Proposed Solution to #U650 Undergraduate Problems, Mathematical Reflections 1 (2024)

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Statement of the Problem:

Evaluate :

$$I = \int_1^e \frac{(\ln(x) - 1)^2 - 3}{(\ln(x) + 2)^2} dx$$

Solution of the Problem:

Substitute $\ln(x)$ as x ,

$$I = \int_0^1 \frac{(x-1)^2 - 3}{(x+2)^2} e^x dx$$

$$I = \int_0^1 \frac{(x+2-3)^2 - 3}{(x+2)^2} e^x dx$$

$$I = \int_0^1 \frac{(x+2)^2 - 6(x+2) + 9 - 3}{(x+2)^2} e^x dx$$

$$I = \int_0^1 \left[e^x - \frac{6e^x}{(x+2)} + \frac{6e^x}{(x+2)^2} \right] dx$$

$$I = \int_0^1 e^x dx - \int_0^1 \frac{6e^x}{(x+2)} dx + \int_0^1 \frac{6e^x}{(x+2)^2} dx$$

Using integration by parts in third integral,

$$\int_0^1 \frac{6e^x}{(x+2)^2} dx = -\frac{6e^x}{(x+2)}\bigg|_0^1 + \int_0^1 \frac{6e^x}{(x+2)} dx$$

Thus,

$$I = \int_0^1 e^x dx - \int_0^1 \frac{6e^x}{(x+2)} dx - \frac{6e^x}{(x+2)}\bigg|_0^1 + \int_0^1 \frac{6e^x}{(x+2)} dx$$

$$I = e^x\bigg|_0^1 - \frac{6e^x}{(x+2)}\bigg|_0^1 dx$$

$I = (e-1) - (2e-3) = 2 - e$, which is the answer.