Proposed Solution to #5773 SSMJ

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Statement of the Problem:

Calculate the following integral :

$$I = \int_0^1 \frac{x\ln^2(x)}{x^3 + x\sqrt{x} + 1}dx$$

Solution of the Problem:

$$I = \int_0^1 \frac{x\ln^2(x)}{(x^{\frac{3}{2}})^2 + x^{\frac{3}{2}} + 1}dx$$

Multiplying numerator and denominator by $(1 - x^{\frac{3}{2}})$,

$$I = \int_0^1 \frac{x\ln^2(x)(1 - x^{\frac{3}{2}})}{(1 - x^{\frac{9}{2}})}dx$$

Since $|x^{\frac{9}{2}}| < 1$ as x goes from 0 to 1 , we can use infinite geometric series expansion,

$$I = \int_0^1 x\ln^2(x)(1 - x^{\frac{3}{2}})\sum_{n=0}^{\infty} x^{\frac{9n}{2}}dx$$

Taking constants inside the sum and interchanging sum and interval using dominated convergence theorem,

$$I = \sum_{n=0}^{\infty}\int_0^1 x\ln^2(x)(1 - x^{\frac{3}{2}})x^{\frac{9n}{2}}dx = \sum_{n=0}^{\infty}\int_0^1 \left(\ln^2(x)x^{\frac{9n}{2}+1} - \ln^2(x)x^{\frac{9n}{2}+\frac{5}{2}}\right)dx$$

Using $\int_0^1 \ln^n(x)x^m dx = (-1)^n\frac{\Gamma(n+1)}{(m+1)^{n+1}}$,

$$I = \sum_{n=0}^{\infty}\frac{\Gamma(3)}{(\frac{9n}{2}+2)^3} - \frac{\Gamma(3)}{(\frac{9n}{2}+\frac{7}{2})^3} = \frac{8}{729}\sum_{n=0}^{\infty}\frac{2}{(n+\frac{4}{9})^3} - \frac{2}{(n+\frac{7}{9})^3}$$

Using $\psi''(x) = \sum_{n=0}^{\infty}\frac{-2}{(n+x)^3}$,

$$I = \frac{8}{729}\left(\psi''(\frac{7}{9}) - \psi''(\frac{4}{9})\right) = 0.20744$$