

*Proposed Solution to #5753 SSMJ*

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**Statement of the Problem:** Let  $x_1, \dots, x_n > 0$  be real numbers and  $s = \sum_{i=1}^n x_i$ . Prove

$$\prod_{i=1}^n x_i^{x_i} \geq \left(\frac{s}{n+s}\right)^s \prod_{i=1}^n (1+x_i)^{x_i}.$$

When does equality occur?

**Solution of the Problem:** We first rearrange the problem as follows:

$$\prod_{i=1}^n \left(\frac{x_i}{1+x_i}\right)^{x_i} \geq \left(\frac{s}{n+s}\right)^s$$

We further modify the problem taking log on both sides,

$$\sum_{i=1}^n x_i \ln\left(\frac{x_i}{1+x_i}\right) \geq s \ln\left(\frac{s}{n+s}\right) \quad \dots\dots\dots(1)$$

Now, we will focus on proving this statement.

Then, consider a function  $f(x) = x \ln\left(\frac{x}{1+x}\right)$ . Then  $f''(x) = \frac{1}{x(1+x)^2} > 0$   $\forall x > 0$ . So, the function is convex in our required interval.

Now, using Jensen's inequality,

$$\frac{\sum_{i=1}^n x_i \ln\left(\frac{x_i}{1+x_i}\right)}{n} \geq \frac{\sum_{i=1}^n x_i}{n} \ln\left(\frac{\frac{\sum_{i=1}^n x_i}{n}}{1 + \frac{\sum_{i=1}^n x_i}{n}}\right)$$

which on simplification gives

$$\sum_{i=1}^n x_i \ln\left(\frac{x_i}{1+x_i}\right) \geq s \ln\left(\frac{s}{n+s}\right)$$

which is what was to be proved from (1). And the equality holds when  $x_1 = x_2 = \dots = x_n$ .