Chapter 5
PERT/CPM

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5.1 Introduction

- PERT (program evaluation and review technique) and CPM (critical path method are available to assist the project manager in carrying out these responsibilities.
- PERT and CPM have been used for a variety of projects, including the following types,

  Construction of a new plant, Research and development of a new product, NASA space exploration projects, Movie productions, Building a ship, Government-sponsored projects for developing a new weapons system, Relocation of a major facility, Maintenance of a nuclear reactor, Installation of a management information system, Conducting an advertising campaign, etc.

- Network models can be used as an aid in scheduling large complex projects that consist of many activities.
- If *the duration of each activity is known with certainty*, then the critical path method (CPM) can be used *to determine the length of time* required to complete a project.
- CPM also can be used to determine how long each activity in the project can be *delayed* without delaying the completion of the project.

- If *the duration of the activities is not known with certainty*, the Program Evaluation and Review Technique (PERT) can be used *to estimate the probability that the project will be completed by a given deadline*.
- To apply CPM and PERT, we need a list of the activities that make up the project.
- The project is considered to be completed when all the activities have been completed. For each activity, there is a set of activities (called the **predecessors** of the activity) that must be completed before the activity begins.

- A project network is used to represent the precedence relationships between activities.
- All activities will be represented by **directed arcs**, and **nodes** will be used to represent the completion of a set of activities.
- For this reason, we often refer to the nodes in our project network as events. This type of project network is called an **AOA** (activity on arc) network.

- To understand how an AOA network represents precedence relationships, suppose that activity A is a predecessor of activity B. Each node in an AOA network represents the completion of one or more activities.

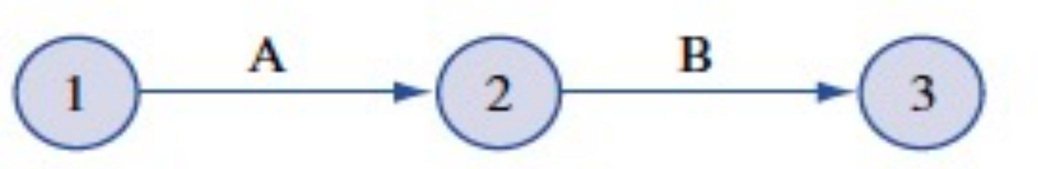
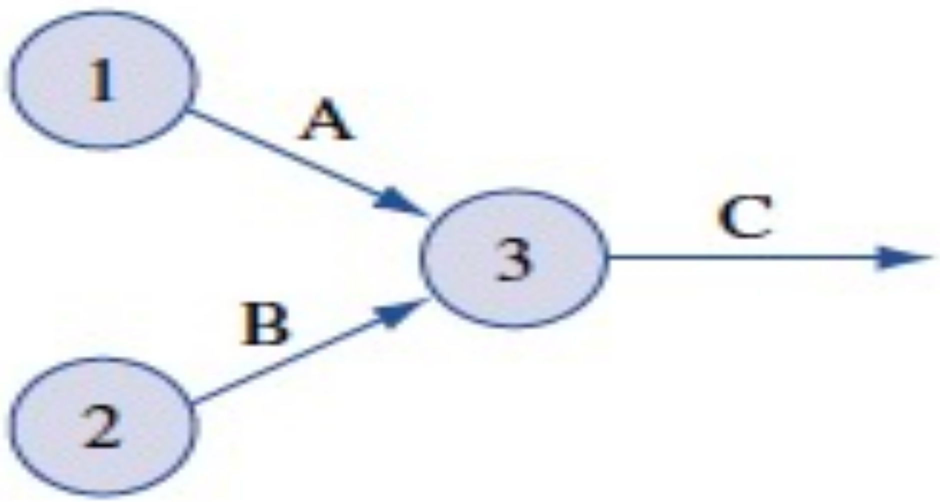


Figure 5.1 represents the completion of activity A and the beginning of activity B.



In Figure 5.2 Node 3 represents the event that activities A and B are completed

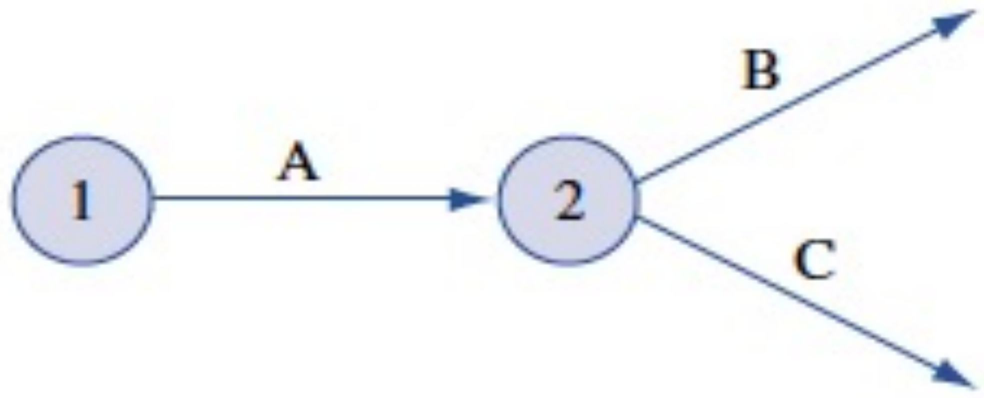


Figure 5.3 Shows activity A as a predecessor of both activities B and C.

Given a list of activities and predecessors, an AOA representation of a project (called a project network or project diagram) can be constructed by using the following rules:

1) Node 1 represents the start of the project. An arc should lead from node 1 to represent each activity that has no predecessors.
2) A node (called the finish node) representing the completion of the project should be included in the network.
3) Number the nodes in the network so that the node representing the completion of an activity always has a larger number than the node representing the beginning of an activity (there may be more than one numbering scheme that satisfies rule 3).
4) An activity should not be represented by more than one arc in the network.
5) Two nodes can be connected by at most one arc.

- To avoid violating rules 4 and 5, it is sometimes necessary to utilise a dummy activity that takes zero time.
- For example, suppose activities A and B are both predecessors of activity C and can begin at the same time. In the absence of rule 5, we could represent this by Figure 5.4.

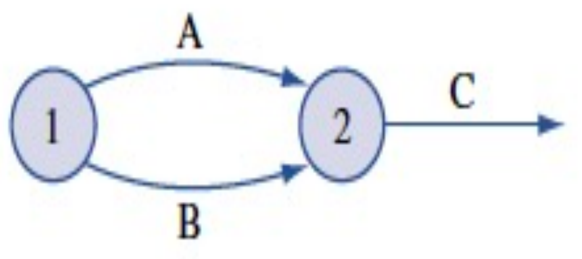


Figure 5.4 Violation of Rule 5

- By using a dummy activity (indicated by a dotted arc), as in Figure 5.5, we may represent the fact that A and B are both predecessors of C.
- Figure 5.5 ensures that activity C cannot begin until both A and B are completed, but it does not violate rule 5.

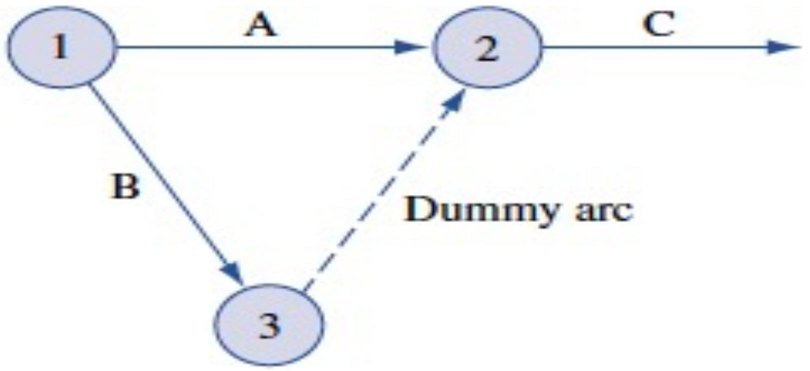


Figure 5.5 Use of Dummy Activity

Rules for Constructing an AOA Project Diagram

1. Node 1 represents the start of the project. An arc should lead from node 1 to represent each activity that has no predecessors.
2. A node (called the finish node) representing the completion of the project should be included in the network.
3. Number the nodes in the network so that the node representing the completion of an activity always has a larger number than the node representing the beginning of an activity (there may be more than one numbering scheme that satisfies rule 3).
4. An activity should not be represented by more than one arc in the network.
5. Two nodes can be connected by at most one arc.

- To avoid violating rules 4 and 5, it is sometimes necessary to utilise a dummy activity that takes zero time.
- For example:

Suppose activities A and B are both predecessors of activity C and can begin at the same time. In the absence of rule 5, we could represent this by Figure 5.4.

However, because nodes 1 and 2 are connected by more than one arc, Figure 5.4 violates rule 5.

By using a dummy activity (indicated by a dotted arc), as in Figure 5.6, we may represent the fact that A and B are both predecessors of C. Figure 5.6 ensures that activity C cannot begin until both A and B are completed, but it does not violate rule 5.

Drawing a Project Network

Widget Co. is about to introduce a new product (product 3). One unit of product 3 is produced by assembling 1 unit of product 1 and 1 unit of product 2. Before production begins on either product 1 or 2, raw materials must be purchased and workers must be trained. Before products 1 and 2 can be assembled into product 3, the finished product 2 must be inspected. A list of activities and their predecessors and of the duration of each activity is given in Table 1. Draw a project diagram for this project.

Table 1 Duration of Activities and Predecessor Relationships for Widget Co.

Activity	Predecessors	Duration (Days)
A = train workers	—	6
B = purchase raw materials	—	9
C = produce product 1	A, B	8
D = produce product 2	A, B	7
E = test product 2	D	10
F = assemble products 1 and 2	C, E	12

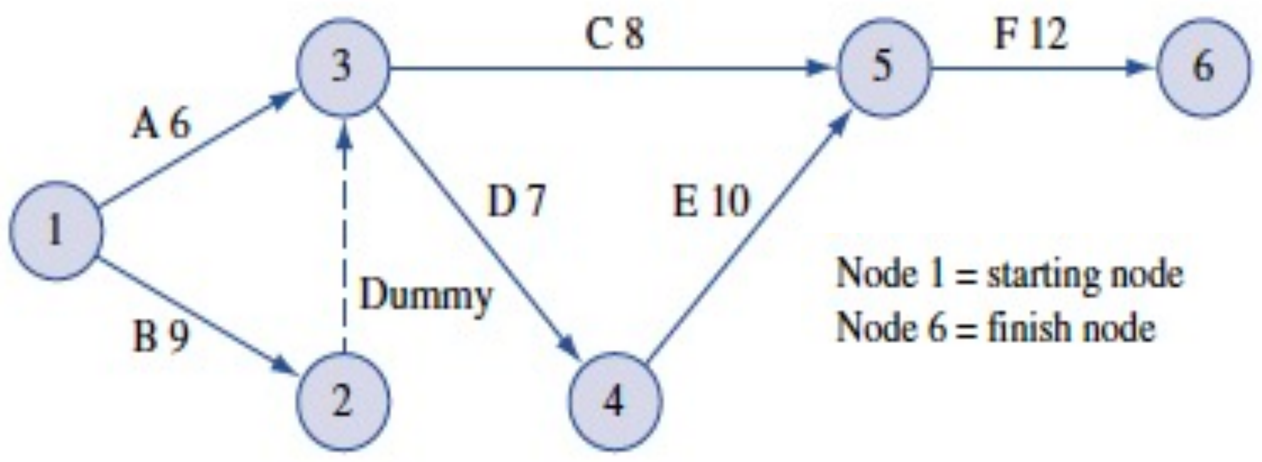


Figure 5.6 Project Diagram for Widget Co.

Solution

Observe that although we list only C and E as predecessors of F, it is actually true that activities A, B, and D must also be completed before F begins. C cannot begin until A and B are completed, and E cannot begin until D is completed, however, so it is redundant to state that A, B, and D are predecessors of F. Thus, in drawing the project network, we need only be concerned with the immediate predecessors of each activity.

Solution

The AOA network for this project is given in Figure 31 (the number above each arc represents activity duration in days). Node 1 is the beginning of the project, and node 6 is the finish node representing completion of the project. The dummy arc (2, 3) is needed to ensure that rule 5 is not violated.

5.3 CPM (Critical Path Method)

- Critical path method (CPM) can be used to determine the length of time required to complete a project.
- CPM also can be used to determine how long each activity in the project can be delayed without delaying the completion of the project.
- Assuming the duration of each activity is known, the critical path method (CPM) may be used to find the duration of a project.

- The two key building blocks in CPM are the concepts of early event time (ET) and late event time (LT) for an event.

DEFINITION ■ The early event time for node i, represented by $ET(i)$, is the earliest time at which the event corresponding to node i can occur. ■

The late event time for node i, represented by $LT(i)$, is the latest time at which the event corresponding to node i can occur without delaying the completion of the project. ■

Computation of Early Event Time

- To find the early event time for each node in the project network, we begin by noting that because node 1 represents the start of the project, $ET(1) = 0$.
- We then compute $ET(2)$, $ET(3)$, and so on, stopping when $ET(\text{finish node})$ has been calculated.
- To illustrate how $ET(i)$ is calculated, suppose that for the segment of a project network in Figure 5.7, we have already determined that $ET(3) = 6, ET(4) = 8, and\ ET(5) = 10$.

- To determine $ET(6)$, observe that the earliest time that node 6 can occur is when the activities corresponding to arc (3, 6), (4, 6), and (5, 6) have all been completed.

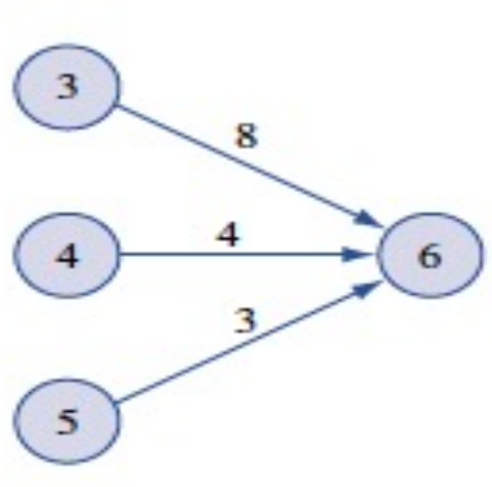


Figure 5.7 Determination of $E\,T(6)$

$$ET(6) = \max \begin{cases} ET(3) + 8 = 14 \\ ET(4) + 4 = 12 \\ ET(5) + 3 = 13 \end{cases}$$

- Thus, the earliest time that node 6 can occur is 14, and $ET(6) = 14$.
- From this example, it is clear that computation of $ET(i)$ requires (for $j < i$) knowledge of one or more of the $ET(j)$'s. This explains why we begin by computing the predecessor ETs.

- In general, if $ET(1), ET(2), \ldots, ET(i-1)$ have been determined, then we compute $ET(i)$ as follows:

Step 1 Find each prior event to node i that is connected by an arc to node i. These events are the **immediate predecessors** of node i.

Step 2 To the ET for each immediate predecessor of the node i add the duration of the activity connecting the immediate predecessor to node i.

Step 3 $ET(i)$ equals the maximum of the sums computed in step 2.

- We now compute the $ET(i)$'s for Example. We begin by observing that $ET(1) = 0$. Node 1 is the only immediate predecessor of node 2, so $ET(2) = ET(1) + 9 = 9$. The immediate predecessors of node 3 are nodes 1 and 2. Thus,

$$ET(3) = \max \begin{cases} ET(1) + 6 = 6 \\ ET(2) + 0 = 9 \end{cases} = 9$$

Node 4's only immediate predecessor is node 3. Thus, $ET(4) = ET(3) + 7 = 16$. Node 5's immediate predecessors are nodes 3 and 4. Thus,

$$ET(5) = \max \begin{cases} ET(3) + 8 = 17 \\ ET(4) + 10 = 26 \end{cases} = 26$$

Finally, node 5 is the only immediate predecessor of node 6. Thus, $ET(6) = ET(5) + 12 = 38$. Because node 6 represents the completion of the project, we see that the earliest time that product 3 can be assembled is 38 days from now.

It can be shown that $ET(i)$ is the length of the longest path in the project network from node 1 to node i.

Computation of Late Event Time

- To compute the LT(i)'s, we begin with the finish node and work backward (in descending numerical order) until we determine LT(1). The project in Example can be completed in 38 days, so we know that LT(6) 38.
- To illustrate how LT(i) is computed for nodes other than the finish node, suppose we are working with a network (Figure 5.8) for which we have already determined that LT(5) 24, LT(6) 26, and LT(7) 28.
- In this situation, how can we compute LT(4)? If the event corresponding to node 4 occurs after LT(5) 3, node 5 will occur after LT(5), and the completion of the project will be delayed.

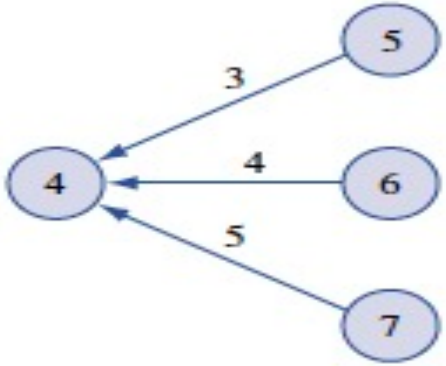


Figure 5.8 Computation of LT (4)

Similarly, if node 4 occurs after $LT(6) - 4$ or if node 4 occurs after $LT(7) - 5$, the completion of the project will be delayed. Thus,

$$LT(4) = \min \begin{cases} LT(5) - 3 = 21 \\ LT(6) - 4 = 22 = 21 \\ LT(7) - 5 = 23 \end{cases}$$

In general, if $LT(j)$ is known for $j > i$, we can find $LT(i)$ as follows:

Step 1 Find each node that occurs after node i and is connected to node i by an arc. These events are the **immediate successors** of node i.

Step 2 From the LT for each immediate successor to node i, subtract the duration of the activity joining the successor the node i.

Step 3 $LT(i)$ is the smallest of the differences determined in step 2.

We now compute the $LT(i)$'s for Example 6. Recall that $LT(6) = 38$. Because node 6 is the only immediate successor of node 5, $LT(5) = LT(6) - 12 = 26$. Node 4's only immediate successor is node 5. Thus, $LT(4) = LT(5) - 10 = 16$. Nodes 4 and 5 are immediate successors of node 3. Thus,

$$LT(3) = \min \begin{cases} LT(4) - 7 = 9 \\ LT(5) - 8 = 18 \end{cases}$$

Node 3 is the only immediate successor of node 2. Thus, $LT(2) = LT(3) - 0 = 9$. Finally, node 1 has nodes 2 and 3 as immediate successors. Thus,

$$LT(1) = \min \begin{cases} LT(3) - 6 = 3 \\ LT(2) - 9 = 0 \end{cases}$$

Table 13 summarizes our computations for Example 6. If $LT(i) = ET(i)$, any delay in the occurrence of node i will delay the completion of the project. For example, because $LT(4) = ET(4)$, any delay in the occurrence of node 4 will delay the completion of the project.

Total Float

- Before the project is begun, the duration of an activity is unknown, and the duration of each activity used to construct the project network is just an estimate of the activity's actual completion time.
- The concept of total float of an activity can be used as a measure of how important it is to keep each activity's duration from greatly exceeding our estimate of its completion time.

Table 2 ET and LT for Widget Co.

Node	ET(i)	LT(i)
1	0	0
2	9	9
3	9	9
4	16	16
5	26	26
6	38	38

DEFINITION ■ For an arbitrary arc representing activity (i, j), the **total float**, represented by $TF(i, j)$, of the activity represented by (i, j) is the amount by which the starting time of activity (i, j) could be delayed beyond its earliest possible starting time without delaying the completion of the project (assuming no other activities are delayed). ■

Equivalently, the total float of an activity is the amount by which the duration of the activity can be increased without delaying the completion of the project.

If we define t_{ij} to be the duration of activity (i, j), then $TF(i, j)$ can easily be expressed in terms of $LT(j)$ and $ET(i)$. Activity (i, j) begins at node i. If the occurrence of node i, or the duration of activity (i, j), is delayed by k time units, then activity (i, j) will be completed at time $ET(i) + k + t_{ij}$. Thus, the completion of the project will not be delayed if

$$ET(i) + k + t_{ij} \leq LT(j) \qquad \text{or} \qquad k \leq LT(j) - ET(i) - t_{ij}$$

Therefore,

$$TF(i,j) = LT(j) - ET(i) - t_{ij}$$

For Example , the $TF(i,j)$ are as follows:

Activity B:	$TF(1, 2) = LT(2) - ET(1) - 9 = 0$
Activity A:	$TF(1, 3) = LT(3) - ET(1) - 6 = 3$
Activity D:	$TF(3, 4) = LT(4) - ET(3) - 7 = 0$
Activity C:	$TF(3, 5) = LT(5) - ET(3) - 8 = 9$
Activity E:	$TF(4, 5) = LT(5) - ET(4) - 10 = 0$
Activity F:	$TF(5, 6) = LT(6) - ET(5) - 12 = 0$
Dummy activity:	$TF(2, 3) = LT(3) - ET(2) - 0 = 0$

Finding a Critical Path

If an activity has a total float of zero, then any delay in the start of the activity (or the duration of the activity) will delay the completion of the project. In fact, increasing the duration of an activity by Δ days will increase the length of the project by Δ days. Such an activity is critical to the completion of the project on time.

DEFINITION ■ Any activity with a total float of zero is a **critical activity.** ■

A path from node 1 to the finish node that consists entirely of critical activities is called a **critical path.** ■

- In Figure 5.6, activities B, D, E, F, and the dummy activity are critical activities and the path 1–2–3–4–5–6 is the critical path (it is possible for a network to have more than one critical path).
- A critical path in any project network is the longest path from the start node to the finish node.
- Any delay in the duration of a critical activity will delay the completion of the project, so it is advisable to monitor closely the completion of critical activities.

Free Float

- As we have seen, the total float of an activity can be used as a measure of the flexibility in the duration of an activity.
- For example, activity A can take up to 3 days longer than its scheduled duration of 6 days without delaying the completion of the project.
- Another measure of the flexibility available in the duration of an activity is free float.

DEFINITION ■ The free float of the activity corresponding to arc (i, j), denoted by $FF(i, j)$, is the amount by which the starting time of the activity corresponding to arc (i, j) (or the duration of the activity) can be delayed without delaying the start of any later activity beyond its earliest possible starting time. ■

Suppose the occurrence of node i, or the duration of activity (i, j), is delayed by k units. Then the earliest that node j can occur is $ET(i) + t_{ij} + k$. Thus, if $ET(i) + t_{ij} + k \leq ET(j)$, or $k \leq ET(j) - ET(i) - t_{ij}$, then node j will not be delayed. If node j is not delayed, then no other activities will be delayed beyond their earliest possible starting times. Therefore,

$$FF(i, j) = ET(j) - ET(i) - t_{ij}$$

- For the example, the $FF(i, j)$ are as follows:

Activity B: $FF(1, 2) = 9 - 0 - 9 = 0$

Activity A: $FF(1, 3) = 9 - 0 - 6 = 3$

Activity D: $FF(3, 4) = 16 - 9 - 7 = 0$

Activity C: $FF(3, 5) = 26 - 9 - 8 = 9$

Activity E: $FF(4, 5) = 26 - 16 - 10 = 0$

Activity F: $FF(5, 6) = 38 - 26 - 12 = 0$

For example, because the free float for activity C is 9 days, a delay in the start of activity C (or in the occurrence of node 3) or a delay in the duration of activity C of more than 9 days will delay the start of some later activity (in this case, activity F).

5.2 PERT (Program Evaluation and Review Technique)

- PERT is an attempt to correct the shortcoming of CPM by modelling the duration of each activity as a random variable.
- If the durations of the project's activities are not known with certainty, then PERT may be used to estimate the probability that the project will be completed in a specified amount of time.

PERT requires that for each activity the following three numbers be specified:

a = estimate of the activity's duration under the most favourable conditions

b = estimate of the activity's duration under the least favourable conditions

m = most likely value for the activity's duration

- If the estimates a, b, and m refer to the activity represented by arc (i, j), then T_{ij} is the random variable representing the duration of the activity represented by arc (i, j).
- T_{ij} has (approximately) the following properties:

$$E(\mathbf{T}_{ij}) = \frac{a + 4m + b}{6}$$

$$\text{var}\mathbf{T}_{ij} = \frac{(b - a)^2}{36}$$

- PERT requires the assumption that the durations of all activities are independent. Then for any path in the project network, the mean and variance of the time required to complete the activities on the path are given by

$$\sum_{(i,\,j)\in\text{path}} E(\mathbf{T}_{ij}) = \text{expected duration of activities on any path}$$

$$\sum_{(i,\,j)\in\text{path}} \text{var}\mathbf{T}_{ij} = \text{variance of duration of activities on any path}$$

- Let CP be the random variable denoting the total duration of the activities on a critical path found by CPM. PERT assumes that the critical path found by CPM contains enough activities to allow us to invoke the Central Limit Theorem and conclude that

$$\mathbf{CP} = \sum_{(i,\, j) \in \text{critical path}} \mathbf{T}_{ij}$$

is normally distributed.

- Assuming (sometimes incorrectly) that the critical path found by CPM is the critical path, and assuming that the duration of the critical path is normally distributed, the preceding equations may be used to estimate the probability that the project will be completed within any specified length of time.
- For the previous example a, b, and m for each activity are shown in Table 3. Now

$$E(T_{12}) = \frac{\{5 + 13 + 36\}}{6} = 9 \qquad \text{var}T_{12} = \frac{(13 - 5)^2}{36} = 1.78$$

$$E(T_{13}) = \frac{\{2 + 10 + 24\}}{6} = 6 \qquad \text{var}T_{13} = \frac{(10 - 2)^2}{36} = 1.78$$

$$E(T_{35}) = \frac{\{3 + 13 + 32\}}{6} = 8 \qquad \text{var}T_{35} = \frac{(13 - 3)^2}{36} = 2.78$$

$$E(T_{34}) = \frac{\{1 + 13 + 28\}}{6} = 7 \qquad \text{var}T_{34} = \frac{(13 - 1)^2}{36} = 4$$

$$E(T_{45}) = \frac{\{8 + 12 + 40\}}{6} = 10 \qquad \text{var}T_{45} = \frac{(12 - 8)^2}{36} = 0.44$$

$$E(T_{56}) = \frac{\{9 + 15 + 48\}}{6} = 12 \qquad \text{var}T_{56} = \frac{(15 - 9)^2}{36} = 1$$

Of course, the fact that arc (2, 3) is a dummy arc yields

$$E(T_{23}) = \text{var } T_{23} = 0$$

Recall that the critical path for Example 6 was 1–2–3–4–5–6.

$$E(CP) = 9 + 0 + 7 + 10 + 12 = 38$$

$$\text{varCP} = 1.78 + 0 + 4 + 0.44 + 1 = 7.22$$

Then the standard deviation for CP is $(7.22)^{1/2} = 2.69$.

Table 3 $a, b,$ and m for Activities in Widget Co.

Activity	a	b	m
(1, 2)	5	13	9
(1, 3)	2	10	6
(3, 5)	3	13	8
(3, 4)	1	13	7
(4, 5)	8	12	10
(5, 6)	9	15	12

- Applying the assumption that CP is normally distributed, we can answer questions such as the following: What is the probability that the project will be completed within 35 days?
- To answer this question, we must also make the following assumption: No matter what the durations of the project's activities turn out to be, 1–2–3–4–5–6 will be a critical path.

- This assumption implies that the probability that the project will be completed within 35 days is just $P(CP \leq 35)$.
- Standardising and applying the assumption that **CP** is normally distributed, we find that Z is a standardised normal random variable with mean 0 and variance 1.

- The cumulative distribution function for a normal random variable is tabulated in
- Table 4. For example, $P(Z \le -1) = 0.1587$ and $P(Z \le 2) = 0.9772$. Thus,

$$P(\mathrm{CP} \le 35) = P\left(\frac{\mathrm{CP} - 38}{2.69} \le \frac{35 - 38}{2.69}\right) = P(\mathrm{Z} \le -1.12) = .13$$

where $F(-1.12) = .13$ may be obtained using the NORMSDIST function in Excel. Entering the formula =NORMSDIST(x) returns the probability that a standard normal random variable with mean 0 and standard deviation 1 is less than or equal to x. For example =NORMDIST(−1.12) yields .1313.

There are several difficulties with PERT:

1) The assumption that the activity durations are independent is difficult to justify.
2) Activity durations may not follow a beta distribution.
3) The assumption that the critical path found by CPM will always be the critical path for the project may not be justified.

- The last difficulty is the most serious. For example, in our analysis of previous example, we assumed that 1–2–3–4–5–6 would always be the critical path.
- If, however, activity A were significantly delayed and activity B were completed ahead of schedule, then the critical path might be 1–3–4–5–6.
- Here is a more concrete example of the fact that (because of the uncertain duration of activities) the critical path found by CPM may not actually be the path that determines the completion date of the project. Consider the simple project network in Figure 5.9.

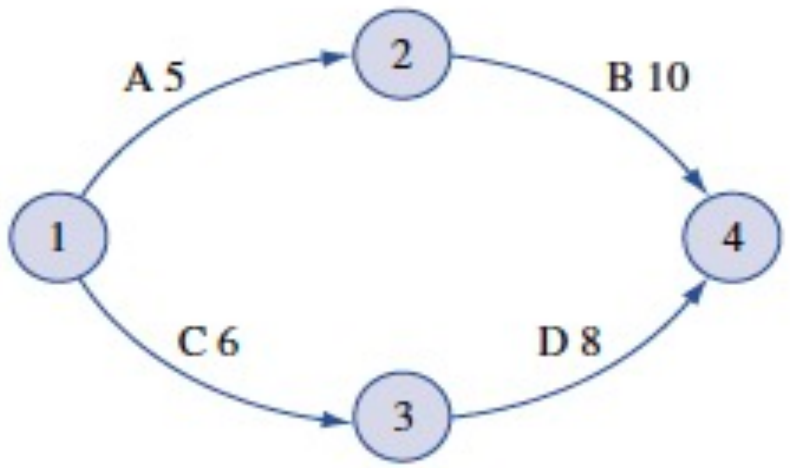


Figure 5.9 Network to Determine Critical Path If Each Activity's Duration Equals m

- Assume that for each activity in Table 3, a, b, and m each occur with probability 1/3.
- If CPM were applied (using the expected duration of each activity as the duration of the activity), then we would obtain the network in Figure 5.9. For this network, the critical path is 1–2–4.
- In actuality, however, the critical path could be 1–3–4. For example, if the optimistic duration of B (6 days) occurred and all other activities had a duration m, then 1–3–4 would be the critical path in the network.

- If we assume that the durations of the four activities are independent random variables, then using elementary, it can be shown that there is a 10/27 probability that 1–3–4 is the critical path, a 15/27 chance that 1–2–4 is the critical path, and a 2/27 chance that 1–2–4 and 1–3–4 will both be critical paths. This example shows that one must be cautious in designating an activity as critical. In this situation, the probability that each activity is actually a critical activity is shown in Table 4.

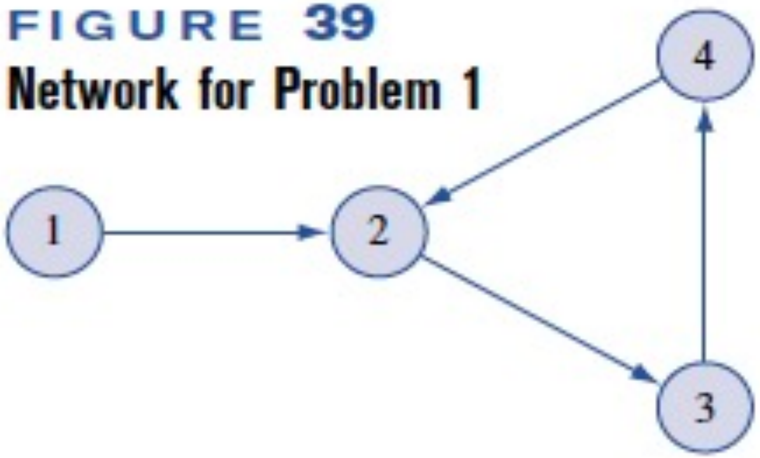
Table 4 Probability That Each Arc is on a Critical Path

Activity	Probability
A	$\frac{17}{27}$
B	$\frac{17}{27}$
C	$\frac{12}{27}$
D	$\frac{12}{27}$

Exercises

1 What problem would arise if the network in Figure 39 were a portion of a project network?

FIGURE 39
Network for Problem 1



2 A company is planning to manufacture a product that consists of three parts (A, B, and C). The company anticipates that it will take 5 weeks to design the three parts and to determine the way in which these parts must be assembled to make the final product. Then the company estimates that it will take 4 weeks to make part A, 5 weeks to make part B, and 3 weeks to make part C. The company must test part A after it is completed (this takes 2 weeks). The assembly line process will then proceed as follows: assemble parts A and B (2 weeks) and then attach part C (1 week). Then the final product must undergo 1 week of testing. Draw the project network and find the critical path, total float, and free float for each activity. Also set up the LP that could be used to find the critical path.

When determining the critical path in Problems 3 and 4, assume that m = activity duration.

4 The promoter of a rock concert in Indianapolis must perform the tasks shown in Table 19 before the concert can be held (all durations are in days).

a Draw the project network.

b Determine the critical path.

c If the advance promoter wants to have a 99% chance of completing all preparations by June 30, when should work begin on finding a concert site?

d Set up the LP that could be used to find the project's critical path.

TABLE 19

Activity	Description	Immediate Predecessors	*a*	*b*	*m*
A	Find site	—	2	4	3
B	Find engineers	A	1	3	2
C	Hire opening act	A	2	10	6
D	Set radio and TV ads	C	1	3	2
E	Set up ticket agents	A	1	5	3
F	Prepare electronics	B	2	4	3
G	Print advertising	C	3	7	5
H	Set up transportation	C	0.5	1.5	1
I	Rehearsals	F, H	1	2	1.5
J	Last-minute details	I	1	3	2