Chapter 1

VECTOR ALGEBRA

1.1 Basic Review of Vectors

■ Definition:

Physical quantities having both magnitude and a definite direction in space. It should follow the law of vector addition.

Example: Velocity, Acceleration, Momentum, Force, Electric Field, Torque, etc.

Note: Current is a physical quantity that has both magnitude and direction but it does not follow the law of vector addition. So, current is a scalar quantity.

■ Various type of vectors:

(1) Equal vectors: Vectors having same magnitude and same direction.

(2) Null Vectors: Vectors having coincident initial and terminal point i.e. its magnitude is zero and it has any arbitrary direction.

(3) Unit Vector: Vector having unit magnitude. Unit vector along $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

(4) Reciprocal Vector: Vector having same direction as $\vec{a}$ but magnitude reciprocal to that of $\vec{a}$, is known as the reciprocal vector of $\vec{a}$. Reciprocal vector of $\vec{a}$ is $\vec{a}^{-1} = \dfrac{1}{|a|}\hat{a}$

(5) Negative Vector: Vectors having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{a}$, is known as the negative vector of $\vec{a}$. Negative vector as $\vec{a}$ is $-\vec{a} = -|a|\hat{a}$

■ Orthogonal Resolution of Vectors:

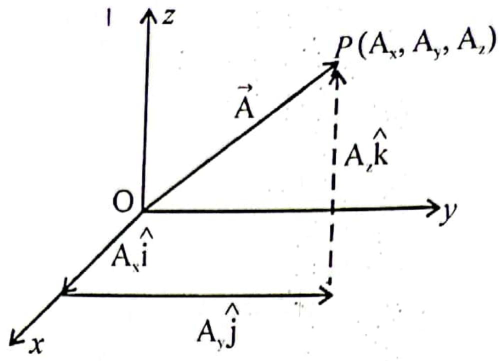
Any vector $\vec{A}$ in the 3-D right- handed rectangular cartesian coordinate system can be represented as

$$\overline{OP} = \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k},$$

where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively and A_x, A_y, A_z are the rectangular components of vector $\vec{A}$ along x, y, z axis.

Magnitude of vector $\vec{A}$ is $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along $\vec{A}$ is $\hat{A} = \vec{A}/|\vec{A}| = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)/\sqrt{A_x^2 + A_y^2 + A_z^2}$



■ **Direction cosines of vector $\vec{A}$:**

If $\vec{A}$ makes angles α, β, γ with x, y and z axes respectively, then direction cosines of $\vec{A}$ are defined as

$$l=\cos\alpha=\frac{A_x}{A};\ m=\cos\beta=\frac{A_y}{A};\ n=\cos\gamma=\frac{A_z}{A}\quad \text{and}\quad l^2+m^2+n^2=1$$

Note: Unit vector along $\vec{A}$ can be written as $\hat{A}=l\hat{i}+m\hat{j}+n\hat{k}$

1.2 Products of Vectors

■ **Scalar Product or Dot Product:**

Dot product of two vectors $\vec{a}$ and $\vec{b}$ are defined as

$$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta \Rightarrow \cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Note:

(i) Dot product is commutative in nature i.e. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

(ii) For two mutually perpendicular vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b}=0$

(iii) $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot i=0$, $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$

(iv) If $\vec{a}=a_z\hat{i}+a_y\hat{j}+a_z\hat{k}$ and $\vec{b}=b_x\hat{i}+b_y\hat{j}+b_z\hat{k}$, then $\vec{a}.\vec{b}=a_xb_x+a_yb_y+a_zb_z$

(v) Projection of $\vec{A}$ on $\vec{B}=\vec{A}.\hat{B}$

■ **Vector Product or Cross Product :**

Cross product of two vectors $\vec{a}$ and $\vec{b}$ are defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note:

(i) Cross product is not commutative i.e. $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$

(ii) For two collinear vectors (parallel or anti-parallel vectors) $\vec{a}\times\vec{b}=0$.

(iii) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$, $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}=\hat{j}$

(iv) If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

■ Scalar Triple Product:

Scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = [abc]$$

Note:

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ i.e. $[abc] = [bca] = [cab]$

(ii) Volume of a parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as concurrent edges is :$V = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$

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(iii) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = 0$

■ Vector Triple Product:

Vector triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

It represents a vector coplanar with $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$.

Note:

(i) $\vec{a}\times\left(\vec{b}\times\vec{c}\right) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times\left(\vec{a}\times\vec{b}\right) = 0$

1.3 Gradient, Divergence and Curl

Gradient of a Scalar Field:

Gradient of a continuously differentiable scalar function $\phi(x, y, z)$ is mathematically defined as:

$$grad\ \phi = \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

where, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \Rightarrow$ 'del' or 'grad' or 'nabla' operator

Physical interpretation: Gradient of scalar function ϕ at any point $P(x, y, z)$ is a vector quantity, whose magnitude is equal to the rate of change of scalar function ϕ with distance along the normal to level surface and its direction is along the normal to the level surface at that point. The direction of $\vec{\nabla}\phi$ is that along which the rate of change of ϕ is maximum.

Level surface: At each point of the level surafce, the value of scalar function ϕ will be same.

Example: Equipotential surface on which value of electrostatic potential is same at all points

Note:

(i) Normal vector to the level surface: $\vec{\nabla}\phi$

(ii) Unit normal vector to the level surface: $\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

Directional Derivative:

Directional derivative of ϕ in the direction of $\vec{A}$ is defined as rate of change of ϕ with distance along the direction of $\vec{A}$. It is mathematically defined as the component of $\vec{\nabla}\phi$ in the direction of vector $\vec{A}$ i.e.

$$\vec{\nabla}\phi \cdot \hat{A} = \vec{\nabla}\phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

Tangent Plane and Normal to the level surface:

Consider $\phi(x, y, z) = c$ be the equation of a level surface. and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point P (x,y,z) on this surface.

Tangent plane at P: $\vec{\nabla}\phi$ is a vector normal to the surface i.e. it is perpendicular to the tangent plane at P. Let, $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of any point on the tangent plane at P to the surface. Therefore, $\vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$ lies in the tangent plane at P and it will be perpendicular to $\vec{\nabla}\phi$

i.e. $(\vec{R} - \vec{r}).\vec{\nabla}\phi = 0$

$$\Rightarrow (X - x)\frac{\partial\phi}{\partial x} + (Y - y)\frac{\partial\phi}{\partial y} + (Z - z)\frac{\partial\phi}{\partial z} = 0$$, which is the tangent plane at point P.

■ Divergence of a vector field:

Divergence of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \text{Scalar quantity}$$

Physical Interpetation: Divergence of $\vec{V}$ at point $P(x, y, z)$ is defined as the outward flux of the vector field $\vec{V}$ per unit volume enclosed by an infinitesimal closed surface surrounding point P.

Note:

(i) If $\vec{\nabla}.\vec{V} = 0$, then $\vec{V}$ is known as solenoidal vector field.

(ii) If $\vec{\nabla}.\vec{V} =$ negative, then $\vec{V}$ is known as sink field i.e. vector lines are going inward.

(iii) If $\vec{\nabla}.\vec{V} =$ positive, then $\vec{V}$ is known as source field i.e. vector lines are the going outward.

(iv) $\vec{\nabla}\cdot\left(\vec{U}+\vec{V}\right) = \vec{\nabla}\cdot\vec{U} + \vec{\nabla}\cdot\vec{V}$ (v) $\vec{\nabla}.\left(u\vec{V}\right) = (\vec{\nabla}u)\cdot\vec{V} + u(\vec{\nabla}.\vec{V})$

■ Curl of a vector field:

Curl of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

It is the measurement of rotation of vector field and the direction of curl of vector is along axis of rotation.

Note:

(i) If $\vec{\nabla} \times \vec{V} = 0$, then $\vec{V}$ is known as an irrotational vector and we can write $\vec{V} = \vec{\nabla}\phi$

(ii) If $\vec{\nabla} \times \vec{V} \neq 0$, then $\vec{V}$ is known as rotational vector.

(iii) $\vec{\nabla} \times (\vec{U} + \vec{V}) = \vec{\nabla} \times \vec{U} + \vec{\nabla} \times \vec{V}$

(iv) $\vec{\nabla} \times (u\vec{V}) = u\vec{\nabla} \times \vec{V} + (\vec{\nabla} u) \times \vec{V}$

Important Vector Identities:

If ϕ, ψ are scalar point functions and $\vec{A}$, $\vec{B}$ are vector point functions in certain region then.

(1) $\vec{\nabla}(\phi+\psi)=\vec{\nabla}\phi+\vec{\nabla}\psi$

(2) $\vec{\nabla}(\phi\psi)=\phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$

(3) $\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$

(4) $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

(5) $\vec{\nabla}(\vec{A}\cdot\vec{B})=\vec{A}\times(\vec{\nabla}\times\vec{B})+(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}$

(6) $\vec{\nabla}\cdot(\phi\vec{A})=\vec{\nabla}\phi\cdot\vec{A}+\phi\left(\vec{\nabla}\cdot\vec{A}\right)$

(7) $\vec{\nabla}\times(\phi\vec{A})=\phi(\vec{\nabla}\times\vec{A})+(\vec{\nabla}\phi)\times\vec{A}$

(8) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$

(9) $\vec{\nabla}\times(\vec{\nabla}\phi)=0$

(10) $\vec{\nabla}\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$

(11) $\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})$

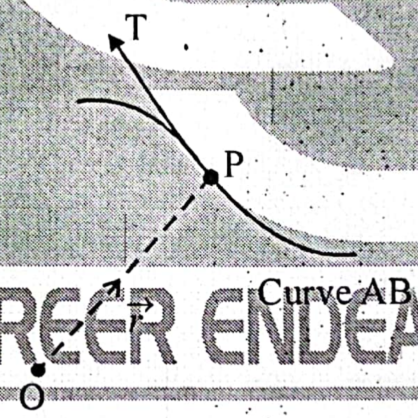
(12) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$

(13) For a constant vector $\vec{A}$, $\vec{\nabla}\left(\vec{A}.\vec{r}\right)=\vec{A}$ and $\vec{\nabla}\times\left(\vec{A}\times\vec{r}\right)=2\vec{A}$

1.4 Line Integration of Vectors

The integration of a vector along a curve is known as line integration of vectors.

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.



Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

Component of $\vec{F}$ along a tangent PT at P = Dot product of $\vec{F}$ and unit vector along PT

$$= \vec{F} \cdot \frac{d\vec{r}}{ds} \left(\frac{d\vec{r}}{ds} \text{ is a unit vector along tangent PT} \right)$$

Line integral of $\vec{F}$ from A to B along the curve C will be $= \int_c \left(\vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int_c \vec{F} \cdot d\vec{r}$

Note:

(i) If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done

$$W_{AB} = \int_A^B \vec{F} \cdot d\vec{r}$$

(ii) If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V} \cdot d\vec{r}$ is called the circulation of $\vec{V}$ round closed curve C.

(iii) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

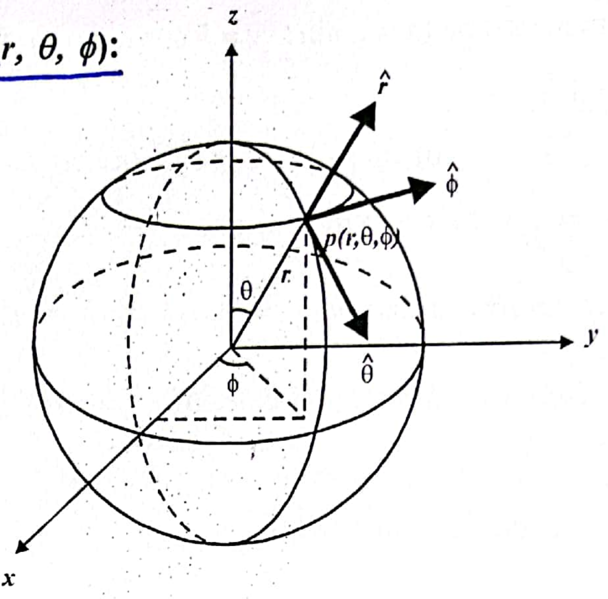
(iv) Work done by a conservative field $\vec{A}$ in moving a particle from point P to Q will be

$$\int_P^Q \vec{F} \cdot \overline{dr} = \int_P^Q \vec{\nabla}\phi \cdot \overline{dr} = \int_P^Q d\phi = \phi_Q - \phi_P = \text{independent of path.}$$

(v) Work done by a conservative field $\vec{A}$ in moving a particle around a closed path C is $\oint_c \vec{F} \cdot \overline{dr} = 0$

1.5 Orthogonal Currvilinear Co-ordinates

(A) Spherical Polar co-ordinates (r, θ, ϕ):



Let the point P (in the above figure) has cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ).

Relation between cartesian and spherical polar co-ordiantes:

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta$$

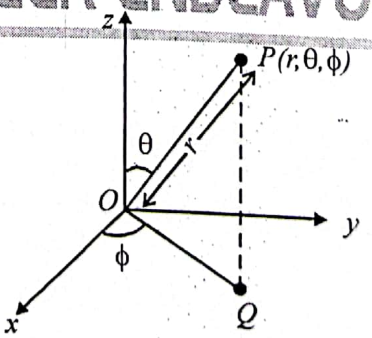
In spherical polar co-ordinates,

$$r = \sqrt{x^2 + y^2 + z^2} = \text{radial distance of the point } P \text{ from the origin}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \text{angle which } OP \text{ makes with } z\text{-axis}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{angle which } OQ \text{ (projection of } OP \text{ in } x\text{-}y \text{ plane) makes with positive } x\text{-axis}$$

Corresponding limits: $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$



■ **Unit vectors in Spherical Polar coordinates:**

The position vector of P is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Putting (x, y, z) in the above relations,

$$\vec{r} = (r\sin\theta\cos\phi)\hat{i} + (r\sin\theta\sin\phi)\hat{j} + r\cos\theta\hat{k}$$

The unit vector along r is given by

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}}{r}$$

$$\Rightarrow \quad \boxed{\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}}$$

The unit vector along θ is given by

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial r/\partial\theta|} = \frac{r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} - r\sin\theta\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi - r^2\sin^2\theta}}$$

$$\Rightarrow \quad \boxed{\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}}$$

The unit vector along ϕ is given by

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial r/\partial\phi|} = \frac{-r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$\Rightarrow \quad \boxed{\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}}$$

Note: The unit vector $\hat{r}, \hat{\theta}, \hat{\phi}$ are orthogonal vectors and it thus follows,

$$\boxed{\hat{r}\times\hat{\theta} = \hat{\phi},\ \hat{\theta}\times\hat{\phi} = \hat{r}, \hat{\phi}\times\hat{r} = \hat{\theta}}$$

■ Line elements:

Line element between r to $r+dr$: $d\vec{l}_r = dr\,\hat{r}$ (along r curve)

Line element between θ to $\theta + d\theta$: $d\vec{l}_\theta = rd\theta\,\hat{\theta}$ (along θ curve)

Line element between ϕ to $\phi + d\phi$: $d\vec{l}_\phi = r\sin\theta d\phi\,\hat{\phi}$ (along ϕ curve)

■ Surface elements:

Constant r surface : $d\vec{s}_r = d\vec{l}_\theta \times d\vec{l}_\phi = r^2\sin\theta d\theta d\phi\,\hat{r}$ (surface of a sphere)

Constant θ surafce : $d\vec{s}_\theta = d\vec{l}_\phi \times d\vec{l}_r = r\sin\theta dr d\phi\,\hat{\theta}$

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_r \times d\vec{l}_\theta = rdrd\theta\,\hat{\phi}$

■ Volume element:

$$dV = d\vec{l}_r.\left(d\vec{l}_\theta \times d\vec{l}_\phi\right) = r^2\sin\theta dr d\theta d\phi$$

Gradient: $\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}$

Curl: $\vec{\nabla}\times\vec{a}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ a_r & ra_\theta & r\sin\theta a_\phi\end{vmatrix}$

Laplacian: $\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

■ Conversion of a vector from Cartesian to Spherical polar co-ordiantes:

$$\hat{r}=\sin\theta\cos\phi\,\hat{i}+\sin\theta\sin\phi\,\hat{j}+\cos\theta\hat{k}$$

$$\hat{\theta}=\cos\theta\cos\phi\,\hat{i}+\cos\theta\sin\phi\,\hat{j}-\sin\theta\hat{k}$$

$$\hat{\phi}=-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}$$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$

Suppose the vector is given in cartesian coordinates i.e. $\vec{r}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r}=a_r\hat{r}+a_\theta\hat{\theta}+a_\phi\hat{\phi}$, the components are related by the following equation.

i.e. $$\begin{bmatrix}a_r\\ a_\theta\\ a_\phi\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}a_x\\ a_y\\ a_z\end{bmatrix}$$

■ Limits for various types of integrations:

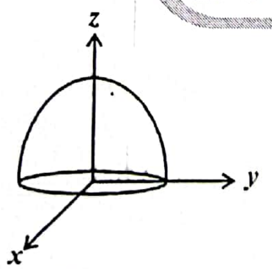


Fig. (a)
Upper hemisphere

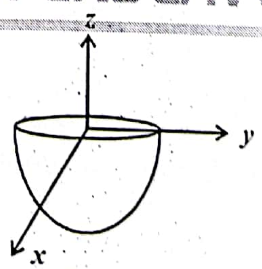


Fig. (b)
Lower hemisphere

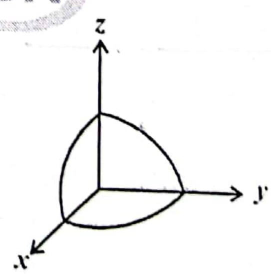


Fig. (c)
First Octant of sphere

Case-I: Full sphere $x^2+y^2+z^2=a^2$

Limit of volume integration : $0\le r\le a;\ \ 0\le\theta\le\pi;\ 0\le\phi\le2\pi$

Limit of surface integration : $r=\text{const}=a;\ \ 0\le\theta\le\pi;\ 0\le\phi\le2\pi$

Case-II: Upper Hemisphere $x^2+y^2+z^2=a^2, z>0$

Limit of volume integration : $0\le r\le a;\ \ 0<\theta\le\frac{\pi}{2};\ 0\le\phi\le2\pi$

Limit of surface integration : $r = \text{const} = a; \ 0 \le \theta \le \frac{\pi}{2}; \ 0 \le \phi \le 2\pi$

Case-III: Lower hemisphere $x^2 + y^2 + z^2 = a^2, z < 0$

Limit of volume integration : $0 \le r \le a; \ \frac{\pi}{2} < \theta \le \pi; \ 0 \le \phi \le 2\pi$

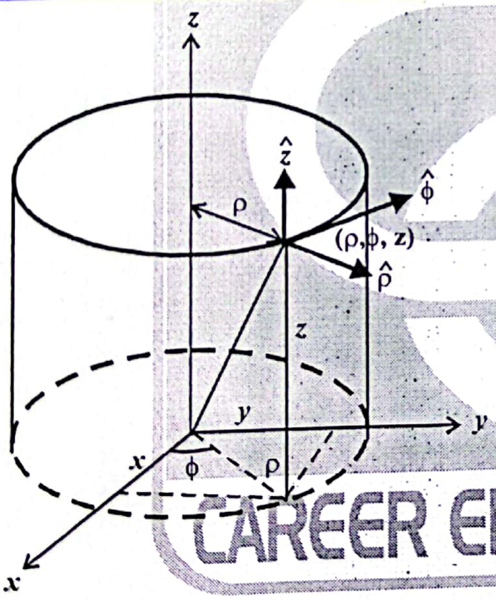
Limit of surface integration : $r = \text{const} = a; \ \frac{\pi}{2} \le \theta \le \pi; \ 0 \le \phi \le 2\pi$

Case-IV: First octant of sphere $x^2 + y^2 + z^2 = a^2, \ x, y, z > 0$

Limit of volume integration : $0 \le r \le a; \ 0 < \theta \le \frac{\pi}{2}; \ 0 \le \phi \le \frac{\pi}{2}$

Limit of surface integration : $r = \text{const} = a; \ 0 \le \theta \le \frac{\pi}{2}; \ 0 \le \phi \le \frac{\pi}{2}$

(B) Cylindrical co-ordinates (ρ, ϕ, z)**:**



Let the point P (in above figure) has cartesian coordinates (x, y, z) and cylinderical coordinates (ρ, ϕ, z).
So, the relation between cartesian and cyllindrical co-ordiantes:

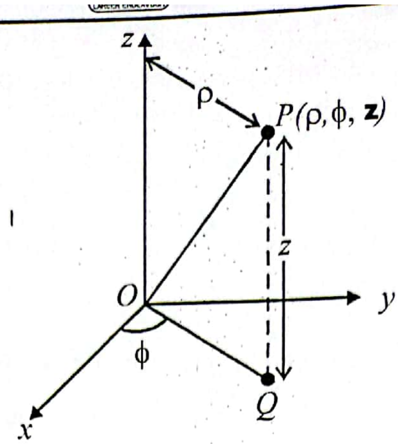
$$x = \rho \cos\phi, \ y = \rho \sin\phi, \ z = z$$

In cylindrical co-ordinates,

$\rho = \sqrt{x^2 + y^2}$ = perpendicular distance point 'P' from the z axis

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ = angle which OQ (projection of OP in x-y plane) makes with positive x-axis

z = height of the point 'P' above the x-y plane



Unit Vector in cylindrical coordinates:

The position vector of the point P is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Putting the value of (x, y, z) in the above relation

$$\vec{r} = \rho\cos\phi\hat{i} + \rho\sin\phi\hat{j} + z\hat{k}$$

The unit vector along ρ is given by

$$\hat{\rho} = \frac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \cos\phi\hat{i} + \sin\phi\hat{j} \quad \Rightarrow \quad \boxed{\hat{\rho} = \cos\phi\hat{i} + \sin\phi\hat{j}}$$

The unit vector along ϕ is given by

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = -\sin\phi\hat{i} + \cos\phi\hat{j} \quad \Rightarrow \quad \boxed{\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}}$$

The unit vector along z is given by

$$\hat{z} = \frac{\partial\vec{r}/\partial z}{|\partial\vec{r}/\partial z|} = \hat{k} \quad \Rightarrow \quad \boxed{\hat{z} = \hat{k}}$$

Note: $\hat{\rho}, \hat{\phi}$ and $\hat{z}$ are orthogonal to each other and they follow

$$\boxed{\hat{\rho}\times\hat{\phi} = \hat{z}, \quad \hat{\phi}\times\hat{z} = \hat{\rho}, \quad \hat{z}\times\hat{\rho} = \hat{\phi}}$$

Line elements:

Line element between ρ to $\rho + d\rho$: $d\vec{l}_\rho = d\rho\,\hat{\rho}$ (along ρ curve)

Line element between ϕ to $\phi + d\phi$: $d\vec{l}_\phi = \rho d\phi\,\hat{\phi}$ (along ϕ curve)

Line element between z to $z + dz$: $d\vec{l}_z = dz\,\hat{z}$ (along z curve)

Surface elements:

Constant ρ surface : $d\vec{s}_\rho = d\vec{l}_\phi \times d\vec{l}_z = \rho d\phi dz\,\hat{\rho}$ (curved surface of a circular cyllinder)

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_z \times d\vec{l}_r = d\rho dz\,\hat{\phi}$

Constant z surafce : $d\vec{s}_z = d\vec{l}_\rho \times d\vec{l}_\phi = \rho d\rho d\phi\,\hat{z}$ (plane surface of a circular cyllinder parallel to x - y plane)

Volume element :

$$dV = d\vec{l}_\rho.\left(d\vec{l}_\phi \times d\vec{l}_z\right) = \rho d\rho d\phi dz$$

Gradient: $\vec{\nabla} = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}$

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$

- **Conversion of a vector from Cartesian to Cylindrical co-ordiantes:**

$$\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\hat{z} = \hat{k}$$

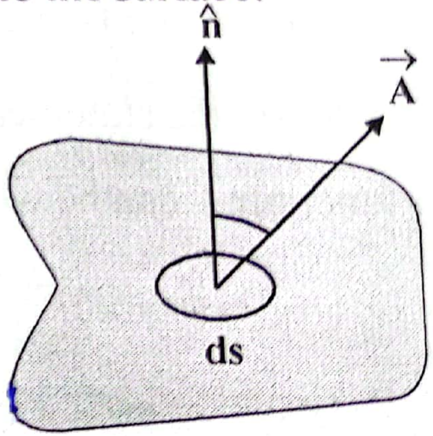
Suppose the vector is given in cartesian coordinates i.e. $\vec{r} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r} = a_\rho\hat{\rho} + a_\phi\hat{\phi} + a_z\hat{z}$, the components are related by the following equation.

$$\begin{bmatrix} a_\rho \\ a_\phi \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

1.6 Surface Integration of Vectors

The integration of vector on an open or closed surface is known as surface integration of vectors.

Let $\vec{A}$ be a vector function and S be the given surface. Surface integral of a vector function $\vec{A}$ over the surface S is defined as the integral of the components of $\vec{A}$ along the normal to the surface.



Components of $\vec{A}$ along the normal

$= \vec{A}.\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds

Surface integral of $\vec{A}$ over S

$$= \Sigma \vec{A}.\hat{n} = \iint_s \left(\vec{A}.\hat{n}\right) ds$$

1.7 Volume Integration of Vectors

Volume integral of a vector field $\vec{A}$ within the volume V can be written as,

$\iiint_V \vec{A} dV$, where, dV is the infinitesimal volume element

1.8 Vector Related Theorems

■ Divergence Theorem:

This theorem is applicable only for closed surfaces and this theorem converts surface integral into volume integral and vice versa.

If V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivatives, then.

$$\oint\!\!\!\oint_S \vec{A}\cdot\vec{ds} = \oint\!\!\!\oint_S \vec{A}\cdot\hat{n}ds = \iiint_V (\vec{\nabla}\cdot\vec{A})dV$$

where $\hat{n}$ is the outward normal to 'S' indicating the positive direction of S.

■ Stokes Theorem:

This theorem is applicable only for open surfaces and this theorem converts surface integral into line integral and vice versa.

If S is an open, two-sided surface bounded by a closed, non-intersecting curve C, and $\vec{A}$ is a vector function of position with continuous derivatives; then

$$\oint_C \vec{A}\cdot\vec{dr} = \iint_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$$

where C is traversed in the positive (counterclockwise) direction.

1.9 Linear Dependency of Vectors

Three vectors $\vec{u}, \vec{v}, \vec{w}$ will be linearly dependent if $\alpha_1\vec{u} + \alpha_2\vec{v} + \alpha_3\vec{w} = 0$, where $\alpha_1, \alpha_2, \alpha_3$ are not all zero. These three dependent vectors do not form the basis of $\mathbb{R}^3$ space.

If $\alpha_1\vec{u} + \alpha_2\vec{v} + \alpha_3\vec{w} = 0$ such that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $\vec{u}, \vec{v}, \vec{w}$ are said to be linearly independent and they do form the basis of $\mathbb{R}^3$ space i.e. any vector in $\mathbb{R}^3$ space can be written as the linear combination of vectors $\vec{u}, \vec{v}, w$ respectively.

How to check:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \begin{cases} = 0 \text{ for linearly dependent vectors} \\ \neq 0 \text{ for linearly independent vectors} \end{cases}$$