COLLEGE OF ENGINEERING, PUNE

(An Autonomous Institute of Government of Maharashtra.)

END Semester Examination
(MA-15001) LINEAR ALGEBRA

Programme : F.Y.B.Tech. **2018-19, Semester-I**
Branches : All **Max. Marks : 60**
Duration : 3 hours **Wednesday, 28$^{\text{th}}$ Nov 2018**
Student PRN No. | | | | | | | | | |

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate course outcomes and maximum marks.
3. Mobile phones and programmable calculators are strictly prohibited.
4. Writing anything on question paper is not allowed.
5. Exchange/Sharing of stationery, calculator etc. is not allowed.
6. All symbols have their usual meaning. An *inner product space* is a vector space together with an inner product (i.e. a positive definite scalar product) defined on it.
7. For every $n \geq 1$, inner product on $\mathbb{R}^n$ will be assumed to be the usual dot product.
8. **Write all subparts of a question together and label each subpart correctly.**

Q.1 (a) Define *rank* of a matrix.

Reduce the following matrix A to a matrix in row-echelon form. Also find a basis for the column space of A. (CO 3) [4]

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{pmatrix}$$

(b) Let S and T be non-empty subsets of $\mathbb{R}^4$. Suppose S contains four elements. Identify the errors (if any) in the following statements and write the corrected statements. (CO 2) [3]

i. $S \cup S^{\perp\perp}$ might not be a subspace of $\mathbb{R}^4$.

ii. $T \cap T^{\perp}$ is always a subspace of $\mathbb{R}^4$.

iii. S is a basis for $\mathbb{R}^4$.

(c) Row equivalent matrices have the same rank. Is the converse true ? Justify your answer. (CO 2) [3]

(d) Let A be a 6×7 matrix whose rank is 4. Is $AX = b$ consistent for every column vector $b \in \mathbb{R}^6$? Explain. (CO 2) [2]

(e) Let V be the vector space of functions which have derivatives of all orders. Let $D : V \to V$ be the derivative map and $I : V \to V$ the identity map. Find $\ker(D)$ and $\ker(L)$, where $L = D - I$. (CO 3) [3]

Q.2 (a) Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a map such that $L(1, -1) = (3, 2)$ and $L(-1, 1) = (2, 3)$. Can you determine $L(0, 0)$? Why ? (CO 3) [2]

(b) Let V be the vector space of continuous real valued functions defined on [0,1] with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$ defined on it. Let $f(t) = t^2 + t - 4$ and $g(t) = t - 1$. Evaluate $\|f - g\|$. (CO 3) [2]

(c) Let V be a vector space and $L : V \to V$ a linear map. Then $L^2 : V \to V$ is the linear map defined by $L^2(v) = L(\,L(v)\,)$, for $v \in V$. Show that $\mathrm{Im}(L^2)$ is a subset of $\mathrm{Im}(L)$. Does there exist any such relation between $\ker(L^2)$ and $\ker(L)$? If yes, write that relation. (CO 1) [4]

(d) Let V be a finite dimensional vector space and $L : V \to V$ a linear map. Show that L is one-to-one if and only if $\ker(L)$ is zero. (CO 4) [3]

(e) Examine whether the following quadratic forms is positive definite.
$Q(x_1, x_2, x_3) = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$. (CO 3) [4]

Q.3 (a) Let $W = \{(x, y, z, w) \in \mathbb{R}^4 | y + z = 0, x = 2w\}$.

i. Show that W is a subspace of $\mathbb{R}^4$.
ii. Find an orthogonal basis B_1 of W.
iii. Find a basis B_2 of $W^{\perp}$.
iv. Apply Gram-Schmidt process to the basis $B_1 \cup B_2$ of $\mathbb{R}^4$ and construct an orthogonal basis of $\mathbb{R}^4$. (CO 3) [2+2+3+3]

(b) Construct a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps (2,0,0) to (2,2) (0,1,0) to (2,3) and (0,0,1) to (3,2). Also find the kernel and and image of T. (CO 3) [5]

Q.4 (a) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Prove that L is one-to-one if and only if it is onto. (CO 5) [3]

(b) Transform the quadratic form $Q(x_1, x_2) = x_1^2 + 24x_1x_2 - 6x_2^2$ to principal axes form or canonical form. What kind of conic section (or pair of straight lines) is represented by the equation $Q = 5$? (CO 3) [4]

(c) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$. Show that A has only one (distinct) eigenvalue, say λ with geometric multiplicity 1.

Construct a basis $\{X_1, X_2, X_3\}$ of generalized eigenvectors for A as described below.

Find an eigenvector X_1 corresponding to λ. Find a non-trivial solution X_2 of $(A-\lambda I)X_2 = X_1$. Find a non-trivial solution X_3 of $(A-\lambda I)X_3 = X_2$. [4]

(d) Consider the system of differential equations:

$$\begin{aligned} \frac{dy_1}{dt} &= a\,y_1 + b\,y_2 \\ \frac{dy_2}{dt} &= c\,y_1 + d\,y_2 \end{aligned}$$

which can be represented in matrix form: $y' = Ay$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Suppose A has two distinct eigenvalues λ_1 and λ_2 with X_1, X_2 as the corresponding eigenvectors. Then the general solution of the above system is given by $y = c_1\, e^{\lambda_1 t} X_1 + c_2\, e^{\lambda_2 t} X_2$, where c_1, c_2 are arbitrary constants. Find the general solution of the following system. (CO 5) [4]

$$\begin{aligned} \frac{dy_1}{dt} &= 5\,y_1 + 2\,y_2 \\ \frac{dy_2}{dt} &= 2\,y_1 + 5\,y_2 \end{aligned}$$

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