Colliding blocks for Quantum Gates in SU(2)

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We show that Brown's analogy, between Grover search and colliding blocks [1], extends to other single qubit quantum gates. We begin with a perturbed version of Grover search in which the oracle only gently marks an item. [1] We show the blocks' analogy for it and other single qubit gates - module a global phase - by using the local isomorphism between SU(2) and SO(3). We end with a speculation on what it would take to show that colliding blocks and gravity are universal as a quantum computing model.

1 Imperfect oracle

Using Brown's [1] notation, we have

$$\sin(\bar{\theta}) = \frac{1}{\sqrt{d}} \text{ and}$$
$$|s\rangle = \sqrt{1-\frac{1}{d}}\,|\bar{s}\rangle + \frac{1}{\sqrt{d}}\,|w\rangle\,, \tag{1}$$

where $|w\rangle$, among $d > 1$ finite choices, is the basis vector whose coefficient is flipped by the perfect Grover oracle and $2\bar{\theta}$ is the angle through which the candidate vector is rotated in each iteration of the Grover search algorithm.

Consider the case when the oracle has, effectively, a coherent error in that the sign for the $|w\rangle$ component is not flipped but is instead multiplied by a phase factor $e^{i\phi}$; $\phi = \pi$ would mean that the Grover oracle is perfect, $\phi = 0$ means that the oracle is the identity matrix (it doesn't mark any item), while an intermediate value of ϕ means that the oracle only marks the $|w\rangle$ coefficient *gently*. This imperfect oracle is modeled by the unitary

$$\hat{U}_{w,\phi} = I + (e^{i\phi} - 1)\,|w\rangle\langle w|\,. \tag{2}$$

Grover's strategy, in the case when $\phi = \pi$, is to start from the initial equal-superposition state $|s\rangle$ and repeatedly apply

$$\begin{aligned}\hat{U}(\bar{\theta},\phi) &:= \hat{U}_s\hat{U}_{w,\phi} \\ &= \begin{bmatrix}\cos(2\bar{\theta}) & e^{i\phi}\sin(2\bar{\theta}) \\ \sin(2\bar{\theta}) & -e^{i\phi}\cos(2\bar{\theta})\end{bmatrix} \\ &= \hat{R}_{2\bar{\theta}}\hat{P}_{\phi+\pi}\end{aligned} \tag{3}$$

[1] These oracles are modeled here as unitaries for the purpose of analysis. It is to be understood that, in practice, one only has blackbox access to them.

for a finite amount of steps where $\hat{U}_s = 2\left|s\right\rangle\!\left\langle s\right| - I$, and $\hat{R}_{2\bar{\theta}}$, $\hat{P}_{\phi+\pi}$ are the unitary rotation and phase matrices

$$\hat{R}_{2\bar{\theta}} \coloneqq \begin{bmatrix} \cos\left(2\bar{\theta}\right) & -\sin\left(2\bar{\theta}\right) \\ \sin\left(2\bar{\theta}\right) & \cos\left(2\bar{\theta}\right) \end{bmatrix} \quad \text{and} \tag{4}$$
$$\hat{P}_{\phi+\pi} \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\phi+\pi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -e^{i\phi} \end{bmatrix}.$$

Each application of $\hat{U}(\bar{\theta},\phi)$ rotates the previous output vector by an angle $2\bar{\theta}$ within the plane determined by $\left|\bar{s}\right\rangle$ and $\left|w\right\rangle$; it takes $\mathcal{O}(\sqrt{d})$ steps to reach $\left|w\right\rangle$ with near certainty when $\phi = \pi$. If it was known that the oracle had such a flaw, a better search strategy would be to estimate ϕ and correct for it. But we are not concerned with that problem here. Instead we show that a fixed $\phi \neq \pi$ corresponds to a "moving wall" in Brown's colliding blocks analogy, which can be used to model the repeated action of any single qubit gate modulo a global phase. But first we describe how Grover's strategy performs when $\phi \neq \pi$; this builds the intuition for the analogy to blocks.

2 Number of Grover updates for imperfect oracle

With repeated applications of $\hat{U}(\bar{\theta},\phi)$, the evolution of the search vector is better tracked on a Bloch sphere (rather than circle with the usual ($\phi = \pi$) Grover) with $\left|\bar{s}\right\rangle$ and $\left|w\right\rangle$ relabeled for convenience as $\left|0\right\rangle$ and $\left|1\right\rangle$ respectively. So we relabel the initial vector $\left|s\right\rangle = \sqrt{1-\frac{1}{d}}\left|0\right\rangle + \frac{1}{\sqrt{d}}\left|1\right\rangle = \cos\left(\bar{\theta}\right)\left|0\right\rangle + \sin\left(\bar{\theta}\right)\left|1\right\rangle$, with the search task now being to approach $\left|1\right\rangle$ as close as possible.

We track the evolution of the search vector by tracking its projection on the eigenvectors of $\hat{U}(\bar{\theta},\phi)$ in each update. The eigenvalues $\lambda_\pm$ and normalized eigenvectors $\left|u_\pm\right\rangle$ of $\hat{U}(\bar{\theta},\phi)$ [2] are

$$\lambda_\pm = e^{i\left(\frac{\phi-\pi}{2}\pm\alpha\right)}$$
$$\left|u_\pm\right\rangle = \left(\frac{1}{2} \mp \frac{\cos(\beta)}{2\sin(\alpha)}\right)^{\frac{1}{2}} \left(\frac{\epsilon \pm \kappa}{2}\left|0\right\rangle + \left|1\right\rangle\right), \tag{5}$$

where $0 < \alpha, \beta < \pi$ satisfy $\cos(\alpha) = \sin\left(\frac{\phi}{2}\right)\cos\left(2\bar{\theta}\right)$, $\cos(\beta) = \cos\left(\frac{\phi}{2}\right)\cos\left(2\bar{\theta}\right)$, and $\epsilon = \cot\left(2\bar{\theta}\right)(1+e^{i\phi})$, $\kappa = \frac{2e^{i\left(\frac{\phi}{2}\right)}\sin(\alpha)}{\sin\left(2\bar{\theta}\right)}$.

After k Grover updates, the search vector $\left|s(k)\right\rangle$ is

$$\begin{aligned} \left|s(k)\right\rangle &= (\hat{U}(\bar{\theta},\phi))^k\left|s\right\rangle \\ &= \lambda_+^k\left\langle u_+|s\right\rangle\left|u_+\right\rangle + \lambda_-^k\left\langle u_-|s\right\rangle\left|u_-\right\rangle, \end{aligned} \tag{6}$$

[2] to reduce clutter, we won't explicitly note that $(\bar{\theta},\phi)$ are also arguments of the eigenvalues and eigenvectors

which makes the probability of measuring 1 after the k^{th} update

$$\begin{aligned}\Pr(1,k) &= |\lambda_+^k \langle u_+|s\rangle \langle 1|u_+\rangle + \lambda_-^k \langle u_-|s\rangle \langle 1|u_-\rangle|^2 \\ &= \underbrace{\frac{\sin^2(\bar{\theta})\left(1+\sin\left(\frac{\phi}{2}\right)\right)}{1-\cos(2\bar{\theta})\sin\left(\frac{\phi}{2}\right)}}_{\text{optimal probability}} - \frac{\sin^2(2\bar{\theta})\sin\left(\frac{\phi}{2}\right)}{\sin^2(\alpha)}\cos^2((k+\frac{1}{2})\alpha).\end{aligned} \tag{7}$$

This shows that the maximum probability for measuring $|1\rangle$ is first achieved at the update $k_* = \lfloor \frac{1}{2}(\frac{\pi}{\alpha} - 1)\rceil$. For $d \gg 1$, we can see its resemblance to Grover's result with

$$k_* = \frac{\frac{\pi\sqrt{d}}{4}}{\left(1 + \frac{d}{4}\cos^2(\frac{\phi}{2})\right)^{1/2}} + c, \tag{8}$$

for some $|c| \leq 2$. Although fewer updates are needed in the $\phi < \pi$ case to reach the optimal probability (see (7)) compared to the usual Grover problem ($\phi = \pi$), it comes with a less than perfect probability of success. This leads us to an intuition that if the system of colliding blocks needs to mimic $\hat{U}(\bar{\theta}, \phi)$, then the blocks should not be allowed to "complete" all the collisions that they did in the case of the perfect ($\phi = \pi$) Grover oracle. One way this can be done is if the wall yields when impinged by the block; we see that a moving wall indeed suffices in the following.

3 Colliding blocks

The main point of this section is that the gentle Grover operation $\hat{U}(\bar{\theta}, \phi)$ (3) corresponds to a certain rotation matrix in SO(3) that describes collisions between three masses moving frictionlessly along a line while colliding elastically. This construction is a generalization of Brown's system where the infinitely heavy wall is replaced by a body with finite mass, as explained in the following.

Consider three masses, as shown in Fig. 1, undergoing two elastic collisions in every round. We track the unit vector

$$|u\rangle := \left(\sqrt{\frac{m_2}{2K}}v_2, \sqrt{\frac{m_w}{2K}}v_w, \sqrt{\frac{m_1}{2K}}v_1\right)^T, \tag{9}$$

where m_1, m_2 are the masses and v_1, v_2 are the velocities of the two blocks on the right, m_w and v_w are the mass and velocity of the block on the left and K is the total kinetic energy of the system. The subscript w stands for wall, which is now assumed to have finite mass (unlike Brown's case) and, like the other two blocks, also slides without friction. The unit vector $|u_{(2)}\rangle$ after the blocks finish a round of collisions - first between m_1 and m_2

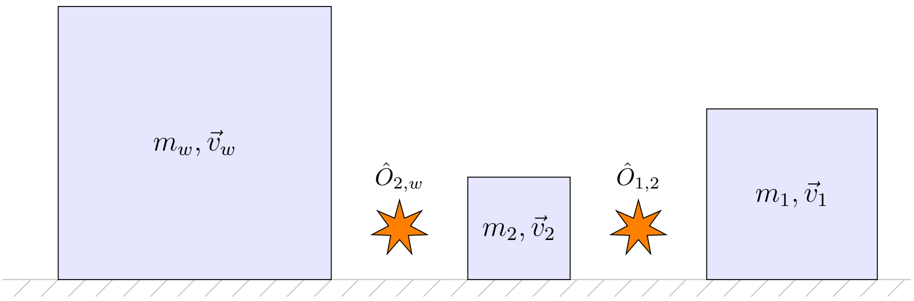

Figure 1: Modulo global phase, the repeated action of any single qubit gate on an initial quantum state can be emulated by the repeated elastic collisions of frictionlessly sliding blocks moving along a common line. The specific gate choice sets the relative sizes of the blocks. The specific initial quantum state sets the initial value of the unit vector $|u\rangle$ which tracks the motion of the blocks.

(modeled as $\hat{O}_{1,2}$), followed by a collision between m_2 and m_w (modeled as $\hat{O}_{1,2}$) - is governed by the rotation

$$\begin{aligned} \left|u_{(2)}\right\rangle &= \hat{O}(\omega,\delta)\left|u\right\rangle \quad \text{where} \\ \hat{O}(\delta,\omega) &= \underbrace{\begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ \sin(\omega) & -\cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\hat{O}_{2,w}} \underbrace{\begin{bmatrix} -\cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ \sin(\delta) & 0 & \cos(\delta) \end{bmatrix}}_{\hat{O}_{1,2}} \end{aligned} \tag{10}$$

is in SO(3), with $\cos(\delta) = \frac{m_1-m_2}{m_1+m_2}$ and $\cos(\omega) = \frac{m_2-m_w}{m_2+m_w}$. Using the classical 2-to-1 homomorphism $f : SU(2) \to SO(3)$ defined by

$$f\left(\begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Re}(x^2-y^2) & \mathrm{Im}(x^2+y^2) & -2\,\mathrm{Re}(xy) \\ -\mathrm{Im}(x^2-y^2) & \mathrm{Re}(x^2+y^2) & 2\,\mathrm{Im}(xy) \\ 2\,\mathrm{Re}(xy^*) & 2\,\mathrm{Im}(xy^*) & |x|^2-|y|^2 \end{bmatrix}, \tag{11}$$

we see that $f(\hat{V}(\delta,\omega)) = (\hat{O}(\delta,\omega))^T$ where

$$\hat{V}(\delta,\omega) = \begin{bmatrix} \cos\left(\frac{\delta}{2}\right)e^{-i\frac{\omega}{2}} & -\sin\left(\frac{\delta}{2}\right)e^{i\frac{\omega}{2}} \\ \sin\left(\frac{\delta}{2}\right)e^{-i\frac{\omega}{2}} & \cos\left(\frac{\delta}{2}\right)e^{i\frac{\omega}{2}} \end{bmatrix}. \tag{12}$$

Comparing (12) with (3), we see that

$$\hat{V}(\delta = 4\bar{\theta}, \omega = \phi) = e^{-i\frac{\phi}{2}} \cdot \hat{U}(\bar{\theta}, \phi). \tag{13}$$

If one is only concerned with repeated applications of $\hat{U}(\bar{\theta},\phi)$ on a quantum state followed by an eventual measurement of the final result, then one can ignore the global phase $e^{-i\frac{\phi}{2}}$ that is accumulated with each step of the process. Consequently, we conclude that tracking the search vector $|s(k)\rangle$ on the Bloch sphere is equivalent to tracking the unit vector $\left|u_{(2k)}\right\rangle$ on the unit sphere in 3 dimensions.

The total number of collisions in the finite-mass-wall case can be related to (7) in a manner analogous to G. Sanderson's explanation [2] for the infinite-mass-wall case, with the wrinkle that one applies the inscribed angle theorem to a sequence of changing circular sections of the sphere on which $|u\rangle$ evolves. We won't do this here; we focus on extending the colliding blocks analogy to other quantum gates.

4 Blocks for SU(2) gates (and more?)

We used the local isomorphism between SU(2) and SO(3) for extending the colliding blocks analogy to the imperfect oracle case in Grover search. However, it is easy to see that any single qubit gate can be expressed in the form of $\hat{U}(\bar{\theta}, \phi)$. Hence the repeated actions of a gate in SU(2) can be emulated by simply adjusting the mass ratios of the blocks. Some examples are

$$\begin{aligned}
\text{Hadamard:}\ \ H &\coloneqq \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix} \\
&= \hat{U}(\bar{\theta} = \frac{\pi}{8}, \phi = 0) \\
\text{Phase:}\ \ P &\coloneqq \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \\
&= \hat{U}(\bar{\theta} = 0, \phi = -\frac{\pi}{2}) \\
R_{\frac{\pi}{8}} &\coloneqq \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{bmatrix} \\
&= \hat{U}(\bar{\theta} = \frac{\pi}{16}, \phi = \pi).
\end{aligned} \tag{14}$$

With an eye on the universal gate set (H, $R_{\frac{\pi}{8}}$, CNOT), one can ask whether the blocks' analogy can be extended to the CNOT gate, and more generally, to any quantum circuit. Two major objections arise:

- When an arbitrary number of gates and qubits are involved in the circuit, the global phase factors that we've ignored become relative phase factors between different qubits. No interference, and hence no quantum computing, is possible without accounting for such relative phases.

- For CNOT gates, and in general for arbitrary quantum circuits, we need to engineer entanglement between systems of colliding blocks.

One could potentially use the local isomorphism between SU(4) and SO(6) to guess a system of colliding blocks for the CNOT gate. But it is not clear how one can engineer entanglement with this approach.

It is interesting to consider whether both of the above issues - engineering relative phase and entanglement between qubits - would benefit from using gravity and by assuming the truth of the ER = EPR conjecture [3]. Perhaps, relative phases between systems of blocks could be engineered by placing these blocks at the appropriate gravitation potentials of a pair of entangled black holes. Entangling such systems of blocks for general quantum circuits would need to ensure that the engineering *does not* require a violation of the monogamy of entanglement between the black holes.

References

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[3] L. Susskind, "ER=EPR, GHZ, and the Consistency of Quantum Measurements", *arXiv e-prints*, arXiv:1412.8483, arXiv:1412.8483, Dec. 2014. arXiv: `1412.8483 [hep-th]` (cit. on p. 6).