Riddler Express (Marbles problem)

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If the number of marbles of each color are p, q, r, then the probability of choosing exactly one of each color is $pqr/\binom{100}{3}$, which equals 1/5 by the hypothesis. Thus

$$p + q + r = 100, \text{ and} \tag{1}$$

$$pqr = 2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11 = 32340. \tag{2}$$

This has the unique (upto exchange) solution $p = 35$, $q = 44$, $r = 21$ among positive integers less than 100. The following argument eliminates the need for a computer-search to prove uniqueness; Maclaurin's inequality and the prime factorization in (2) are used to that end.
Substituting $x = p$, $y = q$, $z = r$ in Maclaurin's inequality

$$\left(\frac{xy + xz + yz}{3}\right)^{1/2} \leqslant \frac{x + y + z}{3} \tag{3}$$

and using (1) and (2), we infer that p must satisfy

$$\frac{980}{11p} + \frac{p(100-p)}{363} \leqslant 9. \tag{4}$$

Inspecting this inequality – in particular, its monotonicity for small values of p – reveals that p (and, by symmetry, q and r) should exceed 18. Since 19 is not one of the factors in the RHS of (2), we have

$$p, q, r \geqslant 20. \tag{5}$$

Substituting $x = pq$, $y = pr$, $z = qr$ in (3), and using (1) we find

$$\left(\frac{pqr(p+q+r)}{3}\right)^{1/2} = \left(\frac{pq^2r + qp^2r + pr^2q}{3}\right)^{1/2} \leqslant \frac{pq + pr + qr}{3} = \frac{p(100-p) + qr}{3}. \tag{6}$$

Since $p(100 - p) \leqslant 2500$ for $1 \leqslant p \leqslant 100$, we can infer that $\frac{32340}{p} = qr \geqslant 614$. This implies that $p \leqslant 53$. Since 51, 52 and 53 have prime factors which are absent in the RHS of (2), we can infer that p (and, by symmetry, q and r) must satisfy

$$p, q, r \leqslant 50. \tag{7}$$

The rest of the analysis relies on using the prime factorization in (2) along with the bounds (5) and (7). Without loss of generality (w.l.o.g.), assume that the prime 5 in (2) is a factor of p. If 2 is a factor of p but 4 is not then, no matter how the other primes in (2) are distributed, $p+q+r$ will be odd, violating (1). This implies that either 4 is a factor of p or p is odd. If 4 is a factor of p, then – since 5 is also a factor – the upper bound (7) implies that $p = 20$. If p is odd, then 4 must divide exactly one of the other two – say q, w.l.o.g. - else $p + q + r$ will be odd, violating (1). Altogether, we have the two cases

$$p = 20, \tag{8}$$

or

$$p = 5 \cdot a, \; q = 4 \cdot b. \tag{9}$$

If (8) holds then, w.l.o.g., let 11 be a factor of q. The bounds (5) and (7) and the factors in (2) then imply that $q = 33$ and thus $r = 49$. This violates (1) and so we rule out (8) as a possibility.
If (9) holds, then the bounds (5) and (7) imply that $a = 7$ and that b is either 7 or 11. Clearly, $b = 11$ is the only choice. The choices $a = 7$, $b = 11$ in (9) leave us with the unique solution $p = 35$, $q = 44$, $r = 21$.