COLLEGE OF ENGINEERING, PUNE
(An Autonomous Institute of Government of Maharashtra.)

# END Semester Examination
# (MA-15001) LINEAR ALGEBRA

**Programme : F.Y.B.Tech.** **2018-19, Semester-I**
**Branches : All** **Max. Marks : 60**
**Duration : 3 hours** **Wednesday, 28th Nov 2018**
**Student PRN No.**

## Instructions:

1. All questions are compulsory.
2. Figures to the right indicate course outcomes and maximum marks.
3. Mobile phones and programmable calculators are strictly prohibited.
4. Writing anything on question paper is not allowed.
5. Exchange/Sharing of stationery, calculator etc. is not allowed.
6. All symbols have their usual meaning. An *inner product space* is a vector space together with an inner product (i.e. a positive definite scalar product) defined on it.
7. For every $n \geq 1$, inner product on $\mathbb{R}^n$ will be assumed to be the usual dot product.
8. **Write all subparts of a question together and label each subpart correctly.**

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**Q.1** (a) Using Gauss elimination method, determine the values of $a$ and $b$ for which the system (CO 3) [5]

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ x_1 + 3x_2 + 5x_3 &= 9 \\ 2x_1 + 5x_2 + a\,x_3 &= b \end{aligned}$$

has (i) no solution, (ii) infinitely many solutions, (iii) unique solution.

(b) Let $V$ be an inner product space. Let $S$ be any non-empty subset of $V$. Define the *orthogonal complement* $S^{\perp}$ of $S$. (CO 1) [1]

(c) Let $V$ be the vector space of all $2 \times 2$ matrices. For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ in $V$, define $\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$. Assume that $\langle \, , \rangle$ is an inner product on $V$. Let $P = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$. Is $P \perp Q$ ? Why ? (CO 2) [2]

(d) Recall that two square matrices $A$ and $B$ of the same size are said to be similar if there exists an invertible matrix $P$ such that $B = P^{-1}AP$. Show that similar matrices have the same characteristic polynomial.
(CO 4) [2]

(e) Let $A$ be a $6 \times 6$ matrix with eigenvalues $\lambda = -2, -1, -1, 0, 0, 0$. Suppose the geometric multiplicities of $-2, -1, 0$ are 1, 2, 1 respectively. Write a Jordan canonical form of $A$.
(Do not find all possible forms.) (CO 3) [2]

(f) Fill in the blanks. (CO 2) [3]
Let $S$ be a subset of $\mathbb{R}^5$ containing seven elements. Then $S$ is certainly .......... If $T : \mathbb{R}^5 \to \mathbb{R}^7$ is a linear map whose ......... is $\text{Span}(S \cap S^{\perp})$, then the rank of $T$ is ..........

**Q.2** (a) Apply Gram-Schmidt process to find an **orthogonal** basis for the subspace $U$ of $\mathbb{R}^4$ spanned by the vectors $(0, 1, 1, 0)$, $(0, 5, -3, -2)$, and $(-3, -3, 5, -7)$. (CO 2) [5]

(b) Examine whether the following quadratic form is positive definite or not. Show all the details of your work.
$Q(x_1, x_2, x_3) = 3x_1^2 + x_3^2 + 2x_1x_2 - 4x_1x_3 - 2x_2x_3.$ (CO 3) [4]

(c) Transform the quadratic form $Q(x_1, x_2) = 17\,x_1^2 - 30\,x_1x_2 + 17\,x_2^2$ to principal axes form or canonical form.
What kind of conic section (or pair of straight lines) is represented by the equation $Q = 128$ ? (CO 3) [4]

(d) Let $\mathbb{R}^\infty$ denote the vector space of all sequences of real numbers.
Find a basis for the kernel of the linear map $L : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by
$L(x_1, x_2, x_3, \ldots\ldots) = (\,x_1+x_2,\ x_2+x_3,\ x_3+x_4, \ldots\ldots,\ x_n+x_{n+1}, \ldots\ldots).$
(CO 3) [2]

**Q.3** (a) Let $V$ be the vector space of twice differentiable functions.

Let $L : V \to V$ be the linear map defined by $L(f(t)) = \dfrac{d^2 f}{dt^2}$, for $f \in V$.
Since vectors in $V$ are functions, eigenvectors of $L$ are also called eigenfunctions.
Show that $\cos t$ and $\sin t$ are eigenfunctions of $L$. What are the eigenvalues corresponding to these eigenfunctions ? (CO 3) [3]

(b) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points with respect to $Y$-axis and then reflects points with respect to the line $y = x$.
Find the matrix of $T$ with respect to the standard basis. Justify your answer. (CO 2) [3]

(c) **(Leontief input-output model)** Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the $3 \times 3$ **consumption matrix**

$$\mathbf{A} = [a_{jk}] = \begin{pmatrix} 0.2 & 0.5 & 0 \\ 0.6 & 0 & 0.3 \\ 0.2 & 0.5 & 0.7 \end{pmatrix}$$

where $a_{jk}$ is the fraction of the output of industry $k$ consumed (purchased) by industry $j$. Let $p_j$ be the price charged by industry $j$ for its total output.
A problem is to find prices so that for each industry, total expenditures equal total income. Assume that this problem leads to $\mathbf{Ap} = \mathbf{p}$, where $\mathbf{p} = (p_1,\, p_2, p_3)^t$ is a column vector.
Find a solution $\mathbf{p}$ of $\mathbf{Ap} = \mathbf{p}$ with non-negative $p_1,\, p_2, p_3$. (CO 5) [3]

(d) Let $V$ be a finite dimensional vector space. (CO 4) [2]
Prove **ANY ONE** of the following statements:

i. Any two bases of $V$ have the same number of elements.
ii. Let $\{v_1, \ldots, v_n\}$ be a maximal linearly independent set of vectors in $V$. Then $\{v_1, \ldots, v_n\}$ is a basis of $V$.

(e) Let $A$ be a $3 \times 3$ matrix with eigenvalues 0, 2 and 7. Suppose $u, v, w$ are the eigenvectors of $A$ corresponding to 0, 2, 7 respectively. Find a basis for the null space of $A$ and a basis for the column space of $A$. Justify your answer. (CO 3) [4]

(Recall that the null space of a matrix $B$ is the set of all solutions of the system $BX = 0$.)

**Q.4** (a) Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation such that
$T(1,0,0) = T(0,1,0) = T(0,0,1) = T(1,1,1)$.
Compute $T(3,2,1)$. (CO 2) [2]

(b) Let $A$ be an $n \times n$ *symmetric* matrix. Let $L_A$ denote the linear transformation defined by $A$. Let $X, Y \in \mathbb{R}^n$ be any two column vectors.
Show that

i. $\langle X, AY \rangle = \langle AX, Y \rangle$

ii. $X \in (\text{Image of } L_A)^{\perp}$ if and only if $X \in$ kernel of $L_A$.

(CO 4) [2+3]

(c) Let $V$ be the vector space of all $2 \times 2$ matrices and let $B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$.
Let $W$ be the subset of $V$ consisting of all $A$ such that $AB = 0$.
Let $f : V \to \mathbb{R}$ be a linear transformation such that $f(A) = 0$ for every $A \in W$.
Suppose $f(I) = 0$ and $f(C) = 3$, where $I$ is the $2 \times 2$ identity matrix and $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

i. Show that $W$ is a subspace of $V$.

ii. Find the dimension of $W$.

iii. Find the nullity of $f$.

iv. Show that $B \in \ker(f)$.

Give justification for your answer. (CO 3, 5) [$2 \times 4$]