Problem Proposed to SSMJ

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Exponential Inequality
Statement of the Problem: Let a and b be real numbers such that $0 < a$ $\leq$ b. Then prove that:

$$\int_a^b e^{x^{2023}} dx > \left(b^{\frac{2025}{2}} a^{\frac{2023}{2}} - a^{\frac{2025}{2}} b^{\frac{2023}{2}}\right)$$

Solution of the problem:
From the series expansion, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$, we can say that for positive x, $e^x \geq 1 + x$ which implies $e^{x^{2023}} \geq 1 + x^{2023}$,Thus,

$$\int_a^b e^{x^{2023}} dx \geq \int_a^b 1 + x^{2023} dx = (b-a) + \frac{b^{2024} - a^{2024}}{2024}$$

$$= (b-a)\left(1 + \frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024}\right)$$

Using AM-GM inequality, we have

$$\frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024} \geq \sqrt[2024]{b^{2023+2022+....+1} a^{1+2+....+2023}} = \sqrt[2024]{b^{\frac{2023\times 2024}{2}} a^{\frac{2023\times 2024}{2}}} = b^{\frac{2023}{2}} a^{\frac{2023}{2}}$$

Thus,

$$1 + \frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024} > b^{\frac{2023}{2}} a^{\frac{2023}{2}}$$

And (b-a)>0,hence,

$$\int_a^b e^{x^{2023}} dx \geq (b-a)\left(1 + \frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024}\right) > (b-a) b^{\frac{2023}{2}} a^{\frac{2023}{2}} = b^{\frac{2025}{2}} a^{\frac{2023}{2}} - b^{\frac{2023}{2}} a^{\frac{2025}{2}}$$

which ends the proof.