Gårding's Inequality

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Abstract

This is an expository paper on the prerequisites for Gårding's inequality. Proposed in [GÅ53] by Lars Gårding, this inequality has applications in the study of weak solutions to elliptic partial differential equations.

This will begin by introducing Lebesgue integration, a stronger form of Riemann integration, and continue into function spaces, specifically L_p and Sobolev spaces. Then we will discuss differential operators. Finally, we will state Gårding's inequality and give one application. Discussion of applications is adapted from [RR04].

Contents

1 Lebesgue integration

1.1 A motivating example

The typical Riemann integration we use works for many functions, but not for all. Consider the *Dirichlet function*

$$1_{\mathbb{Q}} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases} \tag{1}$$

on the interval $[0, 1]$. Since rationals and irrationals are dense in the reals, all upper Darboux sums will come out to 1, and all lower Darboux sums will come out to 0. We conclude that the function is not Riemann integrable. The *Lebesgue integral* seeks to give a larger space of integrable functions, including Equation 1.

1.2 The Lebesgue outer measure

This definition comes from Prof. Rycroft and [nLa22].

Definition 1.1. For an open interval $I = (a, b)$, let $|I| = b - a$. The **Lebesgue outer measure** for a subset E of the real numbers, denoted $\lambda(E)$, is defined as

$$\lambda(E) := \inf \left\{ \sum_{j=1}^{\infty} |I_j| : (I_j)_{j \in \mathbb{N}} \text{ such that } B \subseteq \bigcup_{i=1}^{\infty} I_j \right\}.$$

Sets used in the Lebesgue integral must satisfy the **Cathédory criterion**:

$$\lambda(B) = \lambda(A \cap B) + \lambda(A \cap B^C), \quad \text{for all } A \subseteq \mathbb{R}.$$

1.3 The Lebesgue integral

While Riemann integrals start with splitting the *domain* into smaller and smaller intervals, the Lebesgue integral splits the *range* into smaller and smaller intervals.

Formally, we define *indicator functions* as

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Then we can create *simple function* in terms of sums of these indicator functions and for subsets $S_i \subseteq \mathbb{R}$,

$$s(x) = \sum_n a_n 1_{S_n}.$$

For this simple function, we can define an integral as

$$\int s d\lambda = \sum_n \lambda(S_n) a_n.$$

Definition 1.2. Given that a function is non-negative, we then define the **Lebesgue integral** as

$$\int f d\lambda := \sup \left\{ \int s d\lambda : 0 \leq s \leq f, s \text{ is a simple function} \right\}.$$

If we want to deal with functions with positive and negative ranges, then we can just apply the Lebesgue integral to each part separately.

Example 1.3. For the Dirichlet function,

$$\begin{aligned} \int_0^1 1_{\mathbb{Q}}(x) d\lambda(x) &= \lambda([0,1] \cap \mathbb{Q}) \cdot 1 + \lambda([0,1] \setminus \mathbb{Q}) \cdot 0 \\ &= 0 \cdot 1 + 1 \cdot 0 \\ &= 0. \end{aligned}$$

This matches with out intuition of the "amount" of rationals compared to irrationals on the interval $[0, 1]$. □

Theorem 1.4. *Functions that are Riemann-integrable on a closed interval are Lebesgue-integrable.*

2 The Lp space of functions

2.1 Normed vector spaces

Definition 2.1. A **norm** is a way of formalizing "length" in certain spaces. It is denoted $\|\cdot\|$. There are 4 properties a norm must satisfy:

1. $\|x\| \geq 0$.
2. $\|x\| = 0 \implies x = 0$.
3. For a scalar α, $\|\alpha x\| = |\alpha| \, \|x\|$.
4. The **triangle inequality** holds: $\|x + y\| \leq \|x\| + \|y\|$.

A vector space equipped with a norm is a **normed vector space**. We also have an induced distance function by the norm, $d(x, y) = \|y - x\|$. Therefore, a normed space automatically a metric space.

2.2 The Lp space and norm

Definition 2.2. The **Lebesgue space (L^p space)** is the space of functions from a set that can be assigned a Lebesgue measure $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{R}$ or $\mathbb{C}$ such that the following norm converges:

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}$$

We call this the **L^p norm** of f. The L^p space of functions $f : \Omega \to \mathbb{R}$ is denoted $L^p(\Omega)$.

Proposition 2.3. The Lebesgue space is a vector space, with

$$(f+g)(x) = f(x) + g(x) \in L^p$$
$$(\lambda f)(x) = \lambda f(x) \in L^p$$

Remark 2.1. Most of the norm properties are immediately satisfied, except for the triangle inequality, $\|x+y\| \leq \|x\| + \|y\|$, which follows from *Minkowski's inequality*.

2.3 Bilinear forms and inner products

Definition 2.4. A **bilinear form** on a vector space V is a function $B[\cdot,\cdot] : V \times V \to K$ such that $\langle x, y\rangle$ is linear in x and y. In symbols,

$$B[ax_1 + bx_2, y] = a \cdot B[x_1, y] + b \cdot B[x_2, y],$$
$$B[x, ay_1 + by_2] = a \cdot B[x, y_1] + b \cdot B[x, y_2].$$

K is the field which V is a vector space over. Usually K is $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.5. An **inner product** is an bilinear form $\langle\cdot,\cdot\rangle$ such that

1. $\langle ax_1 + bx_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle$, $\langle x, ay_1 + by_2\rangle = a\langle x, y_1\rangle + b\langle x, y_2\rangle$.
2. *Reflexive:* $\langle y, x\rangle = \overline{\langle x, y\rangle}$, where $\overline{z}$ denotes the *complex conjugate* of z.
3. *Positive Definite:* $\langle x, x\rangle = 0$ if $x = 0$, and $\langle x, x\rangle > 0$ if $x \neq 0$.

Remark 2.2. The complex conjugate is not useful in $\mathbb{R}$, but if the function is mapping to $\mathbb{C}$, it is commonly called a *Hermitian inner product*, and it makes use of the complex conjugate.

A vector space equipped with an inner product is an **inner product space**.

Example 2.6. L^2 has an induced inner product with respect to a measure μ:

$$\langle f, g\rangle_{L^2} = \int_\Omega fg d\mu.$$

In fact, L^2 is the only L^p space with this property.

Remark 2.3. The inner product induces a norm, which therefore induces a metric.

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

$$d(x, y) = \sqrt{\langle x - y, x - y\rangle}.$$

3 Sobolev spaces of functions

3.1 Weak differentiation

Let Ω be an open, connected subset of $\mathbb{R}$. Let ϕ be a differentiable function on $[a, b]$ such that $\phi(a) = \phi(b) = 0$. Then by integration by parts,

$$\int_a^b f\phi' dx = [f(x)\phi(x)]_a^b - \int_a^b f'\phi dx = -\int_a^b f'\phi dx.$$

Even when f' does not exist, we may be able to show that some function can take the place of f'. To generalize, let Ω be an open, connected subset of $\mathbb{R}^n$ (we call this a **domain**). If there exists a function g_i such that

$$\int_\Omega f\partial_i\phi dx = -\int_\Omega g_i\phi dx,$$

for all ϕ that are continuous, smooth (infinitely differentiable), and the set of all values x such that $\phi(x) \neq 0$ is compact, then define the **weak derivative** as $\partial_i f := g_i$.

Definition 3.1. A **multi-index** is shorthand for writing partial derivatives. A multi-index α is a tuple of numbers that correspond to how many times each dimension is partially differentiated. $|\alpha|$ gives the order of the partial derivative.

For higher weak derivatives, if the integrable function u has

$$\int_\Omega u\partial^\alpha \phi = (-1)^{|\alpha|} \int_\Omega v\phi,$$

for some integrable function v, and for all ϕ as defined before, then it is **α-times weakly differentiable**.

3.2 Sobolev spaces

Definition 3.2. For a domain Ω in $\mathbb{R}^n$, the **Sobolev space $W^{k,p}(\Omega)$** is the space of functions $f \in L^p(\Omega)$ that are *weakly* differentiable up to k times. Moreover, we require that each of the derivatives also has a finite L^p norm and be continuous.

Remark 3.1. When $p = 2$, the Sobolev space may be written as $H^k(\Omega)$. Since $H^k(\Omega) \subseteq L^2(\Omega)$, there exists an induced inner product on $H^k(\Omega)$. This is an example of a **Hilbert space**, which is an inner product space that is a complete metric space.

Sobolev spaces admit the norm

$$\|f\|_{W^{k,p}} := \left(\sum_{i=0}^{k} \left\|f^{(i)}\right\|_{L_p}^p\right)^{\frac{1}{p}} = \left(\sum_{i=0}^{k} \int \left|f^{(i)}(t)\right|^p \, dt\right)^{\frac{1}{p}}.$$

3.3 The k-extension property

While there is a general definition for topological vector spaces, we examine the specific definition for X, Y as normed vector spaces.

Definition 3.3. A linear operator $L : X \to Y$ is **bounded** if there exists $M > 0$ such that for all $x \in X$,

$$\|Lx\|_Y \leq M \|x\|_X .$$

Definition 3.4. If there exists a bounded linear operator $E : W^{k,2}(\Omega) \to W^{k,2}(\mathbb{R}^n)$ such that Eu with its domain restricted to Ω equals u for all functions u in $W^{k,2}(\Omega)$, then Ω satisfies the **k-extension property**.

The k-extension property is a classification of domains that is a necessity for Gårding's inequality and many other results in PDE analysis.

4 Differential Operators

4.1 Elliptic operators

Definition 4.1. Let α represent a multi-index. Let L be a partial differential operator, defined for a function u with domain Ω in $\mathbb{R}^n$.

$$Lu = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha u.$$

L is **elliptic** if for every $x \in \Omega$ and every non-zero $\xi \in \mathbb{R}^n$,

$$(-1)^k \sum_{|\alpha|=2k} a_\alpha(x)\xi^\alpha > 0$$

It can be shown that the order of the PDE must be even to be elliptic ($m = 2k$). We then make a stronger condition, **uniform ellipticity**, for an operator of order $2k$:

$$(-1)^k \sum_{|\alpha|=2k} a_\alpha(x)\xi^\alpha > C\,|\xi|^{2k}.$$

for a positive constant C and for every $\xi \in \mathbb{R}^n$.

4.2 Induced bilinear forms

Differential operators induce their own bilinear form. In order to construct a bilinear form for a differential operator L, we perform integration by parts on L:

$$\int_\Omega \phi L u dx = \sum_{0\le|\sigma|,|\gamma|\le k} \int_\Omega a_{\sigma\gamma}(x)\partial^\gamma u \partial^\sigma \phi dx,$$

for mutli-indices γ and σ. We define the induced bilinear form by L as

$$B[u,v] := \sum_{0\le|\sigma|,|\gamma|\le k} \int_\Omega a_{\sigma\gamma}(x)\partial^\gamma u \partial^\sigma v dx$$

5 Gårding's inequality

5.1 Inequality statement

Theorem 5.1 (Gårding's inequality)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain that has the k-extension property. Let u be a function in $W^{k,2}(\Omega)$ and L be a uniformly elliptic differential operator of order $2k$, written as*

$$(Lu)(x) = \sum_{0\le|\alpha|,|\beta|\le k} (-1)^{|\alpha|}\partial^\alpha\left(A_{\alpha\beta}(x)\partial^\beta u(x)\right).$$

If $|\alpha| = |\beta| = k$, then require $A_{\alpha\beta}$ to be bounded and continuous on the closure of Ω. If $|\alpha|, |\beta| \le k$, then require $A_{\alpha\beta} \in L^\infty(\Omega)$. If all of the above are satisfied, then there exists constants $C > 0$ and $G \ge 0$ such that

$$B[u,u] + G\|u\|_{L^2}^2 \ge C\|u\|_{W^{k,2}}^2 \text{ for all } u \in W_0^{k,2}(\Omega),$$

where $B[u,v]$ is a bilinear form induced by the differential operator L, and $W_0^{k,2}(\Omega)$ is the space of functions $u \in W^{k,2}(\Omega)$ such that u evaluates to 0 on the boundary of Ω.

Gårding's inequality relates the L^2 and Sobolev norm of a function u with the induced bilinear form of a uniformly elliptic differential operator.

5.2 Application: A Dirichlet boundary problem

A Dirichlet boundary problem is a common type of elliptic boundary-value problem. Consider a bounded region $\Omega \subseteq \mathbb{R}^n$, a function f, and an elliptic operator L of order $2k$. The solution to a Dirichlet boundary problem we will be studying is a function $u(x, y)$ that satisfies

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\partial\Omega$ denotes the *boundary* of Ω: the closure minus the interior of a set.

There are three levels of solutions to the Dirichlet problem to discuss. First are the **classical solutions**, which require the function u to be continuous, $2k$ times differentiable, and all of its derivatives to also be continuous.

Lightening our criterion to allow for weak derivatives, there are **strong solutions**, which loosen the requirement of u to being $\sim 2k$ times *weakly* differentiable.

Finally, u is a **weak solution** of this Dirichlet problem if $B[v, u] = \int_\Omega fv dx$ for every $v \in W_0^{k,2}(\Omega)$. Even fewer weak derivatives are required, effectively reducing the "smoothness" needed for a function to be considered a solution.

As a general hierarchy,

$$\{\text{classical sol'ns}\} \subseteq \{\text{strong sol'ns}\} \subseteq \{\text{weak sol'ns}\}.$$

Since our classification of weak solutions to the boundary problem requires the induced bilinear form B, it should be clear that the Gårding inequality is useful here. Indeed, the Gårding inequality is used to show the existence of weak solutions to this Dirichlet problem.

Theorem 5.2. *Let L be a differential operator of order $2k$ that satisfies the conditions of 5.1. Then there exists $\lambda \geq 0$ such that for $\lambda_0 \geq \lambda$, the Dirichlet problem for the operator $L + \lambda_0$ has a unique weak solution.*

Remark 5.1. The proof for 5.2 requires both the Gårding inequality and the *Lax–Milgram lemma*, a result that guarantees the existence of unique weak solutions to the Dirichlet problem. This result was proven in [LM55].

References

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