Text

COEP Technological University Pune
Department of Mathematics
(MA- 20004) - VECTOR CALCULUS AND PARTIAL DIFFERENTIAL EQUATIONS
S.Y. B.Tech. Semester III (All Branches)
Tutorial 4 (AY: 2023-24)

01. Find the directional derivative of f at P in the direction of $\overline{a}$ where

(a) $f = \ln(x^2 + y^2), \quad P : (4, 0), \quad \overline{a} = \mathbf{i} - \mathbf{j}$
Ans: $1/2\sqrt{2}$

(b) $f = xyz, \quad P : (-1, 1, 3), \quad \overline{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$
Ans: $7/3$

02. Let $f = xy - yz$, $\overline{v} = [2y, 2z, 4x + z]$, $\overline{w} = [3z^2, 2x^2 - y^2, y^2]$. Find
(a) div(grad f) (b) grad(div $\overline{w}$) (c) div(curl$\overline{v}$) (d) $D_{\overline{w}}f$ at (1,1,0) (e) $[(\text{curl } \overline{v}) \times \overline{w}] \cdot \overline{w}$

03. Prove the following:

(i) $(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$ Hence prove that a non zero vector of constant length is perpendicular to its derivative.

(ii) $(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$

04. For $f = x^2 - y^2$ and $g = e^{x+y}$, verify div $(f\nabla g)-$div $(g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

05. Find the direction and magnitude of the force in an electrostatic field f at the point P:

$f = ln(x^2 + y^2); P(4, 2)$
Ans: $[2/5, 1/5], \sqrt{5}/5$

$f = 2x^2 + 4y^2 + 9z^2; P(-1, 2, -4)$
Ans: $[-4, 16, -72], 1/\sqrt{341}[-4, 16, -72]$

06. For what points $P(x, y, z)$ does gradient of $f = 25x^2 + 9y^2 + 16z^2$ have the direction from P to origin?
Ans: Points along the coordinate axes

07. The flow of heat in a temperature field $T = e^{x^2-y^2} \sin 2xy$ takes place in the direction of maximum decrease of temperature T. Find the direction of the flow in general and at the point $P(1, 1)$.
Ans: -grad(T) and -grad(T) at P

08. Find the unit normal vector for the surface Text $x^2 + y^2 + 2z^2 = 26$ at the point $P(2, 2, 3)$. Also find the equation of tangent plane and normal line through that point.
Ans: $1/\sqrt{11}[1, 1, 3], x + y + 3z = 13, \overline{r} = [2 + t, 2 + t, 3 + 3t]$

09. If on a mountain the elevation above sea level is $z(x, y) = 1500 - 3x^2 - 5y^2$[meters], what is the direction of steepest ascent at $P(-0.2, 0.1)$?
Ans: Ans: $[1.2, -1]$

10. What is the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $(x \ln z) - y^2 = -4$ at (-1,2,1)?
Ans: $15/\sqrt{17}$

11. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2,-1,2).
Ans: $cos^{-1}(8/3\sqrt{21})$

12. Find $div\mathbf{v}$ and its value at P:

(i) $\mathbf{v} = [0, sinx^2yz, cosxy^2z], P(1, 1/2, -\pi)$
Ans: $\sqrt{2}/8$

(ii) $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}[-x, -y, -z]$
Ans: 0

13. Find v_3 such that $div\mathbf{v}$ is greater than zero everywhere if $\mathbf{v} = [x, y, v_3]$
Ans: $v_3 = kz, k > -2$

14. Find potential field f for given $\overline{v}$ or state that $\overline{v}$ has no potential.

(a) $[xy, 2xy, 0]$ (b) $[x^2 - yz, y^2 - zx, z^2 - xy]$
Ans: No potential, $f = (x^3 + y^3 + z^3)/3 - xyz + c$

15. If $\overline{u}$ and $\overline{v}$ are irrotational, then show that $\overline{u} \times \overline{v}$ is incompressible.

16. Plot the the given velocity field of a fluid flow in a square centered at the origin with sides parallel to coordinate axis. Recall that the divergence measures outflow minus inflow. By looking at the flow near the sides of the square, can you see whether div $\overline{v}$ must be positive or negative or zero?

(a) $\overline{v} = x\mathbf{i}$ (b) $\overline{v} = x\mathbf{i} + y\mathbf{j}$ (c) $\overline{v} = x\mathbf{i} - y\mathbf{j}$

17. The velocity vector $\overline{v}(x, y, z)$ of an incompressible fluid rotating in a cylindrical vessel is of the form $\overline{v} = \overline{w} \times \overline{r}$, where $\overline{w}$ is the constant rotation vector. Show that div $\overline{v} = 0$.

18. The velocity vector $\overline{v} = [x, y, -z]$ of a fluid motion is given. Is the flow irrotational? Incompressible?
Ans: Yes, No

19. Find the values of the constants a,b,c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at (1,2,-1) has a maximum of magnitude 64 in a direction parallel to z-axis.
Ans: -6, -24, 8 or 6, 24, -8

20. Find the directional derivative of $f = e^{2x}\cos(yz)$ at (0,0,0) in the direction of tangent to the curve $x = a\sin t,\quad y = a\cos t,\quad z = at$ at $t = \pi/4$.
Ans: 1

21. In what directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?
Ans: $\hat{\imath} + \hat{\jmath}$ and $-\hat{\imath} - \hat{\jmath}$

22. Is there a direction $\overline{u}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14 ? Give reasons to your answer.
Ans: No since directional derivative can be at most $\sqrt{185}$

23. The derivative of $f(x, y)$ at point $P_0(1, 2)$ in the direction of $\mathbf{i}+\mathbf{j}$ is $2\sqrt{2}$ and in the direction of $-2\mathbf{j}$ is -3. What is derivative of f at P_0 in the direction of $-\mathbf{i} - 2\mathbf{j}$?
Ans: $-7/\sqrt{5}$

24. We know that the gradient of differentiable function of two variables at a point is always normal to the function's level curve through that point. Further, the line through a point $P_0(x_0, y_0)$ normal(perpendicular) to a vector $A\mathbf{i}+B\mathbf{j}$ has the equation $A(x-x_0)+B(y-y_0) = 0$. Using these facts, find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ at the point (-2,1).
Ans: $2y - x = 4$

25. Find the points (x, y) and the directions for which the directional derivative of $f(x, y) = 3x^2 + y^2$ has its largest value, if (x, y) is restricted to be on the circle $x^2 + y^2 = 1$. Ans: $(\pm 1, 0)$ along x-axis

26. A differentiable scalar field f has at the point $(1, 2)$, directional derivative $+2$ in the direction toward $(2, 2)$ and -2 in the direction toward $(1, 1)$. Determine the gradient vector at $(1, 2)$ and compute the directional derivative in the direction toward $(4, 6)$.
Ans: $[2, 2]$,14/5

Questions on CO4 and CO5

1. Define Directional Derivative of a scalar function f at point P in the direction of $\overline{a}$. Derive the formula $D_{\overline{a}}f = \dfrac{1}{|\overline{a}|}\,\overline{a} \cdot \text{grad } f$.

2. Let f be a differentiable scalar function that represents a surface $S : f(x, y, z) = c$. If the gradient of f at a point P of S is not the zero vector, then prove that it is a normal vector of S at P.

3. Let $f(P) = f(x, y, z)$ be a scalar function having continuous first partial derivatives. Then prove that grad f exists and its length and direction are independent of the particular choice of Cartesian Co-ordinates in space. If at a point P the gradient of f is not the zero vector, then prove that it has the direction of maximum increase of f at P.

Please report any mistakes in the problems here and start a forum on moodle where you can post the solutions to these.