Numerical Integration Methods for Calculating Moment of Inertia of an Annular Disk

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Abstract

This report presents an analysis of numerical integration methods used to calculate the moment of inertia of an annular disk. We implement and compare three numerical integration techniques: Riemann sum, Trapezoidal rule, and Simpson's rule. The accuracy of these methods is evaluated against the analytical solution, and their performance is analyzed under various conditions.

1 Introduction

The moment of inertia is a crucial concept in physics, representing an object's resistance to rotational acceleration. This study focuses on calculating the moment of inertia of an annular disk, which has applications in mechanics.

1.1 Physical Problem Description

We consider an annular disk with inner radius r_{inner}, outer radius r_{outer}, and total mass M. The moment of inertia I for such a disk about its central axis is given by the integral:

$$I = \int_{r_{inner}}^{r_{outer}} r^2 dm \tag{1}$$

where dm is the mass of an infinitesimal ring of the disk.

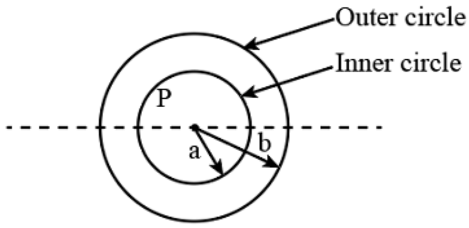

Figure 1: Annular disk

1.2 Analytical Solution

The analytical solution for the moment of inertia of an annular disk is:

$$I = \frac{1}{2}M(r_{outer}^2 + r_{inner}^2) \tag{2}$$

This provides a benchmark for evaluating our numerical methods.

2 Methodology

2.1 Numerical Integration Methods

We implemented three numerical integration methods:

1. Riemann Sum
2. Trapezoidal Rule
3. Simpson's Rule

2.1.1 Riemann Sum

The Riemann sum approximates the integral by summing the areas of rectangles:

$$I \approx \sum_{i=1}^{n} 2\pi\rho r_i^3 \Delta r \tag{3}$$

where ρ is the mass density, r_i is the radius at each step, and Δr is the step size.

2.1.2 Trapezoidal Rule

The Trapezoidal rule uses trapezoids instead of rectangles:

$$I \approx \sum_{i=1}^{n-1} \pi \rho \Delta r (r_i^3 + r_{i+1}^3) \tag{4}$$

2.1.3 Simpson's Rule

Simpson's rule uses parabolic arcs to approximate the integral:

$$I \approx \frac{\Delta r}{3} \sum_{i=0}^{n/2-1} 2\pi\rho(r_{2i}^3 + 4r_{2i+1}^3 + r_{2i+2}^3) \tag{5}$$

2.2 Implementation Details

The following Python code implements the three numerical integration methods:

```
import numpy as np

def riemann_sum(r_inner, r_outer, M, num_steps):
    rho = M / (np.pi * (r_outer**2 - r_inner**2))
    dr = (r_outer - r_inner) / num_steps
    I = 0
    for i in range(num_steps):
        r = r_inner + i * dr
        I += 2 * np.pi * rho * r**3 * dr
    return I

def trapezoidal_rule(r_inner, r_outer, M, num_steps):
    rho = M / (np.pi * (r_outer**2 - r_inner**2))
    dr = (r_outer - r_inner) / num_steps
    I = 0
    for i in range(num_steps):
        r1 = r_inner + i * dr
        r2 = r_inner + (i + 1) * dr
        I += 0.5 * (2 * np.pi * rho * r1**3 + 2 * np.pi * rho * r2**3) * dr
    return I

def simpsons_rule(r_inner, r_outer, M, num_steps):
    rho = M / (np.pi * (r_outer**2 - r_inner**2))
    dr = (r_outer - r_inner) / num_steps
    I = 0
    for i in range(0, num_steps-1, 2):
```

```
        r1 = r_inner + i * dr
        r2 = r_inner + (i + 1) * dr
        r3 = r_inner + (i + 2) * dr
        I += (dr / 3) * (2 * np.pi * rho * r1**3 +
                         8 * np.pi * rho * r2**3 +
                         2 * np.pi * rho * r3**3)
    return I

# Analytical solution
def analytical_solution(r_inner, r_outer, M):
    return 0.5 * M * (r_outer**2 + r_inner**2)
```

Listing 1: Implementation of Numerical Integration Methods

3 Results and Discussion

3.1 Comparison of Numerical Methods

For $r_{inner} = 0.5$, $r_{outer} = 1.0$ and M = 1.0, we get the following results:

Riemann sum: 0.6244167916666669
Trapezoidal method: 0.6250001250000001
Simpson's method: 0.6250000000000001
Analytical method: 0.6250000000000000
Scipy: 0.6250005279153616

3.2 Error Analysis

$$Error = \frac{|I_{analytical} - I_{method}|}{I_{analytical}} \tag{6}$$

For $r_{inner} = 0.5$, $r_{outer} = 1.0$ and M = 1.0, we get the following results:

error in Riemann sum: 0.0009331333333333025
error in Trapezoid method: 2.0000000020559127e-07
error in Simpson rule: 1.7763568394002506e-16

3.2.1 Truncation Error

The theoretical truncation errors for each method are:

- Riemann Sum: $O(\Delta r)$

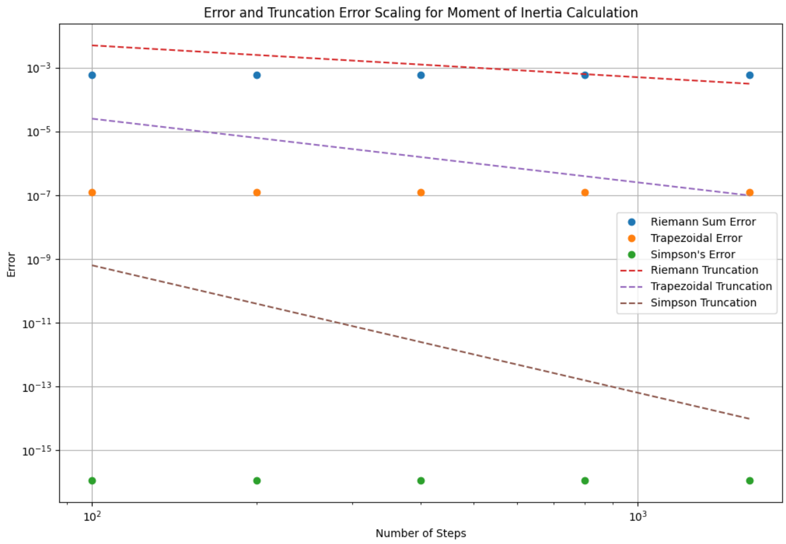

Figure 2: Truncation error with step size

- Trapezoidal Rule: $O(\Delta r^2)$
- Simpson's Rule: $O(\Delta r^4)$

3.3 Limiting Cases

We examined two limiting cases:

1. As r_{inner} approaches r_{outer} (thin ring), I $= mr^2$
2. As r_{inner} approaches 0 (solid disk), I $= mr^2/2$

4 Conclusion

Comparing the results with the analytical solution showed that Simpson's rule works the best followed by Trapezoidal method and Riemann sum rule giving more errors.