## Question [I]

1. Attempt **any two**:

   (a) Determine whether the set of all first-degree polynomials, $ax + b, a \neq 0$ with standard addition and scalar multiplication is a real vector space. Justify your answer. [CO2][2]

   (b) Find a vector parallel to the line of intersection of the two planes: $2x - y + z = 1;\ 3x + y + z = 2.$ [CO1][2]

   (c) Let $V$ be a vector space of all $2 \times 2$ matrices with real entries. Determine whether the set $B = \{A \in V : det(A) = 0\}$ is a subspace of $V$. Justify your answer. [CO2][2]

2. Find eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Hence, find an orthogonal basis for $\mathbb{R}^2$. [CO3][3]

**P.T.O.**

3. An elastic membrane in the $x_1x_2$-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $(x_1, x_2)$ goes over into the point $(y_1, y_2)$ given by $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find the principal directions (eigen vectors). What shape does the boundary circle take under this deformation? [CO5][3]

## Question [II]

1. Let $V$ be a vector space. Prove that if $W_1$ and $W_2$ are subspaces of $V$ then their intersection $W_1 \cap W_2$ is a subspace of $V$.

**OR**

Let $U$ be a subspace of $\mathbb{R}^n$. Let $W$ be a set of elements of $\mathbb{R}^n$ that are perpendicular to every element of $U$, that is, $W = U^{\perp}$. Show that $W$ is a subspace of $\mathbb{R}^n$. [CO2][2]

2. Determine whether $(1, 1, -1)$ can be written as a linear combination of vectors $(2, -1, 3)$ and $(5, 0, 4)$. If yes, find that linear combination. [CO1][2]

3. Determine whether the set $\{1, x^2, x^2+2\}$ spans $P_2(\mathbb{R})$. Is it linearly independent? Justify your answers.

**OR**

Let $S = \{u, v\}$ be a linearly independent set. Prove that the set $\{u + v, u - v\}$ is linearly independent. [CO2][2]

4. Find the eigenvalues and a basis for the eigenspaces for the matrix [CO3][4]

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

## Question [III]

1. Attempt the following:

(a) Let $V$ be a vector space and let $A : V \to V$ be a linear map. Let $v_1, v_2, \ldots, v_n$ be eigenvectors of $A$ with eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_n$ respectively. Assume that these eigenvalues are distinct. Show that the set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. [CO4][3]

?

(b) $\lambda = a$ is an eigenvalue of multiplicity 3. How many linearly independent eigenvectors can it have? [CO3][1]

2. Attempt the following:

(a) Find solution space of the system of linear equations $AX = \mathbf{0}$:

$$x_1 - x_2 + 3x_3 = 0;\ 2x_1 + x_2 + 3x_3 = 0.$$

[CO1][2]

(b) Find the nullity and a basis for the null space of $A$.

**OR**

Find the rank and a basis for the range space of $A$. [CO3][1]

3. Find the rank of matrix $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

**OR**

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (2,-1,4)$, $T(0,2,0) = (1,5,-2)$, $T(0,0,3) = (0,3,1)$. Find $T(2,3,-2)$. [CO3][3]

**Question [IV]**

1. Given $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ Find (i) the characteristic polynomial of $A$, (ii) a basis for eigenspace corresponding to $\lambda = 1$, (iii) the dimension of eigenspace corresponding to $\lambda = 3$. [CO3][4]

2. Attempt **any two**:

(a) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$. Find the matrix of $T$ relative to the basis $B = \{(1,0), (1,1)\}$. [CO3][2]

(b) Determine whether the linear transformation $T(x, y) = (x+4y, x-4y), x, y \in \mathbb{R}$ is invertible. If yes, find it's inverse. [CO3][2]

(c) Let $T_1$, $T_2$ be linear transformations from $\mathbb{R}^3 \to \mathbb{R}^3$ such that $T_1(x, y, z) = (2x+y, y, x+z)$, $T_2(x, y, z) = (x+y, z, y)$. Find the standard matrix for the composition $T_1 \circ T_2$. [CO3][2]

(d) Find conditions on $p$ and $q$ such that the system of linear equations $x+2y = 3$; $px + qy = -9$ has (i) no solution, (ii) a unique solution. [CO1][2]

**P.T.O.**

3. Let $V$ be a vector space and $W_1, W_2$ be subspaces of $V$. If $W_1 \subseteq W_2$ then show that $W_2^{\perp} \subseteq W_1^{\perp}$. [CO3][2]

## Question [V]

1. Attempt **any two**:

   (a) Verify the triangle inequality for $u = (0, 3, 4), v = (2, -2, 2\sqrt{2})$ in the positive definite scalar product space $\mathbb{R}^3$ with usual operations. [CO3][2]

   (b) Consider the vector space of all $2 \times 2$ real matrices with $< A, B >$ defined as the sum of the product of corresponding entries of $A$ and $B$. Is this a positive definite scalar product? Justify your answer. [CO3][2]

   (c) Give a geometrical description of the linear transformation defined by the matrix product $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ [CO3][2]

2. Prove that the inverse of a linear map is linear, if it exists. [CO4][2]

3. Find an orthonormal basis for the vector space of polynomials of degree less equal 2 using Gram Schmidt process starting with the basis $\{-1, 2 + x, x^2 - 1\}$ and using the scalar product $< f, g >= \int_0^2 f(x)g(x)dx$ [CO4][4]

## Question [VI]

1. Attempt **any two**:

   (a) Which of the vectors below are scalar multiples of $u = (1/2, -5/2, 3/7)$?
   $(2/3, -15/8, 9/7), (-5/2, 1/2, -3/35), (-1, 5, 6/7), (0, 0, 0)$ [CO1][2]

   (b) Sketch and identify the image of a square with vertices $(0, 0), (1, 0), (0, 1), (1, 1)$ in $\mathbb{R}^2$ under the shear transformation given by $T(x, y) = (x, y+3x)$. [CO3][2]

   (c) Show that column vectors of the matrix $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ form an orthonormal basis for $\mathbb{R}^3$ with usual operations. Hence find $P^{-1}$. [CO3][2]

2. Let $V = \{T : \mathbb{R}^2 \to \mathbb{R}^2 | \ T \text{ is a linear map } \}$. Give an example of $L_1, L_2 \in V$ such that $\text{nullity}(L_1 + L_2) > \text{nullity}(L_1) + \text{nullity}(L_2)$. [CO3][2]

3. Find matrix of the quadratic form $Q(x, y) = 2x^2 - 3xy - 2y$. Find an orthogonal matrix $P$ such that $P^tAP$ is diagonal. [CO4][4]

*****************