Proposed Solution to #5768 SSMJ

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Statement of the Problem:

Calculate the integral :

$$I = \int_0^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx$$

Solution of the Problem:

$$I_1 = \int_0^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx$$

Substitute x by 1/x, we get

$$I_2 = \int_\infty^0 \frac{\arctan(1/x)(-\ln(x))^2}{1+x+x^2}x^2.\frac{-1}{x^2}dx = \int_0^\infty \frac{arccot(x)\ln^2(x)}{x^2+x+1}dx$$

Adding I_1 and I_2 and using the property that $\arctan(x)$+arccot(x) $= \frac{\pi}{2}$,

$$2I = \int_0^\infty \frac{(\arctan(x) + arccot(x))\ln^2(x)}{x^2+x+1} = \frac{\pi}{2}\int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx$$

Now, to find $\displaystyle\int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx,$ we solve the integral J(a) $= \displaystyle\int_0^\infty \frac{x^a}{x^2+x+1}dx,$

and then calculate J"(0)

$$J(a) = \int_0^1 \frac{x^a}{x^2+x+1}dx + \int_1^\infty \frac{x^a}{x^2+x+1}dx$$

Substituting x as 1/x in the second integral

$$J(a) = \int_0^1 \frac{x^a}{x^2+x+1}dx + \int_1^0 \frac{x^{-a}}{1+x+x^2}x^2.\frac{-1}{x^2}dx = \int_0^1 \frac{x^a+x^{-a}}{x^2+x+1}dx$$

$$Further, J(a) = \int_0^1 \frac{(x^a + x^{-a})(1-x)}{1-x^3}dx$$

Substitute x as $x^{\frac{1}{3}}$,

$$J(a) = \int_0^1 \frac{(x^{\frac{a}{3}} + x^{\frac{-a}{3}})(1-x^{\frac{1}{3}})}{1-x}\frac{x^{\frac{-2}{3}}}{3}dx = \frac{1}{3}\int_0^1 \frac{x^{\frac{a-2}{3}} + x^{\frac{-a-2}{3}} - x^{\frac{a-1}{3}} - x^{\frac{-a-1}{3}}}{1-x}dx$$

$$J(a) = \frac{1}{3}\int_0^1 \frac{1-x^{\frac{-a-1}{3}}}{1-x} + \frac{1-x^{\frac{a-1}{3}}}{1-x} - \frac{1-x^{\frac{a-2}{3}}}{1-x} - \frac{1-x^{\frac{-a-2}{3}}}{1-x}dx$$

Using $\psi(a+1)+\gamma = \int_0^1 \frac{1-x^a}{1-x}dx$, where γ is euler mascheroni constant,

$$J(a) = \frac{1}{3}\left[\psi\left(\frac{-a+2}{3}\right) + \psi\left(\frac{a+2}{3}\right) - \psi\left(\frac{a+1}{3}\right) - \psi\left(\frac{-a+1}{3}\right)\right]$$

$$J(a) = \frac{1}{3}\left[\psi\left(1-\frac{a+1}{3}\right) - \psi\left(\frac{a+1}{3}\right) - \left[\psi\left(1-\frac{a+2}{3}\right) - \psi\left(\frac{a+2}{3}\right)\right]\right]$$

Using the reflection property of digamma function, $\psi(1-x) - \psi(x) = \pi\cot(\pi x)$,

$$J(a) = \frac{1}{3}\left[\pi\cot(\pi(\frac{a+1}{3})) - \pi\cot(\pi(\frac{a+2}{3}))\right]$$

$$J'(a) = \frac{\pi}{3}\left[-cosec^2(\pi(\frac{a+1}{3}))\frac{\pi}{3} + cosec^2(\pi(\frac{a+2}{3}))\frac{\pi}{3}\right]$$

$$J''(a) = \frac{\pi^2}{9}\left[(-2cosec(\pi(\frac{a+1}{3})))(-cosec(\pi(\frac{a+1}{3}))cot(\pi(\frac{a+1}{3})))\frac{\pi}{3} + \right.$$
$$\left.(2cosec(\pi(\frac{a+2}{3})))(-cosec(\pi(\frac{a+2}{3}))cot(\pi(\frac{a+2}{3})))\frac{\pi}{3}\right]$$

$$J''(0) = \frac{2\pi^3}{27}\left[\frac{2}{\sqrt{3}}.\frac{2}{\sqrt{3}}.\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}.\frac{2}{\sqrt{3}}.\frac{-1}{\sqrt{3}}\right]$$

$$J''(0) = \frac{16\pi^3}{81\sqrt{3}}$$

Using $2\text{I} = \frac{\pi}{2}J''(0)$, we get $\text{I} = \frac{\pi}{4} \times \frac{16\pi^3}{81\sqrt{3}} = \frac{4\pi^4}{81\sqrt{3}}$