Proposed Solution to #5778 SSMJ

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Statement of the Problem:

Prove

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n+(-1)^{n-m}\left(3+\frac{m-3n}{n+m}+\frac{n^2}{(m+n)^2}\right)\right)=G$$

where G is catalan's Constant, which is defined as $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}$.

Solution of the Problem: Simplifying the Given Problem,

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n+(-1)^{n-m}\frac{(2m+n)^2}{(m+n)^2}\right)$$

Breaking it into two sums assuming both exist,

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n\right)+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(m+n)^2}\right)$$

Notice that in the first sum, sum goes to 0 when n is even, so

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m(1-(-1))}{(2m+2n-1)^2}+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(m+n)^2}\right)$$

Interchanging the order of sum in the first double sum,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2(-1)^m}{(2m+2n-1)^2}+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(m+n)^2}\right)$$

$$\sum_{n=1}^{\infty}\frac{-2}{(2n+1)^2}+\frac{2}{(2n+3)^2}-\frac{2}{(2n+5)^2}+\frac{2}{(2n+7)^2}...+\sum_{m=1}^{\infty}\frac{1}{m^2}-\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}-\frac{1}{(m+3)^2}.....$$

Noticing that every two pair of terms form a telescoping sum

$$=\frac{-2}{3^2}+\frac{-2}{7^2}+\frac{-2}{11^2}+\frac{-2}{15^2}+\frac{-2}{19^2}....+\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+...$$

$$=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{15^2}+.....=G$$