COEP Technological University Pune
Department of Mathematics
(MA- 20004) - VECTOR CALCULUS AND PARTIAL DIFFERENTIAL EQUATIONS
S.Y. B.Tech. Semester III (All Branches)
Tutorial 6 (AY: 2023-24)

Questions on CO1/CO2

1. Define line integral and surface integral of a vector valued function along a curve or a surface.

2. State the theorems which link line integral with surface integral and surface integral with volume integral,

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Questions on CO2/CO3

1.Derive a representation $z = f(x, y)$ or $g(x, y, z) = 0$ of the following surfaces with parametric representations. Also find the parameter curves (curves u = const and v = const) on the surface and a normal vector $\overline{N} = \overline{r_u} \times \overline{r_v}$ of the surface.

(a) Paraboloid of revolution $\overline{r}(u, v) = [u\cos v, u\sin v, u^2]$.
Ans: $z = x^2 + y^2, x^2 + y^2 = k^2$ in the plane $z = k^2$ i.e. $\overline{r(t)} = [kcost, ksint, k^2]$, $\overline{r(t)} = [tcosk, tsink, t^2]$ this looks like a parabola in 3-D which has axis as z-axis, $\mathbf{N} = [-2u^2cosv, -2u^2sinv, u]$

(b) Cone $\overline{r}(u, v) = [u\cos v, u\sin v, cu]$.
Ans: $z^2 = c^2(x^2 + y^2), x^2 + y^2 = k^2$ in the plane $z = ck, \mathbf{N} = [-cucosv, -cusinv, u]$

You can think of parameter curves as two ways of generating the surface. For example the surface of the paraboloid in (a) above can be thought of to be made up of circles of continuously increasing radii starting from the origin which is a circle of radius zero! These are the parameter curves when the parameter u is treated as constant. Whereas the same surface can be thought of as being made up of continuous parabolic curves in 3-D that are symmetric about the z-axis and passing through the origin. These are the parameter curves when v is treated as constant.

Understanding the parameter curves help you in recognizing the boundary of the surface. For example the paraboloid of revolution in (a) above has no boundary since it extends to infinity in the z-direction. Actually this infinite paraboloid is generated by restricting the parameter v to lie between 0 and 2π (remember that sine and cosine are periodic). Thus $v = 0$ or $v = 2\pi$ will not constitute to the boundary of the paraboloid. If we restrict u to lie between say 0 and 1 then $u = 0$ gives only a point which again is not a boundary but $u = 1$ gives all the points on the circle $x^2 + y^2 = 1$ in the plane $z = 1$ which constitutes the boundary. If we restrict u to lie between 1 and 2 then the surface of the paraboloid has TWO boundaries! To end the discussion, if I close the top of this paraboloid by the plane $z = 2$ then there remains only ONE boundary and if I close also from below by the plane $z = 1$ then there is NO boundary!!!!!!

2.Find a parametric representation of the following surfaces :

(a) Plane $3x + 4y + 6z = 24$
Ans: one out of many $\mathbf{r}(u, v) = [u, v, (24 - 3u - 4v)/6]$

(b) Elliptic cylinder $9x^2 + 4y^2 = 36$.
Ans: one out of many $\mathbf{r}(u, v) = [2cosu, 3sinu, v]$

One advantage of parametrizing the surface by fixing one set of parametric equations is that you have identified a surface normal such that the tangents $\overline{r_u}, \overline{r_v}$ and the surface normal $\overline{N} = \overline{r_u} \times \overline{r_v}$ follow the right hand thumb rule. This gives an orientation to the surface and also to it's boundary, if it has a boundary. For example if the elliptic cylinder above is bounded such that $0 \le z \le 1$ then can you draw the surface and show the orientation of the boundaries? Now do another exercise: Close this cylinder from the bottom. This means that you are adding a surface $S_1 : x^2 + y^2 \le 1; z = 0$ to the curved surface of the cylinder. As remarked above, this new cylinder which is closed at the bottom has only one boundary circle in the plane $z = 1$ while the circle below in the plane $z = 0$ is a boundary which is common to the two surfaces viz. the curved surface and S_1. $\overline{N}$ computed above gives a normal to the curved surface which induces an anticlockwise rotation to the boundary circles and if we take $+\hat{k}$ as the normal to S_1 then this will also induce an anticlockwise rotation to the boundary circle of S_1. Union of these two orientable surfaces (with these normals) is NOT an orientable surface. Union of two orientable surfaces is an orientable surface if and only if the orientation of the common boundary is opposite for the two surfaces having this common boundary and all other boundaries have the same orientation. We must take $-\hat{k}$ to be the normal to S_1 so that the curved surface union S_1 becomes a piecewise smooth orientable surface. In theorems when we say smooth orientable surfaces then it is understood that surface normals should be found accordingly. It should also be understood that the statement "union of two orientable surfaces is an orientable surface" is a true statement. Can you justify??

3.Evaluate $\displaystyle\int\!\!\int_S \overline{F} \cdot \hat{n} \;\; dA$. Indicate the kind of surface in (a) and (b).

(a) $\bar{F} = [3x^2, y^2, 0]$; $S: \; \bar{r} = [u, v, 2u + 3v]$; $0 \le u \le 2, \quad -1 \le v \le 1$.
Ans: -36, plane $z = 2x + 3y; 0 \le x \le 2, -1 \le y \le 1$

(b) $\bar{F} = [\sinh(yz), 0, y^4]$; $S: \; \bar{r} = [u, \cos v, \sin v]$; $\quad -4 \le u \le 4, \quad 0 \le v \le \pi$.
Ans:$-16/5$, Cylinder $y^2 + z^2 = 1, -4 \le x \le 4$

(c) $\bar{F} = [ax, by, cz]$; S : entire surface of the sphere $x^2 + y^2 + z^2 = d^2$.
Ans: $-4\pi d^3(a + b + c)/3$

(d) $\bar{F} = [x, z^2 - zx, -xy]$; S : the triangular surface with vertices $(2, 0, 0), (0, 2, 0)$ and $(0, 0, 4)$.
Ans: $92/3$

4.Evaluate $\displaystyle\int\!\!\int_S \overline{F} \cdot \hat{n} \;\; dA$ by using the divergence theorem.

(a) $\bar{F} = [e^x, e^y, e^z]$; S : the surface of the cube $|x| \le 1, \quad |y| \le 1, \quad |z| \le 1$

(b) $\bar{F} = [x^3, y^3, z^3]$; S : the surface of the sphere $x^2 + y^2 + z^2 = 4$

5. Verify Stoke's Theorem for the following data.

(a) $\bar{F} = [y^2, z^2, x^2]$; S : the portion of the paraboloid $x^2 + y^2 = z, \quad y \geq 0, \quad z \leq 1$
Ans: -4/3

(b) $\bar{F} = [y^2, -x^2, 0]$; S : the circular semidisk $x^2 + y^2 \leq 4; \quad y \geq 0, \quad z = 1$
Ans: -32/3

6. Evaluate the following line integral $\oint_C \overline{F} \cdot \overline{r}'(s)ds$ by using Stoke's Theorem where $\bar{F} = [4z, -2x, 2x]$; C : the ellipse $x^2 + y^2 = 1, \quad z = y + 1$ Ans: 4π

7. Let S denote the hemisphere $x^2 + y^2 + z^2 = 1, \quad z \geq 0$, and let $F = [x, y, 0]$. Let $\hat{n}$ be the unit outward normal of S. Compute the value of surface integral $\int\int_S \overline{F} \cdot \hat{n} \ \ dS$ using

(a) the vector representation $\bar{r} = [\sin u \cos v, \sin u \sin v, \cos u]$;

(b) the explicit representation $z = \sqrt{1 - x^2 - y^2}$. Ans: $-4\pi/3$

8. A fluid flow has flux density vector $\bar{F} = [x, -(2x + y), z]$. Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$. Let $\hat{n}$ be the unit outward normal of S.

Calculate mass of fluid flowing through S in unit time in the direction of $\hat{n}$. Ans: $2\pi/3$

9. Find the flux across the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$ when the velocity vector $\bar{V} = 2y\hat{i} - 3\hat{j} + x^2\hat{k}$ Ans: 132

10. Find the surface area of the plane $x + 2y + 2z = 12$ cut off by $x = 0, y = 0, x = 1, y = 1$.
Ans: $3/2$

11. Verify Gauss Divergence Theorem for $\bar{F} = [2xz, yz, z^2]$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.
Ans: $5\pi a^4/4$

12. Evaluate $\int\int_S (\nabla \times [y, z, x]) \cdot \hat{n} \ \ dS$ over the surface of paraboloid $z = 1 - x^2 - y^2, z \geq 0$.
Ans: π

13. Evaluate $\int\int_S (\nabla \times [xz, -y, x^2y]) \cdot \hat{n} \ \ dS$ where S consists of the three faces not in xz-plane of the tetrahedron bounded by the three co-ordinate planes and the plane $3x + y + 3z = 6$. The normal $\hat{n}$ is the unit normal pointing out of the tetrahedron.
Ans: $4/3$

14. Evaluate :

(a) $\int_C \sin \pi x dx + z dy$; C : boundary of the triangle with vertices $(0,0,0), (1,0,0)(1,1,0)$.
Ans: 0

(b) $\int_C x dx + y dy$; C is the ellipse $x^2 + 4y^2 = 4$.
Ans: 0

(c) $\int_C \sqrt{2 + x^2 + 3y^2} ds$; $C: \bar{r} = [t, t, t^2],\ 0 \leq t \leq 3$
Ans: 42

(d) $\int_C ((2x \ln y - yz)dx + (\frac{x^2}{y} - xz)dy - xydz)$, C : the straight line from $(1,2,1)$ to $(2,1,1)$
Ans: $\ln 2$

(e) $\int_C (y + e^x \ln y)dx + e^x/ydy$, C : the boundary of the region that is bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$.
Ans: -44/15

(f) $\int_C ydx + zdy + xdz$; C : the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.
Ans: $-\pi a^2/\sqrt{2}$

(g) $\int\int_S 6zdydz + (2x + y)dxdz + (-x)dxdy$ S : entire surface of the region bounded by the cylinder $x^2 + z^2 = 9,\ x = 0,\ y = 0,\ z = 0$ and $y = 8$
Ans: -18π

(h) $\int\int_S (x^2 + y^2)^2 - z^2\, dS$, $S : \bar{r} = [u \cos v, u \sin v, 2u]$; $0 \leq u \leq 1, \quad -\pi \leq v \leq \pi$.
Ans: $-5\sqrt{5}\pi/3$

(i) $\int\int_S \cos y dydz + \sin x dxdz + \cos z dxdy$;
S is the closed surface of the cylinder $x^2 + y^2 = 9$; $0 \leq z \leq 2$.
Ans: $9\pi(\cos 2 - 1)$

(j) $\int\int_S [0, x^2, -xz] \cdot \hat{n} dA$ $S: \bar{r} = [u, u^2, v]$; $0 \leq u \leq 1, \quad -2 \leq v \leq 2$.
Ans: -1

(k) $\int_C [x^4, y^4, z^4] \cdot d\bar{r}$; C : the intersection of $x^2 + y^2 + z^2 = a^2$ and $z = y^2$.
Ans: 0

Questions on CO4 and CO5

1. Prove Greens theorem as a special case of Stokes Theorem.

Please report any mistakes in the problems and/or answers given here.