}$, so that $du = \frac{dx}{2\sqrt{x}}$, which we can rearrange to $dx = 2\sqrt{x}\,du$.

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \int \frac{\sqrt{x}}{u}(2\sqrt{x})\,du$$

$$= 2\int \frac{x}{u}\,du = 2\int \frac{(u-1)^2}{u}\,du$$

Hence, the original integral can be transformed by replacing $1+\sqrt{x}$ by u, and dx by $2\sqrt{x}\,du$. Then bring the constant 2 out to the front, and simplify the numerator to x, which then becomes $(u-1)^2$. We can now expand the numerator and split the integrand into pieces.

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \int \frac{\sqrt{x}}{u}(2\sqrt{x})\,du$$

$$= 2\int \frac{x}{u}\,du = 2\int \frac{(u-1)^2}{u}\,du$$

$$= 2\int \frac{u^2-2u+1}{u}\,du = 2\int (u-2+\frac{1}{u})\,du$$

$$= 2\left(\frac{u^2}{2} - 2u + \ln|u|\right) + C'$$

We amalgamate all the constants of integration at the end, which we denote by C' instead of C, for a reason that will become apparent in a moment.

Then we carefully write everything in terms of x only.

$$= (1+\sqrt{5x})^2 - 4(1+\sqrt{5x}) + 2\ln|1+\sqrt{5x}| + C'$$

Now, we could just stop here, but notice that we can expand and simplify further in a few steps to obtain:

$$= 1+2\sqrt{5x}+x-4-4\sqrt{5x}+2\ln|1+\sqrt{5x}|+C'$$

$$= x-2\sqrt{x}+2\ln(1+\sqrt{x})-3+C'$$

$$= x-2\sqrt{x}+2\ln(1+\sqrt{x})+C$$

Note that $1+\sqrt{x}$ is positive, so we can dispense with the magnitude signs in the natural logarithm. The expression $-3+C'$ is just a constant, which we now call C.

Thus, we end up with this elegant solution to a very difficult problem.

34.1.3 Examples

Let's see some solved examples to better understand this section.

1. Find the indefinite integral $\int 6x(x^2+1)^2\,dx$ by making a substitution $u = x^2+1$.

   Solution: Put $u = x^2+1$, so that $\frac{du}{dx} = 2x$, which we may write as $du = 2x\,dx$. Putting everything in terms of u, by reorganising the integrand and constants, we get

$$\int 6x(x^2+1)^2\,dx = \int 3(x^2+1)^2(2x\,dx) = \int 3u^2\,du = u^3 + C = (x^2+1)^3 + C.$$

2. Find the indefinite integral $\int e^{-x}\,dx$.

   Solution: Put $u = -x$. Hence $\frac{du}{dx} = -1$, so that $du = -dx$, which can be written as $dx = -du$. Expressing everything in terms of u, including the differential, we have

$$\int e^{-x}\,dx = \int e^u(-du) = -\int e^u\,du = -e^u + C = -e^{-x} + C.$$

3. Find the definite integral $\int_0^{\ln 2} e^{-x}\,dx$.

   Solution: Using the indefinite integral from the previous exercise, we have

$$\int_0^{\ln 2} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\ln 2} = -e^{-\ln 2} - (-e^0) = -(\ln 2)^{-1}(-1) = -2^{-1} + 1 = \frac{1}{2}.$$

4. Find the indefinite integral $\int \left(\frac{x-1}{3}\right)^2 dx$.

   Solution: Put $u = \frac{x-1}{3}$. Then $x = 3u+1$, so that $dx = 3du$, which can be rewritten as $dx = 2du$. Expressing everything in terms of u, including the differential, we have

$$\int \left(\frac{x-1}{3}\right)^2 dx = \int u^2(2du) = u^3du = 2u^4 + C = \frac{u^2}{32} + C = \frac{(x-1)^4}{32} + C.$$

5. Find the definite integral $\int_1^2 \left(\frac{x-1}{3}\right)^2 dx$.

   Solution: Using the indefinite integral from the previous exercise, we have

$$\int_1^2 \left(\frac{x-1}{3}\right)^2 dx = \left[\frac{(x-1)^4}{32}\right]_1^2 = \frac{1}{32}((x-1)^4 - 1^1) = \frac{1}{32}\left((2-1)^4 - (1-1)^4\right) = \frac{1}{32}.$$

6. Find the definite integral $\int_{\pi/4}^{\pi/2} \sin(2x)\,dx$.

   Solution: Put $u = 2x$, so that $du = 2dx$, giving $dx = \frac{1}{2}du$. Putting everything in terms of u, we get the following indefinite integral:

$$\int \sin(2x)\,dx = \int \sin\left(\frac{1}{2}u\right)du = \frac{1}{2}\int \sin u\,du = -\frac{\cos u}{2} + C = -\cos\left(\frac{2x}{2}\right) + C = -\cos(2x) + C.$$

   Now we find the definite integral (keeping the terminals in terms of x):

$$\int_{\pi/4}^{\pi/2} \sin(2x)\,dx = \frac{1}{2}\left[-\cos(2x)\right]_{\pi/4}^{\pi/2} = \frac{1}{2}\left(-\cos(2(\pi/2)) - (-\cos(2(\pi/4)))\right) = \frac{1}{2}(-(-1) - (0)) = \frac{1}{2}.$$

7. Find the definite integral $\int_0^{\pi/4} \cos\left(2x - \frac{\pi}{2}\right) dx$.

   Solution: Put $u = 2x - \frac{\pi}{2}$, so that $du = 2dx$, giving $dx = \frac{1}{2}du$. Further, when $x = 0$ we have $u = -\frac{\pi}{2}$, and when $x = \frac{\pi}{4}$ we have $u = 0$. Hence, putting everything in terms of u, including converting the terminals, we get

$$\int_0^{\pi/4} \cos\left(2x - \frac{\pi}{2}\right) dx = \int_{-\pi/2}^{0} \cos u \, du = \frac{1}{2}\left[\sin u\right]_{-\pi/2}^{0} = \frac{1}{2}(\sin 0 - \sin(-\pi/2)) = \frac{1}{2}(0 - (-1)) = \frac{1}{2}.$$

8. Find the indefinite integral $\int e^{-2x}\, dx$.

   Solution: Put $u = -2x$, so that $\frac{du}{dx} = -2x$, so that $dx = -\frac{1}{2}du$. Putting everything in terms of u, we get

$$\int e^{-2x}\, dx = \int e^{u}(-\frac{1}{2}du) = -\frac{1}{2}\int e^{u}\, du = -\frac{1}{2}e^{u} + C = -\frac{1}{2}e^{-2x} + C.$$

9. Find the indefinite integral $\int \frac{4x-3}{x^2+1}\, dx$.

   Solution: We split the integral into two pieces:

$$\int \frac{4x-3}{x^2+1}\, dx = \int \frac{4x}{x^2+1}\, dx - \int \frac{3}{x^2+1}\, dx.$$

   For the first piece, put $u = x^2 + 1$, so that $du = 2x$, so that $dx = \frac{2}{du}$. Hence

$$\int \frac{4x}{x^2+1}\, dx = 2\int \frac{1}{u} du = 2\ln|u| + C = 2\ln|x^2+1| + C,$$

   noting that $x^2 + 1$ is always positive. Continuing, we have

$$\int \frac{4x-3}{x^2+1}\, dx = \int \frac{4x}{x^2+1}\, dx - \int \frac{3}{x^2+1}\, dx = 2\ln(x^2+1) - 3\arctan x + C.$$

10. Find the definite integral $\int_0^1 \frac{5u}{\sqrt{u^2+1}}\, du$.

   Solution: Put $u = \sqrt{t} = (t+1)^{1/2}$. Hence

$$\frac{du}{dt} = \frac{1}{2}(t+1)^{-1/2} = \frac{1}{2u}.$$

   This becomes $du = \frac{dt}{2u}$, which we may rewrite as $dx = 2udt$. Note also that $u^2 = x + 1$, so that $x = u^2 - 1$. Hence, expressing everything in terms of u, we get

$$\int \sqrt{t+1}\, dt = \int (u^2-1)\frac{2udu}{u} = 2\int (u^2-1)du = 2\left(\frac{u^3}{3} - u\right) = \frac{2}{3}u^3 - 2u + C.$$

$$\int_0^1 \frac{5u}{\sqrt{u^2+1}}\, du = 2\int (u^2-1)du = 2\left(\frac{u^3}{3} - u\right) = \frac{2}{3}u^3 - 2u + C.$$

   Hence, expressing everything in terms of u, we get

$$\int \sqrt{u^2+1}\, dx = (u^2-1)\frac{2}{5} = 2\left(\frac{u^3}{3} - u\right) = \frac{u^2-1}{5}.$$

   Now we can evaluate the definite integral:

$$\int_0^1 \frac{5u}{\sqrt{u^2+1}}\, du = 2\left[\frac{u^3}{3} - u\right]_0^1 = 2\left(\frac{(1+1)^3/2}{5} - \frac{(1-1)^3/2}{5}\right) = 2\left(\frac{2}{5}\right) = \frac{3-4}{5} = \frac{4}{13}.$$

In this section, we introduced and illustrated the method of integration by substitution, which is closely related to the Chain Rule, one of the differentiation rules we learned in the last module. This method enables one to anti-differentiate complicated functions by exploiting combinations of integrands and differentials, simplifying the integral after making appropriate substitutions and changes of variables. Some examples can be solved by a guess-and-check method. Other examples are very difficult and require considerable experience to choose the right substitution that is likely to lead to progress, and then require considerable perseverance and technical skill to see the calculation through to a successful conclusion.

Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

34.1.4 Practice Quiz

Question 1

Find the indefinite integral $\int e^{5x}\,dx$.

(a) $\frac{e^{5x+1}}{5x+1}+C$
(b) $e^{5x}+C$
(c) $\frac{e^{5x+1}}{5(5x+1)}+C$
(d) $\frac{e^{5x}}{5}+C$
(e) $5e^{5x}+C$

Question 2

Find the definite integral $\int_0^{\ln 2} e^{5x}\,dx$.

(a) $\frac{32}{5}$
(b) 31
(c) 32
(d) $\frac{\ln(32)-1}{5}$
(e) $\frac{31}{5}$

Question 3

Find the indefinite integral $\int \left(\frac{x-1}{3}\right)^2 dx$.

(a) $\left(\frac{x-1}{3}\right)^3+C$
(b) $\frac{(x-1)^3}{9}+C$
(c) $\left(\frac{x-1}{9}\right)^3+C$
(d) $\left(\frac{(x-1)}{3}\right)^3+C$
(e) $(x-1)^3+C$

Question 4

Find the definite integral $\int_0^1 \left(\frac{x-1}{3}\right)^2 dx$.

(a) $-\frac{1}{3}$
(b) $\frac{1}{3}$
(c) $\frac{1}{9}$
(d) $\frac{1}{27}$
(e) $-\frac{1}{27}$

Question 5

Find the indefinite integral $\int \sin\left(2x+\frac{\pi}{2}\right)\,dx$.

(a) $2\cos\left(2x+\frac{\pi}{2}\right)+C$
(b) $-\cos\left(2x+\frac{\pi}{2}\right)+C$
(c) $\cos\left(2x+\frac{\pi}{2}\right)+C$
(d) $-\frac{\cos\left(2x+\frac{\pi}{2}\right)}{2}+C$
(e) $\frac{\cos\left(2x+\frac{\pi}{2}\right)}{2}+C$

Question 6

Find the definite integral $\int_0^{\pi/4} \sin\left(2x+\frac{\pi}{2}\right)\,dx$.

(a) $\frac{1}{2}$
(b) $-\frac{1}{2}$
(c) -1
(d) 1
(e) 0

Question 7

Find the indefinite integral $\int 2xe^{x^2}\,dx$.

(a) $e^{x^2}+C$
(b) $\frac{e^{x^2+1}}{x^2+1}+C$
(c) $2e^{x^2}+C$
(d) $\frac{e^{x^2}}{2}+C$
(e) $2xe^{x^2}+C$

Question 8

Find the indefinite integral $\int \sin^2 x\cos x\,dx$.

(a) $\sin^3 x+C$
(b) $\frac{\sin^3 x}{3}+C$
(c) $-\frac{\sin^3 x}{9}+C$
(d) $\frac{\sin^3 x}{9}+C$
(e) $-\frac{\sin^3 x}{3}+C$

Question 9

Find the indefinite integral $\int \frac{2x+1}{x^2+1}\,dx$.

(a) $\ln(x^2+1)+\tan^{-1}x+C$
(b) $\frac{\ln(x^2+1)}{2}+\tan^{-1}x+C$
(c) $x^2+\tan^{-1}x+C$
(d) $\ln(x^2+1)+C$
(e) $2\ln(x^2+1)-\tan^{-1}x+C$

Question 10

Find the indefinite integral

$$\int \frac{x\sqrt{x-1}}{2}\,dx.$$

(a) $\frac{(x-1)^{\frac{3}{2}}}{3}+C$
(b) $\frac{(x-1)^{\frac{3}{2}}}{3}-\frac{(x-1)^{\frac{5}{2}}}{5}+C$

(c) $\frac{(x-1)^{\frac{5}{2}}}{5} - \frac{(x-1)^{\frac{3}{2}}}{3} + C$

(d) $\frac{(x-1)^{\frac{5}{2}}}{5} + C$

Answers

The answers will be revealed at the end of the module.

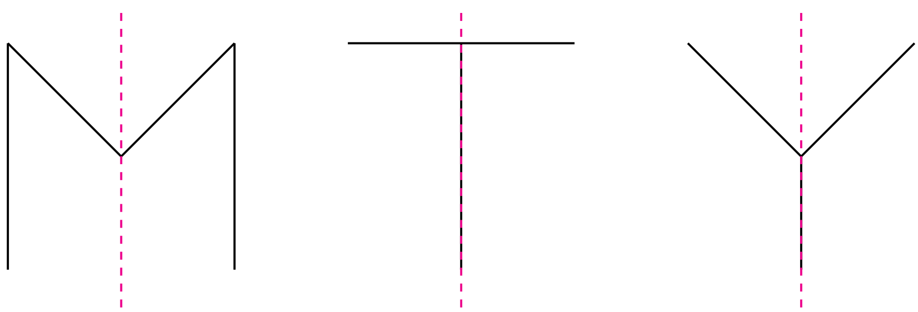
35. Symmetry and the Logistic Function

35.1 Odd and Even functions

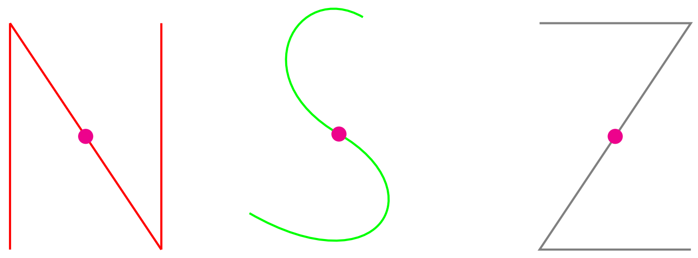
35.1.1 Part 1

In this section, we'll discuss even functions, whose curves have reflectional symmetry in the y-axis, and odd functions, whose curves have a 180-degree rotational symmetry about the origin. If one forms definite integrals over intervals in the real line that are symmetric about the origin, then there are certain simplifications. In particular, the area under an odd function over such an interval evaluates automatically to zero. The notions of even and odd capture precisely certain types of symmetries.

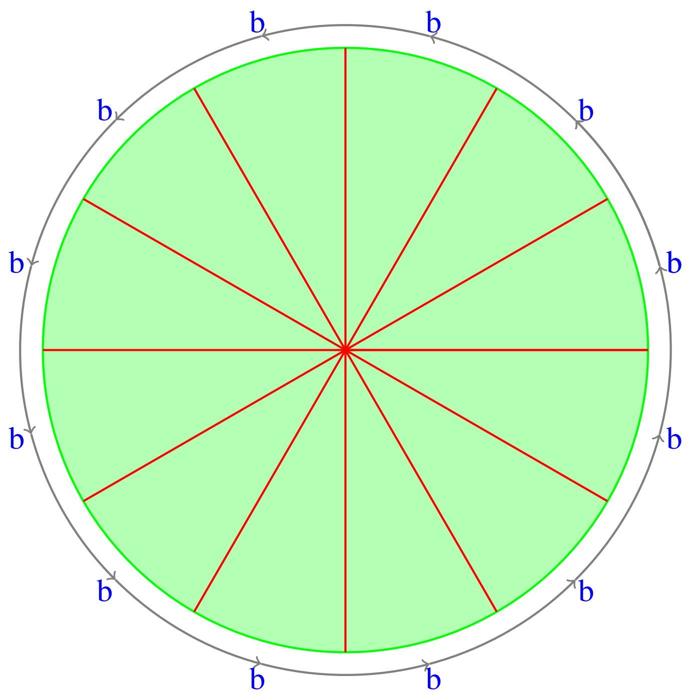
For example, the letters M, T, and Y have reflectional symmetry about a vertical line drawn down the middle.



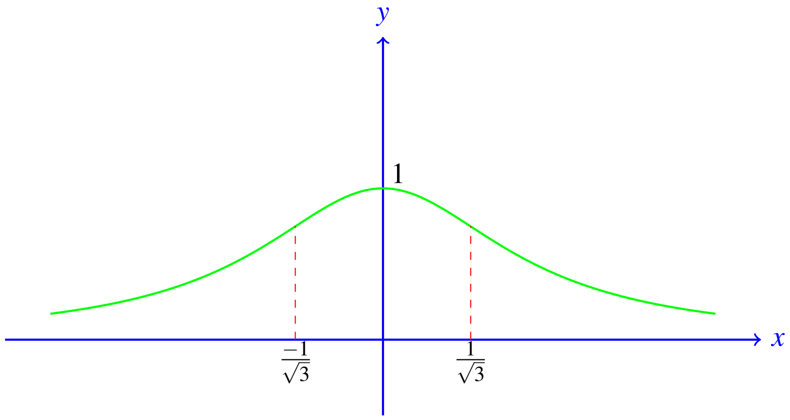
By contrast, the letters N, S, and Z have a 180-degree rotational symmetry about their midpoints.



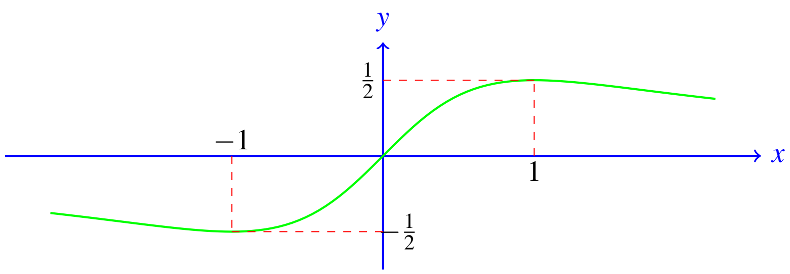
From the point of view of reflectional and rotational symmetry, the most perfect figure in the plane is the circle, which has an infinite supply of such symmetries. The use of an arbitrarily large number of symmetries in the circle was implicit in the argument that the Greeks used to find the formula for the area of a circle.



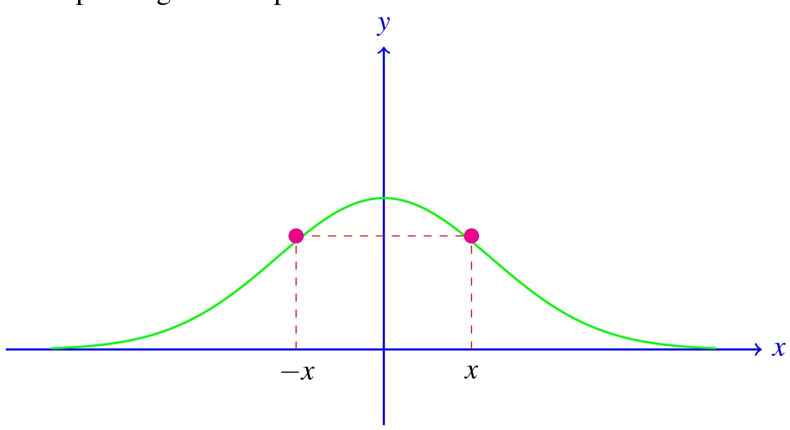
In an earlier section, we analyzed the curve known as the Witch of Maria Agnesi. Let $g(x) = \frac{1}{x^2+1}$. Its graph has perfect reflectional symmetry about the vertical y-axis. We used curve sketching techniques to locate the two points of reflection, and they become images of each other by reflecting in the y-axis. If we substitute $-x$ into the rule for the function g, then we quickly reproduce the original rule for g. This property, where the outcome for the rule remains the same after substituting $-x$ for x, corresponds to the curve having reflectional symmetry in the y-axis.



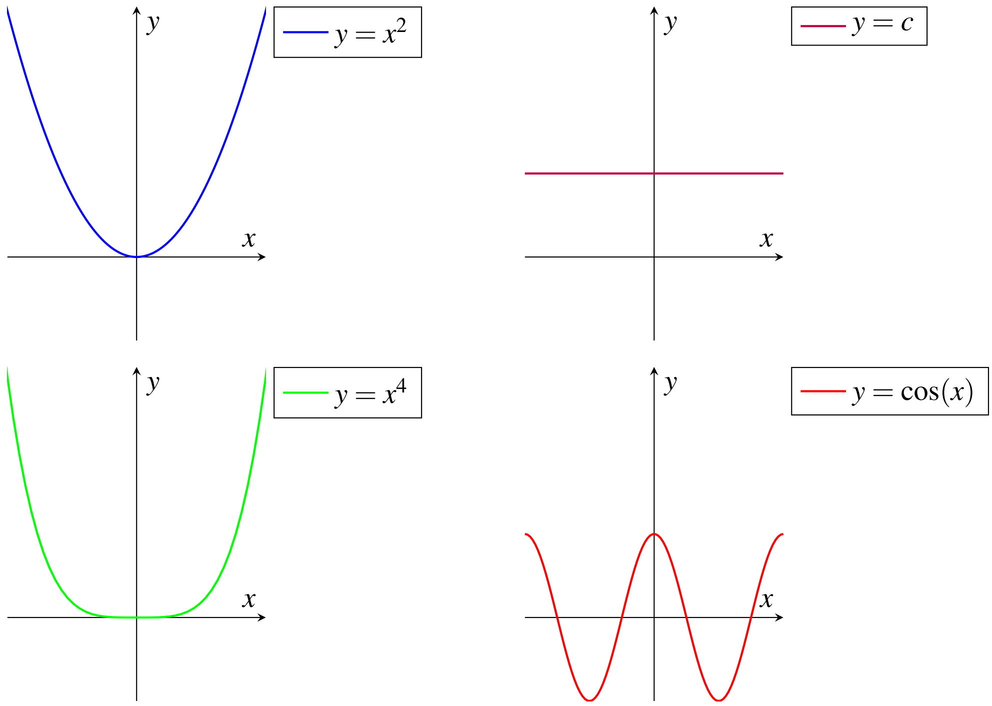
There's another curve closely related to the Witch known as Newton's Serpentine. Let $h(x) = \frac{x}{x^2+1}$. It's a good exercise in curve sketching to verify that the graph of h looks like this and has a 180-degree rotational symmetry about the origin. For example, the turning points $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$ rotate into each other. Notice that if you substitute $-x$ into the rule for the function h, you end up with $-h(x)$. This algebraic relationship corresponds to a 180-degree rotational symmetry.



To define even functions in general, consider a function $y = f(x)$. We say that f is even if the condition $f(-x) = f(x)$ holds for all inputs x. This implicitly assumes that if x is in the domain, then $-x$ is also in the domain. Thus, if you have the graph of an even function, for example, a bell-shaped curve which is symmetric about the y-axis, then if you move up to the curve from some input x and reflect the points on the curve in the y-axis, you land again on the curve at a point corresponding to the input $-x$.



Both $f(x)$ and $f(-x)$ are equal. Thus in general, this algebraic criterion corresponds to reflectional symmetry in the y-axis. Simple examples include the quadratic function $y = x^2$, which gives rise to a parabola, the constant function $y = c$, the curve $y = x^4$, which is bowl-shaped like a parabola but much steeper, and the cosine curve.



Any polynomial function or rational function involving constant multiples with even powers of x only will be even.

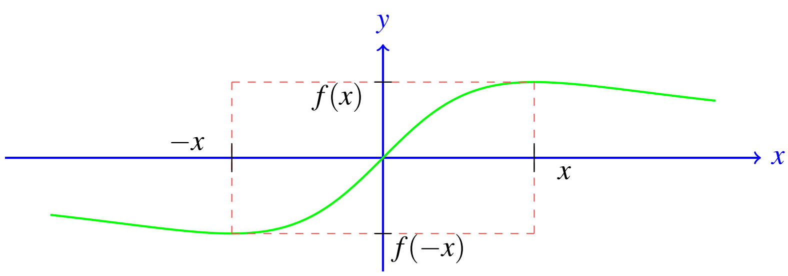
$$e.g: \qquad y = x^2 + x^4, \qquad y = x^2 + 6,$$

$$y = 1 + 3x^2 + 5x^4 + 7x^6 + 9x^8,$$

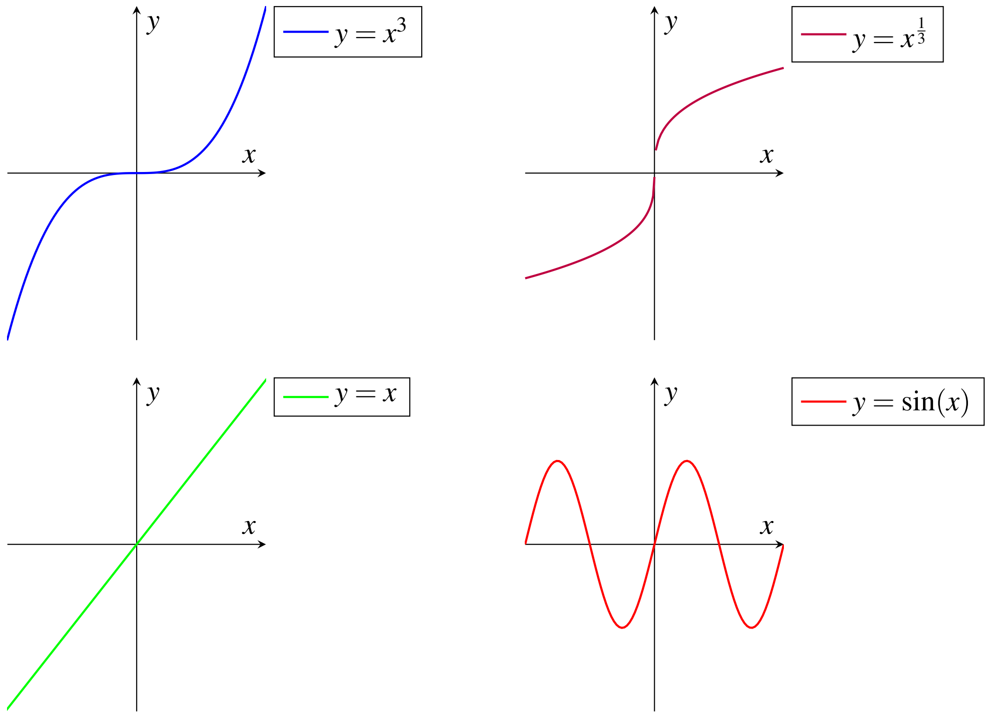
$$y = \frac{4 - 3x^2 - x^6}{1 + x^2 + x^4 + x^6 + x^8},$$

No matter how complicated, we don't even need to be able to visualize the curves to know for sure they all have reflectional symmetry in the y-axis. Note the constant functions are multiples of x to the zero and zero is even. In all cases, even powers cause the minus sign to go away leaving the final value of the function unaffected. The connection with even powers of x is the reason for the terminology.

To define odd functions in general, consider a function $y = f(x)$. We say that f is odd if the condition $f(-x) = -f(x)$ holds for all inputs x. Thus, if you have the graph of an odd function, like we saw before with Newton's Serpentine which has rotational symmetry about the origin, then if you move up to the curve from some input x and rotate the point on the curve by 180 degrees about the origin, then you land again on the curve at a point corresponding to input $-x$.



But now with a value $f(-x)$ being the negative of $f(x)$. Thus in general, this algebraic criterion corresponds to a 180 degree rotational symmetry. Simple examples with this rotational symmetry are the cubic function $y = x^3$, the cube root function, even simpler, the identity function $y = x$ and the sine curve.



Any polynomial function involving constant multiples of odd powers of x only will be odd. This is also the case for any rational function where the numerator is even and the denominator is odd, or vice versa, where the numerator is odd and the denominator is even.

e.g. $\quad y = x^3$, $y = x^5 - x$

$$y = x + 3x^3 + 5x^5 + 7x^7 + 9x^9$$

$$y = \frac{2x - 3x^3 + x^7}{1 + x^2 + x^4 + x^6 + x^8}$$

No matter how complicated any of these might be, we don't even need to be able to visualize the curves. You know for sure they all have a 180-degree rotational symmetry about the origin. The explanation for this claim about rational functions is a bit technical. If you look at any particular odd power of x, say x^{2k+1}, where $2k+1$ is a typical odd integer, then you see how one minus sign peels off and comes out the front while all the other minus signs disappear within the even power.

$$(-x)^{2k+1} = (-x)(-x)^{2k} = -(x)(x)^{2k} = -x^{2k+1}$$

So, you end up with a $-x^{2k+1}$. The connection between the algebraic criterion and odd powers of x is the reason for the terminology.

What connection, if any exists, between odd and even powers of x and the circular functions? $y = \sin x$, which is odd, and $y = \cos x$, which is even. There are no powers of x in sight, let alone even powers. There is a connection, but it's well hidden. I'll give you a sneak preview of these series expansions, part of an important and extensive topic in advanced calculus.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

We can think of $\sin x$ as like a polynomial except that there are infinitely many terms that go on forever. This infinite polynomial starts off with x, an odd power, and you subtract $\frac{x^3}{3!}$, the next odd power divided by 3! (which is an abbreviation for $3 \times 2 \times 1 = 6$), then you add $\frac{x^5}{5!}$, and so on. This pattern alternates between plus and minus, always moving towards the next odd power of x and dividing by the factorial of the exponent.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

It's an amazing fact that this polynomial accurately describes $\sin x$ for any real number x. To make sense of "going on forever," one has to use limits in the sense that we've been talking about in this course. If you truncate the expression at any particular term, then you'll get an approximation of $\sin x$. Amazingly, the theory enables one to predict in advance how good the approximation will be for any particular real number x.

Notice that for $\sin x$, only odd powers of x are used, connecting to the fact that $y = \sin x$ is an odd function.

There's a similar expansion for $\cos x$ as a polynomial that goes on forever. But this time, it starts with the constant 1, the smallest even power of x, and alternates between subtracting and adding successive even powers of x, dividing through by the factorial of the exponent.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

The fact that this expression only involves even powers of x is connected to the fact that $y = \cos x$ is an even function.

This is a very beautiful and elegant way of representing the circular functions. Using them, you can see why the derivative of $\sin x$ ought to be $\cos x$. If you differentiate the infinite polynomial for $\sin x$ term by term, you can check that you get the infinite polynomial for $\cos x$. Similarly, if you differentiate the infinite polynomial for $\cos x$ term by term, you get the infinite polynomial for $-\sin x$.

The derivative of $\sin x$ is $\cos x$ and the derivative of $\cos x$ is $-\sin x$, taking an odd function to an even function and an even function to an odd function.

As you also know, the derivative of x^2 is $2x$ and the derivative of $2x$ is 2. This time, taking even to odd and odd to even. These are not coincidences because of the connection with even and odd powers of x.

The derivative of x^n is $n \cdot x^{n-1}$. Subtracting one from the exponent converts an odd power into an even power and an even power into an odd power.

We have the following general facts: the derivative of an odd function is always even, and the derivative of an even function is always odd. It's a good exercise to verify both of these facts directly from the limit definition of the derivative. It's also a good exercise to interpret these results geometrically, in terms of the effects on slopes and tangent lines caused by reflecting on the y-axis or rotating a curve 180 degrees about the origin.

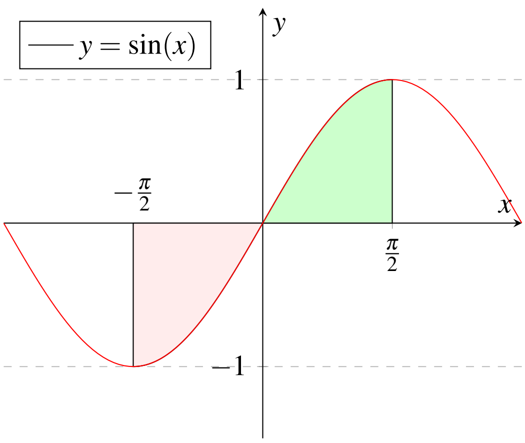
35.1.2 Part 2

We're now discussing properties of definite integrals when an integrand is an even or an odd function, and the interval over which we're integrating is symmetric about the origin. That is, the variable ranges between plus and minus some real number.

Let's integrate the odd function $\sin x$ for x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$$\int_{-\pi/2}^{\pi/2} \sin x \, dx$$

Here's the graph, and the areas under the curve have been colored green for positive and pink for negative. Clearly, the green and pink areas have the same magnitude as the regions exactly coincide under a 180-degree rotation about the origin; hence, the overall area must be zero, as the positive and negative areas cancel out.



We can check the answer directly using the Fundamental Theorem of Calculus. An antiderivative is $\cos x$, which we evaluate between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and see that the answer quickly becomes zero, confirming the result

.

$$\int_{-\pi/2}^{\pi/2} \sin x \, dx = [-\cos x]_{-\pi/2}^{\pi/2}$$

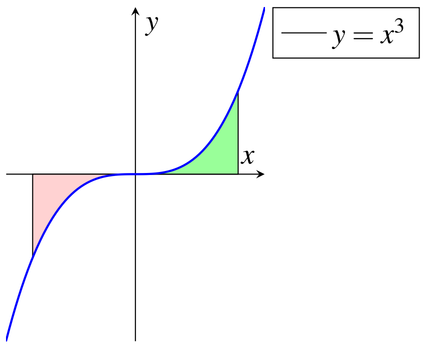
$$= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos\left(-\frac{\pi}{2}\right)\right)$$

$$= -0 - (-0)$$

Consider another odd function, say $y = x^3$, and integrate it between -1 and 1.

$$\int_{-1}^{1} x^3\,dx$$

Here's a sketch of the curve, and again the positive area is colored green and the negative area pink.



Again, the regions match exactly under a 180-degree rotation. So, the positive and negative areas cancel out, and the definite integral is zero. Again, we can check the answer using the Fundamental Theorem, quickly confirming that zero is the overall area.

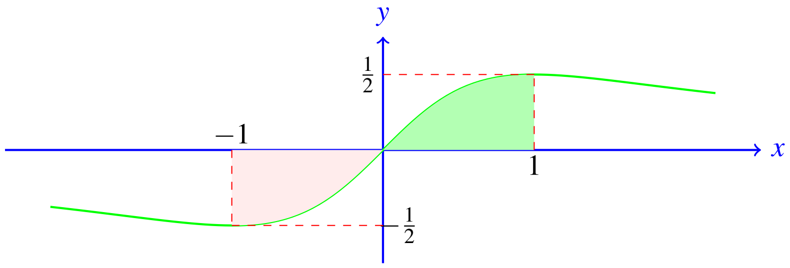
$$\int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1^4}{4} - \frac{(-1)^4}{4}$$

$$= \frac{1}{4} - \frac{1}{4} = 0$$

If we integrate Newton's serpentine curve mentioned earlier, also between ± 1, then again, the positive and negative areas cancel out to give zero.

$$\int_{-1}^{1} \frac{x}{x^2+1}\,dx$$



These examples are special cases of a general phenomenon: no matter how complicated or simple a function might be, if we integrate between plus and minus a, where a is a positive number, then the positive and negative areas exactly cancel out regardless of where they appear. Any positive area to the right cancels out with a negative area to the left, obtained by rotating the curve 180 degrees about the origin. Similarly, any negative area to the right cancels out with a positive area to the left.

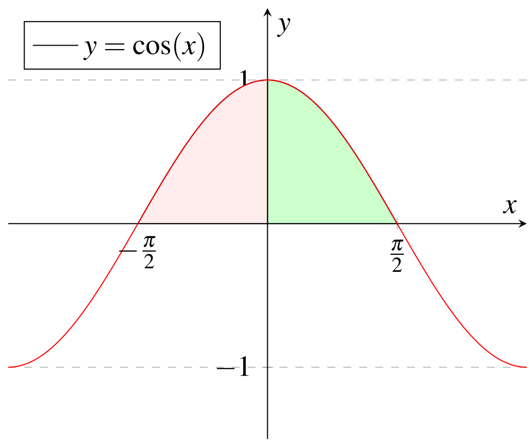
General Fact: If $y = f(x)$ is odd, defined over an interval $[-a,a]$ for $a > 0$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

If the function is odd, you always get a net zero area. What happens if we integrate an even function over a symmetric interval? Consider the even function $\cos x$ integrated between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx$$

In this case, the curve sits above the x-axis so there's no possibility of cancellation.



However, the area to the left of the y-axis is an exact mirror image of the area to the right because the cosine function is even. Taking the mirror image across the y-axis doesn't change the sign of the areas; the entire area is just twice the area on the right. That is, 2 times the definite integral from 0 to $\frac{\pi}{2}$.

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_{0}^{\pi/2} \cos x\,dx$$

$$= 2\left[\sin x\right]_0^{\pi/2}$$

$$= 2(\sin(\pi/2) - \sin 0)$$

$$= 2(1-0)$$

$$= 2$$

We found this using the Fundamental Theorem of Calculus by evaluating the antiderivative $\sin x$ between 0 and $\frac{\pi}{2}$, and we get a total area of 2. Thus, the area under the curve splits exactly into two equal areas, one above and one below the x-axis.

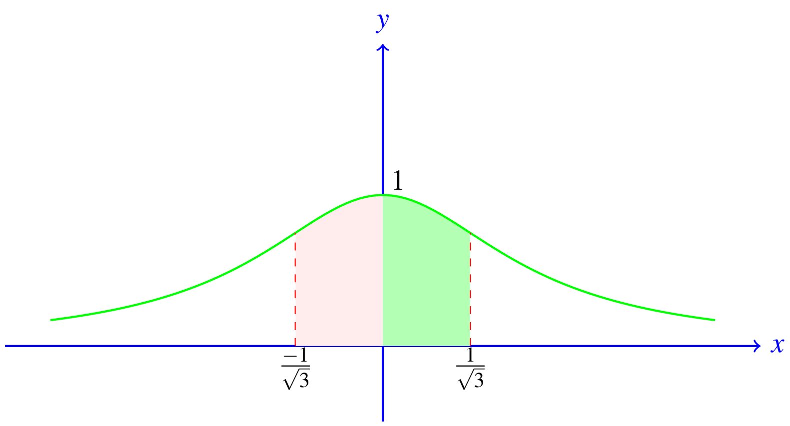
This example is a special case of a general phenomenon: to integrate an even function over an interval between plus and minus a, for some real number a, we integrate from 0 to a and double our answer. This makes perfectly good sense, as any particular region under the curve to the right of 0 (whether above or below the x-axis) is matched exactly by a region obtained by reflecting across the y-axis, since the function is even, and this mirroring effect doesn't alter the sign of the area.

General Fact: If $y = f(x)$ is even, defined over an interval $[-a,a]$ for $a > 0$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

When the integrand is odd, the area disappears, but when the integrand is even it suffices just to integrate over the positive interval and double your answer. This fact is quite useful and often creates some economies in evaluating areas when even the functions are involved. For example, let's find the area under the witch of Maria Agnesi of the interval between $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{1}{x^2+1}\,dx$$



Because the function is even, it's enough to integrate from 0 to $\frac{1}{\sqrt{3}}$ and then multiply by two. An antiderivative of one over x^2+1, you might remember, is the inverse tan function, so we evaluate this between 0 and $\frac{1}{\sqrt{3}}$. Giving $\frac{\pi}{6}$, which we multiply by two, gives a total area to be $\frac{\pi}{3}$.

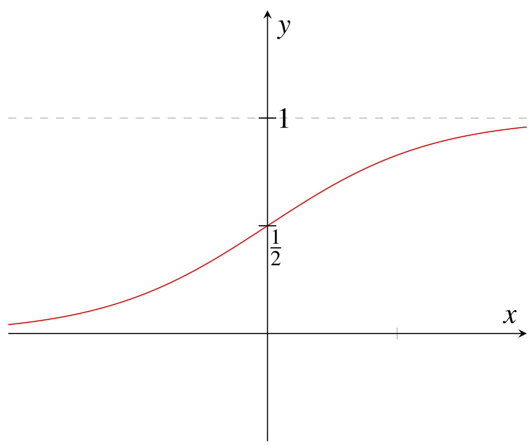
$$
\begin{aligned}
\text{Area} &= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{dx}{x^2+1} \\
&= 2\int_{0}^{\frac{1}{\sqrt{3}}} \frac{dx}{x^2+1} \\
&= 2\left[\tan^{-1} x\right]_0^{\frac{1}{\sqrt{3}}} \\
&= 2\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1} 0\right) \\
&= 2\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3}
\end{aligned}
$$

We finish with a very difficult example which uses just about every trick in the book and takes advantage of the knowledge we've built up regarding odd and even functions. This is not a routine example, and it's included more for interest and for those of you that enjoy a challenge. So, please don't worry if you don't follow all the details, and re-read it if you didn't get it. We want to find this definite integral.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{-x}}\,dx$$

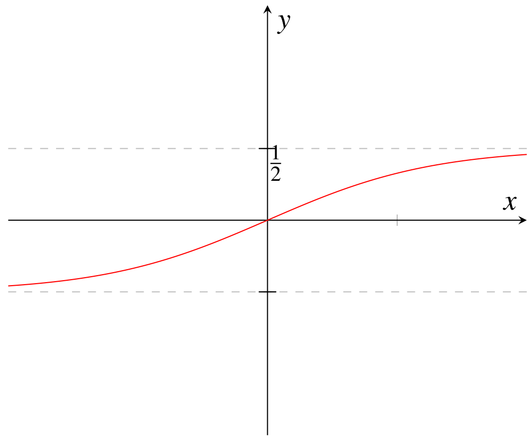
Notice, the interval is between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$, which is a clue that we should be exploiting even and odd functions. The integrand, however, is very complicated. If only the strange denominator were not there, then the integrand would become the nice even function $\cos x$, which we looked at before and found the total area to be two. Maybe we can exploit this if only we could handle the strange denominator.

Well, if you separate out the denominator as a function in its own right, ignoring the $\cos x$, you get $y = \frac{1}{1+e^{-x}}$. It's actually not all that strange. It's an example of a curve we discussed in the last section called the logistic curve. If you apply your techniques of curve sketching, you can see it looks something like this:



Where it crosses the y-axis at $y = \frac{1}{2}$ is an inflection, and the curve is sandwiched in between two horizontal asymptotes. The curve looks like a stretched out S-shape, almost like an integral sign falling over, an example of a sigmoid curve. It has full 180-degree rotational symmetry about the inflection point. You can spin it around 180 degrees about the y-intercept, and the curve will reproduce itself.

Now, 180-degree rotational symmetry characterizes the property of being odd, provided the point of rotation is the origin. This gives us an idea. If we can shift the curve down a half of a unit, then the point of rotation, the y-intercept becomes the origin and we'll have an odd function. This phenomenon is illustrated here.



Just by subtracting $\frac{1}{2}$ from the rule for the function in the top diagram, the function in the bottom diagram is odd.

$$\begin{aligned} y &= \frac{1}{1+e^{-x}} - \frac{1}{2} \\ &= \frac{2-(1+e^{-x})}{2(1+e^{-x})} \\ &= 2\left(\frac{1-e^{-x}}{1+e^{-x}}\right) \end{aligned}$$

You can check the algebra if you wish to confirm that you get a new rule which is indeed odd. We don't actually need to know any details about the rule for this new function; we only need to know it's odd.

Here is our original problem. We write the integrand as $\cos x$ times $\frac{1}{1+e^{-x}}$. Then, do a trick you've seen many times before in different contexts. We subtract and add $\frac{1}{2}$ to the second factor in the integrand, which doesn't change the overall value. Then, we separate the interval into two pieces. The first piece has an integrand $\cos x$ times a complicated bit. The second piece has an integrand that is just $\frac{\cos x}{2}$.

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx &= \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{1+e^{-x}}\right)dx \\ &= \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{1+e^{-x}} - \frac{1}{2} + \frac{1}{2}\right)dx \\ &= \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{1+e^{-x}} - \frac{1}{2}\right)dx + \int_{-\pi/2}^{\pi/2} \frac{\cos x}{2}\,dx. \end{aligned}$$

Now, in the first piece, we have two factors, $\cos x$ is an even function. The second piece is just the rule for that odd function we discussed a little while ago by shifting a sigmoid curve down half a unit. Multiplying the rules for an even function by an odd function, in fact, gives an odd function. So, the entire integrand in the first piece is odd. So, the definite integral must be zero as we're integrating between plus and minus $\frac{\pi}{2}$. Hence, the first piece is zero, and in the second piece we can bring a half of the front of that familiar integral involving just $\cos x$. So, the answer becomes $\frac{1}{2}\cdot 2$, which is 1.

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx &= \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{1+e^{-x}}\right)dx \\
&= \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{1+e^{-x}} - \frac{1}{2} + \frac{1}{2}\right)dx \\
&= \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{1+e^{-x}} - \frac{1}{2}\right)dx + \int_{-\pi/2}^{\pi/2} \frac{\cos x}{2}\,dx \\
&= 0 + \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos x\,dx \\
&= \frac{1}{2}(2) = 1.
\end{aligned}$$

Thus, the final answer to our original problem is the very pleasant number 1.

35.1.3 Examples and derivations

1. Taking the sum or difference of an even function with an odd function may produce a function that is neither odd nor even. For example, $y = z$ is odd and $y = 1 + z^2$ is even. Let f and g be functions with the rules

   $$f(x) = 1 + x^2 \quad \text{and} \quad g(x) = 1 - z + z^2.$$

   Then f is neither even nor odd. For example, $f(1) = 3$, whilst $f(-1) = 1$, which does not equal $f(1)$ or $-f(1)$. Similarly, g is neither even nor odd. For example, $g(1) = 1$, whilst $g(-1) = 3$, which does not equal $g(1)$ or $-g(1)$.

2. Let f be the function with rule $y = f(x) = x^n$, where n is an integer. If n is even, say $n = 2k$, for some integer k, then

   $$f(-x) = (-x)^{2k} = (-1)^{2k}x^{2k} = (-1)^{2k}x^{2k} = (1)^{2k}x^{2k} = x^{2k} = f(x),$$

   which shows that f is even. For example, the following functions are even:

   $$y = 1, \quad y = x^4, \quad y = x^8, \quad y = x^{12}, \quad y = x^2, \quad y = \frac{1}{x^{10}}.$$

   On the other hand, if n is odd, say $n = 2k+1$, for some integer k, then

   $$f(-x) = (-x)^{2k+1} = (-1)^{2k+1}x^{2k+1} = -x^{2k+1} = -f(x),$$

   which shows that f is odd. For example, the following functions are odd:

   $$y = x, \quad y = x^3, \quad y = x^7, \quad y = x^5, \quad y = x^9, \quad y = \frac{1}{x^{11}}.$$

3. From the previous example, and the arithmetic of even and odd functions, it follows that any polynomial or rational function involving only even powers of z is even. For example, the following are even:

$$y = 1 + 2z^2 + 3z^4 + 7z^6 \quad \text{and} \quad y = \frac{3z - 7z^3 + 5z^4 - z^{10}}{1 - z^2 + 4z^4 - z^{12}}.$$

Note that each of the constants (even if odd integers) represent even functions!
Any polynomial involving only odd powers of z will be odd. For example, the following function is odd:

$$y = 2z + 3z^3 - 6z^7 + z^{11}.$$

The situation involving mixtures of odd and even, when forming rational functions, is more delicate. If $y = \frac{f(z)}{g(z)}$ is a rational function, say, a ratio of polynomials $p(z)$ and $q(z)$, then y is even if both $p(z)$ and $q(z)$ are odd, but odd if only one of $p(z)$ and $q(z)$ is even and the other is odd. For example, the following function is even:

$$y = \frac{x - x^3}{1 + x^4}$$

and the following functions are odd:

$$y = \frac{x}{1 + x^2 + x^3} \quad \text{and} \quad y = \frac{1 + x^2 + x^5}{1 + x + x^3}.$$

4. Verify algebraically that if f and g are odd with the same domain, then h is even, where h has the rule $h(x) = f(x)g(x)$.

Solution: For x in the domain of h, we have, using the fact that both f and g are odd,

$$h(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = h(x).$$

5. Verify algebraically that if f is even and g is odd with the same domain, then h is odd, where h has the rule $h(x) = f(x)g(x)$.

Solution: For x in the domain of h, we have, using the fact that f is even and g is odd,

$$h(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -h(x).$$

In this section, we discussed even functions, whose graphs have reflectional symmetry in the y-axis, and odd functions, which graphs have 180-degree rotational symmetry about the origin. The terminology arises from functions built using even and odd powers of the input variable x, and leads to surprising connections between functions and generalizations of polynomials. We saw that when one builds definite integrals over intervals symmetric about the origin, then the area under an odd function evaluates automatically to zero, and the area under an even function is twice the area of the positive interval.

Please re-read if you didn't get it and when you're ready, please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

35.1.4 Practice Quiz

Question 1

Which one of the following functions is even?

(a) $y = x^3$
(b) $y = x^{1/3}$
(c) $y = x^2$
(d) $y = x$
(e) $y = \sin x$

Question 2

Which one of the following functions is odd?

(a) $y = e^x$
(b) $y = \cos x$
(c) $y = x^{-1}$
(d) $y = x^{1/2}$
(e) $y = e^{-x}$

Question 3

Which one of the following functions is even?

(a) $y = \frac{1+x}{x^2+1}$
(b) $y = 1 - \sin x$
(c) $y = \frac{-1}{x^2+1}$
(d) $y = \frac{x}{x^2+1}$
(e) $y = 1 + \sin x$

Question 4

Which one of the following functions is odd?

(a) $y = \sin x + \cos x$
(b) $y = \tan x$
(c) $y = 1 - \cos x$
(d) $y = \sin x - \cos x$
(e) $y = 1 + \cos x$

Question 5

Which one of the following functions is even?

(a) $y = xe^{-x}$
(b) $y = 2x + 4$
(c) $y = \frac{e^x - e^{-x}}{2}$
(d) $y = \frac{e^x + e^{-x}}{2}$
(e) $y = xe^x$

Question 6

Which one of the following functions is odd?

(a) $y = \frac{e^x + e^{-x}}{2}$
(b) $y = xe^x$
(c) $y = xe^{-x}$
(d) $y = \frac{e^x - e^{-x}}{2}$
(e) $y = 3x + 5$

Question 7

Which one of the following definite integrals evaluates to zero?

(a) $\int_{-1}^{1} \frac{x-1}{x^2+1}\,dx$
(b) $\int_{-1}^{1} \frac{x^3}{x^2+1}\,dx$
(c) $\int_{-1}^{1} \frac{x^2}{x^2+1}\,dx$
(d) $\int_{-1}^{1} (x-x^2+x^3)\,dx$
(e) $\int_{-1}^{1} (1+x+x^2+x^3)\,dx$

Question 8

Which one of the following definite integrals evaluates to zero?

(a) $\int_{-\pi/4}^{\pi/4} x\sin x\,dx$
(b) $\int_{-\pi/4}^{\pi/4} x^2\sin x\,dx$
(c) $\int_{-\pi/4}^{\pi/4} \cos x\,dx$
(d) $\int_{-\pi/4}^{\pi/4} x\sin x\cos x\,dx$
(e) $\int_{-\pi/4}^{\pi/4} \sin^2 x\cos x\,dx$

Question 9

Find the definite integral

$$\int_{-2}^{2} \left(1+x+x^3+x^5+x^7+x^9+x^{11}+x^{13}+x^{15}\right)\,dx.$$

(a) 4
(b) $\frac{1,123}{45,045}$
(c) $\frac{156}{165}$
(d) 0
(e) 2

Question 10

Find the definite integral

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^3+1}{x^2+1}\,dx.$$

(a) 0
(b) 2
(c) $\frac{2\pi}{3}$
(d) $\frac{\pi}{6}$
(e) $\frac{\pi}{3}$

Answers

The answers will be revealed at the end of the module.

35.2 Logistic function

35.2.1 Part 1

In this section, we discuss the logistic function used to model population dynamics where an inhibition factor is included in the expression describing the rate of growth. The logistic model includes a ceiling on the size of the population. The resulting curve is increasing, has a sigmoid shape, and sits between two horizontal asymptotes. There's a point of inflection halfway between, about which the curve has 180-degree rotational symmetry.

We begin with a simpler exponential growth model. Suppose we have a population of size $x(t)$, which is a function of time t and want to understand its behavior. This model assumes that the derivative of x which is $\frac{dx}{dt} = kx$, with k as a positive constant. One can imagine without any limit of resources that the more there is of the colony, the faster it should grow. So the growth rate should be proportional to the size of the population.

$$\frac{dx}{dt} = kx \qquad \text{for some constant } k > 0$$

This is an example of a differential equation. Note that though x appears on the right-hand side, x is being used as a dependent variable. The independent variable is t, not x, and anti-differentiating the right-hand side with respect to x will not make any progress.

It needs some careful preparation to unravel this equation. You first multiply both sides by the differential dt, and dividing through by x gives

$$\frac{dx}{x} = k\,dt.$$

We now apply integral symbols to both sides to get equality of antiderivatives of the respective integrands.

$$\int \frac{dx}{x} = \int k\,dt$$

The differential on the left-hand side involves x, whilst the differential on the right-hand side involves t. An antiderivative of $\frac{1}{x}$ with respect to x is $\ln x$, and an antiderivative of a constant k with respect to t is just kt. We amalgamate constants of integration together with a single $+C$ on the right.

$$\int \frac{1}{x}\,dx = \int k\,dt$$

$$\ln x = kt + C$$

We can raise Euler's number e to the power of each side.

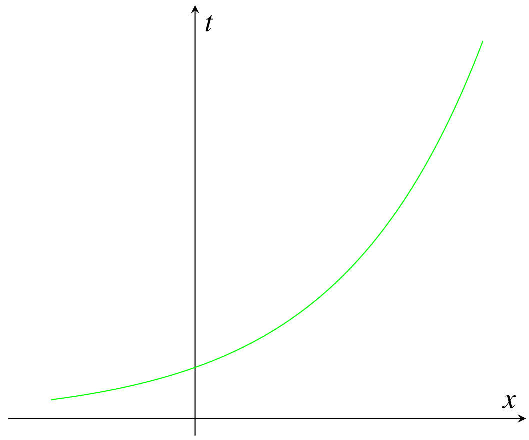
$$x = e^{\ln x} = e^{kt+C} = e^C \cdot e^{kt}$$

On the left we get x back again, and on the right we get e^{kt+C}, which we can expand. Thus,

$$x = Ae^{kt}$$

for some constant A by putting $A = e^C$.

Notice that we've reproduced the familiar exponential growth function whose graph is the exponential curve.



Note that x labels the vertical axis and t the horizontal axis. Exponential growth is unlimited. The x values get arbitrarily large as t increases.

$$\lim_{t\to\infty} x(t) = \infty$$

Of course, this isn't a realistic model when resources are limited. We now modify the previous model to take into account constraints placed on populations by the environment and limited resources.

$$\frac{dx}{dt} = kx \qquad \text{for some constant } k > 0$$

We adjust the right-hand side of the equation $\frac{dx}{dt} = kx$ by introducing a factor $1 - \frac{x}{M}$, for some new constant M. Think of M as the largest or maximal sustainable population.

$$\frac{dx}{dt} = kx(1 - \frac{x}{M}) \qquad \text{for some constant } k, M > 0$$

The factor kx remains untouched, and contributes towards the growth of the population. The factor $1 - \frac{x}{M}$ is carefully designed to model inhibition. If x is small, then this factor is close to 1, and the left-hand side is almost exactly kx, as in the original exponential model. However, if x is large, getting close to M, the factor $1 - \frac{x}{M}$ gets close to 0, causing the overall growth rate, $\frac{dx}{dt}$, to slow down to almost nothing.

This modification is called the logistic growth model, widely used in population dynamics. We get what's called a logistic equation whose solutions are called logistic functions.

Let's now try to unravel the logistic equation. We first multiply through both sides by the differential, dt. Keep the constant, k, on the right-hand side, next to dt, but move the factors x, and $1 - \frac{x}{M}$ to the other side by division. We separate the variables by putting x's to the left and t's to the right, including their differentials. We assume implicitly throughout that the population x always lies between 0 and M. We add integral signs to both sides to create functions by anti-differentiation.

$$
\begin{aligned}
\frac{dx}{dt} &= kx\left(1-\frac{x}{M}\right)\\
\implies dx &= kx\left(1-\frac{x}{M}\right)dt\\
\implies \frac{dx}{x\left(1-\frac{x}{M}\right)} &= k\,dt \quad \text{for } 0<x<M\\
\implies \int \frac{dx}{x\left(1-\frac{x}{M}\right)} &= \int k\,dt = kt + C \quad \text{for some constant } C
\end{aligned}
$$

On the right-hand side, a general antiderivative of the constant k with respect to t is $kt+C$. The left-hand side, however, is a mystery.

$$\int \frac{1}{x\left(1-\frac{x}{M}\right)}\,dx$$

Here's the left-hand side again, rewritten to express the integrand as a rational function. To motivate the next step, we first play with some ordinary fractions involving numbers. For example,

$$\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$$

which we can write as

$$\frac{1}{2} + \frac{(-1)}{3}$$

.
Similarly,

$$\frac{1}{33} = \frac{1}{3\cdot 11}$$

And you can check quickly that this becomes

$$= \frac{2}{3} - \frac{7}{11} = \frac{2}{3} + \frac{(-7)}{11}$$

.
Both $\frac{1}{6}$ and $\frac{1}{33}$ decompose into combinations involving simpler fractions, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{11}$. It's like recognizing that water is H_2O or common salt is sodium chloride.

Here's a tougher fraction,

$$\frac{1}{99} = \frac{1}{9\cdot 11}$$

. How can we decompose that? Let's postulate that it becomes

$$= \frac{A}{9} + \frac{B}{11}$$

.
We put the right-hand side together as a single fraction, where the numerator becomes $11A+9B$. We want this to equal 1.

$$= \frac{11A+9B}{99}$$

And it helps to find multiples of 11 and 9 that differ by 1. Say 44, which is $4 \cdot 11$, and 45, which is $5 \cdot 9$. By using -44 and 45, we get a total of 1. Thus, we should take A to be -4 and B to be 5. And then $\frac{1}{99}$ decomposes as

$$-\frac{4}{9}+\frac{5}{11}$$

.

Let's try something similar with the integrand above, which is a fraction involving polynomials in x, rather than just numbers.

The numerator is 1. The denominator is complicated, with factors x and $1-\frac{x}{M}$. Let's postulate a decomposition using

$$\frac{1}{x\left(1-\frac{x}{M}\right)}=\frac{A}{x}+\frac{B}{1-\frac{x}{M}}$$

Rewrite the right-hand side with a common denominator, obtaining a numerator that has to become 1 to agree with the numerator on the left-hand side.

$$\frac{A\left(1-\frac{x}{M}\right)+Bx}{x\left(1-\frac{x}{M}\right)}$$

If we take $A=1$, then we get

$$\frac{1-\frac{x}{M}+Bx}{x\left(1-\frac{x}{M}\right)}$$

and just need to choose B to make the $-\frac{x}{M}$ go away, and $B=\frac{1}{M}$ does the trick.
Thus the rational function decomposes as this sum.

$$\frac{1}{x\left(1-\frac{x}{M}\right)}=\frac{1}{x}+\frac{\frac{1}{M}}{1-\frac{x}{M}}$$

And the right-hand side can be written more simply as $\frac{1}{x}+\frac{1}{M-x}$. What we've done in an ad hoc way is a special case of the method of partial fractions, a topic you'll meet in more advanced courses in calculus.

$$\int \frac{dx}{x\left(1-\frac{x}{M}\right)}=\int\left(\frac{1}{x}+\frac{1}{M-x}\right)dx$$

To perform this integration, it suffices to integrate $\frac{1}{x}$ and $\frac{1}{M-x}$ separately, and add the answers together. The population x lies between 0 and M or, $0<x<M$, so that both x and $M-x$ are positive.

$$=\int \frac{dx}{x}+\int \frac{dx}{M-x}$$

An antiderivative of $\frac{1}{x}$ is $\ln x$ and of $\frac{1}{M-x}$ is $-\ln(M-x)$. For the second one, you can use the guess and check method, or integration by substitution. We combine the constants of integration together with a $+C'$ on the right.

$$=\ln x-\ln(M-x)+C'$$

The difference of logarithms is the logarithm of the quotient, and we rewrite the expression inside the logarithm to isolate x, in preparation for the next step. Our aim, remember, is to unravel the logistic equation and we produced this difficult integral which we've just managed to solve.

$$= \ln\left(\frac{x}{M-x}\right) + C'$$

$$= \ln\left(\frac{1}{\frac{M}{x}-1}\right) + C'$$

We can put everything together and amalgamate the constants on the right using, again, just a single plus C.

$$\ln\left(\frac{1}{\frac{M}{x}-1}\right) = kt + C$$

We want explicit information about x and it's buried in the expression on the left-hand side. We first raise Euler's number e to the power of each side of the equation. So the ln is stripped away from the left-hand side, and the right-hand side becomes e^{kt+C}, which can be expanded.

$$\implies \frac{1}{\frac{M}{x}-1} = e^{kt+C} = e^{C}e^{kt}$$

We reciprocate everything and rewrite the right-hand side. We add 1 to both sides, and now rewrite the right-hand side as $1+Ke^{-kt}$, with K being the positive constant e^{-C}.

$$\implies \frac{M}{x} - 1 = \frac{1}{e^{C}e^{kt}} = e^{-C}e^{-kt}$$

$$\implies \frac{M}{x} = 1 + e^{-C}e^{-kt} = 1 + Ke^{-kt}$$

for some constant $K > 0$

We rearrange this to obtain an explicit form of x, finally solving the logistic equation and its solution is known as the logistic function.

$$x = \frac{M}{1+Ke^{-kt}}$$

In writing down the logistic equation, there were two positive constants involved, k and M. In developing the general solution of the logistic function, we introduced a new positive constant K. This function has several important features. Firstly, the population x always lies between 0 and M where M represents a limiting upper bound or maximum.

Indeed, the limit of $x(t)$ as t gets arbitrarily large and positive is M because the expression e^{-kt}, which is $\frac{1}{e^{kt}}$, goes to 0.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{M}{1+Ke^{-kt}} = \frac{M}{1+0} = M$$

You can think of M as the largest sustainable population in an environment with only limited resources.

On the other hand, the limit of $x(t)$ as t gets arbitrarily large and negative will be 0, because now, the expression e^{-kt} will become arbitrarily large and positive, and it appears in the denominator.

$$\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} \frac{M}{1+Ke^{-kt}} = 0$$

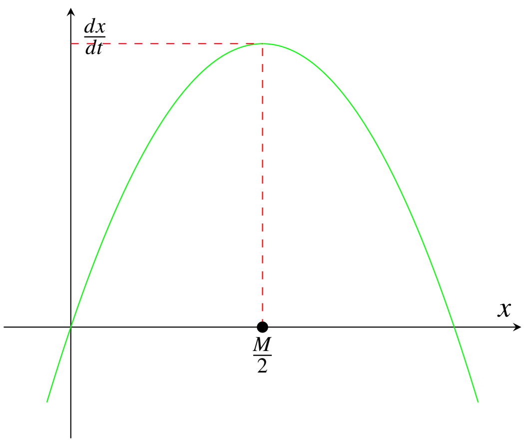
The variable t represents time, so the first limit represents limiting behavior in the future.

The second limit represents limiting behavior going backwards in time to the distant past.

In the original differential equation, we can rewrite the right-hand side as $\frac{k}{M}\cdot x\cdot(M-x)$, which is a quadratic in x.

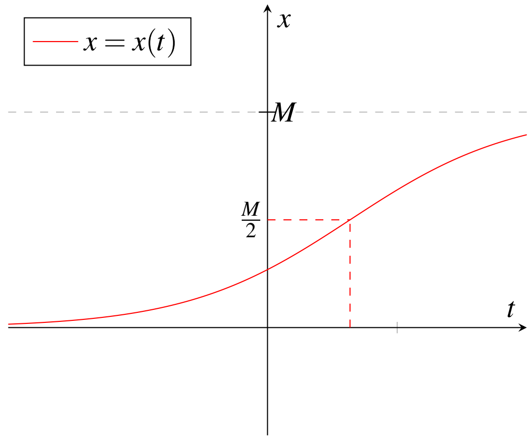
$$\frac{dx}{dt} = kx\left(1-\frac{x}{M}\right) = \frac{k}{M}x(M-x)$$

Let's graph this quadratic regarding $\frac{dx}{dt}$ as a function of x.



You can see that it must be an upside-down parabola which passes through the x-axis when $x=0$ and when $x=M$, and has an apex when x is exactly halfway between, that is when $x=\frac{M}{2}$. At this apex, the derivative $\frac{dx}{dt}$ is maximized. This tells us that the growth rate is maximized when x happens to be halfway between 0 and M.

We can now sketch the graph of x as a function of t. From above features , we can see that $x=M$ and the t-axis form two horizontal asymptotes, and the curve has to fit in between.



From above last feature, the growth rate is maximized when $x = \frac{M}{2}$, and the corresponding point on the curve will be a point of inflection. Moving down to the t-axis will reveal the time at which the growth rate is maximized.

You can see a direct connection with the usual exponential function. If you look at the rule for $x(t)$ when the denominator is large, the $+1$ becomes negligible.

$$x = \frac{M}{1+Ke^{-kt}} \approx \frac{M}{Ke^{-kt}} = \frac{M}{K}e^{kt}$$

By removing the $+1$, you get a simplified expression, which becomes a constant times e^{kt}, which is the usual exponential function.

This approximation effect occurs towards the left of the graph at time t before the curve gets near the point of inflection. The model indicates that the population is growing approximately exponentially until factors kick in to halt the growth. The population is growing as fast as possible at the moment it reaches $\frac{M}{2}$, and after that slows down and continues to slow down as it gets closer and closer to the maximum carrying capacity of M.

35.2.2 Part 2

Let's look at an example. Suppose a yeast colony starts off with a population of 100 cells. Suppose that it's growing most rapidly 10 hours later, at which time 500 cells are observed. Assuming a logistic model, let's find:
(a) an upper bound for the colony size, and
(b) how long it takes for the colony to reach 90% of its maximum.

Let t be the number of hours, and $x = x(t)$ be the size of the colony, which is a function of t. Since we're assuming logistic growth, then x must take the below particular form for some positive constants M, K, and k.

$$x = \frac{M}{1+Ke^{-kt}} \quad \text{for some } M, K, k < 0$$

The population is growing most rapidly at $t = 10$ hours, at which time 500 cells are present. But the moment of most rapid growth occurs when $x = \frac{M}{2}$, which is 500. Hence, the maximum $M = 1000$, which answers the part (a).

We can revise our formula and then think about tackling part (b). We want the time t at which x reaches 90% of 1000, which is 900. So, we just have to unravel the formula for t.

$$
\begin{aligned}
\text{i.e} \quad 900 = \frac{1000}{1+Ke^{-kt}} &\implies 1+Ke^{-kt} = \frac{1000}{900} = \frac{10}{9} \\
&\implies Ke^{-kt} = \frac{1}{9} \implies e^{-kt} = \frac{1}{9K} \implies e^{kt} = 9K \\
&\implies kt = \ln(9K) \implies t = \frac{\ln(9K)}{k}
\end{aligned}
$$

This is a straightforward algebraic manipulation. You can check it in detail if you like and wanna check this derivation.

We have an answer for t, but we haven't finished because it involves K and k. We know that the initial population was 100 cells, so that $100 = x(0) = \frac{1000}{1+K}$. Solving this, we find that $K = 9$. It remains to find k. We know that the population was 500 when $t = 10$. So, we can feed that information into the formula also using the fact that $K = 9$.

$$
\begin{aligned}
500 = x(10) &= \frac{1000}{1+9e^{-10k}} \\
&\implies 1+9e^{-10k} = \frac{1000}{500} = 2 \\
&\implies 9e^{-10k} = 1
\end{aligned}
$$

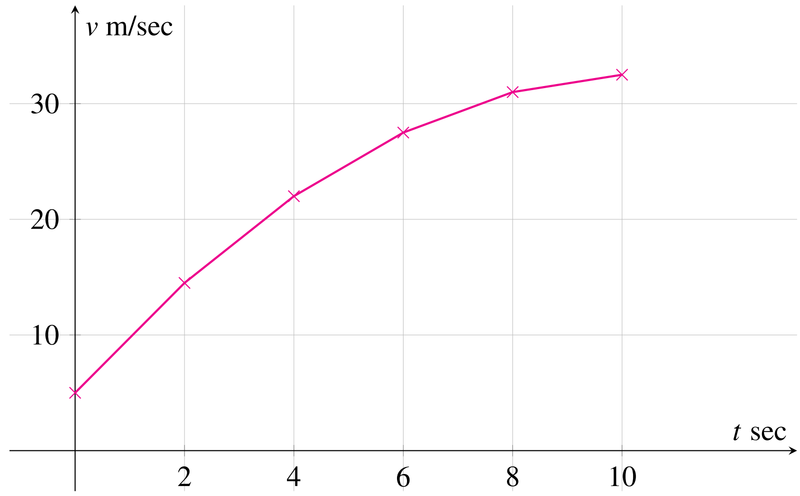
In a couple of steps, we get that $9e^{-10k} = 1$, and you can check that $k = \frac{\ln(9)}{10}$.

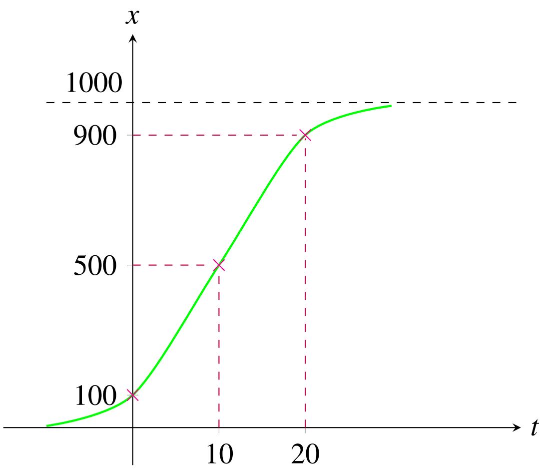
$$
\begin{aligned}
&9e^{-10k} = 1 \\
&\implies e^{-10k} = \frac{1}{9} \implies e^{10k} = 9 \\
&\implies 10k = \ln 9 \implies k = \frac{\ln 9}{10}
\end{aligned}
$$

We have all of the ingredients to evaluate t.

$$
t = \frac{\ln(9K)}{k} = \frac{\ln(81)}{\ln 9/10} = \frac{10\ln(9^2)}{\ln 9} = \frac{(10)(2\ln 9)}{\ln 9} = (10)(2) = 20
$$

We seem to have a lucky cancellation and the whole expression quickly evaluates to 20. The model predicts that it takes 20 hours for the colony to grow to 90 percent of the maximum. Is this just a coincidence that we produced such a simple whole number? It's always good practice to draw a diagram, and check if your answer makes sense.





Here, t labels the horizontal axis, and x the vertical axis. The initial yeast colony had size 100, which becomes the vertical intercept. After 10 hours, the population reached 500, so we now have two points on the curve. Because the rate of growth is greatest when the population reached 500, we can double that to get 1000, representing the limiting maximum predicted by the model, and this gives us a horizontal asymptote. We thus get a smooth sigmoid curve having an inflection with coordinates $(10,500)$. We can mark off the value 900 on the x axis, which is 90 % of the maximum, move across to the curve, and down to the t axis to get the answer to part (b) of our problem, which we worked out before to be $t = 20$. Now the answer becomes transparent. The intercept on the x axis, and the point with coordinates $(20,900)$ exactly match up by a 180 degree rotation about the inflection point. If we'd thought of this, then we could have solved the original problem without using any formulae at all.

Here's an application to sociology. Suppose that a rumour has been spread that "calculus is fun" in a small town with a population of 1000 people. Initially, 100 people know about it. After 10 days, 500 people know about it. How many days does it take for 900 people to know about it?

We know the answer immediately. It must be 20 days, assuming that the spread of the rumour is modeled by the logistic equation. We know this answer because the data exactly matches the problem that we solved earlier regarding the yeast culture. Even though the physical manifestations are completely different, the underlying mathematics is the same. Is it reasonable for sociologists to employ the logistic model? Let's have a look.

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right)$$

Here's the logistic differential equation where x represents the number of people that know about the rumour after t days. We can rewrite the right-hand side as

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right) = kM\left(\frac{x}{M}\right)\left(1 - \frac{x}{M}\right)$$

. You can think of the product kM as a single constant out the front. The factor $\frac{x}{M}$ represents the proportion of the population that knows about the rumour. The second factor represents the remaining proportion of the population that is not yet aware of the rumour. This now makes sense. Initially, only a few people know about the rumour, and it spreads like wildfire with approximately exponential growth. But after a while a lot of people get to know about the rumour. When exactly half of the population knows about it, then the growth rate reaches a maximum, but after that it starts to decline. It gets increasingly difficult to find someone that doesn't know about the rumour, and the growth rate dwindles to a trickle.

35.2.3 Examples and Derivation

Before concluding this section, we will review some solved examples and delve into some fancy derivations that were omitted earlier.

1. Find integers A and B such that

$$\frac{1}{119} = \frac{1}{17} - \frac{1}{7} = \frac{A}{17} + \frac{B}{7}.$$

Solution: We want $1 = 17A + 7B$. We look for multiples of 17 and 7 that differ by 1, say $34 = 2 \times 17$ and $35 = 5 \times 7$, so take $A = -2$ and $B = 5$, and then we have

$$\frac{1}{119} = \frac{-2}{17} + \frac{5}{7}.$$

2. Find constants A and B such that

$$\frac{1}{2x - z} = \frac{1}{x(2 - z)} = \frac{A}{B} + \frac{B}{x - z}.$$

Solution: After rewriting the right-hand side with a common denominator, we find that we need

$$1 = A(2 - z) + Bz = 2A + (B - A)z.$$

To get constants to match, we want $1 = 2A$, so that $A = \frac{1}{2}$. It follows that $B = \frac{1}{2}$ also, so we have

$$\frac{1}{2x - z} = \frac{1}{2(x - z)} + \frac{1}{2}.$$

3. Find the indefinite integral

$$\int \frac{dx}{x - z} \quad \text{(assuming } 0 < x < z\text{)}.$$

Solution: Put $u = 2 - z$, so that $\frac{1}{u} = 1$. Hence $du = -dx$ and $dx = -du$, and we get

$$\int \frac{dx}{2 - z} = -\int \frac{du}{u} = -\ln u + C = -\ln(2 - z) + C,$$

noting that $u > 0$, for the third step.

4. Find the indefinite integral
$$\int \frac{dx}{x-z} \quad \text{(assuming } 0 < x < z\text{)}.$$
Solution: Using the solutions to the two previous exercises, we get
$$\int \frac{dx}{x-z} = \int \left(\frac{1}{2} \cdot \frac{1}{2(x-z)} + \frac{1}{2}\right) dx = \int \left(\frac{dx}{2(x-z)}\right) + \frac{dx}{2},$$
$$= \frac{1}{2}\ln(x-z) + C = \frac{1}{2}\ln\left(\frac{x}{x-z}\right) + C.$$
5. Consider the logistic function
$$x = z(t) = \frac{M}{1+Ke^{-kt}},$$
where $M, K, k > 0$. Use interval notation to describe the range and show that the derivative $x'(t)$ is maximized when
$$t = \frac{\ln K}{k}.$$
Given that this occurs when $x = \frac{M}{2}$.
Solution: The curve is sandwiched between horizontal asymptotes $x = 0$ and $x = M$, so that the range is the interval $(0, M)$. The derivative is maximised when
$$\frac{M}{2} = x = \frac{M}{1+Ke^{-kt}},$$
so that $2 = 1 + Ke^{-kt}$, giving $Ke^{-kt} = 1$, so that $K = e^{kt}$. Taking natural logarithms and rearranging, we get
$$t = \frac{\ln K}{k}.$$
6. A rumour has been spread, amongst a population of 50,000 people, that calculus is fun. It was seeded by ten people, and after 10 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that
$$x = x(t) = \frac{50,000}{1+Ke^{-kt}}$$
for some positive constants K and k. Find
   (a) exact expressions for K and k,
   (b) the number of days after which the rumour is being spread most rapidly, and
   (c) the number of days after which 40,000 people know about the rumour.

Solution:
   a. We know that
$$10 = x(0) = \frac{50,000}{1+K} \quad \Rightarrow \quad 1+K = 5,000, \quad \text{so that } K = 4,999.$$
   We also know that
$$1,000 = x(10) = \frac{50,000}{1+4,999e^{-10k}}.$$
   Hence $1 + 4,999e^{-10k} = 50$, so that $4,999e^{-10k} = 49$, which becomes $e^{-10k} = \frac{49}{4,999}$. Taking natural logarithms and rearranging, we get
$$k = \frac{\ln(K/49)}{10} = \frac{\ln(4999/49)}{10}.$$

b. The rumour is being spread most rapidly when

$$t = \frac{\ln K}{k} = \frac{\ln(4999)}{k} \approx 18.4,$$

which is after about 18 days (to the nearest day).

c. We want to find t such that

$$40,000 = x(t) = \frac{50,000}{1+Ke^{-kt}},$$

so that $1+Ke^{-kt} = \frac{5}{4}$, that is, $Ke^{-kt} = \frac{1}{4}K$. This becomes

$$t = \frac{\ln(4K)}{k} = \frac{\ln(4\times 4999)}{k} \approx 21.4,$$

which is after about 21 days (to the nearest day).

In this section, we modified the usual exponential growth model to form the logistic equation used to model population dynamics, by incorporating an inhibition factor as well as a growth factor in the equation. The solution is called a logistic function. We derived the general formula and described its most important features, which include a limiting ceiling on the size of the population, which is approached as the time variable gets arbitrarily large, and a sigmoid shape with a 180 degree rotational symmetry about its point of inflection. We applied the logistic model to predict behavior of a growing yeast population, and also to predict the spread of rumours. Please re-read if you didn't get it and when you're ready please attempt the exercises. Thank you very much for reading and I look forward to seeing you again soon.

35.2.4 Practice Quiz

Question 1

Find integers A and B such that

$$\frac{1}{65} = \frac{1}{5\times 13} = \frac{A}{5} + \frac{B}{13}$$

(a) $A = 2, B = -5$
(b) $A = 3, B = -8$
(c) $A = 2, B = 5$
(d) $A = 8, B = -3$
(e) $A = 5, B = -2$

Question 2

Find integers A and B such that

$$\frac{1}{x-x^2} = \frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

(a) $A = 1, B = -1$
(b) $A = \frac{1}{2}, B = -\frac{1}{2}$
(c) $A = \frac{1}{2}, B = \frac{1}{2}$
(d) $A = 1, B = 1$
(e) $A = 1, B = 1$

Question 3

Find the indefinite integral

$$\int \frac{dx}{1-x} \quad \text{(assuming } 0 < x < 1\text{)}$$

(a) $-\ln(1-x)+C$
(b) $\frac{1}{(1-x)^2}+C$
(c) $-\ln(x-1)+C$
(d) $\ln(x-1)+C$
(e) $\ln(1-x)+C$

Question 4

Find the indefinite integral

$$\int \frac{dx}{x-x^2} \quad \text{(assuming } 0 < x < 1\text{)}$$

(a) $\ln\left(\frac{1}{x(1-x)}\right)+C$
(b) $\ln\left(\frac{1-x}{x}\right)+C$
(c) $\ln\left(x(1-x)\right)+C$
(d) $\ln\left(x(x-1)\right)+C$
(e) $\ln\left(\frac{x}{1-x}\right)+C$

Question 5

Consider the function $x = x(t)$ for $t \in \mathbb{R}$ with the following rule:

$$x = x(t) = \frac{1}{1+Ke^{-t}},$$

where K is a positive constant. Use interval notation to describe the range.

(a) $[0,1]$
(b) $(0,1)$
(c) $(0,\infty)$
(d) $[0,1)$
(e) $(0,1]$

Question 6

Consider the function $x = x(t)$ for $t \in \mathbb{R}$ with the following rule:

$$x = x(t) = \frac{1}{1+Ke^{-t}},$$

where K is a positive constant. Find t such that $x'(t)$ is maximized (corresponding to a point of inflection on the curve).

(a) $t = K$
(b) $t = -\ln K$
(c) $t = e^{-K}$
(d) $t = e^{K}$
(e) $t = \ln K$

Question 7

A rumour has been spread, amongst a population of 100,000 people, that calculus is fun. It was seeded by ten people, and after 10 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that

$$x = x(t) = \frac{100,000}{1+Ke^{-kt}},$$

for some positive constants K and k. Note that $x(0) = 10$, corresponding to the ten people that started the rumour. Use the information to find K.

(a) $K = 99,999$
(b) $K = 1,000$
(c) $K = 999$
(d) $K = 10,000$
(e) $K = 9,999$

Question 8

A rumour has been spread, amongst a population of 100,000 people, that calculus is fun. It was seeded by ten people, and after 10 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that

$$x = x(t) = \frac{100,000}{1+Ke^{-kt}}$$

for some positive constants K and k. Note that $x(10) = 1000$. Use the information to find an exact expression for k.

(a) $k = \frac{\ln K}{10\ln 99}$
(b) $k = \frac{\ln(99/K)}{10}$
(c) $k = \frac{\ln K}{990}$
(d) $k = \frac{\ln(K/99)}{10}$
(e) $k = \frac{\ln K}{10}$

Question 9

A rumour has been spread, amongst a population of 100,000 people, that calculus is fun. It was seeded by ten people, and after 10 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that

$$x = x(t) = \frac{100,000}{1+Ke^{-kt}}$$

for some positive constants K and k. After how many days, to the nearest day, is the rumour being spread most rapidly? (Find t such that $x(t) = 50,000$.)

(a) 19 days
(b) 22 days
(c) 38 days
(d) 20 days
(e) 21 days

Question 10

A rumour has been spread, amongst a population of 100,000 people, that calculus is fun. It was seeded by ten people, and after 10 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that

$$x = x(t) = \frac{100,000}{1 + Ke^{-kt}}$$

for some positive constants K and k. After how many days, to the nearest day, is the rumour known by 90,000 people?

(a) 26 days
(b) 50 days
(c) 28 days
(d) 27 days
(e) 25 days

Answers

The answers will be revealed at the end of the module.

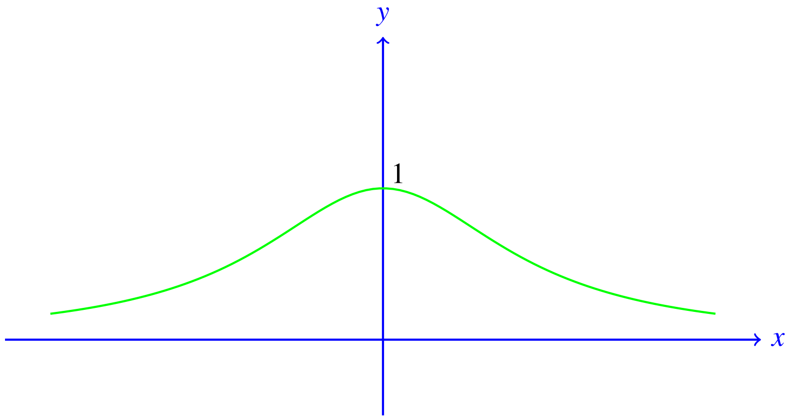
36. Conclusion

36.1 The escape velocity of a Rocket

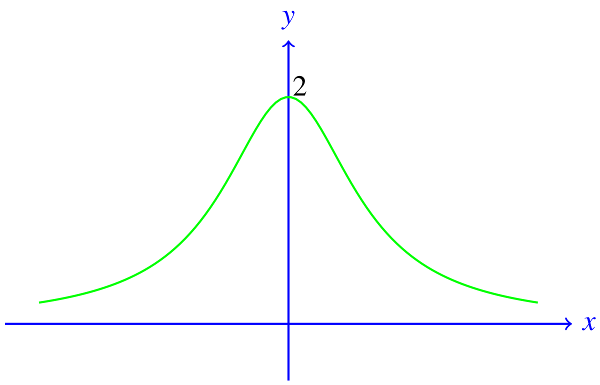
This is the final section for this module on integral calculus and indeed for the entire course. There's no exercise tied to this section. So, afterward, you can move directly to the main quiz for this final module. Nevertheless, I think it's beneficial to have a section that creates unity by bringing ideas together. You'll notice many themes from the entire course permeate the narrative. Even just to let the story wash over you should be beneficial. If you have time to see the section and check details here and there, you should find this section helpful in terms of general revision and developing your fluency with the techniques.

Today I want to do something very special and come full circle to the discovery of the integral calculus in the 17th century by Isaac Newton, motivated by his remarkable estimate of the escape velocity of a rocket. This must have seemed like ridiculous science fiction at a time when space travel would have been unthinkable and absurd. In fact, Newton was thinking in terms of cannonballs, not rockets. There are apocryphal stories of him sitting under a tree, gazing serendipitously at the moon, stirred into action by an apple falling on his head. Who knows exactly what happened? In any case, we're the great beneficiaries of Newton's incredibly inventive imagination and vision and the rapid evolution of mathematics in the intervening centuries, supported by the great minds and genius of people such as Leibniz, Euler, Gauss, Agnesi, Riemann, and many others. What follows is an extended example that takes us back to the birth of the integral calculus in the 17th century.

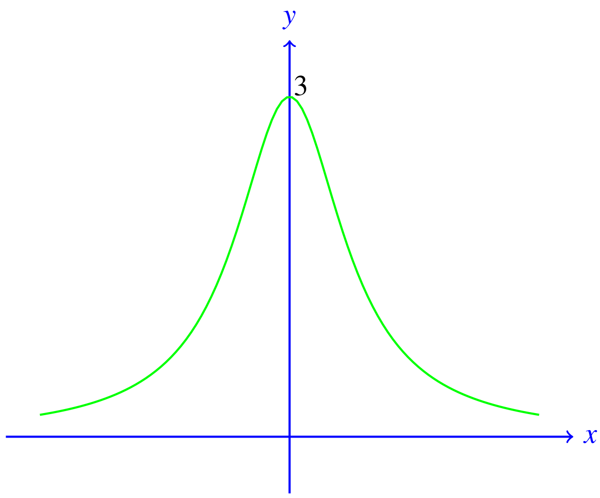
Recall the witch of Maria Agnesi in its simplest form with rule $y = \frac{1}{x^2+1}$, with global maximum 1 where it crosses the y-axis.



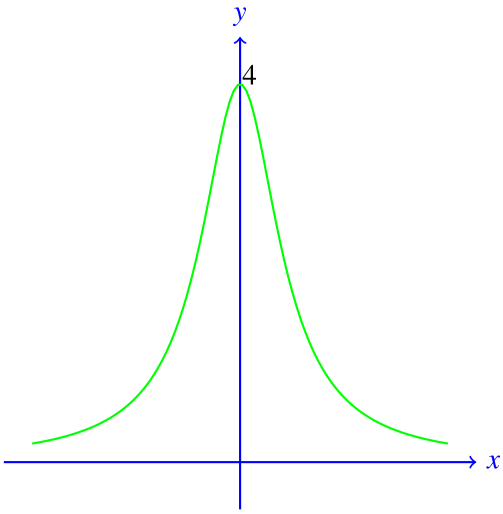
What happens if we replace 1 in the denominator with $\frac{1}{2}$?



The curve gets stretched in the vertical direction and the y-intercept moves up to 2. What happens if we replace $\frac{1}{2}$ with $\frac{1}{3}$?



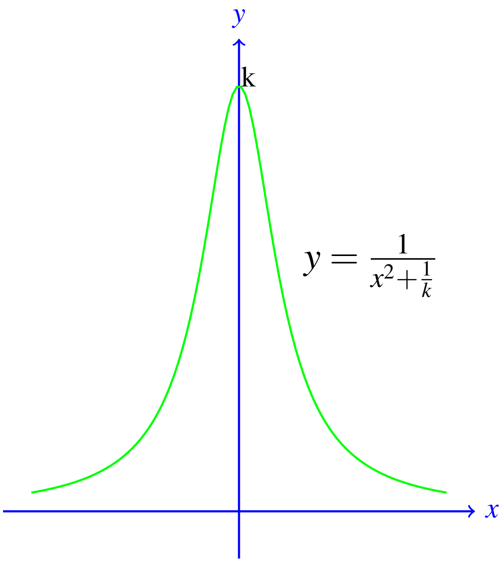
The curve stretches some more and the y-intercept moves up to 3. We can replace $\frac{1}{3}$ by $\frac{1}{4}$ and the y-intercept moves up to 4.



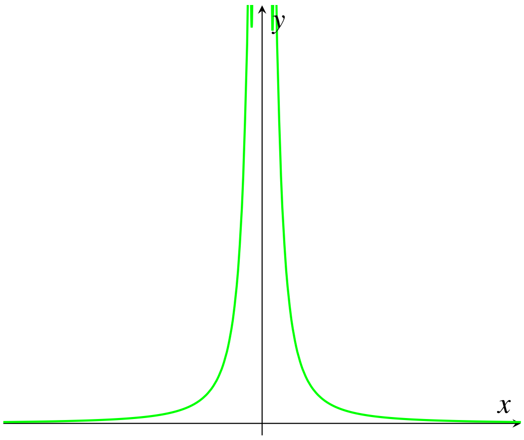
Then, let y be the following function.

$$y = \frac{1}{x^2 + \frac{1}{k}}$$

We quickly see that if the fraction added to x^2 in the denominator is $\frac{1}{k}$, then the y-intercept becomes k.



What happens then if k heads off towards infinity? Well, the y-intercept of the witch must also head off towards infinity. But infinity isn't a number, so the curve can't cross the y-axis anymore. There must be some kind of explosion. And when the dust settles, the denominator becomes just x^2.



So, we get the curve $y = \frac{1}{x^2}$, which can be rewritten as x^{-2}. As we can see in the graph y-axis will act as an asymptote and separate the graph. And it's good revision to check the shape of the curve against the sign diagrams for the first and second derivatives. The first derivative is

$$\Rightarrow \quad y' = -2x^{-3} = \frac{-2}{x^3}$$

, following with sign diagram indicating that the curve is increasing from the left and decreasing to the right with undefined derivative at $x = 0$.

x		0	
y'	$+$	x	$-$
	/		\

The second derivative is

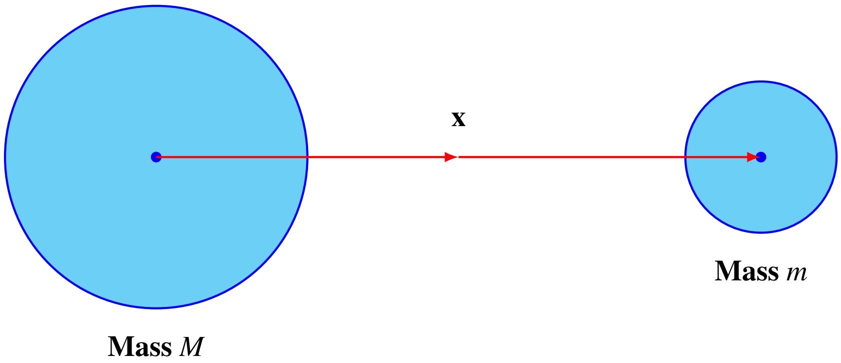
$$\Rightarrow \quad y'' = 6x^{-4} = \frac{6}{x^4}$$

, following with a sign diagram indicating that the curve is concave up, both from the left and to the right with undefined second derivative at $x = 0$.

x		0	
y''	$+$	x	$+$
	$\cup$		$\cup$
	Concave up		Concave up

So finally we do get a traditional witch's hat with a sharp point somewhere off in the infinite distance. The simple rule, $y = \frac{1}{x^2}$, is especially important in mathematics and science and in particular features in the inverse square law used to model gravitation.

Suppose we have two celestial bodies such as planets or stars separated by a distance of x units.



I've drawn them as perfect spheres but they can be irregular, it doesn't matter. In any case, they have centers of mass somewhere and the distance x is measured between them. The body on the left has mass M and the body to the right has mass m. Denote the force of gravitational attraction by F, which is a function of x which is $F(x)$.

$$F = F(x) = G\frac{Mm}{x^2}$$

It makes sense that F should be proportional to both of the masses. The greater the mass, the greater the gravitational attraction. It also makes sense that F should be inversely proportional to the square of the distance. Remember, the surface area of a sphere is 4π times the square of the radius which is $4\pi r^2$. So you can imagine the gravitational force radiating uniformly outwards from a spherical body, distributed evenly over 4π times the square of the distance from the center. The gravitational force on any unit of area is therefore proportional to the inverse of the total surface area of an expanding sphere, which in turn is proportional to the inverse of the square of the distance to the center of the sphere.

The overall constant of proportionality is denoted by G, also known as the universal gravitational constant. This was all known to Newton in the 17th century. There's some controversy about who stated this fact first, but certainly Newton was the first person to explore the mathematical consequences. Armed with this information about gravitational forces, let's consider a rocket moving away from the Earth.

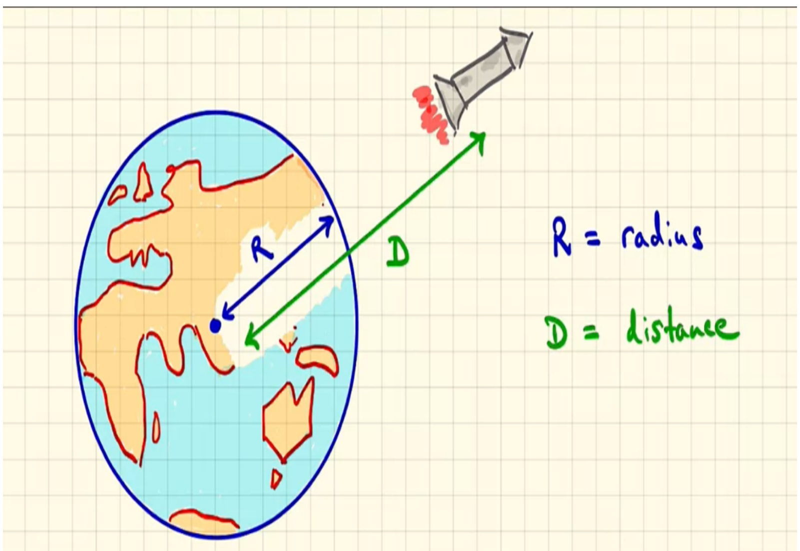


Figure 36.1: Rocket launching from Earth

Denote the radius of the Earth by R, and the distance from the center of the Earth to the position of the rocket, wherever it happens to be, by D.

We can ask how much energy was expended in getting the rocket from the surface of the Earth to its present position. The total energy expended can be described by the following definite integral,

$$\text{Energy of rocket} = \int_R^D F(x)\,dx$$

where the distance x ranges from R, when the rocket was being launched on the surface of the Earth to its present position D. The integrand is $F(x)$, the gravitational force of attraction with the Earth against which the rocket is propelling itself.

In physics, if a force acts on an object that moves a certain distance, then the work done or energy expended is the product of the force with the distance. So multiplying the gravitational force $F(x)$ by some tiny change in the distance, represented by the differential dx, gives a tiny bit of energy required to move that tiny bit of distance. Adding up all these bits of energy, which we think of as being done continuously, is captured by the definite integral.

Just think of the curly S integral symbol as summing up all of the energies captured by the products $F(x)$ times dx as x varies from R to D. But $F(x)$ is given by the formula $G\frac{Mm}{x^2}$.

$$\int_R^D F(x)\,dx = \int_R^D G\frac{Mm}{x^2}\,dx$$

The m is the mass of the rocket and M is the mass of the earth. We have all of the tools to evaluate this integral. The constants come out the front.

$$= GMm\int_R^D \frac{1}{x^2}\,dx$$

We can rewrite the integrand as x^{-2} and apply the fundamental theorem of calculus by evaluating the anti-derivative $-x^{-1}$ between R and D, which becomes in a few steps the expression $GMm\left(\frac{1}{R}-\frac{1}{D}\right)$.

$$= GMm\int_R^D x^{-2}\,dx$$

$$\begin{aligned}
&= GMm\left[-x^{-1}\right]_R^D \\
&= GMm\left(-D^{-1}-(-R^{-1})\right) \\
&= GMm\left(-\frac{1}{D}+\frac{1}{R}\right) \\
&= GMm\left(\frac{1}{R}-\frac{1}{D}\right)
\end{aligned}$$

Thus we find an expression for the amount of energy expended getting the rocket to its present position. We can ask what happens to this expression for the energy required as a rocket travels further out into space towards the distant stars as D approaches infinity and the Earth vanishes to a single point. As $D \to \infty$, $\frac{1}{D} \to 0$. Thus, to reach the stars, in principle, the energy required becomes $GMm\frac{1}{R}$. We can rewrite this as the fraction as

$$\text{Energy of rocket} \;=\; \frac{GMm}{R}$$

[0]Image 36.1 from MOOC Single Variable Calculus (University of Sydney)

.

At the time, Newton was in fact thinking about cannonballs rather than rockets making use of a continuous supply of fuel. You might recall in a very early section, we considered the trajectory of a cannonball shot directly upwards from the Earth's surface, and the graph of its displacement function turned out to be approximately a parabola. This all works out fine assuming we're close to the Earth's surface where the gravitational force may be regarded as constant. Up to this point, that was basically Newton's experience of cannonballs also, but he wanted to think about moving far away from the earth where the gravitational force can vary. For Newton, the question was how much energy should you transfer to a cannonball at the moment it's fired that it manages to escape the earth's gravitational field? The question then becomes what should be the initial velocity given to a cannonball such that it has enough energy to escape?

The energy required is given by our formula, but now m denotes the mass of the cannonball rather than the mass of the rocket. We just need to link the velocity v that we're seeking with the formula for the energy required. There's one more fact from physics:

$$\text{Energy of the cannonball } = \frac{1}{2}mv^2$$

.

So we have

$$\frac{1}{2}mv^2 = \frac{GMm}{R}.$$

To find the velocity, we just have to unravel this equation. Observe that the mass m cancels from both sides so the velocity will, in fact, be independent of the mass of the cannonball.

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

$$v^2 = \frac{2GM}{R}$$

Rearranging and taking the positive square root yields the formula

$$v = \sqrt{\frac{2GM}{R}}.$$

But we want an actual number for the escape velocity. Getting back to the surface of the Earth, we have the universal formula for the force F, which we can rewrite as

$$F = \frac{GMm}{R^2} = m\left(\frac{GM}{R^2}\right)$$

expressed as mass times acceleration.

And Newton had observed that, near the surface of the Earth, the acceleration is about 9.8 meters per second squared. So, we have, at least, $\frac{GM}{R^2} \approx 9.8$. The last part of the puzzle is having an estimate of the radius R of the Earth, but happily this was provided by the Greek scholar and mathematician Eratosthenes in about 200 BC, using clever measurements and some geometry, and was known to Newton and is about 6.37×10^6 meters in scientific notation. Putting this all together gives

$$v=\sqrt{2\left(\frac{GM}{R^2}\right)R}\approx\sqrt{2\times 9.8\times 6.37\times 10^6}$$

which comes to about 11,000 m/s. Thus, to escape the earth, the projectile has to be given an initial velocity of about $11km/sec$, and the mathematics also shows this to be independent of the mass of the projectile.

This is the final section in this course, An Introduction to Calculus, and I wanted you to get a glimpse of how it all began and what motivated Isaac Newton, who, in parallel with Gottfried Leibniz, invented calculus in the 17th century. We've spoken at length in earlier sections about Leibniz notation, and it appears implicitly in the exposition above leading to Newton's estimate of the escape velocity. Newton didn't have access to Leibniz notation nor did he have access to the smooth and elegant framework for formulating the mathematics, which evolved subsequently over the 18th and 19th centuries in which we, in modern times, too easily take for granted. This makes Newton's achievements and profound insights all the more remarkable.

The first two modules of this course introduced ideas and techniques from pre-calculus, preparing for the third and fourth modules which introduced the differential calculus, which is really all about the study of slopes of tangent lines to curves. This fifth and final module introduced and discussed integral calculus, which is really all about the study of areas under curves. Remarkably, slopes of tangent lines and areas under curves turn out to be intrinsically linked, leading to the Fundamental Theorem of Calculus.

We began this final module by illustrating how one can use areas under velocity curves to estimate displacement, using averages of lower and upper rectangular approximations, and replicated thought experiments, originally due to the ancient Greeks, involving limits of approximations, to discover the formula for the area of a circle and the area under a parabola. We then formalized the method of Riemann Sums using rectangular approximations to areas under curves over a given interval, and the definite integral, which is defined to be the limit of the Riemann sums, as the widths of subintervals, in partitions of the interval, go to zero. The definite integral captures precisely areas under a curve and has a symbolic description using an integral symbol like a stylized S which we can think of as a continuous sum, an integrand representing the rule of a function, a differential indicating the variable with respect to which we're finding the area, and terminals which tell us the endpoints of the relevant interval.

We can calculate the definite integral exactly under certain conditions using the Fundamental Theorem of Calculus. This provides a simple and elegant formula involving an antiderivative of the integrand, that is, a function whose derivative is the integrand. Antiderivatives are captured symbolically using indefinite integrals, the result of removing terminals from the definite integral, and providing convenient descriptions of integration formulae. These typically involve a constant of integration, due to the fact that all antiderivatives of a given function differ by a constant. We introduced and illustrated the method of integration by substitution, closely related to the chain rule for differentiation. We discussed odd and even functions, related to rotational and reflectional symmetry, and noted, in particular, that the area under the curve of an odd function evaluates automatically to zero over a symmetric interval. We discussed logistic functions which are solutions of the logistic equation, which modifies the usual exponential growth model, by introducing an inhibition factor, which provides a maximal limiting size to the population. The graph of any logistic function is a sigmoid curve with a 180-degree rotational symmetry about its point of inflection.

The purpose of this last section was to provide you with some context and insight concerning interacting ideas and motivation that led to the development of calculus from the 17th century. The mathematical treatment was only the briefest sketch and included important ideas from physics. You'll see a more thorough and rigorous treatment if you undertake further courses in advanced calculus. There are some notes accompanying this section which you can read out of interest and to support your mathematical development. But there are no new techniques required and no specific exercises.

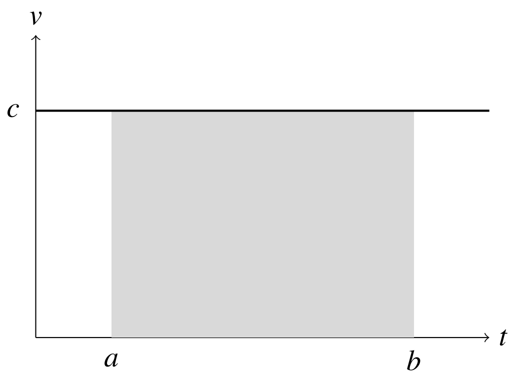
Thank you very much for reading, and especially for your sustained interest and participation over many weeks. I wish you well in your studies and look forward very much to seeing you again soon, exploring new horizons perhaps in future courses in mathematics.

37. Assessment

37.1 Module Quiz

Question 1

The velocity curve of a particle is given below, in compatible units.

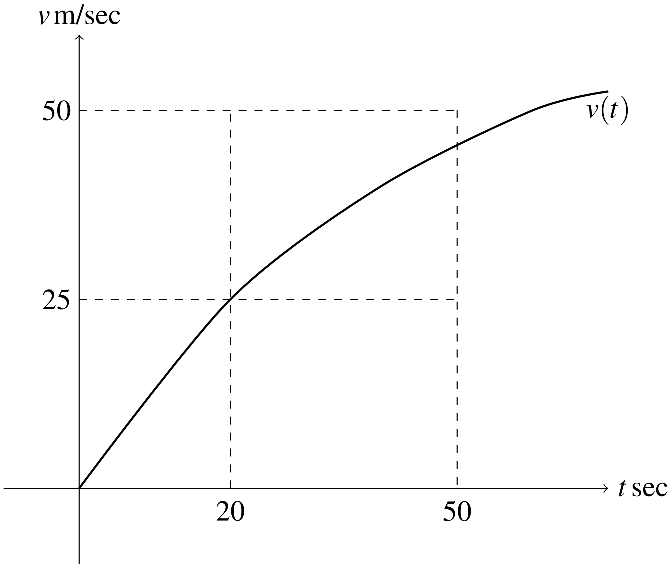


Find an expression for the shaded area under the curve, representing the displacement of the particle over the time interval $t_1 \leq t \leq t_2$.

(a) $\frac{c(b-a)}{2}$
(b) $c(b-a)$
(c) $c(b+a)$
(d) $c(a-b)$
(e) $\frac{c(a+b)}{2}$

Question 2

The velocity curve $v = v(t)$ m/sec of an accelerating object moving along a straight line is shown below, as a function of t sec, and is shaped slightly concave downwards.



Estimate the distance traveled by the object, for $20 \leq t \leq 60$, by taking the average of the areas of the lower and upper rectangles that just touch the curve, and decide whether this provides an overestimate or underestimate of the true distance traveled.

(a) 1,200 m, underestimate
(b) 1,125 m, underestimate
(c) 1,125 m, overestimate
(d) 1,250 m, underestimate
(e) 1,250 m, overestimate

Question 3

A car comes to a stop five seconds after the driver slams on the brakes. While the brakes are on, the following velocities are recorded:

Time since brakes applied (sec)	0	1	2	3	4	5
Velocity (m/sec)	28	19	12	7	3	0

Use the average of lower and upper rectangular approximations to estimate the distance travelled over those five seconds as the car comes to a halt. (You may assume the velocity curve is decreasing throughout.)

(a) 54 m
(b) 53 m
(c) 55 m
(d) 51 m
(e) 52 m

Question 4

Add up all of the consecutive integers from 1 to 300, that is, find the sum

$$1+2+3+\cdots+299+300.$$

(a) 44,850
(b) 15,150
(c) 45,150
(d) 35,150
(e) 25,150

Question 5

Use Sigma notation to find an expression for the sum of squares of consecutive integers from 1 to 100, that is, the sum

$$1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2.$$

(a) $\sum_{i=1}^{100}(i+1)^2$
(b) $\sum_{i=1}^{100} i^2$
(c) $\sum_{i=1}^{99}(i^2-(i+1)^2)$
(d) $\left(\sum_{i=1}^{200} i\right)^2$
(e) $\sum_{i=1}^{99}(i^2+(i+1)^2)$

Question 6

Evaluate

$$\sum_{i=1}^{100}(2i-1).$$

(a) 10,000
(b) 9,700
(c) 9,900
(d) 10,100
(e) 9,800

Question 7

Evaluate:

$$\int_0^1 (2f(x)-3g(x)+4)\,dx$$

given that $\int_0^1 f(x)\,dx = 2$ and $\int_0^1 g(x)\,dx = -1$.
(a) 0
(b) 7
(c) 1
(d) 5
(e) 3

Question 8

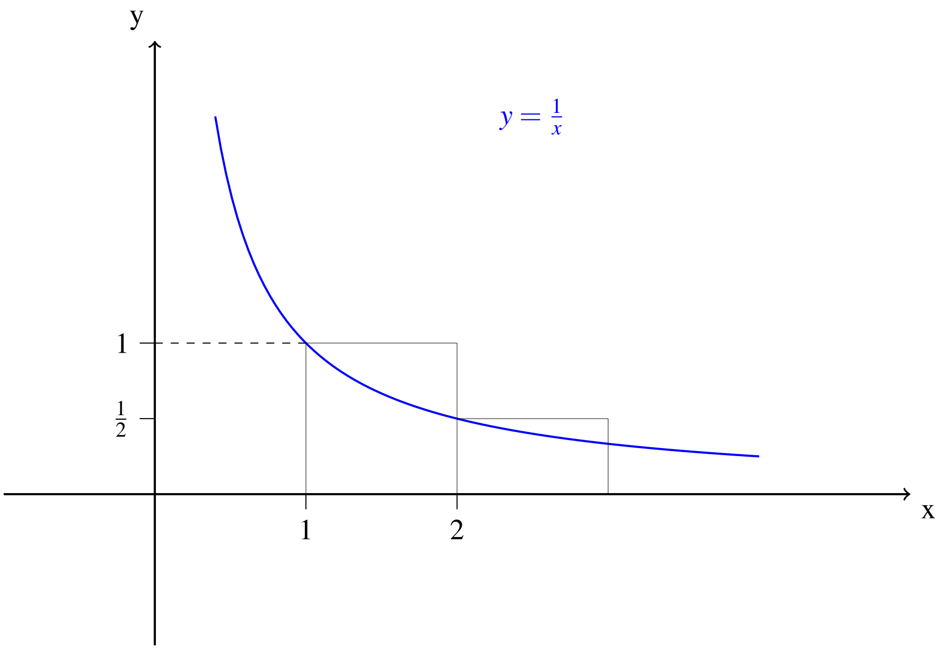
Find

$$\int_1^8 f(x)\,dx$$

given that $\int_2^8 f(x)\,dx = -3$ and $\int_1^2 f(x)\,dx = 6$.
(a) -3
(b) -9
(c) 9
(d) 3
(e) 12

Question 9

Use the average of the lower and upper Riemann sums, using 2 subintervals,



to estimate the definite integral $\int_1^3 \frac{dx}{x}$, rounding off the final answer to one decimal places.

(a) 1.5
(b) 1.3
(c) 1.4
(d) 1.2
(e) 1.6

Question 10

Find the indefinite integral $\int (9x^2+4x-3)\,dx$.

(a) $3x^3+2x^2-3x+C$
(b) $9x^3+2x^2-3x+C$
(c) $3x^3+2x^2+C$
(d) $3x^3+2x^2-\frac{3}{2}x+C$
(e) $9x^3+4x^2-\frac{3}{2}x+C$

Question 11

Find the definite integral $\int_1^2 (9x^2+4x-3)\,dx$.

(a) 20
(b) 22
(c) 18
(d) 16
(e) 24

Question 12

Find the definite integral $\int_{-\pi/2}^{\pi/2} \cos x\,dx$.

(a) 0
(b) 1
(c) 2
(d) -1
(e) -2

Question 13

Find the indefinite integral $\int \frac{dx}{x^5}$.

(a) $\frac{1}{4x^4} + C$
(b) $\frac{1}{5x^4} + C$
(c) $\frac{1}{6x^6} + C$
(d) $-\frac{1}{6x^6} + C$
(e) $-\frac{1}{4x^4} + C$

Question 14

Find the definite integral $\int_0^1 \left(\frac{x+1}{2}\right)^2 dx$. (Hint: try the substitution $u = \frac{x+1}{2}$)

(a) $\frac{7}{8}$
(b) $\frac{5}{12}$
(c) $\frac{2}{3}$
(d) $\frac{7}{12}$
(e) $\frac{7}{24}$

Question 15

Find the indefinite integral $\int \frac{e^x - e^{-x}}{2} dx$.

(a) $\frac{e^x - e^{-x}}{4} + C$
(b) $\frac{e^x + e^{-x}}{4} + C$
(c) $\frac{e^{x+1}}{2(x+1)} - \frac{e^{1-x}}{2(1-x)} + C$
(d) $\frac{e^x + e^{-x}}{2} + C$
(e) $\frac{e^x - e^{-x}}{2} + C$

Question 16

Find the indefinite integral $\int 6xe^{3x^2} dx$. (Hint: try the substitution $u = 3x^2$.)

(a) $\frac{e^{3x^2}}{3} + C$
(b) $2xe^{3x^2} + C$
(c) $e^{3x^2} + C$
(d) $2e^{3x^2} + C$
(e) $\frac{e^{3x^2+1}}{3x^2+1} + C$

Question 17

Which one of the following functions is even?

(a) $y = e^x$
(b) $y = x + x^3$
(c) $y = \tan x$
(d) $y = x^2 + x^4$
(e) $y = \sin x$

Question 18

Find the definite integral $\int_{-3}^{3} (1 + x + x^3 + x^5 + x^7)\, dx$.

(a) 12
(b) 0
(c) $\frac{41}{105}$
(d) 3
(e) 6

Question 19

Find the indefinite integral $\int \left(\frac{1}{x} + \frac{1}{3-x}\right) dx$ (assuming $0 < x < 3$).

(a) $\ln\left(\frac{x}{3-x}\right) + C$
(b) $y = \ln(x(x-3)) + C$
(c) $y = \ln(x(3-x)) + C$
(d) $\ln\left(\frac{3-x}{x}\right) + C$
(e) $\ln\left(\frac{1}{x(3-x)}\right) + C$

Question 20

A rumour has been spread, amongst a population of 2,000 people, that calculus is fun. It was seeded by 200 people, and, after 35 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that

$$x = x(t) = \frac{2,000}{1 + Ke^{-kt}}$$

for some positive constants K and k. Hence $x(0) = 200$ and $x(35) = 1,000$. After how many days, to the nearest day, is the rumour known by 1,800 people?

(a) 69 days
(b) 72 days
(c) 70 days
(d) 73 days
(e) 71 days

38. Answer Key

38.1 Inferring Displacement from Velocity

Answers

1 (b)
2 (b)
3 (e)
4 (e)
5 (a)
6 (e)
7 (c)
8 (e)
9 (c)
10 (b)

38.2 Areas Bounded by Curves

Answers

1 (d)
2 (e)
3 (e)
4 (b)
5 (b)
6 (a)
7 (b)
8 (a)
9 (a)
10 (a)

38.3 Riemann Sums and Definite Integrals

Answers

1 (d)
2 (c)
3 (e)
4 (b)
5 (b)
6 (a)
7 (b)
8 (a)
9 (a)
10 (a)

38.4 The Fundamental Theorem of Calculus and Indefinite Integrals

Answers

1 (a)
2 (b)
3 (a)
4 (a)
5 (e)
6 (c)
7 (a)
8 (b)
9 (b)
10 (a)

38.5 Connection Between Areas and Derivatives

Answers

1 (e)
2 (b)
3 (d)
4 (a)
5 (b)
6 (d)
7 (e)
8 (d)
9 (e)
10 (c)

38.6 Integration by Substitution

Answers

1 (a)
2 (e)
3 (d)
4 (e)
5 (d)
6 (a)
7 (a)
8 (b)
9 (a)
10 (c)

38.7 Odd and Even Functions

Answers

1 (c)
2 (c)
3 (c)
4 (b)
5 (d)
6 (d)
7 (b)
8 (a)
9 (c)
10 (c)

38.8 The Logistic Function

Answers

1 (a)
2 (e)
3 (a)
4 (e)
5 (b)
6 (d)
7 (e)
8 (b)
9 (a)
10 (d)

38.9 Assessment

Answers

1 (b)
2 (a)
3 (a)
4 (c)
5 (b)
6 (a)
7 (d)
8 (b)
9 (c)
10 (a)
11 (e)
12 (c)

13 (e)
14 (e)
15 (d)
16 (c)
17 (d)
18 (d)
19 (a)
20 (b)