§1 December 1st, 2022

This is a subfile package test.

§1.1 Symmetry groups

I will assume knowledge of permutation groups and the definition of a group action. We denote the group of all permutations of $\{1, 2, \ldots, n\}$ as S_n.

Recall that groups are (1,2) closed under an associative binary operation, (3) each element has an inverse, and (4) the group has an indentity. Composition of permutations is read right to left; $g \circ f$ means do f first, then g.

Example 1.1 (Permutation groups)**.** Here are the two simplest permutation groups.

1. $C_n = \langle \rho_n \rangle$ is a cyclic permutation group, if ρ_n is a symmetrical rotation of $360°/n$ degrees.
2. The dihedral permutation group $D_n = \langle \rho_n \rangle \cup \{\tau_1, \ldots, \tau_n\}$ consists of all rotations and reflections of an n-gon is a group.

§1.2 Colorings

Let $\mathbf{c} = (c(1), c(2), \ldots c(n))$ be a coloring of $\{1, 2, \ldots, n\}$. Given a permutation

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \in S_n.$$

$f * \mathbf{c}$ is the **coloring** of f, and is defined as

$$(f * \mathbf{c})(i_k) := c(k), \quad k \in \{1, 2, \ldots, n\}.$$

or

$$(f * \mathbf{c})(l) = c(f^{-1}(l)).$$

Definition 1.2. The **set of colorings** ($\mathcal{C}$) requires the property that for all $f \in G \leq S_n$, and for all $\mathbf{c} \in \mathcal{C}$, $f * \mathbf{c} \in \mathcal{C}$.

We note that for $f, g \in S_n$,

$$(fg) * \mathbf{c} = (g \circ f) * \mathbf{c} = g * (f * \mathbf{c}),$$

as is expected of group actions.

Definition 1.3. Two colorings $\mathbf{c}_1$ and $\mathbf{c}_2$ are **equivalent** if $\exists f \in G$ s.t.

$$f * \mathbf{c}_1 = \mathbf{c}_2.$$

We can show that $\mathbf{c}_1 \sim \mathbf{c}_2$ as defined before is an *equivalence relation.*

§1.3 Burnside's Lemma

Definition 1.4. Given a permutation group G and a set of colorings $\mathcal{C}$, the **stabilizer** is

$$G(\mathbf{c}) := \{f : f \in G, f * \mathbf{c} = \mathbf{c}\}.$$

the **fixed points** by $f \in G$ are

$$\mathcal{C}(f) := \{\mathbf{c} : \mathbf{c} \in \mathcal{C}, f * \mathbf{c} = \mathbf{c}\}.$$

Proposition 1.5

$G(\mathbf{c})$ is a group, and $g * \mathbf{c} = f * \mathbf{c} \iff f^{-1} \circ g \in G(\mathbf{c})$.

Proof. Proving $G(\mathbf{c})$ is a group is omitted. Assume that $g * \mathbf{c} = f * \mathbf{c}$, this is true iff

$$(f^{-1} \circ g) * \mathbf{c} = f^{-1} * (g * \mathbf{c}) = f^{-1} * (f * \mathbf{c}) = (f^{-1} \circ f) * \mathbf{c} = 1 * \mathbf{c} = \mathbf{c}.$$

Thus $f^{-1} \circ g$ does not change $\mathbf{c}$,

$$\iff f^{-1} \circ g \in G(\mathbf{c}).$$

□

Corollary 1.6

$$|\{f * \mathbf{c} : f \in G\}| = \frac{|G|}{|G(\mathbf{c})|}.$$

Proof. The permutations g that satisfy $g * \mathbf{c} = f * \mathbf{c}$ are the permutations in $H = \{f \circ h : h \in G(\mathbf{c})\}$.

Since $f \circ h = f \circ h' \implies h = h'$, the number of permutations of H

$$|\{f \circ h : h \in G(\mathbf{c})\}| = \frac{|G|}{|G(\mathbf{c})|}.$$

□

Theorem 1.7 (Burnside's Lemma)

Let G be a group of permutations of a set X, and let $\mathcal{C}$ be the set of colorings of X s.t. $f * \mathbf{c} \in \mathcal{C}\ \forall f \in G, \mathbf{c} \in \mathcal{C}$. the number of non-equivalent colorings in $\mathcal{C}$ (denoted $N(G, \mathcal{C})$) is

$$N(G, \mathcal{C}) = \frac{1}{|G|} \sum_{f \in G} |\mathcal{C}(f)|.$$

Proof. We count the number of pairs $(f, \mathbf{c})$ such that $f * \mathbf{c} = \mathbf{c}$. One way to get this value is

$$\sum_{f \in G} |\mathcal{C}(f)|.$$

This is equivalent looking at each $\mathbf{c}$ and finding all permutations such that $f * \mathbf{c} = \mathbf{c}$.

Therefore

$$\sum_{f \in G} |\mathcal{C}(f)| = \sum_{\mathbf{c} \in \mathcal{C}} |G(\mathbf{c})|.$$

From the last corollary,

$$\sum_{\mathbf{c} \in \mathcal{C}} |G(\mathbf{c})| = |G| \sum_{\mathbf{c} \in \mathcal{C}} \frac{1}{\text{colorings equivalent to } \mathbf{c}} = |G| \cdot N(G, \mathcal{C})$$

The RHS results from the contribution of every equivalence class being 1. Thus

$$N(G, \mathcal{C}) = \frac{1}{|G|} \sum_{f \in G} |\mathcal{C}(f)|.$$

□

§1.4 Applications of Burnside's lemma

Example 1.8 (Circular permutations). How many ways are there to arrange n distinct objects in a circle?

Consider the permutation group $C_n = \langle \rho_n \rangle$, where ρ_n is a $360°/n$ rotation. Let $\mathcal{C}$ be the $n!$ ways to color the n corners. We use Burnside's lemma to get

$$N(C_n, \mathcal{C}) = \frac{1}{n}(n! + 0 + 0 + \cdots + 0) = (n-1)!$$

Example 1.9 (Necklace beads). How many ways are there to arrange $n \geq 3$ differently colored beads into a necklace?

Instead we have the permutation group $D_n = \langle \rho_n, \tau \rangle$,

$$N(D_n, \mathcal{C}) = \frac{1}{2n}(n! + 0 + \cdots + 0) = \frac{(n-1)!}{2}.$$

Example 1.10 (Infinite multiset). Consider $S = \{\infty \cdot r, \infty \cdot b, \infty \cdot g, \infty \cdot y\}$. How many n permutations are there if we consider a left-to-right reading the same as a right-to-left one.

This has the group $G = \{1, \tau\}$, where 1 is the identity and

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Considering odd and even values of n, we find that

$$|\mathcal{C}(\tau)| = 4^{\lfloor \frac{n+1}{2} \rfloor}.$$

By Burnside's lemma,

$$N(G, \mathcal{C}) = \frac{4^n + 4^{\lfloor \frac{n+1}{2} \rfloor}}{2}.$$

§2 December 4th, 2022

§2.1 Cycle factorization

Example 2.1 (Factorizing $f \in S_8$).

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 4 & 1 & 3 & 2 & 7 \end{pmatrix} = [1,6,3,5] \circ [2,8,7] \circ [4].$$

This is because when we plug in 1, we go to 6, then if we plug in 6, we go to 3, and then to 5, and back to 1.

Every $f \in S_n$ can be written as the product of cycles. In an abstract algebra class, we may exclude the trivial cycles like the [4] in the example above. However, they will prove to be useful later, so we keep them.

For notation let $\#(f)$ for $f \in S_n$ be the number of cycles in the cycle factorization of f.

Theorem 2.2

For $f \in S_n$ for a set X, if we have k colors and want to find the number of fixed colorings under f, then it is given by

$$|\mathcal{C}(f)| = k^{\#(f)}$$

This is because every element in a cycle becomes dependent on each other, so each cycle can only be colored 1 color if it is to be fixed by the permutation.

§2.2 Cycle generating function

We want to find a way of getting the number of each cycles for a permutation. Let $f \in S_n$ be a permutation of X. If it has e_1 1-cycles, e_2 2-cycles, …, and e_n n-cycles, then

$$\sum_{i=1}^{n} i \cdot e_i = n.$$

Definition 2.3. Call the n-tuple $(e_1, e_2, \ldots, e_n)$ the **type** of f, denoted

$$\text{type}(f) := (e_1, e_2, \ldots, e_n).$$

Note that

$$\#(f) = \sum_{i=1}^{n} e_i.$$

To distinguish different permutations of the same type, we introduce n indeterminates $z_1, \ldots, z_n$, where z_k corresponds to a k-cycle ($k = 1, 2, \ldots, n$).

Definition 2.4. For f with $\text{type}(f) = (e_1, \ldots, e_n)$ the **monomial** of f is

$$\text{mon}(f) := z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n} = \prod_{i=1}^{n} z_i^{e_i}.$$

Definition 2.5. The **cycle index** of G is

$$P_G(z_1, z_2, \ldots, z_n) := \frac{1}{|G|} \sum_{f \in G} \text{mon}(f).$$

Example 2.6 (Cycle index)**.** The cycle index of D_4 is

$$P_{D_4}(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2).$$

To apply this polynomial to getting the number of each cycle,

Theorem 2.7

Let $|X| = n$ with k colors, and $\mathcal{C}$ is the set of all k^n colorings of X. Let G be a permutation group of X. Then

$$N(G, \mathcal{C}) = P_G(k, k, \ldots, k).$$

Proof. Apply Burnside's lemma and previous theorems. □

§3 December 9th, 2022

§3.1 Fixed amount of colors

We already know how to solve the following example:

Example 3.1. How many nonequivalent colorings are there of the corners of a regular 5-gon in which there corners are colored red and two are colored blue?

Proof. For the identity, there are 10 fixed colorings. None are fixed for rotations, and 2 fixed for each reflection. By Burnside's theorem, the number of colorings is 2. □

We seek to find a way to solve the general problem. Consider only having two colors. Let $\mathcal{C}_{p,q}$ be the set of colorings of X with p red, q blue, and $p + q = n$ total.

Suppose t_i i-cycles get assigned red for $1 \leq i \leq n$. For the number of red elements to be p, we need

$$p = \sum_{i=1}^{n} t_i \cdot i.$$

A solution to this satisfies

$$0 \le t_i \le e_i, \text{ for } 1 \le i \le n,$$

times

$$\binom{e_1}{t_1} \cdots \binom{e_n}{t_n}.$$

We find that the number of nonequivalent colorings can be found by the monomial of f by considering the coefficient of $r^p b^q$ in

$$(r+b)^{e_1} \cdots (r^n + b^n)^{e_n}.$$

Hence the number of nonequivalent colorings with red and blue is

$$P_G(r+b, r^2+b^2, \ldots, r^n+b^n).$$

§3.2 Pólya's theorem

Theorem 3.2 (Pólya's theorem)

Let X be a set, G be a permutation group on X and $\{u_1, \ldots, u_k\}$ be a set of colors. Let $\mathcal{C}$ be the set of all colorings of X. The generating function for the number of nonequivalent colorings of $\mathcal{C}$ according to the number of colors is

$$P_G(u_1 + \cdots + u_k, u_1^2 + \cdots + u_k^2, \ldots, u_1^n + \cdots + u_k^n).$$

The coefficient of $u_1^{p_1} \cdots u_k^{p_k}$ is the number of nonequivalent colorings in $\mathcal{C}$ with p_i elements colored u_i for $1 \le i \le k$.

If we let $u_i = 1$ for all i, then the equation matches with the lighter case from before,

$$P_G(k, k, \ldots, k).$$

Indeed, the sum is over all possible colorings of X with no limits on k colors.

Example 3.3. Calculate the generating function for the number of nonequivalent colorings of the corners of a square with two colors and three colors.

Proof. We found that

$$P_{D_4} = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2)$$

For two colors $r, d,$, the generating function can be calculated to

$$P_{D_4}(r+b, r^2+b^2, r^3+b^3, r^4+b^4) = r^4 + r^3 b + 2r^2 b^2 + r b^3 + b^4.$$

There are 6 nonequivalent colorings. Similarly, we can calculate P_{D_4} with $r^i + b^i + g^i$. □

Example 3.4 (Coloring faces and corners of a cube)**.** The symmetry group of a cube consists of

1. The indentity rotation with corner type $(8, 0, \ldots, 0)$ and face type $(6, 0, \ldots, 0)$.
2. $90, 180, 270$ degree rotation around opposite faces (3 of each) with corner types $(0, 0, 0, 2, 0, 0, 0, 0)$, $(0, 4, 0, 0, 0, 0, 0, 0)$, $(0, 0, 0, 2, 0, 0, 0, 0)$ respectively, and face types $(2, 0, 0, 1, 0, 0)$, $(2, 2, 0, 0, 0, 0)$, $(2, 0, 0, 1, 0, 0)$ respectively.
3. 180 degree rotation around opposite edges (6) with corner type $(0, 4, 0, 0, \ldots, 0)$, and face type $(0, 3, 0, \ldots, 0)$.
4. $120, 240$ degree rotations about opposite corners (4 of each) with corner types $(2, 0, 2, 0, \ldots, 0)$ for each, and face types $(0, 0, 2, 0, 0, 0)$ for each.

Hence the cycle index for the corner group G_C is

$$P_{G_C}(z_1, \ldots, z_8) = \frac{1}{24}(z_1^8 + 6z_4^2 + 9z_2^4 + 8z_1^2 z_3^2),$$

and for the face group is

$$P_{G_F}(z_1, \ldots, z_8) = \frac{1}{24}(z_1^6 + 6z_1^2 z_4 + 3z_1^2 z_2^2 + 6z_2^3 + 8z_3^2).$$

Hence the number of nonequivaent colorings of the corners with two colors is

$$r^8 + r^7 b + 3r^6 b^2 + 3r^5 b^3 + 7r^4 b^4 + 3r^3 b^5 + 3r^2 b^6 + rb^7 + b^8$$

and for the faces is

$$r^6 + r^5 b + 2r^4 b^2 + 3r^3 b^3 + 2r^2 b^4 + rb^5 + b^6.$$

We verify that the total number of nonequivalent colorings for the corners is 23 and for the faces is 10.

§3.3 Nonisomorphic graphs

Determine the number of nonisomorphic graphs of order (number of vertices) 4 with each possible number of edges.

Proof. The set of edges of any graph of order 4 ($V = \{1, 2, 3, 4\}$) is a subset of

$$X = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

Let $\mathcal{G}_4$ be the set of all nonisomorphic graphs of order 4. We can consider $E_1 \subseteq X$ as a "coloring" of its associated graph $H = (V, E_1)$ by letting it be the color 'y' if the edge is present and the color 'n' if not.

Consider $H_2 = (V, E_2)$. Let $f \in S_4$ permute the 4 vertices. The graphs are isomorphic $\iff$ $\{i, j\}$ is an edge of E_1 $\iff$ $\{f(i), f(j)\}$ is an edge of E_2. Consider the permutation group $H \leq S_6$, which permutes the 6 edges.

This problem is bijected to asking colorings of the set X that are equivalent.

The table summarizes the monomials.

Type	Monomial	# of permutations in H
$(6,0,0,0,0,0)$	z_1^6	1
$(2,2,0,0,0,0)$	$z_1^2z_2^2$	9
$(0,0,2,0,0,0)$	z_3^2	8
$(0,1,0,1,0,0)$	z_2z_4	6

The cycle index of H is

$$P_H = \frac{1}{24}\left(z_1^6 + 9z_1^2z_2^2 + 8z_3^2 + 6z_2z_4\right).$$

Now we substitute $z_j = y^j + n^j$. We get that

$$y^6 + y^5n + 2y^4n^2 + 3y^3n^3 + 2y^2n^4 + yn^5 + n^6.$$

The total number of nonisomorphic graphs of order 4 is 11. Indeed, $P_H(2,2,2,2,2,2) = 11$. $\square$