Proposed Solution to #5779 SSMJ

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Statement of the Problem:

$$\text{If } 0 < a \leq b \text{ then:}$$

$$e^{ab} + e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab}+\frac{a+b}{2}-\frac{2ab}{a+b})^2}$$

Solution of the Problem: Let

$$x = \frac{a+b}{2}, y = \sqrt{ab}, \& z = \frac{2ab}{a+b}$$

Since a and b are positive numbers, x(A.M), y(G.M.) & z(H.M) will be positive numbers. Also notice that since $A.M. \geq G.M. \geq H.M.$, thus $x \geq y \geq z$.
Rewriting the original problem in terms of x,y & z,

$$e^{y^2} + e^{x^2} \leq e^{z^2} + e^{(x+y-z)^2}$$

$$Or, e^{(x+y-z)^2} + e^{z^2} \geq e^{x^2} + e^{y^2}$$

which is what we will prove.
d Here, the sequence $\{x+y-z, z\} \succ \{x, y\}$ because
i) $x+y-z \geq z$ and $x \geq y$
ii) $(x+y-z)+(z) = (x)+(y)$
iii) $x+y-z \geq x$ and $(x+y-z)+z \geq x+y$
And, for $f(t) = e^{t^2}$, $f''(t) = e^{t^2}(4t^2+2)$ is always positive.
Now,
Karamata Inequality states that if sequence (a_i) majorizes sequence (b_i) and f(x) is a convex for all a_i and b_i then $\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i)$. Thus,

$$e^{(x+y-z)^2} + e^{z^2} \geq e^{x^2} + e^{y^2}$$

which proves the original problem.