COEP Technological University Pune

Department of Mathematics

(MA- 20004) - VECTOR CALCULUS AND PARTIAL DIFFERENTIAL EQUATIONS

S.Y. B.Tech. Semester III (All Branches)

Tutorial 5 (AY: 2023-24)

Questions on CO1 / CO2

1. What is a line integral, and how is it different from a regular integral?.
2. What is the mathematical definition of a smooth curve, and why is it essential to require smoothness in defining line integrals along a curve?
3. Define the line integral for a vector field and write a formula to calculate it.
4. Define the line integral for a scalar field and write a formula to calculate it.
5. What is the concept of path independence in the context of line integrals?
6. How does path independence relate to conservative vector fields?
7. State the theorem that link the line integral along a closed path and the double integral.

Questions on CO2 and CO3

1. Calculate $\int_C \overline{F}(r) \cdot \overline{dr}$ for the following data -

   (a) $\overline{F} = [xy,\ x^2y^2]$, C is the quarter circle from $(2,0)$ to $(0,2)$ with center at $(0,0)$.
   Ans: 8/5

   (b) $\overline{F} = [xy,\ x^2y^2]$, C is the straight line from $(2,0)$ to $(0,2)$.
   Ans: - 4/15

   (c) $\overline{F} = [x-y,\ y-z,\ z-x]$, C : $[2\cos t,\ t,\ 2\sin t]$ from $(2,0,0)$ to $(2,2\pi,0)$.
   Ans: $2\pi^2 - 8\pi$

   (d) $\overline{F} = [coshx,\ sinhy,\ e^z]$, C : $[t, t^2, t^3]$ from $(0,0,0)$ to $(2,4,8)$.
   Ans: $sinh2 + cosh4 + e^8 - 2$

   (e) $\overline{F} = [ze^{xz},\ 2\sinh(2y), xe^{xz}]$, C is the parabola $y = x, z = x^2, -1 \leq x \leq 1$.
   Ans: $e - e^{-1}$

2. Evaluate $\int_c f(r)\, ds$ with arc length as parameter. (Recall that, $\int_c f(r)\, ds = \int_a^b f(r)|r'(t)|\, dt$)

   (a) $f = x^2 + y^2$, $C:\ y = 3x$ from (0,0) to (2,6).
   Ans: $80\sqrt{10}/3$

   (b) $f = x^2 + (xy)^{1/3}$ C is the hypocycloid with $\overline{r} = [\cos^3 t, \sin^3 t]$, $0 \leq t \leq \pi$
   Ans: $3\pi/8$

3. Check the following integrals for path independence. In the case of independence evaluate them.

   (a) $\int_{(\pi/2,-\pi)}^{(\pi/4,0)} (\cos x \cos 2y dx - 2\sin x \sin 2y dy)$
   Ans: path independent. $\frac{\sqrt{2}-2}{2}$

   (b) $\int_{(0,0,0)}^{(1,1,1)} (ye^z dy - ze^y dz)$
   Ans: path dependent

   (c) $\int_{(\pi,\pi/2,2)}^{(0,\pi,1)} (-z\sin(xz)dx + \cos y dy - x\sin(xz)dz)$
   Ans: -1

4. Using Green's theorem, evaluate the line integral $\oint_C \overline{F}(r) \cdot \overline{dr}$ counterclockwise around the boundary C of the region R, where

(a) $\overline{F} = [3y^2, x - y^4]$, R is the square with vertices (1,1), (-1,1), (-1,-1), (1,-1).
Ans: 4

(b) $\overline{F} = [2xy^3, 3x^2y^2], C : x^4 + y^4 = 1$. (Sketch the curve).
Ans: 0

(c) $\overline{F} = [xcosh2y, 2x^2sinh2y]$, $R : x^2 \leq y \leq x$.
Ans: $(sinh2 - cosh2 + 1)/4$

(d) $\overline{F} = [xe^{-y^2}, -x^2ye^{-y^2}]$, R : region that is bounded by the square of side $2a$ determined by the inequalities $|x| \leq a, |y| \leq a$
Ans: 0

5. Using Green's theorem, find the area of the region under one arch of a cycloid $\overline{r} = [a(t - \sin t), a(1 - \cos t)]; \quad 0 \leq t \leq 2\pi$.
Ans: $3\pi a^2$

6. (a) Evaluate $\int_c x^2ydx + 2xy^2dy$ from (0,0) along a straight line segment to (1,1/2), and then along a straight line segment to (1,1).

(b) Evaluate $\int_c x^2ydx + 2xy^2dy$ from (0,0) along a straight line segment (1/2,1), and then along a straight line segment to (1,1).

Is $I = \int_c x^2ydx + 2xy^2dy$ path dependent?

Ans: 37/48 ; 55/96 ; Yes

7. Show that the work done by a constant force field $\bar{F} = [a, b, c]$ in moving a particle along any path from A to B is $W = \bar{F} \cdot \bar{AB}$

8. (a) Find a potential function for the gravitational field $\bar{F} = -GmM\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}$; G, m, M are constants.
Ans: $\phi = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$

(b) Let P_1 and P_2 be two points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part(a) in moving a particle from P_1 to P_2 is $GmM\left(\dfrac{1}{s_2} - \dfrac{1}{s_1}\right)$

9 Verify Green's theorem for $F_1 = 3x^2 - 8y^2, F_2 = 4y - 6xy$ and C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$.
Ans: 3/2

Questions on CO4/CO5

1. Prove: $\oint \overline{F}\cdot d\bar{r} = \iint\limits_R \dfrac{\partial F_2}{\partial x}\; dxdy$ where $\overline{F} = [0, F_2]$ and R is the region $a \le y \le b;\; u(y) \le x \le v(y)$ having boundary C.

2. **Fundamental theorem for line integral:** If $\overline{F} = [F_1, F_2, F_3]$ is continuous on a domain D in space and is a conservative vector field, and C is any smooth curve/path from point A to B given by the vector functionn $\bar{r}(t) = [x(t), y(t), z(t)], a \le b$ in D, then $\displaystyle\int_C \overline{F}\cdot d\bar{r} = f(B) - f(A)$, where f is the potential of $\overline{F}$.

3. Prove that a line integral $\displaystyle\int_C \overline{F}\cdot d\bar{r} = \int_C (F_1dx + F_2dy + F_3dz)$ is independent of path in a domain D if and only if its value around every closed path in D is zero

4. State and prove Green's theorem in the plane.

Please report any mistakes in the problems and/or answers given here.