• **5738** *Proposed by Goran Conar, Varazdin, Croatia.*

Let $x_1, ...x_n > 0$ be real numbers and $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geq (\frac{s}{n+s})^s \prod_{i=1}^{n} (1+x_i)^{x_i}.$$

When does equality occur?

Solution by Prakash Pant, Mathematics Initiatives in Nepal (MIN)

We first rearrange the problem as follows:

$$\prod_{i=1}^{n} (\frac{x_i}{1+x_i})^{x_i} \geq (\frac{s}{n+s})^s$$

We further modify the problem taking log on both sides,

$$\sum_{i=1}^{n} x_i \ln(\frac{x_i}{1+x_i}) \geq s \ln(\frac{s}{n+s}) \qquad ....(1)$$

Now, we will focus on proving this statement.

Then, consider a function $f(x) = x \ln(\frac{x}{1+x})$. Then f"(x) = $\frac{1}{x(1+x)^2} > 0 \forall x \geq 0$. So, the function is convex in our required interval.
Now, using Jensen's inequality,

$$\frac{\sum_{i=1}^{n} x_i \ln(\frac{x_i}{1+x_i})}{n} \geq \frac{\sum_{i=1}^{n} x_i}{n} \ln(\frac{\frac{\sum_{i=1}^{n} x_i}{n}}{1+\frac{\sum_{i=1}^{n} x_i}{n}})$$

which on simplification gives

$$\sum_{i=1}^{n} x_i \ln(\frac{x_i}{1+x_i}) \geq s \ln(\frac{s}{n+s})$$

which is what was to be proved from (1). And the equality holds when $x_1 = x_2 = ... = x_n$.