1 On the properties of the expansion of the an irrational and rational

1.1 Introduction

This paper attempts to look into some of the properties of an expansion of the form $\left(a+\sqrt{b}\right)^n$ and investigate some of the interesting applications.

First we will write an expansion say, $\left(a+\sqrt{b}\right)^n$. The two variables a and b are subject to condition that,

$$a-\sqrt{b} \in (0,1)$$

Using the normal binomial expansion we get,

$$\left(a+\sqrt{b}\right)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\cdot\sqrt{b} + \ldots + \binom{n}{n-1}a\cdot\sqrt{b}^{n-1} + \binom{n}{n}\sqrt{b}^n$$

Let us say this is equal to,

$$I+f \iff f \in (0,1)\,, I \in \mathbb{I}^+$$

Now, if we write the expansion as,

$$\left(a-\sqrt{b}\right)^n = \binom{n}{0}a^n - \binom{n}{1}a^{n-1}\cdot\sqrt{b} + \ldots + \binom{n}{n-1}a\cdot\sqrt{b}^{n-1} - \binom{n}{n}\sqrt{b}^n$$

Let us say this is equal to,

$$f^{'} \iff f^{'} \in (0.1)$$

Adding the two, we get,

$$I+f+f^{'} = 2\left[a^n + a^{n-2}\cdot b + \ldots + a^2\sqrt{b}^{n-2} + \sqrt{b}^n\right]$$

Thus,

$$I+f+f^{'} = 2j \iff j \in \mathbb{I}^+$$

Now, since

$$f \in (0,1)$$

and

$$f^{'} \in (0,1)$$

. Adding the two,

$$f+f^{'} \in (0,2)$$

But, we also have

$$I+f+f^{'} = 2j \iff j \in \mathbb{I}^+$$

Now, since $I \in \mathbb{I}^+$ thus, $f + f^{'} \in \mathbb{I}^+$ So, the only integer between 0 and 2 is 1. So,

$$f + f^{'} = 1$$

and,

$$I + f + f^{'} = 2j \iff j \in \mathbb{I}^+$$

So,

$$I + 1 = 2j \iff j \in \mathbb{I}^+$$

So,

$$I = 2j - 1 \iff j \in \mathbb{I}^+$$

Thus, I is an odd integer.

1.2 Conclusion

Thus we can clearly see that, in the expansion of

$$\left[a + \sqrt{b}\right]^n = 2j - 1$$

where, $[x]$= integral part of x. Regardless of the expansion and the numbers, this property will hold subject only to condition,

$$a - \sqrt{b} \in (0, 1)$$

. Should this condition not exist then, we will get a different result, namely,

$$\left[a + \sqrt{b}\right]^n + \left[a - \sqrt{b}\right]^n = 2j^{'} - 1$$

where, $[x]$= integral part of x. This proof is left up to the reader. Also we get,

$$f = 1 - f^{'}$$

as a result of this proof, which is useful in substitutions.