Assignment - 1

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1 Problem

1.1. Find the areas of the triangles formed by the triads of points (4,3), (1,-3), (-3,1), and (4,3), (-3,1), (1,-3) and explain the difference of signs in the two cases.

Solution: Let the points be-

$$\mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.1.1}$$

$$\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.1.2}$$

Area of a $\triangle$ with the vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is

$$\mathbf{\Delta} = \frac{\mathbf{1}}{\mathbf{2}} \begin{vmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \tag{1.1.3}$$

For $\mathbf{A} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$,

$$\mathbf{\Delta} = \frac{\mathbf{1}}{\mathbf{2}} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{1.1.4}$$

$$\therefore \Delta ABC = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & -3 \\ 3 & -3 & 1 \end{vmatrix}$$

$$\xrightarrow{C1 \leftarrow C1 - C3} \frac{1}{2} \begin{vmatrix} 0 & 1 & 1 \\ 7 & 1 & -3 \\ 2 & -3 & 1 \end{vmatrix}$$

$$\xrightarrow{C2 \leftarrow C2 - C3} \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 7 & 4 & -3 \\ 2 & -4 & 1 \end{vmatrix}$$

taking 4 common from $C2$,

$$\rightarrow \frac{4}{2} \begin{vmatrix} 0 & 0 & 1 \\ 7 & 1 & -3 \\ 2 & -1 & 1 \end{vmatrix}$$

Expanding along the first row,

$$\Delta ABC = 2\left[1(7(-1) - 2(1))\right]$$
$$= 2(-7 - 2)$$
$$= 2(-9)$$

$$\therefore \Delta ABC = -18 \tag{1.1.5}$$

For any matrix X, product with I (identity matrix), gives matrix X itself:

$$XI = IX = X \tag{1.1.6}$$

Also, exchanging the columns of I in the product, will exchange columns of X too.

Let J be the matrix obtained by exchanging $C2$ and $C3$ of I, so that

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, where \left|J\right| = 1(0-1) = -1 \tag{1.1.7}$$

$\therefore \triangle ABC$ can be transformed to $\triangle PQR$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & -3 \\ 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & -3 & 1 \\ 3 & 1 & -3 \end{pmatrix} = PQR \tag{1.1.8}$$

We know, $\left|X\right|\left|Y\right| = \left|XY\right|$. From (1.1.5), (1.1.7), $\left|ABC\right|\left|J\right| = \left|(ABC)J\right| = (-18)(-1) = 18$

$$\therefore \triangle PQR = \left|PQR\right| = 18 \tag{1.1.9}$$

Hence, the difference in the sign in the areas of $\triangle ABC$ and $\triangle PQR$.