Colliding blocks and Elementary Quantum Gates

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Abstract

We extend Brown's analogy, between Grover search and classical elastic collisions, to single qubit quantum gates. In particular, we show that the repeated action of a quantum gate on a single qubit can be emulated by repeated elastic collisions between three collinearly moving blocks. We show that this analogy is essentially a manifestation of the local ismorphism between SU(2) and SO(3).

Grover's algorithm [1] provides a quantum speedup for searching an unstructured database when allowed the use of a certain oracular function The algorithm has a clear geometric interpretation in terms of rotations of the search vector around a unit circle. Recently, Brown [2] constructed an analogy between this interpretation and the motion of two colliding objects moving along a straight line.

In this note, we show that Brown's analogy can be extended to the repeated action of *any* single qubit quantum gate. We describe a perturbed version of Grover search in which the oracle only *gently* marks an item. We show the colliding blocks' analogy for it - modulo a possible global phase - by using the local isomorphism between SU(2) and SO(3). We see that such a perturbed Grover search can model the repeated action of any other single qubit gate. We end with a speculation on what it would take to extend this analogy beyond single qubit gates.

1 Imperfect oracle

Consider a qudit $|\psi\rangle := \sum_{k=1}^{k=d} c_k |k\rangle$, i.e., the wavefunction of a general $d-$state system ($d > 1$) with $\langle\psi|\psi\rangle = 1$. Suppose one is given a quantum gate $\hat{U}_w := I - 2|w\rangle\langle w|$, with $w \in (1, 2, 3, ..., d)$ but otherwise unknown. Such a "blackbox oracle" acts on $|\psi\rangle$ by flipping the sign of the coefficient $c_w = \langle w|\psi\rangle$ but leaving all other coefficients of $|\psi\rangle$ unchanged. Grover's algorithm for determining w works by starting with the unbiased vector

$$|s\rangle := \frac{1}{\sqrt{d}} \sum_{k=1}^{k=d} |k\rangle \,, \tag{1}$$

and rotating it in the plane formed by $|w\rangle$ and $|\bar{s}\rangle := \sqrt{\frac{1}{d-1}} \sum_{k \neq w} |k\rangle$. The search vector is repeatedly rotated by $2\bar{\theta}$, where $\sin(\bar{\theta}) := \frac{1}{\sqrt{d}}$, with successive applications of the quantum gate $\hat{U}(\bar{\theta}) := \hat{U}_s \hat{U}_w$, where $\hat{U}_s := 2|s\rangle\langle s| - I$. Rotating $\mathcal{O}(\sqrt{d})$ times maximizes the probability of determining w by a subsequent measurement of the search vector.

Consider the case when the oracle has, effectively, a coherent error in that the sign for the $|w\rangle$ component is not flipped but is instead multiplied by a phase factor $e^{i\phi}$; $\phi = \pi$ would mean that the Grover oracle is perfect, $\phi = 0$ means that the oracle is the identity matrix (it doesn't mark any item), while an intermediate value of ϕ means that the oracle only marks the $|w\rangle$ coefficient *gently*. This imperfect oracle is modeled by the unitary

$$\hat{U}_{w,\phi} := I + (e^{i\phi} - 1)|w\rangle\langle w|. \tag{2}$$

Grover's strategy, when extended to the case when $\phi \neq \pi$, is to start from the initial equal-superposition state $|s\rangle$ and repeatedly apply

$$\begin{aligned} \hat{U}(\bar{\theta}, \phi) &:= \hat{U}_s \hat{U}_{w,\phi} \\ &= \begin{bmatrix} \cos(2\bar{\theta}) & e^{i\phi}\sin(2\bar{\theta}) \\ \sin(2\bar{\theta}) & -e^{i\phi}\cos(2\bar{\theta}) \end{bmatrix} \\ &= \hat{R}_{2\bar{\theta}} \hat{P}_{\phi+\pi} \end{aligned} \tag{3}$$

for a finite amount of steps where $\hat{R}_{2\bar{\theta}}$, $\hat{P}_{\phi+\pi}$ are the unitary rotation and phase matrices

$$\begin{aligned} \hat{R}_{2\bar{\theta}} &:= \begin{bmatrix} \cos(2\bar{\theta}) & -\sin(2\bar{\theta}) \\ \sin(2\bar{\theta}) & \cos(2\bar{\theta}) \end{bmatrix} \quad \text{and} \\ \hat{P}_{\phi+\pi} &:= \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\phi+\pi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -e^{i\phi} \end{bmatrix}. \end{aligned} \tag{4}$$

When $\phi = \pi$, each application of $\hat{U}(\bar{\theta}, \phi)$ rotates the previous output vector by an angle $2\bar{\theta}$ within the plane determined by $|\bar{s}\rangle$ and $|w\rangle$, taking $\mathcal{O}(\sqrt{d})$ steps to reach $|w\rangle$ with near certainty. [1] In the following, we relate the action of $\hat{U}(\bar{\theta}, \phi)$ to a system of elastically colliding blocks.

2 Colliding blocks

The main point of this section is that the gentle Grover operation $\hat{U}(\bar{\theta}, \phi)$ (3) corresponds to a certain rotation matrix in SO(3) that describes elastic collisions between three masses moving frictionlessly along a line. This construction is a generalization of Brown's system where the infinitely heavy wall is replaced by a body with finite mass, as explained in the following.

[1] If it was known that the oracle had $\phi \neq \pi$, a better search strategy would be to estimate and compensate for such a ϕ. But we are not concerned with that problem here.

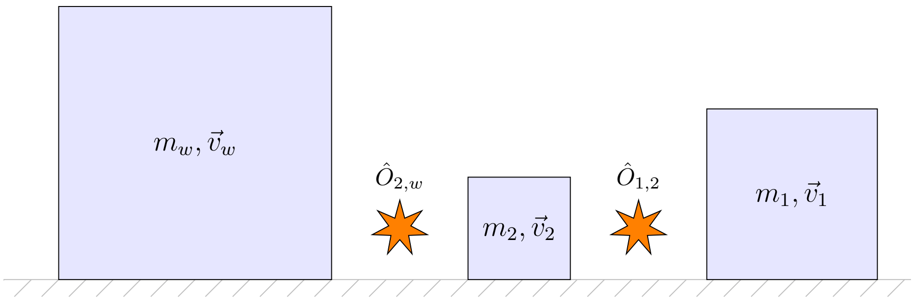

Figure 1: Modulo global phase, the repeated action of any single qubit gate on an initial quantum state can be emulated by the repeated elastic collisions of frictionlessly sliding blocks moving along a common line. The specific gate choice sets the relative sizes of the blocks. The specific initial quantum state sets the initial value of the unit vector $|u\rangle$ which tracks the motion of the blocks.

Consider three masses, as shown in Fig. 1, undergoing two elastic collisions in every round. We track the unit vector

$$|u\rangle := \left(\sqrt{\frac{m_2}{2K}}v_2, \sqrt{\frac{m_w}{2K}}v_w, \sqrt{\frac{m_1}{2K}}v_1\right)^T, \tag{5}$$

where m_1, m_2 are the masses and v_1, v_2 are the velocities of the two blocks on the right, m_w and v_w are the mass and velocity of the block on the left and K is the total kinetic energy of the system. The subscript w stands for wall, which is now assumed to have finite mass (unlike Brown's case) and, like the other two blocks, also slides without friction. This implies that K is conserved. The unit vector $\left|u_{(2)}\right\rangle$ after the blocks finish a round of collisions - first between m_1 and m_2 (modeled as $\hat{O}_{1,2}$), followed by a collision between m_2 and m_w (modeled as $\hat{O}_{1,2}$) - is governed by the rotation

$$\begin{aligned}
&\left|u_{(2)}\right\rangle = \hat{O}(\omega,\delta)\,|u\rangle \quad \text{where} \\
&\hat{O}(\delta,\omega) = \underbrace{\begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ \sin(\omega) & -\cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\hat{O}_{2,w}} \underbrace{\begin{bmatrix} -\cos(\delta) & 0 & \sin(\delta) \\ 0 & 1 & 0 \\ \sin(\delta) & 0 & \cos(\delta) \end{bmatrix}}_{\hat{O}_{1,2}}
\end{aligned} \tag{6}$$

is in SO(3), with $\cos(\delta) = \frac{m_1-m_2}{m_1+m_2}$ and $\cos(\omega) = \frac{m_2-m_w}{m_2+m_w}$. Using the classical 2-to-1 homomorphism $f : \mathrm{SU}(2) \to \mathrm{SO}(3)$ defined by

$$f\left(\begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Re}(x^2-y^2) & \mathrm{Im}(x^2+y^2) & -2\,\mathrm{Re}(xy) \\ -\mathrm{Im}(x^2-y^2) & \mathrm{Re}(x^2+y^2) & 2\,\mathrm{Im}(xy) \\ 2\,\mathrm{Re}(xy^*) & 2\,\mathrm{Im}(xy^*) & |x|^2-|y|^2 \end{bmatrix}, \tag{7}$$

we see that $f(\hat{V}(\delta,\omega)) = (\hat{O}(\delta,\omega))^T$ where

$$\hat{V}(\delta,\omega) = \begin{bmatrix} \cos\left(\frac{\delta}{2}\right)e^{-i\frac{\omega}{2}} & -\sin\left(\frac{\delta}{2}\right)e^{i\frac{\omega}{2}} \\ \sin\left(\frac{\delta}{2}\right)e^{-i\frac{\omega}{2}} & \cos\left(\frac{\delta}{2}\right)e^{i\frac{\omega}{2}} \end{bmatrix}. \tag{8}$$

Comparing (8) with (3), we see that

$$\hat{V}(\delta = 4\bar{\theta}, \omega = \phi) = e^{-i\frac{\phi}{2}} \cdot \hat{U}(\bar{\theta}, \phi). \tag{9}$$

If one is only concerned with repeated applications of $\hat{U}(\bar{\theta}, \phi)$ on a quantum state followed by an eventual measurement of the final result, then one can ignore the global phase $e^{-i\frac{\phi}{2}}$ that is accumulated in each step of the process - the probability of measuring $|w\rangle$ is not changed by ignoring this global phase. Consequently, we conclude that tracking the search vector $|s(k)\rangle$ on the Bloch sphere is equivalent to tracking the unit vector $\left|u_{(2k)}\right\rangle$ on the unit sphere in 3 dimensions. In particular, by setting the mass of the wall m_w to be infinite one recovers Brown's analogy: $\left|u_{(2k)}\right\rangle$ evolves on the two-dimensional unit circle.

We will conclude by noting how the colliding blocks analogy extends to other single qubit gates and speculate on what it would take to extend the analogy to any quantum circuit.

3 Blocks for single qubit gates (and beyond?)

We used the local isomorphism between SU(2) and SO(3) for extending the colliding blocks analogy to the imperfect oracle case in Grover search. However, if one ignores the global phase that accumulates from the repeated action of *any* single qubit gate (as in the imperfect Grover oracle above), then it is easy to see that any gate in U(2) can be expressed in the form of $\hat{U}(\bar{\theta}, \phi)$ by simply adjusting the mass ratios of the blocks in the corresponding rotation in SO(3). Some examples are

$$\begin{aligned} \text{Hadamard:}\ \ H &:= \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix} \\ &= \hat{U}(\bar{\theta} = \frac{\pi}{8}, \phi = 0) \\ \text{Phase:}\ \ P &:= \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \\ &= \hat{U}(\bar{\theta} = 0, \phi = -\frac{\pi}{2}) \\ R_{\frac{\pi}{8}} &:= \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{bmatrix} \\ &= \hat{U}(\bar{\theta} = \frac{\pi}{16}, \phi = \pi). \end{aligned} \tag{10}$$

With an eye on the universal gate set (H, $R_{\frac{\pi}{8}}$, CNOT), one can ask whether the blocks' analogy can be extended to the CNOT gate, and more generally, to any quantum circuit. Two major objections arise:

- When an arbitrary number of gates and qubits are involved in the circuit, the global phase factors that we have ignored become relative phase factors between distinct qubits. No interference, and hence no quantum computing, is possible without accounting for such relative phases.
- For CNOT gates, and in general for arbitrary quantum circuits, one needs to engineer entanglement between systems of colliding blocks.

One could potentially use the local isomorphism between SU(4) and SO(6) to guess a system of colliding blocks for the CNOT gate, but it is not clear if one can engineer relative phases and entanglement with this approach.

References

[1] L. K. Grover, "A fast quantum mechanical algorithm for database search," *arXiv e-prints*, May 1996. arXiv: `quant-ph/9605043 [quant-ph]` (cit. on p. 1).

[2] A. R. Brown, "Playing Pool with ψ: from Bouncing Billiards to Quantum Search," *arXiv e-prints*, Dec. 2019. arXiv: `1912.02207 [quant-ph]` (cit. on p. 1).