Proposed Solution to #5761 SSMJ

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Statement of the Problem:

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]}-H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)}+\int_0^{\frac{\pi}{4}}\frac{4y\sec(y)dy}{\sqrt{9\cos(2y)}}=\zeta(2)$$

where $H_{[n]}=\int_0^1\frac{1-x^n}{1-x}\text{dx}$ and $\zeta(n)=\sum_{k=1}^{\infty}\frac{1}{k^n}$ is Reimann zeta function for n > 1.

Solution of the Problem: Name sum as S and integral as I such that our answer is S + I.

$$S=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]}-H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)}$$

$$I=\int_0^{\frac{\pi}{4}}\frac{4y\sec(y)dy}{\sqrt{9\cos(2y)}}$$

Let us find try to find I.

Using $\cos(2y)=\cos^2(y)-\sin^2(y)$, we get

$$I=\int_0^{\frac{\pi}{4}}\frac{4y\sec(y)dy}{3\sqrt{\cos^2(y)-\sin^2(y)}}$$

Dividing numerator and denominator by $\cos(y)$

$$I=\frac{4}{3}\int_0^{\frac{\pi}{4}}\frac{y\sec^2(y)dy}{\sqrt{1-\tan^2(y)}}$$

Making an u-substitution as u = $\tan(y)$ => $du=\sec^2(y)$ dy . And the integral goes from 0 to 1.

$$I = \frac{4}{3}\int_0^1 \frac{\arctan(u)du}{\sqrt{1-u^2}}$$

Using Integration by Parts,

$$I = \frac{4}{3}\left(\arctan(u)\arcsin(u)\Big|_0^1 - \int_0^1 \frac{\arcsin(u)}{1+u^2}du\right)$$

$$I = \frac{4}{3}\times\frac{\pi^2}{8} - \frac{4}{3}\int_0^1 \frac{\arcsin(u)}{1+u^2}du$$

$$I = \frac{\pi^2}{6} - \frac{4}{3}\int_0^1 \frac{\arcsin(u)}{1+u^2}du$$

Now, since $\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$, the problem can be written as:

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]} - H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)} + \frac{\pi^2}{6} - \frac{4}{3}\int_0^1 \frac{\arcsin(x)}{1+x^2}dx = \frac{\pi^2}{6}$$

which can further be modified as

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]} - H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)} = \frac{4}{3}\int_0^1 \frac{\arcsin(x)}{1+x^2}dx$$

Now, we can use the taylor expansion of $\arcsin(x)$ for $|x| < 1$:

$$\arcsin(x) = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{x^{2n+1}}{2n+1}$$

we get,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]} - H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)} = \frac{4}{3}\int_0^1 \frac{1}{1+x^2}\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{x^{2n+1}}{2n+1}\,dx$$

Since the sum is in the world of n, we can take $\frac{1}{1+x^2}$ inside the sum as a constant.

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]} - H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)} = \frac{4}{3}\int_0^1 \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{1}{1+x^2}\frac{x^{2n+1}}{2n+1}dx$$

Interchanging sum and integral using dominated convergence theorem,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]}-H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)}=\frac{4}{3}\sum_{n=0}^{\infty}\int_0^1\frac{1}{4^n}\binom{2n}{n}\frac{1}{1+x^2}\frac{x^{2n+1}}{2n+1}dx$$

Taking constants out of the integration

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{[\frac{n}{2}]}-H_{[\frac{n-1}{2}]}\right)}{4^n(6n+3)}=4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^n(6n+3)}\int_0^1\frac{x^{2n+1}}{1+x^2}dx$$

Now, it's enough to prove that:

$$4\int_0^1\frac{x^{2n+1}}{1+x^2}=H_{[\frac{n}{2}]}-H_{[\frac{n-1}{2}]}$$

$$\text{Let } y=x^2 => y^{\frac{1}{2}}=x=>\frac{1}{2}y^{-\frac{1}{2}}dy=dx$$

$$=2\int_0^1\frac{y^{n+\frac{1}{2}}}{1+y}y^{-\frac{1}{2}}dy$$

$$=2\int_0^1\frac{y^n}{1+y}dy$$

Using the formula for geometric sum as $|y|<1$

$$=2\int_0^1 y^n\sum_{r=0}^{\infty}(-y)^r dy$$

Since the sum is in the world of r, we can take the constant y^n inside the sum,

$$=2\int_0^1\sum_{r=0}^{\infty}(-1)^r y^{n+r}dy$$

Interchanging the sum and integral using dominated convergence theorem,

$$=2\sum_{r=0}^{\infty}\int_0^1(-1)^r y^{n+r}dy$$

$$=2\sum_{r=0}^{\infty}(-1)^r\left(\frac{y^{n+r+1}}{n+r+1}\right)\Big|_0^1$$

$$= 2\left(\sum_{r=0}^{\infty}(-1)^r\frac{1}{n+r+1}\right)$$

$$= 2\left(\sum_{r=0}^{\infty}\frac{1}{n+2r+1}-\frac{1}{n+2r+2}\right)$$

Changing the starting point of sum from r=0 to r=1

$$= 2\left(\sum_{r=1}^{\infty}\frac{1}{n+2r-1}-\frac{1}{n+2r}\right)$$

$$= \sum_{r=1}^{\infty}\frac{1}{r+\frac{n-1}{2}}-\frac{1}{r+\frac{n}{2}}$$

Now, we know that $\displaystyle\int_0^1\frac{1-t^x}{1-t}dt = H_x = \sum_{r=1}^{\infty}\frac{1}{r}-\frac{1}{r+x}$

$$\sum_{r=1}^{\infty}\frac{1}{r+\frac{n-1}{2}}-\frac{1}{r+\frac{n}{2}} = \sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+\frac{n}{2}}\right)-\left(\sum_{r=1}^{\infty}\frac{1}{r}-\frac{1}{r+\frac{n-1}{2}}\right)$$

$$= H_{[\frac{n}{2}]} - H_{[\frac{n-1}{2}]}$$

which proves the problem.