OLYMPIAD CORNER

$No.413$

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OC635. Let n $\geq$ 3 be an integer. Prove that for all positive real numbers $x_1, ..., x_n$.

$$\frac{1+x_1^2}{x_2+x_3}+\frac{1+x_2^2}{x_3+x_4}+......+\frac{1+x_{n-1}^2}{x_n+x_1}+\frac{1+x_n^2}{x_1+x_2}\geq n$$

Proof: We first break the sum as follows:

$$=\frac{1}{x_2+x_3}+\frac{1}{x_3+x_4}+......+\frac{1}{x_n+x_1}+\frac{1}{x_1+x_2}+\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_4}+......+\frac{x_{n-1}^2}{x_n+x_1}+\frac{x_n^2}{x_1+x_2}$$

Now, using Titu's Lemma,

$$\geq\frac{(1+1+..+1)^2}{(x_2+x_3)+..+(x_n+x_1)+(x_1+x_2)}+\frac{(x_1+x_2+..+x_n)^2}{(x_2+x_3)+..+(x_n+x_1)+(x_1+x_2)}$$

$$=\frac{n^2}{2\sum_{i=1}^n x_i}+\frac{(\sum_{i=1}^n x_i)^2}{2\sum_{i=1}^n x_i}$$

$$=\frac{1}{2}(\frac{n^2}{\sum_{i=1}^n x_i}+\sum_{i=1}^n x_i)$$

Then, using AM-GM Inequality,

$$\geq\frac{1}{2}\times 2\sqrt{\frac{n^2}{\sum_{i=1}^n x_i}\times\sum_{i=1}^n x_i}=n$$

Thus, we have proved that

$$\frac{1+x_1^2}{x_2+x_3}+\frac{1+x_2^2}{x_3+x_4}+......+\frac{1+x_{n-1}^2}{x_n+x_1}+\frac{1+x_n^2}{x_1+x_2}\geq n$$