Colliding blocks for more Quantum Gates

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Suppose the blackbox oracle in Grover search [1] does an imperfect job and only *gently* marks a single item. What is the consequence of following Grover's strategy? Is there a better strategy using adaptive control that compensates for that gentle marking? After all, if it's too gentle, then it might not be worth proceeding with Grover at all! What is the classical analogue to such gentle Grover (for example, using billiards following Brown [1])? [2]

Using Brown's [1] notation for Grover search, we have

$$\begin{aligned} \sin(\bar{\theta}) &= \frac{1}{\sqrt{d}}, \\ \sin(2\bar{\theta}) &= \frac{2\sqrt{d-1}}{d}, \\ \cos(2\bar{\theta}) &= 1 - \frac{2}{d}, \quad \text{and} \\ |s\rangle &= \sqrt{1-\frac{1}{d}}\,|\bar{s}\rangle + \frac{1}{\sqrt{d}}\,|w\rangle\,, \end{aligned} \tag{1}$$

where $|w\rangle$, among $d > 1$ finite choices, is the basis vector whose coefficient is flipped by the perfect Grover oracle and $2\bar{\theta}$ is the angle through which the candidate vector is rotated in each iteration of the Grover search algorithm.

Consider the case when the oracle has, effectively, a coherent error in that the sign for the $|w\rangle$ component is not flipped but is instead multiplied by a phase factor $e^{i\phi}$; $\phi = \pi$ would mean that the Grover oracle is perfect, $\phi = 0$ means that the oracle is the identity matrix (it doesn't mark any item), while an intermediate value of ϕ means that the oracle only marks the $|w\rangle$ coefficient *gently*. The imperfect Grover oracle would be implemented by the unitary

[1] These oracles are modeled here as unitaries purely for the purpose of analysis. It is to be understood that, in practice, one has only blackbox access to them; the term "blackbox" will be omitted henceforth.

[2] It might be possible to understand Shor's algorithm using billiards ball too: if $p+q$ balls are moving in a straight line, the first p with velocity $+1$ and the next q with velocity -1, then there will be $N = pq$ collisions...just consider all the balls "going through" each other as in my long-ago balls-on-a-ring problem or Galperin's paper (which motivated Brown and which has a footnote regarding the same). The "going through" notion indicates its relation to permutation; in particular it may be possible a (at least, one-way?) reduction between computational complexity of factoring/multiplication and permanent/determinant.

$$\hat{U}_{w,\phi} = I + (e^{i\phi} - 1) |w\rangle\langle w| . \tag{2}$$

Grover's strategy, in the case when $\phi = \pi$, is to start from the initial equal-superposition state $|s\rangle$ and repeatedly apply

$$\begin{aligned}
\hat{U}(\bar{\theta}, \phi) &\coloneqq \hat{U}_s \hat{U}_{w,\phi} \\
&= \begin{bmatrix} \cos(2\bar{\theta}) & e^{i\phi}\sin(2\bar{\theta}) \\ \sin(2\bar{\theta}) & -e^{i\phi}\cos(2\bar{\theta}) \end{bmatrix} \\
&= \hat{R}_{2\bar{\theta}} \hat{P}_{\phi+\pi}
\end{aligned} \tag{3}$$

for a finite amount of steps where $\hat{U}_s = 2|s\rangle\langle s| - I$, and $\hat{R}_{2\bar{\theta}}$, $\hat{P}_{\phi+\pi}$ are the unitary rotation and phase matrices [3]

$$\begin{aligned}
\hat{R}_{2\bar{\theta}} &\coloneqq \begin{bmatrix} \cos(2\bar{\theta}) & -\sin(2\bar{\theta}) \\ \sin(2\bar{\theta}) & \cos(2\bar{\theta}) \end{bmatrix} \quad \text{and} \\
\hat{P}_{\phi+\pi} &\coloneqq \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\phi+\pi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -e^{i\phi} \end{bmatrix} .
\end{aligned} \tag{4}$$

So each step (one application of $\hat{U}(\bar{\theta}, \phi)$) rotates the previous output vector by an angle $2\bar{\theta}$ within the plane determined by $|\bar{s}\rangle$ and $|w\rangle$; it takes $\mathcal{O}(\sqrt{d})$ steps to reach $|w\rangle$ with near certainty when $\phi = \pi$.

What happens with Grover's strategy when $\phi \neq \pi$? One would need to apply some kind of adaptive control to rotate the output closer to $|w\rangle$ in subsequent steps, although this is constrained by the imperfect knowledge of ϕ. Does the optimal strategy involve starting from $|s\rangle$ (sounds reasonable...it reflects our ignorance about the oracle)?

1 Number of Grover updates for imperfect oracle

With the complex coefficients implicit in the repeated applications of $\hat{U}(\bar{\theta}, \phi)$, the motion of the search vector is better tracked on a Bloch sphere (rather than circle with the usual ($\phi = \pi$) Grover) with $|\bar{s}\rangle$ and $|w\rangle$ relabeled for convenience as $|0\rangle$ and $|1\rangle$ respectively. So we relabel the initial vector $|s\rangle = \sqrt{1 - \frac{1}{d}}\,|0\rangle + \frac{1}{\sqrt{d}}\,|1\rangle = \cos(\bar{\theta})\,|0\rangle + \sin(\bar{\theta})\,|1\rangle$, with the search task now being to approach $|1\rangle$ as close as possible.

We track the evolution of the search vector by tracking its projection on the eigenvectors of $\hat{U}(\bar{\theta}, \phi)$ in each update. The eigenvalues $\lambda_{\pm}$ and

[3] As long as the input vector $|\psi\rangle$ is of the type $|\psi\rangle = a\,|\bar{s}\rangle + b\,|w\rangle$, the output vector $|\psi'\rangle = \hat{U}\,|\psi\rangle$ will also be spanned by $|\bar{s}\rangle$ and $|w\rangle$ (i.e., $|\psi'\rangle = a'\,|\bar{s}\rangle + b'\,|w\rangle$ where a, b, a', b' are complex numbers) and the decomposition of $\hat{U}$ into the rotation and phase unitaries in (3) is valid. If the input vector has a component along $|\perp\rangle$ that is orthogonal to $|\bar{s}\rangle$ and $|w\rangle$, then it is passed through untouched by $\hat{U}_{w,\phi}$, $\hat{U}_s$ as well as $\hat{R}_{2\bar{\theta}}$, $\hat{P}_{\phi+\pi}$.

normalized eigenvectors $|u_\pm\rangle$ of $\hat{U}(\bar{\theta},\phi)$ [4] are

$$\begin{aligned}\lambda_\pm &= e^{i\left(\frac{\phi-\pi}{2}\pm\alpha\right)}\\ |u_\pm\rangle &= \left(\frac{1}{2}\mp\frac{\cos(\beta)}{2\sin(\alpha)}\right)^{\frac{1}{2}}\left(\frac{\epsilon\pm\kappa}{2}|0\rangle+|1\rangle\right),\end{aligned} \tag{5}$$

where $0<\alpha,\beta<\pi$ satisfy $\cos(\alpha)=\sin\left(\frac{\phi}{2}\right)\cos\left(2\bar{\theta}\right)$, $\cos(\beta)=\cos\left(\frac{\phi}{2}\right)\cos\left(2\bar{\theta}\right)$, and $\epsilon=\cot\left(2\bar{\theta}\right)(1+e^{i\phi})$, $\kappa=\frac{2e^{i\left(\frac{\phi}{2}\right)}\sin(\alpha)}{\sin\left(2\bar{\theta}\right)}$.

After k Grover updates, the search vector $|s(k)\rangle$ is

$$\begin{aligned}|s(k)\rangle &= (\hat{U}(\bar{\theta},\phi))^k|s\rangle\\ &= \lambda_+^k\langle u_+|s\rangle|u_+\rangle+\lambda_-^k\langle u_-|s\rangle|u_-\rangle,\end{aligned} \tag{6}$$

which makes the probability of measuring 1 after the k^{th} update

$$\begin{aligned}\Pr(1,k) &= |\lambda_+^k\langle u_+|s\rangle\langle 1|u_+\rangle+\lambda_-^k\langle u_-|s\rangle\langle 1|u_-\rangle|^2\\ &= \frac{\sin^2(\bar{\theta})\left(1+\sin\left(\frac{\phi}{2}\right)\right)}{1-\cos\left(2\bar{\theta}\right)\sin\left(\frac{\phi}{2}\right)}-\frac{\sin^2(2\bar{\theta})\sin\left(\frac{\phi}{2}\right)}{\sin^2(\alpha)}\cos^2((k+\frac{1}{2})\alpha),\end{aligned} \tag{7}$$

which shows that the maximum probability for measuring $|1\rangle$ is achieved first at the update $k_*=\lfloor\frac{1}{2}(\frac{\pi}{\alpha}-1)\rceil$. For $d\gg 1$, we can see its resemblance to Grover's result with

$$k_*=\frac{\frac{\pi\sqrt{d}}{4}}{\left(1+\frac{d}{4}\cos^2(\frac{\phi}{2})\right)^{1/2}}+c, \tag{8}$$

for some $|c|\leq 2$. Although fewer updates are needed in the $\phi<\pi$ case to reach the optimal probability (first term of (7)) compared to the usual Grover problem ($\phi=\pi$), it comes with a less than perfect probability of success.

2 Colliding blocks

The gentle Grover operation $\hat{U}(\bar{\theta},\phi)$ (3) corresponds to a certain rotation matrix that describes collisions between three masses moving along a line and undergoing elastic collisions. The rotation matrix is in fact, upto a certain global phase factor defined later, isomorphic to $\hat{U}(\bar{\theta},\phi)$; this isomorphism stems from the 2-to-1 homomorphism from SU(2) to SO(3). This construction is a generalization of Brown's system where the infinitely heavy wall is replaced by a body with finite mass, as explained in the following.

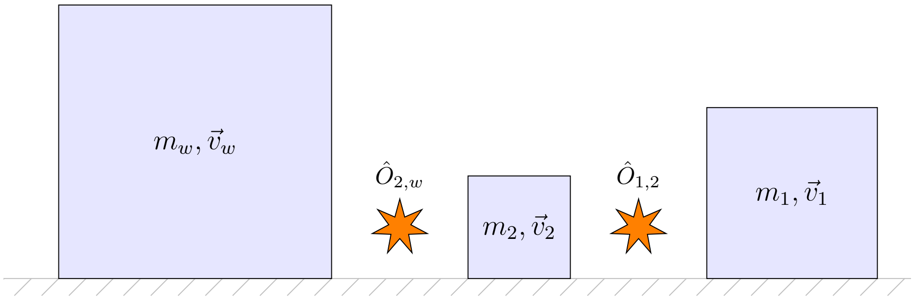

Figure 1: Modulo global phase, the repeated action of any single qubit gate on an initial quantum state can be emulated by the repeated elastic collisions of frictionlessly sliding blocks moving along a common line. The specific gate choice sets the relative sizes of the blocks. The specific initial quantum state sets the initial value of the unit vector $|u\rangle$ (9) which tracks the motion of the blocks.

Consider three masses, as shown in Fig. 1, undergoing two elastic collisions in every round. We track the unit vector

$$|u\rangle := \left(\sqrt{\frac{m_2}{2K}}v_2, \sqrt{\frac{m_w}{2K}}v_w, \sqrt{\frac{m_1}{2K}}v_1\right)^T, \tag{9}$$

where m_1, m_2 are the masses and v_1, v_2 are the velocities of the two blocks on the right, m_w and v_w are the mass and velocity of the block on the left and K is the total kinetic energy of the system. The subscript w stands for wall, which is now assumed to have finite mass (unlike Brown's case) and, like the other two blocks, also slides without friction. The unit vector $\left|u_{(2)}\right\rangle$ after the blocks finish a round of collisions - first between m_1 and m_2 (modeled as $\hat{O}_{1,2}$), followed by a collision between m_2 and m_w (modeled as $\hat{O}_{1,2}$) - is governed by the rotation [5]

$$\begin{aligned} \left|u_{(2)}\right\rangle &= \hat{O}(\omega,\delta)\,|u\rangle \quad \text{where} \\ \hat{O}(\delta,\omega) &= \underbrace{\begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ \sin(\omega) & -\cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\hat{O}_{2,w}} \underbrace{\begin{bmatrix} -\cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ \sin(\delta) & 0 & \cos(\delta) \end{bmatrix}}_{\hat{O}_{1,2}} \end{aligned} \tag{10}$$

[4] to reduce clutter, we won't explicitly note that $(\bar{\theta}, \phi)$ are also arguments of the eigenvalues and eigenvectors

[5] see p.77 of Notebook N190325, where $\hat{O}(\omega, \delta)$ is denoted by $R(u_2.u_w, u_1)$

is in SO(3), with $\cos(\delta) = \frac{m_1 - m_2}{m_1 + m_2}$ and $\cos(\omega) = \frac{m_2 - m_w}{m_2 + m_w}$. Using the standard 2-to-1 homomorphism $f : SU(2) \to SO(3)$ defined by

$$f\left(\begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Re}(x^2 - y^2) & \mathrm{Im}(x^2 + y^2) & -2\,\mathrm{Re}(xy) \\ -\mathrm{Im}(x^2 - y^2) & \mathrm{Re}(x^2 + y^2) & 2\,\mathrm{Im}(xy) \\ 2\,\mathrm{Re}(xy^*) & 2\,\mathrm{Im}(xy^*) & |x|^2 - |y|^2 \end{bmatrix}, \tag{11}$$

we see that $f(\hat{V}(\delta, \omega)) = (\hat{O}(\delta, \omega))^T$ where

$$\hat{V}(\delta, \omega) = \begin{bmatrix} \cos\left(\frac{\delta}{2}\right) e^{-i\frac{\omega}{2}} & -\sin\left(\frac{\delta}{2}\right) e^{i\frac{\omega}{2}} \\ \sin\left(\frac{\delta}{2}\right) e^{-i\frac{\omega}{2}} & \cos\left(\frac{\delta}{2}\right) e^{i\frac{\omega}{2}} \end{bmatrix}. \tag{12}$$

Define action of $\hat{V}((\delta, \omega)$ and $\hat{O}((\delta, \omega)$ on the corresponding vectors on the Bloch sphere and unit sphere in 3D respectively: If needed, this homomorphism can be spelled out in more detail by using p.76 and 78 of my notebook N190325, or with pp.245-246 of A.Zee's book "Group theory in a nutshell for physicists.. Comparing (12) with (3), we see that

$$\hat{V}(\delta = 4\bar{\theta}, \omega = \phi) = e^{-i\frac{\phi}{2}} \cdot \hat{U}(\bar{\theta}, \phi). \tag{13}$$

So, ignoring the global phase $e^{-i\frac{\phi}{2}}$ we conclude that tracking the search vector $|s(k)\rangle$ on the Bloch sphere is equivalent to tracking the unit vector $|u_{(2k)}\rangle$ on the unit sphere in 3 dimensions.

The total number of collisions in the finite-mass-wall case can be related to (7) in a manner analogous to G. Sanderson's explanation for the infinite-mass-wall case, with the wrinkle that one applies the inscribed angle theorem to a sequence of changing circular sections of the sphere on which $|u\rangle$ evolves. We won't do this here, preferring instead to focus on extending the colliding blocks analogy to other quantum gates.

Complete this next: We use the relation between (special?) unitary matrices and (special) orthogonal groups for the above analogy. Some other single unitary gates that can be expressed in the form of $\hat{U}(\bar{\theta}, \phi)$ are (include masses of the three billiard blocks for each of these gates)

$$\begin{aligned} \text{Hadamard:}\ \ H &:= \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix} \\ &= \hat{U}(\bar{\theta} = \frac{\pi}{8}, \phi = 0) \\ \text{Phase:}\ \ P &:= \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \\ &= \hat{U}(\bar{\theta} = 0, \phi = -\frac{\pi}{2}) \\ R_{\frac{\pi}{8}} &:= \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{bmatrix} \\ &= \hat{U}(\bar{\theta} = \frac{\pi}{16}, \phi = \pi). \end{aligned} \tag{14}$$

3 So what?

With an eye on the universal gate set (H, $R_{\frac{\pi}{8}}$, CNOT: cite Aaronson's lecture 16 in undergrad course [6]):
The 2-qubit CNOT gate also admits a billiards analogy due to the relation between (special?) U(4) and ????rotation group.

Then this: An obstruction to making the "composing" billiards circuits to complete the analogy is that each qubit picks up a phase after a unitary in a general quantum circuit, which becomes a relative phase when composed with outputs of gates acting on other qubits. This crucial aspect is hard to accommodate in the billiards model (This was not a problem in Grover because "no. of collisions - oracle calls" analogy did not need to consider the overall (not relative) phase because that is unimportant in the final probability).
Can geometry of general relativity help in velocities of billiard balls picking up relative phase? Quantum phase is equivalent to delay due to slow clocks at different gravitational potential? Think of the simplest gate that may require entanglement: CNOT.
It may be that provided SU(4) is locally isomorphic to SO(6) one can arrange CNOT gate's action on some 2-qubit state with a pair of systems, each containing three balls which represent a single qubit as above. Look at Einstein-Rosen paper from 1930s to see if the two states can be entangled through a wormhole. Also L. Susskind's papers, including p.24 of https://arxiv.org/pdf/1604.02589.pdf. Basically, I want this (vague) property: correlation in terms of the motions of both systems, but without any component of motion that changes the relative spatial separation between the systems. Everything else that I can think of - "simple" gravitation/electromagnetic forces - can induce some kind of correlation in motion (which doesn't look like entanglement though) but also makes the systems move towards, or away from, each other.

[6]https://www.scottaaronson.com/qclec/16.pdf

References

[1] A. R. Brown, "Playing Pool with ψ: from Bouncing Billiards to Quantum Search", *arXiv e-prints*, arXiv:1912.02207, arXiv:1912.02207, Dec. 2019. arXiv: `1912.02207 [quant-ph]` (cit. on p. 1).