Applied Estimation Lab 1—Extended Kalman Filter

Michal Staniaszek

November 24, 2013

1 Part I—Preparatory Questions

1.1 Kalman Filter

1. A control is something that is applied to the system to modify its state. The control can be controlled by us, or it can be another process which we model, but have no control over. A measurement is information that we receive about the state of the environment based on some sensor reading, or other measuring device, which does not necessarily give direct information about the state of the system. The state is a set of parameters which represent the system that we are working with. The state is what we are trying to estimate the evolution of, given the controls and measurements received.

2. It is not possible for the uncertainty to increase during an update. It naturally increases when a control is applied due to the uncertainty of the state transition, but since information is *gained* when we receive a measurement, the uncertainty will always decrease. This can be verified by looking at the covariance matrix update equation:

$$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$$

   We know that the product $K_t C_t$ will produce a matrix which is positive semidefinite, and therefore subtracting this from the identity will result in the multiplication of $\bar{\Sigma}_t$ by a fractional value. Thus, Σ_t will have smaller values in it than $\bar{\Sigma}_t$. The smaller the values in Σ_t, the tighter the Gaussians, and the lower the uncertainty. It should be noted that the estimate is different from the true state. Even if the uncertainty in the estimate goes down, the error between the true state and the estimated state can increase due to a sub-optimal model.

3. The weighting between measurements and belief is the Kalman gain K_t. The gain is computed and used in the update step to modify both the new mean μ_t and covariance matrix Σ_t. The measurement update is done using

$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t)$$

   The part of the measurement z_t added to μ_t is proportional to K_t, and therefore K_t defines how much it is taken into consideration. The size of K_t is influenced by $\bar{\Sigma}$ and Q_t, the predicted covariance and measurement noise respectively, which means that the size of the uncertainty, and the unreliability of measurements affect the gain.

4. The effect of changes in Q_t, the measurement noise, can be investigated by looking at the computation of K_t:

$$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

   It is easier to see the effects of a large Q_t in the scalar case:

$$\begin{aligned} k_t &= c_t \bar{\sigma}_t (c_t^2 \bar{\sigma}_t + q_t)^{-1} \\ &= \frac{c_t \bar{\sigma}_t}{c_t^2 \bar{\sigma}_t + q_t} \end{aligned}$$

   As q_t tends to infinity, we get

$$k_t = \lim_{q_t \to \infty} \frac{c_t \bar{\sigma}_t}{c_t^2 \bar{\sigma}_t + q_t} \approx 0$$

   So, as q_t increases, the Kalman gain tends towards zero, meaning that measurements will not be considered at all in the update step. This is also the case in the non-scalar case.

5. For the measurements to have an increased effect, the Kalman gain must increase. As seen above, as $q_t \to \infty$, $k_t \to 0$. It is easy to see that as $q_t \to 0$, $k_t \to \frac{1}{c_t}$. This indicates that reducing the uncertainty Q_t in the measurements will increase the gain. In addition, the value of c_t (or structure of C_t) may also have some effect on the gain.

6. During prediction, the belief uncertainty increases. Intuitively, this happens because we are uncertain about the state transitions that are being made, depending on the uncertainty ϵ_t, modeled by R_t. Because we could transition to any number of states after making a transition, there is no way that we could *decrease* the uncertainty.

$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

   Since Σ_{t-1} is positive semidefinite, multiplying it by A_t, which has the same property, results in larger values in the matrix. Adding the measurement covariance R_t further increases this uncertainty.

7. If the true distributions of the measurement and state transition noise are indeed Gaussians as we have assumed, then it follows that there can be no distribution that better estimates the true distribution. It can be shown that if instead of using the means predicted by the Gaussians we use a different mean, then a worse estimate results.

8. In a MLE, we assume that nothing is known about the distribution of the initial state, which means that the distribution is uniform. In a MAP, a prior is used in the computation of the posterior. Generally, when using a Kalman filter, we do not know anything about the initial distribution, and so ideally we would use a uniform distribution. However, since the KF uses Gaussian representations, we cannot do this. As a result, the KF is really a MAP estimator, but in some sense it could be said to be both.

1.2 Extended Kalman Filter

9. The extended Kalman filter is an extension of the Kalman filter to systems with non-linear state transitions. This extension destroys some of the good properties of the Kalman filter, but is used much more in practice because most systems are non-linear in some way. In the EKF, the state transition matrix A_t is replaced with a Jacobian G_t based on the linearisation of the state transition function $g(u_t, \mu_{t-1})$, which approximates the effect of noise on the non-linear state transition as a linear function. The measurement equation is also modified to use a Jacobian H_t instead of C_t, again representing the linearised measurement function.

10. The EKF is not guaranteed to converge. Divergence can be caused by various factors, such as an incorrect model, measurement correlations or bias, disturbances, or problems with the matrix structures. The convergence also depends on the linearisation. If the state transition function is highly nonlinear, then the linearisation may be inaccurate when the uncertainty is large due to variation in the nonlinear function.

11. To reduce the divergences, we can increase the uncertainty parameters that are being used. This means that the filter will be less certain about the evolution of the system, and less likely to diverge because of an estimate that has relatively low uncertainty, but is in fact incorrect. If the divergence is due to bad data association, then we can change the matching threshold that is being used.

1.3 Localisation

12. If the robot is completely unsure of its location, then the prior is a Gaussian with a high covariance. After seeing the landmark, all that can be deduced is that the robot must be somewhere on the circumference of the circle of radius r centred on the landmark, with the added Gaussian noise creating a ring, with probabilities decreasing towards the edges of the ring. Because it is not possible to represent this ring shape with a Gaussian, the estimate will likely place the mean on top of the landmark, with a large covariance to cover the whole ring. A single landmark is not enough to actually localise, since it does not give enough information.

13. If the measurement also includes a bearing, we can deduce that the robot must be facing in a specific direction, but we do not gain any additional information about its position, as that is determined by the range measurement alone in the situation with which we are presented. The distribution of possible positions is still a ring, but we now know that the orientation of the robot is dependent on its position on the ring.

14. Since we have good motion estimation, we know approximately how far we will move in the direction of motion. However, because the initial bearing is uncertain, the uncertainty on the Gaussian perpendicular to the direction of motion would be large, indicating that in actual fact the position could be quite far from the mean in that direction. The ideal distribution would be banana shaped (depending on the initial uncertainty), but since we must represent it as a Gaussian, it must be approximated as best as possible. The heading will be correlated with the position on the arc.

15. There are numerous reasons for why an EKF update could go wrong due to a measurement. The most obvious of these is a spurious sensor reading, which appears to detect

a feature when it is not in fact there. Issues could also be caused by data processing errors which again result in the detection of a feature which does not exist. If such a measurement was received by the EKF, particularly one using a sequential update, then if it is taken into account then the update becomes inconsistent; something has been measured and taken into account, modifying the belief, when in fact it should have been discarded. Since the EKF uses Gaussians, if the landmark is not unique, we are not necessarily able to update the posterior to reflect the true state. All that we can say is that we are near some landmark, and we must compromise to choose only the one which has the maximum likelihood at the time. If we started off with very little information about bearing, then the landmark that we end up choosing as the one we thing we are close to could be the wrong one, and the posterior will be moved to an incorrect location. In general, because it is not possible to represent the posterior with a Gaussian, the linearisation is not accurate and this could lead to a bad update.

2 Part II—Matlab Exercises

2.1 Kalman Filter

1. The dimensions of ϵ_k correspond to the number of parameters required to represent the state x_t. Each element of ϵ_k represents the noise inherent to that parameter of the state during the state transition. In the example of the car, ϵ_k is a 2×1 vector, with noise for both position and velocity. δ_k is the noise that is present in the measurement. Its dimensions depend on the length of the measurement vector. If we consider each element of this vector to be a measurement from a different sensor, it is natural to have different noise for each. In the car example, δ_k is a scalar value, as there is only one thing that is being measured, the position of the car. In the scalar case, a Gaussian, or normal distribution is characterised by a mean μ and a variance σ^2 around that mean, and is generally represented by $\mathcal{N}(\mu, \sigma^2)$. In the vector case, μ becomes a vector, and σ^2 a matrix. The notation for μ is the same, but the variance is represented by the *covariance matrix* Σ, and as such a multivariate Gaussian is represented by $\mathcal{N}(\mu, \Sigma)$. A white Gaussian is one with zero mean and a diagonal covariance matrix, such that the noise in one parameter is not correlated with any other.

2. x A 2×1 vector representing the true state of the system. Contains the position and velocity of the car at each time step.

   $\hat{x}$ A 2×1 vector representing the state of the car estimated by the KF. Contains an estimate of the position and velocity of the car at each time step. This is the state vector that is used by the KF—we have no access to the true state.

   P A 2×2 covariance matrix representing the uncertainty of the current EKF estimate. Contains either the predicted covariance $\bar{\Sigma}$ or the updated covariance Σ, depending on which point in the code execution is at. Used to compute both the Kalman gain K and the covariance after prediction or update.

   G A multiplier on the noise model for the state transitions to get the correct matrix dimensionality.

   D A multiplier on the noise model for the measurements to get the correct matrix dimensionality.

Q A 2×2 matrix. The noise model for the state transition. This indicates the estimates that we have made of the true noise in the system for each parameter in the state. In the ideal case, this matrix contains the values of $wStdP^2$ and $wStdV^2$. If the noise is lower than the true noise, then the estimator will be optimistic, and if the noise is higher then it will be pessimistic. Both of these can cause problems, either with overconfidence in bad estimates, or too little confidence in quite a good estimate.

R A scalar value. The noise model for the measurements. Indicates an estimate of the true noise on the measurements that we will receive from our simulated sensor. The ideal value for this is the actual noise on the measurements, which is represented by $vStd$.

$wStdP$ The true variance of the noise in the position. The noise is added to the system by generating a normally distributed random number with a zero mean and variance $wStdP$.

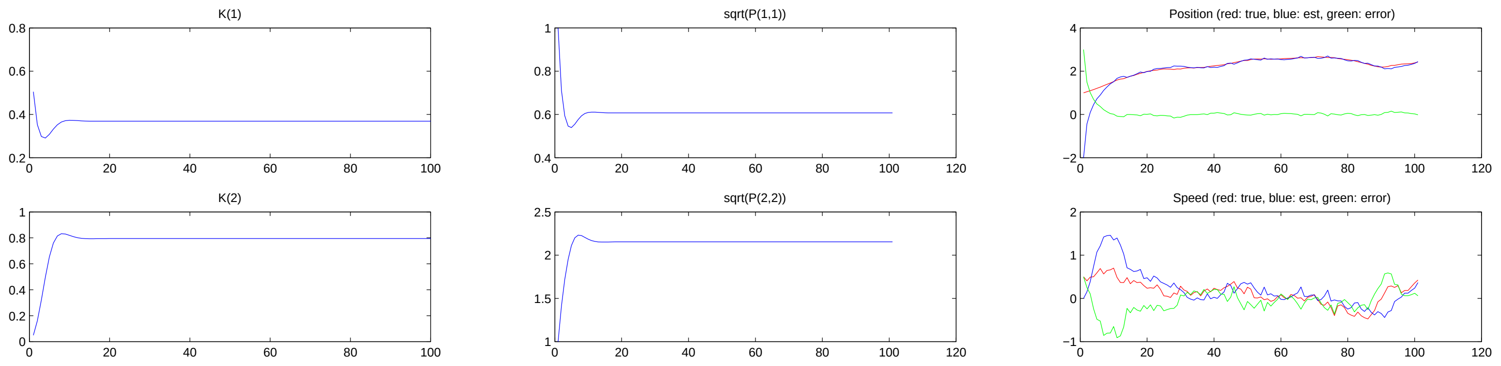
$wStdV$ The true variance of the noise in the velocity. The noise is added to the system by generating a normally distributed random number with zero mean and variance $wStdP$.

$vStd$ The true variance of the noise in the measurement. This is added to the measurement at each time step by generating a normally distributed random number with zero mean and variance of $vStd$.

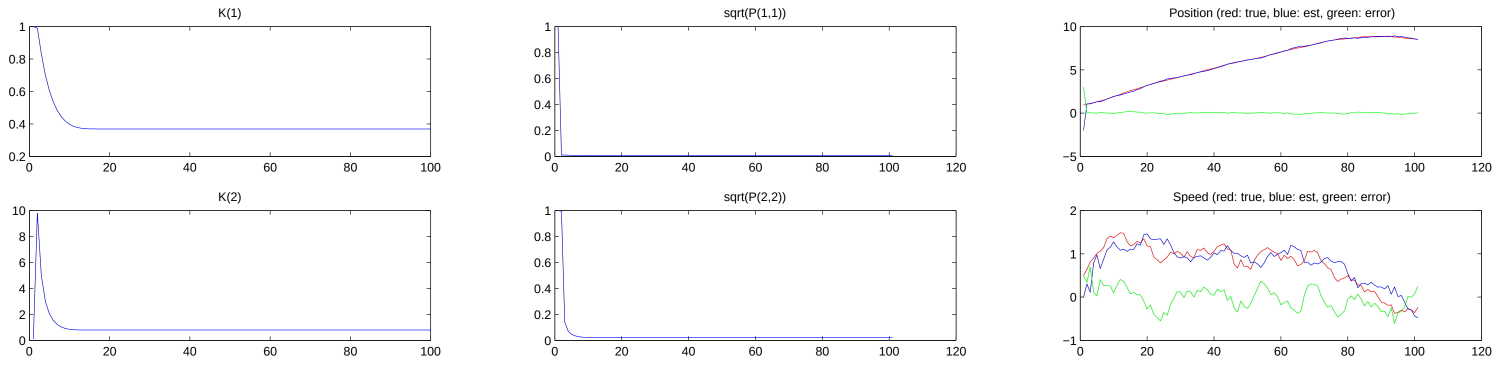
u The control applied to the system at each time step, that is, the instantaneous acceleration of the car, which is assumed to be constant between time steps. Setting this to zero will mean that the system is driven by the noise $wStdP$.

PP Stores the values of the covariance matrix after the update step at each time step such that they can be plotted later to examine the evolution of the covariance over time.

3. In Figure 2(b), we increase Q, giving a higher process noise estimate. From this we would expect that our measurements would be trusted more, as our motion is uncertain. This is confirmed by comparing the Kalman gain with Figure 2(a). We see that the steady state gain for both range $K1$, and speed $K2$ is higher, so we give the measurements higher weight. As a result of the higher process noise, the tracking of the speed suffers, it is much more difficult to predict. With low process noise, we expect measurements to receive a lower weight, as our process is more predictable. This is clear from the low gain in both position and velocity in Figure 2(c). Additionally, we have a lower steady state covariance as we are more certain about how the system evolves. As a result of the low gain, tracking of the true state is smoother, and has a slight lag. R represents the measurement noise model. A higher measurement noise means that we put less weight on our measurements as they are unreliable. High measurement noise keeps the gain low, and has a large effect on the convergence time of the system. Tracking of the true state is smooth, as the measurements do not force the estimate to move much (Figure (d)). With low measurement noise, the gain is high, and so measurements have a large effect on the system. This results in a rapidly changing estimate of both the position and the speed. When both Q and R are high, we are uncertain about both the process and the measurements. As a result, the Kalman gain is low, and the steady state covariance is high. The result is slow convergence to the true state, as the update is more cautious about updates. With both low, we have more contribution from the



(a) High noise: $Q \times 100$, $R \times 100$

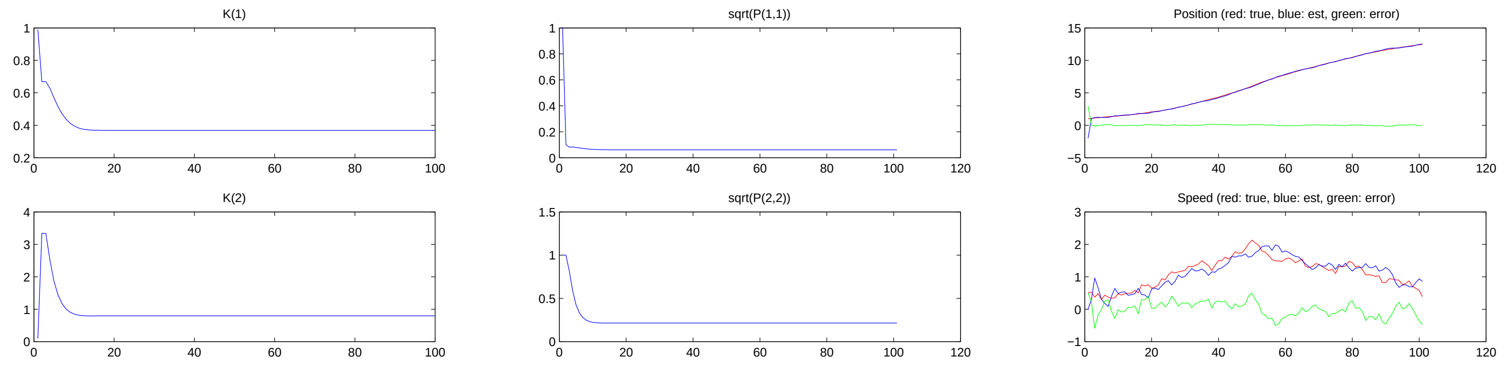


(b) Low noise: $Q/100$, $R/100$

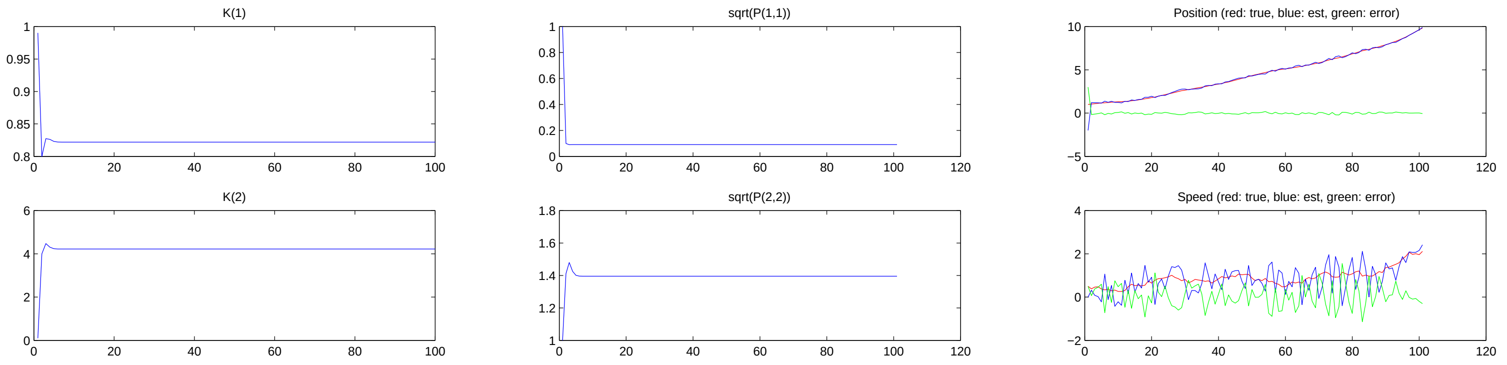
Figure 1: Graphs for different process and measurement noise models, from left to right the Kalman gain, covariance and error. By default, $Q = \texttt{diag}(0.0001, 0.01)$, $R = 0.01$. Here, we change both Q and R by the same proportion.

measurements, and a very low steady state covariance. In both cases, the tracking of the speed does not undergo any significant change, which can be seen in Figure 1.

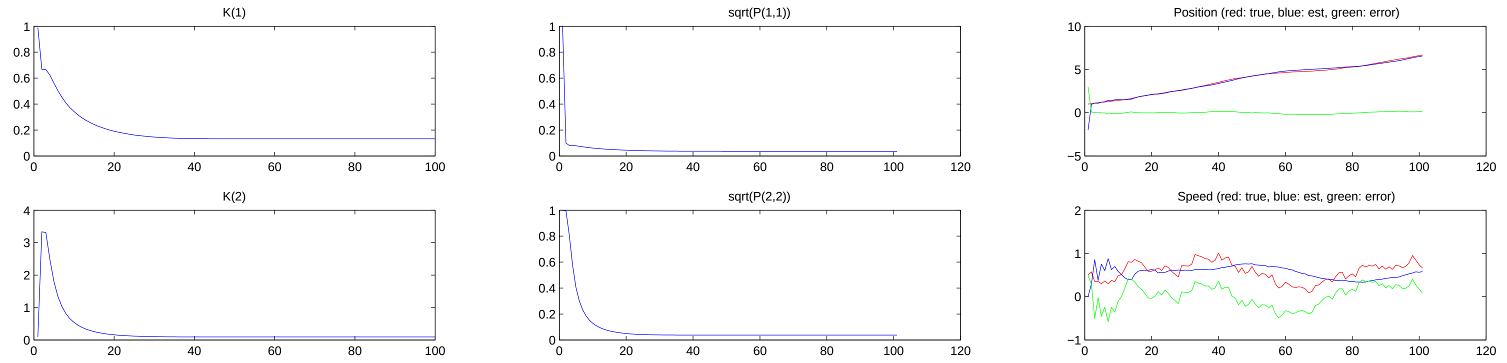
4. Changes in the values in P affect the belief state in the initial position. If P is very high, the initial uncertainty in our estimated position $\hat{x}$ is very large. When this is the case, the Kalman gain rises rapidly to take into account the measurements, and the systems converges in around the same number of time steps as with some lower values of P. Particularly noticeable is a peak in the gain for the velocity at time step 2, after which the gain decays quickly, indicating that the state uncertainty is decreasing, so it can be taken into account more. For $K > 100$, the maximum value of the peak does not exceed 10. If P is low, then we believe that our initial $\hat{x}$ is a very good estimate of the true state. This can have an effect on the convergence of the system—if the true state is actually far from the initial estimate of $\hat{x}$, then the state transition is trusted more than it should be, leading to a longer convergence time, due to the need to compensate for the excessive initial certainty.



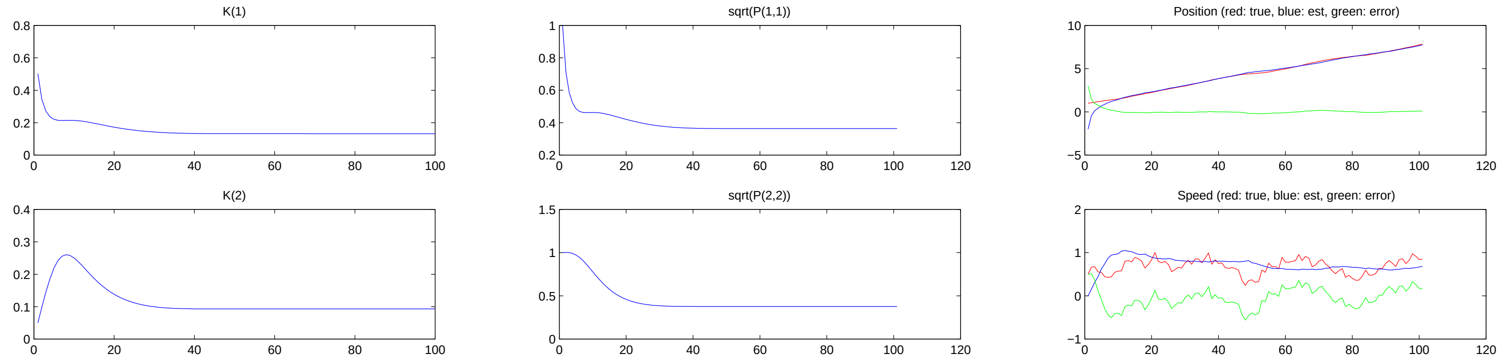
(a) Normal: $Q =$`diag`$(0.0001, 0.01)$, $R = 0.01$.



(b) High process noise: $Q \times 100$, R

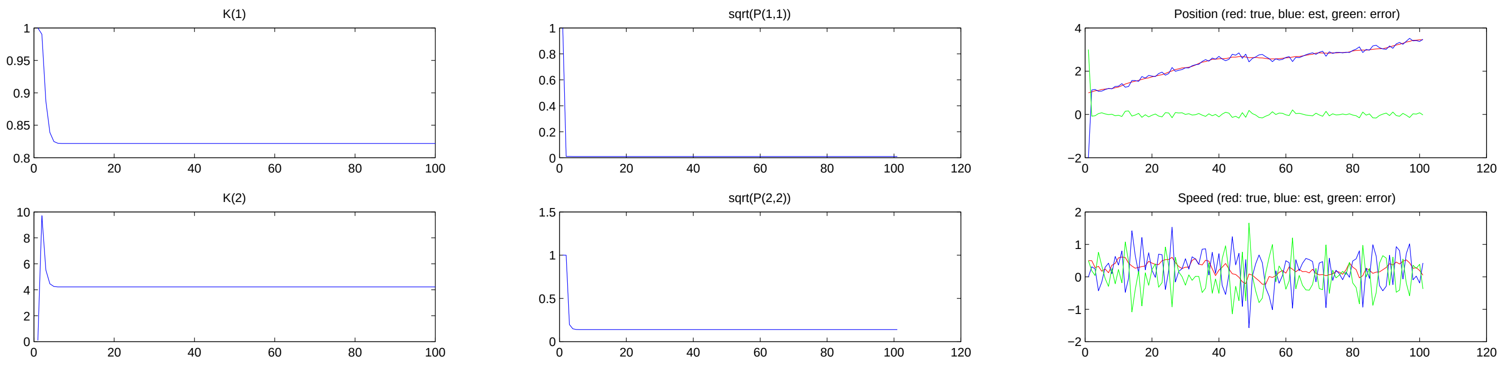


(c) Low process noise: $Q/100$, R



(d) High measurement noise: Q, $R \times 100$

Figure 2: Graphs for different process and measurement noise models, from left to right the Kalman gain, covariance and error. By default, $Q =$`diag`$(0.0001, 0.01)$, $R = 0.01$.



(e) Low measurement noise: $Q, R/100$

Figure (cont.) 2: Graphs for different process and measurement noise models, from left to right the Kalman gain, covariance and error. By default, $Q =$`diag`$(0.0001, 0.01)$, $R = 0.01$.

2.2 EKF Localisation

5. The prediction step is used to generate the prior $\overline{bel}(x_t)$, which is our belief about the state of the robot before the measurements are taken into account. The update step integrates $\overline{bel}(x_t)$ and the probability of making the observed measurement.

$$\overline{bel}(x_t) = \int p(x_t \mid u_{1:t}, x_{t-1}) bel(x_{t-1}) dx_{t-1} \tag{1}$$

Prediction step

$$bel(x_t) = \eta p(z_t \mid x_t, M) \overline{bel}(x_t) \tag{2}$$

Update step

$$p(x_t \mid u_{1:t}, z_{1:t}, \bar{x}_0, M) = \underbrace{\eta p(z_t \mid x_t, M) \underbrace{\int p(x_t \mid u_t, x_{t-1}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}, \bar{x}_0, M) dx_{t-1}}_{\text{Prediction Step}}}_{\text{Update step}} \tag{3}$$

In (3), the integral is equivalent to $\overline{bel}(x_t)$, and forms the prediction step. Multiplying this by the normalisation constant and measurement probability $\eta p(z_t \mid x_t, M)$ gives the update step.

6. The assumption that measurements are independent of each other is valid, so long as some conditions are fulfilled. If we know that the noise is uncorrelated, then the measurements are unbiased, as the noise on each measurement is white. This means that we can expect measurements to be evenly distributed between values which are too small and too large. If this holds, then as the number of measurements tends towards infinity, we will be closer and closer to the true measurement value. If the sensor noise is correlated, then we cannot make an independence assumption. In general, the measurements at a single time step have a single dependency, the true state at that time step. However, each measurement is unaffected by the previous ones—assuming the sensor behaves in the same way for each measurement.

7. The value of δ_m is within the bounds $[0, 1]$. The value of δ_m defines the confidence level in the sensor return values. A high value indicates high confidence in the measurements,

which means that only values at the very ends of the tails will be rejected. A low value will result in more rejected associations, as there is less confidence in the measurements, so values closer to the centre of the distribution will also be rejected. If all measurements are reliable, only coming from actual features, then we should set λ_m to be high, such that only measurements at the extreme tails will be rejected. If the measurements are unreliable, then we should use a lower value of λ_m so as to ensure that only those measurements in which we have a high confidence are kept and integrated into our belief.

8. The sequential update has problems with noisy measurements, as they can cause inconsistencies to arise in the belief state. That is, if we receive a noisy measurement, and believe that it is a good measurement, we will integrate it into our belief, and the mean and covariance will be adjusted accordingly, with the mean moving towards the measurement, and the covariance being reduced. However, in actual fact, we have not really decreased the covariance, as the measurement is noisy—we have moved the mean into a position which could actually be *less* likely than the previous estimate. As a result, all subsequent measurements are affected, and the mean jumps around, with noise having a large effect. This effect can be mitigated by using the batch update, which compensates for this by averaging the noise by doing a single update for all of the measurements.

9. In the pseudocode, we perform a computation of the expected measurement $\hat{z}_t^k$, the linearised measurement model H_t^k, and the measurement uncertainty S_t^k. These computations need to be done only a single time for each landmark k, but they are instead done repeatedly for each observation i. This results in ki computations of these values, whereas only k computations are needed. Pre-computing these values and then performing the computation of $\nu_t^{i,k}$, $\nu_t^{i,k}$, and $\psi_t^{i,k}$ will result in a significant reduction in the computation time required, especially if there are a large number of observations.

10. In the batch algorithm, $\bar{\nu}_t$ and $\bar{H}_t$ have dimensions $2i \times 1$ and $2i \times 3$ respectively, where i is the number of observations, compared to 2×1 and 2×3 for the sequential algorithm. This indicates that in the sequential update, each observation is factored into the update individually, whereas in the batch case, the observations are grouped together and then K_t, μ_t and Σ_t are computed in one go with the single large matrix. As a result, the batch update is more expensive than the multiple sequential updates, as the matrix dimensions are much larger. The inversion, which has approximately $\mathcal{O}(n^3)$ complexity in the number of elements, becomes particularly costly.

3 Results

Plots in this section are colour coded. Green is the ground truth, the actual state of the robot. Red is the state of the robot estimated by the EKF. Blue is the state of the robot based on the odometry readings.

The results of running the EKF with sequential update on the first map can be seen in Figure 3. We can see from the covariance graph that between 200 and 300 time steps there are two peaks in the uncertainty. This roughly corresponds to the two corners on the right side of the map. Since this part of the map is symmetric, the uncertainty increases as it is possible to be in either of those two places. The overall error is very small, indicating that the EKF performs very well in this example.

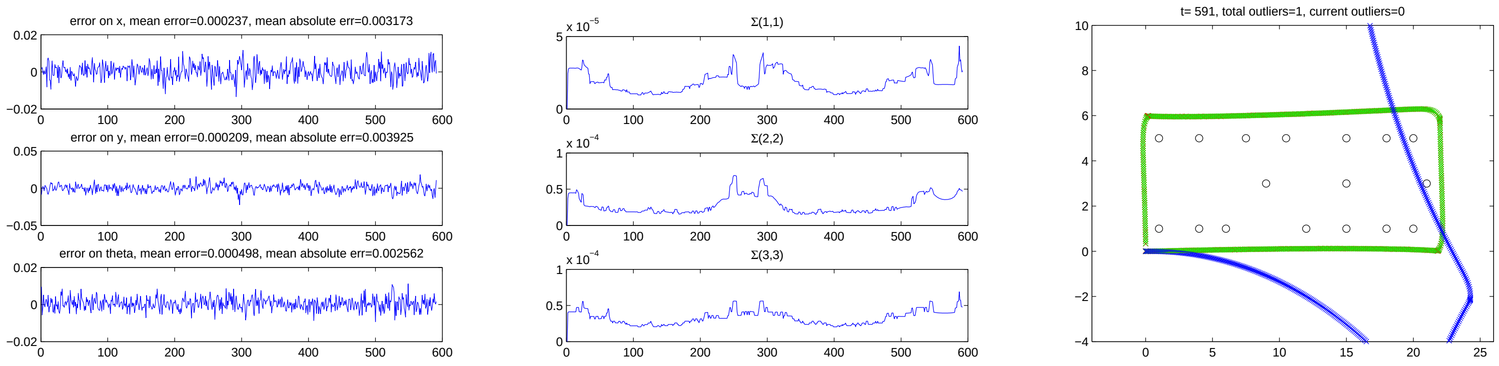


Figure 3: A run of the EKF on the first test map. This gives an indication of the poor quality of odometry data. The noise models used were R =`diag`$(0.01^2, 0.01^2, 0.0175^2)$, Q =`diag`$(0.01^2, 0.0175^2)$.

On the second map, seen in Figure 4, we have the problem of outliers in the measurements, coming from spurious measurements or sensor error. While the odometry is of bad quality as always, it is very important that we properly determine the uncertainty, as if we are overly reliant on measurements, the filter will diverge, with large deviations from the true path of the robot. With an 100 times larger uncertainty in the process model, the filter diverges very quickly and gets stuck at the bottom of the ellipse. With 10 times larger uncertainty, the tracking is reasonable until the top left part, where the filter gets stuck. When the noise model is correctly chosen, outliers are correctly detected and the filter tracks the true state well. We see an interesting pattern in the covariance, which corresponds to encountering each of the landmarks. Each time we see one, the uncertainty increases, as the position and heading are somewhat correlated with the measured position of the landmark.

The third map has no outliers, but instead has problems with high noise measurements affecting the sequential update. Figure 5(a) and (b) show the runs for sequential and batch updates of the EKF respectively. Using the sequential update, the noisy measurements result in a large deviation from the true state. In contrast, with the batch update there is only slight deviation from the true state. We again see a sawtooth pattern in the covariance, corresponding to each landmark being encountered. The x and y covariances appear to follow a sine curve. This may be as a result of the computation of the position from the measurements, which include sinusoids.

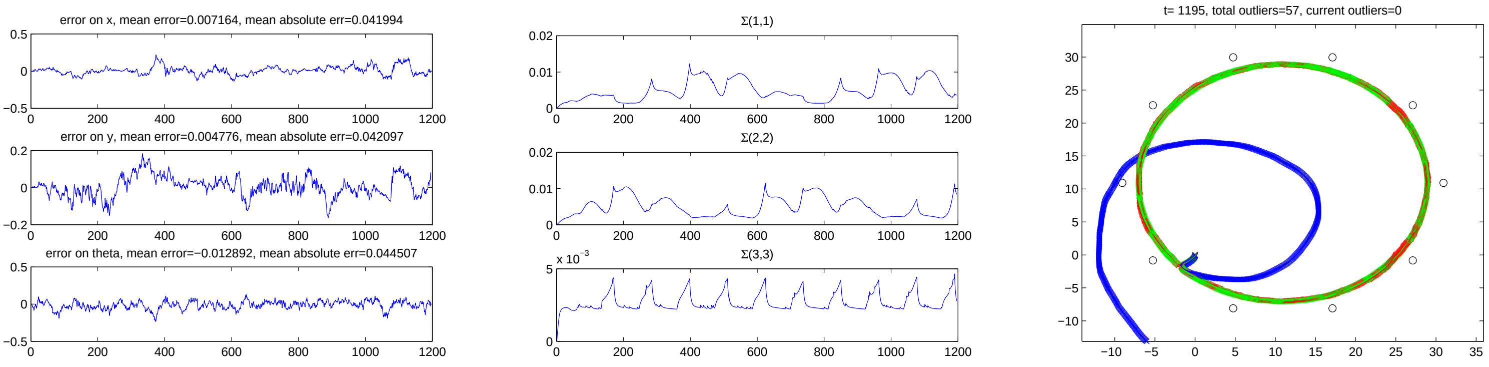
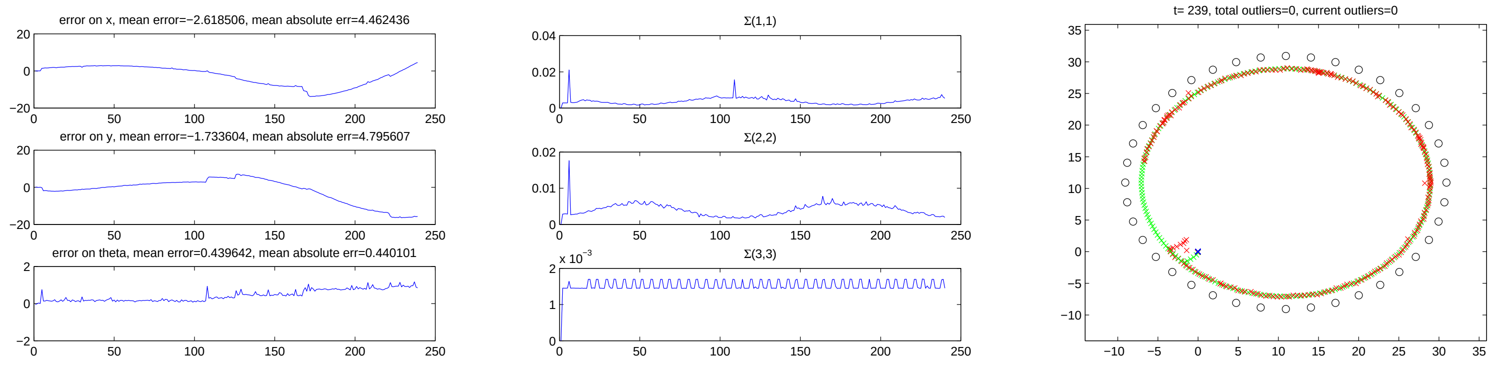
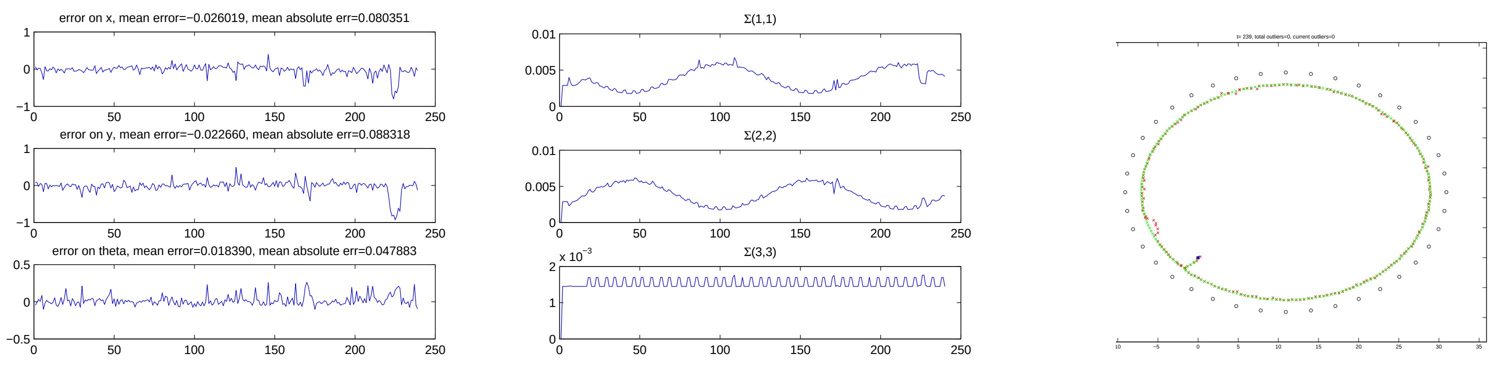


Figure 4: Results of running the EKF on the second test map. The noise models used were $R =$`diag`$(0.01^2, 0.01^2, 0.0175^2)$, $Q =$`diag`$(0.2^2, 0.2^2)$. In this case, the noise models are important as outliers can cause divergence if not correctly chosen.



(a) Sequential update



(b) Batch update

Figure 5: Results of running the EKF on the third test map. The noise models used were $R =$`diag`$(1, 1, 1)$, $Q =$`diag`$(0.1^2, 0.1^2)$.