Proposed Solution to #5744 SSMJ

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Statement of the Problem:
Show that:

$$\left(\int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}}\,dx\right)^2+\left(\int_1^\infty \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}}\,dx\right)^2=\frac{\pi}{\sqrt{5}}$$

Solution of the Problem:

Let us begin by considering the first integral. The second integral has a similar solution :

$$I_1=\left(\int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}}\,dx\right)$$

Since $e^{i\theta}=\cos\theta+i\sin\theta$, we have $\mathrm{Re}\big\{e^{i\theta}\big\}=\cos\theta$. Thus,

$$\begin{aligned}I_1&=\mathrm{Re}\left\{\left(\int_1^\infty \frac{e^{i\ln x^2}}{x^2\sqrt{\ln x}}\,dx\right)\right\}\\&=\mathrm{Re}\left\{\left(\int_1^\infty \frac{x^{2i}}{x^2\sqrt{\ln x}}\,dx\right)\right\}\\&=\mathrm{Re}\left\{\left(\int_1^\infty x^{2i-2}(\ln x)^{-\frac{1}{2}}dx\right)\right\}\end{aligned}$$

Now, make a u-substitution with u=$\ln x$. Then, e^u=x which implies dx=e^u du. At x=1, u=0 and at x=$\infty, u=\infty$.

$$=\mathrm{Re}\left\{\left(\int_0^\infty e^{u(2i-2)}u^{-\frac{1}{2}}e^u\,du\right)\right\}$$

$$= \mathrm{Re}\left\{ \left(\int_0^\infty e^{-u(1-2i)} u^{-\frac{1}{2}}\, du \right) \right\}$$

Now, we know,

$$\int_0^\infty e^{-st}\, t^{m-1}\, dt = \frac{\Gamma(m)}{s^m}$$

Thus,

$$\mathrm{Re}\left\{ \left(\int_0^\infty e^{-u(1-2i)} u^{-\frac{1}{2}}\, du \right) \right\} = \mathrm{Re}\left\{ \frac{\Gamma(1/2)}{(1-2i)^{\frac{1}{2}}} \right\}$$

$$= \mathrm{Re}\left\{ \frac{\Gamma(1/2)}{(\frac{1}{\sqrt{5}} + \frac{-2}{\sqrt{5}} i)^{\frac{1}{2}}\, (\sqrt{5})^{\frac{1}{2}}} \right\}$$

Say $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{-2}{\sqrt{5}}$.

$$= \mathrm{Re}\left\{ \frac{\Gamma(1/2)}{(\cos\alpha + i\sin\alpha)^{\frac{1}{2}}\, (\sqrt{5})^{\frac{1}{2}}} \right\}$$

$$= \mathrm{Re}\left\{ \frac{\Gamma(1/2)\, e^{-\frac{1}{2}\alpha}}{(\sqrt{5})^{\frac{1}{2}}} \right\}$$

$$= \frac{\Gamma(1/2)\cos\left(-\frac{1}{2}\alpha\right)}{(\sqrt{5})^{\frac{1}{2}}}$$

$$= \frac{\Gamma(1/2)\cos\left(\frac{\alpha}{2}\right)}{(\sqrt{5})^{\frac{1}{2}}}$$

Since $\cos(\alpha) = \frac{1}{\sqrt{5}}$, we have $cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos(\alpha)}{2}}$. Thus,

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}} = \sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}$$

Hence,

$$I_1 = \frac{\Gamma(1/2)}{(\sqrt{5})^{\frac{1}{2}}} \sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}$$

Using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$I_1^2 = \frac{\pi(\sqrt{5}+1)}{10}$$

Now, using the fact that $\sin(\theta) = \mathrm{Im}\{e^{i\theta}\}$ and with similar procedure, we evaluate second integral to get:

$$I_2^2 = \frac{\pi(\sqrt{5}-1)}{10}$$

Finally,

$$I_1^2 + I_2^2 = \frac{\pi(\sqrt{5}-1)}{10} + \frac{\pi(\sqrt{5}+1)}{10}$$

$$= \frac{\pi}{\sqrt{5}}$$

Q.E.D