# Applied Estimation Lab 1—Extended Kalman Filter

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November 18, 2013

## 1 Part I—Preparatory Questions

### 1.1 Kalman Filter

1. A control is something that is applied to the system to modify its state. The control can be controlled by us, or it can be another process which we model, but have no control over. A measurement is information that we receive about the state of the environment based on some sensor reading, or other measuring device, which does not necessarily give direct information about the state of the system. The state is a set of parameters which represent the system that we are working with. The state is what we are trying to estimate the evolution of, given the controls and measurements received.

2. It is not possible for the uncertainty to increase during an update. It naturally increases when a control is applied due to the uncertainty of the state transition, but since information is *gained* when we receive a measurement, the uncertainty will always decrease. This can be verified by looking at the covariance matrix update equation:

$$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$$

   We know that the product $K_t C_t$ will produce a matrix which is positive semidefinite, and therefore subtracting this from the identity will result in the multiplication of $\bar{\Sigma}_t$ by a fractional value. Thus, $\Sigma_t$ will have smaller values in it than $\bar{\Sigma}_t$. The smaller the values in $\Sigma_t$, the tighter the Gaussians, and the lower the uncertainty. It should be noted that the estimate is different from the true state. Even if the uncertainty in the estimate goes down, the error between the true state and the estimated state can increase due to a sub-optimal model.

3. The weighting between measurements and belief is the Kalman gain $K_t$. The gain is computed and used in the update step to modify both the new mean $\mu_t$ and covariance matrix $\Sigma_t$. The measurement update is done using

$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t)$$

   The part of the measurement $z_t$ added to $\mu_t$ is proportional to $K_t$, and therefore $K_t$ defines how much it is taken into consideration. The size of $K_t$ is influenced by $\bar{\Sigma}$ and $Q_t$, the predicted covariance and measurement noise respectively, which means that the size of the uncertainty, and the unreliability of measurements affect the gain.

4. The effect of changes in $Q_t$, the measurement noise, can be investigated by looking at the computation of $K_t$:

   $$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

   It is easier to see the effects of a large $Q_t$ in the scalar case:

   $$\begin{aligned} k_t &= c_t \bar{\sigma}_t (c_t^2 \bar{\sigma}_t + q_t)^{-1} \\ &= \frac{c_t \bar{\sigma}_t}{c_t^2 \bar{\sigma}_t + q_t} \end{aligned}$$

   As $q_t$ tends to infinity, we get

   $$k_t = \lim_{q_t \to \infty} \frac{c_t \bar{\sigma}_t}{c_t^2 \bar{\sigma}_t + q_t} \approx 0$$

   So, as $q_t$ increases, the Kalman gain tends towards zero, meaning that measurements will not be considered at all in the update step. This is also the case in the non-scalar case.

5. For the measurements to have an increased effect, the Kalman gain must increase. As seen above, as $q_t \to \infty$, $k_t \to 0$. It is easy to see that as $q_t \to 0$, $k_t \to \frac{1}{c_t}$. This indicates that reducing the uncertainty $Q_t$ in the measurements will increase the gain. In addition, the value of $c_t$ (or structure of $C_t$) may also have some effect on the gain.

6. During prediction, the belief uncertainty increases. Intuitively, this happens because we are uncertain about the state transitions that are being made, depending on the uncertainty $\epsilon_t$, modeled by $R_t$. Because we could transition to any number of states after making a transition, there is no way that we could *decrease* the uncertainty.

   $$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

   Since $\Sigma_{t-1}$ is positive semidefinite, multiplying it by $A_t$, which has the same property, results in larger values in the matrix. Adding the measurement covariance $R_t$ further increases this uncertainty.

7. If the true distributions of the measurement and state transition noise are indeed Gaussians as we have assumed, then it follows that there can be no distribution that better estimates the true distribution. It can be shown that if instead of using the means predicted by the Gaussians we use a different mean, then a worse estimate results.

8. In a MLE, we assume that nothing is known about the distribution of the initial state, which means that the distribution is uniform. In a MAP, a prior is used in the computation of the posterior. Generally, when using a Kalman filter, we do not know anything about the initial distribution, and so ideally we would use a uniform distribution. However, since the KF uses Gaussian representations, we cannot do this. As a result, the KF is really a MAP estimator, but in some sense it could be said to be both.

## 1.2 Extended Kalman Filter

9. The extended Kalman filter is an extension of the Kalman filter to systems with non-linear state transitions. This extension destroys some of the good properties of the Kalman filter, but is used much more in practice because most systems are non-linear in some way. In the EKF, the state transition matrix $A_t$ is replaced with a Jacobian $G_t$ based on the linearisation of the state transition function $g(u_t, \mu_{t-1})$, which approximates the effect of noise on the non-linear state transition as a linear function. The measurement equation is also modified to use a Jacobian $H_t$ instead of $C_t$, again representing the linearised measurement function.

10. The EKF is not guaranteed to converge. Divergence can be caused by various factors, such as an incorrect model, measurement correlations or bias, disturbances, or problems with the matrix structures. The convergence also depends on the linearisation. If the state transition function is highly nonlinear, then the linearisation may be inaccurate when the uncertainty is large due to variation in the nonlinear function.

11. To reduce the divergences, we can increase the uncertainty parameters that are being used. This means that the filter will be less certain about the evolution of the system, and less likely to diverge because of an estimate that has relatively low uncertainty, but is in fact incorrect. If the divergence is due to bad data association, then we can change the matching threshold that is being used.

## 1.3 Localisation

12. If the robot is completely unsure of its location, then the prior is a Gaussian with a high covariance. After seeing the landmark, all that can be deduced is that the robot must be somewhere on the circumference of the circle of radius $r$ centred on the landmark, with the added Gaussian noise creating a ring, with probabilities decreasing towards the edges of the ring. Because it is not possible to represent this ring shape with a Gaussian, the estimate will likely place the mean on top of the landmark, with a large covariance to cover the whole ring. A single landmark is not enough to actually localise, since it does not give enough information.

13. If the measurement also includes a bearing, we can deduce that the robot must be facing in a specific direction, but we do not gain any additional information about its position, as that is determined by the range measurement alone in the situation with which we are presented. The distribution of possible positions is still a ring, but we now know that the orientation of the robot is dependent on its position on the ring.

14. Since we have good motion estimation, we know approximately how far we will move in the direction of motion. However, because the initial bearing is uncertain, the uncertainty on the Gaussian perpendicular to the direction of motion would be large, indicating that in actual fact the position could be quite far from the mean in that direction. The ideal distribution would be banana shaped (depending on the initial uncertainty), but since we must represent it as a Gaussian, it must be approximated as best as possible. The heading will be correlated with the position on the arc.

15. There are numerous reasons for why an EKF update could go wrong due to a measurement. The most obvious of these is a spurious sensor reading, which appears to detect

a feature when it is not in fact there. Issues could also be caused by data processing errors which again result in the detection of a feature which does not exist. If such a measurement was received by the EKF, particularly one using a sequential update, then if it is taken into account then the update becomes inconsistent; something has been measured and taken into account, modifying the belief, when in fact it should have been discarded. Since the EKF uses Gaussians, if the landmark is not unique, we are not necessarily able to update the posterior to reflect the true state. All that we can say is that we are near some landmark, and we must compromise to choose only the one which has the maximum likelihood at the time. If we started off with very little information about bearing, then the landmark that we end up choosing as the one we thing we are close to could be the wrong one, and the posterior will be moved to an incorrect location. In general, because it is not possible to represent the posterior with a Gaussian, the linearisation is not accurate and this could lead to a bad update.

# 2 Part II—Matlab Exercises

## 2.1 Kalman Filter

1. The dimensions of $\epsilon_k$ correspond to the number of parameters required to represent the state $x_t$. Each element of $\epsilon_k$ represents the noise inherent to that parameter of the state during the state transition. In the example of the car, $\epsilon_k$ is a $2 \times 1$ vector, with noise for both position and velocity. $\delta_k$ is the noise that is present in the measurement. Its dimensions depend on the length of the measurement vector. If we consider each element of this vector to be a measurement from a different sensor, it is natural to have different noise for each. In the car example, $\delta_k$ is a scalar value, as there is only one thing that is being measured, the position of the car. In the scalar case, a Gaussian, or normal distribution is characterised by a mean $\mu$ and a variance $\sigma^2$ around that mean, and is generally represented by $\mathcal{N}(\mu, \sigma^2)$. In the vector case, $\mu$ becomes a vector, and $\sigma^2$ a matrix. The notation for $\mu$ is the same, but the variance is represented by the *covariance matrix* $\Sigma$, and as such a multivariate Gaussian is represented by $\mathcal{N}(\mu, \Sigma)$. A white Gaussian is one with zero mean and a diagonal covariance matrix, such that the noise in one parameter is not correlated with any other.

2. $x$ A $2 \times 1$ vector representing the true state of the system. Contains the position and velocity of the car at each time step.

   $\hat{x}$ A $2 \times 1$ vector representing the state of the car estimated by the KF. Contains an estimate of the position and velocity of the car at each time step. This is the state vector that is used by the KF—we have no access to the true state.

   $P$ A $2 \times 2$ covariance matrix representing the uncertainty of the current EKF estimate. Contains either the predicted covariance $\bar{\Sigma}$ or the updated covariance $\Sigma$, depending on which point in the code execution is at. Used to compute both the Kalman gain $K$ and the covariance after prediction or update.

   $G$ A multiplier on the noise model for the state transitions.

   $D$ A multiplier on the noise model for the measurements.

   $Q$ A $2 \times 2$ matrix. The noise model for the state transition. This indicates the estimates that we have made of the true noise in the system for each parameter in the state. In the ideal case, this matrix contains the values of $wStdP^2$ and $wStdV^2$. If the noise is lower than the true noise, then the estimator will be optimistic, and if the noise is higher then it will be pessimistic. Both of these can cause problems, either with overconfidence in bad estimates, or too little confidence in quite a good estimate.

$R$ A scalar value. The noise model for the measurements. Indicates an estimate of the true noise on the measurements that we will receive from our simulated sensor. The ideal value for this is the actual noise on the measurements, which is represented by $vStd$.

$wStdP$ The true variance of the noise in the position. The noise is added to the system by generating a normally distributed random number with a zero mean and variance $wStdP$.

$wStdV$ The true variance of the noise in the velocity. The noise is added to the system by generating a normally distributed random number with zero mean and variance $wStdP$.

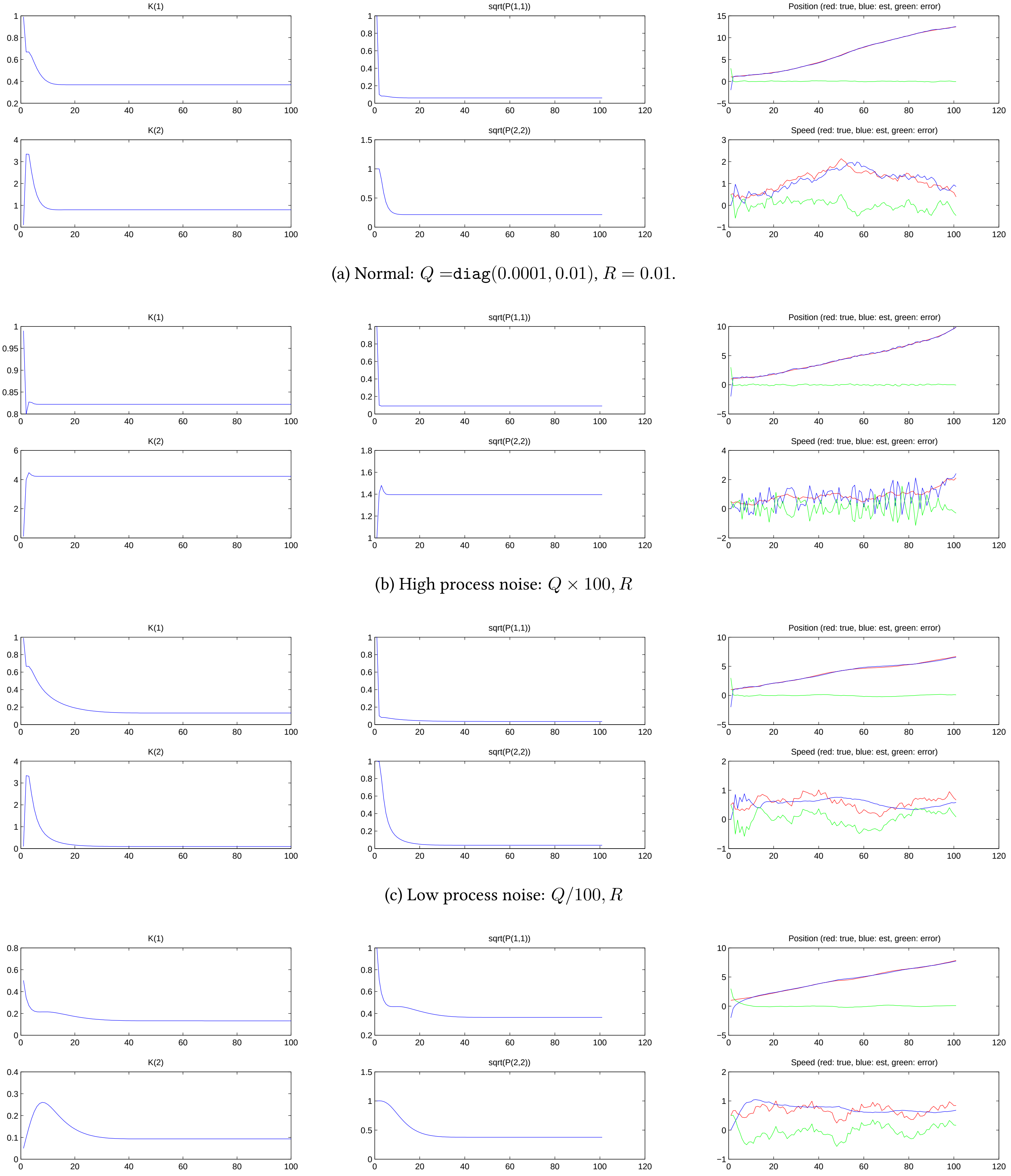
$vStd$ The true variance of the noise in the measurement. This is added to the measurement at each time step by generating a normally distributed random number with zero mean and variance of $vStd$.

$u$ The control applied to the system at each time step, that is, the instantaneous acceleration of the car, which is assumed to be constant between time steps. Setting this to zero will mean that the system is driven by the noise $wStdP$.

$PP$ Stores the values of the covariance matrix after the update step at each time step such that they can be plotted later to examine the evolution of the covariance over time.

3.

4. Changing the value of $P$ and $\hat{x}$ has no noticeable effect on the standard deviation of the error in position or velocity. However, making $P$ very small, so that the uncertainty at the start is low, makes the time to convergence increase in the case of the position. Convergence for the velocity seems to take approximately the same amount of time. Making the uncertainty very large seems to have no effect on convergence or error.



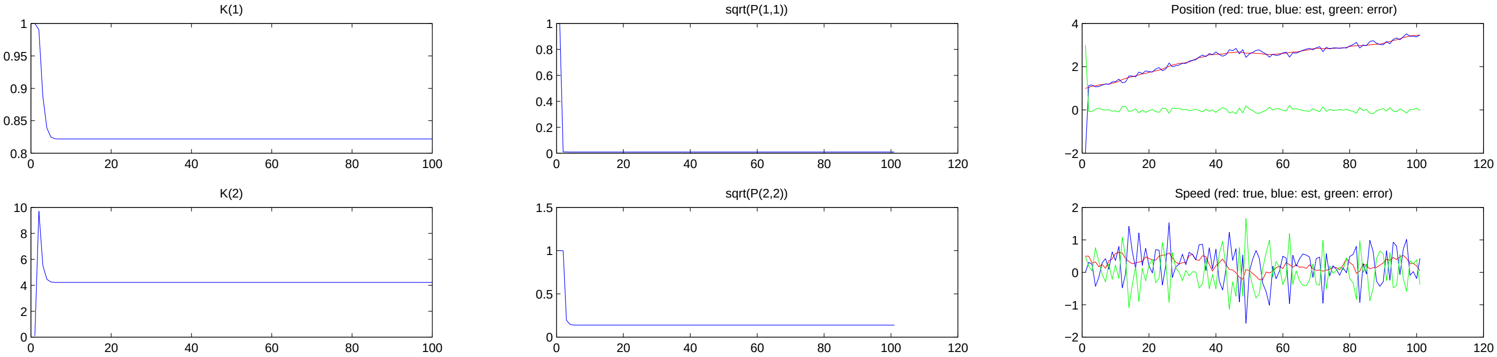
(a) Normal: $Q = \texttt{diag}(0.0001, 0.01)$, $R = 0.01$.

(b) High process noise: $Q \times 100$, $R$

(c) Low process noise: $Q/100$, $R$

(d) High measurement noise: $Q$, $R \times 100$

Figure 1: Graphs for different process and measurement noise models, from left to right the Kalman gain, covariance and error.



(e) Low measurement noise: $Q, R/100$

Figure (cont.) 1: Graphs for different process and measurement noise models, from left to right the Kalman gain, covariance and error.

## 2.2 EKF Localisation

5. The prediction step is used to generate the prior $\overline{bel}(x_t)$, which is our belief about the state of the robot before the measurements are taken into account. The update step integrates $\overline{bel}(x_t)$ and the probability of making the observed measurement.

$$\overline{bel}(x_t) = \int p(x_t \mid u_{1:t}, x_{t-1}) bel(x_{t-1}) dx_{t-1} \tag{1}$$

Prediction step

$$bel(x_t) = \eta p(z_t \mid x_t, M) \overline{bel}(x_t) \tag{2}$$

Update step

$$p(x_t \mid u_{1:t}, z_{1:t}, \bar{x}_0, M) = \eta p(z_t \mid x_t, M) \underbrace{\int p(x_t \mid u_t, x_{t-1}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}, \bar{x}_0, M) dx_{t-1}}_{\text{Prediction Step}} \tag{3}$$

In (3), the integral is equivalent to $\overline{bel}(x_t)$, and forms the prediction step. Multiplying this by the normalisation constant and measurement probability $\eta p(z_t \mid x_t, M)$ gives the update step.

# 3 Results