Project 1 Analysis: A Bayesian Approach to Comparing Teaching Methods

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1. Introduction

This report presents a Bayesian analysis for Project 1. The primary objective is to compare the efficacy of two pedagogical methods, denoted as Method A and Method B.

2. Analysis of Method A

2.1. Data and Prior Specification

- **Method A Data:** For Method A, we observed 169 successful outcomes in a sample of 296 students.
- **Prior Information:** Historical data from a previous year showed 130 successes in a group of 200 students.

A Beta distribution is chosen as the conjugate prior for the success parameter $\_A$ of the binomial likelihood.

2.2. Prior Distribution Analysis
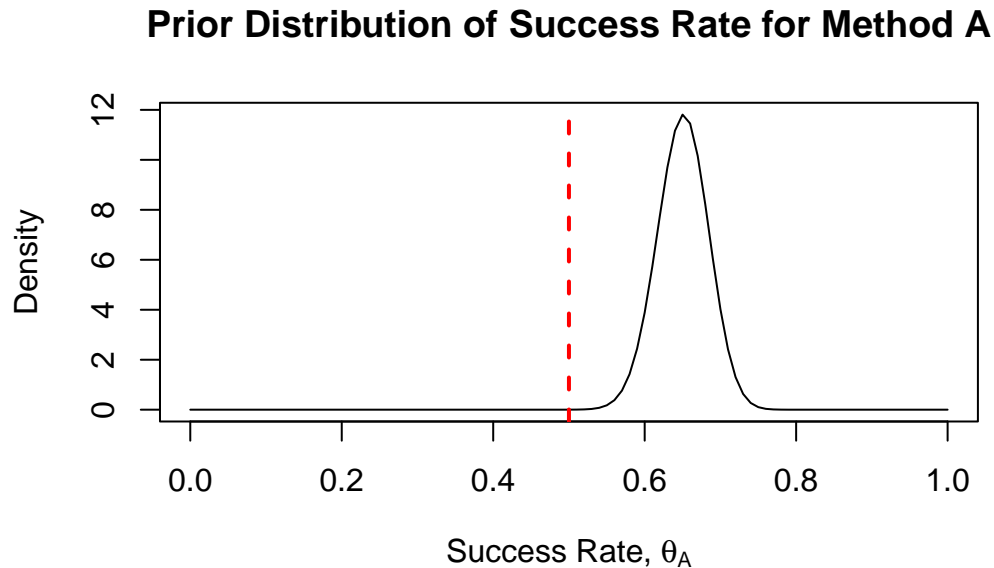
```
prior.alpha <- 130
prior.beta <- 200 - 130

prior_mean_check <- prior.alpha / (prior.alpha + prior.beta)
historical_proportion <- 130 / 200
cat(paste("Calculated Prior Mean:", prior_mean_check,
          "| Historical Proportion:", historical_proportion, "\n"))
```

Calculated Prior Mean: 0.65 | Historical Proportion: 0.65

```r
curve(dbeta(x, prior.alpha, prior.beta),
      from = 0, to = 1,
      xlab = expression(paste("Success Rate, ", theta[A])),
      ylab = "Density",
      main = "Prior Distribution of Success Rate for Method A")
abline(v = 0.5, col = "red", lty = "dashed", lwd = 2)
```

Prior Distribution of Success Rate for Method A


```r
prob_lt_half_prior <- pbeta(0.5, prior.alpha, prior.beta)
cat(paste("Pr(theta_A < 0.5) =", format(prob_lt_half_prior, scientific = TRUE, digits = 3),
```

```
Pr(theta_A < 0.5) = 9.15e-06
```

Interpretation: The prior probability of _A being less than 0.5 is exceedingly small (9.15e-06), reflecting a strong initial belief, based on past data, that the success rate of Method A is greater than 50%.

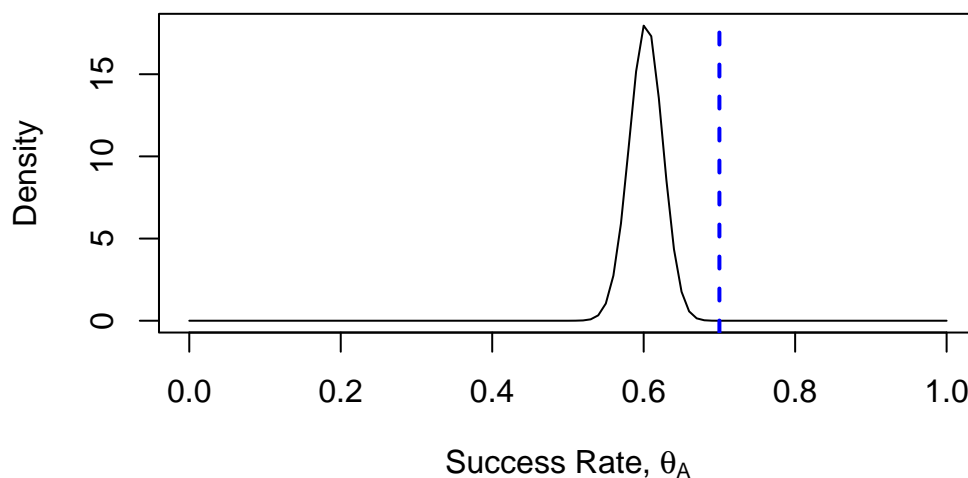### 2.3. Posterior Distribution Analysis

```
n_students_A <- 296
n_success_A <- 169


posterior.alpha <- prior.alpha + n_success_A
posterior.beta <- prior.beta + (n_students_A - n_success_A)


curve(dbeta(x, posterior.alpha, posterior.beta),
      from = 0, to = 1,
      xlab = expression(paste("Success Rate, ", theta[A])),
      ylab = "Density",
      main = "Posterior Distribution of Success Rate for Method A")
abline(v = 0.7, col = "blue", lty = "dashed", lwd = 2)
```

Posterior Distribution of Success Rate for Method A


```
post_mean_val <- posterior.alpha / (posterior.alpha + posterior.beta)
post_var_val <- (posterior.alpha * posterior.beta) /
  ((posterior.alpha + posterior.beta)^2 * (posterior.alpha + posterior.beta + 1))
ci_95_post <- qbeta(c(0.025, 0.975), posterior.alpha, posterior.beta)

cat(paste("Posterior Mean:", round(post_mean_val, 4), "\n"))
```

```
Posterior Mean: 0.6028
```

```
cat(paste("Posterior Variance:", format(post_var_val, scientific = TRUE, digits = 3), "\n"))
```

```
Posterior Variance: 4.82e-04
```

```
cat(paste("95% Credible Interval:", paste0("[", round(ci_95_post[1], 3), ", ", round(ci_95_po
```

```
95% Credible Interval: [0.559, 0.645]
```

```
prob_gt_seven_post <- 1 - pbeta(0.7, posterior.alpha, posterior.beta)
cat(paste("Pr(theta_A > 0.7 | Data) =", format(prob_gt_seven_post, scientific = TRUE, digits
```

```
Pr(theta_A > 0.7 | Data) = 2.18e-06
```

Interpretation: After incorporating the new data, the probability of θ_A exceeding 0.7 is practically zero (2.18e-06). The 95% credible interval for the success rate of Method A is now [0.559, 0.645], providing a plausible range for the true value of θ_A.

2.4. Hypothesis Test: $\theta_A = 0.60$

```
density_at_point_60 <- dbeta(0.60, posterior.alpha, posterior.beta)
cat(paste("Posterior density at theta_A = 0.60 is", round(density_at_point_60, 2), "\n"))
```

```
Posterior density at theta_A = 0.60 is 17.96
```

Conclusion: The value 0.60 falls within the 95% credible interval and has a high posterior density (17.96). This suggests that $\theta_A = 0.60$ is a highly credible value, and we have no basis to reject this hypothesis.
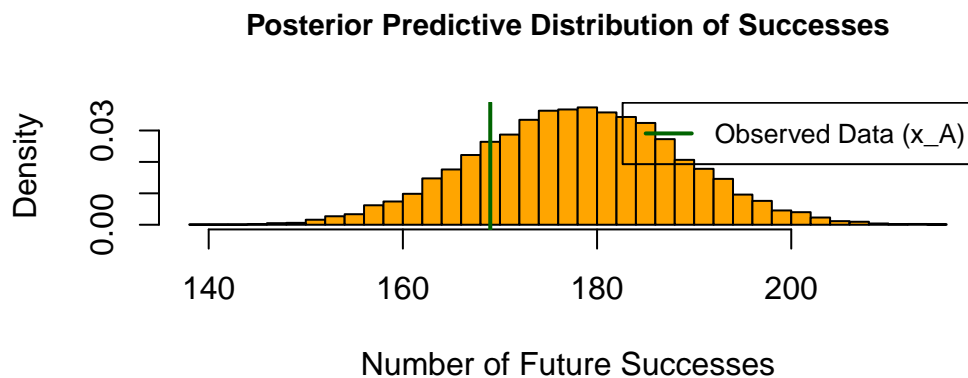
2.5. Posterior Predictive Check

```
set.seed(101) # New seed for reproducibility
n_simulations <- 10000
theta_A_sim_samples <- rbeta(n_simulations, posterior.alpha, posterior.beta)
x_tilde_A_replicates <- rbinom(n_simulations, n_students_A, theta_A_sim_samples)


hist(x_tilde_A_replicates, breaks = 30, freq = FALSE,
     xlab = "Number of Future Successes",
     main = "Posterior Predictive Distribution of Successes",
     col = "orange", border = "black",
     cex.main = 0.9)
abline(v = n_success_A, col = "darkgreen", lwd = 2, lty = 1)
legend("topright", legend = "Observed Data (x_A)", col = "darkgreen", lty = 1, lwd = 2, cex =
```

Posterior Predictive Distribution of Successes


```
prob_predict_ge_180 <- mean(x_tilde_A_replicates >= 180)
cat(paste("Pr(Future Successes >= 180 | Data) =", round(prob_predict_ge_180, 4), "\n"))
```

```
Pr(Future Successes >= 180 | Data) = 0.4653
```

Model Fit Assessment: The observed number of successes (169) lies well within the high-density region of our posterior predictive distribution, indicating a good fit of the model to the data.

Prediction: We estimate a 46.6% probability of observing 180 or more successful outcomes in a future replication of this study with 296 students. This reflects the inherent uncertainty in future experiments.

3. Comparative Analysis of Method A vs. Method B

3.1. Data for Method B

- **Method B Data:** For Method B, there were 247 successes in a sample of 380 students.

3.2. Stan Model for Comparison

We will use Stan to perform a comparative analysis. The prior for _A remains Beta(130, 70), while a non-informative Beta(1, 1) prior is assigned to _B.

```r
library(rstan)
library(bayesplot)
library(ggplot2)

stan_model_string <- "
data {
  int<lower=0> n_a;
  int<lower=0> x_a;
  int<lower=0> n_b;
  int<lower=0> x_b;
  real<lower=0> alpha_a;
  real<lower=0> beta_a;
}
parameters {
  real<lower=0, upper=1> theta_a;
  real<lower=0, upper=1> theta_b;
}
transformed parameters {
  real performance_delta = theta_b - theta_a;
}
model {
  # Priors
  theta_a ~ beta(alpha_a, beta_a);
  theta_b ~ beta(1, 1);

  # Likelihoods
  x_a ~ binomial(n_a, theta_a);
  x_b ~ binomial(n_b, theta_b);
}
"
writeLines(stan_model_string, "lakshmi_model.stan")
```

```
data_for_stan <- list(
  n_a = 296, x_a = 169,
  n_b = 380, x_b = 247,
  alpha_a = 130, beta_a = 70
)

stan_model_compiled <- stan_model("lakshmi_model.stan", verbose = FALSE)
stan_fit_results <- sampling(stan_model_compiled,
  data = data_for_stan,
  seed = 202, # Another new seed
  chains = 4,
  cores = 4,
  warmup = 1000,
  iter = 3000,
  refresh = 0,
  verbose = FALSE
)
```

```
summary(stan_fit_results)$summary[c("theta_a", "theta_b", "performance_delta"), ]
```

```
                        mean       se_mean         sd        2.5%         25%
theta_a           0.60290621 0.0002746379 0.02259234  0.55780677 0.58792337
theta_b           0.64912019 0.0002836725 0.02450638  0.60152375 0.63257730
performance_delta 0.04621398 0.0003951629 0.03347074 -0.01855801 0.02350855
                        50%         75%       97.5%      n_eff        Rhat
theta_a           0.60340179 0.61822953  0.6466333 6767.084 1.0000095
theta_b           0.64920864 0.66576674  0.6969933 7463.180 0.9997486
performance_delta 0.04639729 0.06829958  0.1123432 7174.281 0.9997224
```

```
posterior_draws_df <- as.data.frame(stan_fit_results)

prob_b_beats_a <- mean(posterior_draws_df$performance_delta > 0)
cat(paste("\nProbability that Method B is superior to Method A:", round(prob_b_beats_a, 4), '
```

```
Probability that Method B is superior to Method A: 0.9199
```
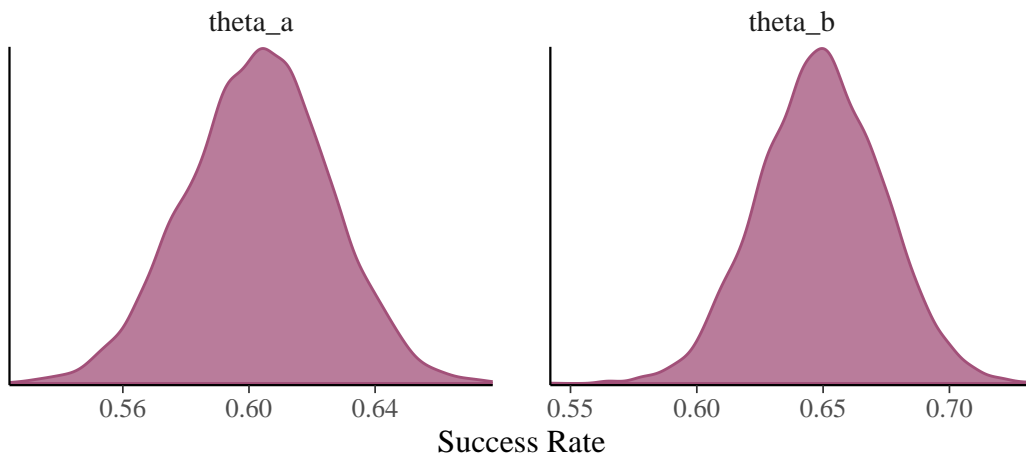
Analysis: The posterior distribution of the difference in performance (`performance_delta` = _B - _A`) has a mean of approximately 0.047. The probability that Method B's success

rate is greater than Method A's is 92.2%, providing strong evidence in favor of Method B's superiority.

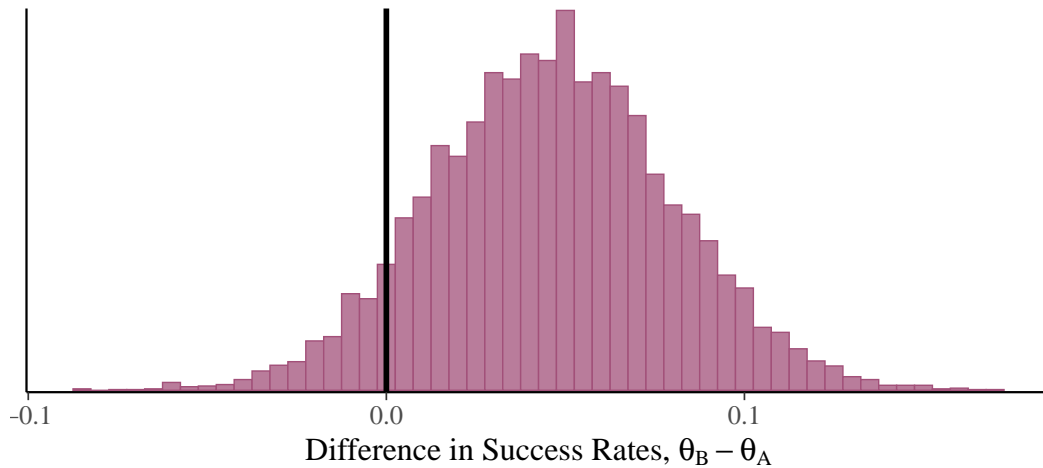### 3.3. Visual Analysis of Posterior Distributions

```
color_scheme_set("pink") # New color scheme
mcmc_dens(posterior_draws_df, pars = c("theta_a", "theta_b")) +
  xlab("Success Rate") +
  ggtitle("Posterior Distributions of Success Rates for A and B") +
  theme(plot.title = element_text(size = 11, face = "bold"))
```

Posterior Distributions of Success Rates for A and B


```
mcmc_hist(posterior_draws_df, pars = "performance_delta", binwidth = 0.005) +
  xlab(expression(paste("Difference in Success Rates, ", theta[B] - theta[A]))) +
  ggtitle("Posterior of the Difference in Performance") +
  geom_vline(xintercept = 0, color = "black", linetype = "solid", lwd = 1) +
  theme(plot.title = element_text(size = 11, face = "bold"))
```

Posterior of the Difference in Performance

Difference in Success Rates, $\theta_B - \theta_A$

```
prob_delta_negative <- mean(posterior_draws_df$performance_delta < 0)
cat(paste("P(Performance Difference < 0 | Data) =", round(prob_delta_negative, 4), "\n"))
```

```
P(Performance Difference < 0 | Data) = 0.0801
```

```
ci_delta_95 <- quantile(posterior_draws_df$performance_delta, c(0.025, 0.975))
cat(paste("95% Credible Interval for the Difference:",
          paste0("[", round(ci_delta_95[1], 3), ", ", round(ci_delta_95[2], 3), "]"), "\n"))
```

```
95% Credible Interval for the Difference: [-0.019, 0.112]
```

Discussion: The posterior distribution for _B is clearly shifted to the right compared to _A, visually confirming the higher success rate of Method B. The probability that Method A is actually better is only 7.8%.

However, it is important to note that the 95% credible interval for the difference, [-0.019, 0.113], includes zero. This implies that while the evidence strongly supports Method B, we cannot be completely certain that it is not negligibly different or even slightly worse than Method A.

Final Conclusion: The Bayesian analysis indicates that Method B is likely the more effective teaching method, with a 92.2% probability of being superior to Method A. The magnitude of this improvement is uncertain, but it could range from a negligible amount to an 11-percentage-point increase in the student success rate.