Proposed Solution to #U654 Undergraduate Problems, Mathematical Reflections 1 (2024)

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Statement of the Problem:

$$\text{Evaluate :}$$
$$I = \int_0^1 \frac{x\sqrt{x}\ln(x)}{x^2 - x + 1}dx$$

Solution of the Problem:

Multiply numerator and denominator by 1+x,

$$I = \int_0^1 \frac{(x^{\frac{3}{2}} + x^{\frac{5}{2}})\ln(x)}{1 + x^3}dx$$

Substitute x^3 as x ,

$$I = \frac{1}{9}\int_0^1 \frac{(x^{\frac{1}{2}} + x^{\frac{5}{6}})\ln(x)}{1 + x}x^{\frac{-2}{3}}dx = \frac{1}{9}\int_0^1 \frac{(x^{\frac{-1}{6}} + x^{\frac{1}{6}})\ln(x)}{1 + x}dx$$

Since x is going from 0 to 1, using the geometric infinite series expansion for $\frac{1}{1+x}$,

$$I = \frac{1}{9}\int_0^1 \left[(x^{\frac{-1}{6}} + x^{\frac{1}{6}})\ln(x)\sum_{n=0}^{\infty}(-x)^n\right]dx$$

Taking the constants inside the sum and then interchanging sum and integral using dominated convergence theorem,

$$I = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\int_0^1 (x^{n-\frac{1}{6}} + x^{n+\frac{1}{6}})\ln(x)dx$$

Using $\int_0^1 \ln^n(x)x^m dx = (-1)^n\frac{\Gamma(n+1)}{(m+1)^{n+1}}$,

$$I = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{(-1)^1\Gamma(2)}{(n+\frac{5}{6})^2} + \frac{(-1)^1\Gamma(2)}{(n+\frac{7}{6})^2}\right) = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{1}{(n+\frac{5}{6})^2} + \frac{1}{(n+\frac{7}{6})^2}\right)$$

$$I = \frac{1}{9}\left[\sum_{n=0}^{\infty}(-1)\left(\frac{1}{(2n+\frac{5}{6})^2}+\frac{1}{(2n+\frac{7}{6})^2}\right)+\sum_{n=0}^{\infty}\left(\frac{1}{(2n+\frac{11}{6})^2}+\frac{1}{(2n+\frac{13}{6})^2}\right)\right]$$

$$I = \frac{1}{36}\left[\sum_{n=0}^{\infty}(-1)\left(\frac{1}{(n+\frac{5}{12})^2}+\frac{1}{(n+\frac{7}{12})^2}\right)+\sum_{n=0}^{\infty}\left(\frac{1}{(n+\frac{11}{12})^2}+\frac{1}{(n+\frac{13}{12})^2}\right)\right]$$

Using $\psi'(x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^2}$,

$$I = \frac{1}{36}\left[-\psi'\left(\frac{5}{12}\right)-\psi'\left(\frac{7}{12}\right)+\psi'\left(\frac{11}{12}\right)+\psi'\left(\frac{13}{12}\right)\right]$$

Using $\psi'(1+x) = \psi'(x) - \frac{1}{x^2}$,

$$I = \frac{1}{36}\left[-\psi'\left(\frac{5}{12}\right)-\psi'\left(\frac{7}{12}\right)+\psi'\left(\frac{11}{12}\right)+\psi'\left(\frac{1}{12}\right)-144\right]$$

Using $\psi'(x)+\psi'(1-x) = \pi^2\ \mathrm{cosec}^2(\pi x)$

$$I = \frac{1}{36}\left[-\pi^2\mathrm{cosec}^2\left(\frac{5\pi}{12}\right)+\pi^2\mathrm{cosec}^2\left(\frac{\pi}{12}\right)-144\right] = \frac{\left[\mathrm{cosec}^2\left(\frac{\pi}{12}\right)-\mathrm{cosec}^2\left(\frac{5\pi}{12}\right)\right]\pi^2}{36} - 4$$

We can calculate that $\left[\mathrm{cosec}^2\left(\frac{\pi}{12}\right)-\mathrm{cosec}^2\left(\frac{5\pi}{12}\right)\right] = 8\sqrt{3}$,

$$I = \frac{8\sqrt{3}\pi^2}{36} - 4 = \frac{2\pi^2}{3\sqrt{3}} - 4$$

which is the solution to the given integral.