COEP Technological University Pune
(A Public University of Government of Maharashtra, Pune-411005)
Department of Mathematics
(MA- 20004) - VECTOR CALCULUS AND PARTIAL DIFFERENTIAL EQUATIONS
S.Y. B.Tech. Semester III (All Branches)
**Tutorial 7 (AY: 2023-24)**

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1. Identify the following equations by their name (if any), order, linear or nonlinear, homogeneous or non-homogeneous :

$(i)$ $\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = e^{s+t}$

Ans : 2-D Poisson equation, second order, linear, non-homogeneous differential equation.

$(ii)$ $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial t^2}$

Ans : second order, linear, homogeneous differential equation that can be solved as an ODE.

$(iii)$ $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Ans : One dimensional wave equation, second order, linear, homogeneous differential equation.

$(iv)$ $(\frac{\partial^2 y}{\partial x^2}) siny + \frac{\partial y}{\partial t} = 0$

Ans : second order, non-linear, homogeneous differential equation.

$(v)$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ Ans: Same as $(i)$.

2. Write the general one dimensional heat equation and show that $u = e^{-9t} sin\omega x$ is a solution for some suitable constant $c$.

3. State the ideal physical conditions which are assumed to model the

Partial Differential Equation of a vibrating string ?

Ans: String is homogeneous i.e. density is uniform and perfectly elastic i.e. no resistance to bending, tightly stretched so that tension is so large that gravitational force can be neglected, motion of the particles is only in one direction i.e. vertical so that deflection and slope at any point is always small in absolute value.

4. Derive the one-dimensional wave equation which governs the transverse vibrations of an elastic string of length $L$ with standard assumptions stated in question 3 above. Discuss what will happen if some of the assumptions are not satisfied. Also, what additional assumption will simplify the calculations ?

5. Write a general second order homogeneous partial differential equation in two independent variables. For this prove the fundamental theorem of superposition of solutions.

6. Find solutions $u(x, y)$ of the following equations by separating variables.

$(i)$ $u_x + u_y = (x + y)u$ Ans : $ce^{(x^2+y^2)/2+k(x-y)}$
$(ii)$ $y^2u_x - x^2u_y = 0$ Ans : $ce^{m(x^3+y^3)}$
$(iii)$ $u_x = 2u_t + u$ Ans : $ce^{(1+2k)x+kt}$
$(iv)$ $2xz_x - 3yz_y = 0$ Ans : $cx^{k/2}y^{k/3}$

7. Find the deflection $u(x, t)$ of the string of length $L = 1$ when $c = 1$, the initial velocity is zero, and the initial deflection is $2x - x^2, 0 < x < 1/2$ and is zero if $1/2 < x < 1$ Express $u(x, t)$ as superposition of two functions.

8. Find the deflection $u(x, t)$ of the string of length $L$ when $c = 1$, the initial velocity is zero, and the initial deflection in the interval $[0, L]$ is

$(i)$ $f(x) = kx(1 - x^2)$ with $L = 1$

$$\text{Ans}: \ u(x,t) = \sum_{n=1}^{\infty} \frac{12k(-1)^{n+1}}{n^3\pi^3} \cos n\pi t \ \sin n\pi x$$

$(ii)$ $f(x) = k(\sin(\pi x) - 1/3 \sin(3\pi x))$ with $L = 1$

$$\text{Ans}: \ u(x,t) = k \cos \pi t \sin \pi x - (k/3) \cos 3\pi t \sin 3\pi x$$

$$(iii) \ \ f(x) = \begin{cases} \frac{3b}{L}x & ; 0 \le x \le L/3 \\ \frac{3b}{L}(L-2x) & ; L/3 \le x \le 2L/3 \\ \frac{3b}{L}(x-L) & ; 2L/3 \le x \le L \end{cases}$$

$$\text{Ans}: \ u(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x; \quad B_n = \frac{18b \sin n\pi/3}{n^2\pi^2}(1 - 2\cos n\pi/3)$$

9. Find the deflection $u(x,t)$ of the string of length $L$ when $c = 1$, the initial deflection is zero, and the initial velocity is

$(i)$ $g(x) = 0.01x$ if $0 \le x \le \pi/2$ and $g(x) = 0.01(\pi - x)$ if $\pi/2 \le x \le \pi$. Take $L = \pi$.

$(ii)$ $g(x) = b \sin(3\pi x/L) \cos(2\pi x/L)$. Take $L = \pi$.

Ans : $\frac{b}{2} \sin t \sin x + \frac{b}{5} \sin 5t \sin 5x$

For a vibrating string there is only one set of boundary conditions viz. zero boundary conditions. And hence when we solve the ODE we get only the trivial solution for two choices of the constant ($k = 0$ and $k > 0$) and only the case when $k < 0$ gives us a solution which is harmonic in nature and then we can get the complete solution by applying the initial conditions.
But this is not so for the one dimensional heat equation. Mathematically you get a solution for two cases - viz. when $k = 0$ , solution is $u(x,t) = Ax + B$; and when $k < 0 = -\lambda^2$, the solution is $u(x,t) = (acos\lambda x + bsin\lambda x)e^{-c^2\lambda^2 t/l^2}$ (Note that the third choice of $k$ is physically

impossible since the temperature cannot go to infinity as t goes to infinity). Now using the principle of superposition we get the solution of the problem as $u(x,t) = Ax + B + (acos\lambda x + bsin\lambda x)e^{-c^2\lambda^2 t/l^2}$. Observe that this tends to $Ax + B$ as $t \to \infty$ and hence is called the steady state solution. What happens to the temperature distribution till the steady state is reached is governed by the term $(acos\lambda x + bsin\lambda x)e^{-c^2\lambda^2 t/l^2}$ and hence this part of the solution is called the transient solution. Now to get the values of $A, B$ and $\lambda$ we must have three conditions and they are - the two boundary conditions and one initial condition. We can have different sets of boundary conditions and initial condition may also be given in different ways. Following problems (5 to 11) illustrate the different sets of boundary conditions and initial conditions. Starting from the solution $u(x,t) = Ax + B + (acos\lambda x + bsin\lambda x)e^{-c^2\lambda^2 t/l^2}$ one can find the constants by using the given conditions.

**Important tips :**

**1. Always think of the physical nature of the problem.**
**2. Note that it is always easier to use the zero conditions first.**
**3. Any zero condition, for example $u(0,t) = 0$ must be understood as to be true for all $t$**
**4. Insulated ends or adiabatic conditions mean that there is no flow of heat in the direction of the gradient which for one dimensional heat equation means that $\partial u/\partial x$ is zero at the ends at all times.**
**5. You could have mixed boundary conditions.**

10. Find the temperature $u(x,t)$ in a bar of length $L$ that is perfectly insulated laterally, whose ends are kept at temperature $0\ ^0c$ and whose initial temperature is $f(x)$ where

$(i)$ $f(x) = k\sin(0.2\pi x); L = 10cm, \rho = 10.6gm/cm^3$, thermal conductivity $= 1.04cal/(cmsec^o c)$, $\sigma = 0.056cal/gm^0 c$.

Ans : $B_2 = k$, all other coefficients are 0.

$(ii)$ $f(x) = x$ if $0 < x < 2.5; f(x) = 2.5$ if $2.5 < x < 7.5; f(x) = 10 - x$ if $7.5 < x < 10; L = 10cm, \rho = 10.6gm/cm^3$, thermal conductivity $=$ $1.04cal/(cmsec^oc)$, $\sigma = 0.056cal/gm^0c$.

$(iii)$ $f(x) = 2 - 0.4|x - 5|; L = 10cm$

11. Find the temperature $u(x,t)$ in a bar that is perfectly insulated laterally, whose ends are insulated and whose initial temperature is $f(x)$ where

$(i)$ $f(x) = \cos(2x), L = \pi, c = 1$

$(ii)$ $f(x) = 1 - (x/\pi), L = \pi, c = 1$

$(iii)$ $f(x) = (x - \pi/2)^2, L = \pi, c = 1.$

Ans :

$$u(x,t) = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{1}{k^2} \cos 2kxe^{-4k^2t}$$

12. If the ends $x = 0$ and $x = L$ of laterally insulated bar of length $L$ are kept at constant temperatures 10 $^0c$ and 50 $^0c$ respectively, what is the temperature $u(x)$ in the bar after a long time ( $t \to \infty$).

Ans : $u(x) = 10 + 40x/L$

13. A bar of 10 cm long with ends A and B kept at 20 $^0c$ and 40 $^0c$ respectively until steady state conditions prevail. The temperature at A is then suddenly raised to 50 $^0c$ and at the same time at B lowered to 10 $^0c$ and these are maintained. Find the subsequent temperature distribution. Show that the temperature at the middle point of the bar remains unaltered for all time.

Ans:

$$u(x,t) = -4x + 50 - \frac{60}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi x}{5}e^{-tk^2c^2\pi/25}; u(5,t) = 30^0C$$

14. A rod of 100 cm. length has its ends kept at 0 $^0c$ and 100 $^0c$ until the steady state conditions prevail. The two ends are then suddenly insulated and maintained so. Find the temperature in the rod. Show that the sum of the temperatures at any two points equidistant from the centre is always 100 $^0c$.

Ans :

$$u(x,t) = 50 - \frac{400}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{100}e^{-t(2k-1)^2\pi^2c^2/(100)^2}$$

15. Obtain the temperature in a bar of length 1, and with lateral surface insulated where, $u_x(1,t) = 0, u(0,t) = 10, u(x,0) = 10 - x, 0 < x < 1$.

Ans :

$$10 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}e^{-t(2n-1)^2c^2\pi^2/4}$$

16. The heat flow in a bar of length $\pi$ with $c = 1$ which is laterally insulated and whose one end $x = 0$ is kept at 0 $^0c$ and at the another end heat is flowing into air of constant temperature 0 $^0c$ is governed by the one dimensional heat equation with boundary conditions $u(0,t) = 0; u_x(\pi,t) = -u(\pi,t)$. Find a solution $u(x,t)$. Can you find infinitely many solutions ? If so, what are they ?

Ans : $u(x,t) = A\sin px e^{-p^2t}$ where $p = -\tan p\pi$

17. Show that the problem consisting of $u_t - c^2u_{xx} = Ne^{-\alpha x}$ subject to $u(0,t) = 0 = u(L,t); u(x,o) = f(x)$ can be reduced to a problem for the homogeneous heat equation by setting $u(x,t) = v(x,t) + w(x)$ and determining $w$ so that $v$ satisfies the homogeneous equation and the conditions

$v(0,t) = v(L,t) = 0; v(x,0) = f(x) - w(x).$

18. The faces of a thin square copper plate of side 24 cm. are perfectly insulated. The right side is kept at 20 $^0c$ and the other sides are kept at 0 $^0c$. Find the steady state temperature $u(x,y)$ in the plate.

Ans:

$$\frac{80}{\pi} \sum_{n \text{odd}} \frac{1}{n \sinh n\pi} \sinh \frac{n\pi x}{24} \sin \frac{n\pi y}{24}$$

19. Find the steady state temperature distribution in a thin rectangular metal plate $0 < x < a, 0 < y < b$ with its two faces insulated with the following boundary conditions prescribed on the four edges - $u(0,y) = u(a,y) = u(x,0) = 0; u(x,b) = \sin(n\pi x/a)$.

Ans:

$$u(x,y) = \sum_{n=1}^{\infty} cosech \frac{n\pi b}{a} \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

20. Obtain the deflection of a vibrating elastic string $u(x,t)$ at any point $x$ and at any time $t > 0$ where the vibrations are governed by the PDE $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ subject to boundary conditions $u(0,t) = u(L,t) = 0$ for all $t$ and initial conditions $u(x,0) = f(x); \dfrac{\partial u}{\partial t}_{|t=0} = g(x)$.

21. Can you express solution of one dimensional wave equation as $u(x,t) = \frac{1}{2}(g^*(x - ct) + g^*(x + ct))$ where $g(x)$ is initial velocity and $g^*(x)$ is its odd/even extension, initial deflection is zero.

22. Derive the one-dimensional heat equation which governs the heat flow in a long thin bar of length $L$ with insulated lateral surface.

23. Obtain the temperature $u(x,t)$ at any point $x$ and at any time $t > 0$

in a thin long bar of length $L$ where the heat flow is governed by the PDE $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ if both the ends are kept at $0\ ^0c$ and initial temp. is given by $u(x, 0) = f(x)$.

24. Obtain the temperature $u(x, t)$ at any point $x$ and at any time $t > 0$ in a thin long bar of length $L$ where the heat flow is governed by the PDE $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ if both the ends are insulated and initial temp. is given by $u(x, 0) = f(x)$.

25. Obtain the temperature $u(x, y)$ at any point $(x, y)$ in a rectangular plate of sides $a$ and $b$ where the heat flow is steady, faces are completely insulated and if initial temperature is given by $u(0, y) = f(y),\ u(x, 0) = u(a, y) = u(x, b) = 0$.

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Please report any mistakes in the problems and/or answers given here.

***Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country. - David Hilbert***