Example : Calculate time complexity for $T(n) = 8T(n/2) + n^2$

Solution: For the given recurrence relation,

$$f(n) = n$$

$$a = 8, \quad b = 2$$

$$\log_b a = \log_2 8 = 3$$

$$\Rightarrow n^{\log_b a} = n^3$$

$$n^2 < n^3$$

$$\Rightarrow f(n) < n^{\log_b a} \quad \text{(Case 1)}$$

$$\text{For Case 2, Time complexity} = \Theta(n^{\log_b a})$$

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$$\Rightarrow \text{Time complexity} = \Theta(n^3)$$

Example : Calculate time complexity for $T(n) = 4T(n/2) + n^2$

Solution: For the given recurrence relation,

$$f(n) = n$$

$$a = 4, \quad b = 2$$

$$\log_b a = \log_2 4 = 2$$

$$\Rightarrow n^{\log_b a} = n^2$$

$$n^2 = n^2$$

$$\Rightarrow f(n) = n^{\log_b a} \quad \text{(Case 2)}$$

$$\text{For Case 2, Time complexity} = \Theta(n^{\log_b a} log\ n)$$

$$\Rightarrow \text{Time complexity} = \Theta(n^2 log\ n)$$

Example : Calculate time complexity for $T(n) = 2T(n/4) + n^{0.5}$

Solution: For the given recurrence relation,

$$f(n) = n^{0.5}$$

$$a = 2, \quad b = 4$$

$$\log_b a = \log_4 2 = 0.5$$

$$\Rightarrow n^{\log_b a} = n^{0.5}$$

$$n^{0.5} = n^{0.5}$$

$$\Rightarrow f(n) = n^{\log_b a} \quad \text{(Case 2)}$$

$$\text{For Case 2, Time complexity} = \Theta(n^{\log_b a} log\ n)$$

$$\Rightarrow \text{Time complexity} = \Theta(n^{0.5} log\ n)$$

Example : Calculate time complexity for $T(n) = 2T(\sqrt{n}) + 1$

Solution:

$$\text{Let } n = 2^m$$

$$\Rightarrow \log n = m \text{ and } \sqrt{n} = 2^{m/2}$$

$$\Rightarrow T(2^m) = 2T(2^{m/2}) + 1$$

$$\text{Let } T(2^m) = S(m)$$

$$\Rightarrow T(2^{m/2}) = S(m/2)$$

$$\text{New equation: } S(m) = 2S(m/2) + 1$$

$$\text{For the above equation, } f(m) = 1$$

$$\text{where } a = 2, \quad b = 2$$

$$\Rightarrow m^{\log_b a} = m^{\log_2 2} = m$$

$$1 < m$$

$$\Rightarrow f(m) < m^{\log_b a} \quad \text{(Case 1)}$$

$$\text{For Case 1, Time complexity} = \Theta(m^{\log_b a})$$

$$\Rightarrow \text{Time complexity} = \Theta(m)$$

$$\Rightarrow \text{Time complexity} = \Theta(log\ n)$$

Example : Calculate time complexity for $T(n) = 2T(\sqrt{n}) + log\ n$

Solution:

$$\text{Let } n = 2^m$$
$$\Rightarrow \log n = m \text{ and } \sqrt{n} = 2^{m/2}$$
$$\Rightarrow T(2^m) = 2T(2^{m/2}) + m$$
$$\text{Let } T(2^m) = S(m)$$
$$\Rightarrow T(2^{m/2}) = S(m/2)$$
$$\text{New equation: } S(m) = 2S(m/2) + m$$
$$\text{For the above equation, } f(m) = m$$
$$\text{where } a = 2, \quad b = 2$$
$$\Rightarrow m^{\log_b a} = m^{\log_2 2} = m$$
$$m = m$$
$$\Rightarrow f(m) = m^{\log_b a} \quad \text{(Case 2)}$$
$$\text{For Case 2, Time complexity} = \Theta(m^{\log_b a} log\ m)$$
$$\Rightarrow \text{Time complexity} = \Theta(m\ log\ m)$$
$$\Rightarrow \text{Time complexity} = \Theta(log\ n\ log\ Log\ n)$$

Example : Calculate time complexity for $T(n) = 2T(\sqrt{n}) + n$

Solution:

$$\text{Let } n = 2^m$$
$$\Rightarrow \log n = m \text{ and } \sqrt{n} = 2^{m/2}$$
$$\Rightarrow T(2^m) = 2T(2^{m/2}) + 2^m$$
$$\text{Let } T(2^m) = S(m)$$
$$\Rightarrow T(2^{m/2}) = S(m/2)$$
$$\text{New equation: } S(m) = 2S(m/2) + 2^m$$
$$\text{For the above equation, } f(m) = 2^m$$
$$\text{where } a = 2, \quad b = 2$$
$$\Rightarrow m^{\log_b a} = m^{\log_2 2} = m$$
$$2^m > m$$

$$\Rightarrow f(m) > m^{\log_b a} \quad \text{(Case 3)}$$

$$\text{For Case 3, Time complexity} = \Theta(f(m))$$

$$\Rightarrow \text{Time complexity} = \Theta(2^m)$$

$$\Rightarrow \text{Time complexity} = \Theta(n)$$

Example : Calculate time complexity for $T(n) = T(n/3) + 1$

Solution: For the given recurrence relation,

$$f(n) = 1$$

$$a = 1, \quad b = 3$$

$$\log_b a = \log_3 1 = 0$$

$$\Rightarrow n^{\log_b a} = n^0 = 1$$

$$1 = 1$$

$$\Rightarrow f(n) = n^{\log_b a} \quad \text{(Case 2)}$$

$$\text{For Case 2, Time complexity} = \Theta(n^{\log_b a} log\ n)$$

$$\Rightarrow \text{Time complexity} = \Theta(1 log\ n)$$

Example : Calculate time complexity for T(n) = T(n-1) + n

Solution: T(n) = T(n-1) + n

$$\Rightarrow T(n-1) = T(n-2) + n - 1$$

$$\Rightarrow T(n-2) = T(n-3) + n - 2$$

$$\vdots$$

$$\Rightarrow \ T(1) = T(0) + 1$$

$$\Rightarrow T(n) = T(0) + (1 + 2 + 3 + \cdots + n)$$
$$= T(0) + \frac{n(n+1)}{2}$$

$$\Rightarrow \text{Time complexity} = O(n^2)$$

Example : Calculate time complexity $T(n) = 8T(n/2) + n$

Solution: For the given recurrence relation,

$$f(n) = n$$
$$a = 8, \quad b = 2$$
$$\log_b a = \log_2 8 = 3$$
$$\Rightarrow n^{\log_b a} = n^3$$
$$n < n^3$$
$$\Rightarrow f(n) < n^{\log_b a} \quad \text{(Case 1)}$$
$$\text{For Case 1, Time complexity} = \Theta(n^{\log_b a})$$
$$\Rightarrow \text{Time complexity} = \Theta(n^3)$$

Example : Calculate time complexity in small o notation for f(n) = $8n^3 + n$

Solution: For the above equation,

$$\text{Let } g(n) = n^4$$
$$\text{For small-o notation,} \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{8n^3 + n}{n^4}$$
$$= \lim_{n\to\infty} \frac{8}{n} + \lim_{n\to\infty} \frac{1}{n^3}$$
$$= 0 + 0$$
$$= 0$$
$$\Rightarrow \text{Time complexity} = o(n^4)$$

Example : Calculate time complexity in small ω notation for f(n) = $8n^3 + n$

Solution: For the above equation,

$$\text{Let } g(n) = n^2$$
$$\text{For small-o notation,} \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{8n^3 + n}{n^2}$$
$$= \lim_{n\to\infty} \frac{8n}{1} + \lim_{n\to\infty} \frac{1}{n}$$
$$= \infty + 0$$
$$= \infty$$
$$\Rightarrow \text{Time complexity} = \omega(n^2)$$

Example : Solve the recurrence relation $T(n) = 2T(n-1) + n$, T(1) = 1.

Solution: For the given recurrence relation,

$$\text{Characteristic equation: } r - 2 = 0$$
$$\Rightarrow r = 2$$
$$\Rightarrow T(n) = A * 2^n + Q_0 + Q_1 n$$

Substituting particular solution in original equation,

$$Q_0 + Q_1 n = 2(Q_0 + Q_1(n-1)) + n$$
$$\Rightarrow Q_0 + Q_1 n = 2Q_0 + 2Q_1(n-) + n$$

Comparing coefficients,

$$Q_0 = 2Q_0 - 2Q_1$$
$$\Rightarrow 2Q_1 = Q_0$$
$$\Rightarrow Q_0 = -2$$
$$Q_1 = 2Q_1 + 1$$
$$\Rightarrow Q_1 = -1$$
$$\Rightarrow T(n) = A * 2^n - 2 - n$$
$$T(1) = A * 2^1 - 2 - 1 and T(1) = 1$$
$$\Rightarrow A = 2$$
$$T(n) = 2 * 2^n - 2 - n$$

Question : Solve the recurrence relation $S(k) - S(k-1) - 6S(k-2) = -30$, S(0) = 20 and S(1) = -5.

Solution: For the given recurrence relation,

$$\text{Characteristic equation: } r^2 - r - 6 = 0$$
$$\Rightarrow (r-3)(r+2) = 0$$
$$\Rightarrow r = 3, -2$$
$$\Rightarrow S(k) = A * 3^k + B(-2)^k + Q_0$$

Substituting particular solution in original equation,

$$Q_0 - Q_0 - 6Q_0 = -30$$
$$\Rightarrow -6Q_0 = -30$$

$$\Rightarrow Q_0 = 5$$

$$\Rightarrow S(k) = A * 3^k + B(-2)^k + 5$$

$$S(0) = A + B + 5 and S(0) = 20$$

$$\Rightarrow A + B = 15$$

$$S(1) = 3A - 2B + 5 and S(1) = -5$$

$$\Rightarrow 3A - 2B = -10$$

On solving, A = 4 and B = 11

$$\Rightarrow S(k) = 4 * 3^k + 11(-2)^k + 5$$

Question : Solve $a_r + 5a_{r-1} = 9, \quad a_0 = 6.$

Solution: For the given recurrence relation,

$$\text{Characteristic equation: } r + 5 = 0$$

$$\Rightarrow r = -5$$

$$\Rightarrow a_n = A * (-5)^n + Q_0$$

Substituting the particular solution in the original equation,

$$Q_0 + 5Q_0 = 9$$

$$\Rightarrow 6Q_0 = 9$$

$$\Rightarrow Q_0 = 1.5$$

$$\Rightarrow a_n = A * (-5)^n + 1.5$$

$$\text{Using the initial condition, } a_0 = A + 1.5 \quad \text{and} \quad a_0 = 6$$

$$\Rightarrow A = 4.5$$

$$\Rightarrow a_n = 4.5 * (-5)^n + 1.5$$

Question : Find particular solution

$$f(n) = 5f(n-1) + 6f(n-2)$$
$$= 3n^2 - 2n + 1$$

Solution : For the above recurrence relation,

$$\text{Characteristic equation: } r^2 - 5r - 6 = 0$$

$$\Rightarrow (r-6)(r+1) = 0$$

$$\Rightarrow r = 6, -1$$

$$\Rightarrow f(n) = A*6^n + B(-1)^n + Q_0 + Q_1 n + Q_2 n^2$$

Substituting particular solution in original equation,

$$(Q_0 + Q_1 n + Q_2 n^2) - 5(Q_0 + Q_1(n-1) +$$

$$Q_2(n-1)^2) - 6(Q_0 + Q_1(n-2) + Q_2(n-2)^2) = 3n^2 - 2n + 1$$

Comparing coefficients,

$$-10Q_2 = 3$$

$$\Rightarrow Q_2 = -3/10$$

$$34Q_2 - 10Q_1 = -2$$

$$\Rightarrow Q_1 = -82/100$$

$$-29Q_2 + 17Q_1 - 10Q_0 = 1$$

$$\Rightarrow Q_0 = -1317/100$$

Particular solution,

$$= -\frac{1317}{100} - \frac{82}{100}n - \frac{3}{10}n^2$$

Question : Calculate time complexity $T(n) = 8T(n/2) + n$ using change variable method.

Solution: For the given recurrence relation,

$$Let n = 2^k$$

$$\Rightarrow T(2^k) = 4T(2^k/2) + 2^k$$

$$Let T(2^k) = T(k)$$

$$\Rightarrow T(k) = 4T(k-1) + 2^k$$

$$\Rightarrow T(k-1) = 4T(k-2) + 2^{k-1}$$

$$\Rightarrow 2T(k-1) = 8T(k-2) + 2^k$$

$$\Rightarrow T(k) - 6T(k-1) + 8T(k-2) = 0$$

Characteristic equation,

$$r^2 - 6r + 8 = 0$$

$$\Rightarrow (r-4)(r-2) = 0$$

$$\Rightarrow r = 2, 4$$

$$\Rightarrow T(k) = A*2^k + B*4^k$$

$$\Rightarrow T(k) = An + Bn^2$$

$\Rightarrow$ Time complexity = $O(n^2)$