Proposed Solution to #5756 SSMJ

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Statement of the Problem:

$$\text{Calculate } T = \int_0^\infty \frac{dx}{x^2(\tan^2(x) + \cot^2(x))}$$

Solution of the Problem:

$$\text{Using} \tan(x) = \frac{\sin(x)}{\cos(x)} \text{ and} \cot(x) = \frac{\cos(x)}{\sin(x)}, \text{ we get}$$

$$= \int_0^\infty \frac{\sin^2(x)\cos^2(x)}{x^2} . \frac{dx}{(\sin^4(x) + \cos^4(x))}$$

$$\text{Using} \sin(2x) = 2\sin(x)\cos(x) \text{ and } a^2 + b^2 = (a+b)^2 - 2ab, \text{ we get}$$

$$= \int_0^\infty \frac{\sin^2(2x)}{(2x)^2} . \frac{dx}{(\sin^2(x) + \cos^2(x))^2 - 2\sin^2(x)\cos^2(x)}$$

$$\text{Using} \sin^2(x) + \cos^2(x) = 1 \text{ and } \sin(2x) = 2\sin(x)\cos(x), weget$$

$$= \int_0^\infty \frac{\sin^2(2x)}{(2x)^2} . \frac{dx}{\left(1 - \frac{1}{2}\sin^2(2x)\right)}$$

Now, we make a u-substitution such that u = 2x . This implies $\frac{du}{2} = dx$. The integral now goes from 0 to infinity.

$$= \int_0^\infty \frac{\sin^2(u)}{u^2} . \frac{du}{\left(2 - \sin^2(u)\right)}$$

$$\text{Using } \sin^2(x) + \cos^2(x) = 1, \text{ we get}$$

$$= \int_0^\infty \frac{\sin^2(u)}{u^2} . \frac{du}{(1 + \cos^2(u))}$$

Lobachevsky's Formula states that if $0 \leq u < \infty$, f(u)=f(-u) and f(u+π k) = f(u), then

$$\int_0^\infty \frac{\sin^2(u)}{u^2} f(u)du = \int_0^{\frac{\pi}{2}} f(u)du$$

Here, $f(u) = \dfrac{1}{1+\cos^2(u)}$ satisfies the conditions of Lobachevsky's Formula. Therefore,

$$\int_0^\infty \frac{\sin^2(u)}{u^2} . \frac{du}{(1+\cos^2(u))} = \int_0^{\frac{\pi}{2}} \frac{du}{1+\cos^2(u)}$$

Multiplying numerator and denominator by $\sec^2(u)$, we get

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2(u)du}{\sec^2(u)+1}$$

Using $\sec^2(u) - \tan^2(u) = 1$, we get

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2(u)du}{2+\tan^2(u)}$$

Now, we make a y-substitution such that y = tan(u). This implies dy= $\sec^2(u)$ du . The integral now goes from 0 to infinity.

$$= \int_0^\infty \frac{dy}{2+y^2}$$

$$\text{We know, } \int \frac{1}{a^2+x^2} dx = \frac{1}{a} arctan\left(\frac{x}{a}\right) + c$$

Thus,

$$\int_0^\infty \frac{dy}{2+y^2} = \frac{1}{\sqrt{2}} arctan\left(\frac{x}{\sqrt{2}}\right)\Big|_0^\infty$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2\sqrt{2}}$$

$$\text{Hence, } T = \int_0^\infty \frac{dx}{x^2(\tan^2(x)+\cot^2(x))} = \frac{\pi}{2\sqrt{2}}$$