Problem proposed to SSMJ

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Inequality Involving Exponents

Statement of the Problem:

Let x,y,z be positive real numbers such that x+y+z=3. Prove that:

$$\prod_{x,y,z}(x^{1/x}e^{e}) \leq e^{\sum_{x,y,z} e^{x}}$$

At what values of x,y and z does equality occur?

Solution of the Problem:

We modify the problem by taking ln on both sides:

$$\frac{\ln x}{x}+\frac{\ln y}{y}+\frac{\ln z}{z}+3e \leq e^{x}+e^{y}+e^{z}$$

We then rearrange it as:

$$\sum_{x,y,z}(\frac{\ln x}{x}-e^{x}) \leq -3e \quad ....(1)$$

Now, check that for f(x) $=\frac{\ln x}{x}-e^{x}$, f"(x) is negative $\forall$ x,y,z >0. We now use Jensen's Inequality.

$$\frac{\sum_{x,y,z}\frac{\ln x}{x}-e^{x}}{3} \leq \frac{\ln\left(\frac{x+y+z}{3}\right)}{\frac{x+y+z}{3}}-e^{\frac{x+y+z}{3}}$$

Using x+y+z=3,

$$\sum_{x,y,z}\frac{\ln x}{x}-e^{x} \leq -3e$$

which is what was to be proved from (1). And the equality holds when x=y=z=1.