Module 1: Basics of Digital Signal Processing

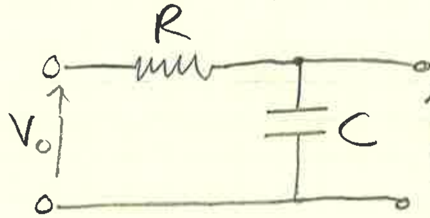
1.1 Introduction to digital signal processing

<u>Signal</u>: Description of evolution of a physical phenomenon

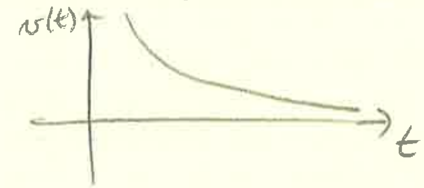
- Weather → temperature
- Sound → pressure
- Sound → magnetic deviation
- Light intensity → gray level on paper

<u>Analysis</u>: understanding the information carried by the signal

<u>Synthesis</u>: creating a signal to contain the given information



$$v(t) = V_0(1 - e^{-t/RC})$$

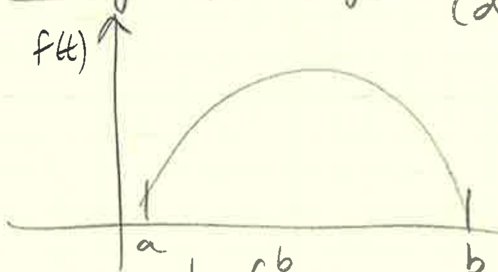


2 dof R, C

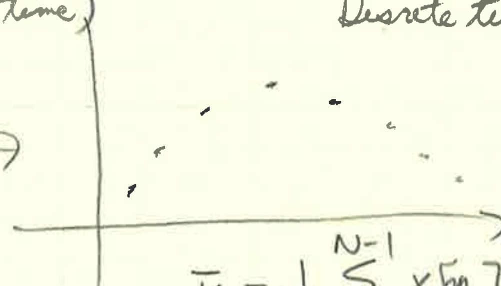
<u>Analog signals</u> $f: \mathbb{R} \to \mathbb{V}$

<u>From analog to digital</u>: $f(t) \to$ sample

<u>The digital paradigm</u> (discrete time)



Discrete time



$$\bar{x} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$\bar{x} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

"time"

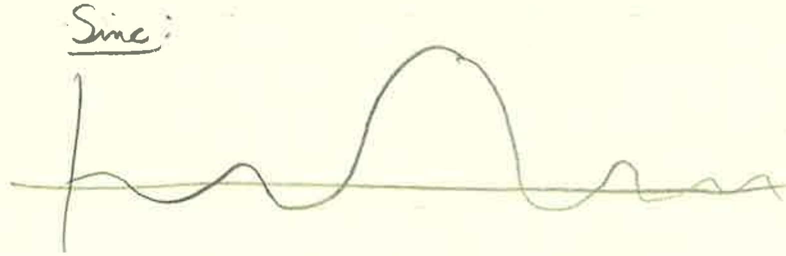
$x: \mathbb{Z} \to \mathbb{V}$ ($\mathbb{Z}$ or $\mathbb{R}$)

<u>The Sampling Theorem (1920)</u>

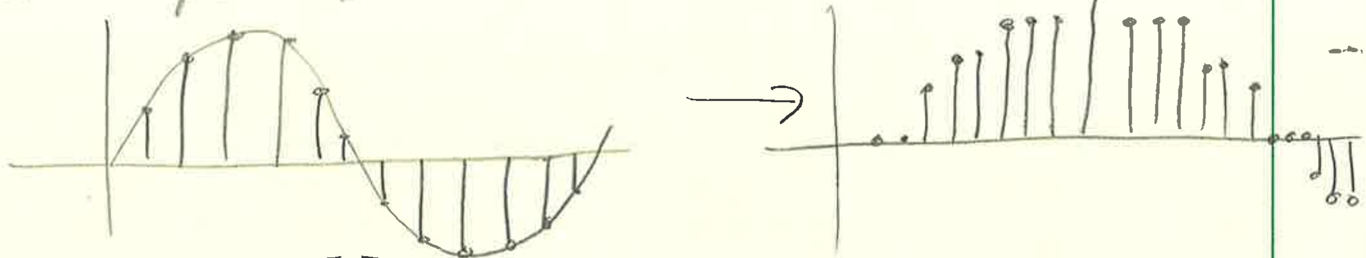
$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

<u>Sinc</u>:

Infinite support



(discrete amplitude)



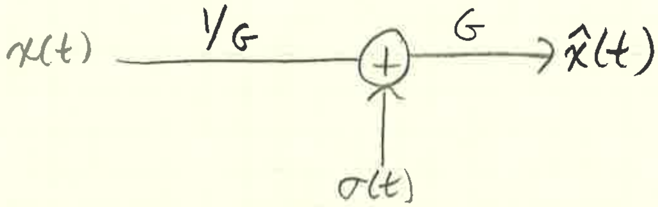
$x[n]$ → $\hat{x}[n]$ Countable set of levels

Why is it important?
- storage
- processing
- transmission

<u>Digital Storage</u> : $\{0, 1\}$

<u>Data Transmission</u>

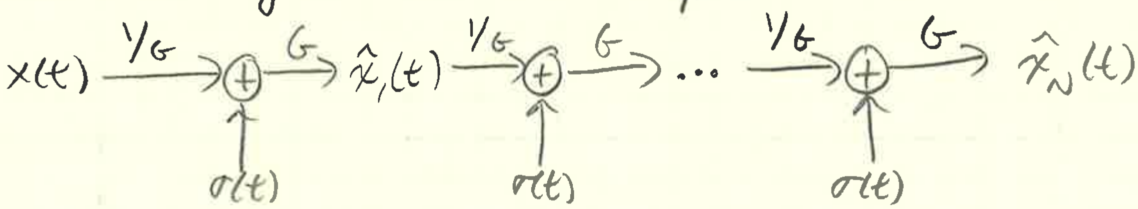
TX → $\boxed{\text{channel}}$ → RX



$$\hat{x}(t) = x(t)/G + G\sigma(t)$$

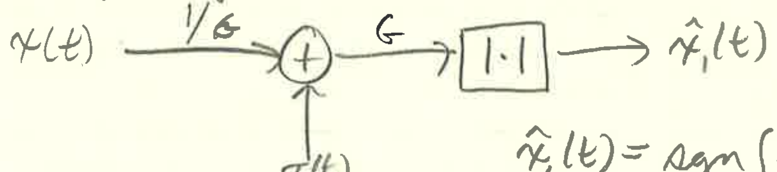
now, it is noisy

For a long channel, we need repeaters

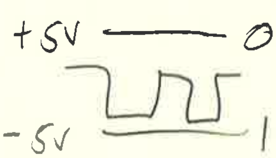


$$\hat{x}_N(t) = x(t) + NG\sigma(t)$$

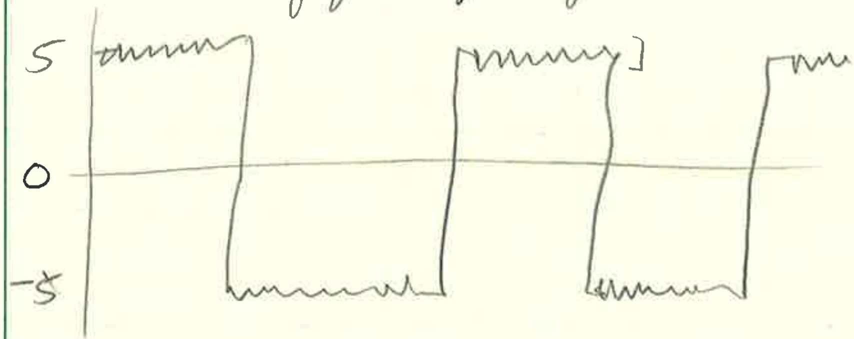
In digital signals, we can threshold



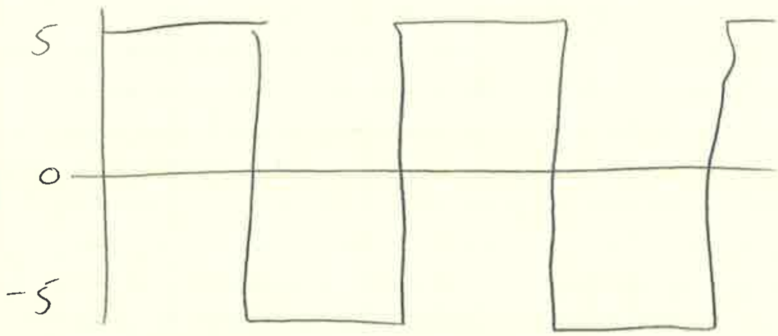
$$\hat{x}_1(t) = \text{sgn}\left[x(t) + G\sigma(t)\right]$$



Transmission of quantized signals



$$G\left(x(t)/G + \sigma(t)\right) = x(t) + G(\sigma(t))$$



$$\hat{x}(t) = G \operatorname{sgn}\left[x(t) + G\sigma(t)\right]$$

(after thresholding operator)

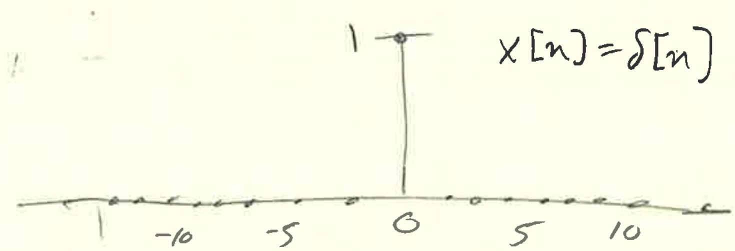
Digital Signal Processing: Key Ideas

- Discretization of time:
  - samples replace idealized models
  - simple maths replaces calculus
- Discretization of values:
  - general-purpose of storage
  - general-purpose processing (CPU)
  - noise can be controlled

1.2 Discrete-time signals

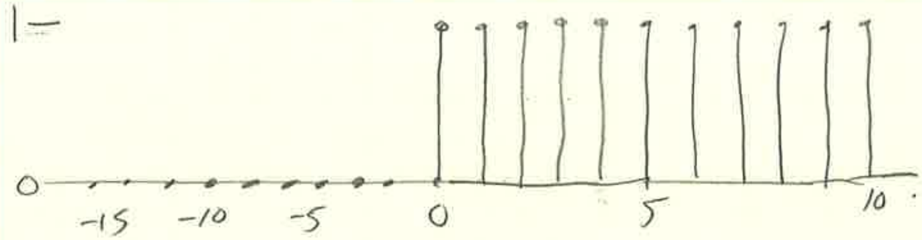
discrete-time signal : a sequence of complex numbers

- One dimension (for now)
- notation : $x[n]$
- two-sided sequences : $x: \mathbb{Z} \to \mathbb{C}$
- n is a-dimensional "time"
- analysis : periodic measurement
- synthesis : stream of generated samples



$x[n] = \delta[n]$

Ex: Used to synchronize audio and video in a movie



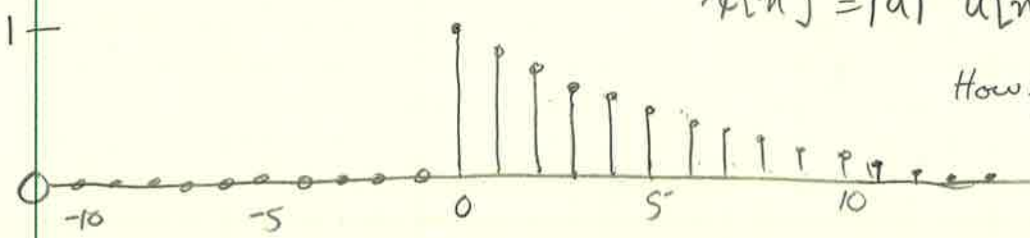
$x[n] = u[n]$

$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

(switch)

Exponential decay

$x[n] = |a|^n u[n]$, $|a| < 1$



How fast does coffee get cold?

Newton's Law of cooling $\frac{dT}{dt} = -c(T - T_{env}) \Rightarrow T(t) = T_{env} + (T_0 - T_{env})e^{-ct}$

Sinusoid $x[n] = \sin(\omega_0 n + \theta)$, ω_0, θ in rad

Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

Finite-length signals

- sequence notation: $x[n]$, $n = 0, 1, \ldots, N-1$
- vector notation: $\bar{x} = [x_0\ x_1 \ldots x_{N-1}]^T$
- practical entities, good for numerical packages (e.g. numpy)

Infinite-length signals

- sequence notation: $x[n]$, $n \in \mathbb{Z}$
- abstraction, good for theorems

Periodic signals

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN]$, $n, k, N \in \mathbb{Z}$
- same information as finite-length of length N
- "natural" bridge between finite and infinite lengths

Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n], & 0 \le n < N \\ 0, & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

- same information as finite-length of length N
- another bridge between finite and infinite lengths

Elementary operators

- scaling: $y[n] = \alpha x[n]$, $\alpha \in \mathbb{C}$
- sum: $y[n] = x[n] + z[n]$
- product: $y[n] = x[n] \cdot z[n]$

$\left.\right\} \quad 0 \le n \le N-1$ (for scaling, sum and product)

- shift by k (delay): $y[n] = x[n-k]$, $k \in \mathbb{Z}$

Shift of a finite-length: finite-support

$$\ldots 0\ 0\ 0\ \underbrace{(x_0\ x_1\ \ldots,\ x_7)}_{\bar{x}[n]}\ 0\ 0\ 0\ \ldots$$

$$\cdot 0\ 0\ 0\ \overbrace{(0\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)}^{\bar{x}[n-1]}\ x_7\ 0\ 0\ \ldots$$

$$\cdot 0\ 0\ 0\ \overbrace{(0\ 0\ 0\ 0\ x_0\ x_1\ x_2\ x_3)}^{\bar{x}[n-4]}\ x_4\ x_5\ x_6\ x_7\ 0\ 0\ 0\ \ldots$$

Shift of a finite length: periodic extension

$$\ldots \overbrace{(x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7)}^{x[n]} \ldots$$

$$\ldots x_5\ x_6\ x_7\ \overbrace{(x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7)}^{\tilde{x}[n]}\ x_0\ x_1\ x_2\ \ldots$$

$$\ldots x_4\ x_5\ x_6\ \overbrace{(x_7\ x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)}^{\tilde{x}[n-1]}\ x_7\ x_0\ x_1\ \ldots$$

$$\cdots x_1\, x_2\, x_3\, \overbrace{\boxed{x_4\, x_5\, x_6\, x_7\, x_0\, x_1\, x_2\, x_3}}^{\tilde{x}[n-4]}\, x_4\, x_5\, x_6 \cdots$$

<u>Energy and power</u>

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

<u>Energy and power: periodic signals</u>

$$E_{\tilde{x}} = \infty$$

$$P_{\tilde{x}} \equiv \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

1.3 Basic signal processing

1.3.a How your PC plays discrete-time sounds

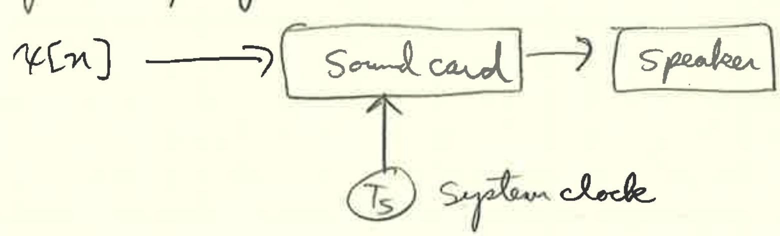
<u>The discrete-time sinusoid</u>

$$x[n] = \sin(\omega_0 n + \theta)$$

<u>Digital vs. physical frequency</u>

- **Discrete time**:
  - n: no physical dimension (just a counter)
  - periodicity: how many samples before pattern repeats
- Physical world:
  - periodicity: how many seconds before pattern repeats
  - frequency measured in Hz (s^{-1})

<u>How your PC plays sounds</u>



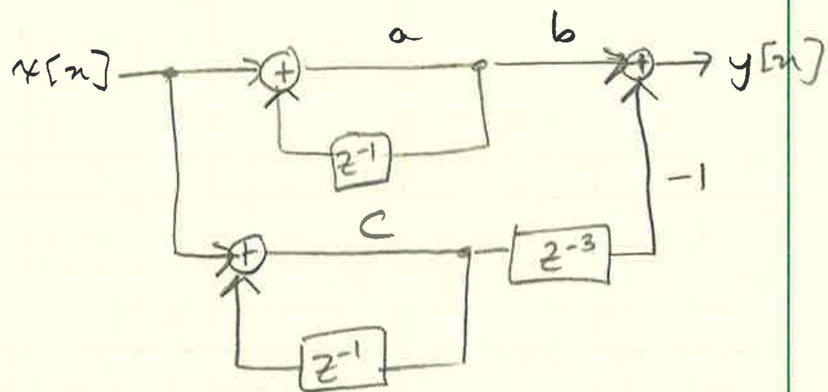
- set T_s, time in seconds between samples
- periodicity of M samples → periodicity of MT_s seconds
- real world frequency: $f = \dfrac{1}{MT_s}$ Hz

- usually we choose F_S, the number of samples per second
- $T_S = 1/F_S$

  Eg. for a typical value, $F_S = 48000$ Hz, $T_S \approx 20.8 \mu s$.
  If $M = 110$, $f \approx 440$ Hz

1.3b The Karplus-Strong algorithm

DSP as Meccano



Building blocks:

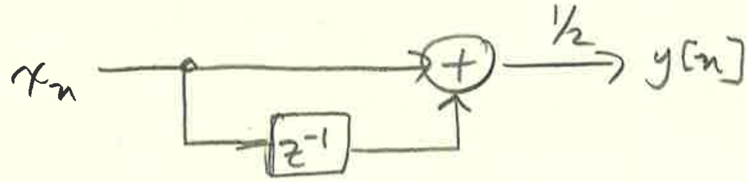
- Adder: $x[n], y[n] \rightarrow \oplus \rightarrow x[n]+y[n]$
- Multiplier: $x[n] \xrightarrow{\alpha} \alpha x[n]$
- Unit Delay: $x[n] \rightarrow \boxed{z^{-1}} \rightarrow x[n-1]$
- Arbitrary Delay: $x[n] \rightarrow \boxed{z^{-N}} \rightarrow x[n-N]$

The 2-point Moving Average

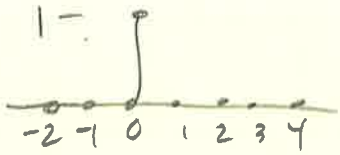
- simple average: $m = \frac{a+b}{2}$
- moving average: take a "local" average

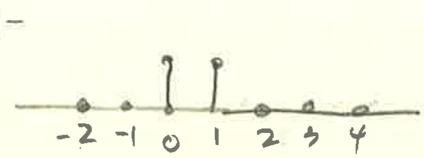
$$y[n] = \frac{x[n] + x[n-1]}{2}$$

- DSP Blocks:



Ex: $x[n] = \delta[n]$





$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1}{2}$$

$$y[1] = \frac{x[1] + x[0]}{2} = \frac{1}{2}$$

– $x[n] = u[n]$

$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1+0}{2} = \frac{1}{2}$$

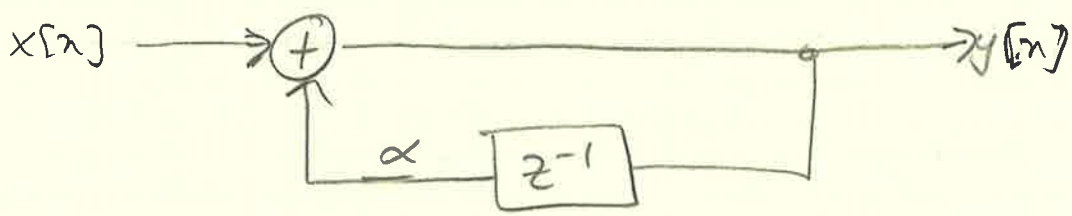
$$y[1] = \frac{x[1] + x[0]}{2} = \frac{1+1}{2} = 1$$

– $x[n] = \cos(\omega n)$, $\omega = \pi/10$

$$y[n] = \frac{\cos \omega n - \cos \omega(n-1)}{2} = \cos(\omega n + \theta)$$

– $x[n] = (-1)^n \Rightarrow y[n] = 0, \; \forall n$

What if we reverse the loop?



$$y[n] = x[n] + \alpha y[n-1] \quad , \; \alpha \in \mathbb{R}$$

(recursion)

<u>How we solve the chicken-and-egg problem</u>

Zero Initial conditions

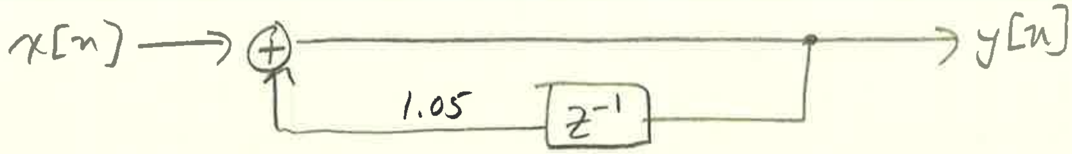
- set a start time (usually $n_0 = 0$)
- assume input and output are zero for all time before n_0

<u>Ex</u>: A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

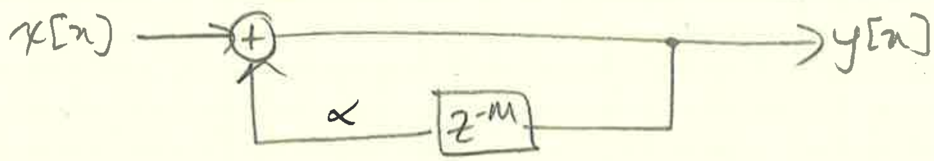


$$y[n] = 1.05\, y[n-1] + x[n]$$

Ex: One-time investment $x[n] = 100\,\delta[n]$

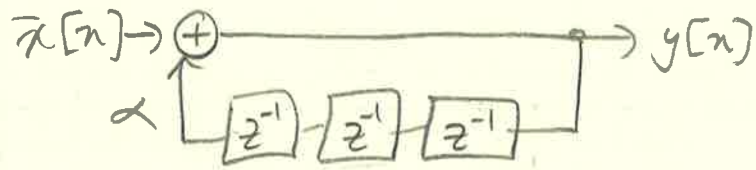
- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25$, $y[3] = 115.7625$, etc.
- In general: $y[n] = (1.05)^n \cdot 100\, u[n]$

An interesting generalization



$$y[n] = \alpha\, y[n-M] + x[n]$$

- Creating loops



$$y[n] = \alpha\, y[n-3] + \bar{x}[n]$$

Ex: $M = 3$, $\alpha = 0.7$, $x[n] = \delta[n]$

- $y[0] = 1$, $y[1] = 0$, $y[2] = 0$
- $y[3] = 0.7$, $y[4] = 0$, $y[5] = 0$
- $y[6] = 0.7^2$, $y[7] = 0$, $y[8] = 0$, etc.

Ex: $M = 3$, $\alpha = 1$, $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

- $y[0] = 1$, $y[1] = 2$, $y[2] = 3$
- $y[3] = 1$, $y[4] = 2$, $y[5] = 3$
- $y[6] = 1$, $y[7] = 2$, $y[8] = 3$, etc.

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \le n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

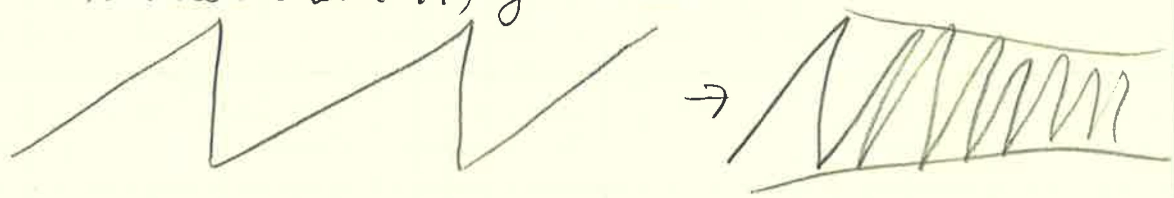
Ex: $M=100$, $\alpha=1$, $\bar{x}[n] = \sin(2\pi n/100)$ for $0 \le n < 100$ and zero elsewhere

$F_S = 48\,\text{kHz} \rightarrow 480\,\text{Hz}$

Introducing some realism

- M controls frequency (pitch)
- α controls envelope (decay)
- $\bar{x}[n]$ controls color (timbre)

Proto-violin: $M=100$, $\alpha=0.95$, $\bar{x}[n]$: zero-mean sawtooth wave between 0 and 99, zero elsewhere



The Karplus-Strong Algorithm

$M=100$, $\alpha=0.9$, $\bar{x}[n]$: 100 random values in $[-1,1]$ between 0 and 99, zero elsewhere.

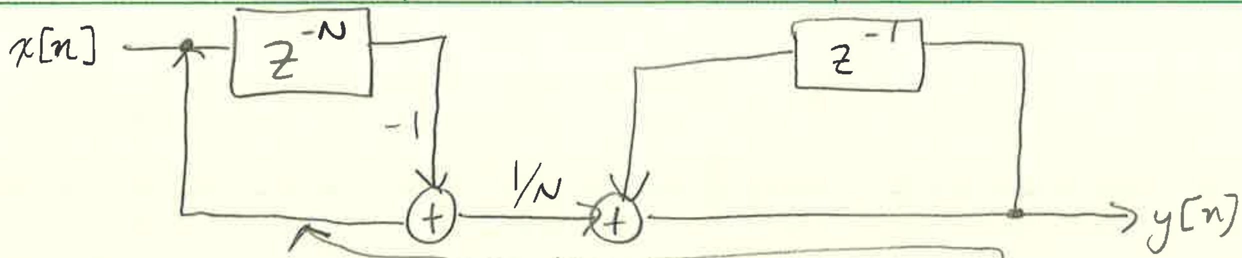
Similar to a harpsichord.

Signal of the Day: Goethe's Temperature Measurement

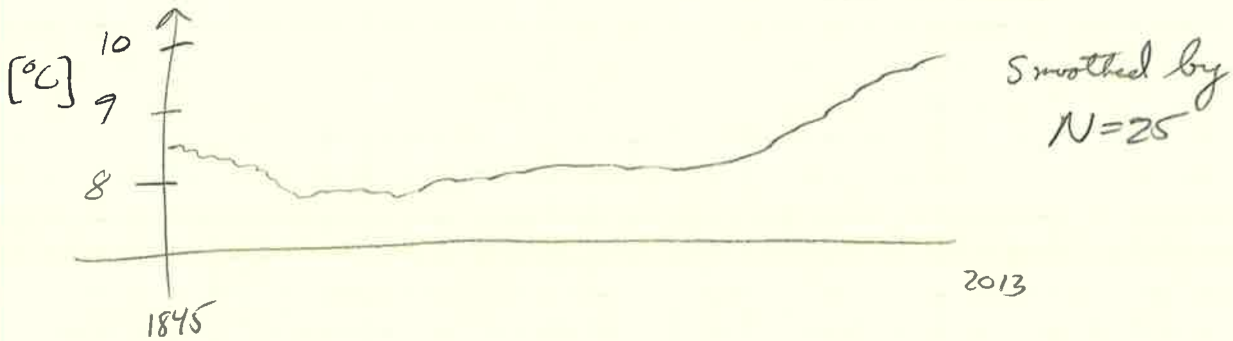
Smoothing
- Moving average: $y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$
- N: window of last observations over which the average is computed

A recursive method

$$y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$$
$$= \frac{1}{N}x[n] + \underbrace{\frac{1}{N}\sum_{m=1}^{N-1} x[n-m] + \frac{1}{N}x[n-N]}_{y[n-1]} - \frac{1}{N}x[n-N]$$
$$= y[n-1] + \frac{1}{N}\left(x[n] - x[n-N]\right)$$



$$y[n] = y[n-1] + \boxed{1/N\left(x[n] - x[n-N]\right)}$$

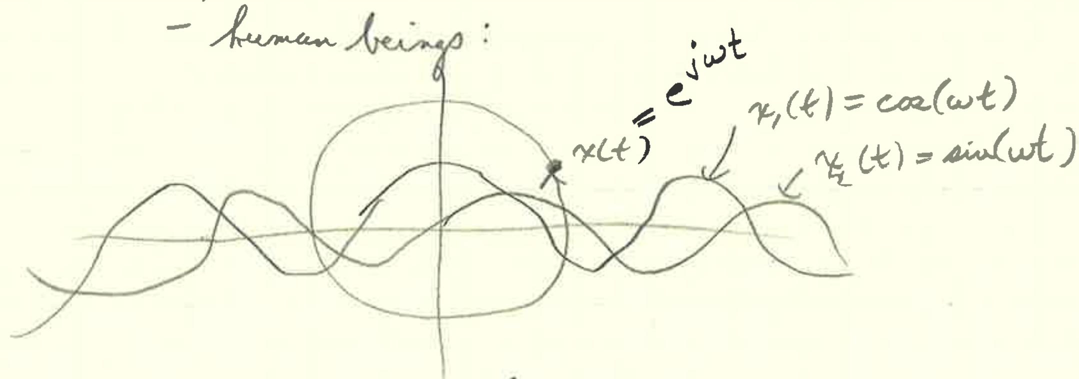


1.4 Complex exponentials

$$j = \sqrt{-1}$$

Oscillations are everywhere!

- Sustainable **dynamic systems exhibit** oscillatory behavior
- Intuitively: things that don't move in circles can't last:
  - bombs
  - rockets
  - human beings:



- The discrete-time oscillatory heartbeat

  Ingredients:
  - a frequency ω (units: radians)
  - an initial phase ϕ (units: radians)
  - an amplitude A

$$x[n] = Ae^{j(\omega n+\phi)}$$
$$= A\left[\cos(\omega n+\phi) + j\sin(\omega n+\phi)\right]$$

Why complex exponentials?

- we can use complex numbers in digital systems, so why not?
- it makes sense: every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra

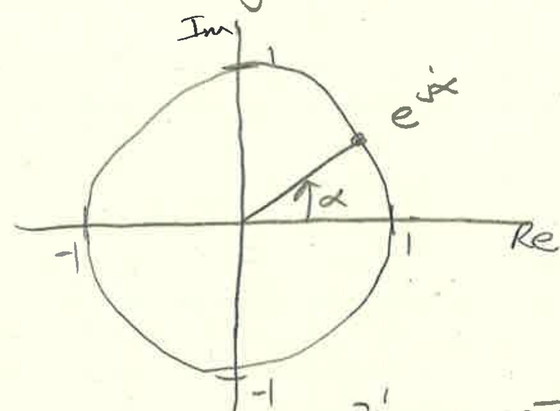
Ex: change the phase of a cosine the "old-school" way

$$\cos(\omega n+\phi) = a\cos(\omega n) - b\sin(\omega n), \quad a=\cos\phi,\ b=\sin\phi$$

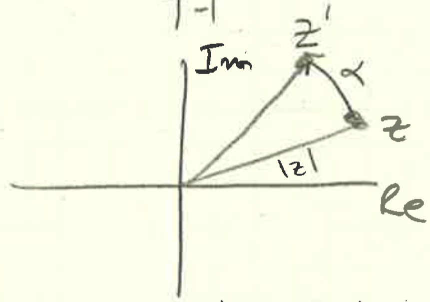
use complex exponentials

$$\cos(\omega n+\phi) = Re\left[e^{j(\omega n+\phi)}\right] = Re\left[e^{j\omega n}e^{j\phi}\right]$$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$



$$|e^{j\alpha}| = 1$$

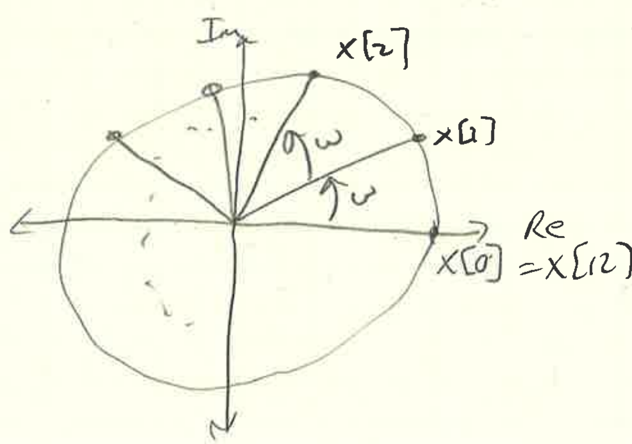


rotation $z' = ze^{j\alpha}$

The complex exponential generating machine

$$x[n] = e^{j\omega n}; \quad x[n+1] = e^{j\omega}x[n]$$

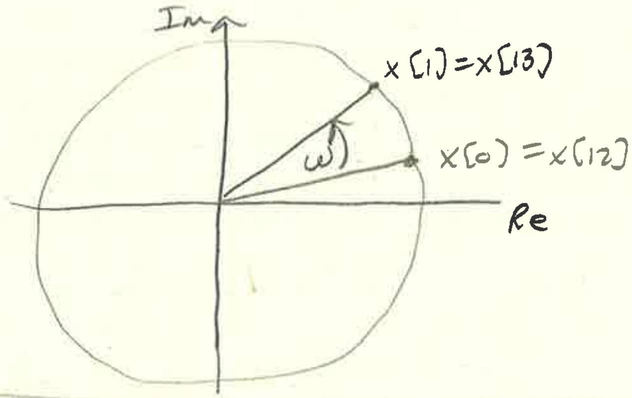
$$x[0]=1$$



$$\omega = \frac{2\pi}{12}$$

Initial phase

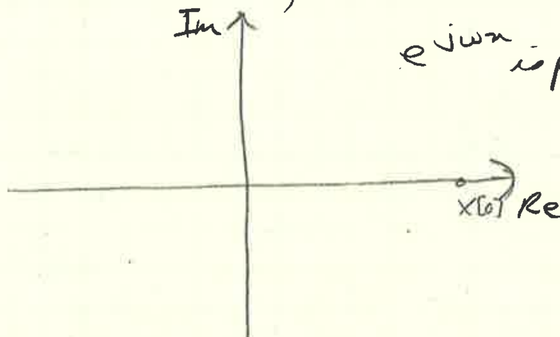
$$x[n] = e^{j(\omega n+\phi)}; \quad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



$\omega = \frac{2\pi}{12}$

$\boxed{\text{Careful: not every sinusoid is periodic in discrete time}}$

$$x[n] = e^{j\omega n}; \quad x[n+1] = e^{j\omega} x[n]$$



$e^{j\omega n}$ is periodic in $n \iff \omega = \frac{M}{N} 2\pi$, $M, N \in \mathbb{Z}$

$$x[n] = x[n+N]$$

$$e^{j(\omega n+\phi)} = e^{j(\omega(n+N)+\phi)}$$

$$e^{j\omega n} e^{j\phi} = e^{j\omega n} e^{j\omega N} e^{j\phi}$$

$$e^{j\omega N} = 1 \iff \omega N = 2M\pi, \ M \in \mathbb{Z}$$

$$\omega = \frac{M}{N} 2\pi$$

<u>2π-periodicity</u>: one point, many names

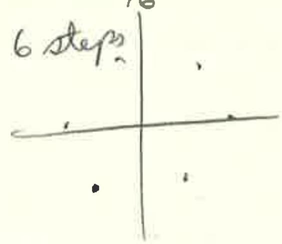
$$e^{j\alpha} = e^{j(\alpha+2\pi k)}, \ \forall k \in \mathbb{Z}$$

One point, many names: Aliasing

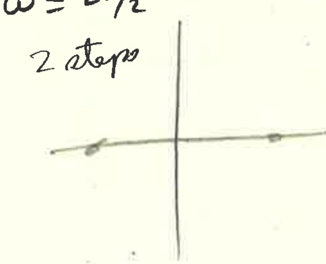
How "fast" can we go?

$\omega = \frac{2\pi}{12}$

$\omega = 2\pi/6$

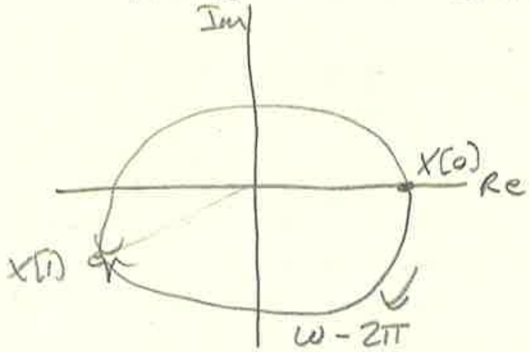


$\omega = 2\pi/2$



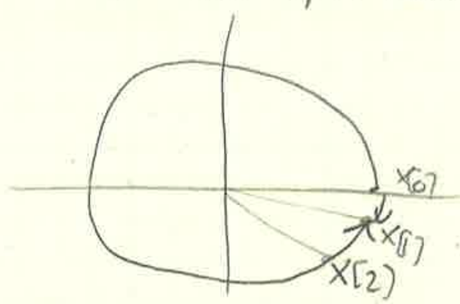
What if we go faster?

$\pi < \omega < 2\pi$ corresponds to going slower in opposite direction



$\omega = 2\pi - \alpha$, α small

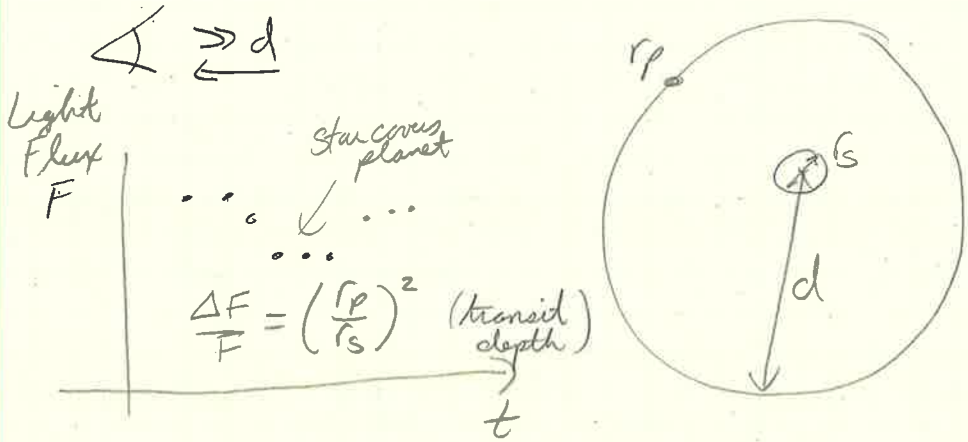
very slow in opposite direction



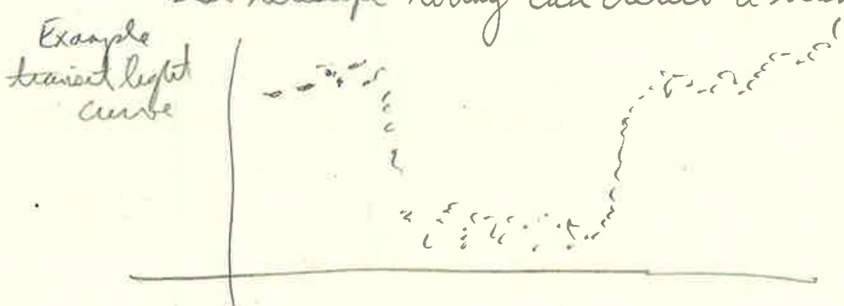
2.1 Signal processing and vector spaces

Common framework: vector space
- vector spaces are very general objects
- vector spaces are defined by their properties
- once you know the properties are satisfied, you can use all the tools for the space

Signal of the day: exoplanet hunting



- Earth: $\frac{\Delta F}{F} = \left(\frac{r_p}{r_s}\right)^2 = \left(\frac{6{,}371}{696{,}000}\right)^2 \approx 0.01\%$
- Jupiter: $\frac{\Delta F}{F} = \left(\frac{69{,}911}{696{,}000}\right)^2 \approx 1\%$
- Best telescope today can detect a transit depth of 0.1%



2.2 Vector Spaces

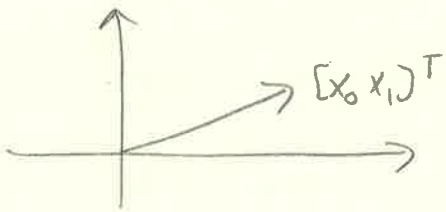
2.2a Vector space

Some familiar examples
- $\mathbb{R}^2, \mathbb{R}^3$: Euclidean space
- $\mathbb{R}^N, \mathbb{C}^N$: linear algebra

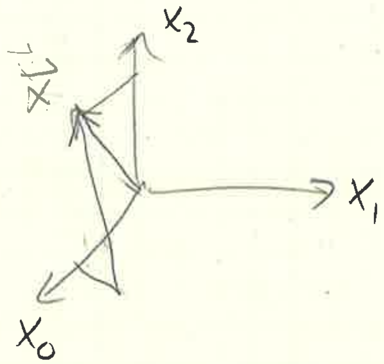
Other examples
- $\ell_2(\mathbb{Z})$: space of square-summable infinite sequences
- $L_2([a,b])$: space of square-integrable <u>functions</u> over an interval

Some can be represented geometrically

$\mathbb{R}^2$: $\vec{x} = [x_0 \ x_1]^T$

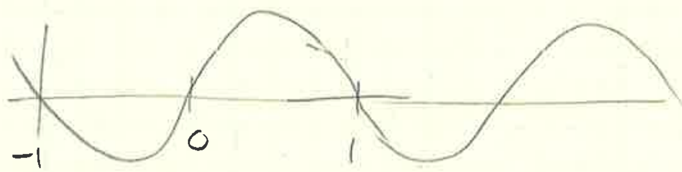


$\mathbb{R}^3$: $\vec{x} = [x_0 \ x_1 \ x_2]^T$



$L_2([-1,1])$: $\vec{x} = x(t)$, $t \in [-1,1]$

$\vec{x} = \sin(\pi t)$



Can't plot $\mathbb{R}^N$, $N > 3$ or $\mathbb{C}^N$, $N > 1$

Ingredients

- the set of vectors V
- a set of scalars (say $\mathbb{C}$)

We need at least to be able to:

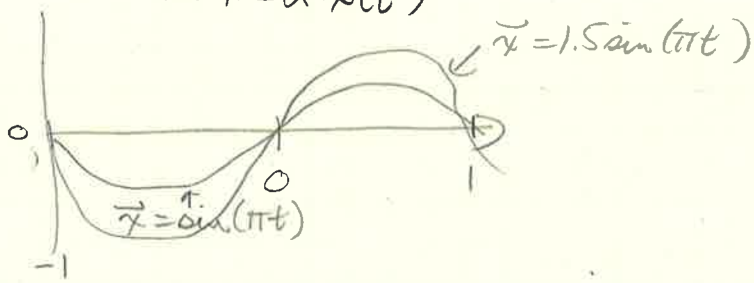
- resize vectors, i.e., multiply a vector by a scalar
- combine vectors together, i.e., sum them

Formal properties: For $\vec{x}, \vec{y}, \vec{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- $\alpha(\vec{x} + \vec{y}) = \alpha\vec{y} + \alpha\vec{x}$
- $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
- $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$
- $\exists 0 \in V : \vec{x} + 0 = 0 + \vec{x} = \vec{x}$
- $\forall \vec{x} \in V, \exists(-\vec{x}) : \vec{x} + (-\vec{x}) = 0$

Scalar multiplication in $L_2[-1,1]$

$$\alpha\vec{x} = \alpha x(t)$$



We need something more: inner product (aka dot product)

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$$

- measure of similarity between vectors
- inner product is zero? vector are orthogonal (maximally different)

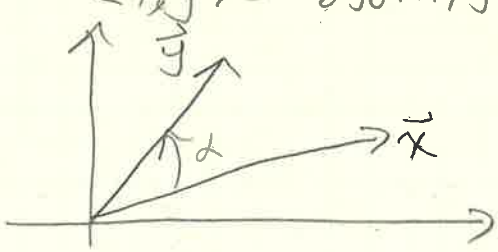
Formal properties of the inner product

For $\vec{x}, \vec{y}, \vec{z} \in V$, $\alpha \in \mathbb{C}$:

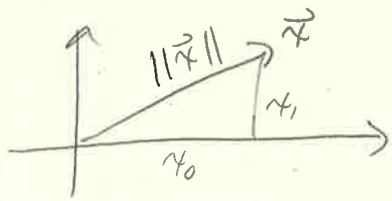
- $\langle \vec{x}+\vec{y}, \vec{z}\rangle = \langle \vec{x}, \vec{z}\rangle + \langle \vec{y}, \vec{z}\rangle$
- $\langle \vec{x}, \vec{y}\rangle = \langle \vec{y}, \vec{x}\rangle^*$
- $\langle \alpha\vec{x}, \vec{y}\rangle = \alpha^*\langle \vec{x}, \vec{y}\rangle$
- $\langle \vec{x}, \alpha\vec{y}\rangle = \alpha\langle \vec{x}, \vec{y}\rangle$
- $\langle \vec{x}, \vec{x}\rangle \geq 0$

- $\langle \vec{x}, \vec{x}\rangle = 0 \Leftrightarrow \vec{x} = 0$
- If $\langle \vec{x}, \vec{y}\rangle = 0$ and $\vec{x}, \vec{y} \neq 0$, then $\vec{x}$ and $\vec{y}$ are called orthogonal

$$\langle \vec{x}, \vec{y}\rangle = x_0 y_0 + x_1 y_1 = \|\vec{x}\|\,\|\vec{y}\| \cos\alpha$$



$$\langle x, x\rangle = x_0^2 + x_1^2 = \|\vec{x}\|^2$$

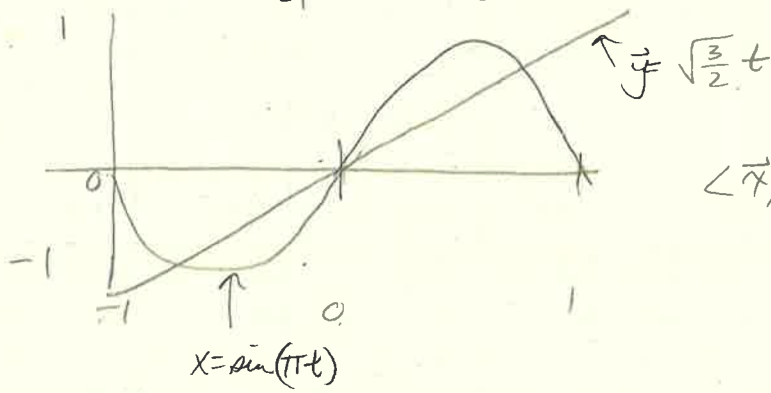


Inner product in $L_2[-1,1]$

$$\langle \vec{x},\vec{y}\rangle = \int_{-1}^{1} x(t)\,y(t)\,dt$$

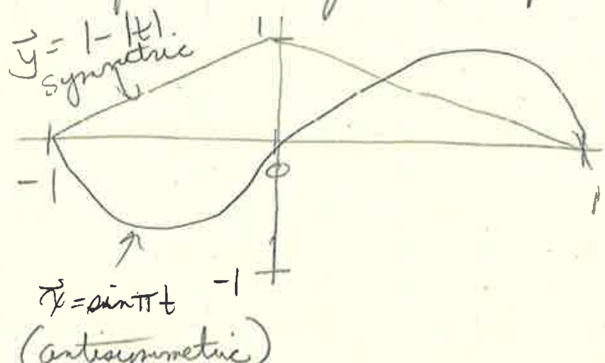
$$\|\sin(\pi t)\|^2 = \int_{-1}^{1} \sin^2\pi t\,dt = 1$$

$\vec{y}=t$: $\|\vec{y}\|^2 = \int_{-1}^{1} t^2 dt = \frac{2}{3}$



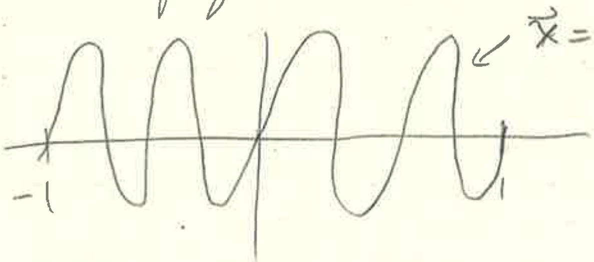
$$\langle \vec{x},\vec{y}\rangle = \int_{-1}^{1}\sqrt{\tfrac{3}{2}}\,t\sin\pi t\,dt = \frac{2}{\pi}\sqrt{\frac{3}{2}} \approx 0.78$$

$\vec{x},\vec{y}$ from orthogonal subspaces:



$\langle \vec{x},\vec{y}\rangle = 0$

Sinusoids with frequencies that are integer multiples of a fundamental



$\vec{y}=\sin 5\pi t$

$\langle \vec{x},\vec{y}\rangle = 0$

Norm vs Distance

- inner product defines a norm: $\|\vec{x}\| = \sqrt{\langle \vec{x},\vec{x}\rangle}$
- norm defines a distance: $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

Distance in $L_2[-1,1]$: the Mean Square Error

$$\|\vec{x}-\vec{y}\|^2 = \int_{-1}^{1} |x(t)-y(t)|^2 dt$$

$\vec{x} = \sin 4\pi t$, $\vec{y} = \sin 5\pi t$, , $\|\vec{x}-\vec{y}\|^2 = \int_{-1}^{1} |\sin 4\pi t - \sin 5\pi t|^2 dt = 2$

2.2.b Signal Spaces

Finite-Length Signals

finite-length and periodic signals live in $\mathbb{C}^N$

- vector notation : $\vec{x} = [x_0 \ x_1 \ \dots \ x_{N-1}]^T$
- all operations well-defined and intuitive
- space of N-periodic signals sometimes indicated by $\tilde{\mathbb{C}}^N$

Inner product for signals

$$\langle \vec{x}, \vec{y} \rangle = \sum_{n=0}^{N-1} x^*[n]\, y[n]$$

well-defined for all finite-length vectors

Infinite Signals?

$$\langle \vec{x}, \vec{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]\, y[n]$$

We require sequences to be square-summable : $\sum |x[n]|^2 < \infty$

i.e. in $\ell_2(\mathbb{Z})$ (finite-energy)

many interesting signals are not in $\ell_2(\mathbb{Z})$, such as,

$x[n] = 1$, $x[n] = \cos(\omega n)$, etc.

Completeness

Limiting operations must yield vector space elements

An incomplete space : $\mathbb{Q}$ $\quad x_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$,

but $\lim_{n\to\infty} x_n = e \notin \mathbb{Q}$

Hilbert Space

1. a vector space : $H(V, \mathbb{C})$
2. an inner product : $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$
3. Complete

Bases

Linear combination is the basic operation in vector spaces:

$$\vec{g} = \alpha \vec{x} + \beta \vec{y}$$

Can we find a set of vectors $\{\vec{w}^{(k)}\}$ so that we can write any vector as a linear combination of the $\{\vec{w}^{(k)}\}$?

Canonical $\mathbb{R}^2$ basis

$$\vec{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Another $\mathbb{R}^2$ basis

$$\vec{v}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \alpha_0 = x_0 - x_1, \quad \alpha_1 = x_1$$

Not a basis for $\mathbb{R}^2$

$$\vec{g}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{g}^{(1)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ not linearly independent

What about infinite-dimensional spaces?

$$\vec{x} = \sum_{k=0}^{\infty} \alpha_k \vec{w}^{(k)}$$

a basis for $\ell_2(\mathbb{Z})$

$$\vec{e}^{(k)} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad 1 \text{ in } k^{th} \text{ position}, \ k \in \mathbb{Z}$$

What about function vector spaces?

$$f(t) = \sum_k \alpha_k h^{(k)}(t)$$

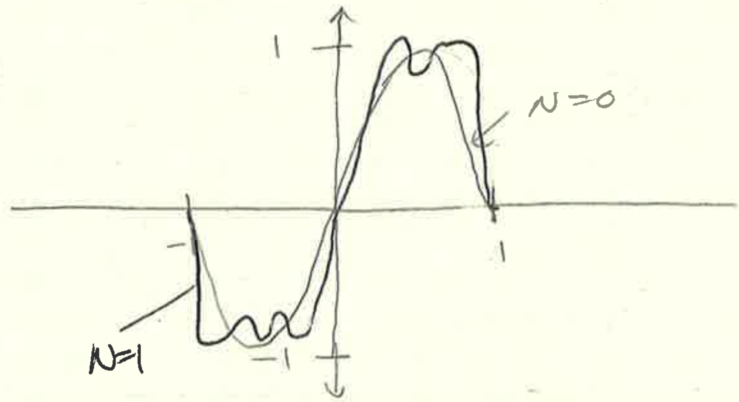
A basis for the functions over an interval?

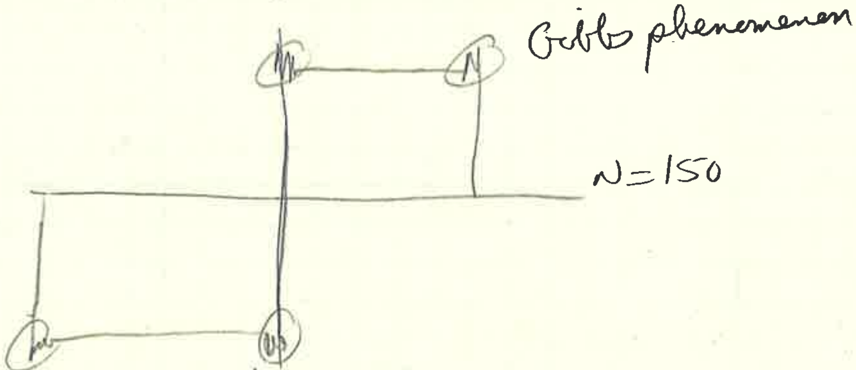
the Fourier basis for $[-1,1]$

$$\left\{ \frac{1}{\sqrt{2}}, \cos \pi t, \sin \pi t, \cos 2\pi t, \sin 2\pi t, \cos 3\pi t, \sin 3\pi t, \ldots \right\}$$

Using the Fourier Basis (approximating a square wave)

$$\sum_{k=0}^{N} \frac{\sin(2k+1)\pi t}{2k+1} = \sum_{k=0}^{N} \frac{w^{(4k+2)}}{2k+1}$$





Bases: formal definition

Given:

- a vector space H
- a set of K vectors from H: $W = \{\vec{w}^{(k)}\}_{k=0,1,\ldots,K-1}$

W is a basis for H if:

1. We can write for all $x \in H$:

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

2. the coefficents α_k are unique

Uniqueness implies linear independence

$$\sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} = 0 \iff \alpha_k = 0, \quad k = 0, 1, \ldots, K-1$$

Special bases

Orthogonal basis:

$$\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = 0, \quad k \neq n$$

Orthonormal basis: $\langle \vec{w}^{(k)}, \vec{w}^{(n)} \rangle = \delta[n-k]$

We can use Gram-Schmidt to normalize any orthogonal basis

Basis expansion

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)}$$, how do we find the α's?

Orthonormal bases are the best: $\alpha_k = \langle \vec{w}^{(k)}, \vec{x} \rangle$

Change of basis

$$\vec{x} = \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \vec{v}^{(k)}$$

If $\{\vec{v}^{(k)}\}$ is orthonormal:

$$\beta_h = \langle \vec{v}^{(h)}, \vec{x} \rangle$$

$$= \langle \vec{v}^{(h)}, \sum_{k=0}^{K-1} \alpha_k \vec{w}^{(k)} \rangle = \sum_{k=0}^{K-1} \alpha_k \langle \vec{v}^{(h)}, \vec{w}^{(k)} \rangle$$

$$= \sum_{k=0}^{K-1} \alpha_k c_{hk}$$

$$= \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0(K-1)} \\ & & \vdots & \\ c_{(K-1)0} & c_{(K-1)1} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix}$$

Change of basis: example

- canonical basis $E = \{\vec{e}^{(0)}, \vec{e}^{(1)}\}$
- $\vec{x} = \alpha_0 \vec{e}^{(0)} + \alpha_1 \vec{e}^{(1)}$
- new basis $V = \{\vec{v}^{(0)}, \vec{v}^{(1)}\}$ with $\vec{v}^{(0)} = [\cos\theta \;\; \sin\theta]^T$, $\vec{v}^{(1)} = [-\sin\theta \;\; \cos\theta]^T$

$$\vec{x} = \beta_0 \vec{v}^{(0)} + \beta_1 \vec{v}^{(1)}$$

- new basis is orthonormal: $c_{hk} = \langle \vec{v}^{(h)}, \vec{e}^{(k)} \rangle$
- in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = R\vec{\alpha}$$

- R: Rotation matrix
- $R^T R = I$

2.4 Subspaces and approximations

Vector Subspace

- A subset of vectors closed under addition and scalar multiplication
- Example: $\mathbb{R}^2 \subset \mathbb{R}^3$

- Subspace of symmetric functions over $L_2[-1,1]$

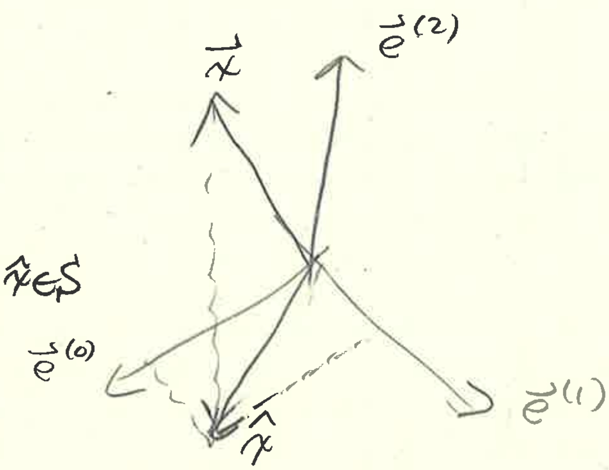
$\vec{x} = \cos \pi t$
$\vec{y} = \cos 5\pi t$ to name a couple

- Subspaces have their own bases

$$\left\{ \vec{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ basis for a plane

Approximation

Problem:
- vector $x \in V$
- subspace $S \subseteq V$
- approximate $\vec{x}$ with $\hat{x} \in S$



Least-Squares Approximation

- $\{\vec{s}^{(k)}\}_{k=0,1,\dots,K-1}$ orthonormal basis for S
- orthogonal projection:

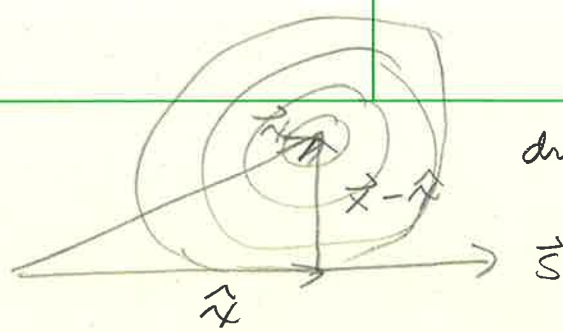
$$\hat{x} = \sum_{k=0}^{K-1} \langle \vec{s}^{(k)}, \hat{x} \rangle \vec{s}^{(k)}$$

- orthogonal projection is the "best" approximation over S
- orthogonal projection has minimum-norm error:

$$\arg\min_{\vec{y} \in S} \|\vec{x} - \vec{y}\| = \hat{x}$$

- error is orthogonal to approximation:

$$\langle \vec{x} - \hat{x}, \hat{x} \rangle = 0$$



draw concentric circles until hitting $\vec{S}$. This radius vector is $\vec{x}-\hat{x}$.

Example: polynomial approximation

- vector space $P_N[-1,1] \subset L_2[-1,1]$
- $\vec{p} = a_0 + a_1 t + \ldots + a_{N-1} t^{N-1}$
- a self-evident, naive basis: $\vec{S}^{(k)} = t^k, \quad k = 0, 1, \ldots, N-1$
- naive basis is not orthonormal

goal: approximate $\vec{x} = \sin t \in L_2[-1,1]$ over $P_3[-1,1]$

- build orthonormal basis from naive basis
- project $\vec{x}$ over the orthonormal basis
- compute approximation error
- compare error to Taylor approximation (well known but not optimal over the interval)

Building an orthonormal basis

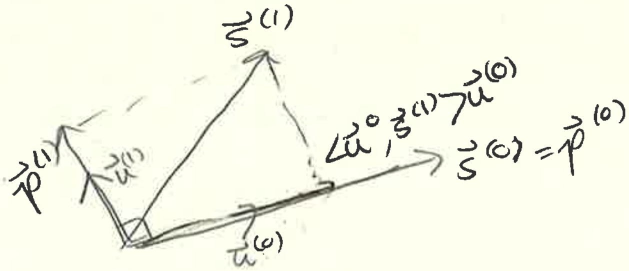
Gram-Schmidt orthonormalization procedure:

$$\{\vec{s}^{(k)}\} \longrightarrow \{\vec{u}^{(k)}\}$$

original set $\qquad$ orthonormal set

Algorithmic procedure: at each step k

1. $\vec{p}^{(k)} = \vec{s}^{(k)} - \sum_{n=0}^{k-1} \langle \vec{u}^{(n)}, \vec{s}^{(k)} \rangle \vec{u}^{(n)}$
2. $\vec{u}^{(k)} = \vec{p}^{(k)} / \| \vec{p}^{(k)} \|$



Apply Gram-Schmidt to $S = \{1, t, t^2, t^3, \dots\}$

$$\langle \vec{x}, \vec{y} \rangle = \int_{-1}^{1} x(t)\, y(t)\, dt$$

→ $\vec{s}^{(0)} = 1$

- $\vec{p}^{(0)} = \vec{s}^{(0)} = 1$
- $\|\vec{p}^{(0)}\|^2 = 2$
- $\vec{u}^{(0)} = \vec{p}^{(0)} / \|\vec{p}^{(0)}\| = \frac{1}{\sqrt{2}}$

− $\vec{s}^{(1)} = t$

- $\langle \vec{u}^{(0)}, \vec{s}^{(1)} \rangle = \int_{-1}^{1} \frac{t}{\sqrt{2}} = 0$
- $\vec{p}^{(1)} = \vec{s}^{(1)} = t$
- $\|\vec{p}^{(1)}\|^2 = \frac{2}{3}$
- $\vec{u}^{(1)} = \sqrt{3/2}\, t$

− $\vec{s}^{(2)} = t^2$

- $\langle \vec{u}^{(0)}, \vec{s}^{(2)} \rangle = \int_{-1}^{1} \frac{t^2}{\sqrt{2}} dt = \frac{2}{3\sqrt{2}}$
- $\langle \vec{u}^{(1)}, \vec{s}^{(2)} \rangle = \int_{-1}^{1} t^3/\sqrt{2} = 0$
- $\vec{p}^{(2)} = \vec{s}^{(2)} - \frac{2}{3\sqrt{2}} \vec{u}^{(0)} = t^2 - \frac{1}{3}$
- $\|\vec{p}^{(2)}\|^2 = 8/45$
- $\vec{u}^{(2)} = \sqrt{\frac{5}{8}}\,(3t^2 - 1)$

Legendre Polynomials

The Gram-Schmidt algorithm leads to an orthonormal basis for $P_N([-1,1])$

$$\vec{u}^{(0)} = \sqrt{\frac{1}{2}}, \quad \vec{u}^{(1)} = \sqrt{\frac{3}{2}}\, t, \quad \vec{u}^{(2)} = \sqrt{\frac{5}{8}}\,(3t^2-1), \quad \vec{u}^{(3)} = \dots$$

Orthogonal projection over $P_3[-1,1]$

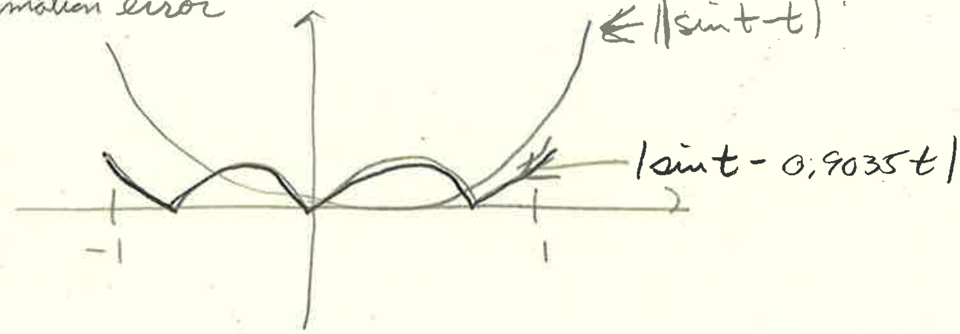
$$\alpha_k = \langle \vec{u}^{(k)}, \vec{x} \rangle = \int_{-1}^{1} u_k(t) \sin t \, dt$$

- $\alpha_0 = \langle \frac{1}{\sqrt{2}}, \sin t \rangle = 0$
- $\alpha_1 = \langle \sqrt{\frac{3}{2}}\, t, \sin t \rangle \approx 0.7377$
- $\alpha_2 = \langle \sqrt{\frac{5}{8}}\,(3t^2-1), \sin t \rangle = 0$

$\sin t \to \alpha_1 \vec{u}^{(1)} \approx 0.9035\, t$

Taylor Series: $\sin t \approx t$

Approximation error



Error norm:

Orthogonal projection over $P_3[-1,1]$:

$$\|\sin t - \alpha_1 \vec{u}^{(1)}\| \approx 0.0337$$

Taylor series: $\|\sin t - t\| \approx 0.0857$

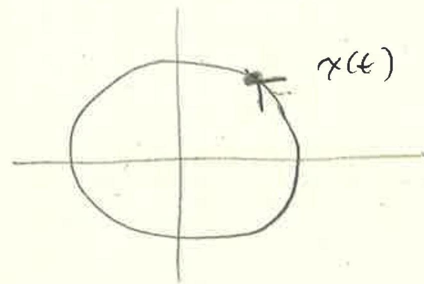
Module 3: Part 1 - Basics of Fourier Analysis

3.1.a The frequency domain

- Oscillations are everywhere
  - Sustainable dynamic systems exhibit oscillatory behavior
  - Intuitively: things that don't move in circles don't last:
    - bombs
    - rockets
    - human beings...

Period P

Frequency $f = \frac{1}{P}$



- The intuition
  - humans analyze complex signals (audio, images) in terms of their sinusoidal components
  - We can build instruments that "resonate" at one or multiple frequencies (tuning fork vs. piano)
  - the "frequency domain" seems to be as important as the time domain

- Fundamental question: can we decompose any signal into sinusoidal elements? Yes, using Fourier analysis

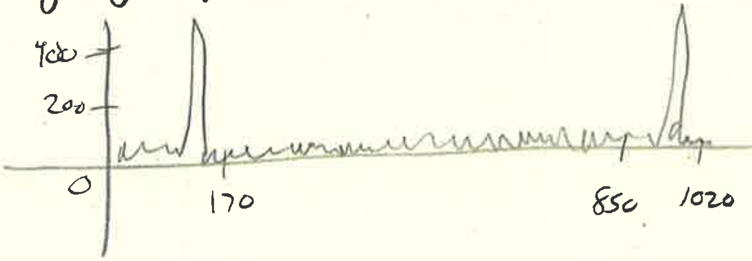
Analysis	Synthesis
- from time domain to frequency domain	- from frequency domain to time domain
- find the contribution of different frequencies	- create signals with known frequency content
- discover "hidden" signal properties	- fit signals to specific frequency regions

3.1b The DFT as a change of basis

- The mathematical setup
  - let's start with finite-length signals (i.e. vectors in $\mathbb{C}^N$)
  - Fourier analysis is a simple change of basis
  - a change of basis is a change of perspective

- Mystery signal in time domain

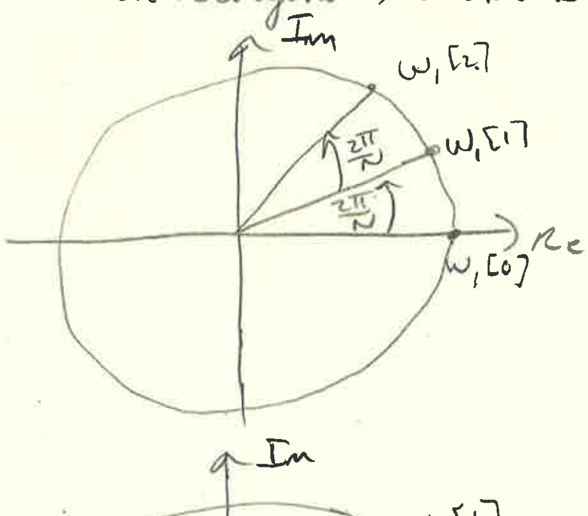
- Mystery signal in the Fourier basis



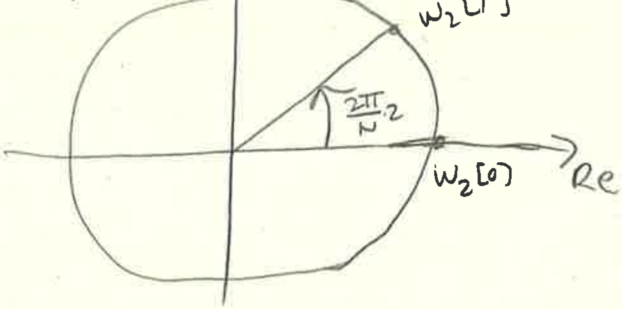
- The Fourier Basis for $\mathbb{C}^N$

Claim: the set of N signals in $\mathbb{C}^N$
$w_k[n] = e^{j\frac{2\pi}{N}nk}$, $n,k = 0,1,\ldots,N-1$ is an orthogonal basis in $\mathbb{C}^N$. $\omega = \frac{2\pi}{N}k$

In vector notation: $\{\vec{w}^{(k)}\}_{k=0,1,\ldots,N-1}$ with $w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$ is an orthogonal basis in $\mathbb{C}^N$

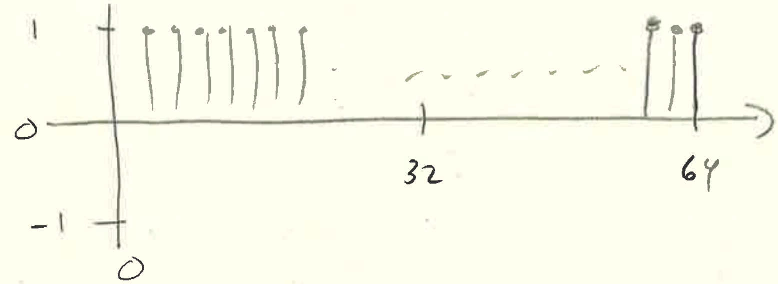


$w_n^{(1)} = e^{j\frac{2\pi}{N}k}$

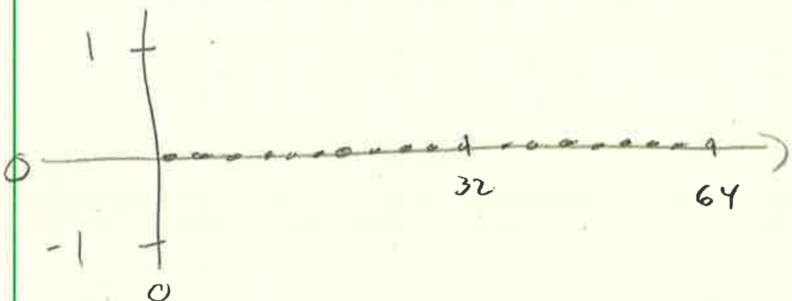


$w_n^{(2)} = e^{j\frac{2\pi}{N}2n}$

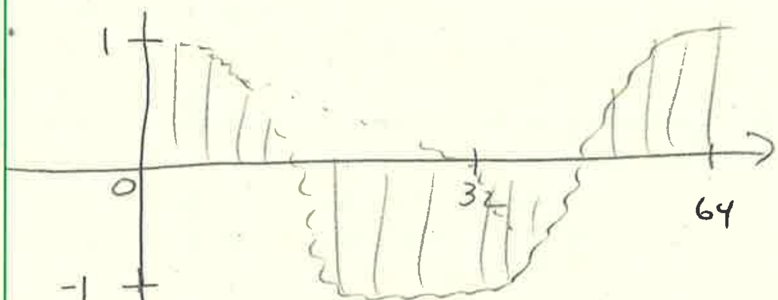
Basis vector $\vec{w}^{(0)} \in \mathbb{C}^{64}$



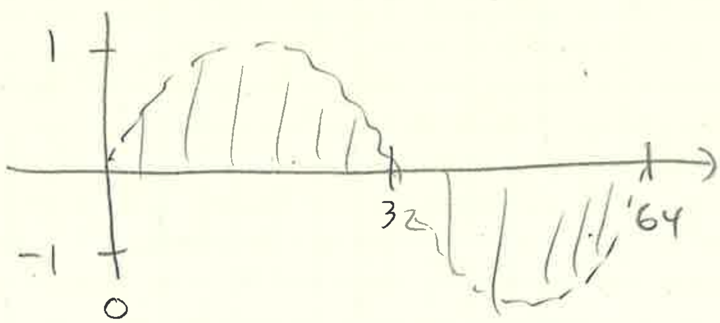
$$\vec{w}_n^{(0)} = e^{i\frac{2\pi}{N}0n} = 1$$



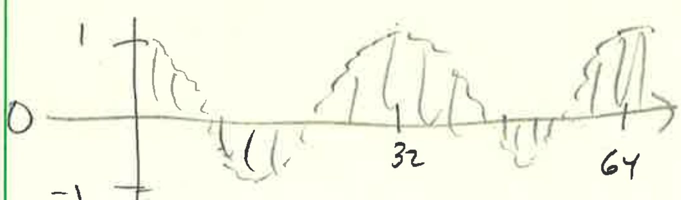
Basis vector $\vec{w}^{(1)} \in \mathbb{C}^{64}$, $\vec{w}_n^{(1)} = e^{i\frac{2\pi}{N}1n}$



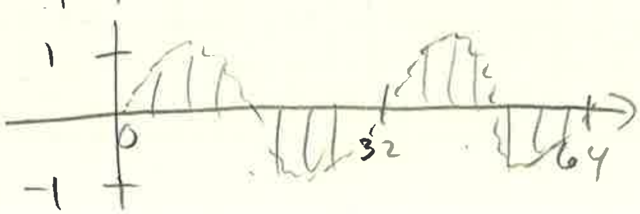
$\omega = \frac{2\pi}{N}$



Basis vector $\vec{w}^{(2)} \in \mathbb{C}^{64}$



$\omega = \frac{2\pi}{N} \cdot 2$



$\vec{w}^{(3)} \in \mathbb{C}^{64}; \quad \omega = \frac{2\pi}{N} 3 = \frac{2\pi}{64} 3$

⋮

$\vec{w}^{(16)} \in \mathbb{C}^{64}; \quad \omega = \frac{2\pi}{64} 16 = \frac{\pi}{2}$

⋮

$\vec{w}^{(32)} \in \mathbb{C}^{64}; \quad \omega = \frac{2\pi}{64} 32 = \pi$

⋮

$\vec{w}^{(62)} \in \mathbb{C}^{64}$ has same real part as $\vec{w}^{(2)}$ but the imaginary part is inverted

⋮

$Re(\vec{w}^{(63)}) = Re(\vec{w}^{(1)})$ but imaginary parts are inverted

- Proof of orthogonality

$$\langle \vec{w}^{(k)}, \vec{w}^{(h)} \rangle = \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}nk}\right)^* e^{j\frac{2\pi}{N}nh}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(h-k)n}$$

$$\left(\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}\right) \quad = \begin{cases} N & , h=k \\ \dfrac{1-e^{j2\pi(h-k)}}{1-e^{j\frac{2\pi}{N}(h-k)}} = 0 & , \text{otherwise} \end{cases}$$

$$h-k \in \mathbb{N} \Rightarrow e^{j2\pi(h-k)} = 1$$

- Remarks
  - N orthogonal vectors $\rightarrow$ basis for $\mathbb{C}^N$
  - vectors are not orthonormal. Normalization factor would be $1/\sqrt{N}$

3.2 The Discrete Fourier Transform (DFT)

3.2a DFT definition

- Basis expansion
  - Analysis formula: $X_k = \langle \vec{w}^{(k)}, \vec{x} \rangle$
  - Synthesis formula: $\vec{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \vec{w}^{(k)}$

- Change of basis in matrix form
  - Define $W_N = e^{-j\frac{2\pi}{N}}$ (or simply W when N is evident)

    Change of basis matrix $\underline{W}$ with $\underline{W}[n,m] = W_N^{nm}$

$$\underline{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & & & & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix}$$

  - Analysis formula: $\underline{X} = \underline{W}\,\vec{x}$

    Synthesis formula: $\vec{x} = \frac{1}{N}\underline{W}^H \underline{X}$

- Basis expansion (signal notation)

  Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, \ldots, N-1$$

  N-point signal in the frequency domain

  Synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}, \quad n = 0, 1, \ldots, N-1$$

  N-point signal in the "time" domain

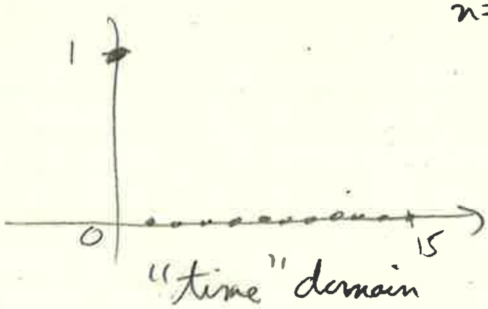
3.2b Examples of DFT calculation

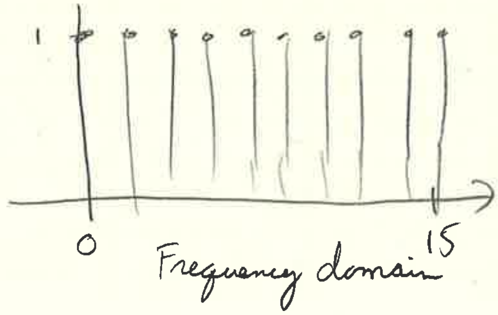
- DFT is obviously linear

$$\mathrm{DFT}\{\alpha x[n] + \beta y[n]\} = \alpha\,\mathrm{DFT}\{x[n]\} + \beta\,\mathrm{DFT}\{y[n]\}$$

- DFT of $x[n] = \delta[n]$, $x[n] \in \mathbb{C}^N$

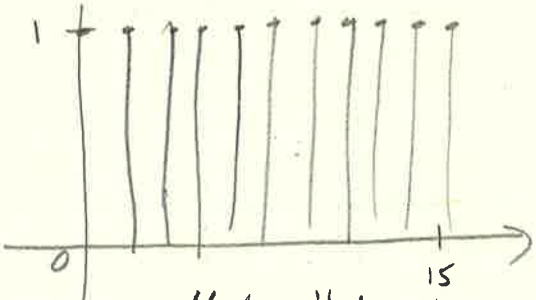
$$X[k] = \sum_{n=0}^{N-1} \delta[n] e^{-j\frac{2\pi}{N}nk} = 1$$



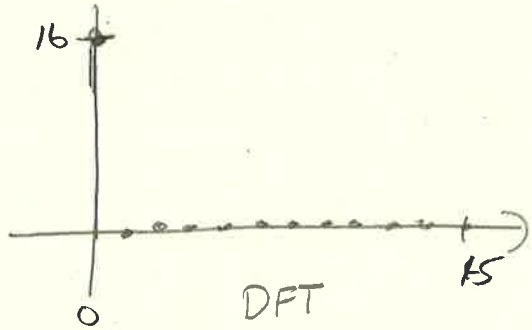


DFT of $x[n] = 1$, $x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = N\delta[k]$$



"time" domain



DFT

DFT of $x[n] = 3\cos(2\pi/16n)$, $x[n] \in \mathbb{C}^{64}$

$$x[n] = 3\cos\left(\frac{2\pi}{16n}\right) = 3\cos\left(\frac{2\pi}{64}4n\right) \qquad \omega = \frac{2\pi}{64}$$

$$= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right]$$

$$= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right] \qquad -j\frac{2\pi}{64}4n = j\frac{2\pi}{64}60n$$

$$= \frac{3}{2}\left[w_4[n] + w_{60}[n]\right]$$

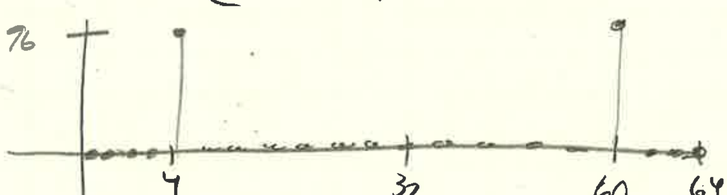
$$X[k] = \langle w_k[n], x[n]\rangle$$

$$= \langle w_k[n], \tfrac{3}{2}(w_4[n] + w_{60}[n])\rangle$$

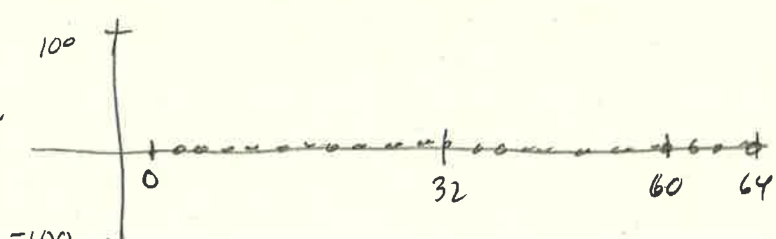
$$= \frac{3}{2}\langle w_k[n], w_4[n]\rangle + \frac{3}{2}\langle w_k[n], w_{60}[n]\rangle$$

$$= \begin{cases} \frac{3}{2}\cdot 64 = 96, & k = 4, 60 \\ 0, & \text{otherwise} \end{cases}$$

Re



Im



- DFT of $x[n] = 3\cos\left(\frac{2\pi}{16}n + \pi/3\right)$, $x[n] \in \mathbb{C}^{64}$

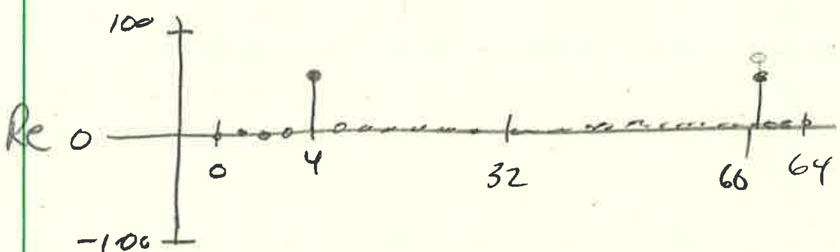
$$x[n] = 3\cos\left(\frac{2\pi}{16}n + \frac{\pi}{3}\right)$$

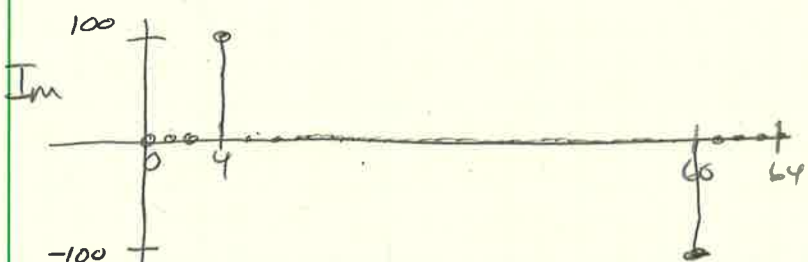
$$= 3\cos\left(\frac{2\pi}{64}4n + \frac{\pi}{3}\right)$$

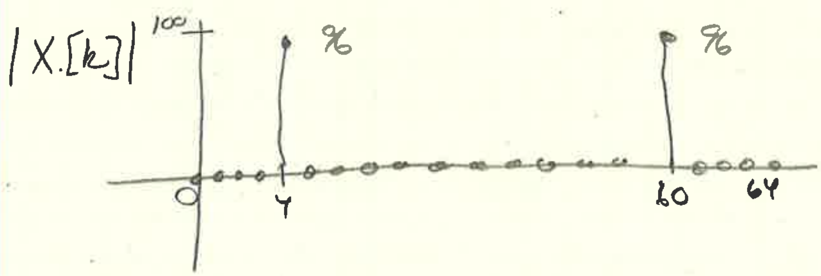
$$= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi}{64}4n} e^{-j\frac{\pi}{3}}\right]$$

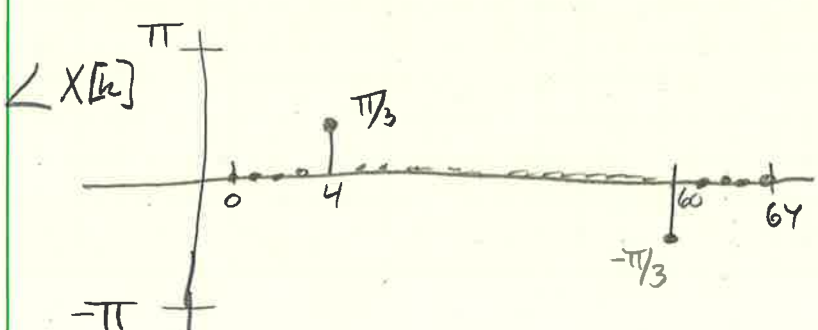
$$= \frac{3}{2}\left[e^{j\frac{\pi}{3}} w_4[n] + e^{-j\frac{\pi}{3}} w_{60}[n]\right]$$

$$X[k] = \langle w_k[n], x[n] \rangle = \begin{cases} 96e^{j\frac{\pi}{3}}, & k=4 \\ 96e^{-j\frac{\pi}{3}}, & k=6 \\ 0, & \text{otherwise} \end{cases}$$



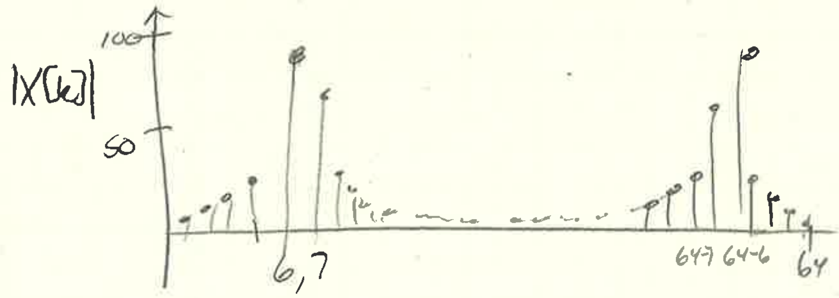


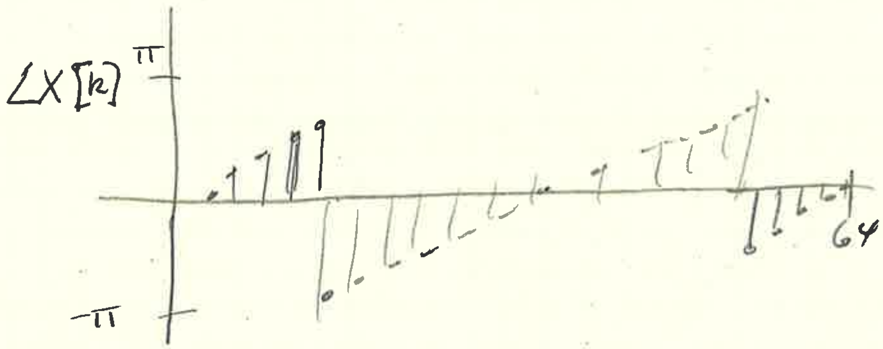




- DFT of $x[n] = 3\cos\left(\frac{2\pi}{10}n\right)$, $x[n] \in \mathbb{C}^{64}$

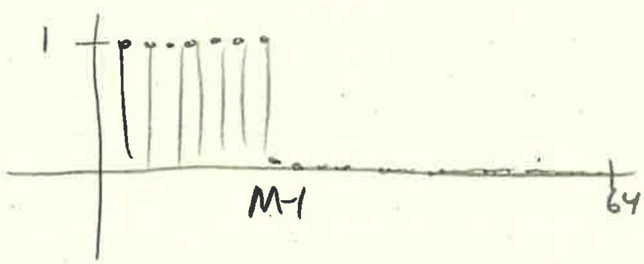
$$\frac{2\pi}{64}6 < \frac{2\pi}{10} < \frac{2\pi}{64}7$$





- DFT of length-M step in $\mathbb{C}^N$

$$x[n] = \sum_{h=0}^{M-1} \delta[n-h], \quad n = 0, 1, \ldots, N-1$$



$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{M-1} e^{-j\frac{2\pi}{N}nk}$$

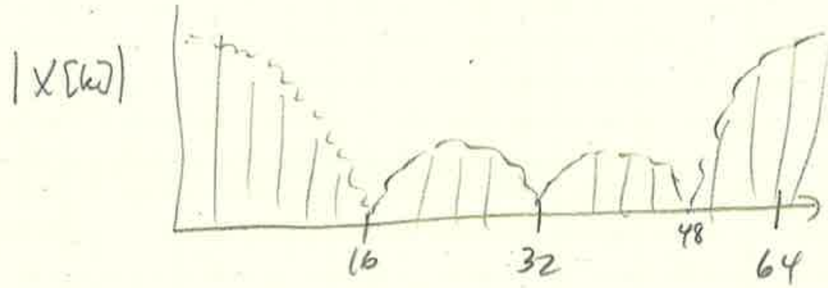
$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}} \qquad \left(1 - e^{-j\alpha} = e^{-j\frac{\alpha}{2}}\big(\underbrace{e^{j\frac{\alpha}{2}} - e^{-j\frac{\alpha}{2}}}_{2j\sin\frac{\alpha}{2}}\big)\right)$$

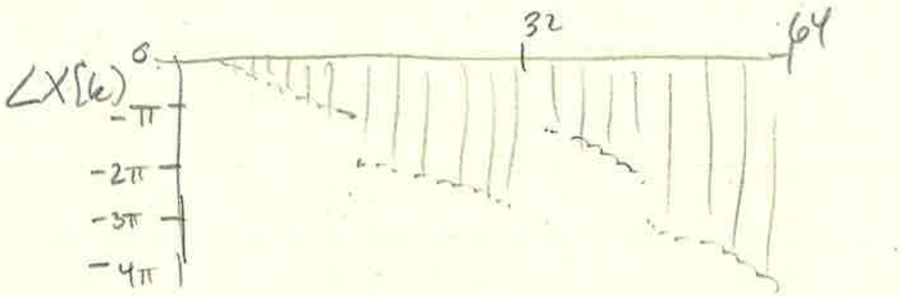
$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM} - e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right]}$$

$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

- $X[0] = M$, from the definition of the sum
- $X[k] = 0$, if Mk/N is an integer $(0 \le k < N)$
- $\angle X[k]$ is linear in k (except at sign changes for the real part)

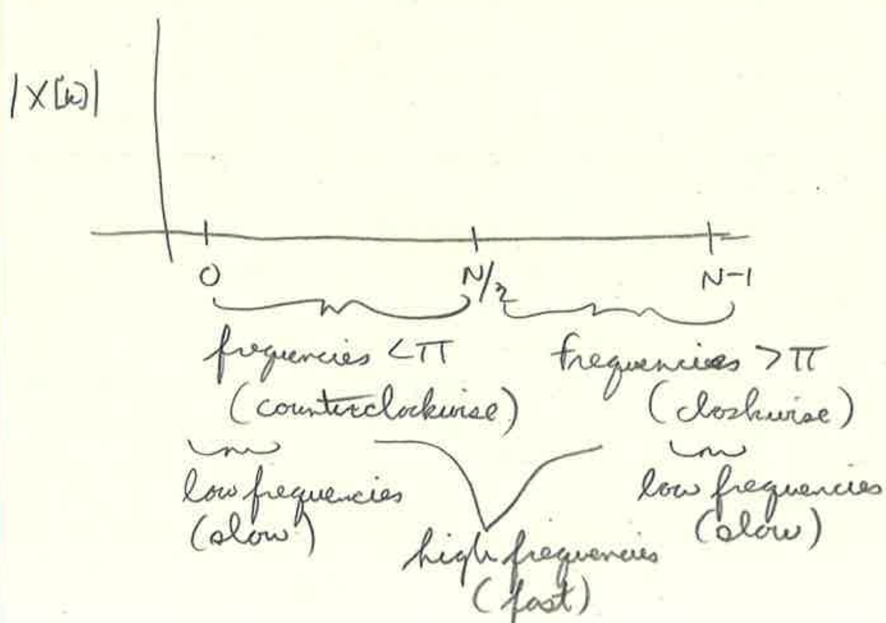
— DFT of length-4 step in $\mathbb{C}^{64}$



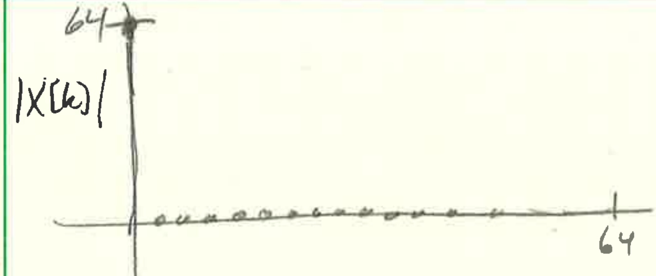


- Wrapping the phase
  - Often the phase is displayed "wrapped" over the $[-\pi, \pi)$ interval.
    - most numerical packages return wrapped phase
    - phase can be unwrapped by adding multiples of 2π.

3.2c Interpreting a DFT plot

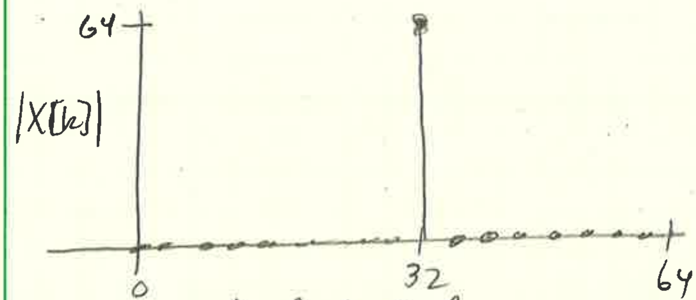


$X[n] = 1$ (slowest signal)



only lowest frequency

$x[n] = \cos \pi n = (-1)^n$ (fastest signal)



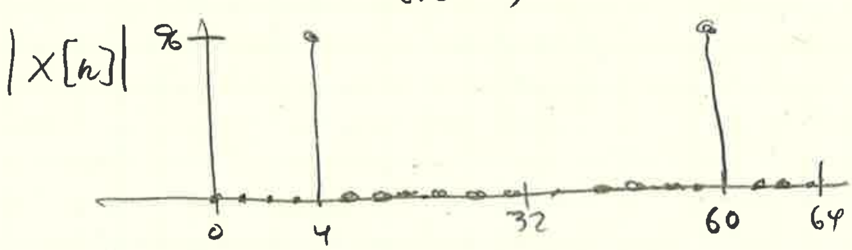
only highest frequency

- Energy distribution
  - Parseval : $\|\vec{x}\|^2 = \sum |\alpha_k|^2$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

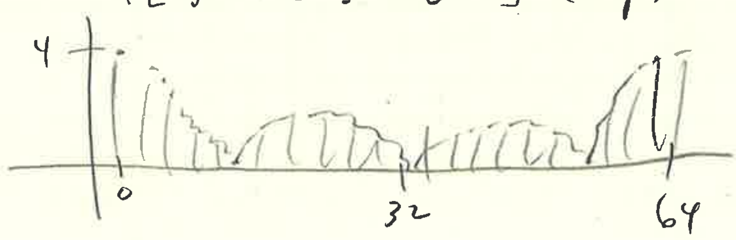
  - square magnitude of k-th DFT coefficient proportional to signal's energy at frequency $\omega = \frac{2\pi}{N} k$

$x[n] = 3\cos(\frac{2\pi}{16} n)$ (sinusoid)



energy concentrated on single frequency (counterclockwise and clockwise combine to give real signal)

$x[n] = u[n] - u[n-4]$ (step)

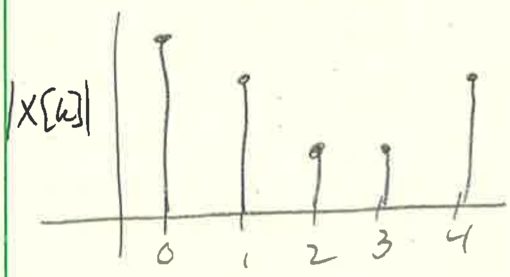


energy mostly in low frequencies

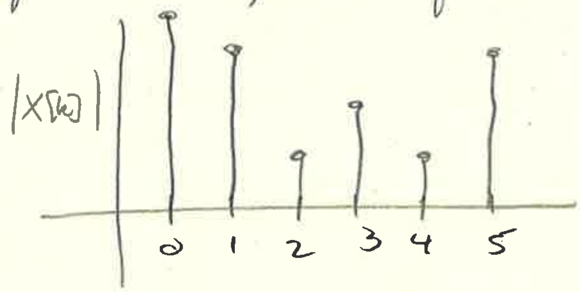
– DFT of real signals

- For real signals the DFT is "symmetric" in magnitude:

$$|X[k]| = |X[N-k]| \quad \text{for } k = 1, 2, \ldots, \lfloor N/2 \rfloor \text{ (floor)}$$

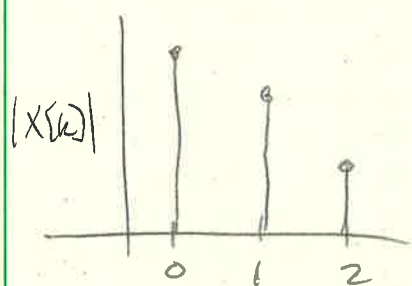


N=5, odd length

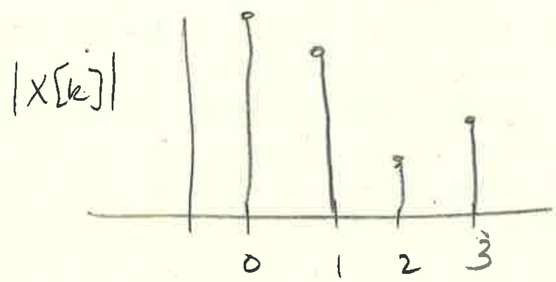


N=6, even length

- For real signals, magnitude plots need only $\lfloor N/2 \rfloor + 1$ points



N=5, odd length



N=6, even length

3.3: The DFT in practice

3.3a DFT analysis

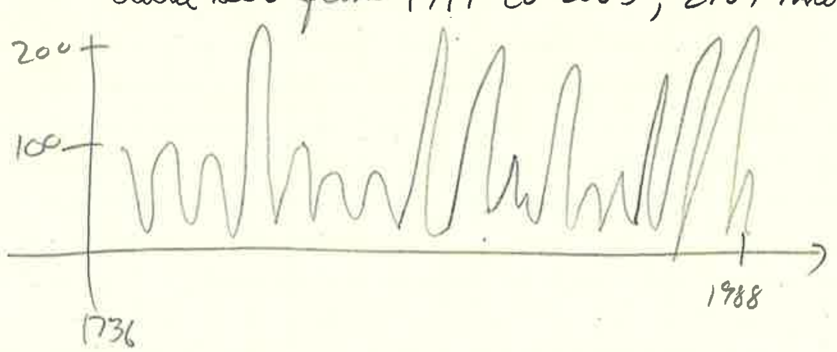
– Mystery signal revisited

$$x[n] = \cos(\omega n + \phi) + \eta[n] \quad \text{with}$$

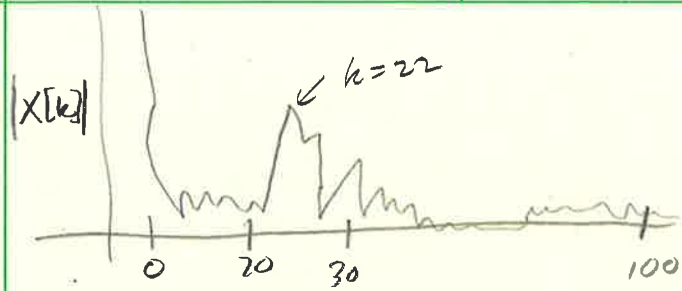
$$\phi = 0, \quad \omega = \frac{2\pi}{1024} 64$$ ↖ peak at $k = 64$

– Solar spots

- sunspot number: $S = 10 \times \#$ of clusters $+ \#$ of spots
- data set from 1749 to 2003, 2904 months

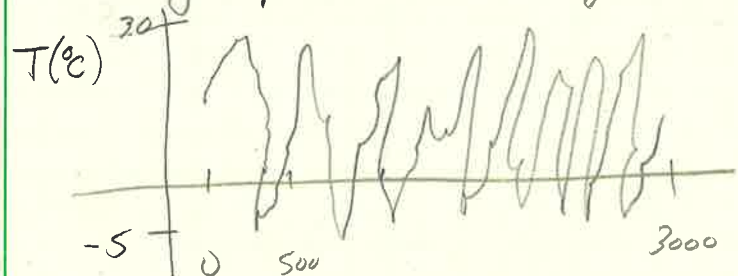


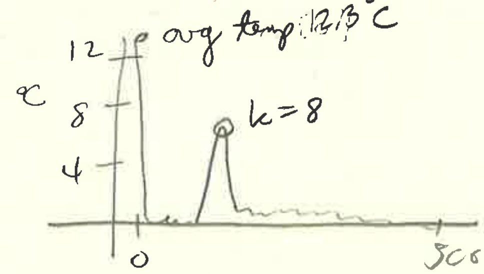
DFT of solar spots signal



- DFT main peak for k = 22
- 22 cycles over 2904 months
- period : $\frac{2904}{22} \approx 11$ years

– Daily temperature (2920 days)





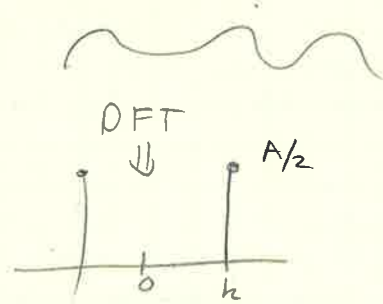
first few hundred DFT coefficients (in magnitude and normalized by the length of the temperature vector)

- average value (0-th DFT coefficient normalized) : 12.3°C

$$X[0] = \sum_{n=0}^{N-1} x(n)$$

- DFT main peak for k = 8, value 6.4°C
- 8 cycles over 2920 days
- period : $\frac{2920}{8} = 365$ days
- temperature excursion : 12.3°C ± 12.8°C

$A = \cos(\omega n)$



– Labeling the frequency axis
- If we know the "clock" of the system T_s
  - fastest (positive) frequency is $\omega = \pi$
  - sinusoid at $\omega = \pi$ needs two samples to do a full revolution
  - time between samples: $T_s = \frac{1}{F_s}$ seconds

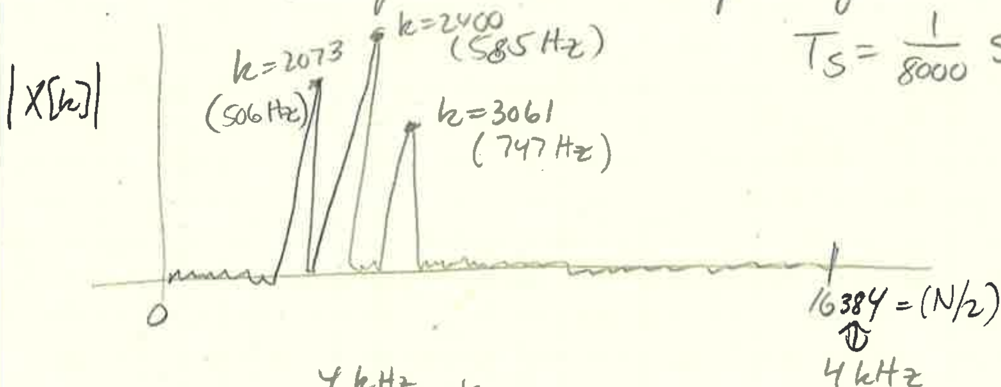
– real-world period for fastest sinusoid : $2T_s$ seconds

– real-world frequency for fastest sinusoid : $F_s/2$ Hz

- Example: train whistle

32768 samples (the "clock" of the system $F_S = 8000\,Hz$)

$T_S = \frac{1}{8000}\,s$



$F_S/2 \Updownarrow k = N/2$

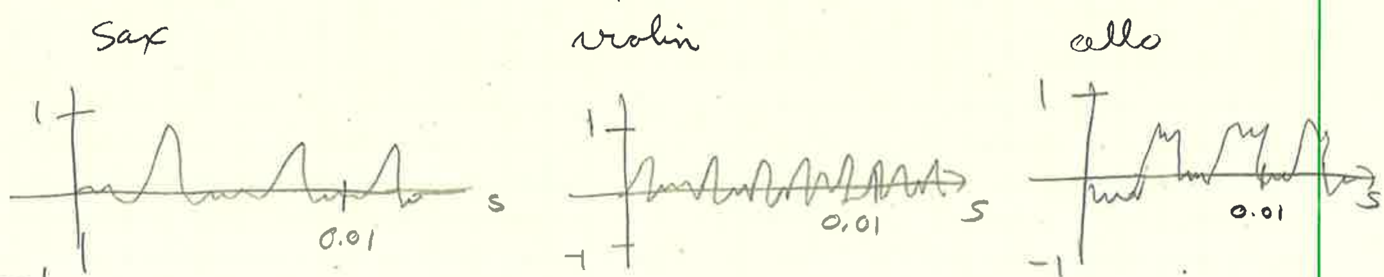
$$\text{Frequency} = \frac{4\,kHz}{16384} \cdot k$$

If we look up the frequencies:

B minor chord

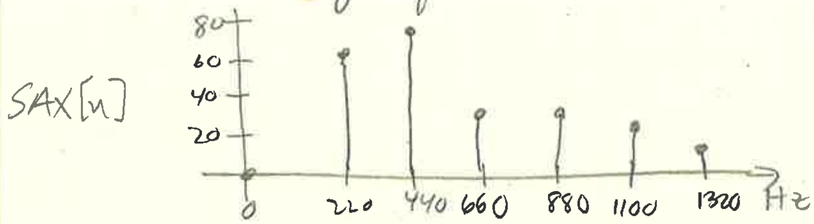
3.36 DFT example - analysis of musical instruments

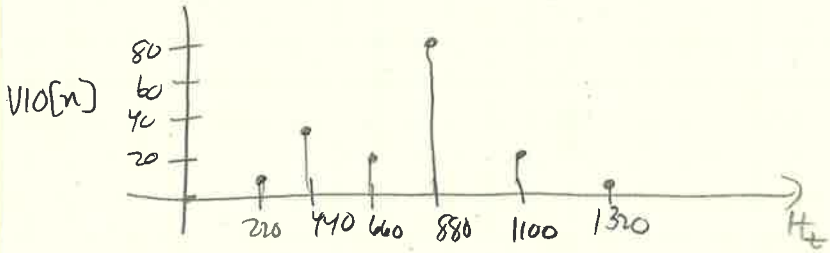
- Analysis of musical instruments
  - "We all know about pitch...
    - It is really about frequency, or cycles per seconds
    - How about harmonics?
      - That is what gives the timbre of an instrument!
- A difficult temporal analysis of musical instruments

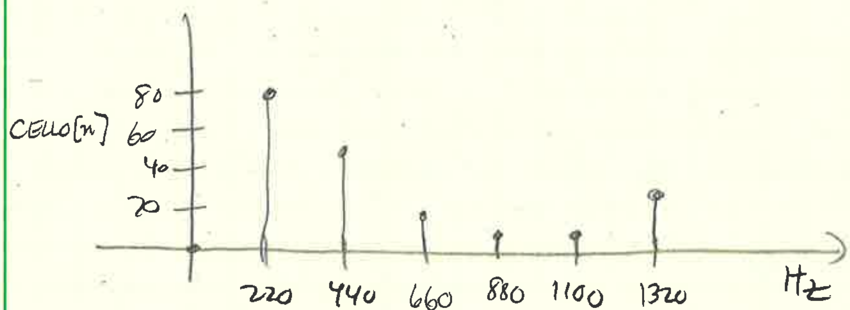


- What is the note?
- Could you guess the instrument from the temporal plot?
- In the time domain it is hard to process information of the sound

- A Fourier analysis of musical instruments





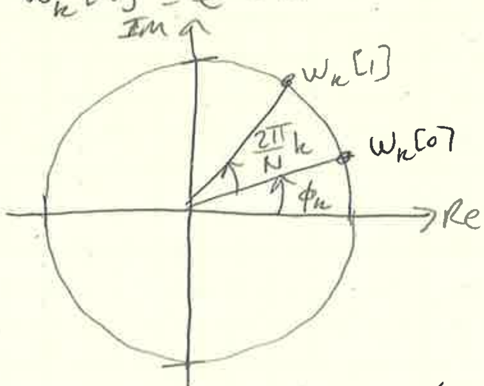


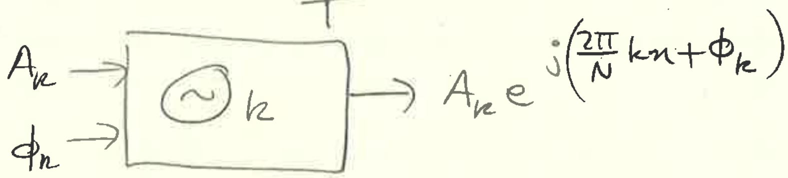
- The played note is the frequency of the first peak: 220 Hz in this case
- Other peaks are called harmonics: they define the typical sound of the instrument
- Without Fourier, we would have been lost

3.3c DFT synthesis

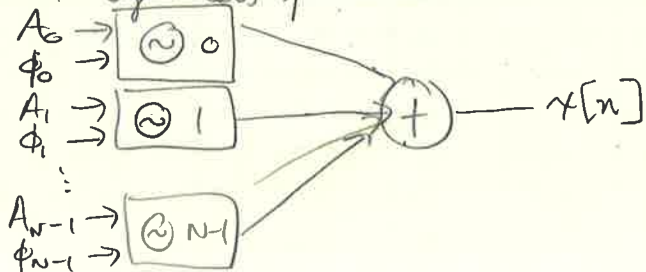
- Synthesis: the sinusoidal generator

$$w_k[n] = e^{j\left(\frac{2\pi}{N}kn + \phi_k\right)}$$





- DFT Synthesis formula

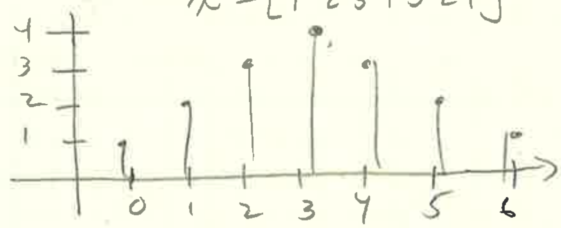


- Initializing the machine

$$A_k = |X[k]|/N$$

$$\phi_k = \angle X[k]$$

$\vec{x} = [1\ 2\ 3\ 4\ 3\ 2\ 1]^T$

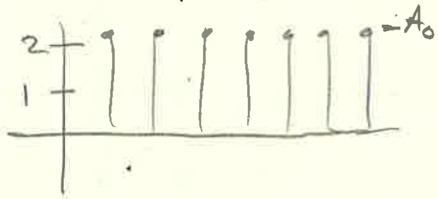


k	A_k	ϕ_k
0	2.2857	0
1	0.7213	-2.6928
2	0.0440	0.8976
3	0.0919	-1.7952
4	0.0919	-1.7952
5	0.0440	-0.8976
6	0.7213	2.6928

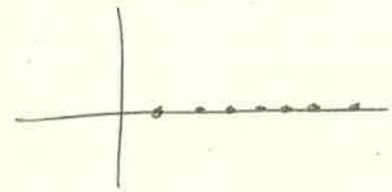
$\underline{k=0}$

$A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$

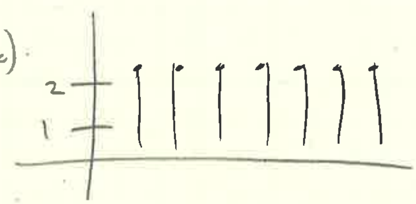
Re

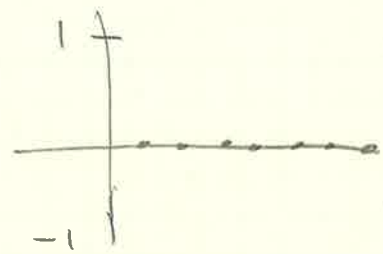


Im



$\sum A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$

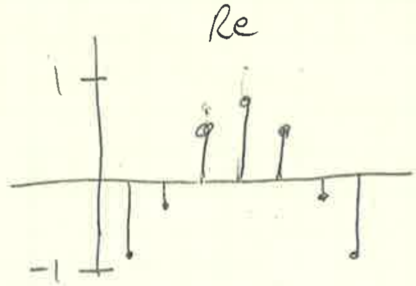




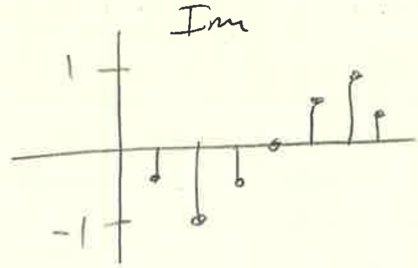
$k=1$

$A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$

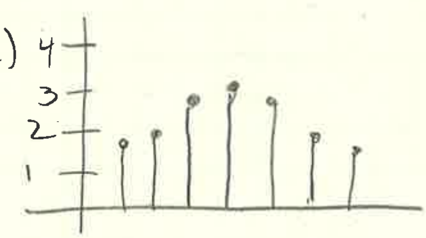
Re

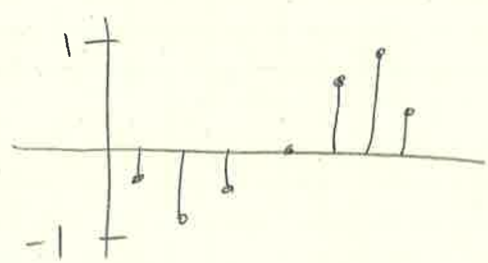


Im



$\sum A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$





at $k=6$, we have reconstructed the signal

- Running the machine too long...

$$x[n+N] = x[n]$$ - Output signal is N-periodic!

- Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}, \quad \underline{n \in \mathbb{Z}}$$, produces an

N-periodic signal in the time domain

the analysis formula:

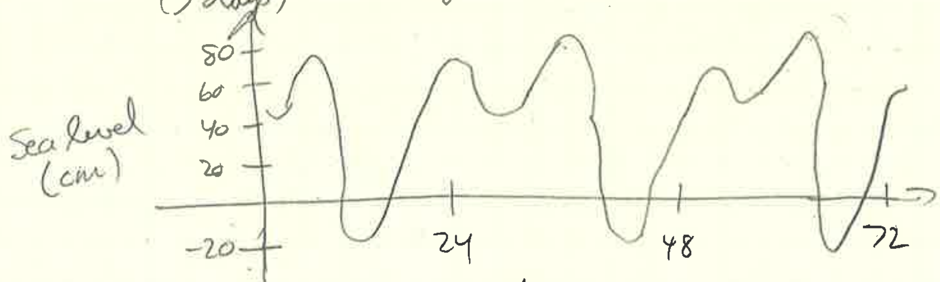
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, \quad k \in \mathbb{Z}$$, produces N-periodic

signal in the frequency domain.

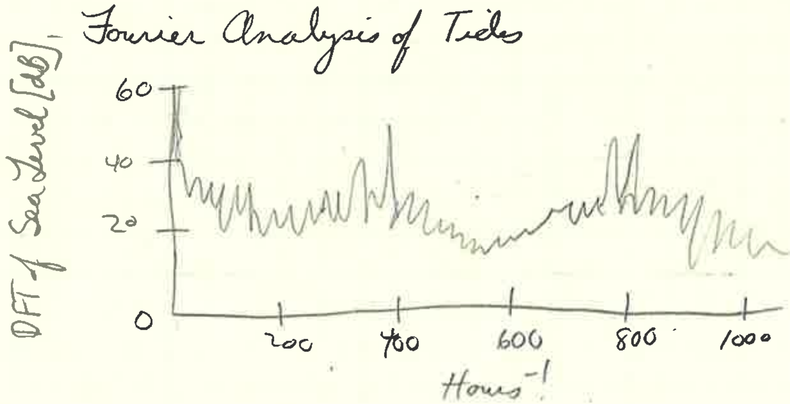
3.3d DFT example tide prediction in Venice

- Tides are due mostly to periodic astronomical phenomena
- Can we predict tides using Fourier? The first step is to approximate them
- We consider hourly measurements taken in Canal Grande during 2011

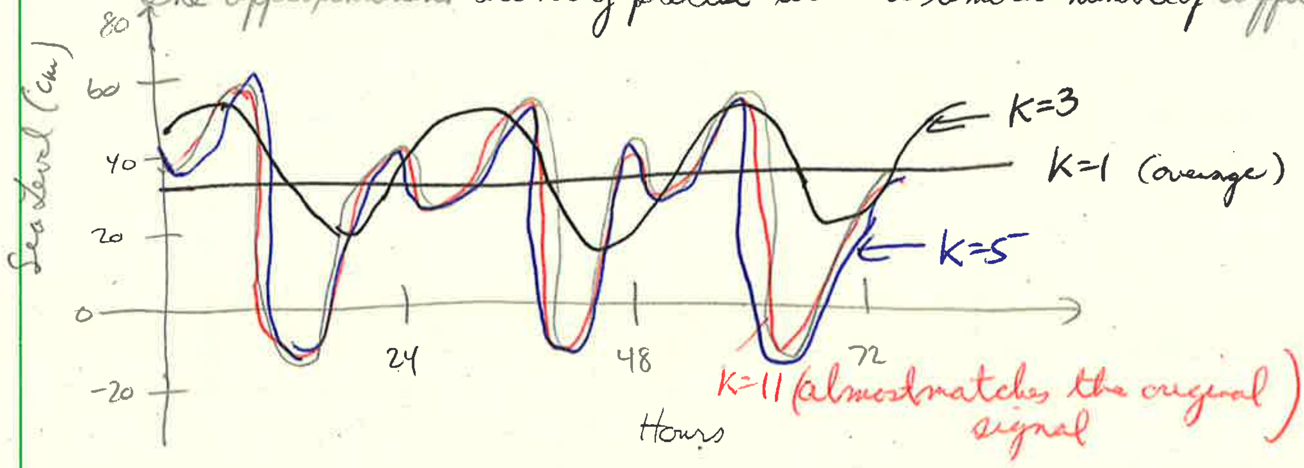
(3 days)



- Fourier Analysis of Tides



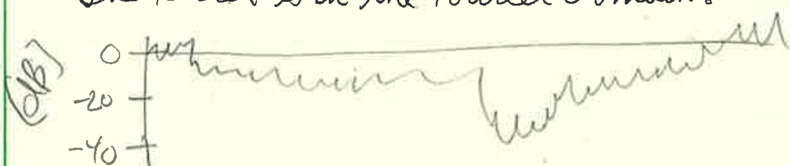
- Let us approximate tides using few Fourier coefficients
  - We consider on K = 1, 3, 5, 11 Fourier coefficients
  - Darker gray represents approximation with more coefficients
  - The approximations are very precise with a limited number of coefficients

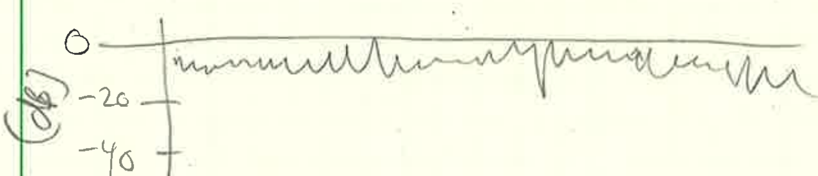


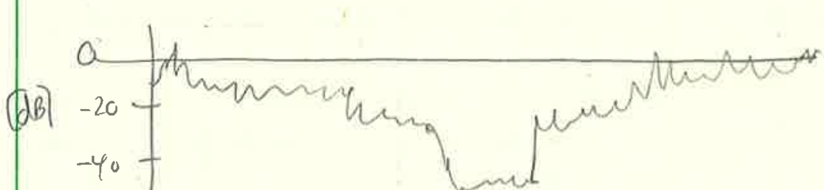
3.3e DFT example - MP3 compression

- MP3 Compression trick
  - How can an MP3 song sound so good while being so compressed?
  - Compression introduces noise
  - The trick is to shape the noise!

The secret is in the Fourier domain!

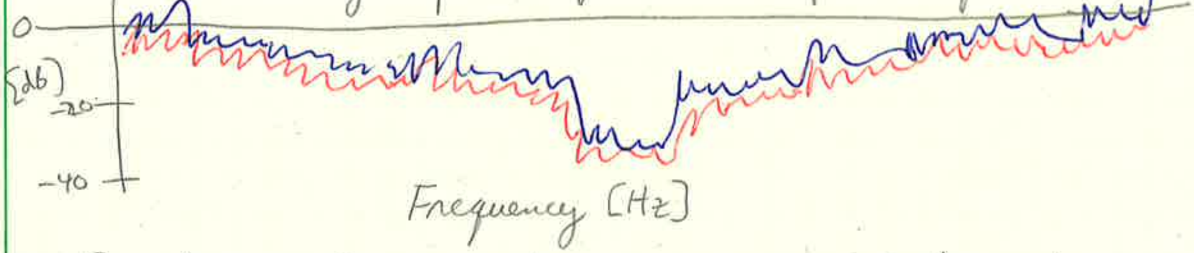




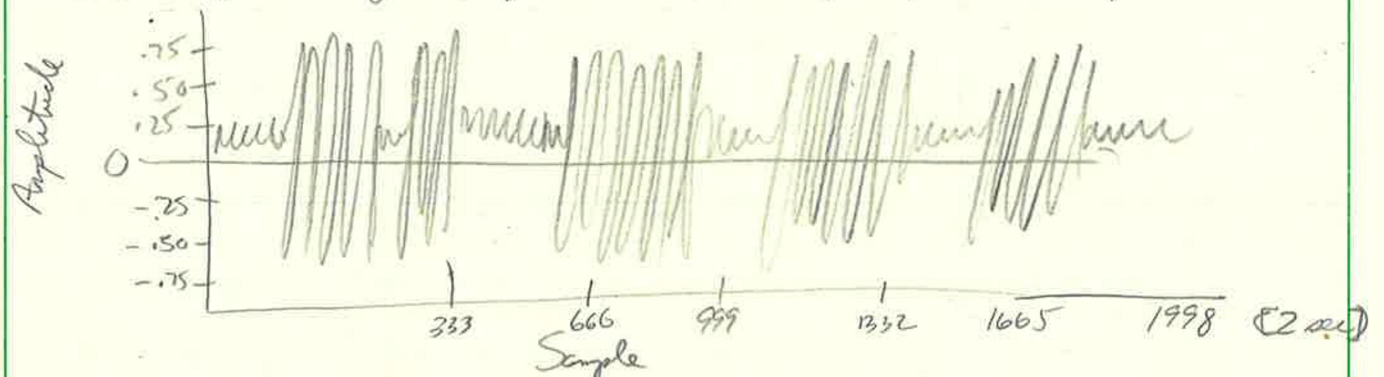


- Conclusions
  - MP3: complex compression algorithm that introduces errors
  - Errors shaped as the song in the Fourier domain → higher perceived quality
  - MP3 minimizes the perceived quality decay by shaping the compression errors

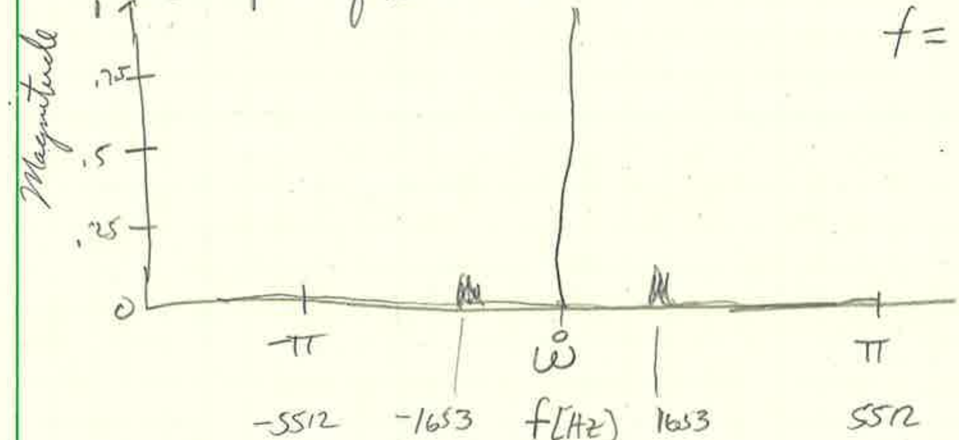
Fourier analysis of the song [blue] and of the shaped noise [red]



Signal of the Day: The first man-made signal from outer space (Sputnik)

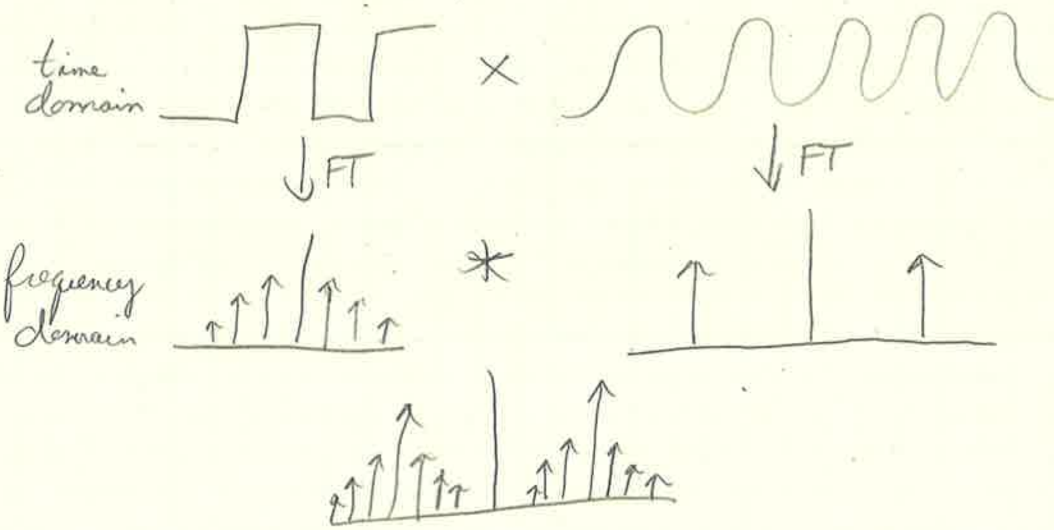


Another point of view: DFT



$$f = \frac{\omega f_s}{2\pi}$$ ← frequency at which we measure the signal

- Let's understand this...



Summary of Lesson 3.3

The DFT can be used as an analysis tool to understand the frequency components that a signal contains. If a signal has an associated system clock T_S (or a frequency $F_S = 1/T_S$), we can map the index k of the DFT coefficients to real frequencies. The largest digital frequency $N/2$ is associated with the largest continuous-time frequency $F_S/2$. Thus, the continuous frequency corresponding to index k is given by $\frac{kF_S}{N}$ and is measured in Hz.

The DFT synthesis can be seen as a series of up to N coupled sinusoidal generators:

- sinusoidal generator k has frequency $\frac{2\pi k}{N}$
- the amplitude of sinusoidal generator k is given by the magnitude of the DFT coefficient $|X[k]|$
- the phase of sinusoidal generator k is given by the phase of the DFT coefficient $\angle X[k]$

If we let the DFT synthesis run beyond $N-1$, we obtain an N-periodic signal, $x[n+N] = x[n]$. Likewise, the analysis formula produces also an N-periodic series of Fourier coefficients. This side comment will be very important when we study another form of Fourier transform for periodic sequences, namely discrete Fourier Series (DFS).

3.4 The Short-Time Fourier Transform (STFT)

3.4a The STFT

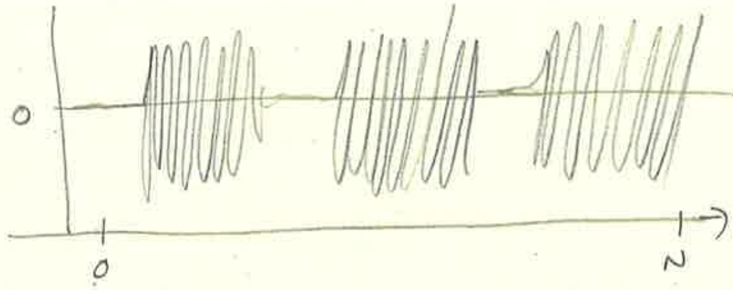
– Dual-Tone Multi Frequency dialing (DTMF)

	1209 Hz	1336 Hz	1477 Hz
697 Hz	1	2	3
770 Hz	4	5	6
852 Hz	7	8	9
941 Hz	*	0	#

Analog telephone

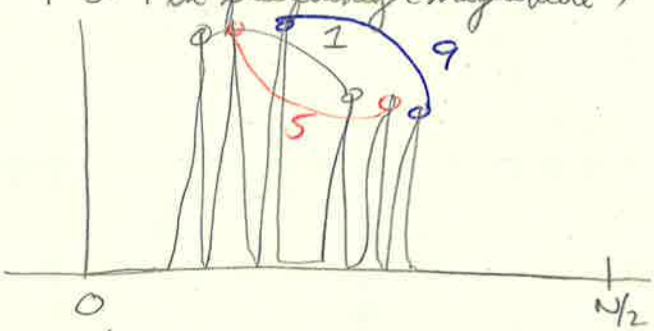
1) Frequencies are co-prime
2) No sum or difference of frequencies is in the set

1-5-9 in time



Can't tell the digit in time domain

1-5-9 in frequency (magnitude)



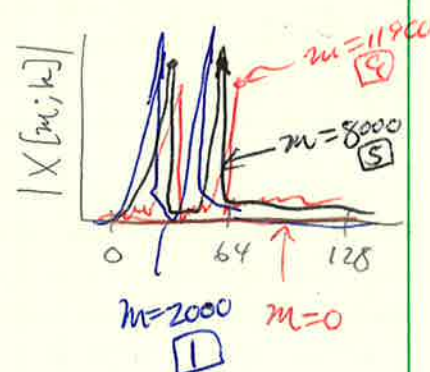
- The fundamental tradeoff
  - time representation obfuscates frequency
  - frequency representation obfuscates time

- Short-term Fourier Transform

  Idea:
  - take small signal pieces of length L
  - look at the DFT of each piece:

$$X[\underset{\substack{\uparrow\\ \text{starting point}}}{m};k] = \sum_{n=0}^{L-1} x[m+n]\, e^{-j\frac{2\pi}{L}nk}$$

STFT (L=256)

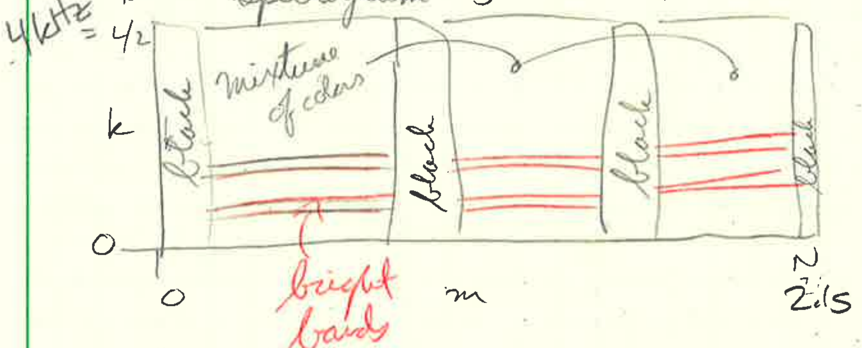


3.4b The spectogram

Idea:

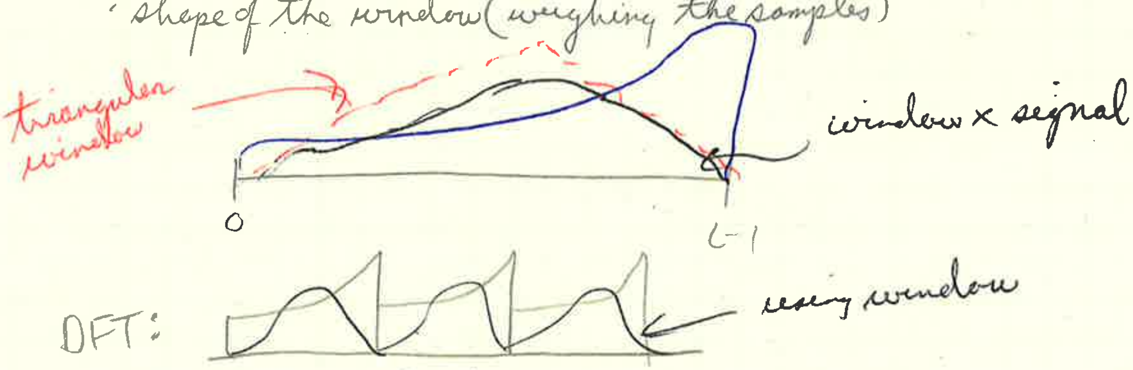
- Color-code the magnitude: dark is small, white is large
- use $10 \log_{10}(|X[m;k]|)$ to see better (power in dB)
- plot spectral slices one after another

DTMF spectogram ($F_S = 8000$ Hz)



- Labeling the Spectogram
  - If we know the "system clock" $F_S = 1/T_S$, we can label the axis
    - highest positive frequency: $F_S/2$ Hz
    - frequency resolution: F_S/L Hz
    - width of time slices: LT_S seconds

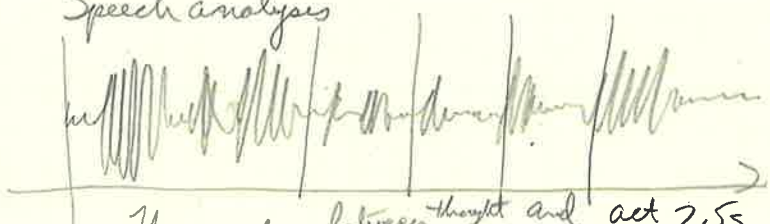
- The Spectogram
  Questions:
  - width of the analysis window?
  - position of the windows (overlapping?)
  - shape of the window (weighing the samples)



- Wideband vs. Narrowband
  - Long window: narrowband spectogram
    - long window ⇒ more DFT points ⇒ more frequency resolution $\frac{F_S}{L}$
    - long window ⇒ more "things can happen" ⇒ less precision in time
  - Short window: wideband spectogram
    - short window ⇒ many time slices ⇒ precise location of transitions
    - short window ⇒ fewer DFT points ⇒ poor frequency resolution

3.4c Time-frequency tiling

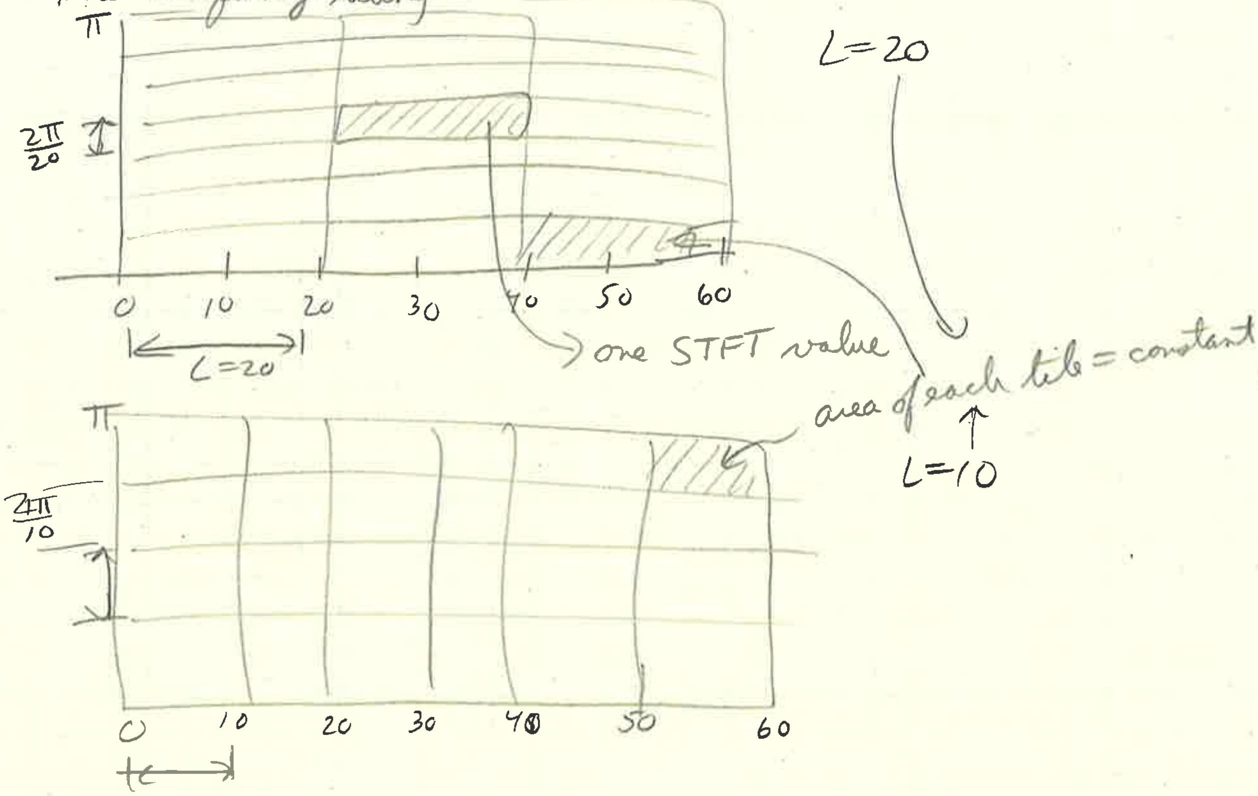
Speech analysis



8 ½ kHz

There is a lag between thought and act 2.5s

- Time-Frequency tiling



- Food for thought
  - time "resolution" $\Delta t = L$
  - frequency "resolution" $\Delta f = 2\pi/L$
  - $\Delta t \Delta f = 2\pi$

    uncertainty principle!

- Even more food for thought
  - more sophisticated tilings of time-frequency planes can be obtained with the <u>wavelet</u> transform.

3.5 Discrete Fourier Series

3.5a Discrete Fourier Series

DFS = DFT with periodicity explicit

- The DFS maps an N-periodic signal onto an N-periodic sequence of Fourier coefficients
- The inverse DFS maps an N-periodic sequence of Fourier coefficients onto an N-periodic signal
- The DFS of an N-periodic signal is mathematically equivalent to the DFT of one period

– Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

• For an N-periodic sequence $\tilde{x}[n]$

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\mathrm{DFS}\{\tilde{x}[n-M]\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$, $\tilde{X}[k] = \mathrm{DFS}\{\tilde{x}[n]\}$

– $\mathrm{IDFS}\{\underbrace{e^{-j\frac{2\pi}{N}Mk}}_{\text{delay factor}}\tilde{X}[k]\} = \tilde{x}[n-M]$

• For an N-point signal $x[n]$:

- $x[n-M]$ is not well-defined
- build $\tilde{x}[n] = x[n \bmod N] \Rightarrow \tilde{X}[k] = X[k]$
- $\mathrm{IDFT}\{e^{-j\frac{2\pi}{N}Mk}X[k]\} = \mathrm{IDFS}\{e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]\}$
  $= \tilde{x}[n-M] = x[(n-M) \bmod N]$

– Shifts for finite-length signals are "naturally" circular

3.5b Karplus-Strong revisited and DFS

– Periodic sequences: a bridge to infinite-length signals

- N-periodic sequence: N degrees of freedom
- DFS: only N Fourier coefficients capture all of the information

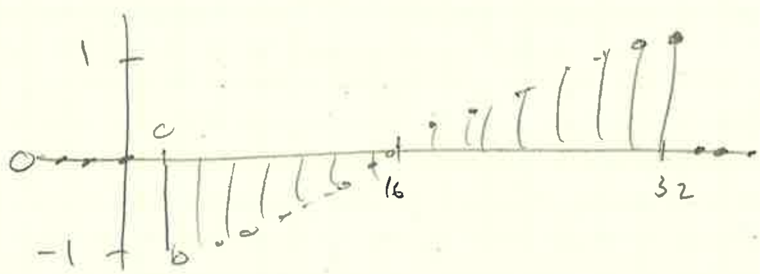
- Karplus-Strong revisited

$$y[n] = \alpha y[n-M] + x[n]$$

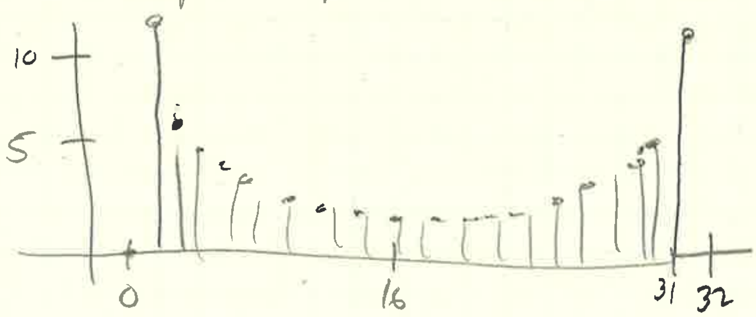
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \le n < M$
- $\alpha = 1$ (for now)

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{st}\text{ period}}, \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{2^{nd}\text{ period}}, \underbrace{\bar{x}[0], \bar{x}[1], \ldots}_{\ldots}$$

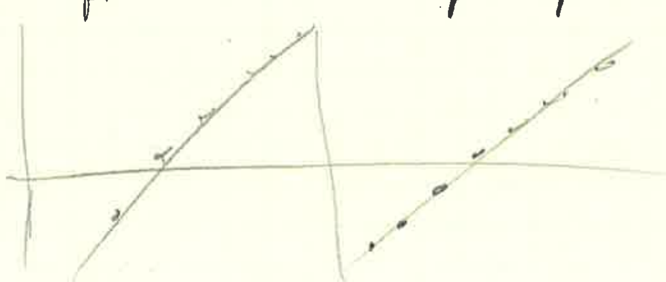
- Example: 32-tap sawtooth wave



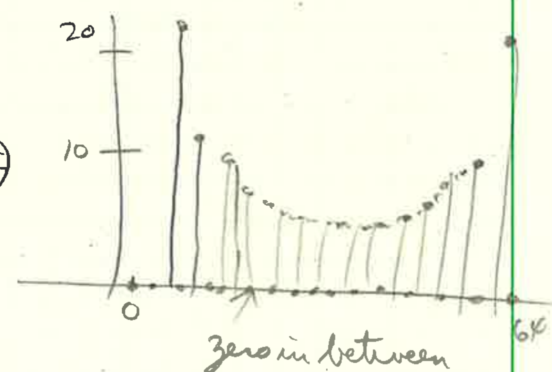
DFT of 32-tap sawtooth wave



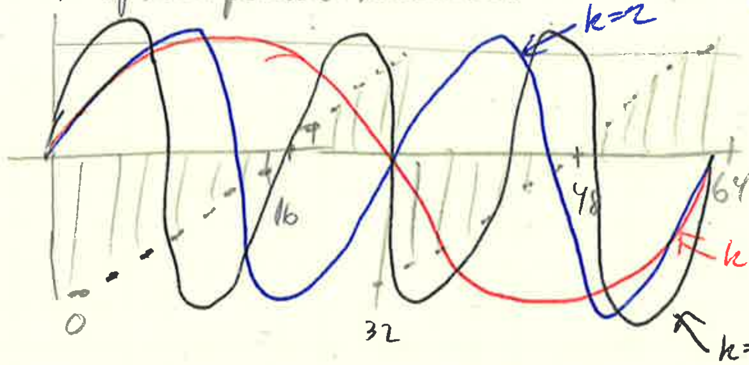
- What if we take the DFT of two periods?



$\Rightarrow$



- DFT of two periods: intuition



$k=2$ inner product gives ~~one~~ DFT coeff twice the original

$k=0$ inner product sums to average

$k=1$ inner product sums to 0

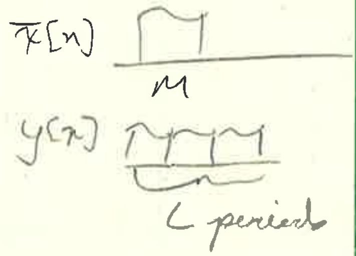
$k=3$ antisymmetry $\Rightarrow$ inner product sums to 0 (same for all odd k)

- DFT of L periods

$$X_L[k] = \sum_{n=0}^{LM-1} y[n]\, e^{-j\frac{2\pi}{LM}nk}, \quad k = 0,1,2,\ldots, LM-1$$

$(k = n + pM)$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} y[n+pM]\, e^{-j\frac{2\pi}{LM}(n+pM)k}$$



$$= \sum_{p=0}^{L-1} \sum_{n=0}^{M-1} y[n]\, e^{-j\frac{2\pi}{LM}nk}\, e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{M-1} \bar{x}[n]\, e^{-j\frac{2\pi}{LM}nk}$$

- We've seen this before

$$\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} = \begin{cases} L, & \text{if } k \text{ is a multiple of } L \\ 0, & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

- DFT of L periods

$$X_L[k] = \begin{cases} L\bar{X}[k/L], & \text{if } k = 0, L, 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- DFT and DFS
  - again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period.
  - To stress the periodicity of the underlying signal, we use the term DFS

Summary of Lesson 3.5

The discrete Fourier series is just a different flavor of the DFT applied to periodic series. N-periodic sequences in the time domain are mapped onto N-periodic sequences in the frequency domain. Furthermore, the definition of the DFS retrospectively better justifies the use of circular shifts as the natural extension of shifts for finite-length sequences.

Later in the lesson, we have revisited the Karplus-Strong algorithm to illustrate a key point about the DFS. If we take the DFT of L repetitions of a finite length sequence of length N, we obtain a series which is non-zero only at multiple integers of L. Moreover, these non zero coefficients are just scaled versions of the DFT coefficients of the original finite-length sequence. Therefore, all the spectral information of a N-periodic sequence is entirely captured by the DFT coefficients of one period.

3.6 The Discrete-Time Fourier Transform

- The situation so far

  - Fourier representation for signal classes:
    - N-point finite-length : DFT
    - N-point periodic : DFS
    - infinite length: ?

- Karplus-Strong revisited, part 2

  - consider now $\alpha < 1$
  - generated signal is infinite-length but not periodic:

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{st}\text{ period}}, \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1]}_{2^{nd}\text{ period}}, \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1]}_{\ldots} \ldots$$

  - What is a good spectral representation?

- DFT of increasingly long signals

  - Start with the DFT. What happens when $N \to \infty$?
  - $(2\pi/N)k$ becomes denser in $[0, 2\pi]$ ... $\quad \mathbb{C}^N, \omega = \frac{2\pi}{N}$
  - In the limit $(\frac{2\pi}{N})k \to \omega$: $\displaystyle\sum_n x[n]e^{-j\omega n}, \omega \in \mathbb{R}$

- Discrete-Time Fourier Transform (DTFT)

  - Formal definition:
    - $x[n] \in \ell_2(\mathbb{Z})$

- define a function of $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- inversion (when $F(\omega)$ exists):

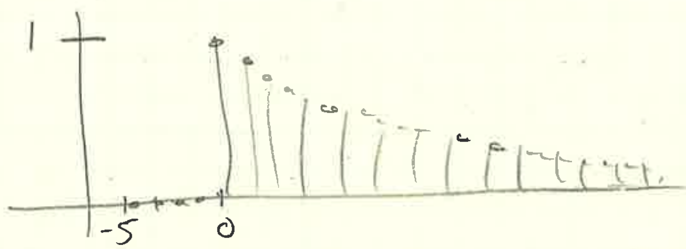
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \quad n \in \mathbb{Z}$$

- DTFT periodicity and notation
  - $F(\omega)$ is 2π-periodic
  - to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  - by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

- $x[n] = \alpha^n u[n]$, $|\alpha| < 1$



- DTFT of $x[n] = \alpha^n u[n]$, $|\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

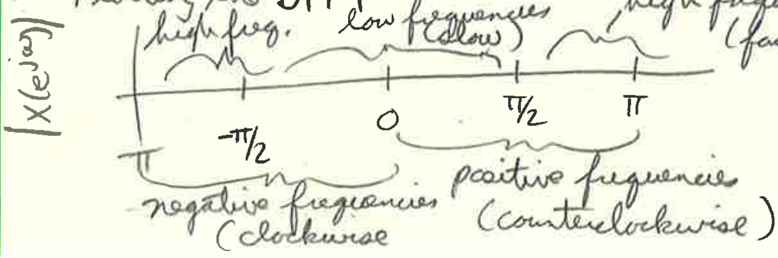
$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

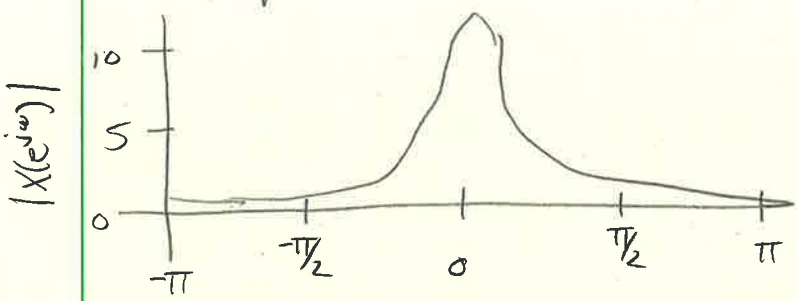
$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

$$|X(e^{j\omega})|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos\omega}$$

- Plotting the DTFT

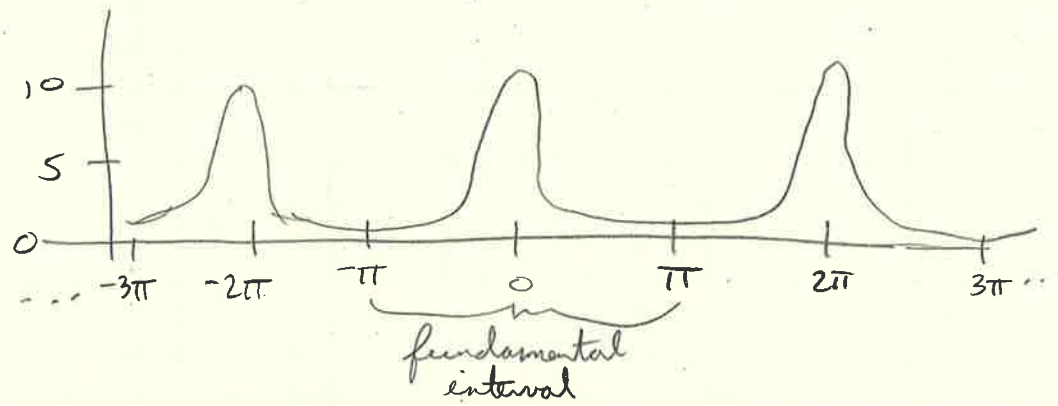


- DTFT of $x[n] = \alpha^n u[n]$, $|\alpha| < 1$



$\alpha = 0.9$

- Remember the periodicity!

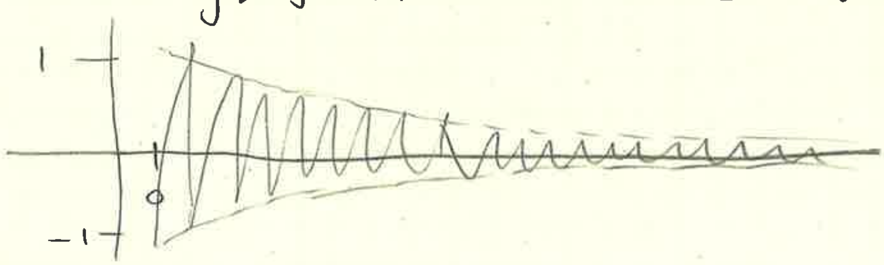


- KS revisited, part 2: 32-tap sawtooth wave

$$x[n] = \frac{2n}{M-1} - 1, \quad n = 0, 1, \ldots, M-1 \qquad (M=32)$$

- KS revisited, part 2: decay $\alpha = 0.9$

$$y[n] = \alpha^{\lfloor n/M \rfloor}\, \bar{x}[n \bmod M]\, u[n]$$



- DTFT of KS signal

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

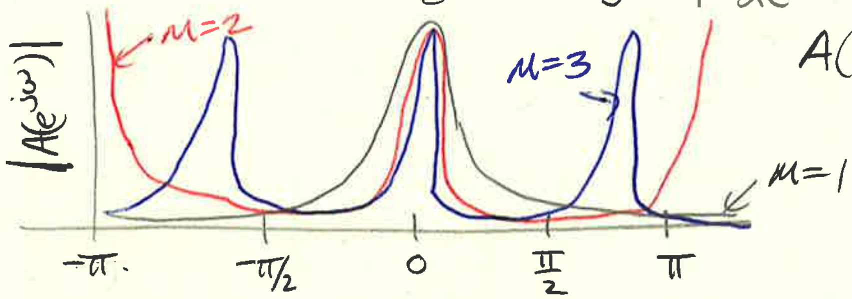
$$= \sum_{p=0}^{\infty} \sum_{n=0}^{M-1} \alpha^p \bar{x}[n] e^{-j\omega(pM+n)}$$

$$= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{n=0}^{M-1} \bar{x}[n] e^{-j\omega n}$$

$$= A(e^{j\omega M})\, \bar{X}(e^{j\omega})$$

- We know the first term

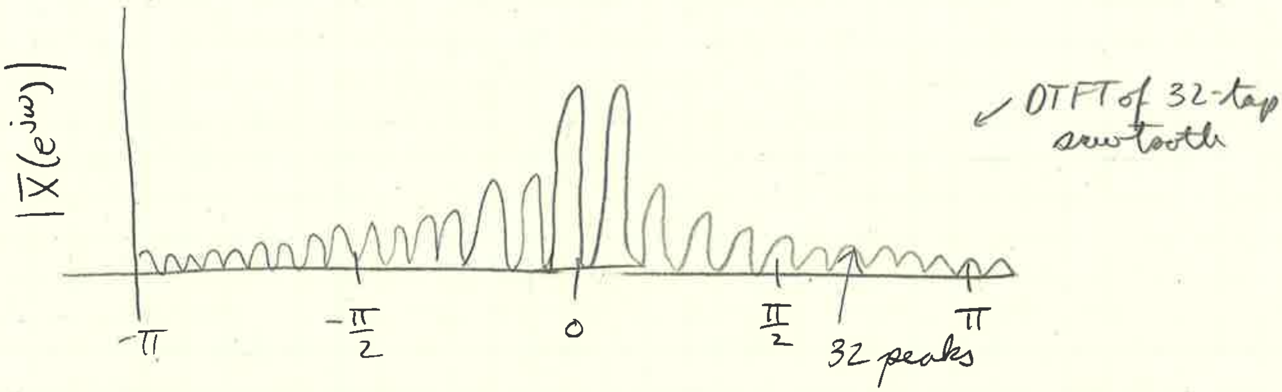
$$A(e^{j\omega}) = \text{DTFT}\{\alpha^n u[n]\} = \frac{1}{1-\alpha e^{-j\omega}} \quad (M=1)$$



$A(e^{j\omega M})$ rescales the frequency axis: periodicity!

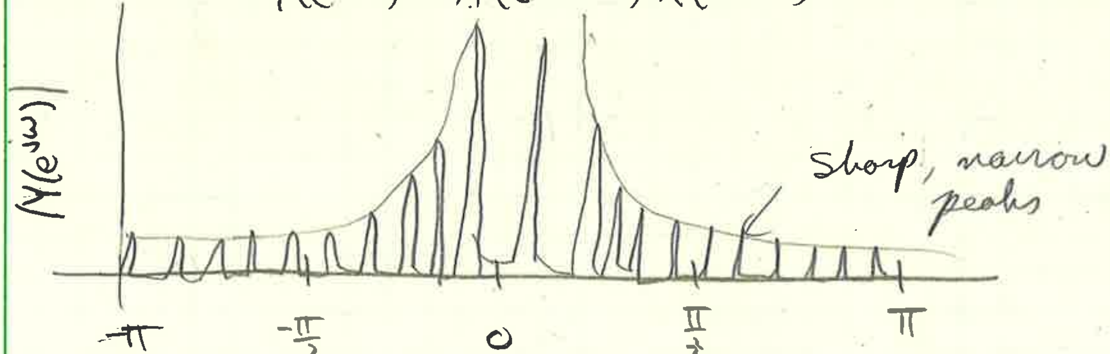
- Second term

$$\bar{X}(e^{j\omega}) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$



- DTFT of KS with decay

$$Y(e^{j\omega}) = A(e^{j\omega M})\,\bar{X}(e^{j\omega})$$



3.6b Existence and properties of the DTFT

- Existence easy for absolutely summable sequences

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\le \sum_{n=-\infty}^{\infty} |x[n]e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Inversion easy for absolutely summable sequences

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}\Big) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega}_{=0 \text{ unless } n=k}$$

$$= x[n]$$

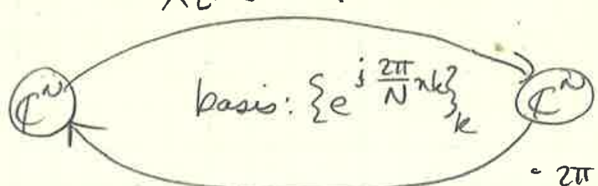
- A formal change of bases
  - Formally DTFT is an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

  - "basis" is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega\in\mathbb{R}}$
  - something "breaks down": we start with sequences but the transform is a function
  - we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

- Review: DFT

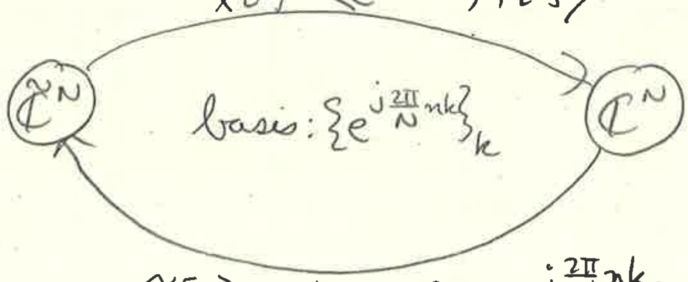
$$X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n] \rangle$$



$$x[n] = \frac{1}{N}\sum X[k] e^{j\frac{2\pi}{N}nk}$$

- Review: DFS

$$\tilde{X}[k] = \langle e^{j\frac{2\pi}{N}nk}, \tilde{x}[n] \rangle$$



$$\tilde{x}[n] = \frac{1}{N}\sum \tilde{X}[k] e^{j\frac{2\pi}{N}nk}$$

- DTFT

$$\begin{tikzcd} \ell_2(\mathbb{Z}) \arrow[rr, bend left, "X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle"] & \text{"basis"}: \{e^{j\omega n}\}_\omega & L_2([-\pi,\pi]) \arrow[ll, bend left, "x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega"] \end{tikzcd}$$

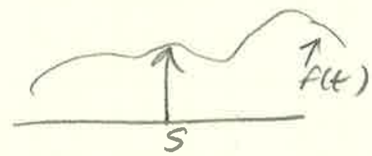
- DTFT properties
  - linearity : $DTFT\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
  - time shift : $DTFT\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$
  - modulation (dual) : $DTFT\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$
  - time reversal : $DTFT\{x[-n]\} = X(e^{-j\omega})$
  - conjugation : $DTFT\{x^*[n]\} = X^*(e^{-j\omega})$
- Some particular cases:
  - if $x[n]$ is symmetric, the DTFT is symmetric:
    $$x[n] = x[-n] \Leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$$
  - if $x[n]$ is real, the DTFT is Hermitian-symmetric:
    $$x[n] = x^*[n] \Leftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$
  - special case: if $x[n]$ is real, the magnitude of the DTFT is symmetric:
    $$x[n] \in \mathbb{R} \Rightarrow |X(e^{j\omega})| = |X(e^{-j\omega})|$$
  - more special case: if $x[n]$ is real and symmetric, $X(e^{j\omega})$ is also real and symmetric.

3.6c The DTFT as a change of basis

- DTFT as basis expansion
  - Some things are OK:
    - $DFT\{\delta[n]\} = 1$
    - $DTFT\{\delta[n]\} = \langle e^{j\omega n}, \delta[n] \rangle = 1$

- Some things aren't:
  - $\mathrm{DFT}\{1\} = N\delta[k]$
  - $\mathrm{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$
    - problem: too many interesting sequences are not square-summable!

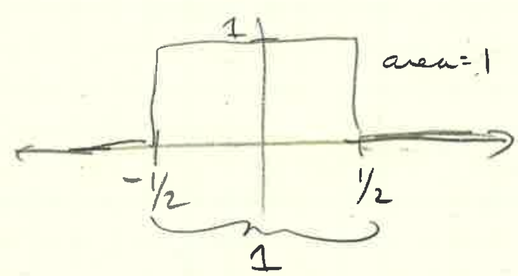
- The Dirac delta functional
  - Defined by the "sifting" property:



$$\int_{-\infty}^{\infty} \delta(t-s) f(t)\,dt = f(s), \quad \forall \text{ functions of } t \in \mathbb{R}, \; s \in \mathbb{R}$$

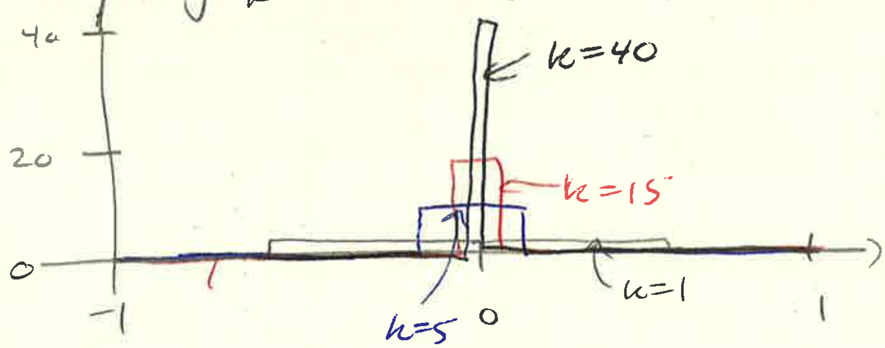
- Intuition
  - Family of localizing functions $r_k(t)$ with $k \in \mathbb{N}$ and $t \in \mathbb{R}$
  - Support inversely proportional to k
  - constant area

$$\mathrm{rect}(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$



  - Consider the localizing family $r_k(t) = k\,\mathrm{rect}(kt)$:
    - nonzero over $\left[-\frac{1}{2k}, \frac{1}{2k}\right]$, i.e., support is $\frac{1}{k}$
    - area is 1

- The family $r_k(t) = k\,\mathrm{rect}(kt)$



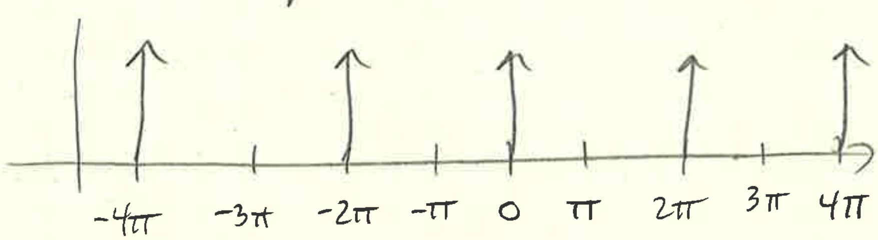
- Extracting a point value
  - By the mean value theorem:

$$\int_{-\infty}^{\infty} r_k(t) f(t)\,dt = k \int_{-\frac{1}{2k}}^{\frac{1}{2k}} f(t)\,dt = f(\gamma)\Big|_{\gamma \in \left[-\frac{1}{2k}, \frac{1}{2k}\right]}$$

and so:
$$\lim_{k\to\infty} \int_{-\infty}^{\infty} r_k(t) f(t)\,dt = f(0)$$

- The Dirac delta functional
  - The delta functional shorthand. Instead of writing
  $$\lim_{k\to\infty} \int_{-\infty}^{\infty} r_k(t-s) f(t)\,dt$$
  we write $\int_{-\infty}^{\infty} \delta(t-s) f(t)\,dt$, as if $\lim_{k\to\infty} r_k(t) = \delta(t)$.

- The "pulse train"
$$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$
just a technicality to use the Dirac delta in the space of 2π-periodic functions

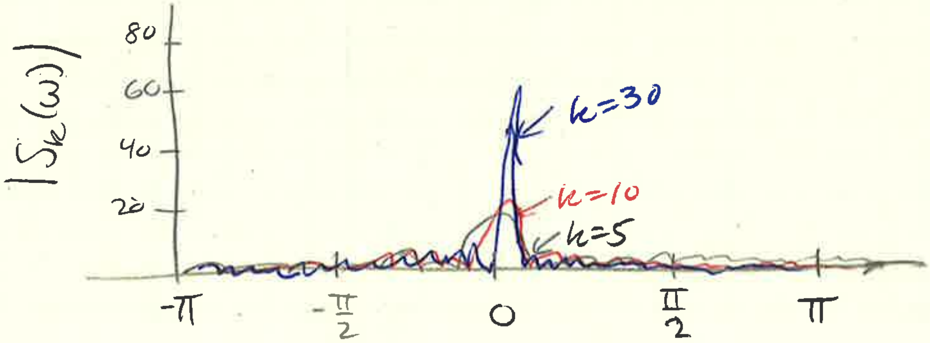


$$\text{IDTFT}\{\tilde{\delta}(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\delta}(\omega) e^{j\omega n}\,d\omega$$
$$= \int_{-\pi}^{\pi} \delta(\omega)\, e^{-j\omega n}\,d\omega = e^{j\omega n}\big|_{\omega=0} = 1$$

$$(\text{IDFT}\{N\delta[k]\} = 1)$$

$$\Rightarrow \text{DTFT}\{1\} = \tilde{\delta}(\omega)$$

- Does this make sense?
  - Partial DTFT sum: $S_k(\omega) = \sum_{n=-k}^{k} e^{-j\omega n}$



- Using the same technique

$$\text{IDTFT}\{\tilde{\delta}(\omega-\omega_0)\} = e^{j\omega_0 n}$$

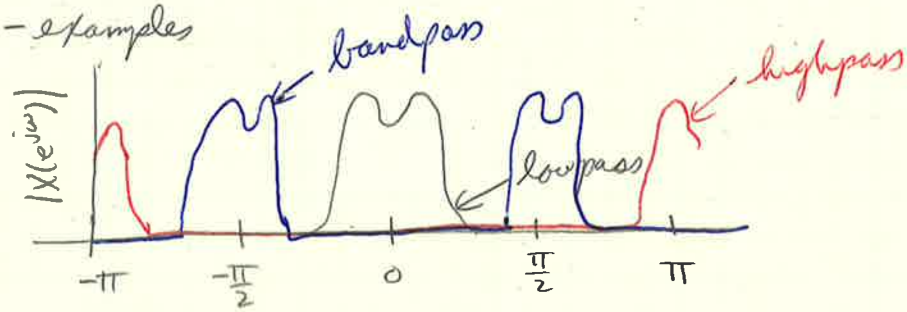
$\Rightarrow$
- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\{e^{j\omega_0 n}\} = \tilde{\delta}(\omega-\omega_0)$
- $\text{DTFT}\{\cos\omega_0 n\} = [\tilde{\delta}(\omega-\omega_0)+\tilde{\delta}(\omega+\omega_0)]/2$
- $\text{DTFT}\{\sin\omega_0 n\} = -j[\tilde{\delta}(\omega-\omega_0)-\tilde{\delta}(\omega+\omega_0)]/2$

3.7: Sinusoidal Modulation

3.7a Sinusoidal modulation

- Classifying signals in frequency
  - Three broad categories according to where most of the spectral energy resides:
    - lowpass signals (also known as "baseband" signals)
    - highpass signals
    - bandpass signals

- examples



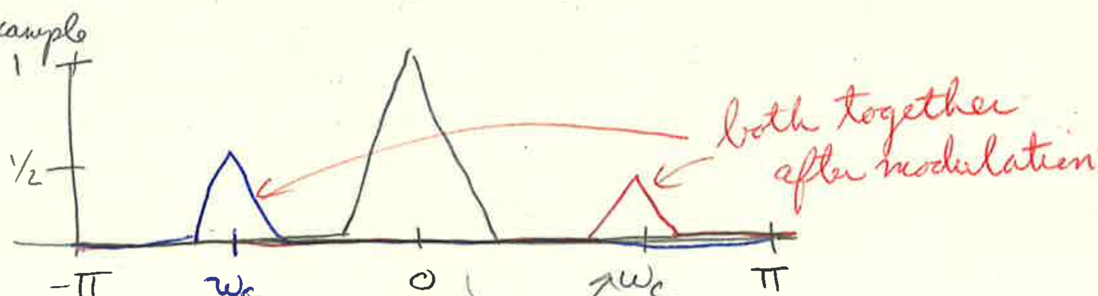
- Sinusoidal modulation

$$\text{DTFT}\{x[n]\cos(\omega_c n)\} = \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\}$$

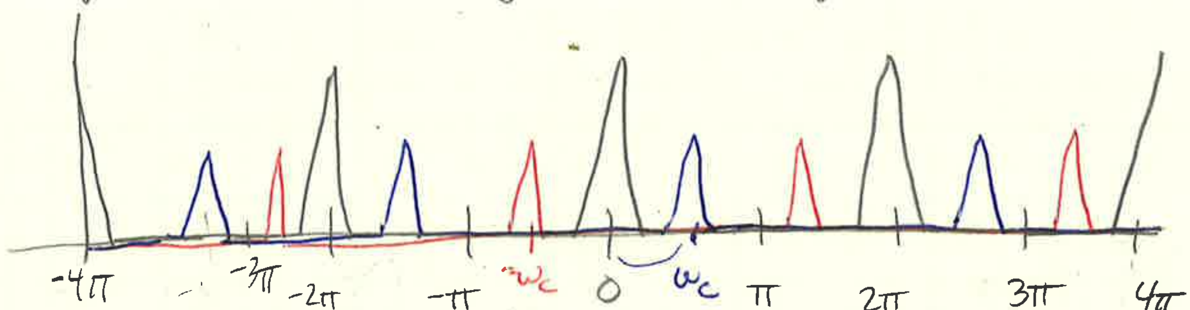
$$= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

- usually $x[n]$ baseband
- ω_c is the carrier frequency

- example

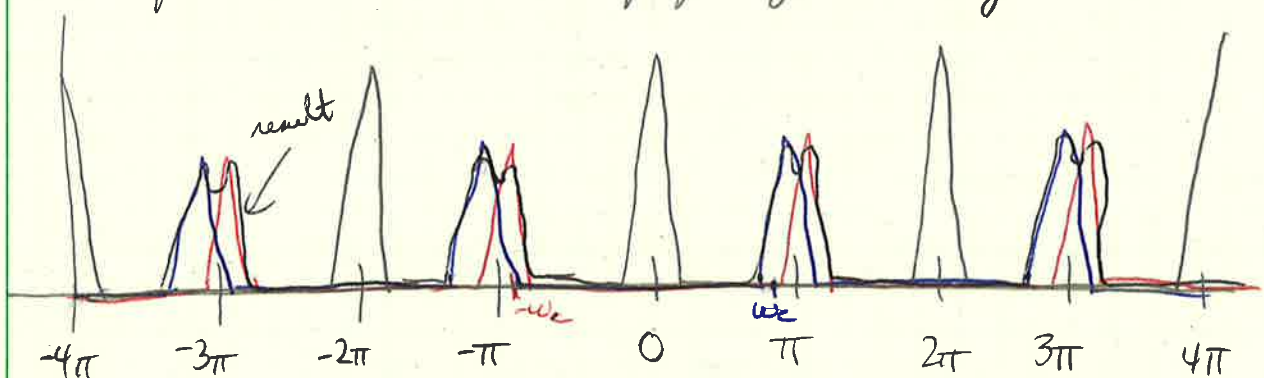


- Again, explicitly showing the periodicity of the spectrum



combine red and blue to get the resulting modulated spectrum

- Careful when the modulation frequency is too large!



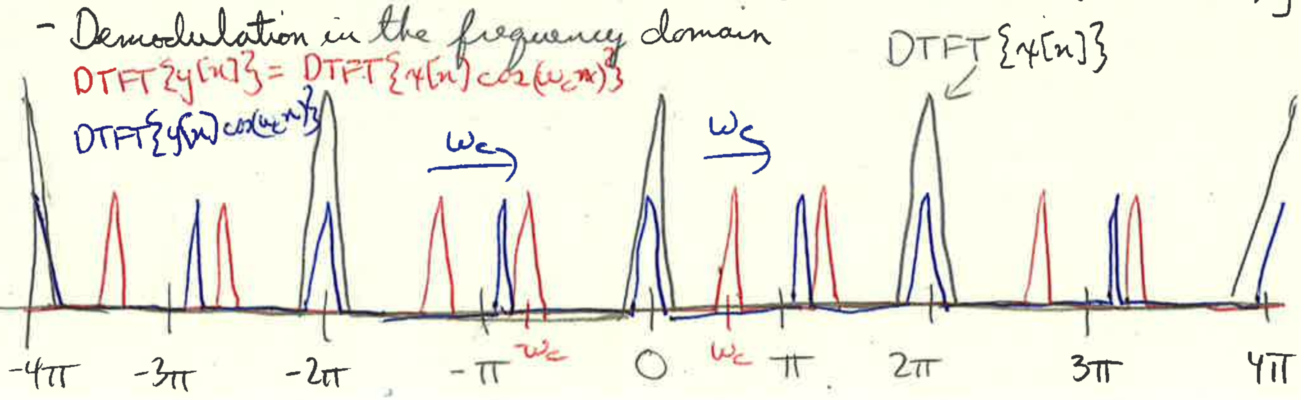
- Sinusoidal modulation: applications
  - voice and music are lowpass signals
  - radio channels are bandpass, in much higher frequencies
  - modulation brings the baseband signal in the transmission band
  - demodulation at the receiver brings it back

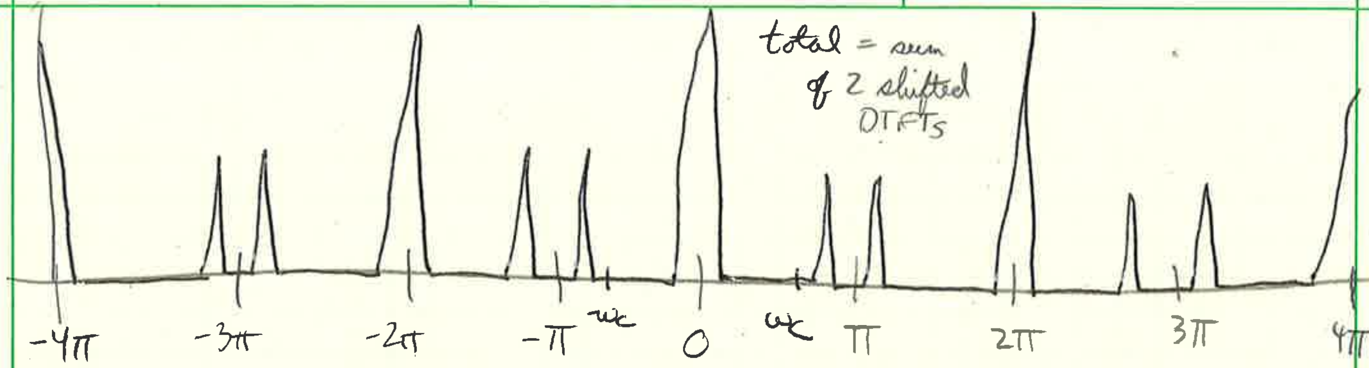
- Sinusoidal demodulation
  - just multiply the received signal by the carrier again

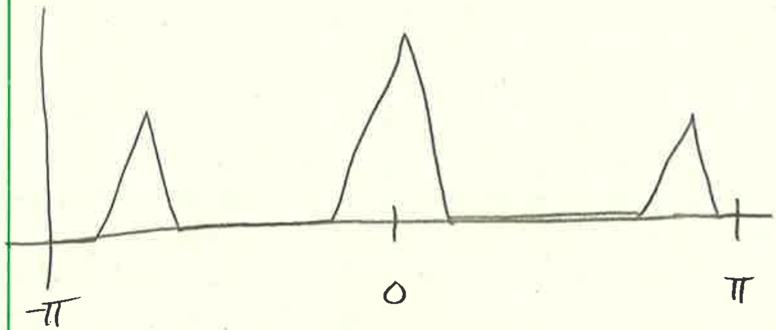
$$y[n] = x[n]\cos(\omega_c n), \quad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}
\mathrm{DTFT}\{y[n]\cdot 2\cos(\omega_c n)\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\
&= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j\omega}) + X(e^{j\omega}) + X(e^{j(\omega+2\omega_c)})\right] \\
&= X(e^{j\omega}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]
\end{aligned}$$

- Demodulation in the frequency domain







DTFT of demodulated signal

- we recovered the baseband signal exactly ...
- but we have some spurious high-frequency components
- in the next Module we will learn how to get rid of them!

3.7b Tuning a guitar

- Problem (abstraction):
  - reference sinusoid at frequency ω_0
  - tunable sinusoid at frequence ω
  - make $\omega = \omega_0$ "by ear"

- The procedure
  1. bring ω close to ω_0 (easy)
  2. when $\omega \approx \omega_0$ play both sinusoids together
  3. trigonometry comes to the rescue

$$x[n] = \cos(\omega_0 n) + \cos(\omega n)$$
$$= 2\cos\left(\frac{\omega_0+\omega}{2} n\right)\cos\left(\frac{\omega_0-\omega}{2} n\right)$$
$$\approx 2\cos(\Delta_\omega n)\cos(\omega_0 n)$$

- Let's see what's happening

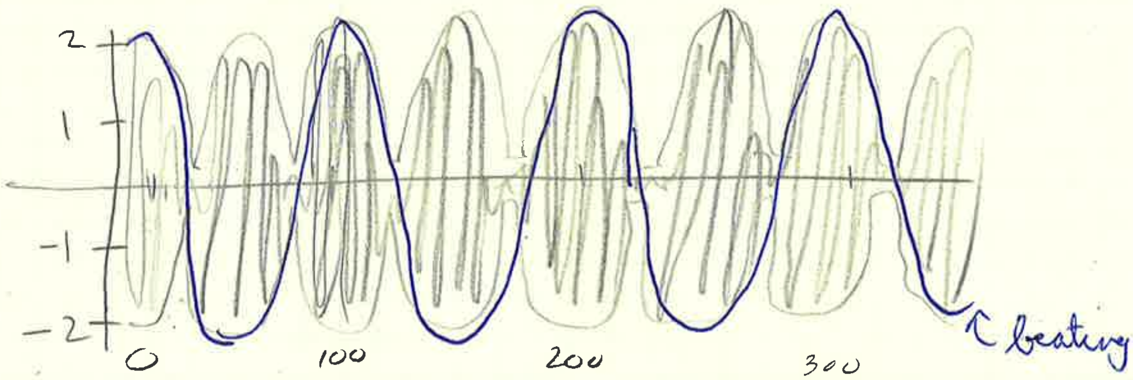
$$x[n] \approx \underbrace{2\cos(\Delta_\omega n)}_{\text{"error" signal}}\ \underbrace{\cos(\omega_0 n)}_{\text{modulation at } \omega_0}$$

"error" signal

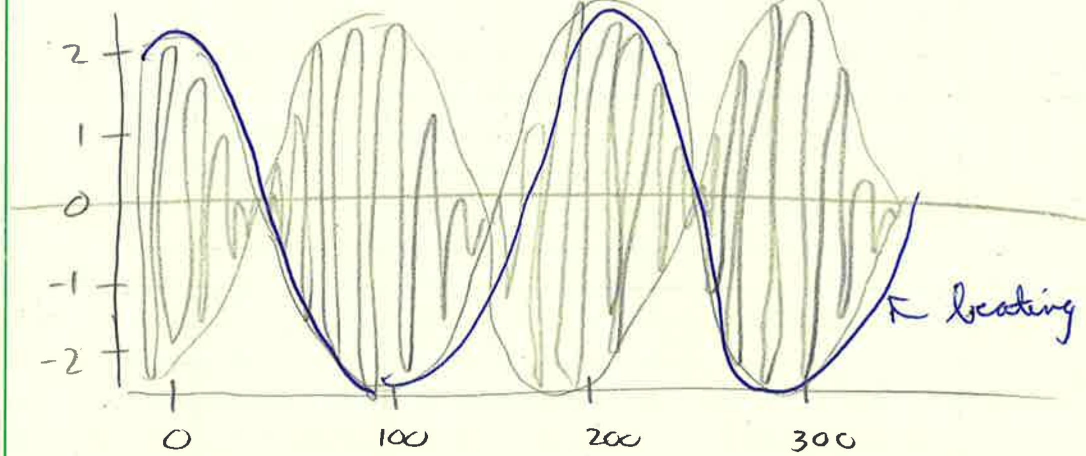
· modulation at ω_0

- when $\omega \approx \omega_0$, the error signal is too low to be heard; modulation brings it up to hearing range and we perceive it as amplitude oscillations of the carrier frequency

- In the time domain ...



$$\omega_0 = 2\pi(0.2), \quad \omega = 2\pi(0.22), \quad \Delta_\omega = 2\pi(0.0100)$$



$$\omega_0 = 2\pi(0.2), \quad \omega = 2\pi(0.21), \quad \Delta_\omega = 2\pi(0.0050)$$

- A Detour on Western Music Conventions

  - Each note has a unique frequency, $A_3 = 220\,Hz$, $A_4 = 440\,Hz$ (reference note), $A_5 = 880\,Hz$.
  - An octave corresponds to doubling/halving frequency
  - Octaves are separated into 12 evenly spaced half-tones

$$f_h = f_0 \cdot 2^{h/12}$$

3.8 Relationship between transforms

- Overview
  - DFT, DFS, DTFT
  - DTFT of periodic sequences
  - DTFT of finite-support sequences
  - Zero padding

- Transforms
  - DFT, DFS: change of basis in $\mathbb{C}^N$
  - DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
  - basis vectors are "building blocks" for any signal
  - DFT: numerical algorithm (computable)
  - DTFT: mathematical tool (proofs)
- Embedding finite-length signals
  - N-tap signal $x[n]$
  - natural spectral representation: DFT $X[k]$
  - two ways to embed $x[n]$ into an infinite sequence:
    - periodic extension: $\tilde{x}[n] = x[n \bmod N]$
    - finite support extension: $\bar{x}[n] = \begin{cases} x[n], & 0 \le n < N \\ 0, & \text{otherwise} \end{cases}$
  - how does $X[k]$ relate to the DTFT of the embedded signals?
- DTFT of periodic signals

$$\tilde{x}[n] = x[n \bmod N]$$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n}$$

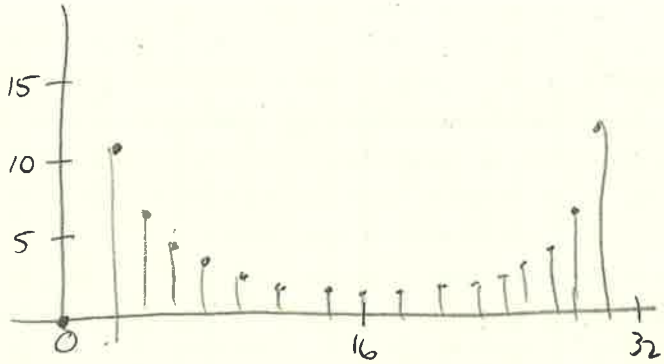
$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)$$

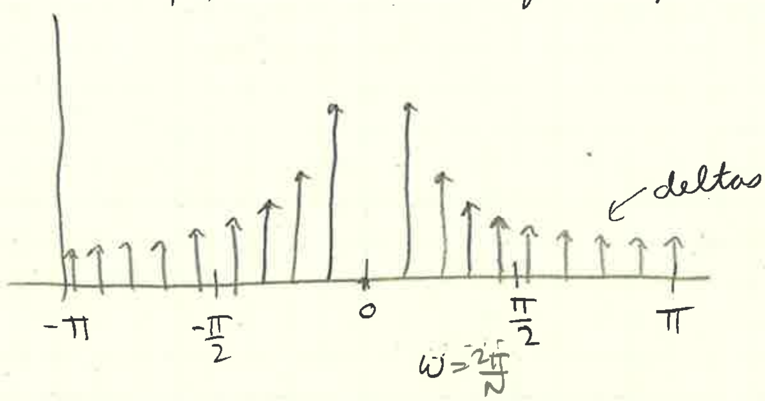
$$\left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} = \mathrm{DTFT}\left\{ e^{j\frac{2\pi}{N}nk} \right\} = \tilde{\delta}\left(\omega - \frac{2\pi}{N}k\right) \right)$$

$$\tilde{X}(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \tilde{\delta}\left(\omega - \frac{2\pi}{N}k\right)$$

- DFT of 32-tap sawtooth



- DTFT of periodic extension of 32-tap sawtooth



- DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n], & 0 \le n < N \\ 0, & \text{otherwise} \end{cases}$$

$$\bar{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)$$

$\left[\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} = \bar{R}(e^{j(\omega - \frac{2\pi}{N}k)}) \right.$, where $\bar{R}(e^{j\omega})$ is the DTFT of $r[n]$, the interval indicator signal: $\left. r[n] = \begin{cases} 1, & 0 \le n < N \\ 0, & \text{otherwise} \end{cases} \right]$

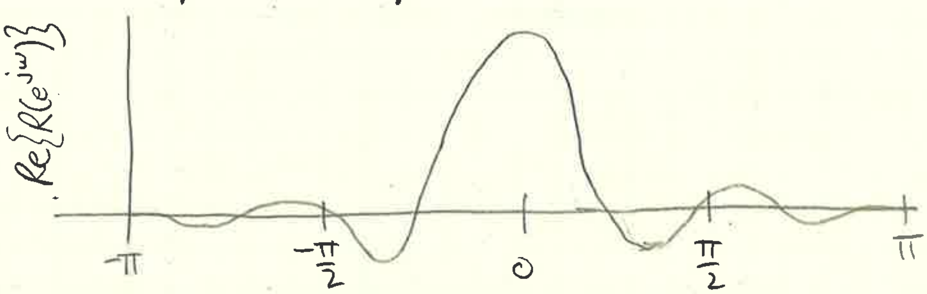
- DTFT of interval signal

$$\bar{R}(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

$$= \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}$$

- DTFT of interval signal ($N=9$)

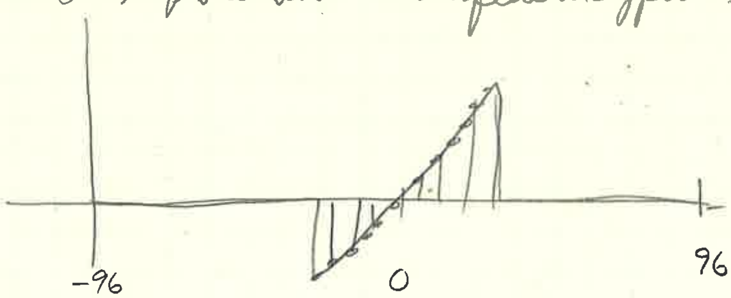


- DTFT of finite-support signals cont'd

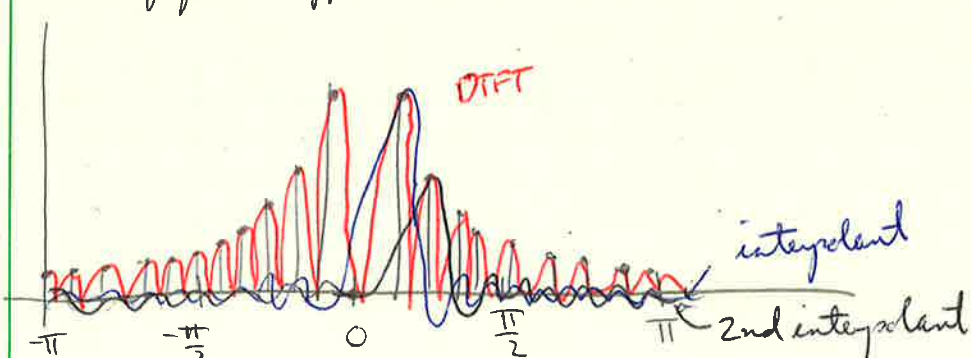
$$\bar{X}(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\,\Lambda\left(\omega - \tfrac{2\pi}{N}k\right), \text{ where } \Lambda(\omega) = \frac{1}{N}\bar{R}(e^{j\omega}):$$

smooth interpolation of DFT values.

- 32 tap sawtooth with finite support extension



- DTFT of finite support extension (sketch)



- About zero-padding

"When computing the DFT numerically one may "pad" the data vector with zeros to obtain "nicer" plots"

$$X_M[h] = \sum_{n=0}^{M-1} x'[n] e^{-j\frac{2\pi}{M}nh} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}nh}$$

extension of $x[n]$

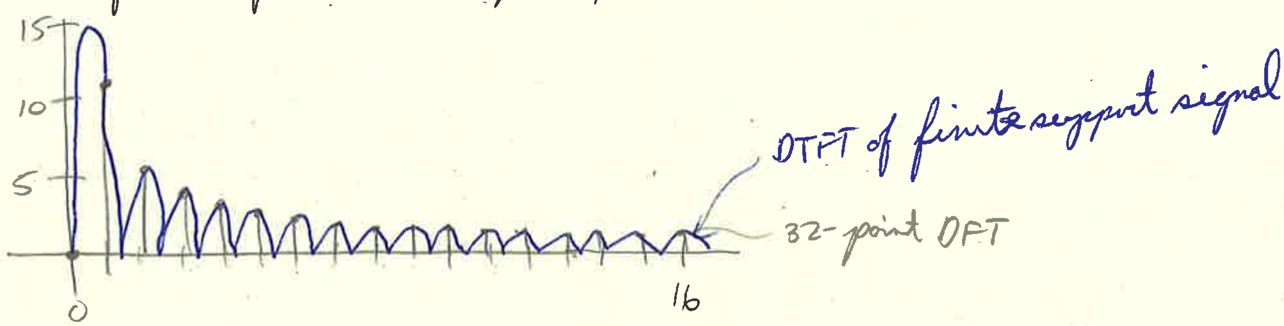
$$= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X_N[k] e^{j\frac{2\pi}{N}nk}\right) e^{-j\frac{2\pi}{M}nh}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_N[k]\left(\sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{M}h-\frac{2\pi}{N}k\right)n}\right)$$

$$= \bar{X}(e^{j\omega})\Big|_{\omega=\frac{2\pi}{M}h}$$

- zero padding does not add information
- a zero-padded DFT is simply a sampled DTFT of the finite-support extension

- DFT of 32-tap sawtooth, zero-padded



4.1.a Linear time-invariant filters

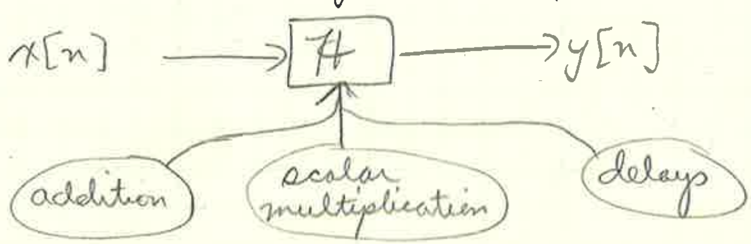
- A generic signal processing device

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$$

$$y[n] = \mathcal{H}\{x[n]\}$$

- Linearity : $\mathcal{H}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha \mathcal{H}\{x_1[n]\} + \beta \mathcal{H}\{x_2[n]\}$
- Time Invariance : $y[n] = \mathcal{H}\{x[n]\} \Longleftrightarrow \mathcal{H}\{x[n-n_0]\} = y[n-n_0]$

- Linear, time-invariant systems (LTI)



$$y[n] = H(x[n], x[n-1], x[n-2], \ldots, y[n-1], y[n-2], \ldots) \quad \text{(causal LTI)}$$

with $H(\cdot)$ a linear function of its arguments

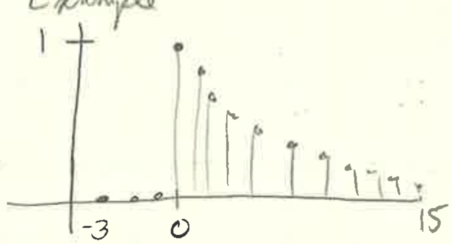
4.1.b Convolution

- Impulse response

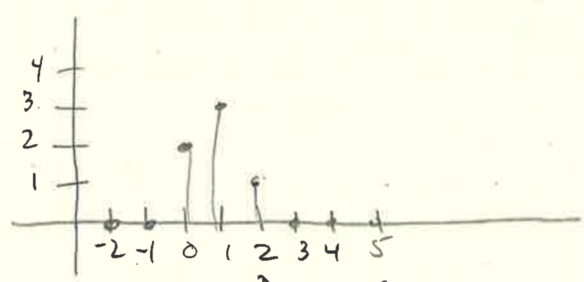
$$h[n] = \mathcal{H}\{\delta[n]\}$$

- Fundamental result: impulse response fully characterizes the LTI system!

- Example



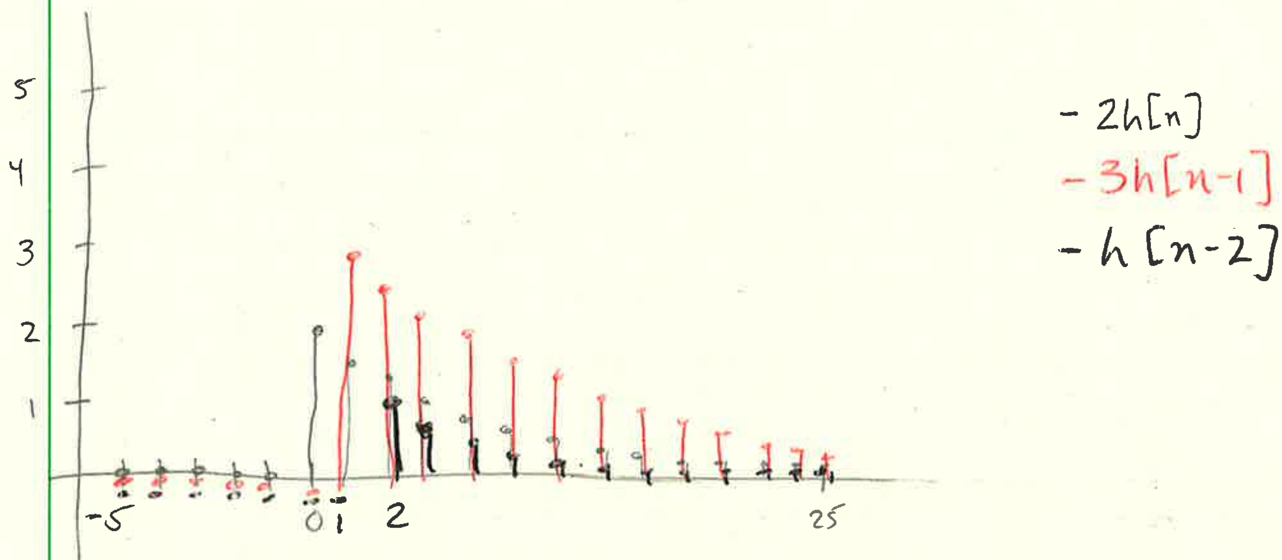
$$h[n] = \alpha^n u[n]$$

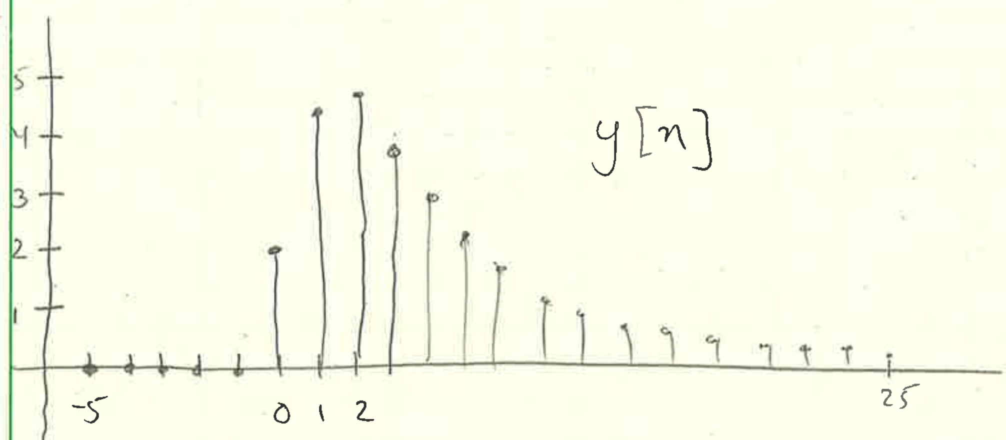


$$x[n] = \begin{cases} 2, & n=0 \\ 3, & n=1 \\ 1, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

- $x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$
- we know the impulse response $h[n] = \mathcal{H}\{\delta[n]\}$
- compute $y[n] = \mathcal{H}\{x[n]\}$ exploiting linearity and time-invariance

$$y[n] = \mathcal{H}\{2\delta[n] + 3\delta[n-1] + \delta[n-2]\}$$
$$= 2\mathcal{H}\{\delta[n]\} + 3\mathcal{H}\{\delta[n-1]\} + \mathcal{H}\{\delta[n-2]\}$$
$$= 2h[n] + 3h[n-1] + h[n-2]$$





- Convolution

We can always write (Module 3.2):

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

by linearity and time invariance: $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = x[n] * h[n]$

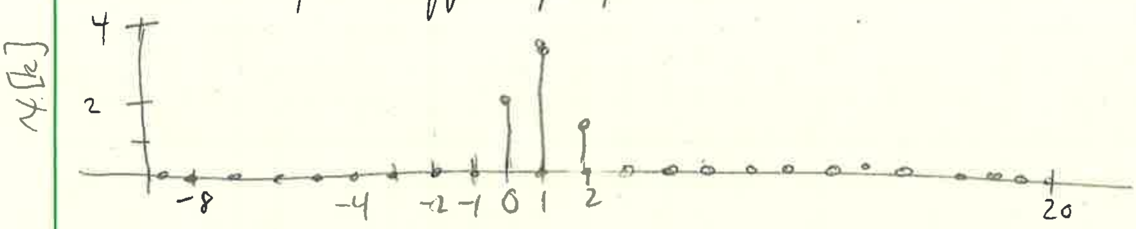
- Performing the convolution algorithmically

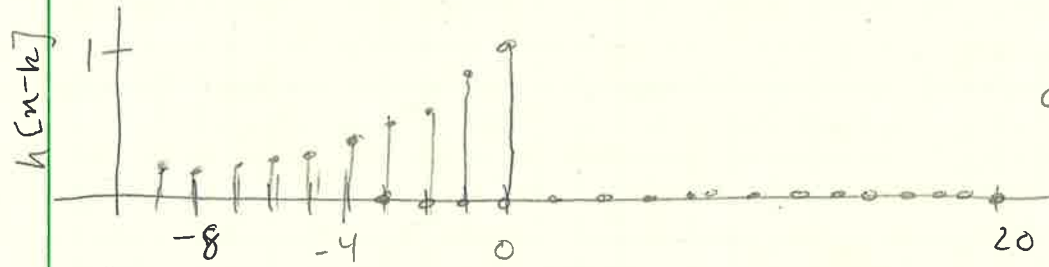
$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

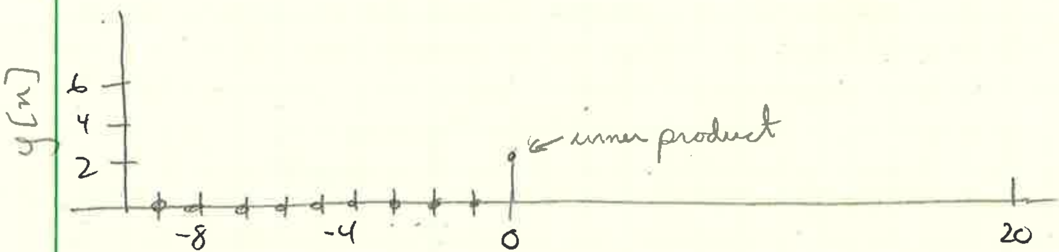
Ingredients:
- a sequence $x[m]$
- a second sequence $h[m]$

The recipe:
- time reverse $h[m]$
- at each step n (from $-\infty$ to ∞)
  - center the time-reversed $h[m]$ in n (i.e. shift by $-n$)
  - compute the inner product

- Same example, different perspective







- Convolution properties
  - linearity and time invariance (by definition)
  - commutativity: $(x*h)[n] = (h*x)[n]$
  - associativity for absolutely and square-summable sequences:
    $((x*h)*w)[n] = (x*(h*w))[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{w[n]} \longrightarrow y[n]$$

$$x[n] \rightarrow \boxed{(h*w)[n]} \longrightarrow y[n]$$

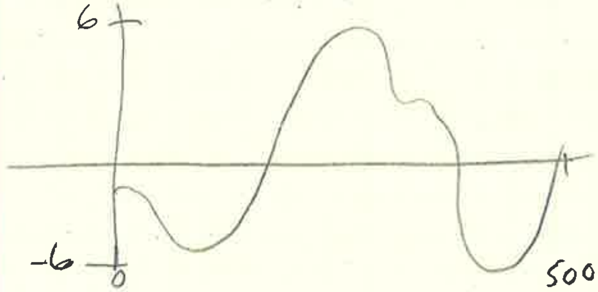
Signal of the Day: Can one hear the shape of a room?

- Our Model: Room Impulse Response (RIR)
  - Linear model
    - Sound level in room acoustics is low
    - Linear model is good approximation
    - Entirely characterized by impulse response, i.e., response to Dirac impulse
  - Room impulse response
    - Describe audio channel between sender S and receiver R
    - Sum up effect of direct path transmission and subsequent attenuated echoes.

4.2 Filtering by Example

4.2.a The moving average filter

- Typical filtering scenario: denoising



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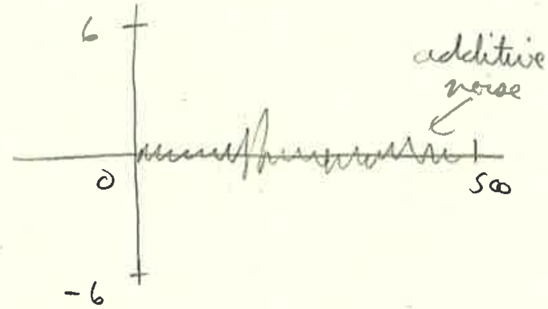

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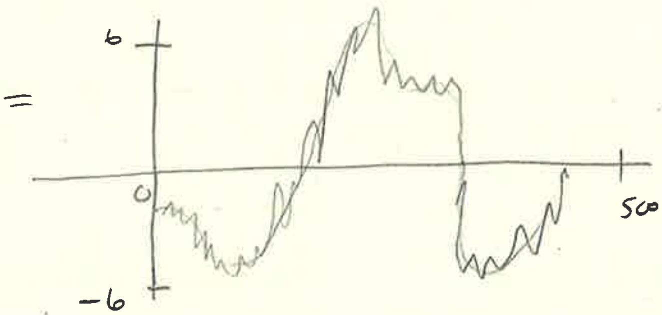
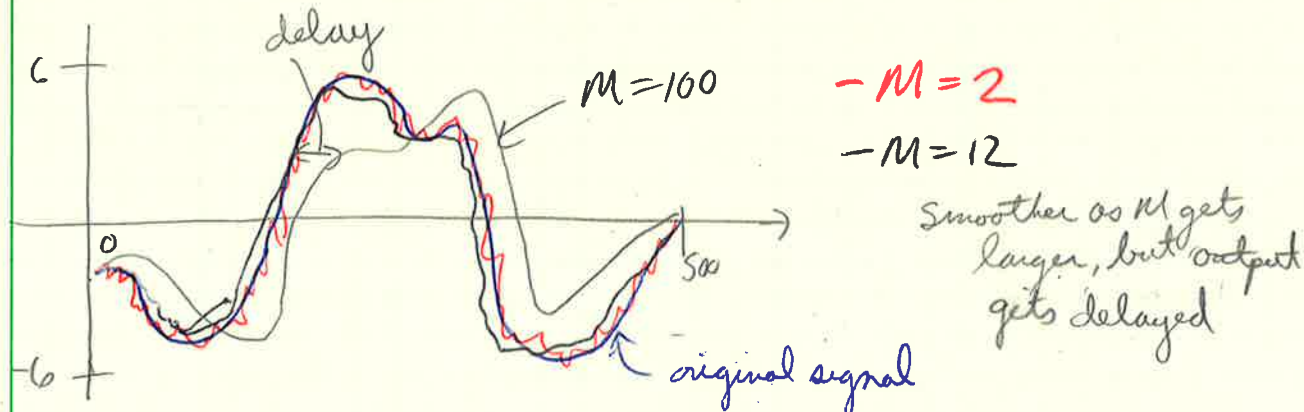


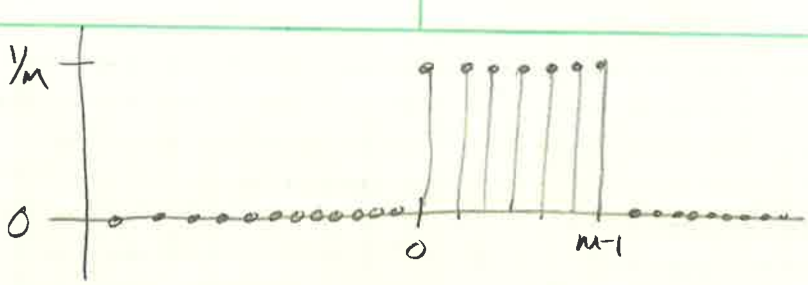
noisy signal

- Denoising by Moving Average (MA)
  - idea: replace each sample by the local average
  - for instance: $y[n] = (x[n] + x[n-1])/2$
  - more generally: $$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$



- MA: impulse response

$$h[n] = \frac{1}{M}\sum_{k=0}^{M-1} \delta[n-k] = \begin{cases} 1/M, & 0 \le n < M \\ 0, & \text{otherwise} \end{cases}$$



- MA: analysis
  - smoothing effect proportional to M
  - number of operations and storage also proportional to M

- From the MA to a first-order recursion

$$y_M[n] = \frac{1}{M}\left(x[n] + x[n-1] + \dots + x[n-M+1]\right)$$ – Moving average over M points

$$y_M[n] = \frac{1}{M}x[n] + \underbrace{\frac{1}{M}\left(x[n-1] + \dots + x[n-M+1]\right)}$$

"almost" $y_{M-1}[n-1]$, i.e. moving average over $M-1$ points, delayed by one

Formally:

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M}\sum_{k=0}^{M-1} x[(n-1)-k] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-(k+1)]$$

$$= \frac{1}{M}\sum_{k=0+1}^{M-1+1} x[n-k] = \frac{1}{M}\sum_{k=1}^{M} x[n-k]$$

$$y_{M-1}[n] = \frac{1}{M-1}\sum_{k=0}^{M-2} x[n-k]$$

$$y_{M-1}[n-1] = \frac{1}{M-1}\sum_{k=1}^{M-1} x[n-k]$$

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \qquad y_{M-1}[n-1] = \frac{1}{M-1}\sum_{k=1}^{M-1} x[n-k]$$

$$\sum_{k=0}^{M-1} x(n-k) = x[n] + \sum_{k=1}^{M-1} x[n-k]$$

$$M y_M[n] = x[n] + (M-1)\, y_{M-1}[n-1]$$

$$\Leftrightarrow y_M[n] = \frac{M-1}{M} y_{M-1}[n-1] + \frac{1}{M}x[n]$$

$$y_M[n] = \lambda\, y_{M-1}[n-1] + (1-\lambda)\, x[n], \quad \lambda = \frac{M-1}{M}$$

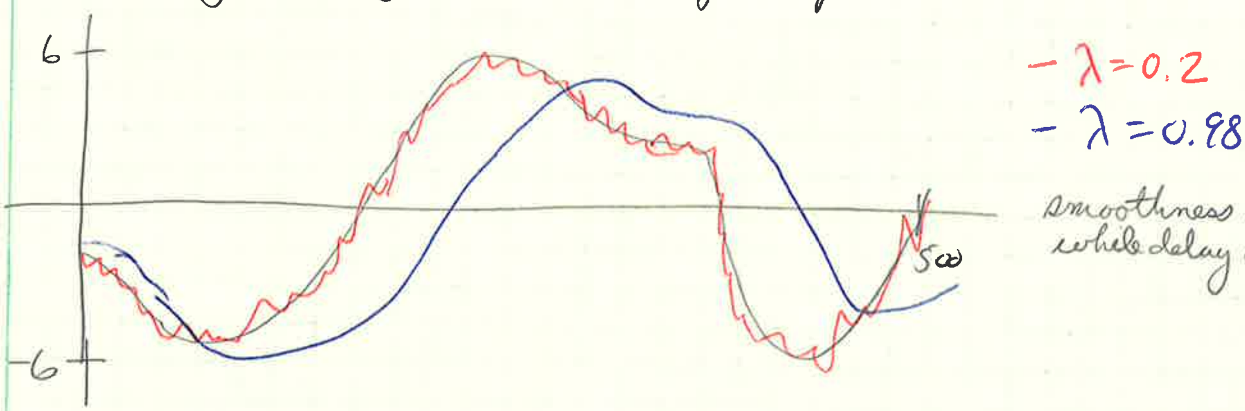
4.2.6 The leaky integrator

- From the MA to a first-order recursion

$$y_M[n] = \frac{M-1}{M} y_{M-1}[n-1] + \frac{1}{M} x[n]$$

$$y_M[n] = \lambda y_{M-1}[n-1] + (1-\lambda) x[n], \quad \lambda = \frac{M-1}{M}$$

- The Leaky Integrator
  - When M is large, $y_{M-1}[n] \approx y_M[n]$ (and $\lambda \approx 1$)
  - try the filter
$$y[n] = \lambda y[n-1] + (1-\lambda) x[n]$$
  - filter is now recursive, since it uses its previous output value
- Denoising recursively with the Leaky Integrator

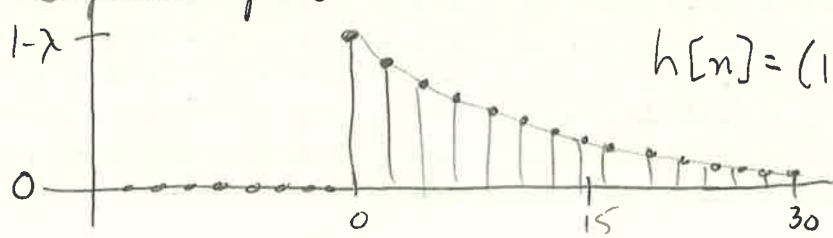


smoothness increases while delay increases

- What about the impulse response?
$$y[n] = \lambda y[n-1] + (1-\lambda)\delta[n]$$
  - $y[n] = 0, \ \forall n < 0$
  - $y[0] = \lambda y[-1] + (1-\lambda)\delta[0] = 1-\lambda$
  - $y[1] = \lambda y[0] + (1-\lambda)\delta[1] = \lambda(1-\lambda)$
  - $y[2] = \lambda y[1] + (1-\lambda)\delta[2] = \lambda^2(1-\lambda)$

$\vdots$

- Impulse Response



$$h[n] = (1-\lambda)\lambda^n u[n]$$

- Leaky Integrator: why the name?

Discrete-time integrator is a boundless accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

We can rewrite the integrator as

$$y[n] = y[n-1] + x[n]$$

To prevent "explosion", pick $\lambda < 1$

$$y[n] = \lambda y[n-1] + (1-\lambda)\, x[n]$$

keep only a fraction λ of the accumulated value so far and forget ("leak") a fraction $1-\lambda$

add only a fraction $1-\lambda$ of the current value to the accumulator

Summary of Lesson 4.2

In this lesson we have studied two examples of LTI filters: the moving average and the leaky integrator. The moving average is just a local average computed over the last M observations, including the current one. We have seen the formula for its impulse response and derived a simple recursive formula to compute it efficiently.

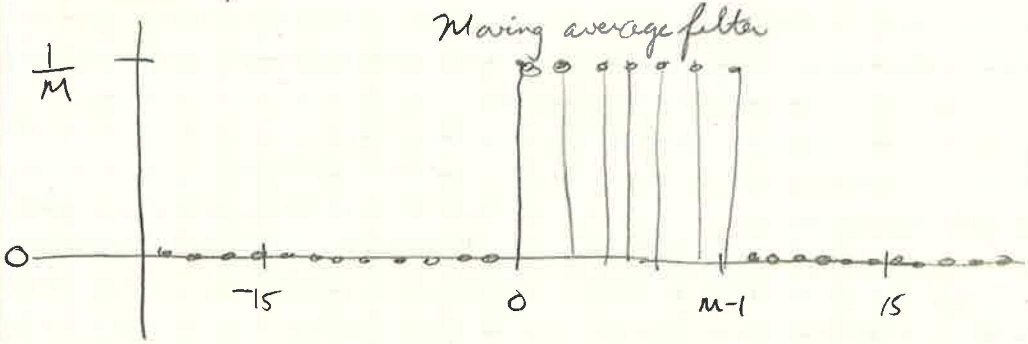
The leaky integrator is a special case of the moving average filter when M becomes large. It is also defined by a recursive formula. The basic idea is to add a portion λ of the past accumulated values so far and a fraction $1-\lambda$ of the current observation. By setting the value of $\lambda < 1$, we ensure that the system never blows up.

4.3 Filter Stability

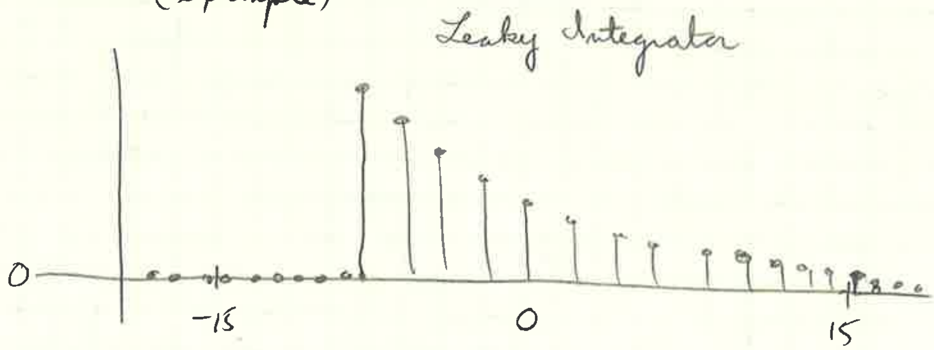
4.3.a Filter classification in the time domain

- Filter types according to impulse response
  - Finite Impulse Response (FIR)
  - Infinite Impulse Response (IIR)
  - causal
  - non causal

- FIR
  - impulse response has finite support
  - only a finite number of samples are involved in the computation of each output sample
- FIR (example)



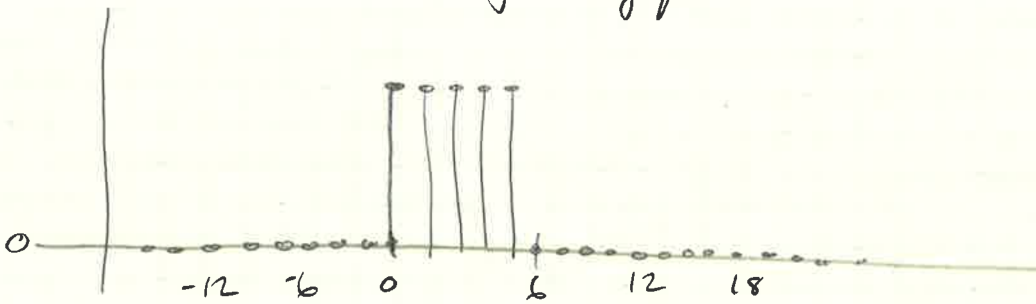
- IIR
  - impulse response has infinite support
  - a potentially infinite number of samples are involved in the computation of each output sample
  - surprisingly, in many cases the computation can still be performed in a finite amount of steps.
- IIR (example)



- Causal vs Noncausal
  - causal:
    - impulse response is zero for $n<0$
    - only past samples (with respect to the present) are involved in the computation of each output sample
    - causal filters can work "on line" since they only need the past
  - non causal:
    - impulse response is nonzero for some (or all) $n<0$
    - can still be implemented in an offline fashion (when all input data are available on storage, e.g., in Image Processing)

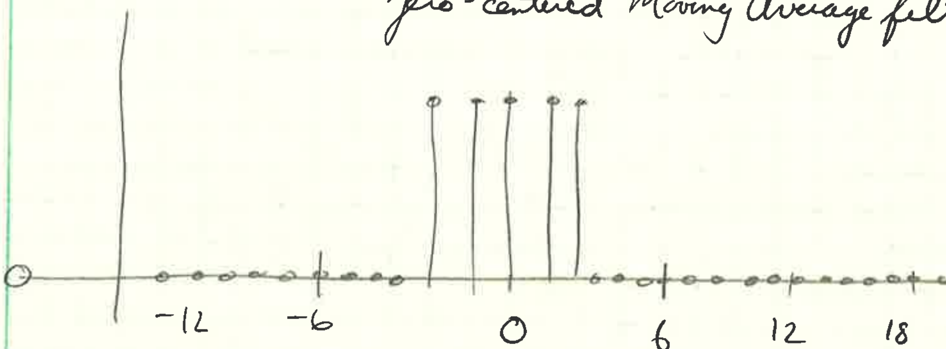
- Causal example

Moving Average filter



- Noncausal example

Zero-centered Moving Average filter



4.3.6 Filter Stability

- Stability
  - key concept: avoid "explosions" if the input is nice
  - a nice signal is a bounded signal: $|x[n]| < M, \forall n$
  - Bounded-Input-Bounded-Output (BIBO) stability: if the input is nice the output should be nice

- Fundamental Stability Theorem

A filter is BIBO stable $\Leftrightarrow$ its impulse response is absolutely summable

proof ($\Rightarrow$) Hypotheses: $|x[n]| < M$, $\sum_n h[n] = L < \infty$
Thesis: $|y[n]|$ bounded

proof: $$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]\,x[n-k]|$$

$$\le M \sum_{k=-\infty}^{\infty} |h[k]| < ML$$

proof ($\Leftarrow$) : Hypotheses : $|x[n]| < M$, $|y[n]| < P$

Thesis : $h[n]$ is absolutely summable

proof by contradiction :

- assume $\sum_n |h[n]| = \infty$
- build $x[n] = \begin{cases} +1, & h[-n] \geq 0 \\ -1, & h[n] < 0 \end{cases}$
- clearly, $x[n]$ is bounded
- however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]\, x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

- The good news : FIR filters are always stable
- Checking the stability of IIRs
  - Let's check the Leaky Integrator :

$$\sum_{n=-\infty}^{\infty} |h[n]| = |1-\lambda| \sum_{n=0}^{\infty} |\lambda|^n$$

$$= \lim_{n\to\infty} |1-\lambda| \frac{1-|\lambda|^{n+1}}{1-|\lambda|} < \infty \quad \text{for } |\lambda| < 1$$

  - Stability is guaranteed for $|\lambda| < 1$
  - We will study indirect methods for filter stability later in this Module.

4.4 Frequency Response

4.4.a The convolution theorem

- A remarkable result

$$e^{j\omega_0 n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow ?$$

$$y[n] = e^{j\omega_0 n} * h[n] = h[n] * e^{j\omega_0 n}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}$$

$$= H(e^{j\omega_0})\, e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow H(e^{j\omega_0})e^{j\omega_0 n}$$

- complex exponentials are eigensequences of LTI systems, i.e., linear filters cannot change the frequency of sinusoids
- DTFT of impulse response determines the frequency character of a filter

- Magnitude and phase

$$\text{If } H(e^{j\omega_0}) = Ae^{j\theta}, \text{ then } \mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n+\theta)}, \quad A \in \mathbb{R}, \ \theta \in [-\pi, \pi]$$

amplitude: amplification $(A>1)$ or attenuation $(0 \le A < 1)$

phase shift: delay $(\theta<0)$ or advancement $(\theta>0)$

- The convolution theorem

- In general: $\text{DTFT}\{x[n] * h[n]\} = ?$
- Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions of the form $X(e^{j\omega})e^{j\omega n}$

$$X(e^{j\omega})e^{j\omega n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow X(e^{j\omega})H(e^{j\omega})e^{j\omega n}$$

$$\begin{aligned}
\text{DTFT}\{x[n]*h[n]\} &= \sum_{n=-\infty}^{\infty} (x*h)[n]\, e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k]e^{-j\omega(n-k)}e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k]e^{-j\omega(n-k)} \\
&= H(e^{j\omega})X(e^{j\omega})
\end{aligned}$$

- Frequency response

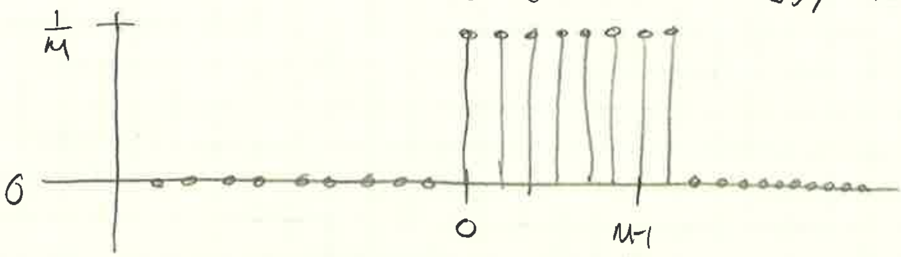
$$H(e^{j\omega}) = \mathrm{DTFT}\{h[n]\}$$

- Two effects:
  - Magnitude: amplification ($|H(e^{j\omega})| > 1$) or attenuation ($|H(e^{j\omega})| < 1$) of input frequencies
  - phase: overall delay and shape changes

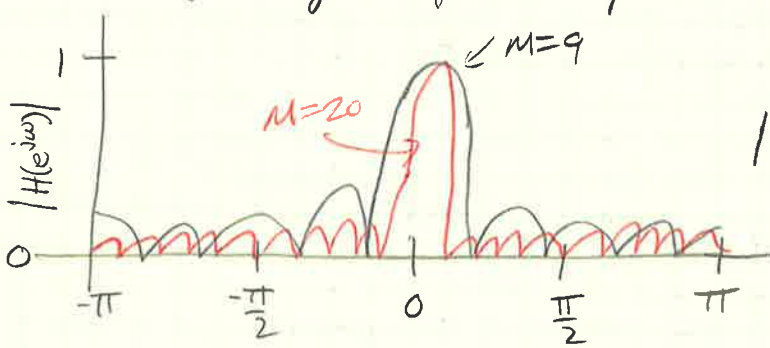
4.4.6 Examples of frequency response

- Moving Average revisited

$$h[n] = (u[n] - u[n-M])/M$$



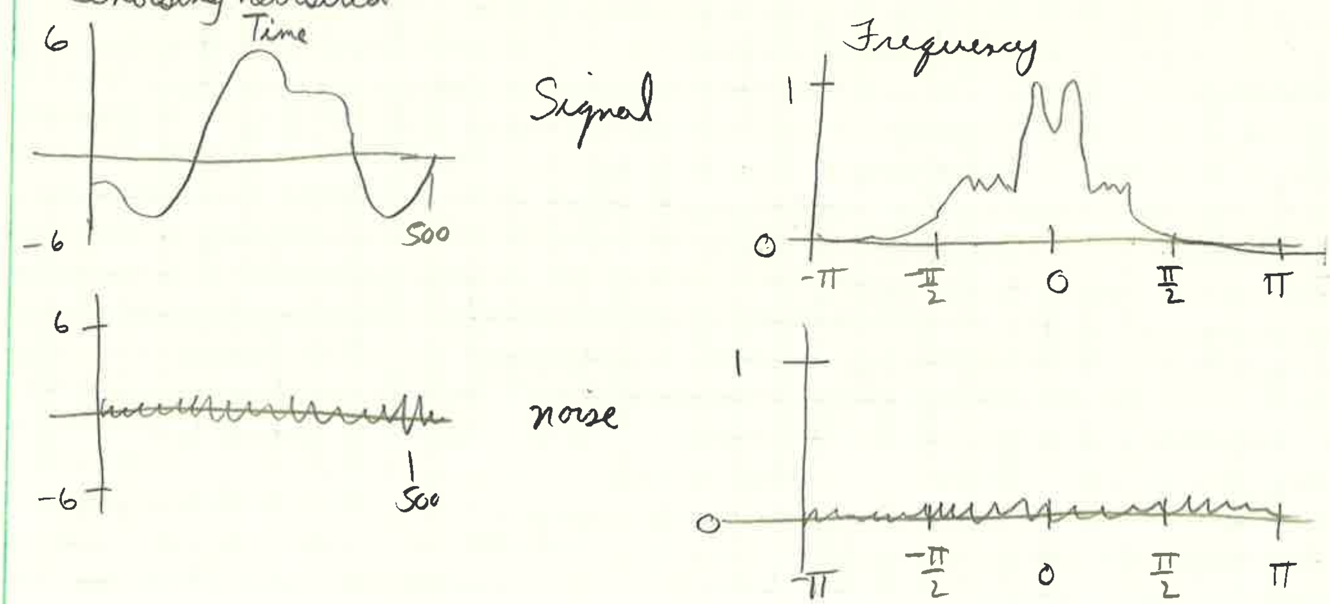
- Moving Average, magnitude response

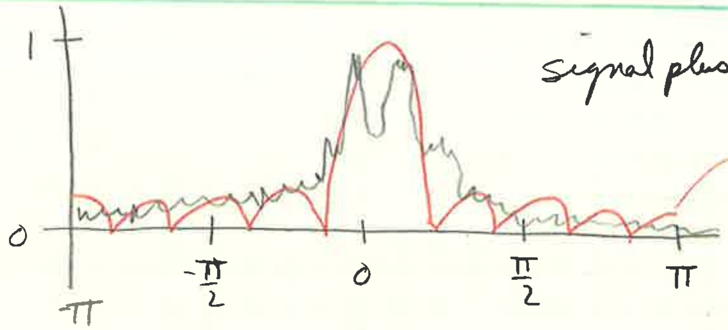


$$|H(e^{j\omega})| = \frac{1}{M}\left|\frac{\sin(\frac{\omega}{2}M)}{\sin(\frac{\omega}{2})}\right|$$

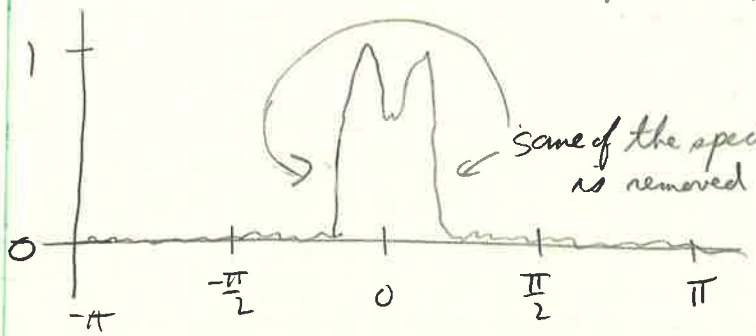
$$|H(e^{j\omega})| = 0, \quad \omega = \frac{2\pi}{M}k, \; k \neq 0$$

- Denoising revisited



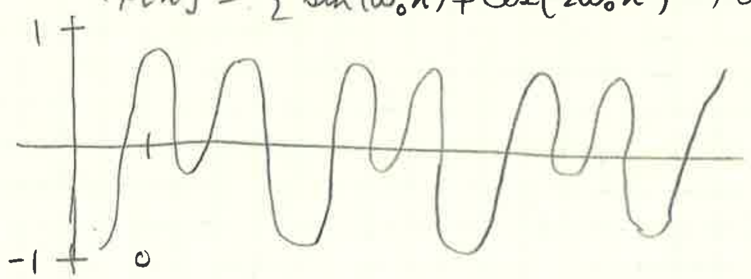


$\Downarrow$ product of signal plus noise and 9-point MA filter

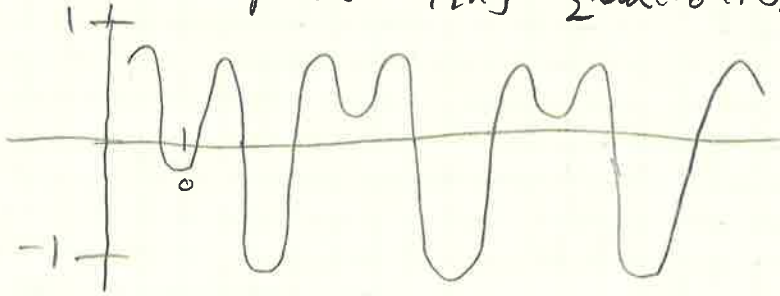


- What about the phase?
  - Assume $|H(e^{j\omega})| = 1$
    - zero phase: $\angle H(e^{j\omega}) = 0$ (spectrum is real)
    - linear phase: $\angle H(e^{j\omega}) = d\omega$, $d \in \mathbb{R}$
    - nonlinear phase
- Phase and signal shape

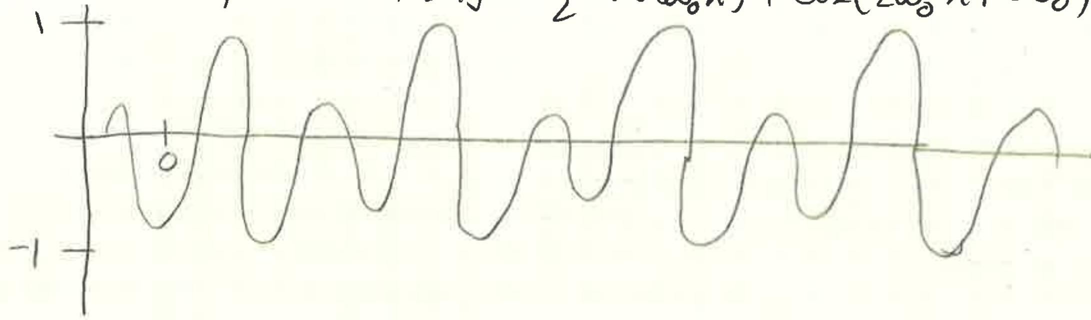
$x[n] = \frac{1}{2}\sin(\omega_0 n) + \cos(2\omega_0 n)$, $\omega_0 = \frac{2\pi}{40}$ (0-phase signal)



- linear phase: $x[n] = \frac{1}{2}\sin(\omega_0 n + \theta_0) + \cos(2\omega_0 n + 2\theta_0)$, $\theta_0 = \frac{8\pi}{5}$



· nonlinear phase : $x[n] = \frac{1}{2}\sin(\omega_0 n) + \cos(2\omega_0 n + 2\theta_0)$



– Linear phase

$$x[n] \rightarrow \boxed{\mathcal{D}} \longrightarrow x[n-d]$$

- $y[n] = x[n-d]$
- $Y(e^{j\omega}) = e^{-j\omega d} X(e^{j\omega})$
- $H(e^{j\omega}) = e^{-j\omega d}$
- linear phase term

In general, if $H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega d}$, $A(e^{j\omega}) \in \mathbb{R}$

$$x[n] \longrightarrow \boxed{A(e^{j\omega})} \longrightarrow \boxed{\text{delay by } d} \longrightarrow y[n]$$

– Moving average is linear phase

$$H(e^{j\omega}) = \frac{1}{M} \frac{\sin\left(\frac{\omega}{2} M\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{M-1}{2}\omega}$$

$d = \frac{M-1}{2}$

– Leaky Integrator revisited : $h[n] = (1-\lambda)\lambda^n u[n]$

$$H(e^{j\omega}) = \frac{1-\lambda}{1-\lambda e^{j\omega}}$$

Finding magnitude and phase requires a little algebra...

[Recall from complex algebra: $\frac{1}{a+jb} = \frac{a-jb}{a^2+b^2}$

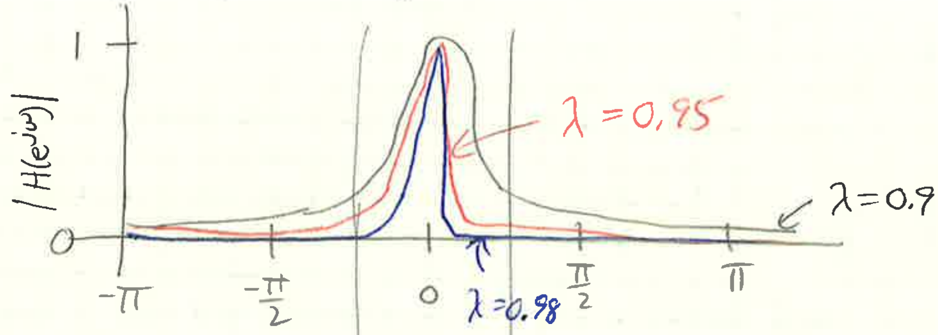
So that if $x = \frac{1}{a+jb}$, $|x|^2 = \frac{1}{a^2+b^2}$, $\angle x = \tan^{-1}\left[\frac{-b}{a}\right]$]

$$H(e^{j\omega}) = \frac{1-\lambda}{(1-\lambda\cos\omega) - j\sin\omega}$$

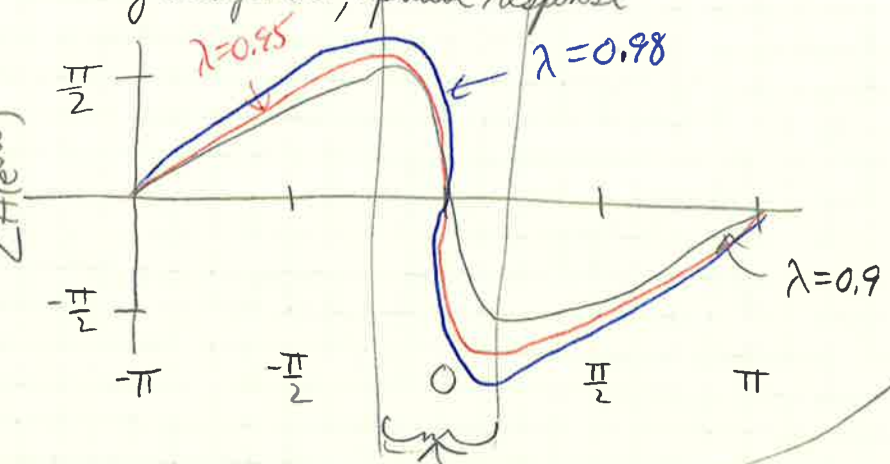
So that:

$$|H(e^{j\omega})|^2 = \frac{(1-\lambda)^2}{1-2\lambda\cos\omega+\lambda^2}, \quad \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{\lambda\sin\omega}{1-\lambda\cos\omega}\right]$$

- Leaky Integrator, magnitude response

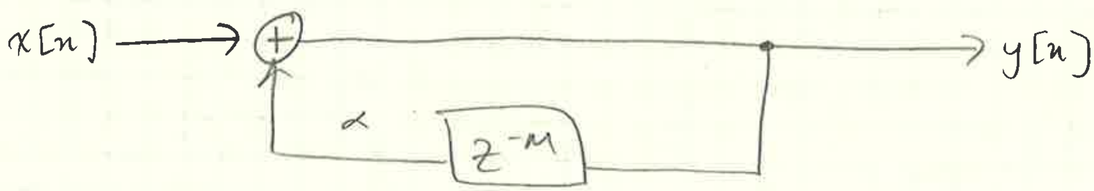


- Leaky Integrator, phase response



Phase is sufficiently linear where it matters!

Karplus - Strong revisited, again!



$$y[n] = \alpha y[n-M] + x[n]$$

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \dots, \bar{x}[M-1]}_{1^{st}\text{ period}}, \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \dots, \alpha\bar{x}[M-1]}_{2^{nd}\text{ period}}, \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \dots}_{\dots}$$

- DTFT of KS signal, using the convolution theorem

Key observation:

$$y[n] = \bar{x}[n] * w[n], \quad w[n] = \begin{cases} \alpha^k, & n = kM \\ 0, & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \bar{X}(e^{j\omega})\, W(e^{j\omega})$$