Proposed Solution to #5729 SSMJ

Solution proposed by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.

Problem proposed by Goran Conar, Varazdin, Croatia.

Statement of the Problem: Let $0 < a < b$ be real numbers. Prove the following inequality:

$$(a+b)^{(a+b)}(b-a)^{(b-a)} < (\frac{a^2+b^2}{b})^{2b}$$

Solution of the Problem: d We modify the problem taking ln on both sides,

$$(a+b)\ln(a+b) + (b-a)\ln(b-a) < 2b\ln(\frac{a^2+b^2}{b}) \qquad ....(1)$$

Now, we will focus on proving this statement.

Then, consider a function $f(x) = \ln(x)$. Then f"(x) $= -\frac{1}{x^2} < 0$ $\forall x > 0$. So, the function is concave $\forall x \neq 0$. Also notice that both a+b and b-a are strictly positive here.
Now, using Jensen's inequality,

$$\frac{(a+b)\ln(a+b) + (b-a)\ln(b-a)}{a+b+b-a} \leq \ln(\frac{(a+b)(a+b) + (b-a)(b-a)}{a+b+b-a})$$

which on simplification gives

$$(a+b)\ln(a+b) + (b-a)\ln(b-a) \leq 2b\ln(\frac{a^2+b^2}{b})$$

And the equality holds when $a+b = b-a \Rightarrow$ a=0. But since $0 < a < b$, the equality does not hold. Hence, we have proved (1).