Proposed Solution to #U649 Undergraduate Problems, Mathematical Reflections 1 (2024)

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Statement of the Problem:

Evaluate :

$$\lim_{n\to\infty}\left(\frac{(1^2+n^2)(2^2+n^2)...(n^2+n^2)}{n!n^n}\right)^{\frac{1}{n}}$$

Solution of the Problem:

Let L be the limit,

$$L=\lim_{n\to\infty}\left(\frac{(1^2+n^2)(2^2+n^2)...(n^2+n^2)}{n!n^n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{k^2+n^2}{kn}\right)^{\frac{1}{n}}$$

Taking log on both sides,

$$\log(L)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(\frac{k^2+n^2}{kn}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\log\left(\frac{k}{n}+\frac{n}{k}\right)$$

We know $\lim_{n\to\infty}\sum_{k=1}^{n}\left[f(a+\frac{(b-a)}{n}k)\frac{b-a}{n}\right]=\int_a^b f(x)dx$,Thus

$$\log(L)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\log\left(\frac{k}{n}+\frac{n}{k}\right)=\int_0^1\log(x+\frac{1}{x})dx=\int_0^1\log(x^2+1)dx-\int_0^1\log(x)dx$$

Using integration by parts for the first integral,

$$I_1=\int_0^1\log(x^2+1)dx=\log(x^2+1)x\Big|_0^1-2\int_0^1\frac{x^2}{x^2+1}dx=\log(x^2+1)x\Big|_0^1-2\int_0^1\left(1-\frac{1}{x^2+1}\right)dx$$

$$I_1=\log(x^2+1)x\Big|_0^1-2\int_0^1 1dx+\int_0^1\frac{2}{x^2+1}dx$$

$$I_1 = \log(x^2+1)x\Big|_0^1 - 2x\Big|_0^1 + 2\arctan(x)\Big|_0^1 = \log(2) - 2 + \frac{\pi}{2}$$

Using integration by parts for the second integral as well,

$$I_2 = \int_0^1 \log(x)dx = (x\log(x) - x)\Big|_0^1 = -1$$

Combining,

$$\log(L) = I_1 - I_2 = \log(2) - 2 + \frac{\pi}{2} - (-1) = \log(2) + \frac{\pi}{2} - 1$$

$$L = 2e^{\frac{\pi}{2}-1}$$