Crux (Numbered) Problem

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4866. Find all functions f:$R \to R$ such that the equation

$$f(xy + f(f(y))) = xf(y) + y$$

holds for all real numbers x and y.
Solution: Let P(x,y) be the stated property. Let f(1)=A and f(f(1))=B.
P(x,1) :

$$f(x \times 1 + f(f(1))) = xf(1) + 1$$
$$f(x + A)) = Bx + 1$$

which concludes that f(x) is linear. Thus, say f(x) = kx+c.
Substituting it back in the original functional equation, we obtain

$$kxy + k^3y + k^2c + kc + c = kxy + xc + y$$

Comparing the coefficient of x, y, xy and constant term, we get

$$k = 1, c = 0$$

Thus, the only solution is f(x)=x which clearly satisfies.