

# Module 1: Basics of Digital Signal Processing

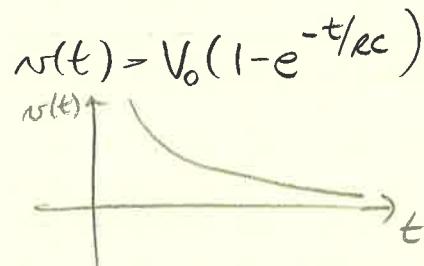
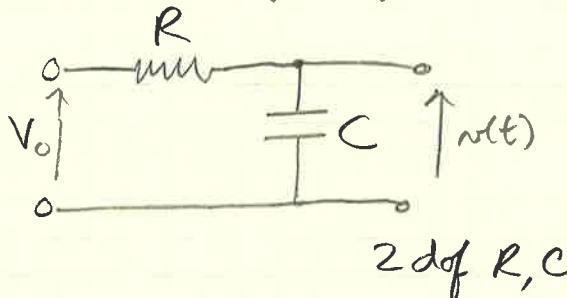
## 1.1 Introduction to digital signal processing

Signal: Description of evolution of a physical phenomenon

- Weather  $\rightarrow$  temperature
- Sound  $\rightarrow$  pressure
- Sound  $\rightarrow$  magnetic deviation
- Light intensity  $\rightarrow$  gray level on paper

Analysis: understanding the information carried by the signal

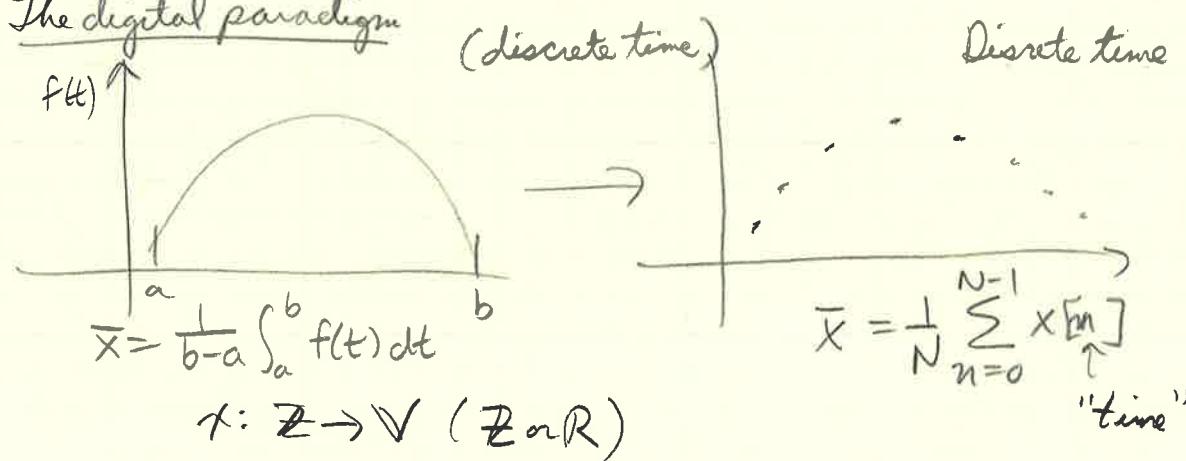
Synthesis: creating a signal to contain the given information



Analog signals  $f: \mathbb{R} \rightarrow \mathbb{V}$

From analog to digital:  $f(t) \rightarrow$  sample

The digital paradigm

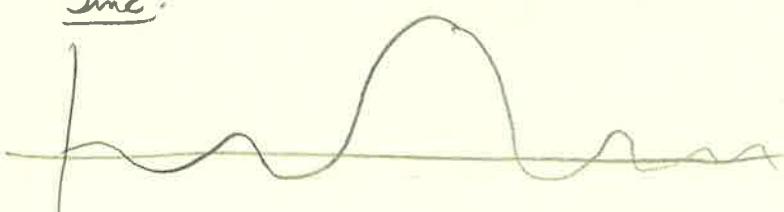


$x: \mathbb{Z} \rightarrow \mathbb{V} (\mathbb{Z} \text{ or } \mathbb{R})$

The Sampling Theorem (1920)

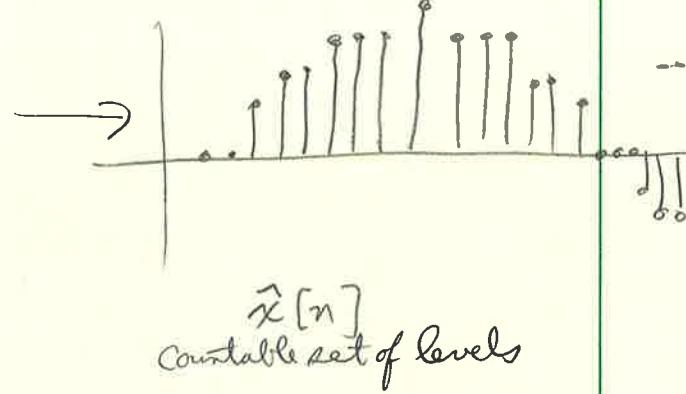
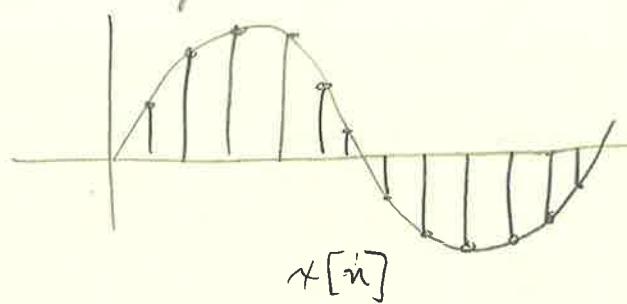
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right)$$

Sinc:



Infinite support

(discrete amplitude)



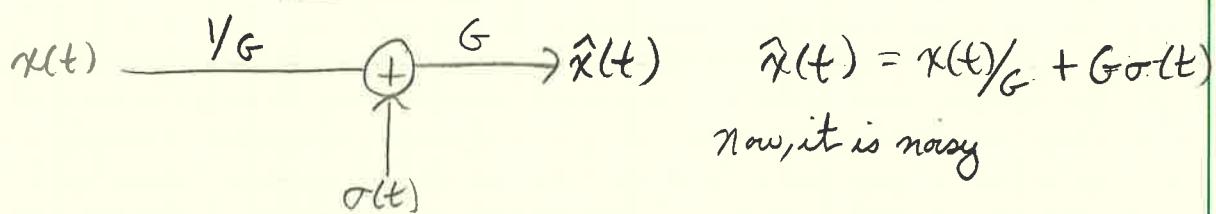
Why is it important?

- storage
- processing
- transmission

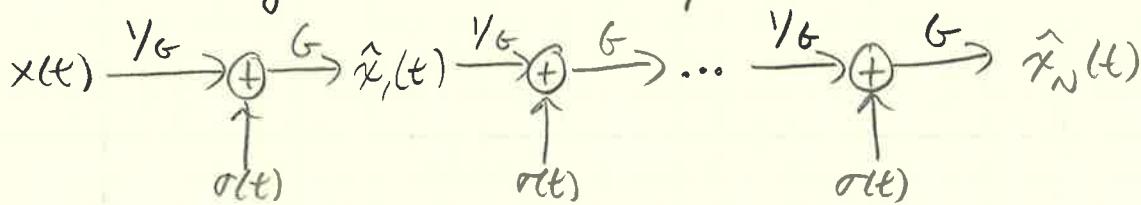
Digital Storage :  $\{0, 1\}$

### Data Transmission

TX  $\rightarrow$  channel  $\rightarrow$  RX

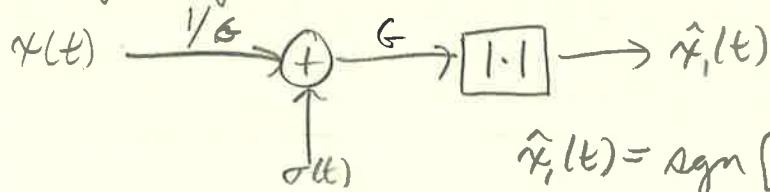


For a long channel, we need repeaters

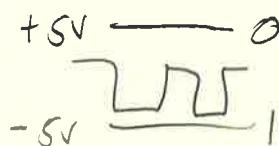


$$\hat{x}_N(t) = x(t) + NG\sigma(t)$$

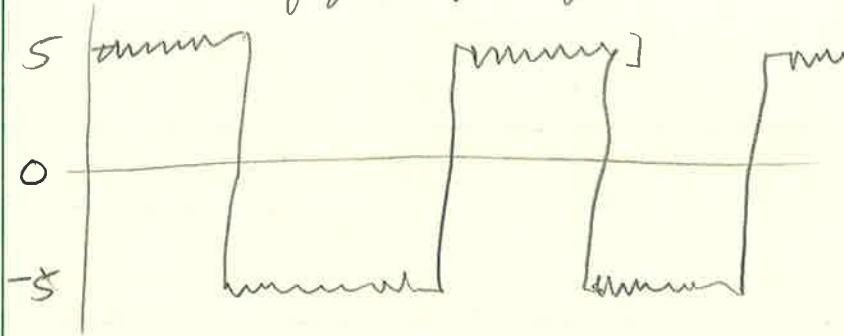
In digital signals, we can threshold



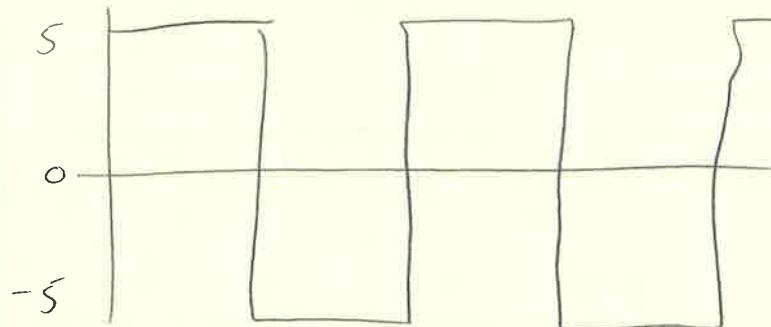
$$\hat{x}_1(t) = \text{sgn} [x(t) + G\sigma(t)]$$



## Transmission of quantized signals



$$G(x(t)/G + \sigma(t)) = x(t) + G(\sigma(t))$$



$$\hat{x}(t) = G \operatorname{sgn}[x(t) + G\sigma(t)]$$

(after thresholding operator)

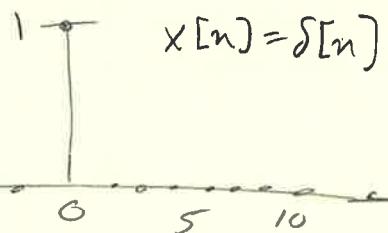
## Digital Signal Processing: Key Ideas

- Discretization of time:
  - samples replace idealized models
  - simple math replaces calculus
- Discretization of values:
  - general-purpose of storage
  - general-purpose processing (CPU)
  - noise can be controlled

## 1.2 Discrete-time signals

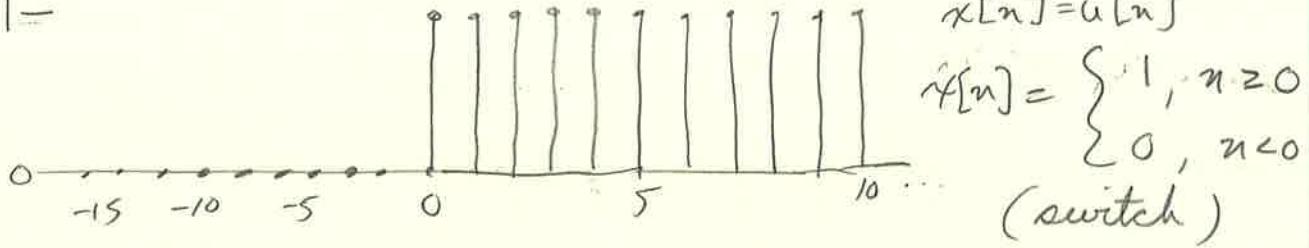
discrete-time signal : a sequence of complex numbers

- One dimension (for now)
- notation :  $x[n]$
- two-sided sequences :  $x: \mathbb{Z} \rightarrow \mathbb{C}$
- $n$  is a-dimensional "time"
- analysis : periodic measurement
- synthesis : stream of generated samples

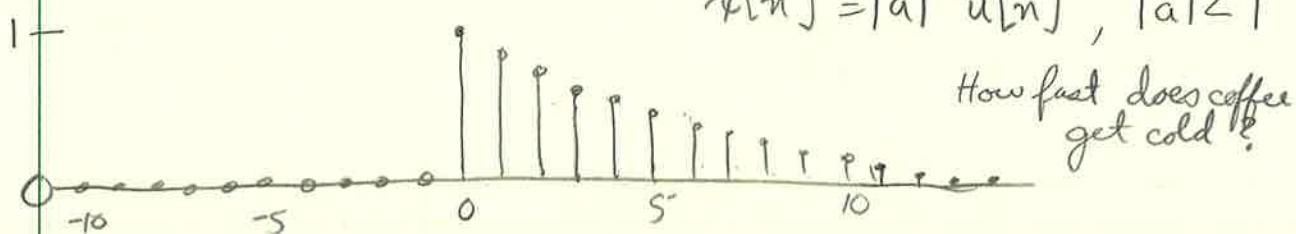


Ex: Used to synchronize audio and video in a movie

1-



Exponential decay



Newton's Law of cooling  $\frac{dT}{dt} = -c(T - T_{\text{env}}) \Rightarrow T(t) = T_{\text{env}} + (T_0 - T_{\text{env}}) e^{-ct}$

Sine wave  $x[n] = \sin(\omega_0 n + \theta)$ ,  $\omega_0, \theta$  in rad

### Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

### Finite-length signals

- sequence notation:  $x[n], n=0, 1, \dots, N-1$
- vector notation:  $\bar{x} = [x_0 \ x_1 \ \dots \ x_{N-1}]^T$
- practical entities, good for numerical packages (e.g. numpy)

### Infinite-length signals

- sequence notation:  $x[n], n \in \mathbb{Z}$
- abstraction, good for theorems

### Periodic signals

- $N$ -periodic sequence:  $\tilde{x}[n] = \tilde{x}[n+kN], n, k, N \in \mathbb{Z}$
- same information as finite-length of length  $N$
- "natural" bridge between finite and infinite lengths

## Finite-support signals

- Finite-support sequence :

$$\bar{x}[n] = \begin{cases} x[n], & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases} \quad n \in \mathbb{Z}$$

- same information as finite-length of length  $N$
- another bridge between finite and infinite lengths

## Elementary operators

- scaling :  $y[n] = \alpha x[n], \alpha \in \mathbb{C}$
  - sum :  $y[n] = x[n] + z[n]$
  - product :  $y[n] = x[n] \cdot z[n]$
  - shift by  $k$  (delay) :  $y[n] = x[n-k], k \in \mathbb{Z}$
- $\left. \begin{matrix} \\ \\ \end{matrix} \right\} 0 \leq n \leq N-1$

Shift of a finite-length : finite-support

$$\dots 000 \boxed{x_0 x_1 \dots x_7} 000 \dots$$

$\bar{x}[n]$

$$\dots 000 \boxed{0 x_0 x_1 x_2 x_3 x_4 x_5 x_6} x_7 0 0 \dots$$

$\bar{x}[n-1]$

$$\dots 000 \boxed{0 0 0 0 x_0 x_1 x_2 x_3} x_4 x_5 x_6 x_7 0 0 \dots$$

$\bar{x}[n-4]$

Shift of a finite length : periodic extension

$$\dots \boxed{x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7} \dots$$

$\bar{x}[n]$

$$\dots x_5 x_6 x_7 \boxed{x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7} x_8 x_9 \dots$$

$\tilde{x}[n]$

$$\dots x_4 x_5 x_6 \boxed{x_7 x_0 x_1 x_2 x_3 x_4 x_5 x_6} x_7 x_8 x_9 \dots$$

$\tilde{x}[n-1]$

$\cdots x_1 x_2 x_3 \boxed{x_4 x_5 x_6 x_7 x_8} x_1 x_2 x_3 x_4 x_5 x_6 \cdots$

### Energy and power

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

### Energy and power: periodic signals

$$\begin{aligned} E_{\tilde{x}} &= \infty \\ P_{\tilde{x}} &= \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2 \end{aligned}$$

## 1.3 Basic signal processing

### 1.3.2 How your PC plays discrete-time sounds

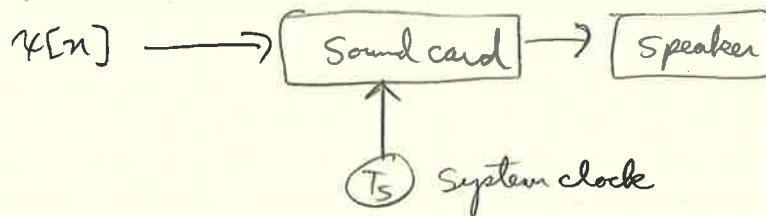
#### The discrete-time sinusoid

$$x[n] = \sin(\omega_0 n + \theta)$$

#### Digital vs. physical frequency

- Discrete time:
  - no: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- Physical world:
  - periodicity: how many seconds before pattern repeats
  - frequency measured in Hz ( $s^{-1}$ )

#### How your PC plays sounds



- set  $T_S$ , time in seconds between samples
- periodicity of  $M$  samples  $\rightarrow$  periodicity of  $MT_S$  seconds
- real world frequency:  $f = \frac{1}{MT_S}$  Hz

- usually we choose  $F_s$ , the number of samples per second

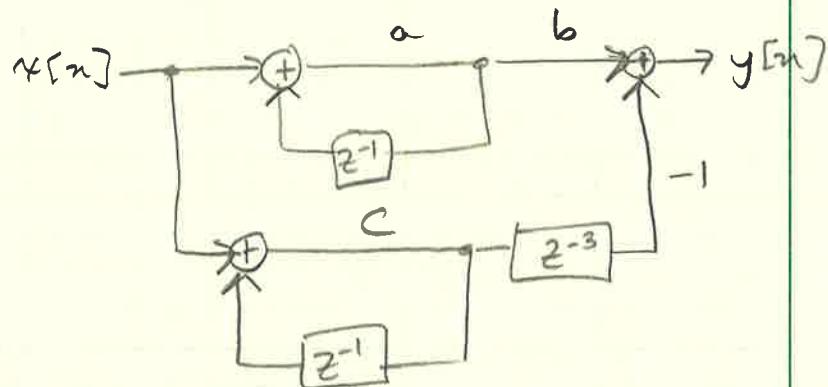
$$\cdot T_s = 1/F_s$$

Eg. for a typical value,  $F_s = 48000 \text{ Hz}$ ,  $T_s \approx 20.8 \mu\text{s}$ .

If  $M = 110$ ,  $f \approx 440 \text{ Hz}$

### 1.3.6 The Kautus-Strong algorithm

#### DSP as Meccano



#### Building blocks:

- Adder:  $x[n]$   $y[n]$   $\rightarrow x[n] + y[n]$

- Multiplier:  $x[n] \xrightarrow{\alpha} \alpha x[n]$

- Unit Delay:  $x[n] \rightarrow [z^{-1}] \rightarrow x[n-1]$

- Arbitrary Delay:  $x[n] \rightarrow [z^{-N}] \rightarrow x[n-N]$

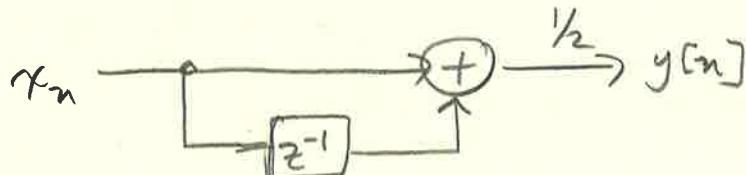
#### The 2-point Moving Average

- simple average:  $M = \frac{a+b}{2}$

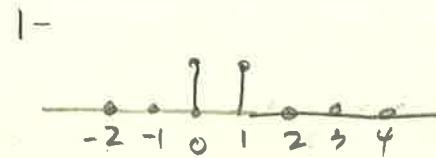
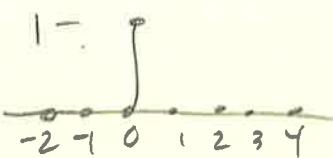
- moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

- DSP Blocks:



Ex:  $x[n] = \delta[n]$



$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$y[1] = \frac{x[1] + x[0]}{2} = \frac{1+1}{2} = 1$$

-  $x[n] = u[n]$

$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1+0}{2} = \frac{1}{2}$$

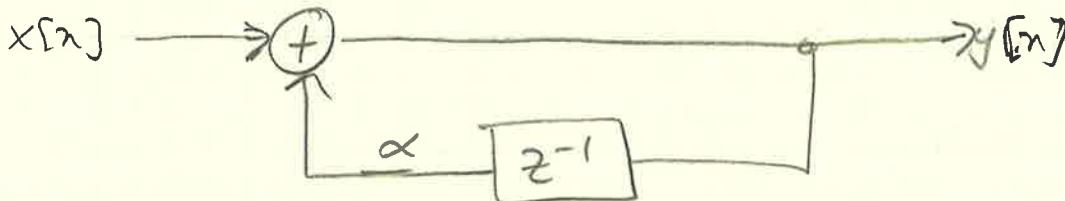
$$y[1] = \frac{x[1] + x[0]}{2} = \frac{1+1}{2} = 1$$

-  $x[n] = \cos(\omega n)$ ,  $\omega = \pi/10$

$$y[n] = \frac{\cos \omega n - \cos \omega(n-1)}{2} = \cos(\omega n + \theta)$$

-  $x[n] = (-1)^n \Rightarrow y[n] = 0, \forall n$

What if we reverse the loop?



$$y[n] = x[n] + \alpha y[n-1], \alpha \in \mathbb{R}$$

(recursion)

How we solve the chicken-and-egg problem

Zero Initial conditions

• set a start time (usually  $n_0 = 0$ )

• assume input and output are zero for all time before  $n_0$

Ex: A simple model for banking

A simple equation to describe compound interest:

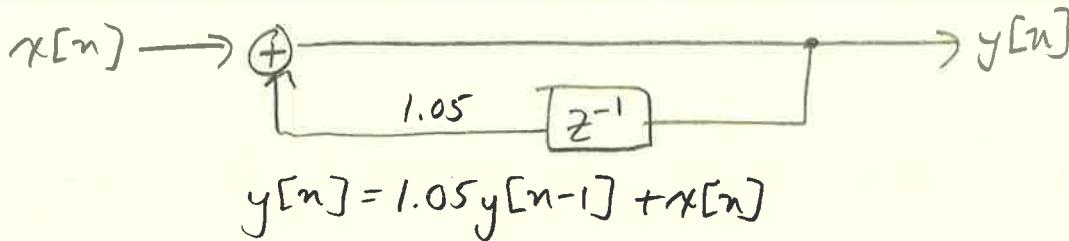
- constant interest/borrowing rate of 5% per year

- interest accrues on Dec 31

- deposits/withdrawals during year  $n$ :  $x[n]$

- balance at year  $n$ :

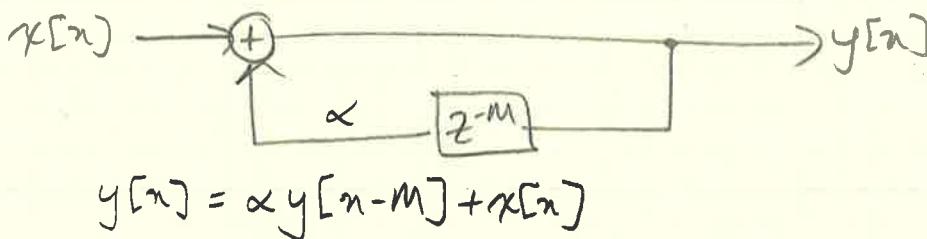
$$y[n] = 1.05y[n-1] + x[n]$$



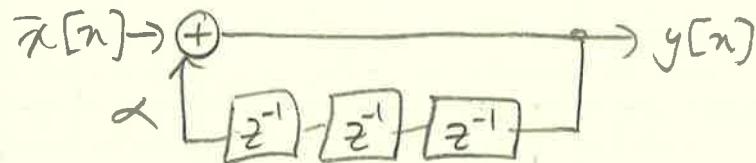
Ex: One-time investment  $x[n] = 100\delta[n]$

- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25, y[3] = 115.7625$ , etc.
- In general:  $y[n] = (1.05)^n / 100 u[n]$

An interesting generalization



• Creating loops



Ex:  $M=3, \alpha=0.7, x[n] = \delta[n]$

- $y[0] = 1, y[1] = 0, y[2] = 0$
- $y[3] = 0.7, y[4] = 0, y[5] = 0$
- $y[6] = 0.7^2, y[7] = 0, y[8] = 0$ , etc.

Ex:  $M=3, \alpha=1, x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

- $y[0] = 1, y[1] = 2, y[2] = 3$
- $y[3] = 1, y[4] = 2, y[5] = 3$
- $y[6] = 1, y[7] = 2, y[8] = 3$ , etc.

(We can make music with that!)

- build a recursion loop with a delay of  $M$
- choose a signal  $\bar{x}[n]$  that is nonzero only for  $0 \leq n < M$
- choose a decay factor
- input  $\bar{x}[n]$  to the system
- play the output

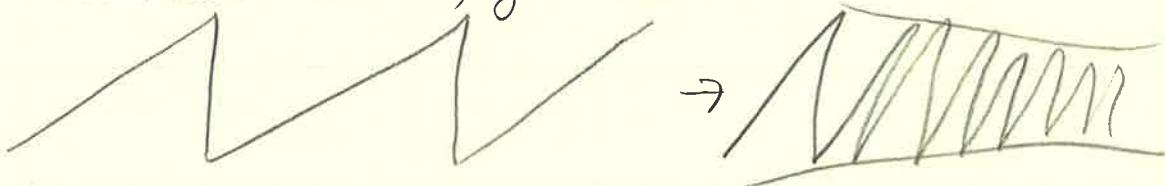
Ex:  $M=100$ ,  $\alpha=1$ ,  $\bar{x}[n] = \sin(2\pi n/100)$  for  $0 \leq n < 100$  and zero elsewhere

$$F_S = 48 \text{ kHz} \rightarrow 480 \text{ Hz}$$

### Introducing some realism

- $M$  controls frequency (pitch)
- $\alpha$  controls envelope (decay)
- $\bar{x}[n]$  controls color (timbre)

Proto-violin:  $M=100$ ,  $\alpha=0.95$ ,  $\bar{x}[n]$ : zero-mean sawtooth wave between 0 and 99, zero elsewhere



### The Karplus - Strong Algorithm

$M=100$ ,  $\alpha=0.9$ ,  $\bar{x}[n]$ : 100 random values between 0 and 99, zero elsewhere  $\stackrel{\text{in } [-1, 1]}{\sim}$

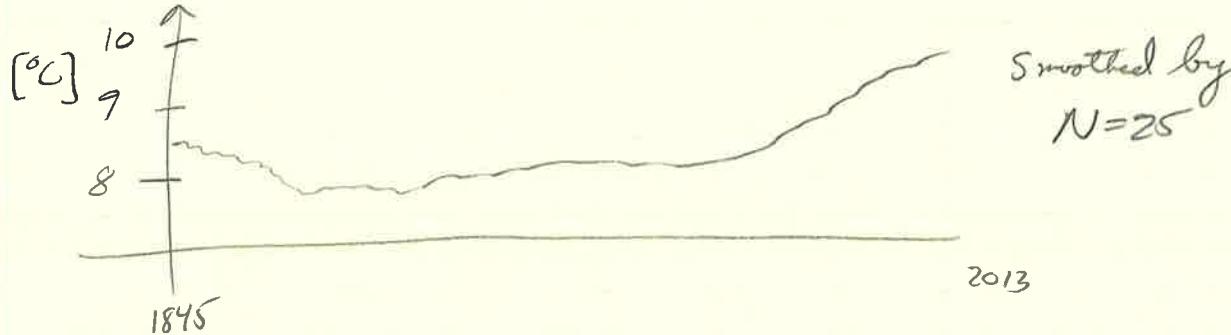
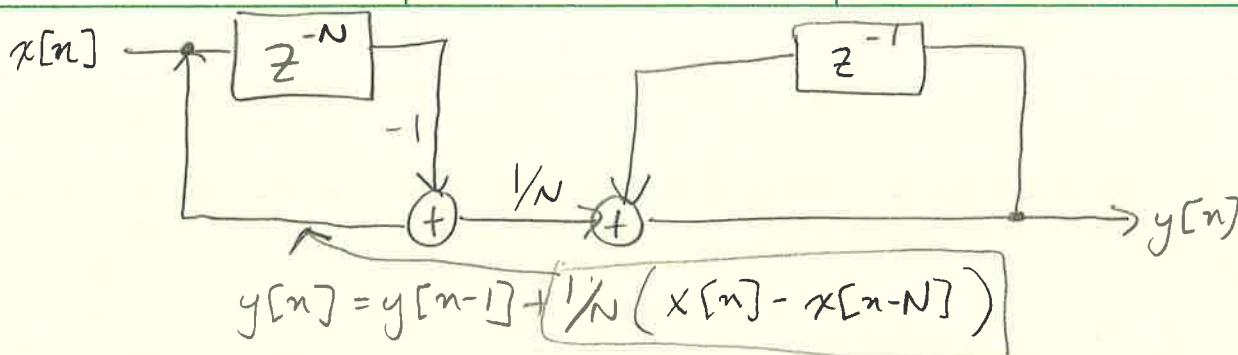
Similar to a harpsichord.

### Signal of the Day: Goethe's Temperature Measurement

Smoothing { Moving average :  $y[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[n-m]$   
 $N$ : window of last observations over which the average is computed

#### A recursive method

$$\begin{aligned} y[n] &= \frac{1}{N} \sum_{m=0}^{N-1} x[n-m] \\ &= \frac{1}{N} x[n] + \frac{1}{N} \underbrace{\sum_{m=1}^{N-1} x[n-m]}_{y[n-1]} + \frac{1}{N} x[n-N] - \frac{1}{N} x[n-N] \\ &= y[n-1] + \frac{1}{N} (x[n] - x[n-N]) \end{aligned}$$

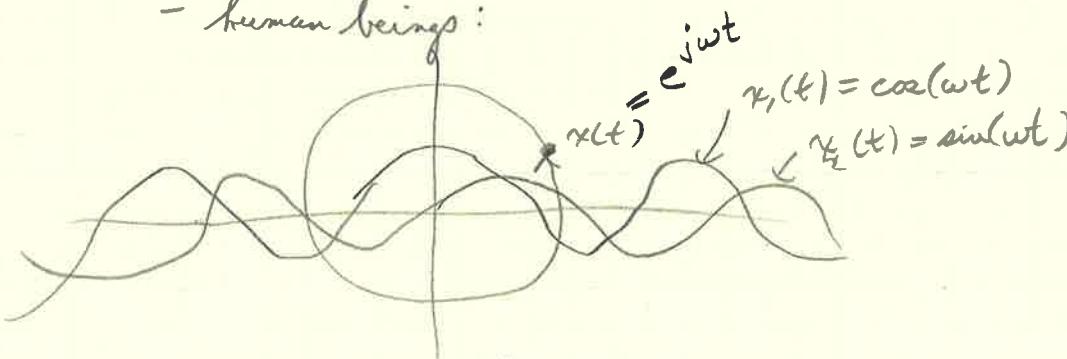


#### 1.4 Complex exponentials

$$j = \sqrt{-1}$$

Oscillations are everywhere!

- Sustainable dynamic systems exhibit oscillatory behavior
- Intuitively: things that don't move in circles can't last:
  - bowls
  - rockets
  - human beings



- The discrete-time oscillatory heartbeat  
Ingredients:

- a frequency  $\omega$  (units: radians)
- an initial phase  $\phi$  (units: radians)
- an amplitude  $A$

$$\begin{aligned} x[n] &= Ae^{j(\omega n + \phi)} \\ &= A [\cos(\omega n + \phi) + j \sin(\omega n + \phi)] \end{aligned}$$

Why complex exponentials?

- we can use complex numbers in digital systems, so why not?
- it makes sense: every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra

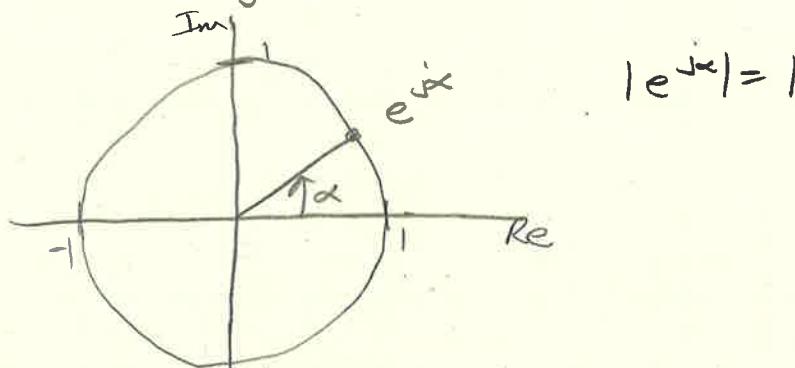
Ex: change the phase of a cosine the "old-school" way

$$\cos(\omega n + \phi) = a \cos(\omega n) - b \sin(\omega n), \quad a = \cos \phi, \quad b = \sin \phi$$

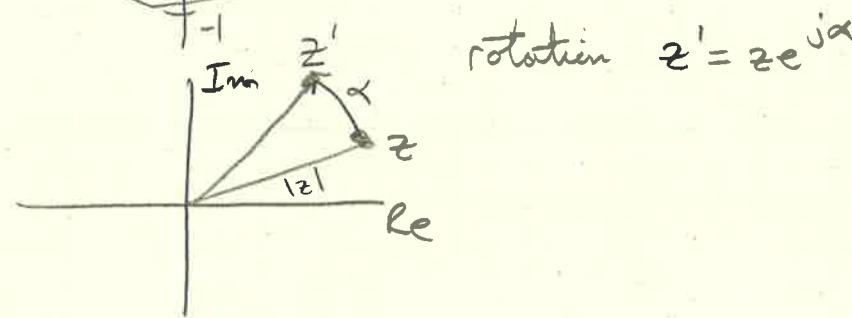
use complex exponentials

$$\cos(\omega n + \phi) = \operatorname{Re}[e^{j(\omega n + \phi)}] = \operatorname{Re}[e^{j\omega n} e^{j\phi}]$$

$$e^{j\alpha} = \cos \alpha + j \sin \alpha$$



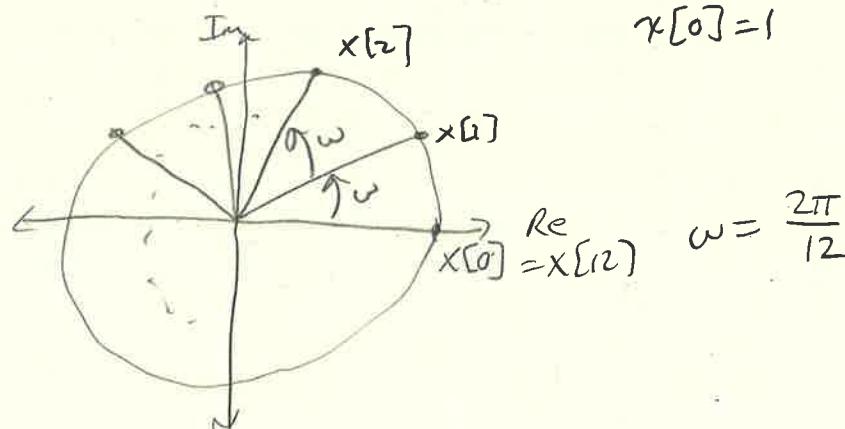
$$|e^{j\alpha}| = 1$$



$$\text{rotation } z' = z e^{j\alpha}$$

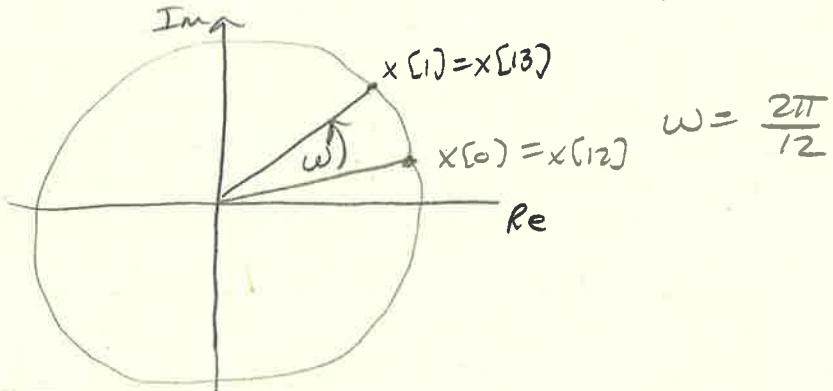
The complex exponential generating machine

$$x[n] = e^{j\omega n}; \quad x[n+1] = e^{j\omega} x[n]$$



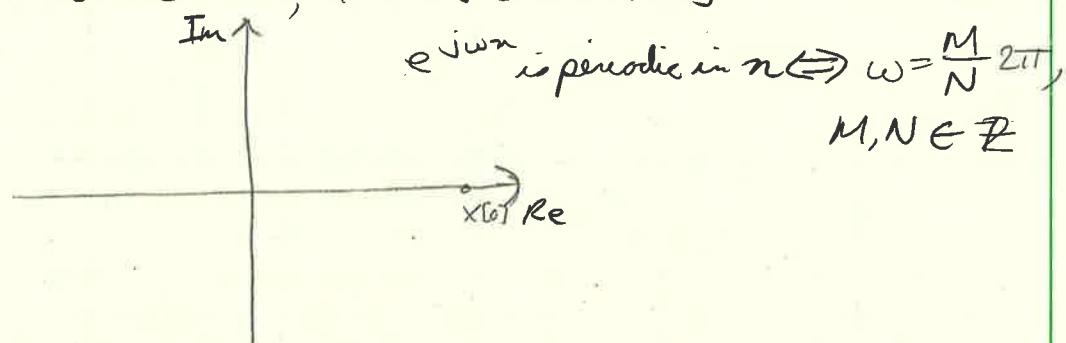
$$x[0] = 1$$

Initial phase  
 $x[n] = e^{j(\omega n + \phi)}$ ;  $x[n+1] = e^{j\omega} x[n]$ ,  $x[0] = e^{j\phi}$



Careful: not every sinusoid is periodic in discrete time

$$x[n] = e^{j\omega n}; x[n+1] = e^{j\omega} x[n]$$



$$x[n] = x[n+N]$$

$$e^{j(\omega n + \phi)} = e^{j(\omega(n+N) + \phi)}$$

$$e^{j\omega n} e^{j\phi} = e^{j\omega n} e^{j\omega N} e^{j\phi}$$

$$e^{j\omega N} = 1 \Leftrightarrow \omega N = 2M\pi, M \in \mathbb{Z}$$

$$\omega = \frac{M}{N} 2\pi$$

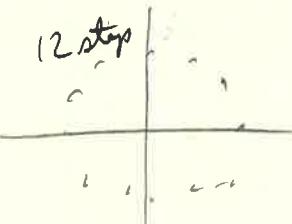
2π-periodicity: one point, many names

$$e^{j\alpha} = e^{j(\alpha + 2\pi k)}, \forall k \in \mathbb{Z}$$

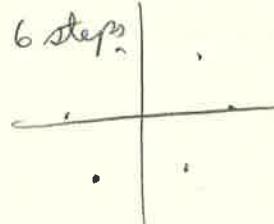
One point, many names: Aliasing

How "fast" can we go?

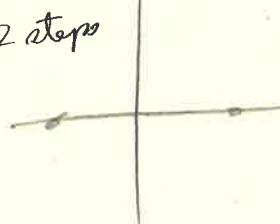
$$\omega = \frac{2\pi}{12}$$



$$\omega = 2\pi/6$$



$$\omega = 2\pi/2$$

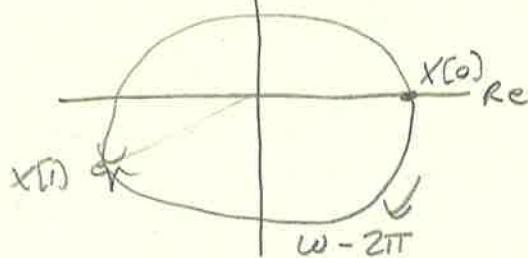


What if we go faster?

$$\pi < \omega < 2\pi$$

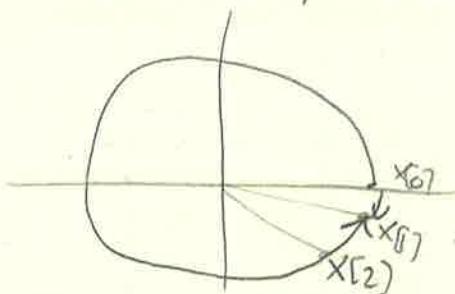
Im

corresponds to going slower  
in opposite direction



$$\omega = 2\pi - \alpha, \alpha \text{ small}$$

very slow in opposite  
direction



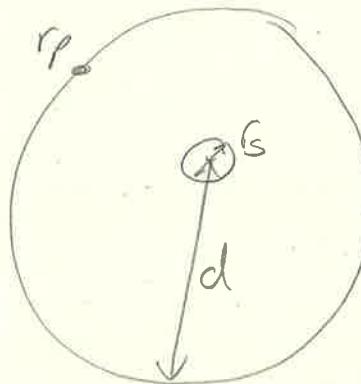
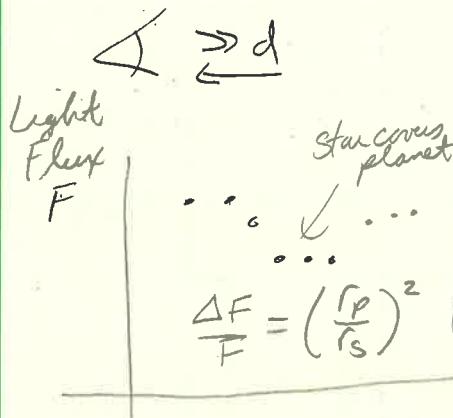
## 2.1 Signal processing and vector spaces

2.1

Common framework: vector space

- vector spaces are very general objects
- vector spaces are defined by their properties
- once you know the properties are satisfied, you can use all the tools for the space

Signal of the day: exoplanet hunting



$$\frac{\Delta F}{F} = \left( \frac{r_p}{r_s} \right)^2 \quad (\text{transit depth})$$

$$\text{- Earth: } \frac{\Delta F}{F} = \left( \frac{r_p}{r_s} \right)^2 = \left( \frac{6,371}{696,000} \right)^2 \approx 0.01\%$$

$$\text{- Jupiter: } \frac{\Delta F}{F} = \left( \frac{69,911}{696,000} \right)^2 \approx 1\%$$

• Best telescope today can detect a transit depth of 0.1%.



## 2.2 Vector Spaces

### 2.2a Vector space

Some familiar examples

•  $\mathbb{R}^2, \mathbb{R}^3$ : Euclidean space

•  $\mathbb{R}^N, \mathbb{C}^N$ : linear algebra

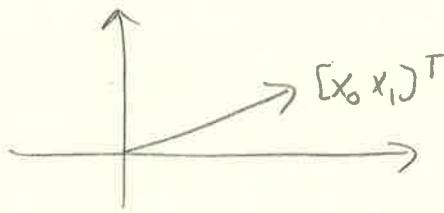
Other examples:

•  $\ell_2(\mathbb{Z})$ : space of square-summable infinite sequences

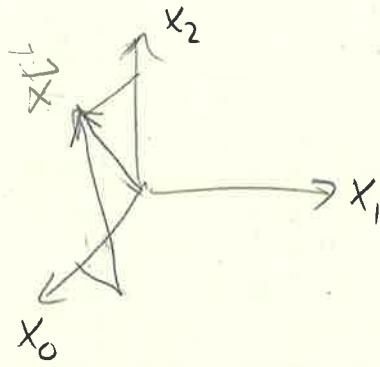
•  $L_2([a,b])$ : space of square-integrable functions over an interval

Some can be represented geometrically

$$\mathbb{R}^2: \vec{x} = [x_0 \ x_1]^T$$

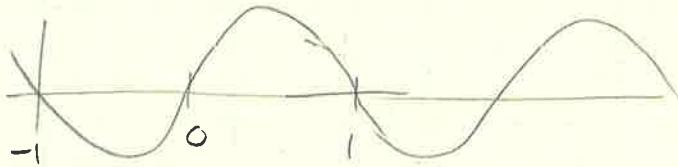


$$\mathbb{R}^3: \vec{x} = [x_0 \ x_1 \ x_2]^T$$



$$L_2([-1, 1]): \vec{x} = x(t), t \in [-1, 1]$$

$$\vec{x} = \sin(\pi t)$$



Can't plot  $\mathbb{R}^N, N > 3$  or  $\mathbb{C}^N, N > 1$

### Ingredients

- the set of vectors  $V$
- a set of scalars (say  $\mathbb{C}$ )

We need at least to be able to:

- resize vectors, i.e., multiply a vector by a scalar
- combine vectors together, i.e., sum them

Formal properties: For  $\vec{x}, \vec{y}, \vec{z} \in V$  and  $\alpha, \beta \in \mathbb{C}$ :

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

$$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$$

$$\alpha(\vec{x} + \vec{y}) = \alpha\vec{y} + \alpha\vec{x}$$

$$(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$$

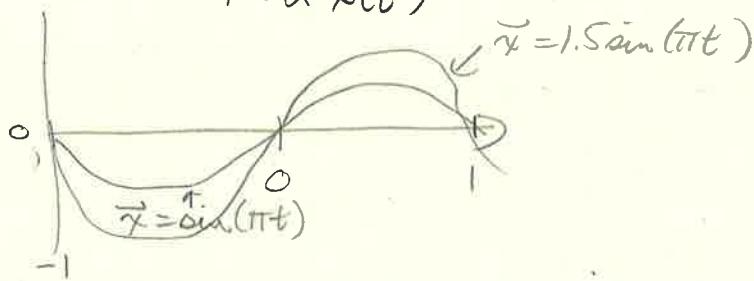
$$\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$$

$$\exists 0 \in V : \vec{x} + 0 = 0 + \vec{x} = \vec{x}$$

$$\forall \vec{x} \in V, \exists (-\vec{x}) : \vec{x} + (-\vec{x}) = 0$$

Scalar multiplication in  $L_2[-1, 1]$

$$\alpha \vec{x} = \alpha x(t)$$



We need something more: inner product (aka dot product)

$$\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$$

- measure of similarity between vectors

- inner product is zero? vectors are orthogonal (maximally different)

Formal properties of the inner product

For  $\vec{x}, \vec{y}, \vec{z} \in V, \alpha \in \mathbb{C}$ :

- $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

- $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$

- $\langle \alpha \vec{x}, \vec{y} \rangle = \alpha^* \langle \vec{x}, \vec{y} \rangle$

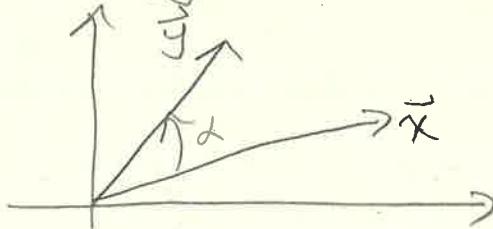
- $\langle \vec{x}, \alpha \vec{y} \rangle = \alpha \langle \vec{x}, \vec{y} \rangle$

- $\langle \vec{x}, \vec{x} \rangle \geq 0$

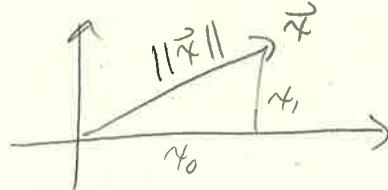
- $\langle \vec{x}, \vec{x} \rangle = 0 \Leftrightarrow \vec{x} = 0$

- If  $\langle \vec{x}, \vec{y} \rangle = 0$  and  $\vec{x}, \vec{y} \neq 0$ , then  $\vec{x}$  and  $\vec{y}$  are called orthogonal

$$\langle \vec{x}, \vec{y} \rangle = x_0 y_0 + x_1 y_1 = \|\vec{x}\| \|\vec{y}\| \cos \varphi$$



$$\langle \vec{x}, \vec{x} \rangle = x_0^2 + x_1^2 = \|\vec{x}\|^2$$

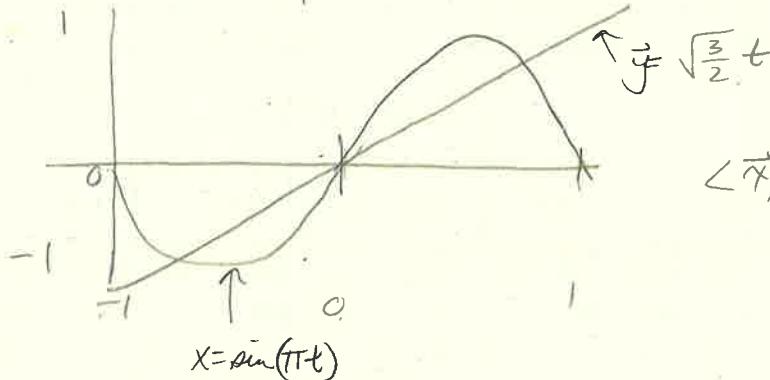


Inner product in  $L_2[-1, 1]$

$$\langle \vec{x}, \vec{y} \rangle = \int_{-1}^1 x(t) y(t) dt$$

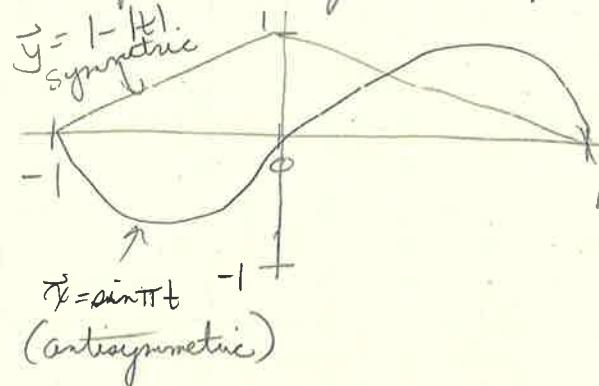
$$\| \sin(\pi t) \| = \sqrt{\int_{-1}^1 \sin^2 \pi t dt} = 1$$

$$\vec{y} = t: \| \vec{y} \| = \sqrt{\int_{-1}^1 t^2 dt} = \frac{2}{3}$$



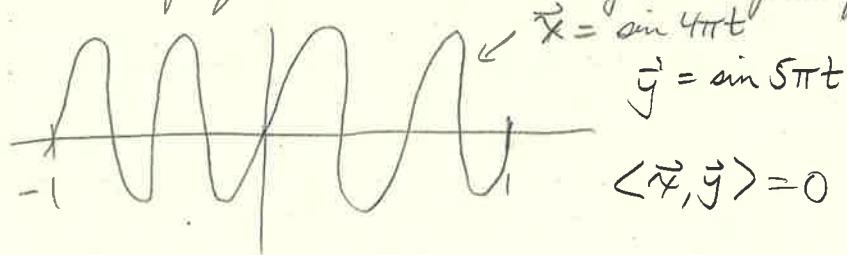
$$\begin{aligned}\langle \vec{x}, \vec{y} \rangle &= \int_{-1}^1 \sqrt{\frac{3}{2}} t \sin \pi t dt \\ &= \frac{2}{\pi} \sqrt{\frac{3}{2}} \approx 0.78\end{aligned}$$

$\vec{x}, \vec{y}$  from orthogonal subspaces:



$$\langle \vec{x}, \vec{y} \rangle = 0$$

Sinusoids with frequencies that are integer multiples of a fundamental



$$\langle \vec{x}, \vec{y} \rangle = 0$$

Norm vs Distance

- inner product defines a norm:  $\| \vec{x} \| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

- norm defines a distance:  $d(\vec{x}, \vec{y}) = \| \vec{x} - \vec{y} \|$

Distance in  $L_2[-1, 1]$ : the Mean Square Error

$$\| \vec{x} - \vec{y} \|^2 = \int_{-1}^1 |x(t) - y(t)|^2 dt$$

$$\vec{x} = \sin 4\pi t, \quad \vec{y} = \sin 5\pi t, \quad \| \vec{x} - \vec{y} \|^2 = \int_{-1}^1 |\sin 4\pi t - \sin 5\pi t|^2 dt = 2$$

## 2.2.b Signal Spaces

### Finite-Length Signals

finite-length and periodic signals live in  $\mathbb{C}^N$

- vector notation:  $\vec{x} = [x_0 \ x_1 \ \dots \ x_{N-1}]^T$

- all operations well-defined and intuitive

- space of  $N$ -periodic signals sometimes indicated by  $\widetilde{\mathbb{C}}^N$

### Inner product for signals

$$\langle \vec{x}, \vec{y} \rangle = \sum_{n=0}^{N-1} x^*[n] y[n]$$

well-defined for all finite-length vectors

Infinite Signals?  $\langle \vec{x}, \vec{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n] y[n]$

We require sequences to be square-summable:  $\sum |x[n]|^2 < \infty$   
i.e. in  $\ell_2(\mathbb{Z})$  (finite-energy)

Many interesting signals are not in  $\ell_2(\mathbb{Z})$ , such as,

$$x[n] = 1, \quad x[n] = \cos(\omega n), \text{ etc.}$$

### Completeness

Limiting operations must yield vector space elements

An incomplete space:  $\mathbb{Q}$      $x_n = \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q},$

but  $\lim_{n \rightarrow \infty} x_n = e \notin \mathbb{Q}$

### Hilbert Space

1. A vector space:  $H(V, \mathbb{C})$

2. An inner product:  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$

3. Complete

Bases

Linear combination is the basic operation in vector spaces:

$$\vec{g} = \alpha \vec{x} + \beta \vec{y}$$

Can we find a set of vectors  $\{\vec{w}^{(k)}\}$  so that we can write any vector as a linear combination of the  $\{\vec{w}^{(k)}\}$ ?

Canonical  $\mathbb{R}^2$  basis

$$\vec{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Another  $\mathbb{R}^2$  basis

$$\vec{v}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \alpha_0 = x_0 - x_1, \alpha_1 = x_1$$

Not a basis for  $\mathbb{R}^2$ 

$$\vec{g}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{g}^{(1)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{not linearly independent}$$

What about infinite-dimensional spaces?

$$\vec{x} = \sum_{k=0}^{\infty} \alpha_k \vec{w}^{(k)}$$

a basis for  $l_2(\mathbb{R})$ 

$$\vec{e}^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad 1 \text{ in } k^{\text{th}} \text{ position, } k \in \mathbb{Z}$$

What about function vector spaces?

$$f(t) = \sum_k \alpha_k h^{(k)}(t)$$

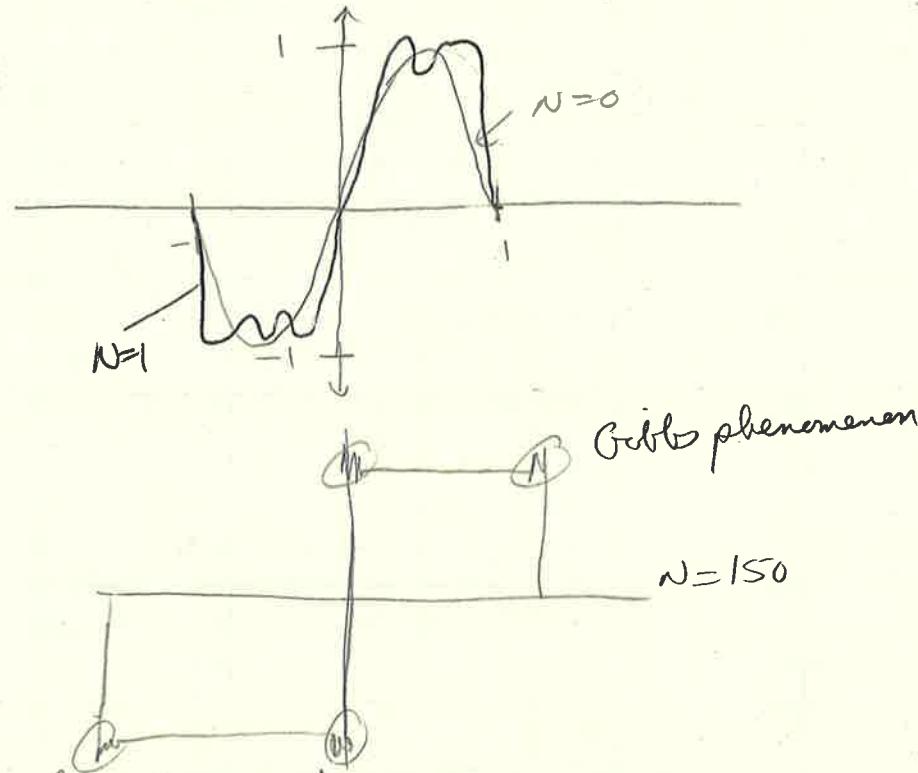
A basis for the functions over an interval?

the Fourier basis for  $[-1, 1]$

$$\left\{ \frac{1}{\sqrt{2}}, \cos \pi t, \sin \pi t, \cos 2\pi t, \sin 2\pi t, \cos 3\pi t, \sin 3\pi t, \dots \right\}$$

Using the Fourier Basis (approximating a square wave)

$$\sum_{k=0}^N \frac{\sin((2k+1)\pi t)}{2k+1} = \sum_{k=0}^N \frac{w^{(4k+2)}}{2k+1}$$



Bases: formal definition

Given:

- a vector space  $H$
- a set of  $K$  vectors from  $H$ :  $W = \{\vec{w}^{(k)}\}_{k=0,1,\dots,K-1}$

$W$  is a basis for  $H$  if:

1. We can write for all  $x \in H$ :

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

2. the coefficients  $\alpha_k$  are unique

Uniqueness implies linear independence

$$\sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} = 0 \Rightarrow \alpha_k = 0, \quad k=0,1,\dots,K-1$$

Special bases

Orthogonal basis:  
 $\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = 0, \quad k \neq n$

Orthonormal bases:  $\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = \delta_{[n-k]}$

We can use Gram-Schmidt to normalize any orthogonal basis

Basis expansion

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)}, \text{ how do we find the } \alpha's?$$

Orthonormal bases are the best:  $\alpha_k = \langle \vec{w}^{(k)}, \vec{x} \rangle$

Change of basis

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \vec{v}^{(k)}$$

If  $\{\vec{v}^{(k)}\}$  is orthonormal:

$$\begin{aligned} \beta_k &= \langle \vec{v}^{(k)}, \vec{x} \rangle \\ &= \left\langle \vec{v}^{(k)}, \sum_{h=0}^{K-1} \alpha_h \vec{w}^{(h)} \right\rangle = \sum_{h=0}^{K-1} \alpha_h \langle \vec{v}^{(k)}, \vec{w}^{(h)} \rangle \end{aligned}$$

$$= \sum_{h=0}^{K-1} \alpha_h C_{hk}$$

$$= \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0(K-1)} \\ & \vdots & & \\ c_{(K-1)0} & c_{(K-1)1} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix}$$

Change of basis: example

- canonical basis  $E = \{\vec{e}^{(0)}, \vec{e}^{(1)}\}$

$$\vec{x} = \alpha_0 \vec{e}^{(0)} + \alpha_1 \vec{e}^{(1)}$$

- new basis  $V = \{\vec{v}^{(0)}, \vec{v}^{(1)}\}$  with  $\vec{v}^{(0)} = [\cos \theta \sin \theta]^T$   
 $\vec{v}^{(1)} = [-\sin \theta \cos \theta]^T$

$$\vec{x} = \beta_0 \vec{v}^{(0)} + \beta_1 \vec{v}^{(1)}$$

- new basis is orthonormal:  $C_{hk} = \langle \vec{v}^{(h)}, \vec{e}^{(k)} \rangle$

- in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = R \alpha$$

-  $R$ : rotation matrix

$$R^T R = I$$

Vector Subspace

- A subset of vectors closed under addition and scalar multiplication.
- Example:  $\mathbb{R}^2 \subset \mathbb{R}^3$
- Subspace of symmetric functions over  $L_2[-1, 1]$   
 $\vec{x} = \cos \pi t$   
 $\vec{y} = \cos 5\pi t$  to name a couple

- Subspaces have their own bases

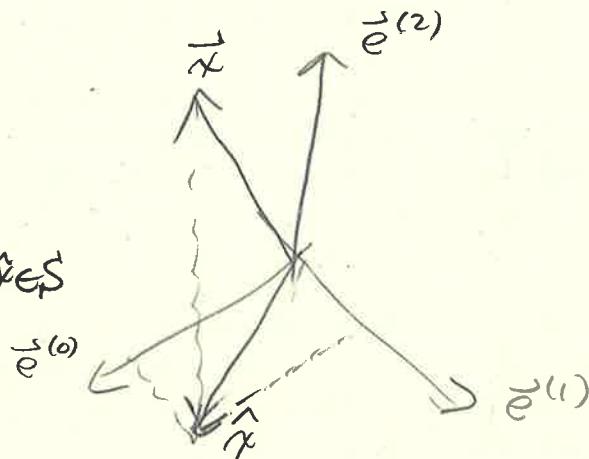
$$\left\{ \vec{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ basis for a plane}$$

Approximation

Problem:

- vector  $x \in V$
- subspace  $S \subseteq V$

- approximate  $\vec{x}$  with  $\vec{x} \in S$

Least-Squares Approximation

- $\{\vec{s}^{(k)}\}_{k=0, \dots, K-1}$  orthonormal basis for  $S$

- orthogonal projection:

$$\hat{x} = \sum_{k=0}^{K-1} \langle \vec{s}^{(k)}, \vec{x} \rangle \vec{s}^{(k)}$$

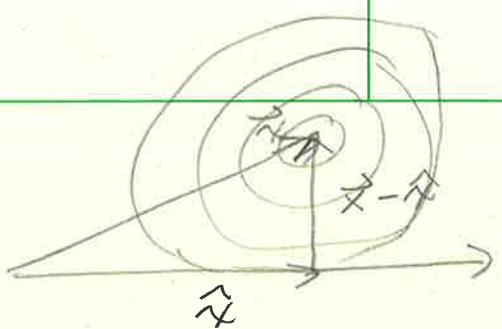
- orthogonal projection is the "best" approximation over  $S$

- orthogonal projection has minimum-norm error:

$$\underset{y \in S}{\operatorname{argmin}} \| \vec{x} - \vec{y} \| = \hat{x}$$

- error is orthogonal to approximation:

$$\langle \vec{x} - \hat{x}, \hat{x} \rangle = 0$$



draw concentric circles  
until hitting  $\vec{s}$ . This radius  
vector is  $\vec{r} - \vec{x}$ .

### Example: polynomial approximation

- vector space  $P_N[-1, 1] \subset L_2[-1, 1]$
- $\vec{p} = a_0 + a_1 t + \dots + a_{N-1} t^{N-1}$
- a self-evident, naive basis:  $\vec{s}^{(k)} = t^k$ ,  $k=0, 1, \dots, N-1$
- naive basis is not orthonormal

goal: approximate  $\vec{x} = \sin t \in L_2[-1, 1]$  over  $P_3[-1, 1]$

- build orthonormal basis from naive basis
- project  $\vec{x}$  over the orthonormal basis
- compute approximation error
- compare errors to Taylor approximation (well known but not optimal over the interval)

### Building an orthonormal basis

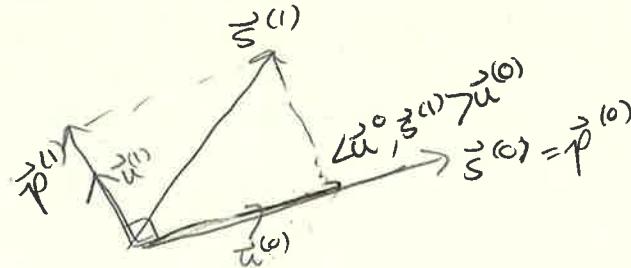
Gram-Schmidt orthonormalization procedure:

$$\left\{ \vec{s}^{(k)} \right\}_{\text{original set}} \rightarrow \left\{ \vec{u}^{(k)} \right\}_{\text{orthonormal set}}$$

Algorithmic procedure: at each step  $k$

$$1. \vec{p}^{(k)} = \vec{s}^{(k)} - \sum_{n=0}^{k-1} \langle \vec{u}^{(n)}, \vec{s}^{(k)} \rangle \vec{u}^{(n)}$$

$$2. \vec{u}^{(k)} = \vec{p}^{(k)} / \| \vec{p}^{(k)} \|$$



Apply Gram-Schmidt to  $S = \{1, t, t^2, t^3, \dots\}$

$$\langle \vec{x}, \vec{y} \rangle = \int_{-1}^1 x(t) y(t) dt$$

$$\rightarrow \vec{s}^{(0)} = 1$$

$$\cdot \vec{p}^{(0)} = \vec{s}^{(0)} = 1$$

$$\cdot \| \vec{p}^{(0)} \|^2 = 2$$

$$\cdot \vec{u}^{(0)} = \vec{p}^{(0)} / \| \vec{p}^{(0)} \| = \frac{1}{\sqrt{2}}$$

$$\rightarrow \vec{s}^{(1)} = t$$

$$\cdot \langle \vec{u}^{(0)}, \vec{s}^{(1)} \rangle = \int_{-1}^1 \frac{t}{\sqrt{2}} dt = 0$$

$$\cdot \vec{p}^{(1)} = \vec{s}^{(1)} = t$$

$$\cdot \| \vec{p}^{(1)} \|^2 = \frac{2}{3}$$

$$\cdot \vec{u}^{(1)} = \sqrt{\frac{3}{2}} t$$

$$\rightarrow \vec{s}^{(2)} = t^2$$

$$\cdot \langle \vec{u}^{(0)}, \vec{s}^{(2)} \rangle = \int_{-1}^1 \frac{t^2}{\sqrt{2}} dt = \frac{2}{3\sqrt{2}}$$

$$\cdot \langle \vec{u}^{(1)}, \vec{s}^{(2)} \rangle = \int_{-1}^1 t^3 / \sqrt{2} dt = 0$$

$$\cdot \vec{p}^{(2)} = \vec{s}^{(2)} - \frac{2}{3\sqrt{2}} \vec{u}^{(0)} = t^2 - \frac{1}{3}$$

$$\cdot \| \vec{p}^{(2)} \|^2 = \frac{8}{45}$$

$$\cdot \vec{u}^{(2)} = \sqrt{\frac{5}{8}} (3t^2 - 1)$$

## Legendre Polynomials

The Gram-Schmidt algorithm leads to an orthonormal basis for  $P_n([-1, 1])$

$$\vec{u}^{(0)} = \sqrt{\frac{1}{2}}, \vec{u}^{(1)} = \sqrt{\frac{3}{2}} t, \vec{u}^{(2)} = \sqrt{\frac{5}{8}} (3t^2 - 1), \vec{u}^{(3)} = \dots$$

## Orthogonal projection over $P_3[-1, 1]$

$$\alpha_k = \langle \vec{u}^{(k)}, \vec{x} \rangle = \int_{-1}^1 u_k(t) \sin t dt$$

$$\cdot \alpha_0 = \langle \frac{1}{\sqrt{2}}, \sin t \rangle = 0$$

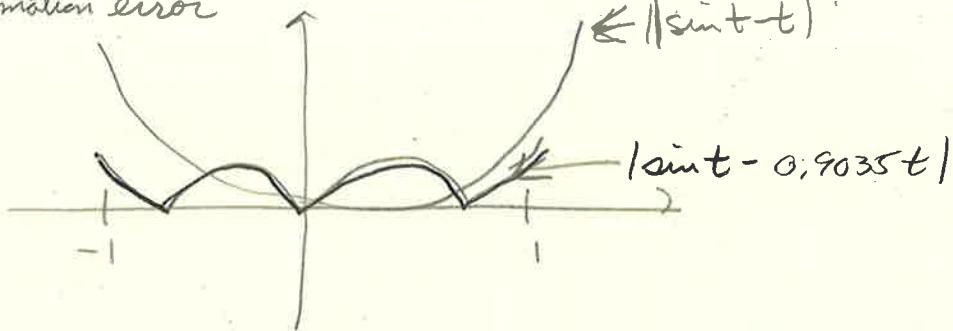
$$\cdot \alpha_1 = \langle \sqrt{\frac{3}{2}} t, \sin t \rangle \approx 0.7377$$

$$\cdot \alpha_2 = \langle \sqrt{\frac{5}{8}} (3t^2 - 1), \sin t \rangle = 0$$

$$\sin t \rightarrow \alpha_1 \vec{u}^{(1)} \approx 0.9035 t$$

Taylor Series:  $\sin t \approx t$

Approximation error



Error norm:

Orthogonal projection over  $P_3[-1, 1]$ :

$$\| \sin t - \alpha_i \bar{u}^{(1)} \| \approx 0.0337$$

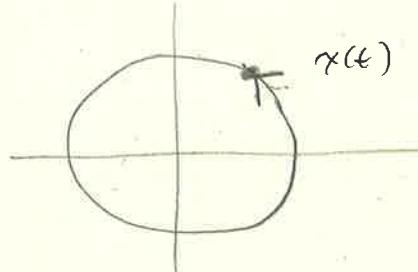
Taylor series:  $\| \sin t - t \| \approx 0.0857$

## 3.1.a The frequency domain

- Oscillations are everywhere

- Sustainable dynamic systems exhibit oscillatory behavior
- Intuitively: things that don't move in circles don't last:
  - bombs
  - rockets
  - human beings...

Period  $P$   
Frequency  $f = \frac{1}{P}$



- The intuition

- humans analyse complex signals (audio, images) in terms of their sinusoidal components
- We can build instruments that "resonate" at one or multiple frequencies (tuning fork vs. piano)
- the "frequency domain" seems to be as important as the time domain
- Fundamental question: can we decompose any signal into sinusoidal elements? Yes, using Fourier analysis

## Analysis

- from time domain to frequency domain
- find the contribution of different frequencies
- discover "hidden" signal properties

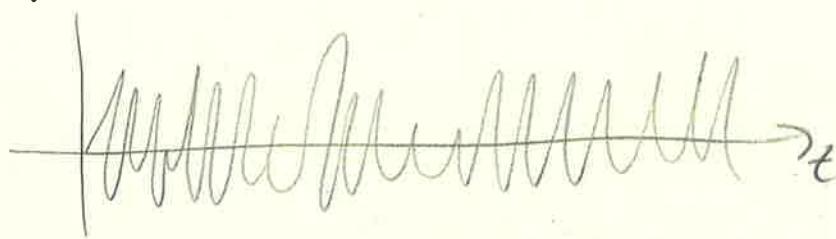
## Synthesis

- from frequency domain to time domain
- create signals with known frequency content
- fit signals to specific frequency regions

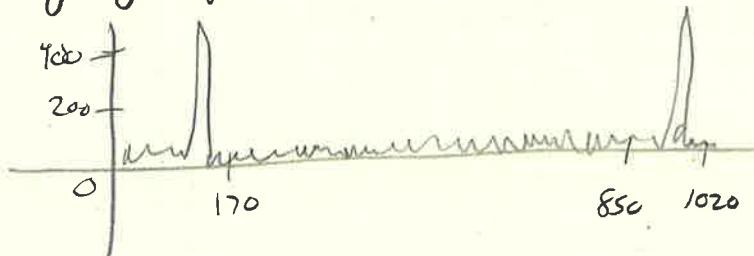
## 3.1.b The DFT as a change of basis

- The mathematical setup
  - let's start with finite-length signals (i.e. vectors in  $\mathbb{C}^N$ )
  - Fourier analysis is a simple change of basis
  - a change of basis is a change of perspective

- Mystery signal in time domain



- Mystery signal in the Fourier basis

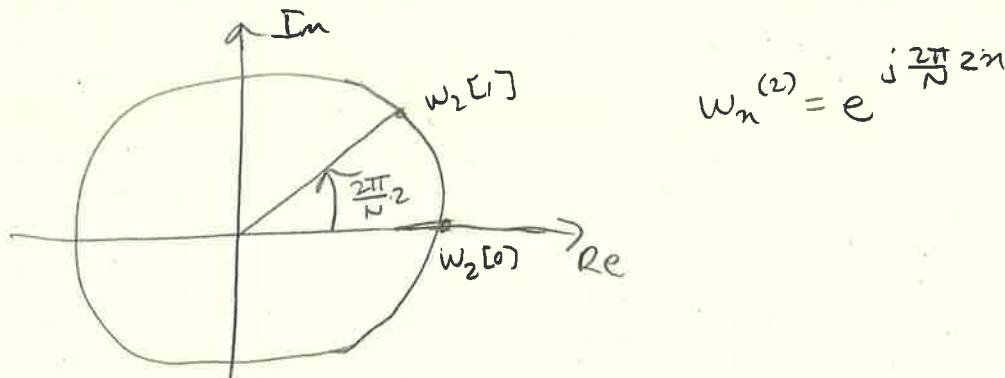
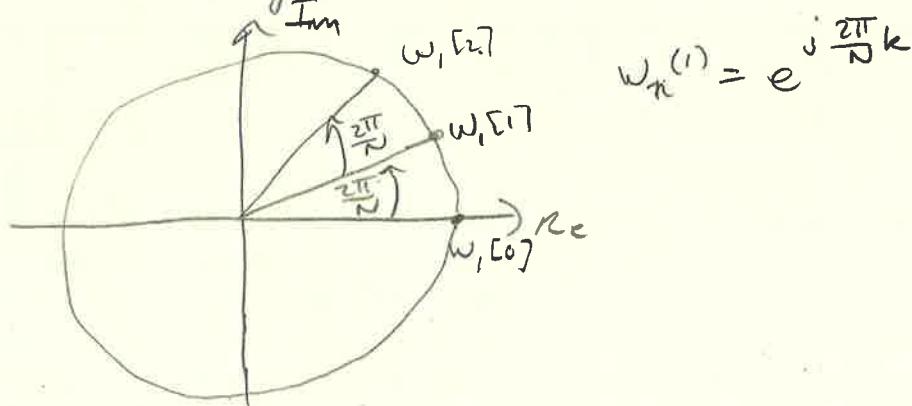


- The Fourier Basis for  $\mathbb{C}^N$

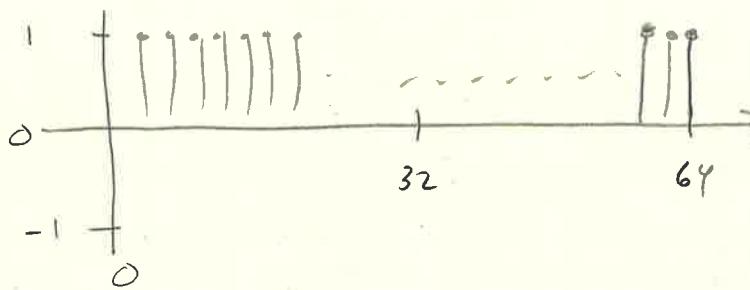
Claim: the set of  $N$  signals in  $\mathbb{C}^N$

$w_n[n] = e^{j \frac{2\pi}{N} nk}$ ,  $n, k = 0, 1, \dots, N-1$  is an orthogonal basis in  $\mathbb{C}^N$ .  $\omega = \frac{2\pi}{N} k$

In vector notation:  $\{\tilde{w}^{(k)}\}_{k=0,1,\dots,N-1}$  with  $w_n^{(k)} = e^{j \frac{2\pi}{N} nk}$   
is an orthogonal basis in  $\mathbb{C}^N$ .

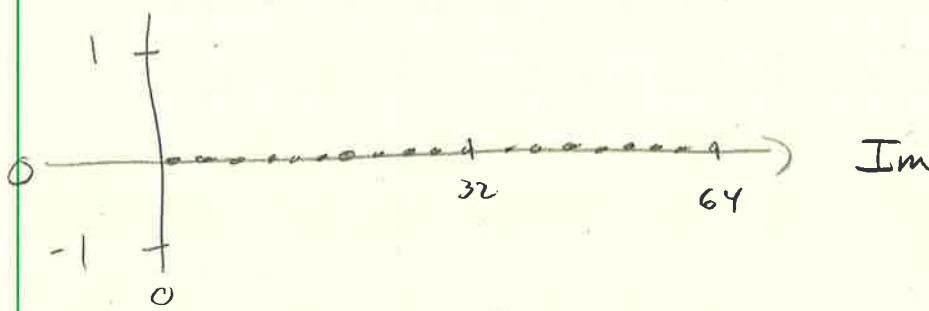


Basis vector  $\vec{w}^{(0)} \in \mathbb{C}^{64}$



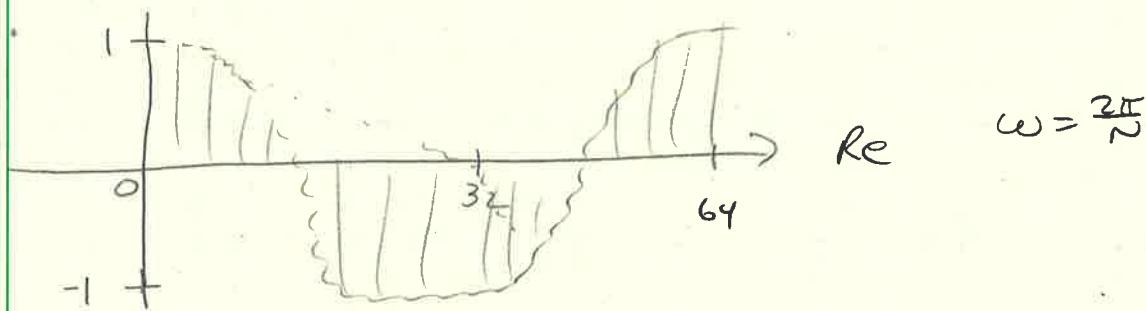
Re

$$\vec{w}_n^{(0)} = e^{j \frac{2\pi}{N} 0n} = 1$$



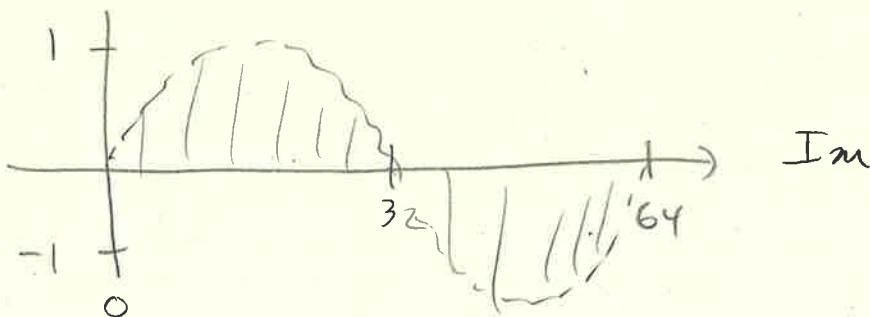
Im

Basis vector  $\vec{w}^{(1)} \in \mathbb{C}^{64}$ ,  $\vec{w}_n^{(1)} = e^{j \frac{2\pi}{N} 1n}$



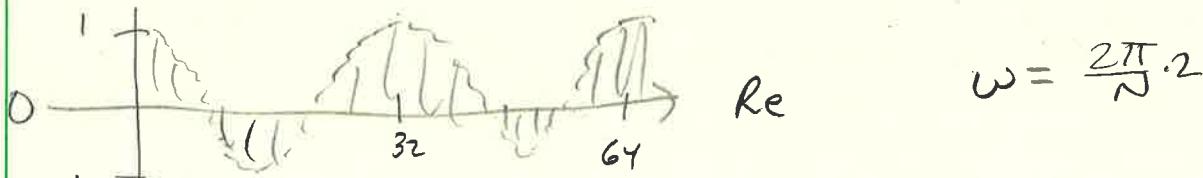
Re

$$\omega = \frac{2\pi}{N}$$



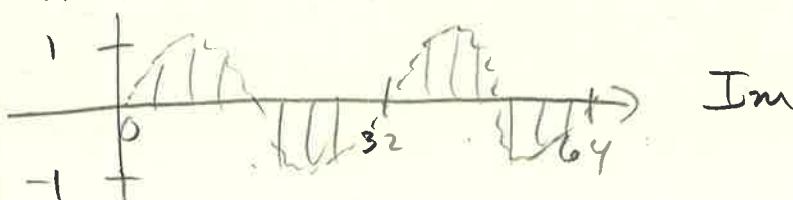
Im

Basis vector  $\vec{w}^{(2)} \in \mathbb{C}^{64}$



Re

$$\omega = \frac{2\pi}{N} \cdot 2$$



Im

$$\vec{w}^{(3)} \in \mathbb{C}^{64}; \quad \omega = \frac{2\pi}{N} 3 = \frac{2\pi}{64} 3$$

⋮

$$\vec{w}^{(16)} \in \mathbb{C}^{64}; \quad \omega = \frac{2\pi}{64} 16 = \frac{\pi}{2}$$

⋮

$$\vec{w}^{(32)} \in \mathbb{C}^{64}; \quad \omega = \frac{2\pi}{64} 32 = \pi$$

⋮

$\vec{w}^{(64)} \in \mathbb{C}^{64}$  has same real part as  $\vec{w}^{(2)}$  but the imaginary part is inverted

$\operatorname{Re}(\vec{w}^{(63)}) = \operatorname{Re}(\vec{w}^{(1)})$  but imaginary parts are inverted

- Proof of orthogonality

$$\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = \sum_{n=0}^{N-1} (e^{j \frac{2\pi}{N} nk})^* e^{j \frac{2\pi}{N} nh}$$

$$= \sum_{n=0}^{N-1} e^{j \frac{2\pi}{N} (n-k)n}$$

$$\left( \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \right) = \begin{cases} N & n=k \\ \frac{1-e^{j \frac{2\pi}{N} (n-k)}}{1-e^{j \frac{2\pi}{N} (n-k)}} & \text{otherwise} \end{cases}$$

$$n-k \in \mathbb{N} \Rightarrow e^{j 2\pi(n-k)} = 1$$

- Remarks

- $N$  orthogonal vectors  $\rightarrow$  basis for  $\mathbb{C}^N$

- vectors are not orthonormal. Normalization factor would be  $\sqrt{N}$

### 3.2 The Discrete Fourier Transform (DFT)

#### 3.2a DFT definition

- Basis expansion

- Analysis formula:  $X_k = \langle \vec{w}^{(k)}, \vec{x} \rangle$

- Synthesis formula:  $\vec{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \vec{w}^{(k)}$

- Change of basis in matrix form

Define  $W_N = e^{-j\frac{2\pi}{N}}$  (or simply  $W$  when  $N$  is evident)

Change of basis matrix  $\underline{W}$  with  $\underline{W}[n,m] = W_N^{nm}$

$$\underline{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)^2} \end{bmatrix}$$

Analysis formula:  $\underline{X} = \underline{W} \vec{x}$

Synthesis formula:  $\vec{x} = \frac{1}{N} \underline{W}^H \underline{X}$

- Basis expansion (signal notation)

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} nk}, \quad k=0, 1, \dots, N-1$$

$N$ -point signal in the frequency domain

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N} nk}, \quad n=0, 1, \dots, N-1$$

$N$ -point signal in the "time" domain

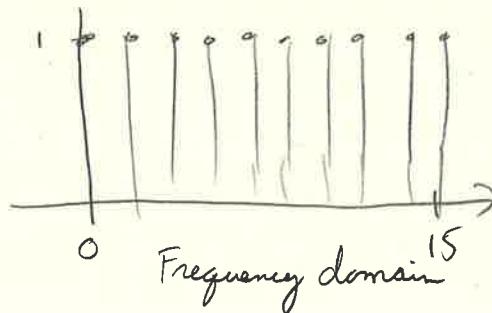
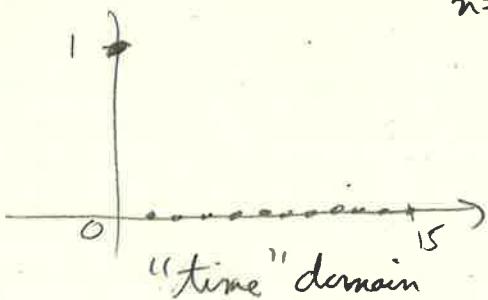
### 3.2 b Examples of DFT calculation

- DFT is obviously linear

$$\text{DFT}\{\alpha x[n] + \beta y[n]\} = \alpha \text{DFT}\{x[n]\} + \beta \text{DFT}\{y[n]\}$$

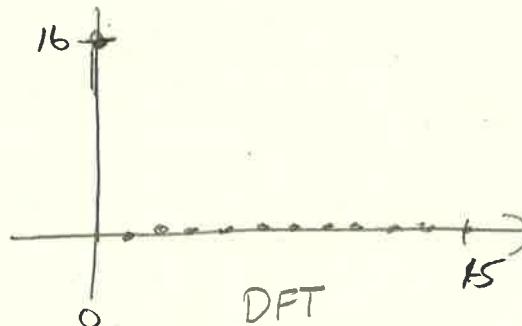
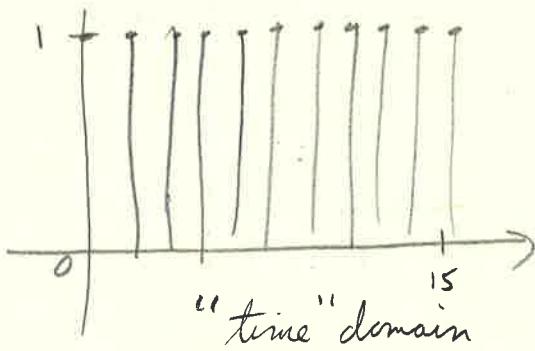
- DFT of  $x[n] = \delta[n]$ ,  $x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} \delta[n] e^{-j\frac{2\pi}{N} nk} = 1$$



DFT of  $x[n] = 1, x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} e^{-j \frac{2\pi}{N} nk} = N \delta[k]$$



DFT of  $x[n] = 3 \cos(\frac{2\pi}{16}n), x[n] \in \mathbb{C}^{64}$

$$x[n] = 3 \cos\left(\frac{2\pi}{16}n\right) = 3 \cos\left(\frac{2\pi}{64}4n\right) \quad \omega = \frac{2\pi}{64}$$

$$= \frac{3}{2} \left[ e^{j \frac{2\pi}{64}4n} + e^{-j \frac{2\pi}{64}4n} \right]$$

$$= \frac{3}{2} \left[ e^{j \frac{2\pi}{64}4n} + e^{j \frac{2\pi}{64}60n} \right] \quad -j \frac{2\pi}{64}4n = j \frac{2\pi}{64}60n$$

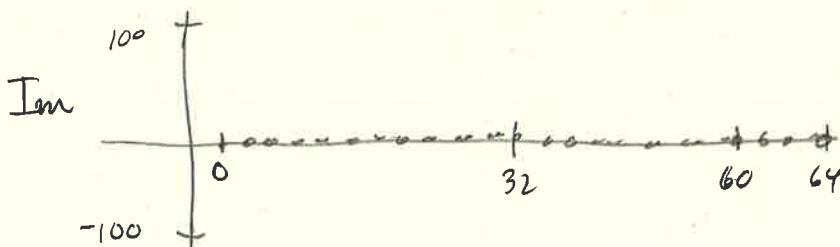
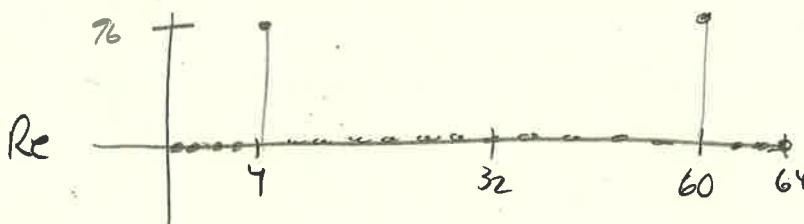
$$= \frac{3}{2} [w_4[n] + w_{60}[n]]$$

$$X[k] = \langle w_k[n], x[n] \rangle$$

$$= \langle w_k[n], \frac{3}{2}(w_4[n] + w_{60}[n]) \rangle$$

$$= \frac{3}{2} \langle w_k[n], w_4[n] \rangle + \frac{3}{2} \langle w_k[n], w_{60}[n] \rangle$$

$$= \begin{cases} \frac{3}{2} \cdot 64 = 96, & k = 4, 60 \\ 0, & \text{otherwise} \end{cases}$$



- DFT of  $x[n] = 3 \cos\left(\frac{2\pi}{16}n + \pi/3\right)$ ,  $x[n] \in \mathbb{C}^{64}$

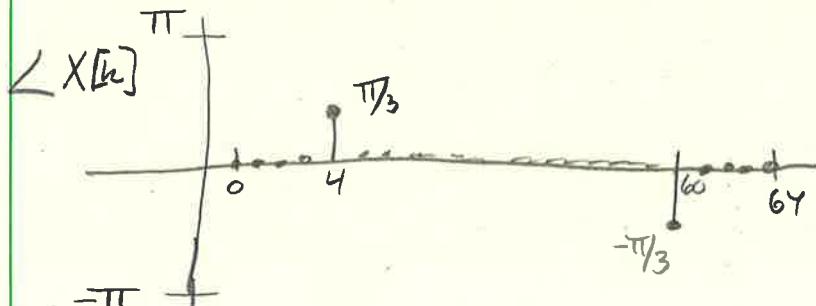
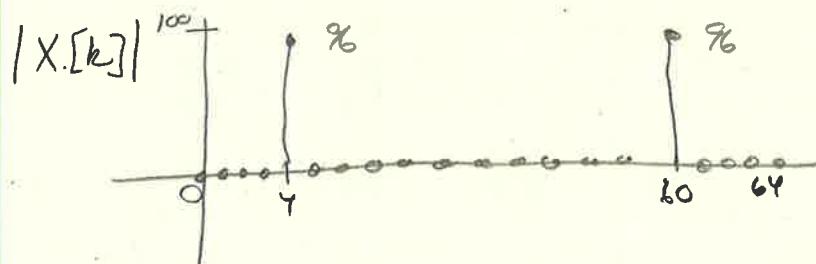
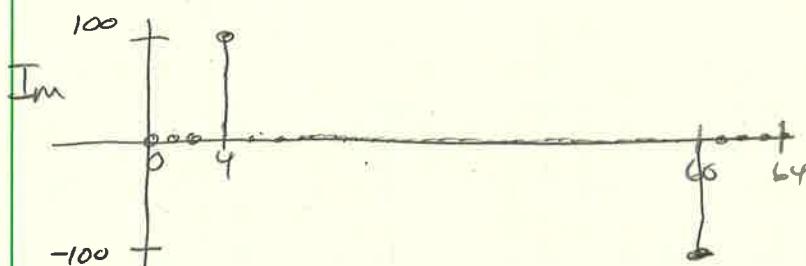
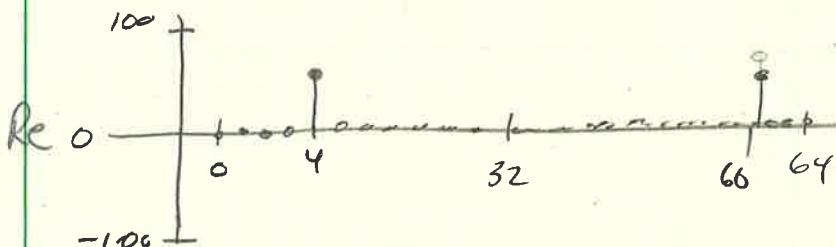
$$x[n] = 3 \cos\left(\frac{2\pi}{16}n + \frac{\pi}{3}\right)$$

$$= 3 \cos\left(\frac{2\pi}{64}4n + \frac{\pi}{3}\right)$$

$$= \frac{3}{2} \left[ e^{j\frac{2\pi}{64}4n} e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi}{64}4n} e^{-j\frac{\pi}{3}} \right]$$

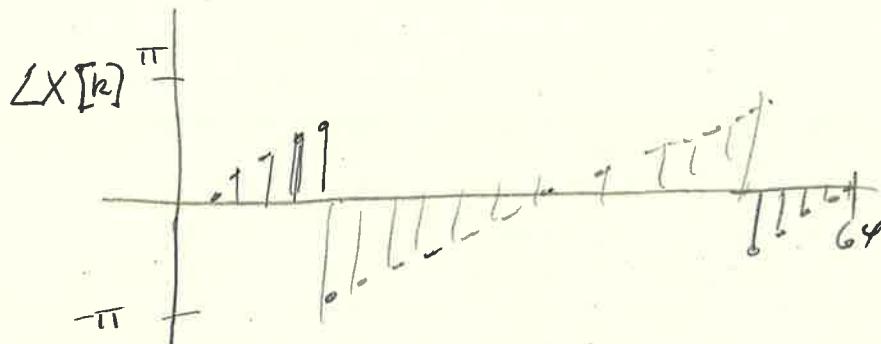
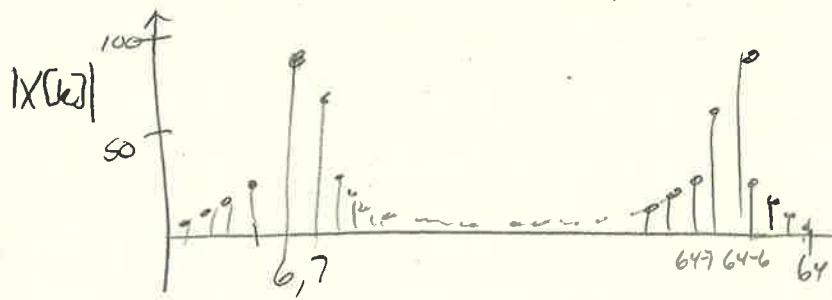
$$= \frac{3}{2} \left[ e^{j\frac{\pi}{3}} w_4[n] + e^{-j\frac{\pi}{3}} w_{60}[n] \right]$$

$$X[k] = \langle w_n[n], x[n] \rangle = \begin{cases} 96e^{j\frac{\pi}{3}}, & k=4 \\ 96e^{-j\frac{\pi}{3}}, & k=6 \\ 0, & \text{otherwise} \end{cases}$$



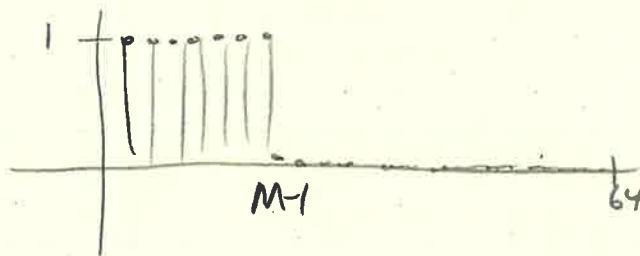
- DFT of  $x[n] = 3 \cos\left(\frac{2\pi}{10}n\right)$ ,  $x[n] \in \mathbb{C}^{64}$

$$\frac{2\pi}{64} 6 < \frac{2\pi}{10} < \frac{2\pi}{64} 7$$



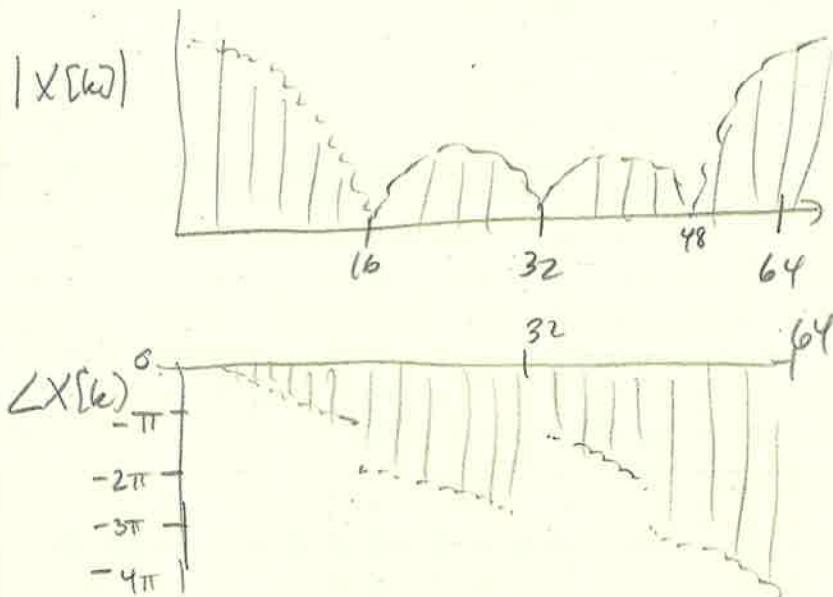
- DFT of length- $M$  step in  $\mathbb{C}^N$

$$x[n] = \sum_{h=0}^{M-1} \delta[n-h], \quad n=0, 1, \dots, N-1$$



$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} nk} = \sum_{n=0}^{M-1} e^{-j \frac{2\pi}{N} nk} \\ &= \frac{1 - e^{-j \frac{2\pi}{N} kM}}{1 - e^{-j \frac{2\pi}{N} k}} \quad \left( 1 - e^{-j\alpha} = e^{-j\frac{\alpha}{2}} (e^{j\frac{\alpha}{2}} - e^{-j\frac{\alpha}{2}}) \right) \\ &= \frac{e^{-j \frac{\pi}{N} kM} [e^{j \frac{\pi}{N} kM} - e^{-j \frac{\pi}{N} kM}]}{e^{-j \frac{\pi}{N} k} [e^{j \frac{\pi}{N} k} - e^{-j \frac{\pi}{N} k}]} \\ &= \frac{\sin\left(\frac{\pi}{N} Mk\right)}{\sin\left(\frac{\pi}{N} k\right)} e^{-j \frac{\pi}{N} (M-1)k} \end{aligned}$$

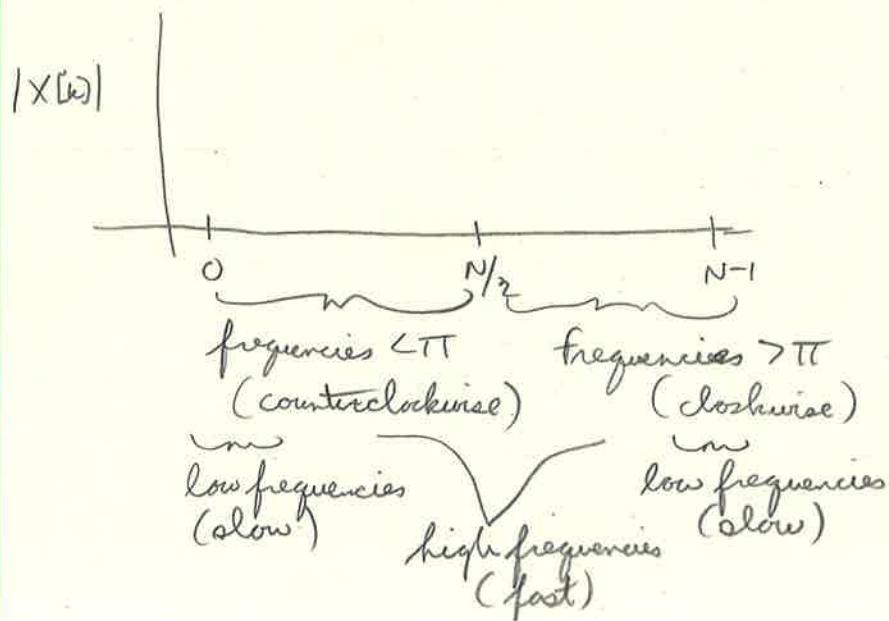
- $X[0] = M$ , from the definition of the sum
- $X[k] = 0$ , if  $M^k/N$  is an integer ( $0 \leq k < N$ )
- $\angle X[k]$  is linear in  $k$  (except at sign changes for the real part)
  - DFT of length-4 step in  $\mathbb{C}^{64}$

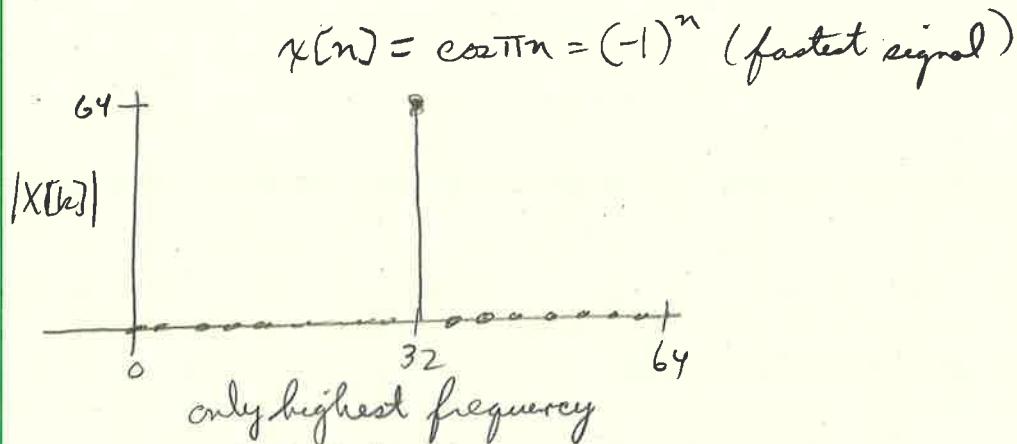
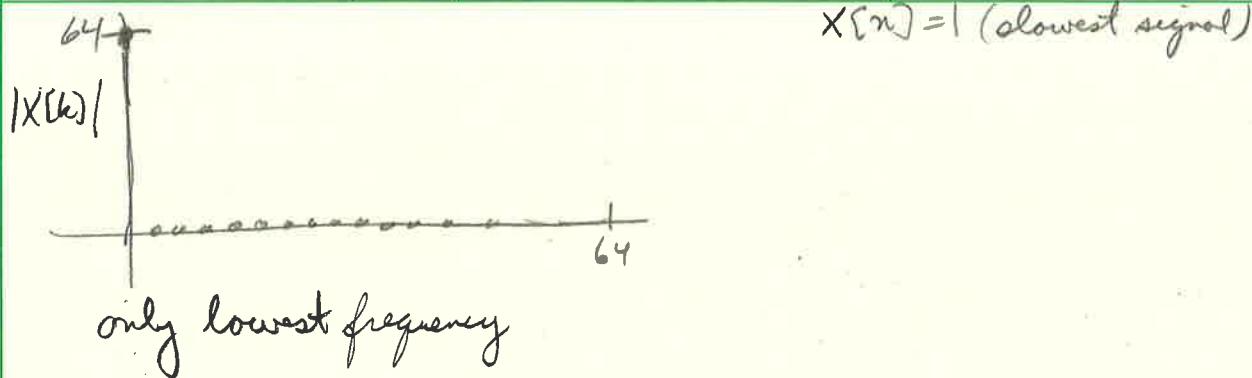


### - Wrapping the phase

- Often the phase is displayed "wrapped" over the  $[-\pi, \pi]$  interval
  - most numerical packages return wrapped phase
  - phase can be unwrapped by adding multiples of  $2\pi$

### 3.2c Interpreting a DFT plot





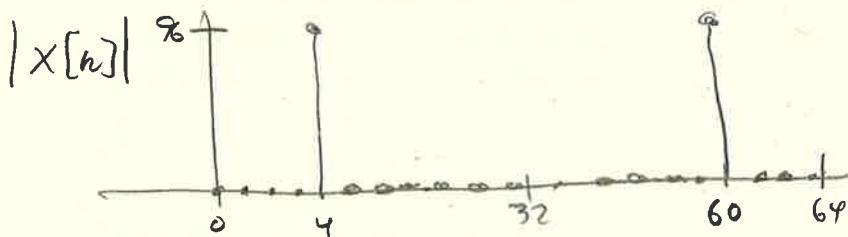
- Energy distribution

- Parseval :  $\|\vec{x}\|^2 = \sum |x_k|^2$

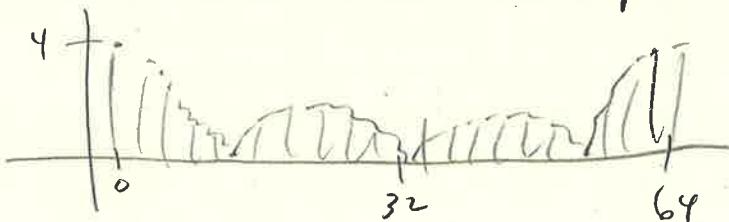
$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

- square magnitude of  $k$ -th DFT coefficient proportional to signals energy at frequency  $\omega = \frac{2\pi}{N} k$ .

$$x(n) = 3 \cos\left(\frac{2\pi}{16} n\right) \quad (\text{sine wave})$$



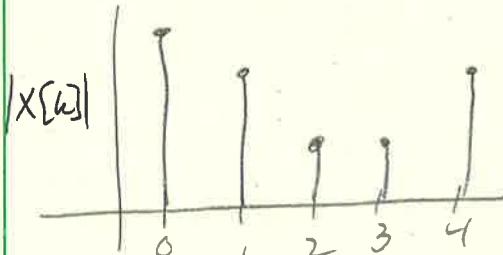
$$x(n) = u[n] - u[n-4] \quad (\text{step})$$



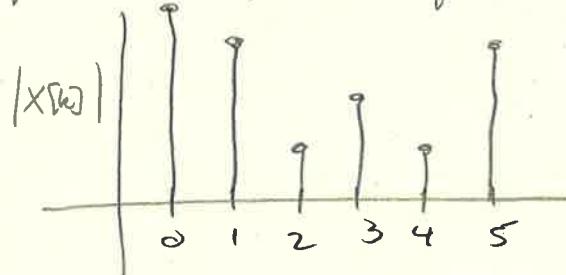
- DFT of real signals

For real signals the DFT is "symmetric" in magnitude:

$$|X[k]| = |X[N-k]| \text{ for } k=1, 2, \dots, \lfloor N/2 \rfloor \text{ (floor)}$$

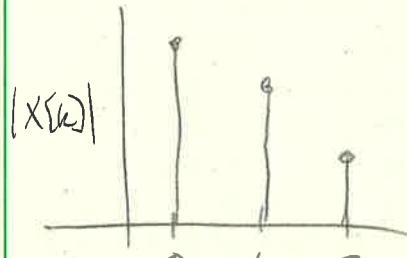


N=5, odd length

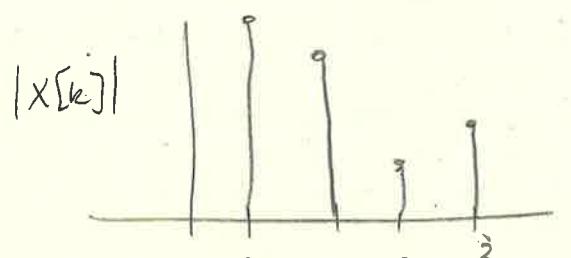


N=6, even length

For real signals, magnitude plots need only  $\lfloor N/2 \rfloor + 1$  points



N=5, odd length



N=6, even length

### 3.3 : The DFT in practice

#### 3.3a DFT analysis

- Mystery signal revisited

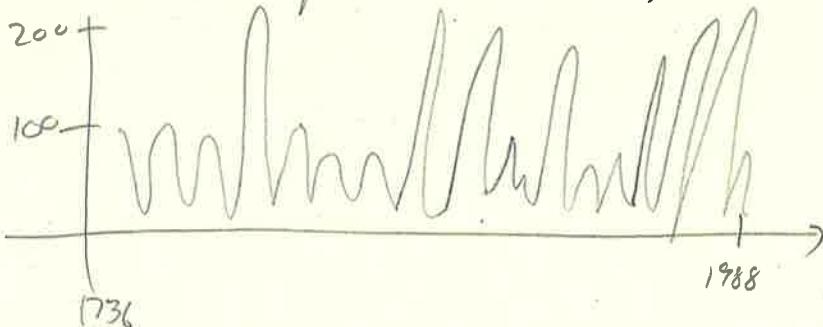
$$x[n] = \cos(\omega n + \phi) + \eta[n] \text{ with}$$

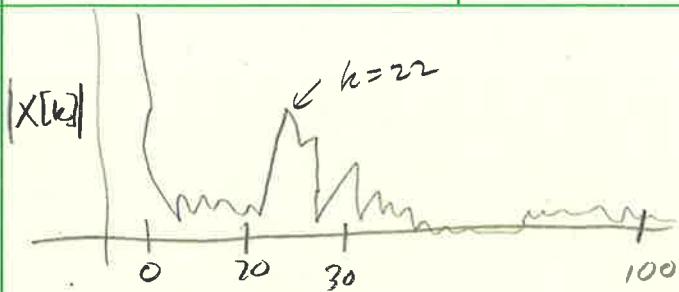
$$\phi=0, \omega = \frac{2\pi}{1024} 64 \quad \text{peak at } k=64$$

- Solar spots

- sunspot number :  $S = 10 \times \# \text{ of clusters} + \# \text{ of spots}$

- data set from 1749 to 2003, 2904 months

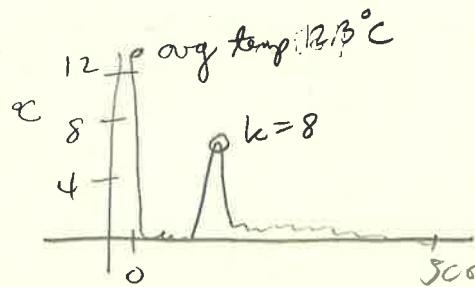
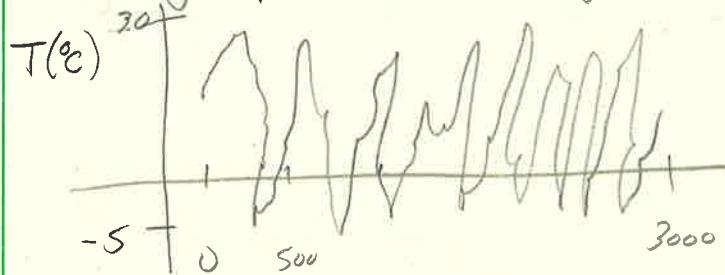




DFT of solar spots signal

- DFT main peak for  $k=22$
- 22 cycles over 2904 months
- period:  $\frac{2904}{22} \approx 11$  years

- Daily temperature (2920 days)

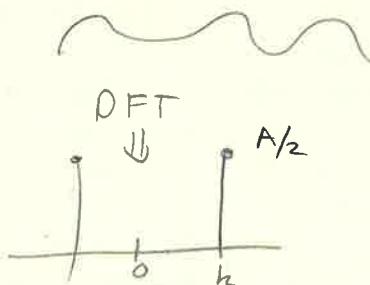


first few bunched DFT coefficients  
(in magnitude and normalized by the length of the temperature vector)

- average value (0-th DFT coefficient):  $12.3^\circ\text{C}$  normalized
- DFT main peak for  $k=8$ , value  $6.4^\circ\text{C}$
- 8 cycles over 2920 days
- period:  $\frac{2920}{8} = 365$  days
- temperature excursion:  $12.3^\circ\text{C} \pm 12.8^\circ\text{C}$

$$X[0] = \sum_{n=0}^{N-1} X(n)$$

$$A = \cos(\omega n)$$



- Labeling the frequency axis

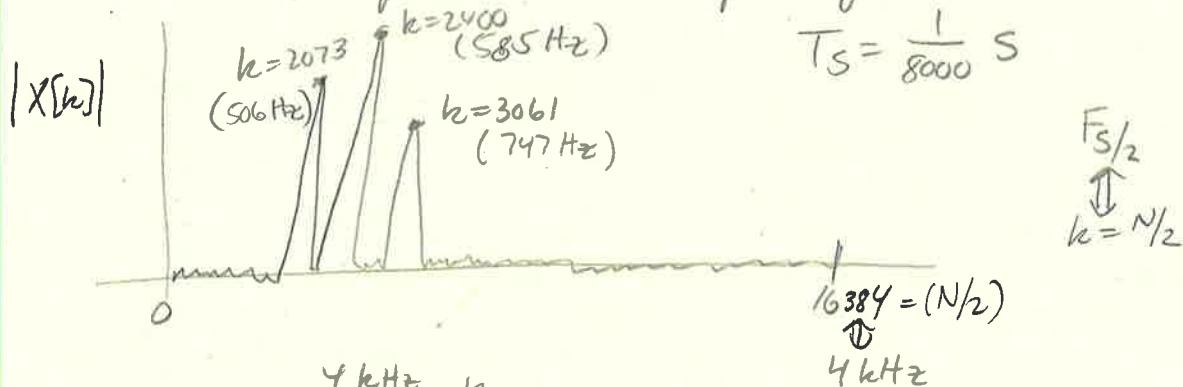
- If we know the "clock" of the system  $T_s$ 
  - fastest (positive) frequency is  $\omega = \pi$
  - sinusoid at  $\omega = \pi$  needs two samples to do a full revolution
  - time between samples:  $T_s = \frac{1}{F_s}$  seconds

- real-world period for fastest sinusoid:  $2T_s$  seconds

- real-world frequency for fastest sinusoid:  $F_s/2$  Hz

- Example: train whistle

32768 samples (the "clock" of the system  $F_S = 8000 \text{ Hz}$ )



$$\text{Frequency} = \frac{4k \text{ Hz}}{16384} \cdot k$$

If we look up the frequencies:



B minor chord

### 3.3b DFT example - analysis of musical instruments

- Analysis of musical instruments

"We all know about pitch..."

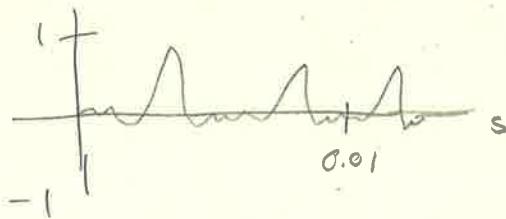
• It is really about frequency, or cycles per seconds

• How about harmonics?

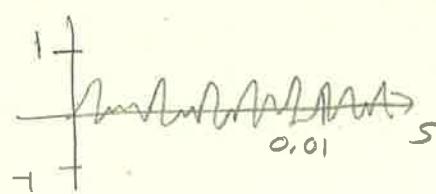
• That is what gives the timbre of an instrument!

- A difficult temporal analysis of musical instruments

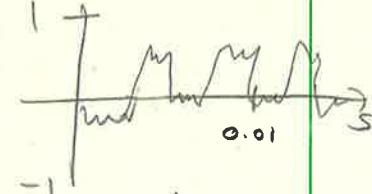
Sax



violin



cello

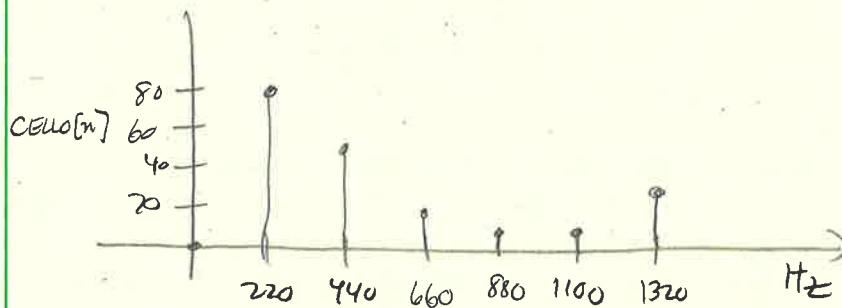
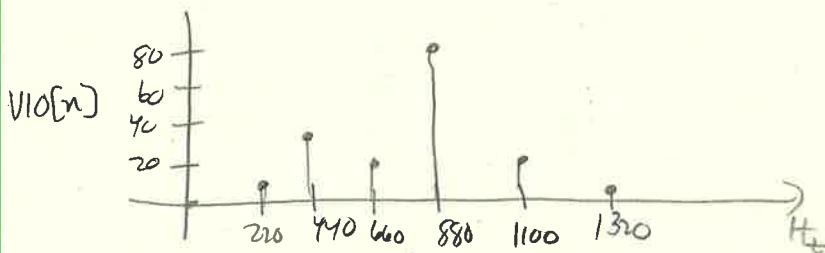
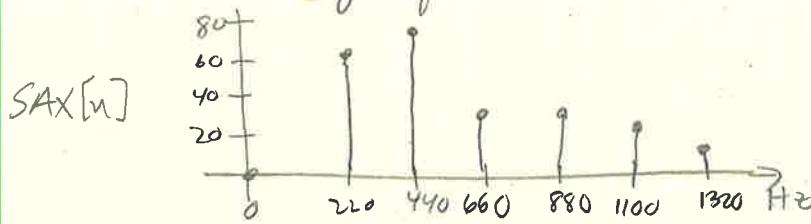


• What is the note?

• Could you guess the instrument from the temporal plot?

• In the time domain it is hard to process information of the sound

- A Fourier analysis of musical instruments

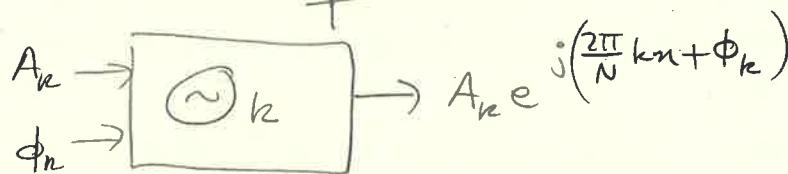
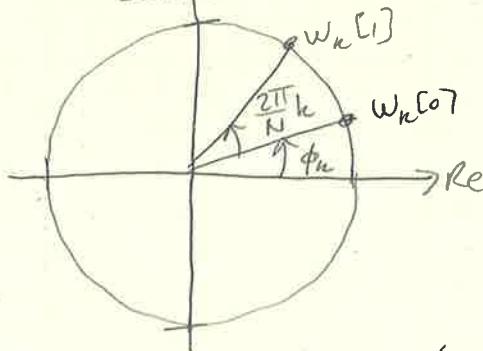


- The played note is the frequency of the first peak : 220 Hz in this case
- Other peaks are called harmonics : they define the typical sound of the instrument
- Without Fourier we would have been lost

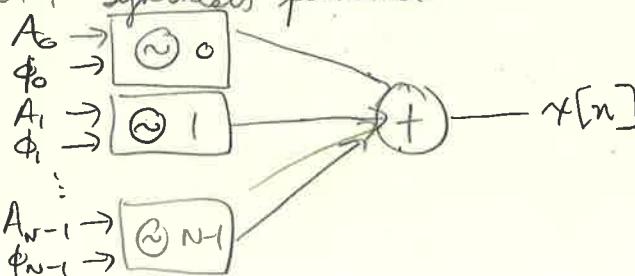
### 3.3c DFT synthesis

- Synthesis : the sinusoidal generator

$$w_k[n] = e^{j\left(\frac{2\pi}{N}kn + \phi_k\right)}$$



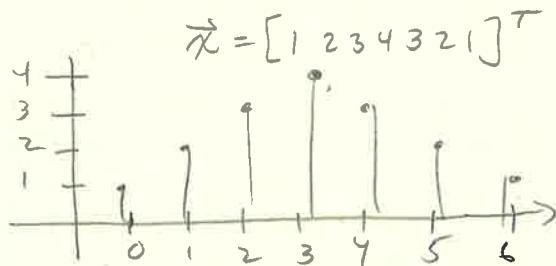
- DFT Synthesis formula



- Initializing the machine

$$A_k = |X[k]| / N$$

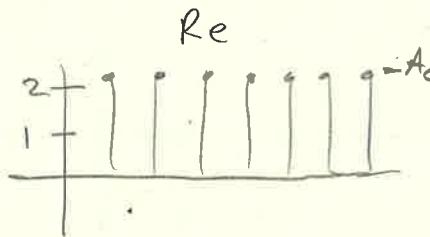
$$\phi_k = \angle X[k]$$



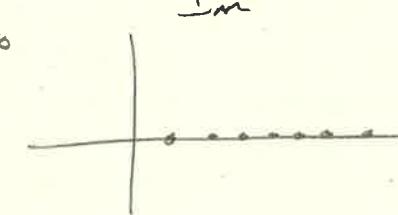
$k$	$A_k$	$\phi_k$
0	2.2857	0
1	0.7213	-2.6928
2	0.0440	0.8976
3	0.0919	-1.7952
4	0.0919	-1.7952
5	0.0440	-0.8976
6	0.7213	2.6928

$$k=0$$

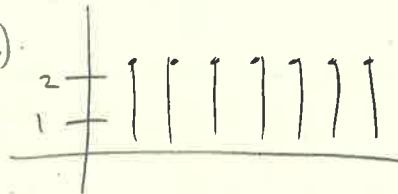
$$A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$$



Im

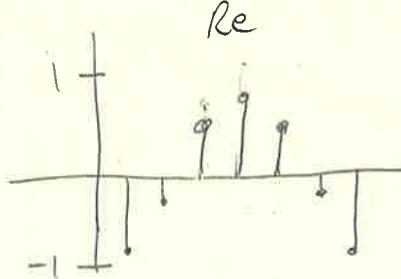


$$\sum A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$$

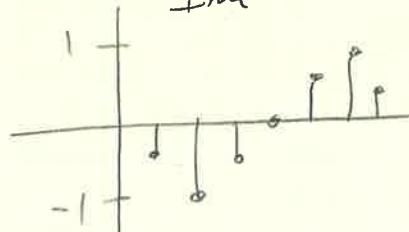


$$k=1$$

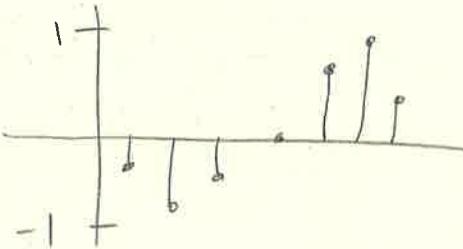
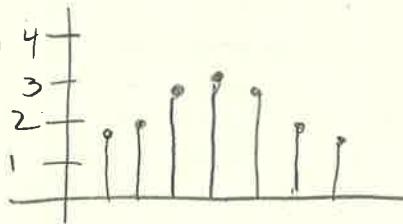
$$A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$$



Im



$$\sum A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$$



at  $k=6$ , we have reconstructed the signal

- Running the machine too long ...

$x[n+N] = x[n]$  - Output signal is  $N$ -periodic!

- Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} nk}, \quad n \in \mathbb{Z}, \text{ produces an}$$

$N$ -periodic signal in the time domain

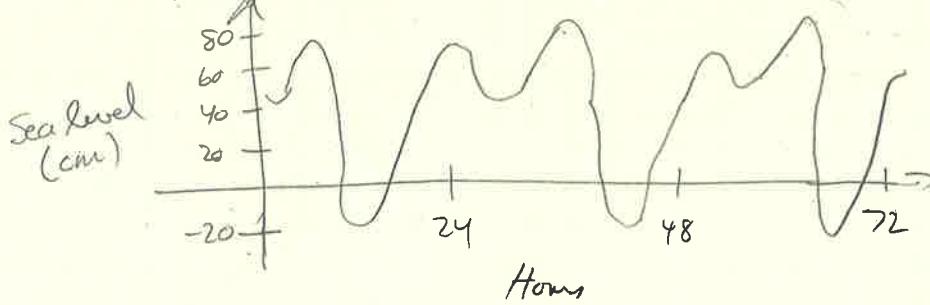
the analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} nk}, \quad k \in \mathbb{Z}, \text{ produces } N\text{-periodic}$$

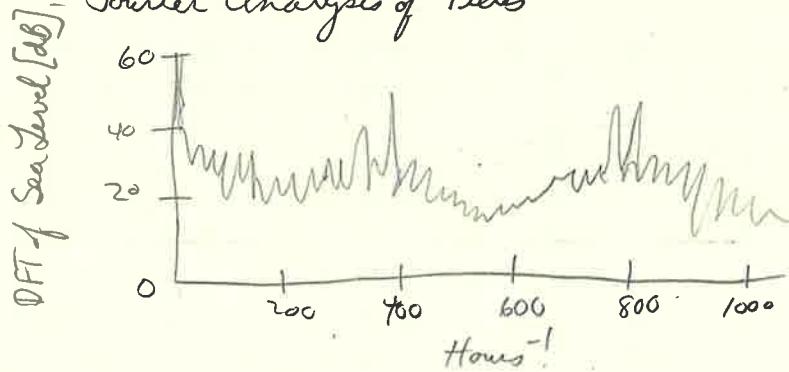
signal in the frequency domain.

### 3.3d DFT example - tide prediction in Venice

- Tides are due mostly to periodic astronomical phenomena
- Can we predict tides using Fourier? The first step is to approximate them
- We consider hourly measurements taken in Canal Grande during 2011  
(3 days)



Fourier Analysis of Tides

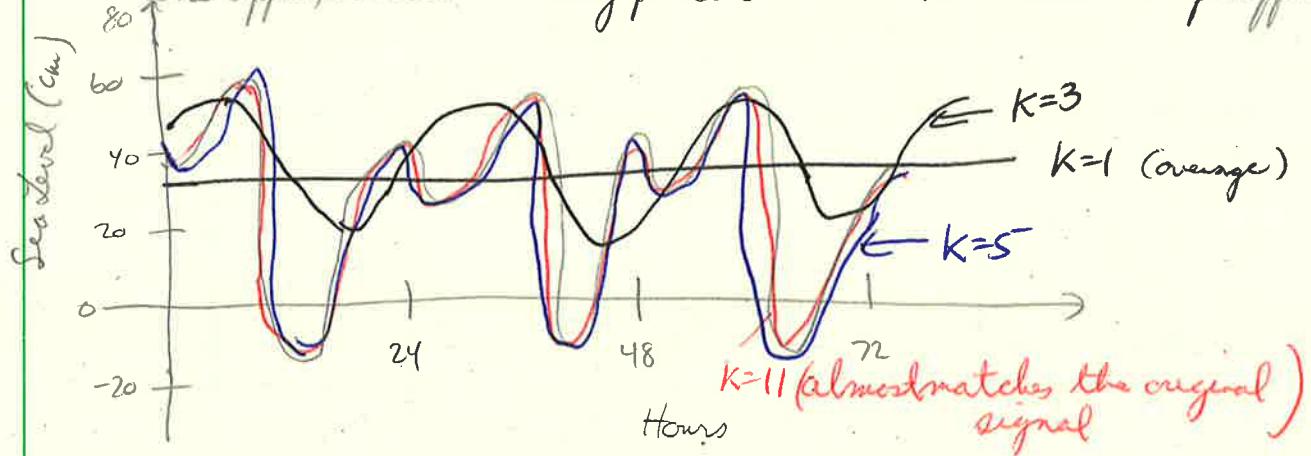


- Let us approximate tides using few Fourier coefficients

- We consider only  $K=1, 3, 5, 11$  Fourier coefficients

- Darker gray represents approximation with more coefficients

- The approximations are very precise with a limited number of coefficients

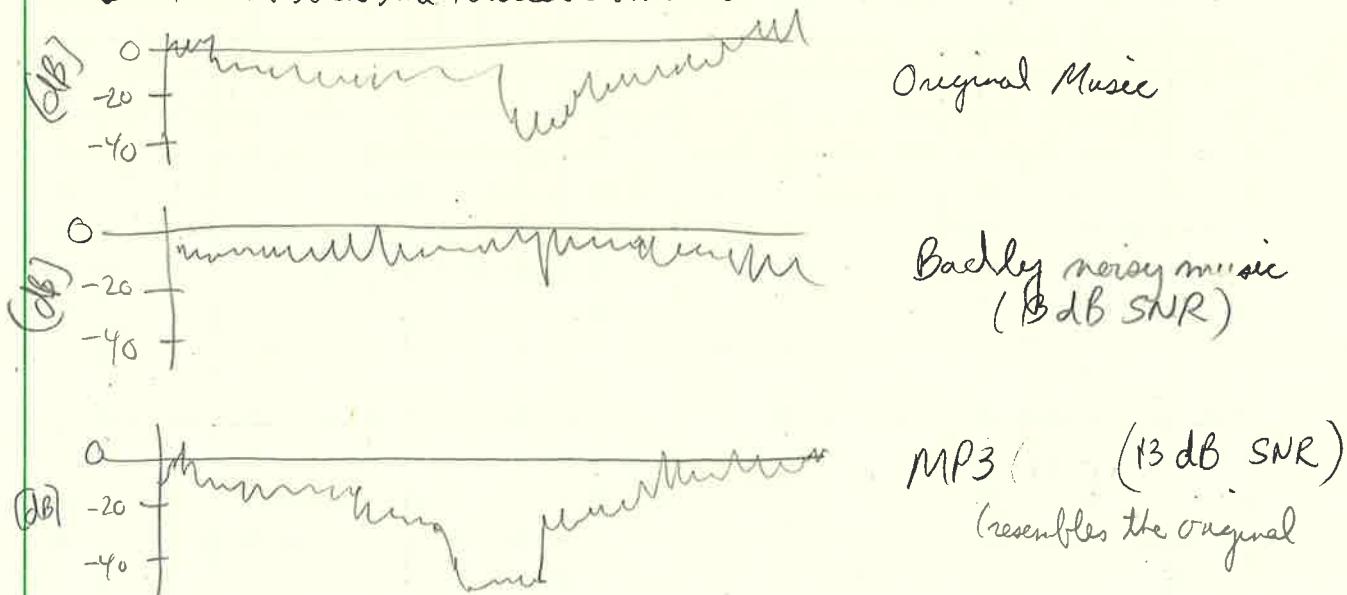


### 3.3e DFT example - MP3 compression

- MP3 Compression trick

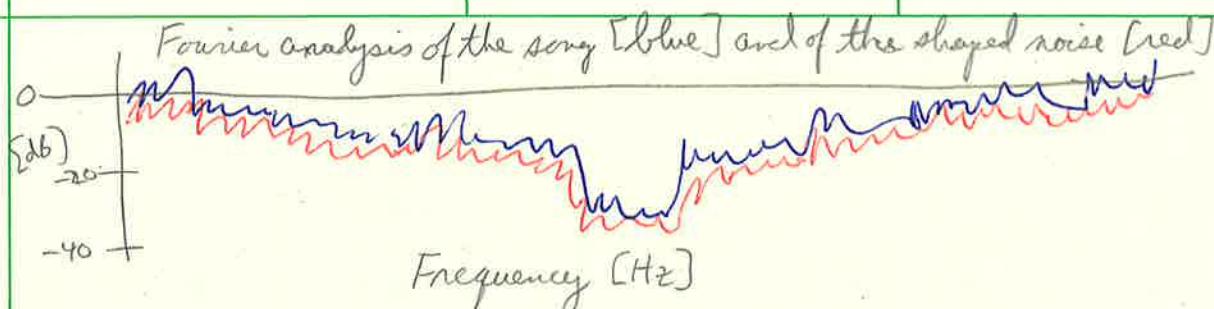
- How can an MP3 song sound so good while being so compressed?
- Compression introduces noise!
- The trick is to shape the noise!

The secret is in the Fourier domain!

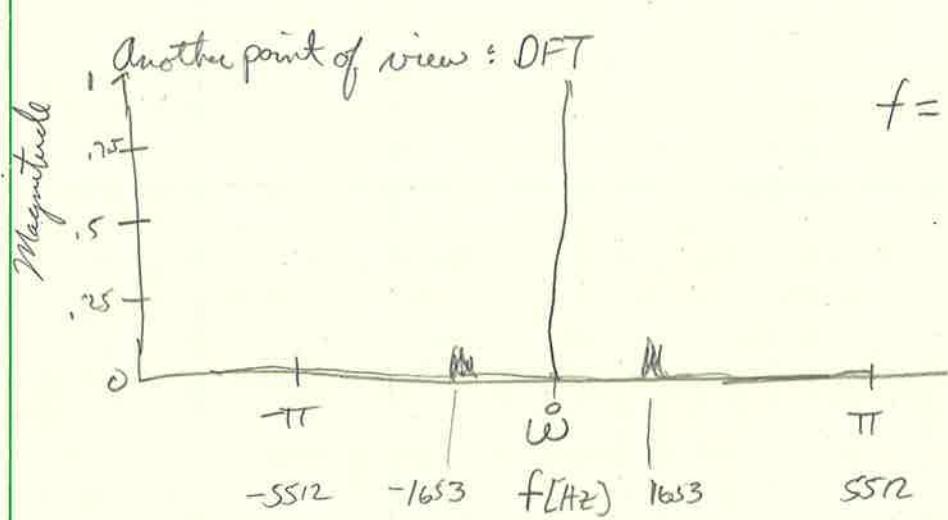
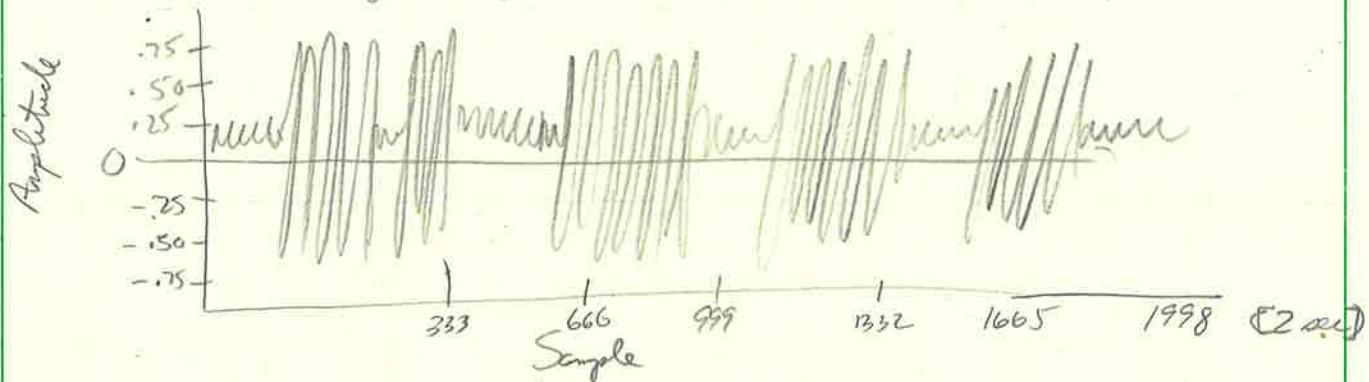


- Conclusions:

- MP3: complex compression algorithm that introduces errors
- Errors shaped as the song in the Fourier domain  $\rightarrow$  higher perceived quality
- MP3 minimizes the perceived quality decay by shaping the compression errors

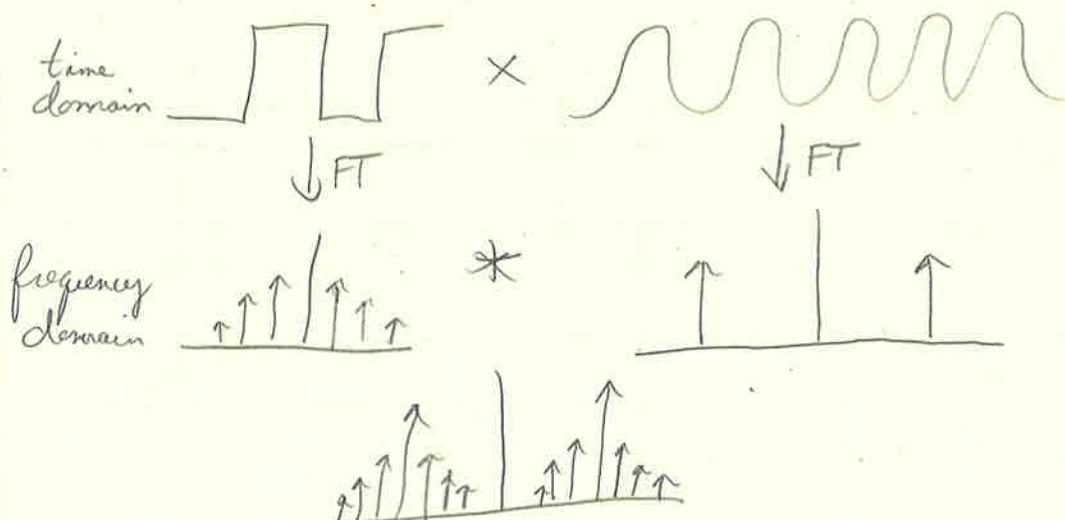


Signal of the Day: The first man-made signal from outer space (Sputnik)



$$f = \frac{\omega f_s}{2\pi} \leftarrow \begin{array}{l} \text{frequency at} \\ \text{which we} \\ \text{measure the signal} \end{array}$$

- Let's understand this...



## Summary of Lesson 3.3

The DFT can be used as an analysis tool to understand the frequency components that a signal contains. If a signal has an associated system clock  $T_s$  (or a frequency  $F_s = 1/T_s$ ), we can map the index  $k$  of the DFT coefficients to real frequencies. The largest digital frequency  $N/2$  is associated with the largest continuous-time frequency  $F_s/2$ . Thus, the continuous frequency corresponding to index  $k$  is given by  $\frac{kF_s}{N}$  and is measured in Hz.

The DFT synthesis can be seen as a series of up to  $N$  coupled sinusoidal generators:

- sinusoidal generator  $k$  has frequency  $\frac{2\pi k}{N}$
- the amplitude of sinusoidal generator  $k$  is given by the magnitude of the DFT coefficient  $|X[k]|$
- the phase of sinusoidal generator  $k$  is given by the phase of the DFT coefficient  $\angle X[k]$

If we let the DFT synthesis run beyond  $N-1$ , we obtain an  $N$ -periodic signal,  $x[n+N] = x[n]$ . Likewise, the analysis formula produces also an  $N$ -periodic series of Fourier coefficients. This side comment will be very important when we study another form of Fourier transform for periodic sequences, namely discrete Fourier Series (DFS).

## 3.4 The Short-Time Fourier Transform (STFT)

### 3.4a The STFT

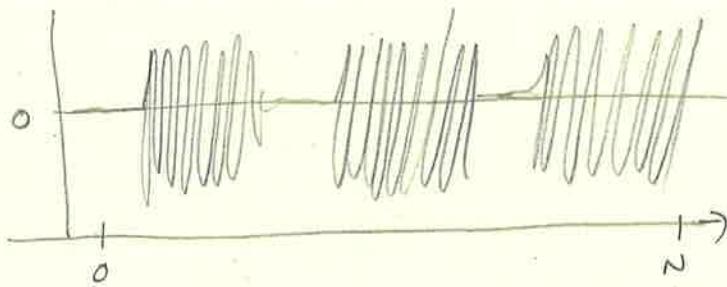
- Dual-Tone Multi Frequency dialing (DTMF)

	1209 Hz	1336 Hz	1477 Hz
697 Hz	1	2	3
770 Hz	4	5	6
852 Hz	7	8	9
941 Hz	*	0	#

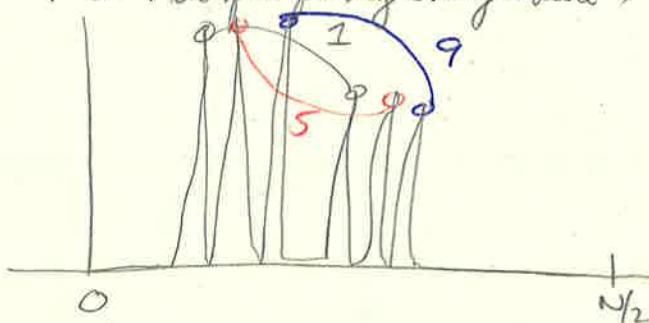
Analog  
telephone

- 1) Frequencies are co-prime
- 2) No sum or difference of frequencies is in the set

1-5-9 in time

Can't tell the digit  
in time domain

1-5-9 in frequency (magnitude)



### - The fundamental tradeoff

- time representation obscures frequency
- frequency representation obscures time

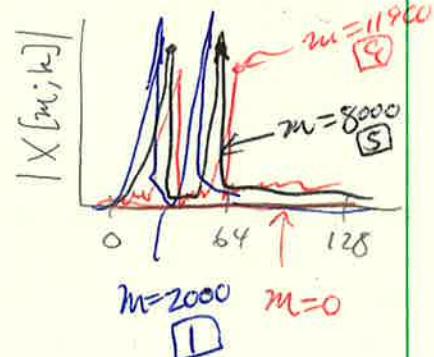
### - Short-term Fourier Transform

Idea:

- take small signal pieces of length L
- look at the DFT of each piece

$$X[m; k] = \sum_{n=0}^{L-1} x[m+n] e^{-j \frac{2\pi}{L} nk}$$

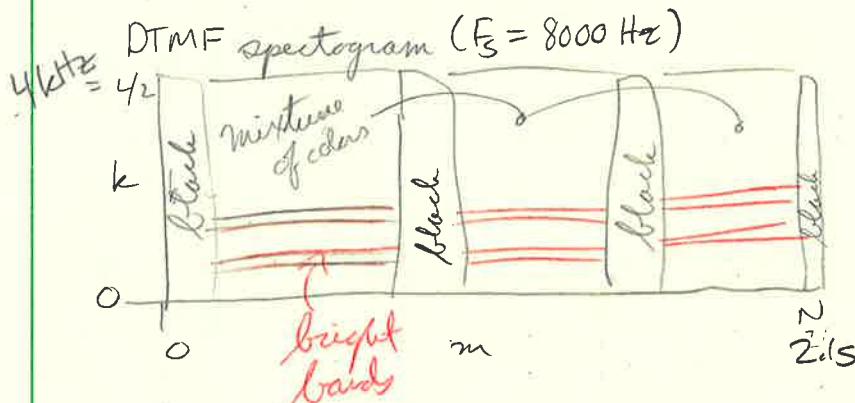
Starting point

STFT ( $L=256$ )

### 3.4b The spectrogram

Idea:

- Color-code the magnitude: dark is small, white is large
- use  $10 \log_{10}(|X[m; k]|)$  to see better (power in dB)
- plot spectral slices one after another



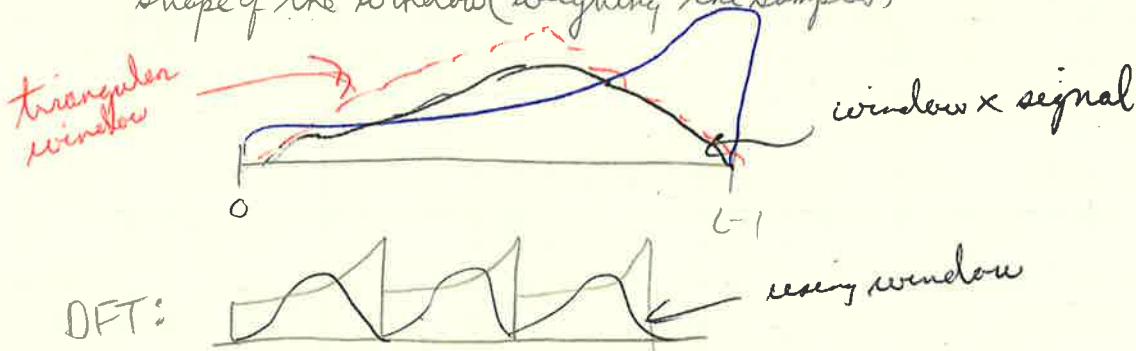
#### - Labeling the Spectrogram

- If we know the "system clock"  $F_s = 1/T_s$ , we can label the axis
  - highest positive frequency:  $F_s/2 \text{ Hz}$
  - frequency resolution:  $F_s/L \text{ Hz}$
  - width of time slices:  $L T_s \text{ seconds}$

#### - The Spectrogram

Questions:

- width of the analysis window?
- position of the windows (overlapping?)
- shape of the window (weighing the samples)



#### - Wideband vs. Narrowband

Long window? narrowband spectrogram

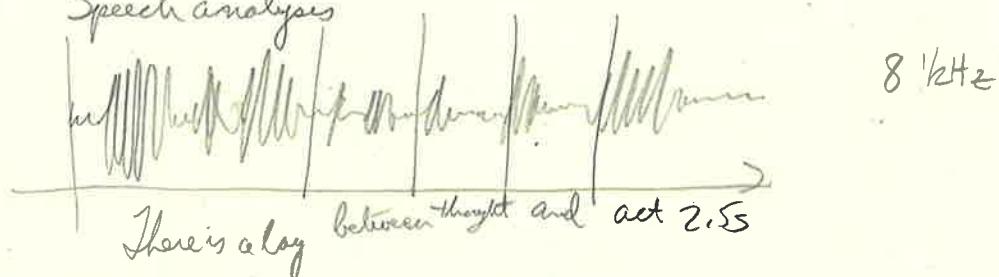
- long window  $\Rightarrow$  more DFT points  $\Rightarrow$  more frequency resolution  $\frac{F_s}{L}$
- long window  $\Rightarrow$  more "things can happen"  $\Rightarrow$  less precision in time

Short window? wideband spectrogram

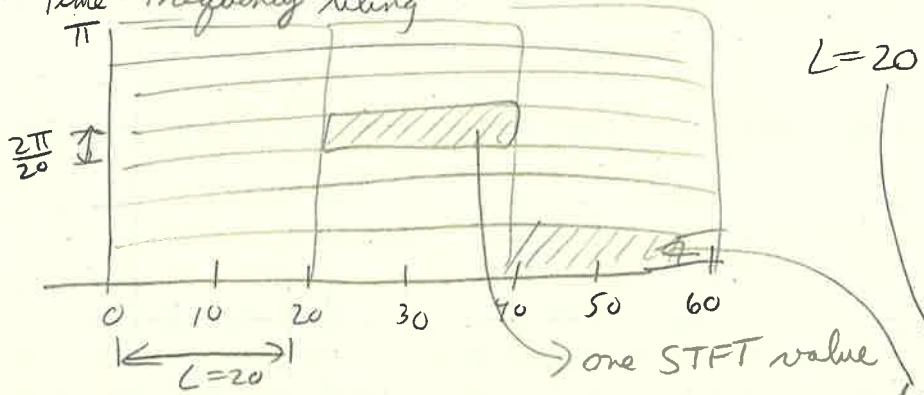
- short window  $\Rightarrow$  many time slices  $\Rightarrow$  precise location of transitions
- short window  $\Rightarrow$  fewer DFT points  $\Rightarrow$  poor frequency resolution

### 3.4c Time - frequency tiling

Speech analysis



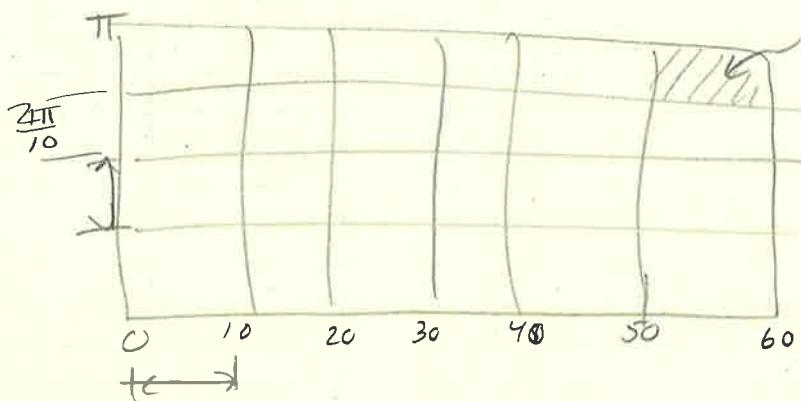
- Time - Frequency tiling



$$L=20$$

area of each tile = constant

$$L=10$$



- Food for thought

- time "resolution"  $\Delta t = L$
- frequency "resolution"  $\Delta f = 2\pi/L$
- $\Delta t \Delta f = 2\pi$

uncertainty principle !

- Even more food for thought

- more sophisticated tilings of time - frequency planes can be obtained with the wavelet transform.

### 3.5 Discrete Fourier Series

#### 3.5a Discrete Fourier Series

DFS = DFT with periodicity explicit

- The DFS maps an  $N$ -periodic signal onto an  $N$ -periodic sequence of Fourier coefficients
- The inverse DFS maps an  $N$ -periodic sequence of Fourier coefficients onto an  $N$ -periodic signal
- The DFS of an  $N$ -periodic signal is mathematically equivalent to the DFT of one period

- Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

- For an  $N$ -periodic sequence  $\tilde{x}[n]$

- $\tilde{x}[n-M]$  is well-defined for all  $M \in \mathbb{N}$

$$\text{DFS} \{ \tilde{x}[n-M] \} = e^{-j \frac{2\pi}{N} M k} \tilde{X}[k], \quad \tilde{X}[k] = \text{DFS} \{ \tilde{x}[n] \}$$

$$-\text{IDFS} \left\{ e^{-j \frac{2\pi}{N} M k} \tilde{X}[k] \right\} = \tilde{x}[n-M]$$

delay factor

- For an  $N$ -point signal  $x[n]$ :

- $x[n-M]$  is not well-defined

- build  $\tilde{x}[n] = x[n \bmod N] \Rightarrow \tilde{X}[k] = X[k]$

$$\begin{aligned} -\text{IDFT} \left\{ e^{-j \frac{2\pi}{N} M k} X[k] \right\} &= \text{IDFS} \left\{ e^{-j \frac{2\pi}{N} M k} \tilde{X}[k] \right\} \\ &= \tilde{x}[n-M] = x[(n-M) \bmod N] \end{aligned}$$

- Shifts for finite-length signals are "naturally" circular

#### 3.5b Karpus-Strong revisited and DFS

- Periodic sequences: a bridge to infinite-length signals

- $N$ -periodic sequence:  $N$  degrees of freedom

- DFS: only  $N$  Fourier coefficients capture all of the information

- Karpbus-Strong revisited



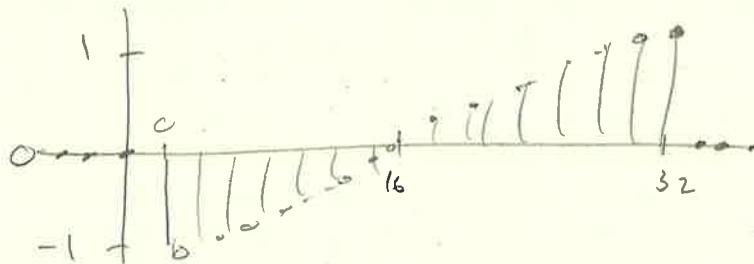
$$y[n] = \alpha y[n-M] + x[n]$$

- choose a signal  $\bar{x}[n]$  that is nonzero only for  $0 \leq n < M$

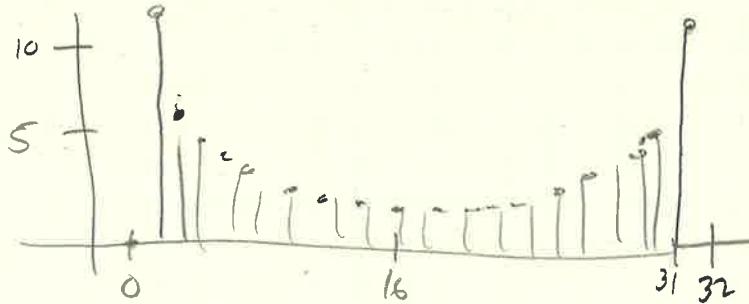
-  $\alpha = 1$  (for now)

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \dots, \bar{x}[M-1]}_{\text{1st period}}, \underbrace{\bar{x}[0], \bar{x}[1], \dots, \bar{x}[M-1]}_{\text{2nd period}}, \bar{x}[0], \bar{x}[1], \dots, \dots$$

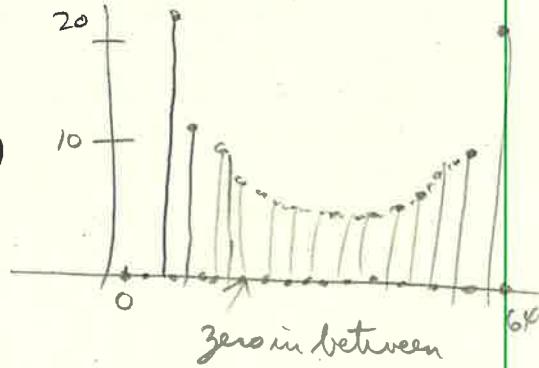
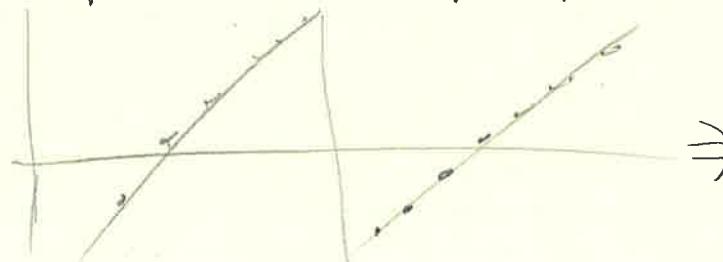
- Example : 32-tap Sawtooth wave

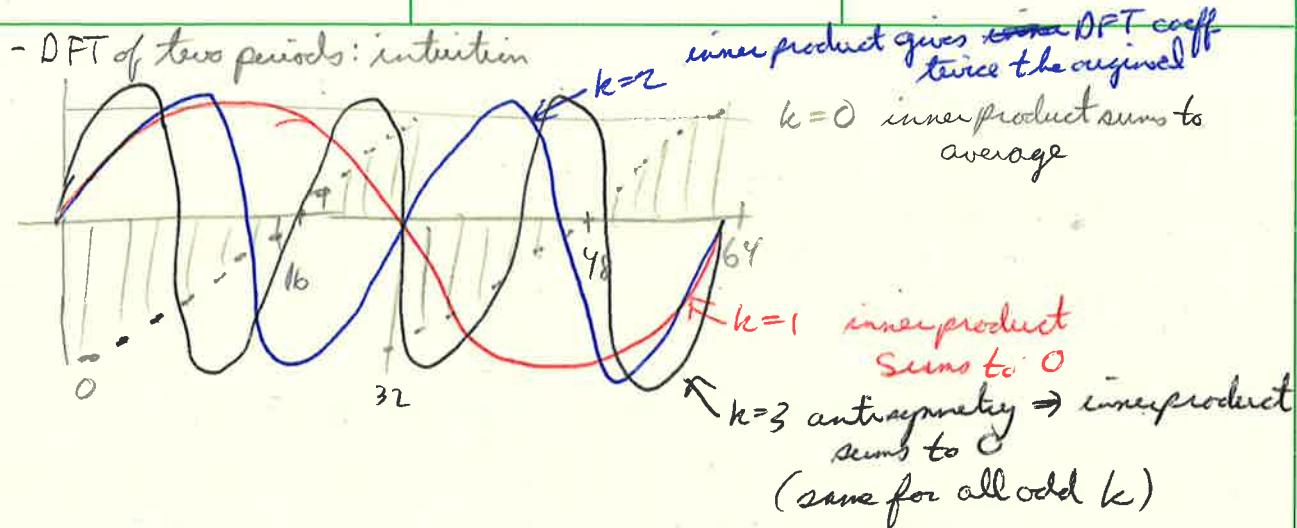


DFT of 32-tap sawtooth wave



- What if we take the DFT of two periods?





- DFT of  $L$  periods

$$X_L[k] = \sum_{n=0}^{M-1} y[n] e^{-j \frac{2\pi}{M} nk}, \quad k=0, 1, 2, \dots, M-1$$

$$(k=n+pM) = \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} y[n+pM] e^{-j \frac{2\pi}{M}(n+pM)k}$$

$$\bar{x}[n] \quad \begin{array}{|c|c|c|c|}\hline & \text{1} & \text{0} & \dots & \text{1} \\ \hline & \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} \quad \frac{M}{m}$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} y[n] e^{-j \frac{2\pi}{M} nk} e^{-j \frac{2\pi}{L} pk}$$

$$y[n] \quad \begin{array}{|c|c|c|c|}\hline & \text{1} & \text{0} & \dots & \text{1} \\ \hline & \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} \quad L \text{ periods}$$

$$= \left( \sum_{p=0}^{L-1} e^{-j \frac{2\pi}{L} pk} \right) \sum_{n=0}^{M-1} \bar{x}[n] e^{-j \frac{2\pi}{M} nk}$$

- We've seen this before

$$\sum_{p=0}^{L-1} e^{-j \frac{2\pi}{L} pk} = \begin{cases} L, & \text{if } k \text{ is a multiple of } L \\ 0, & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

- DFT of  $L$  periods

$$X_L[k] = \begin{cases} L \bar{x}[k/L], & \text{if } k=0, L, 2L, \dots \\ 0, & \text{otherwise} \end{cases}$$

- DFT and DFS

- again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period.

- To stress the periodicity of the underlying signal, we use the term DFS

## Summary of Lesson 3.5

The discrete Fourier series is just a different flavor of the DFT applied to periodic series.  $N$ -periodic sequences in the time domain are mapped onto  $N$ -periodic sequences in the frequency domain. Furthermore, the definition of the DFS retrospectively better justifies the use of circular shifts as the natural extension of shifts for finite-length sequences.

Later in the lesson, we have revisited the Karples-Strong algorithm to illustrate a key point about the DFS. If we take the DFT of  $L$  repetitions of a finite length sequence of length  $N$ , we obtain a series which is non-zero only at multiple integers of  $L$ . Moreover, these non-zero coefficients are just scaled versions of the DFT coefficients of the original finite-length sequence. Therefore, all the spectral information of a  $N$ -periodic sequence is entirely captured by the DFT coefficients of one period.

## 3.6 The Discrete-Time Fourier Transform

### - The situation so far

- Fourier representation for signal classes:

- $N$ -point finite-length: DFT

- $N$ -point periodic: DFS

- infinite length: ?

### - Karples-Strong revisited, part 2

- consider now  $\alpha < 1$

- generated signal is infinite-length but not periodic:

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \dots, \bar{x}[M-1]}_{\text{1st period}}, \underbrace{\alpha \bar{x}[0], \alpha \bar{x}[1], \dots, \alpha \bar{x}[M-1]}_{\text{2nd period}}, \underbrace{\alpha^2 \bar{x}[0], \alpha^2 \bar{x}[1], \dots}_{n}$$

- What is a good spectral representation?

### - DFT of increasingly long signals

- Start with the DFT. What happens when  $N \rightarrow \infty$ ?

- $(2\pi/N)k$  becomes denser in  $[0, 2\pi]$  ...  $C^N$ ,  $\omega = \frac{2\pi}{N}$

- In the limit  $(\frac{2\pi}{N})k \rightarrow \omega$ :  $\sum_n x[n] e^{-j\omega n}$ ,  $\omega \in \mathbb{R}$

### - Discrete-Time Fourier Transform (DTFT)

- Formal definition:

- $x[n] \in l_2(\mathbb{Z})$

- define a function of  $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- inversion (when  $F(\omega)$  exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, n \in \mathbb{Z}$$

- DTFT periodicity and notation

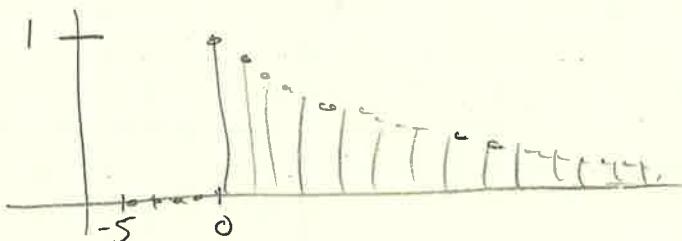
- $F(\omega)$  is  $2\pi$ -periodic

- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- by convention,  $X(e^{j\omega})$  is represented over  $[-\pi, \pi]$

$$- x[n] = \alpha^n u[n], |\alpha| < 1$$



$$- \text{DTFT of } x[n] = \alpha^n u[n], |\alpha| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

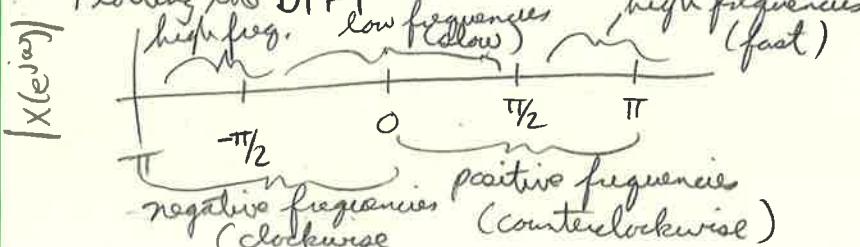
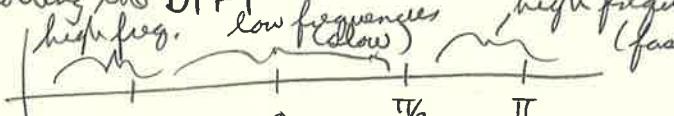
$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

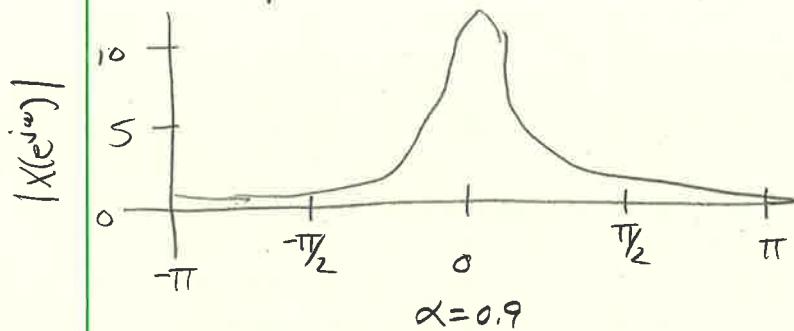
$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

$$|X(e^{j\omega})|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos \omega}$$

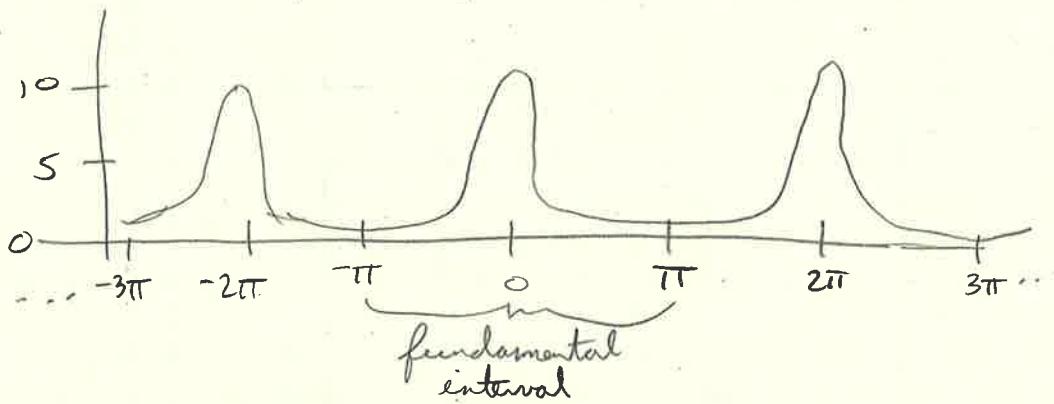
- Plotting the DTFT



- DTFT of  $x[n] = \alpha^n u[n]$ ,  $|\alpha| < 1$



- Remember the periodicity!

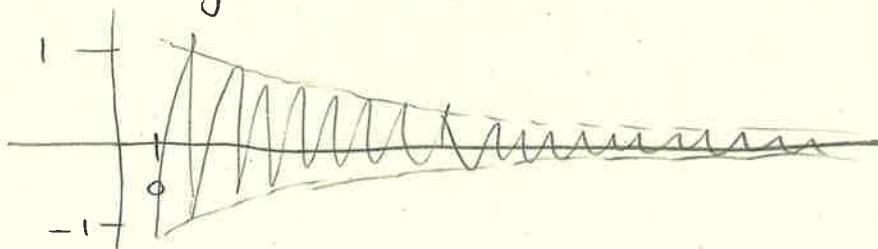


- KS revisited, part 2: 32-tap sawtooth wave

$$x[n] = \frac{2n}{M-1} - 1, \quad n=0, 1, \dots, M-1 \quad (M=32)$$

- KS revisited, part 2: decay  $\alpha = 0.9$

$$y[n] = \alpha^{\lfloor \frac{n}{M} \rfloor} \bar{x}[n \bmod M] u[n]$$

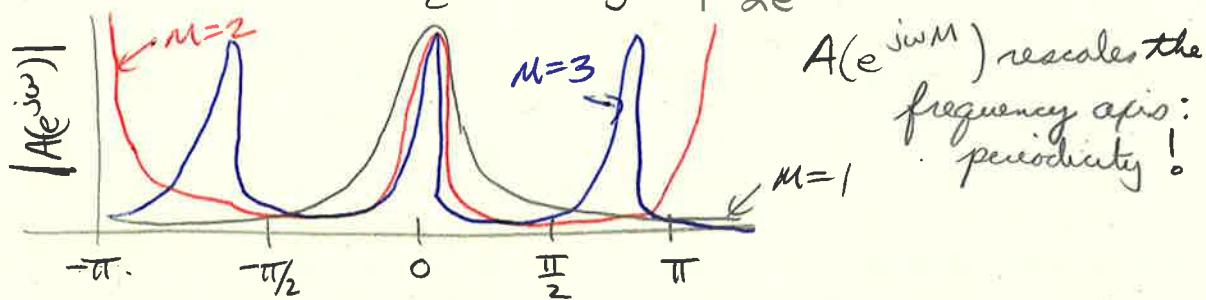


- DTFT of KS signal

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \\ &= \sum_{p=0}^{\infty} \sum_{n=0}^{M-1} \alpha^p \bar{x}[n] e^{-j\omega(pM+n)} \\ &= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega M p} \sum_{n=0}^{M-1} \bar{x}[n] e^{-j\omega n} \\ &= A(e^{j\omega M}) \bar{X}(e^{j\omega}) \end{aligned}$$

- We know the first term

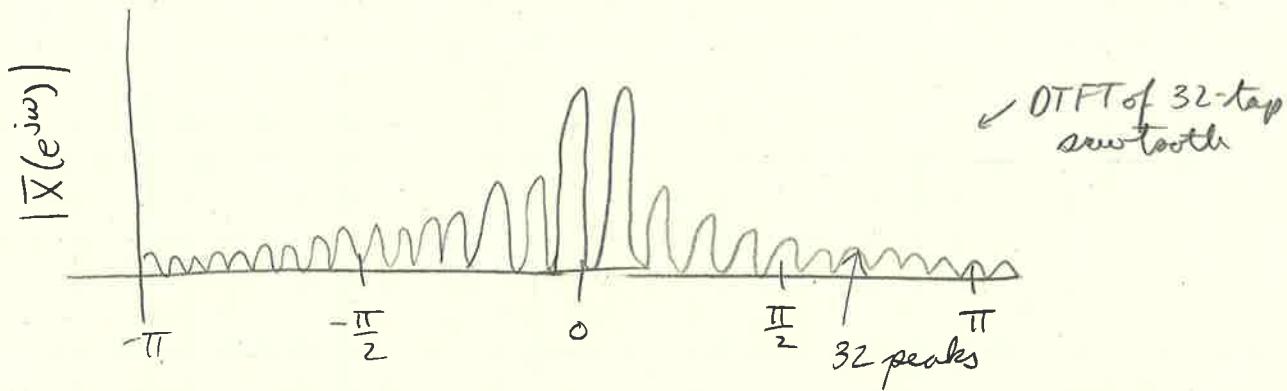
$$A(e^{j\omega}) = \text{DTFT}\{\alpha^n u[n]\} = \frac{1}{1-\alpha e^{-j\omega}} \quad (n=1)$$



$A(e^{j\omega M})$  rescales the frequency axis: periodically!

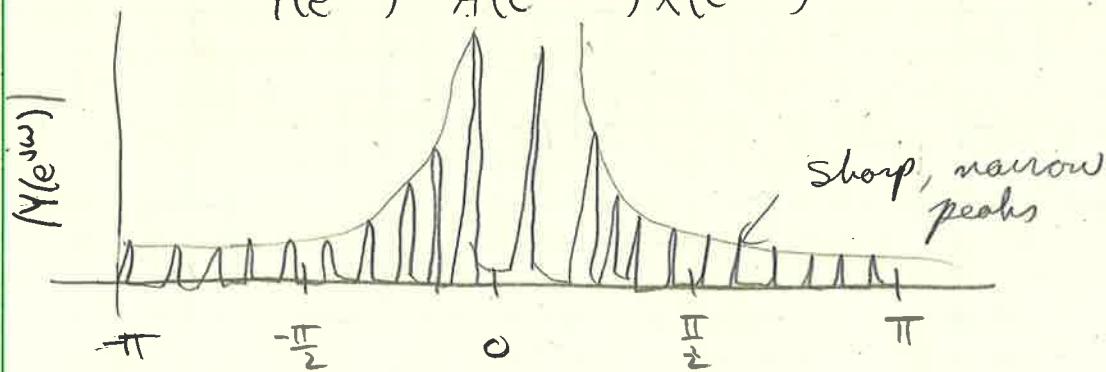
- Second term

$$\bar{X}(e^{j\omega}) = e^{-j\omega} \left( \frac{M+1}{M-1} \right) \frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$



- DTFT of KS with decay

$$Y(e^{j\omega}) = A(e^{j\omega M}) \bar{X}(e^{j\omega})$$



### 3.6b Existence and properties of the OTFT

- Existence easy for absolutely summable sequences

$$\begin{aligned} |\bar{X}(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]| e^{-j\omega n} = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \end{aligned}$$

- Inversion easy for absolutely summable sequences

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} x[k] e^{-jkw} \right) e^{jwn} dw \\ &= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\int_{-\pi}^{\pi} \frac{e^{jw(n-k)}}{2\pi} dw}_{n} \\ &= x[n] \end{aligned}$$

$= 0$  unless  $n=k$

- A formal change of bases

• Formally DTFT is an inner product in  $\mathbb{C}^\infty$ :

$$\sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \langle e^{jwn}, x[n] \rangle$$

- "basis" is an infinite, uncountable basis:  $\{e^{jwn}\}_{w \in \mathbb{R}}$
- something "breaks down": we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

- Review: DFT

$$X[k] = \langle e^{j \frac{2\pi}{N} nk}, x[n] \rangle$$

basis:  $\{e^{j \frac{2\pi}{N} nk}\}_k$

$$x[n] = \frac{1}{N} \sum X[k] e^{j \frac{2\pi}{N} nk}$$

- Review: DPS

$$\tilde{X}[k] = \langle e^{j \frac{2\pi}{N} nk}, \tilde{x}[n] \rangle$$

basis:  $\{e^{j \frac{2\pi}{N} nk}\}_k$

$$\tilde{x}[n] = \frac{1}{N} \sum \tilde{X}[k] e^{j \frac{2\pi}{N} nk}$$

- DTFT

$$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle$$

$\ell_2(\mathbb{R})$

"basis":  $\{e^{j\omega n}\}_\omega$

$\ell_2([-T, T])$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- DTFT properties

- linearity:  $\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
- time shift:  $\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$
- modulation(dual):  $\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$
- time reversal:  $\text{DTFT}\{x[-n]\} = X(e^{-j\omega})$
- conjugation:  $\text{DTFT}\{x^*[n]\} = X^*(e^{-j\omega})$

- Some particular cases:

- if  $x[n]$  is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \Leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$$

- if  $x[n]$  is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \Leftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

- special case: if  $x[n]$  is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \Rightarrow |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- more special case: if  $x[n]$  is real and symmetric,  $X(e^{j\omega})$  is also real and symmetric.

### 3.6c The DTFT as a change of basis

- DTFT as basis expansion

• Some things are OK:

$$\text{DTFT}\{\delta[n]\} = 1$$

$$\text{DTFT}\{\delta[n]\} = \langle e^{j\omega n}, \delta[n] \rangle = 1$$

- Some things aren't:

$$\text{- DFT } \{1\} = NS[k]$$

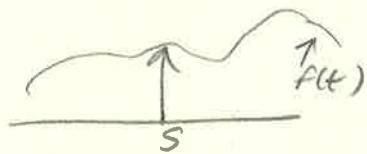
$$\text{- DTFT } \{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$$

- problem: too many interesting sequences are not square-summable!

### - The Dirac delta functional

- Defined by the "sifting" property:

$$\int_{-\infty}^{\infty} \delta(t-s) f(t) dt = f(s), \quad \forall \text{ functions } t \in \mathbb{R}, s \in \mathbb{R}$$



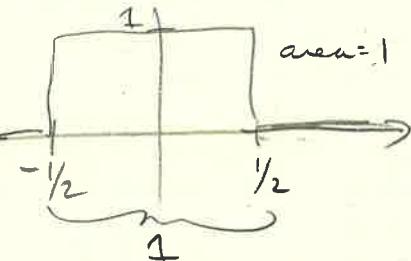
#### - Intuition

- Family of localizing functions  $r_k(t)$  with  $k \in \mathbb{N}$  and  $t \in \mathbb{R}$

- Support inversely proportional to  $k$

- Constant area

$$\text{rect}(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

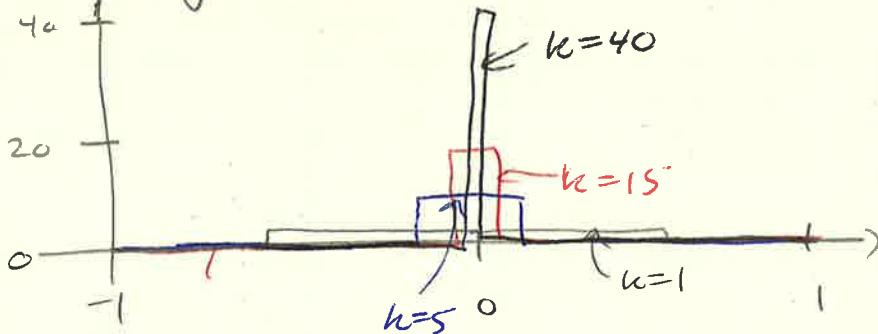


- Consider the localizing family  $r_k(t) = k \text{rect}(kt)$ :

- nonzero over  $[-\frac{1}{2k}, \frac{1}{2k}]$ , i.e., support is  $\frac{1}{k}$

- area is 1

### - The family $r_k(t) = k \text{rect}(kt)$



#### - Extracting a point value

- By the mean value theorem:

$$\int_{-\infty}^{\infty} r_k(t) f(t) dt = k \int_{-\frac{1}{2k}}^{\frac{1}{2k}} f(t) dt = f(\gamma) \Big|_{\gamma \in [-\frac{1}{2k}, \frac{1}{2k}]}$$

and so:

$$\lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} r_k(t) f(t) dt = f(0)$$

### - The Dirac delta functional

The delta functional shorthand. Instead of writing

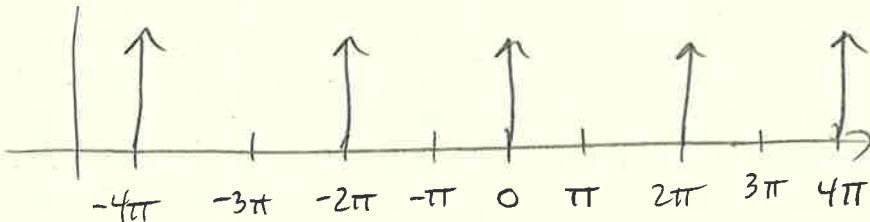
$$\lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} r_k(t-s) f(t) dt$$

we write  $\int_{-\infty}^{\infty} \delta(t-s) f(t) dt$ , as if  $\lim_{k \rightarrow \infty} r_k(t) = \delta(t)$ .

### - The "pulse train"

$$\tilde{S}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

just a technique to use the Dirac delta in the space of  
\$2\pi\$-periodic functions



$$\text{IDFT}\{\tilde{S}(\omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{S}(\omega) e^{j\omega n} d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega = e^{j\omega n} \Big|_{\omega=0} = 1$$

$$(\text{IDFT}\{N \delta[k]\} = 1)$$

$$\Rightarrow \text{DTFT}\{1\} = \tilde{S}(\omega)$$

- Does this make sense?

Partial DTFT sum:  $S_k(\omega) = \sum_{n=-k}^k e^{-j\omega n}$

