Assignment - 1

B21312
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Question : 1

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x_1, x_2) \longmapsto x_1^2 + 0.5\, x_2^2 + x_1 x_2$$

(i) find the direction of greatest increase of f at $x = (1,1)$

$$D_{\vec{v}} f(x_1, x_2) = (\nabla f(x_1, x_2)) \cdot \hat{v}$$

$$= |\nabla f(x_1, x_2)|\, \underbrace{|\hat{v}| \cos\theta}$$

Max when $\cos\theta = 1$
$\theta = 0^\circ$
$\hat{v} \parallel \hat{\nabla} f(x_1, x_2)$

$\therefore$ in dir^n of gradient
$\Downarrow$
greatest increase

$$\nabla f(x_1, x_2) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right]$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\left(x_1^2 + 0.5\, x_2^2 + x_1 x_2 \right)$$

$$= 2x_1 + 0 + x_2$$
$$= 2x_1 + x_2$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}\left(x_1^2 + 0.5\, x_2^2 + x_1 x_2 \right)$$

$$= 0.5(2x_2) + x_1$$
$$= x_1 + x_2$$

$$\nabla f(x_1, x_2) = \left[2x_1 + x_2 , x_1 + x_2 \right]$$

at $(x_1, x_2) = (1, 1)$

$$\nabla f(1,1) = [3, 2] = 3\hat{x}_1 + 2\hat{x}_2$$

dir^n of greatest increase.

unit vector: ($\because$ direction) $(\hat{v})$

$$\hat{\nabla} f(1,1) = \left[\frac{3}{\sqrt{13}} , \frac{2}{\sqrt{13}} \right] = \frac{3}{\sqrt{13}}\hat{x}_1 + \frac{2}{\sqrt{13}}\hat{x}_2$$

(ii) Find the direction of greatest decrease of f at x (1,1).

$$D_{\vec{v}} f(x_1, x_2) = \hat{v} \cdot \nabla f(x_1, x_2)$$

$$= \|\hat{v}\| \, \|\nabla f(x_1, x_2)\| \cos\theta$$

min when $\hat{v} \not\parallel \hat{\nabla} f(x_1, x_2)$

$$\vec{v} = -\vec{\nabla} f(x_1, x_2) \ @ \ (1,1)$$

$$\vec{v} = -[3, 2] = [-3, -2]$$

$$= -3\hat{x}_1 - 2\hat{x}_2$$

unit vector ($\hat{v}$) $= \left[-\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}\right] = -\frac{3}{\sqrt{13}}\hat{x}_1 - \frac{2}{\sqrt{13}}\hat{x}_2$

$\therefore$ dir^n

dir^n of greatest decrease.

(iii) Find the dir^n in which f does not Instantly change at $x = (1,1)$.

A f^n f has no change in any dir^n that is orthogonal to $\nabla f(x_1, x_2)$.

i.e. $D_{\vec{v}} \nabla f(x_1, x_2) = \|\vec{v}\| \, \|\nabla f(x_1, x_2)\| \cos\theta$ $= 0$

0 as $\theta = 90°$

Let dir^n vector : $\vec{u} = \langle u_x, u_y \rangle$

$$\nabla f = \langle 2x_1 + x_2, \ x_1 + x_2 \rangle$$

$$\nabla f \big|_{x=(1,1)} = \langle 3, 2 \rangle$$

$$3u_x + 2u_y = 0$$

also $\vec{u}$ is dir^n vector:

$$\|\vec{u}\| = 1$$

$$u_x^2 + u_y^2 = 1$$

$$u_x = -\frac{2}{3} u_y$$

$$\frac{4}{9} u_y^2 + u_y^2 = 1$$

$$\frac{13}{9} u_y^2 = 1$$

$$u_y = \pm \frac{3}{\sqrt{13}}$$

$$u_x = \mp \frac{2}{\sqrt{13}}$$

$$\vec{u} = \pm \left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$+$: $-\frac{2}{\sqrt{13}} \hat{x}_1 + \frac{3}{\sqrt{13}} \hat{x}_2$

$-$: $+\frac{2}{\sqrt{13}} \hat{x}_1 - \frac{3}{\sqrt{13}} \hat{x}_2$

(iv) $\tilde{x} : \mathbb{R} \longrightarrow \mathbb{R}^2$

$$t \longmapsto \tilde{x}(t)$$

s.t. $\forall t \in \mathbb{R}$

$$f(\tilde{x}(t)) = f(1,1)$$

$$\frac{\partial f(\tilde{x}(t))}{\partial t} = \frac{\partial f(\tilde{x}(t))}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial t} = 0$$

$$\Rightarrow \nabla f(\tilde{x}) \frac{\partial \tilde{x}}{\partial t} = 0$$

$$\therefore \nabla f(\tilde{x}) \perp^r \text{ to } \frac{\partial \tilde{x}}{\partial t}$$

let say $\tilde{x}(t) = \langle x_1(t), x_2(t) \rangle$

$\mathbb{R} \to \mathbb{R}^2$

$$f(\tilde{x}(t)) = f(1,1) = c$$

$$f(x_1(t), x_2(t)) = f(1,1) = c$$

derivation wrt t

$$f'(x_1(t), x_2(t)) = 0$$

$$\frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} = 0$$

dot Product

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) \cdot \left(\frac{dx_1}{dt}, \frac{dx_2}{dt} \right) = 0$$

$$\nabla f(x_1, x_2) \cdot \frac{\partial \tilde{x}}{\partial t} = 0$$

$$\nabla f(\tilde{x}) \cdot \frac{\partial \tilde{x}}{\partial t} = 0$$

$$\boxed{\nabla f(\tilde{x}) \perp^r \text{ to } \frac{\partial \tilde{x}}{\partial t}}$$

$\therefore$ tangent line $\frac{\partial \tilde{x}}{\partial t}$ is $\perp^r$ to the gradient $\nabla f(\tilde{x})$.

(e) Visualisation :

$$f(\tilde{x}(t)) = f(1,1)$$
$$= 1^2 + 0.5 * 1^2 + 1$$
$$= 2.5$$

$$f(\tilde{x}(t)) = 2.5$$

$$\tilde{x}(t) \longrightarrow t \; , \; x_2(t)$$

$$t^2 + 0.5\, x_2^2(t) + t\, x_2(t) = 2.5$$

$$0.5\, x^2 + t\, x + t^2 - 2.5 = 0$$

$$D = t^2 - 4 \times 0.5\,(t^2 - 2.5)$$

$$= t^2 - 2t^2 + 5$$
$$= -t^2 + 5 \quad = 5 - t^2$$

$$\frac{-b \pm \sqrt{5 - t^2}}{1}$$

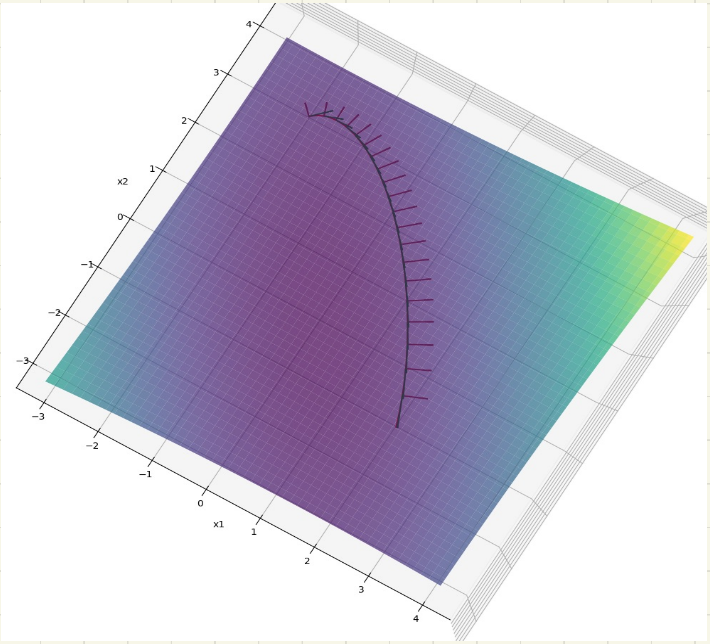
$$= -t + \sqrt{5 - t^2}.$$

$$\therefore\; x_2(t) = -t + \sqrt{5 - t^2}$$

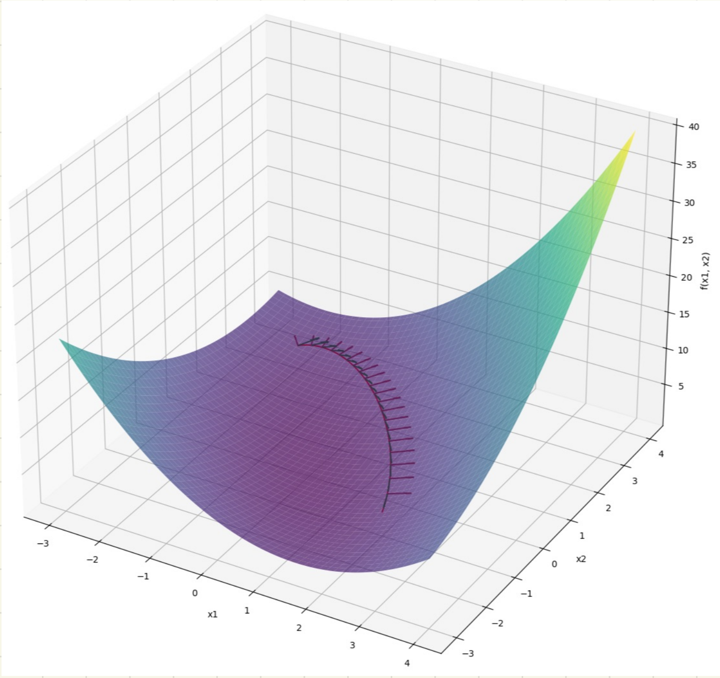
Parametric eq^n $\left(t,\; -t + \sqrt{5 - t^2}\right)$

$$\frac{\partial \tilde{x}}{\partial t} = \left(1 \;,\; -1 + \frac{1(-2t)}{2\sqrt{5 - t^2}}\right) = \left(1 \;,\; -1 - \frac{t}{\sqrt{5 - t^2}}\right)$$

Top View:



Side View:



Question : 2

(a) $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ (convex functions)

To show : $f+g : \mathbb{R} \to \mathbb{R}$

$x \longmapsto f(x) + g(x)$ is convex.

$\because$ f & g are convex functions

let $x, y \in \mathbb{R}$

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) \quad \text{--- (i)}$$

$$g(tx + (1-t)y) \leq t g(x) + (1-t) g(y) \quad \text{--- (ii)}$$

$\forall\, t \in [0,1]$

add (i) & (ii)

$$f(tx + (1-t)y) + g(tx + (1-t)y) \leq t\,(f(x) + g(x)) + (1-t)(f(y) + g(y))$$

let $h = f + g$

$$h(tx + (1-t)y) = f(tx + (1-t)y) + g(tx + (1-t)y)$$

$$h(x) = f(x) + g(x)$$
$$h(y) = f(y) + g(y)$$

$$h(tx + (1-t)y) \leq t\, h(x) + (1-t)\, h(y)$$

$\therefore$ h is a convex f^n
$\therefore f+g$ is a convex f^n.

(b) given: g is additionally non decreasing function

i.e. $g(y) \geq g(x)$ if $y > x$

Let $h = g \circ f$

To Prove h is convex:

$$h(tx + (1-t)y) \leq t h(x) + (1-t) h(y) \quad \forall x, y \in \mathbb{R} \;\&\; t \in [0,1]$$

$$g(f(tx + (1-t)y)) \leq t\, g(f(x)) + (1-t)\, g(f(y))$$

$\because$ f is a convex function:

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) \quad \forall t \in [0,1]$$

$\because$ g is non decreasing function:

$$g(f(tx + (1-t)y)) \leq g(t f(x) + (1-t) f(y)) \quad \text{—— (A)}$$

$\because$ g is also a convex function:

$$g(t f(x) + (1-t) f(y)) \leq t\, g(f(x)) + (1-t)\, g(f(y)) \quad \text{—— (B)}$$

$\forall t \in [0,1]$

from (A) & (B) :

$$g(f(tx + (1-t)y)) \leq t\, g(f(x)) + (1-t)\, g(f(y))$$

$$h(tx + (1-t)y) \leq t\, h(x) + (1-t)\, h(y) \quad \forall\, t \in [0,1] \;\&\; x, y \in \mathbb{R}$$

$\therefore$ h is a convex f^n

$\therefore$ gof is a convex function

Question : 3

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x_1, x_2) \longmapsto e^{\pi x_1} - \sin(\pi x_2) + \pi x_1 x_2$$

(a) gradient of f (∇f) :

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]$$

$$\frac{\partial f}{\partial x_1} = e^{\pi x_1}(\pi) - 0 + \pi x_2$$

$$\frac{\partial f}{\partial x_2} = 0 - \pi\cos(\pi x_2) + \pi x_1$$

$$\nabla f(\bar{x}) = \nabla f(x_1, x_2) = \left[\pi e^{\pi x_1} + \pi x_2 , -\pi\cos(\pi x_2) + \pi x_1\right]$$

$\bar{x} = (x_1, x_2)$

$$\nabla f(\bar{x}) = \nabla f(x_1, x_2) = \left[\pi e^{\pi x_1} + \pi x_2 , -\pi\cos(\pi x_2) + \pi x_1\right]$$

(b) Hessian of f for an arbitrary $\mathbf{x}$:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$f = e^{\pi x_1} - \sin(\pi x_2) + \pi x_1 x_2$$

$$\frac{\partial f}{\partial x_1} = \pi e^{\pi x_1} - 0 + \pi x_2 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = \pi$$

$$\frac{\partial^2 f}{\partial x_1^2} = \pi^2 e^{\pi x_1}$$

$$\frac{\partial f}{\partial x_2} = 0 - \pi \cos(\pi x_2] + \pi x_1$$

$$\frac{\partial^2 f}{\partial x_2^2} = +\pi^2 \sin(\pi x_2) + 0$$

$$\Rightarrow \pi^2 \sin(\pi x_2)$$

$$H = \begin{bmatrix} \pi^2 e^{\pi x_1} & \pi \\ \pi & \pi^2 \sin(\pi x_2) \end{bmatrix}$$

for $\mathbf{x} = (x_1, x_2)$ (arbitrary)

(c) State the first order polynomial $T_{1,a}(\mathbf{x})$ expanded around point $a = (0, 1)$.

$$f \in C^1$$

$$f(x) = \underbrace{f(a) + \nabla f(a)^T (x-a)}_{T_1(x,a) \text{ or } T_{1,a}(x)} + R_1(x,a)$$

$$f(a) = f(x_1,x_2)\Big|_{a=(0,1)} = e^{\pi x_1} - \sin(\pi x_2) + \pi x_1 x_2 \Big|_{a=(0,1)}$$

$$= 1 - \sin(\pi)$$

$$= 1$$

$$\nabla f(x) = \begin{bmatrix} \pi e^{\pi x_1} + \pi x_2 \\ -\pi \cos(\pi x_2) + \pi x_1 \end{bmatrix}$$

$$\nabla f(a) = \nabla f(0,1) = \begin{bmatrix} 2\pi \\ \pi \end{bmatrix}$$

$$(\mathbf{x} - \mathbf{a}) = \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix}$$

$$\nabla f(a)^T = \begin{bmatrix} 2\pi & \pi \end{bmatrix}$$

$$\nabla f(a)^T (x-a) = \begin{bmatrix} 2\pi & \pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix}$$

$$= 2\pi x_1 + \pi (x_2 - 1)$$

$$T_{1,a}(x) = f(a) + \nabla f(a)^T (x-a)$$

$$= 1 + 2\pi x_1 + \pi(x_2 - 1)$$

$$\boxed{T_{1,a}(x) = 1 + 2\pi x_1 + \pi(x_2 - 1)}$$

(d) Determine second order taylor polynomial $T_{2,a}(x)$ expanded around point $a = (0,1)$

$$f(x) = \underbrace{f(a) + \nabla f(a)^T (x-a) + \frac{1}{2}(x-a)^T H(a)(x-a)}_{T_{2,a}(x)} + R_2(x,a)$$

$$H(x) = \begin{bmatrix} \pi^2 e^{\pi x_1} & \pi \\ \pi & \pi^2 \sin(\pi x_2) \end{bmatrix}$$

$$H(a) = H(0,1) = \begin{bmatrix} \pi^2 & \pi \\ \pi & 0 \end{bmatrix}$$

$f(a) = 1$ (calculated in Part d)

$\nabla f(a)^T \cdot (x-a) = 2\pi x_1 + \pi(x_2 - 1)$ (calculated in Part d)

$$(x-a)^T H(a) = \begin{bmatrix} x_1 & x_2 - 1 \end{bmatrix} \begin{bmatrix} \pi^2 & \pi \\ \pi & 0 \end{bmatrix}$$

$$T_{2,a}(x) = 1 + (2\pi \quad \pi)\begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix} + \frac{1}{2}(x_1 \quad x_2 - 1)\begin{pmatrix} \pi^2 & \pi \\ \pi & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix}$$

(e) Determine if $T_{2,a}$ is a convex function ?

$T_{2,a}(x)$ is a multivariate quadratic function

to check where $T_{2,a}$ is a convex f^n or not we need to do the eigen analysis of its Hessian Matrix

$$q(x) = x^T A x + b^T x + c$$

$$\hookrightarrow H = 2A$$

$$\frac{1}{2}\begin{pmatrix} \pi^2 x + \pi(y-1) & \pi x \end{pmatrix}$$

$$\begin{pmatrix} x \\ y-1 \end{pmatrix}$$

Comparing $T_{2,a}(x)$ & $q(x)$

$$A = \frac{1}{2}\begin{bmatrix} \pi^2 & \pi \\ \pi & 0 \end{bmatrix}$$

$$\frac{1}{2}(\pi^2 x + \pi(y-1))x$$

$$+ \frac{1}{2}\pi x(y-1)$$

$$\therefore H = 2A = \begin{bmatrix} \pi^2 & \pi \\ \pi & 0 \end{bmatrix}$$

$$|H - \lambda I| = 0$$

$$\begin{vmatrix} \pi^2 - \lambda & \pi \\ \pi & -\lambda \end{vmatrix} = 0$$

$$(\pi^2-\lambda)(-\lambda) - \pi^2 = 0$$

$$(\pi^2-\lambda)\lambda + \pi^2 = 0$$
$$\pi^2\lambda - \lambda^2 + \pi^2 = 0$$

$$\lambda^2 - \pi^2\lambda - \pi^2 = 0$$

$$\lambda = \frac{\pi^2 \pm \sqrt{\pi^4 + 4\pi^2}}{2}$$

$$= \frac{\pi^2 \pm \pi\sqrt{\pi^2+4}}{2}$$

$\lambda_1 = 10.7847$

$\lambda_2 = -0.9151$

$\therefore$ not +ve semi definite

$\therefore$ $T_{2,a}(x)$ is not convex

Question : 4

$$f : [-1, 2] \to \mathbb{R}$$
$$x \mapsto e^{x^3 - 2x^2}$$

(a) Compute f'

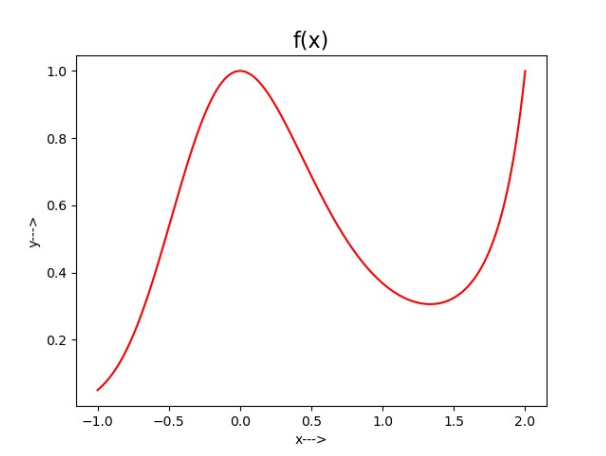
$$f' = \frac{d}{dx}\left(e^{x^3-2x^2}\right)$$
$$= e^{x^3-2x^2}(3x^2 - 4x)$$
$$f' = (3x^2 - 4x)\, e^{x^3-2x^2}$$

(b) Plot f and f' with Python ?

Code for f :

```
f=lambda x: np.exp(x**3-2*x**2)
x=np.linspace(-1,2,1000)
y=f(x)
pt.plot(x,y,color="red")
pt.xlabel("x--->")
pt.ylabel("y--->")
pt.title("f(x)",fontsize=16)
pt.savefig('f(x).png')
```

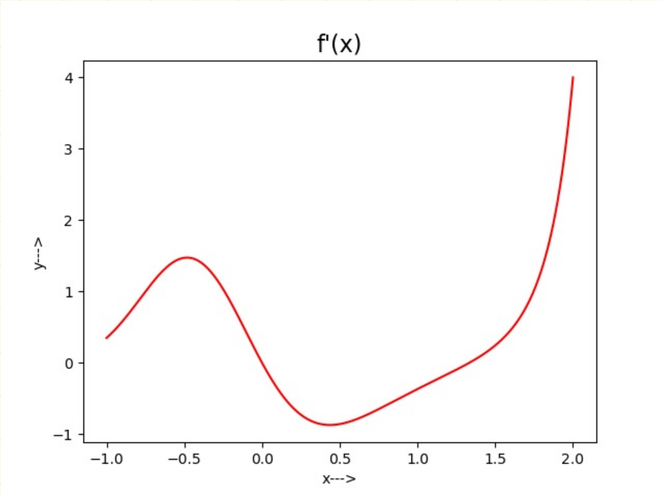
Plot :



Code for f' :

```
f_dash=lambda x: (3*x**2-4*x)*np.exp(x**3-2*x**2)
x=np.linspace(-1,2,1000)
y=f_dash(x)
pt.plot(x,y,color="red")
pt.xlabel("x--->")
pt.ylabel("y--->")
pt.title("f'(x)",fontsize=16)
pt.savefig('f\'(x).png')
```

Plot for f' :



(c) Find all possible candidates x^* for maxima & minima.

possible candidates are end points i.e. $x = -1, 2$ and where $f'(x) = 0$

$$(3x^2 - 4x)\underbrace{e^{x^3 - 2x^2}}_{\neq 0} = 0$$

$$3x^2 - 4x = 0$$

$$x = 0$$
$$x = 4/3$$

$\therefore$ Possible candidates are $-1, 2, 0, 4/3$.

(d) Compute f''

$$f' = (3x^2 - 4x)\, e^{x^3 - 2x^2}$$

$$f'' = \frac{d}{dx}\left((3x^2 - 4x)\, e^{x^3 - 2x^2}\right)$$

$$= (6x - 4)\, e^{x^3 - 2x^2} + (3x^2 - 4x)^2 e^{x^3 - 2x^2}$$

(e) Determine if the candidates are local maxima, minima or neither.

@ $x = -1$

$$f(x) = e^{(x^3 - 2x^2)} = e^{(-1-2)} = e^{-3}$$
$$= 0.0498$$

$$f'(x)\Big|_{x=-1} = (3+4)\, e^{-3} > 0$$

i.e. f^n is increasing to right of $x=-1$
$\therefore$ local minima

@ $x = 0$

$$f'(x) = 0$$

$$f''(x)\Big|_{x=0} = (-4)\, e^{0} + (0)$$
$$= -4$$

$$f''(x) < 0$$

$\therefore$ local maxima.

$$f(x)\Big|_{x=0} = e^{0} = 1$$

@ $x = 4/3$

$$f'(x) = 0$$

$$f''(x)\Big|_{x=4/3} = \left(\overset{2}{\cancel{6}}\cdot\frac{4}{\cancel{3}} - 4\right) e^{(4/3)^3 - 2(4/3)^2} + \left(3\cdot\left(\frac{4}{3}\right)^2 - 4\cdot\frac{4}{3}\right)^2 e^{(4/3)^3 - 2(4/3)^2}$$

$$= 4\, e^{(4/3)^3 - 2(4/3)^2} + \cancelto{0}{\left(\frac{16}{3} - \frac{16}{3}\right)^2} e^{(4/3)^3 - 2(4/3)^2}$$

$$> 0$$

$\therefore$ local minimum

$$f(x) = e^{x^3 - 2x^2}$$

$$= e^{(4/3)^3 - 2(4/3)^2}$$

$$= e^{\frac{64}{27} - 2\cdot\frac{16}{9}}$$

$$= e^{\frac{64}{27} - \frac{32}{9}}$$

$$= e^{\frac{64-96}{27}} = e^{-\frac{32}{27}}$$

$$= 0.306$$

@ $x = 2$

$$f(x) = e^{(x^3 - 2x^2)} = e^0 = 1$$

$f'(x)\Big|_{x=2} > 0$ $\therefore$ f^n is $\uparrow$ as $x = 2$ is approached from Left
$\therefore$ b is a local maxima

(f) global maxima and global minima of f :

global maxima $\Rightarrow$ max $\{ f(-1), f(0), f(\frac{1}{3}), f(2) \}$

$\Rightarrow 1$
at $x = 0$ & 2

global minima = min $\{ f(-1), f(0), f(\frac{1}{3}), f(2) \}$

$= e^{-3} = 0.0498$
@ $x = -1$