Set Theory

Set: A set is an unordered well-defined collection of objects. The objects in a set are called the elements or members of the set. We use capital letters such as A, B, ... etc, to represents sets and lower case letters to represents elements or members of the set.

Ex: $A = \{1, 2, 3, 4, 5\}$. A is a set of numbers containing elements or members 1, 2, 3, 4, 5.

Set Representation: The following two methods are commonly used for describing a set:

(1) The tabulation method or Enumeration method

(2) The Rule method or Set builder method.

(1) **The tabulation method:** Here we list the elements of the set in between braces { }.

Ex: The set A contains elements a, b, c, d then it is expressed as $A = \{a, b, c, d\}$

(2) **The Rule method** In this method, we specify the set by stating a characteristic property which all the elements of the set possess and which no other object possesses.

Ex: $A = \{x \mid x \text{ is a positive odd integer}\}$

$B = \{x \mid x \text{ is a positive integer less than } 4\}$

Standard Notations

$x \in A \rightarrow$ x belongs to A or x is an element or member of set A

$x \notin A \rightarrow$ x does not belong to set A

$\phi \rightarrow$ Empty set or Null set

$U \rightarrow$ Universal set

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$N \rightarrow$ The set of all natural numbers
$I \rightarrow$ The set of all integers
$I_+ \rightarrow$ The set of all positive integers
$I_0 \rightarrow$ The set of all non-zero integers
$C, C_0 \rightarrow$ The set of all complex, non-zero complex numbers respectively.
$Q, Q_0, Q_+ \rightarrow$ The set of rational, non-zero rational, positive rational numbers respectively.
$R, R_0, R_+ \rightarrow$ The set of real, non-zero real, positive real numbers respectively.

If the number of elements in a set is finite, Then we say that the set is a finite set. Set having infinitely many elements are called infinite sets. A set having only one element is called a singleton set.

Ex: (1) $A = \{x \mid x \in N, 3 < x < 9\}$
$B = \{$months of years$\}$ $\rightarrow$ finite set

(2) $A = \{$The set of all integers$\}$
$B = \{x \mid x \in N, x$ is a multiple of $2\}$ $\rightarrow$ infinite set

(3) The set of all integers between 2 and 8 that are squares of integers is the singleton set consisting of the integer 4.

The Null set or Empty set: The set that contains no element is called the null set or the empty set and is denoted by ϕ.

Ex: $A = \{x : x^2 = 4, x$ is odd$\}$
$B = \{x : x^2 = 9, x$ is even$\}$

The set $\phi = \{0\}$ is not a null set because 0 is

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the element of the set.

<u>Equality of sets</u>: Two sets A and B are said to be equal if both have the same elements. Therefore, every element which belongs to A is also an element of the set B and every element which belongs to the set B is also an element of the set A. Symbolically written as

$A = B \iff (x \in A \iff x \in B)$.

<u>Ex:</u> $A = \{1, 2, 3, 4\}$

and $B = \{x \mid x$ is a positive integer with $x^2 < 20\}$

Then $A = B$.

<u>subsets</u>: If A and B are sets, then A is called a subset of B if and only if every element of A is also an element of B.

Symbolically written as $A \subseteq B \iff \forall x$, if $x \in A$ then $x \in B$

The phrases A is contained in B and B contains A are alternative ways of saying that A is a subset of B. <u>Ex:</u> $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ then $A \subseteq B$

The set A is not a subset of a set B if and only if there is atleast one element of A that is not an element of B. Symbolically written as $A \nsubseteq B \iff$ there exists x such that $x \in A$ and $x \notin B$.

<u>Ex:</u> $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 6\}$ then $A \nsubseteq B$

<u>Proper Subset</u>: If A is subset of B and $A \neq B$ then A is said to be <u>proper subset of B</u>. If A is a proper subset of B then B is not subset of A i.e. there is atleast one element in B which is not in A

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Ex: $A = \{2, 3, 4\}$, $B = \{2, 3, 4, 5\}$ then A is a proper subset of B.

Note: (1) The null set ϕ is a subset of every set.

proof: Suppose that ϕ is not a subset of set A ie $\phi \not\subseteq A$

Then there exists an element $x \in \phi$ such that $x \notin A$.

But ϕ is the null set and hence for every x, $x \notin \phi$ because null set does not contain any element.

From above, $x \in \phi$ and $x \notin \phi$ which is contradiction

Hence, ϕ is a subset of A. Hence proved.

~~(2) Two sets A and~~

(2) Every set is a subset of itself ie $A \subseteq A$

proof: Let $x \in A \Rightarrow x \in A \therefore A \subseteq A$.

As every element belonging to set A is also an element of set A. Therefore, A is subset of A itself.

(3) Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

proof: Suppose, the sets A and B are equal. Then from the definition of equality of sets, ~~it clearly~~ implies that $A \subseteq B$ and also $B \subseteq A$.

Suppose, $A \subseteq B$ and $B \subseteq A \Rightarrow A$ and B are equal sets.

(4) For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

proof: Let x be any element of the set A.

Since, $A \subseteq B \therefore x \in A \Rightarrow x \in B$

$B \subseteq C \therefore x \in B \Rightarrow x \in C \quad \therefore x \in A \Rightarrow x \in C$

$\therefore A \subseteq C$.

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(5) For any sets A, B, C, if $A = B$ and $B = C$, then $A = C$.

Proof: Let x be any element of the set A.

Since $A = B$ $\therefore x \in A \Leftrightarrow x \in B$

$B = C$ $\therefore x \in B \Leftrightarrow x \in C$.

$\therefore x \in A \Leftrightarrow x \in C \Rightarrow A = C$.

Universal Set: If all the sets under investigation are subsets of a fixed set U, then the set U is called universal set.

Ex: 1) In plane geometry, the universal set consists of all the points in the plane.

2) In studies concerned with defective parts of a machine the set of all parts of the machine is taken as the universal set.

The universal set varies from one discussion to the other, and the context indicates the choice of the universal set. The universal set is not unique.

Power Set: The power set of any given set A is the set of all subsets of A and is denoted by $P(A)$. If A has n elements then $P(A)$ has 2^n elements.

Ex: 1) If $A = \{r, s\}$ then its subsets are $\phi, \{r\}, \{s\}, \{r, s\}$

$P(A) = \{\phi, \{r\}, \{s\}, \{r, s\}\}$ which has $2^2 = 4$ elements

2) If $A = \{1, 2, 3\}$ then its subsets are $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} = A$.

$\therefore P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

ie $P(A) = 2^3 = 8$ elements etc.

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Result: If a finite set A has n elements then prove that the power set of A has 2^n elements.

~~Sol~~ **proof**: If a finite set A has n elements, then a subset of A can have no element, one element, two elements, three elements, ... $(n-1)$ elements and all the n elements. The subset having no element is the null set and the subset having all the n elements is A itself. There are ${}^nC_1$ subsets each consisting of one of the elements of the ~~given~~ set A. ~~There~~ Similarly, There will be ${}^nC_2$ subsets each consisting of any two of the n elements of the set A and so on. At last, there will be ${}^nC_n$ subset consisting of all the n elements of the set A

Hence, the total number of subsets of A is

$$1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n \qquad \because {}^nC_0 = 1$$

$$= 1 \cdot {}^nC_0 + {}^nC_1 (1)^2 + {}^nC_2 (1)^3 + \cdots + {}^nC_n (1)^n$$

$$= (1+1)^n \quad \text{, using binomial theorem}$$

$$= 2^n$$

Note: 2^n subsets formed from n elements of a set. Out of 2^n subsets $2^n - 1$ subsets will be proper and one subset improper i.e the set itself.

Venn Diagrams: A diagram representation of a set is called a venn diagram. The collection of everything cannot be considered a set. We will assume that for each discussion there is a universal set called universe U, containing all objects for which the discussion is meaningful. Any other set mentioned in the

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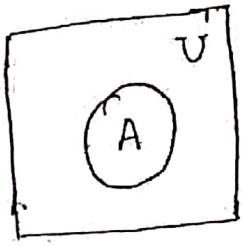
discussion will automatically be assumed to be a subset of U.

The universal set U is represented in a venn diagram by the interior of a rectangle and the set by disks inside the rectangle.



(The venn diagrams are used to understand the relationships between sets). Results from a venn diagram usually are not considered to constitute a proof; but the results in the venn diagram can be converted into a proof.

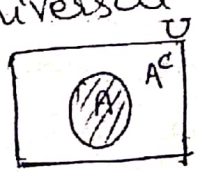
Venn diagrams are used to represent set operations. In these diagrams, shading are used to indicate which regions resent which sets. The results are obtained by determining which regions in the diagram are shaded in which fashion.

To draw venn diagram following points are to be taken into consideration:

(i) All the sets are usually represented by circles

(ii) The universal set is represented by a rectangle

(iii) The complement of the set is represented by that portion of the universal set which is not in the set.



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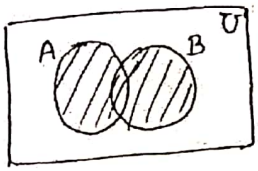
Operations on Sets :

(i) Union of Sets : Union of the sets A and B is defined to be the set of all those elements which belong to A or B or both and is denoted by $A \cup B$. Thus,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The venn-diagram for $A \cup B$ is shown below.



Shaded region represents $A \cup B$.

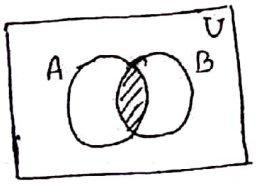
Ex: Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5, 6\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

(ii) Intersection of Sets : Intersection of two sets A and B is the set of all those elements which belong to both A and B and is denoted by $A \cap B$. Thus

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The venn-diagram for $A \cap B$ is shown below.

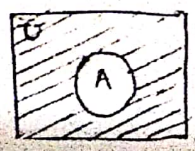


Shaded region represents $A \cap B$

Complement of a Set : The Complement of a set A is a set of all those elements of the universal set which do not belong to A and is denoted by A^c or $\bar{A}$ or $U - A$. Thus,

$$\bar{A} = \{x \mid x \in U \text{ and } x \notin A\}$$

The venn-diagram for $\bar{A}$ is



Shaded region represents $\bar{A}$.

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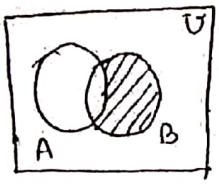
Ex: Let U is the set of all natural numbers $A = \{1, 2, 3\}$ then
$\bar{A} = \{$ all natural numbers except $1, 2, 3\}$.

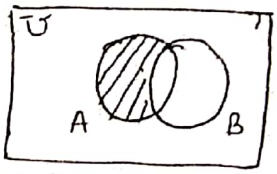
Relative complement: Let A and B be two subsets of a universal set U. Then the set of all elements that belong to B but not to A is called the complement of A relative to B (or relative complement of A in B) and is denoted by $B-A$. That is

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

The venn-diagrams for $B-A$ and $A-B$ are shown below.



The shaded region represent $B-A$.



Shaded region represents $A-B$.

Ex: If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 7, 8\}$ then

$B - A = \{7, 8\}$ and $A - B = \{1, 2\}$.

Note: (1) For any sets A and B, the sets $A-B$ and $B-A$ are disjoint.

(2) If A and B are disjoint, then $A - B = A$ and $B - A = B$.

(3) If U is the universal set and $A \subseteq U$, then $U - A = \bar{A}$

(4) $A = (A \cup B) - (B - A)$, $B = (A \cup B) - (A - B)$

(5) $A = (A \cap B) \cup (A - B)$, $B = (A \cap B) \cup (B - A)$

Symmetric Difference of Sets: The symmetric difference of two sets A and B is the set containing all the elements that are in A or in B but not in both and is denoted by $A \Delta B$ or $A \oplus B$. Thus,

$$A \Delta B = (A \cup B) - (A \cap B)$$
$$= \{x \mid x \in A \cup B \text{ and } x \notin A \cap B\}$$

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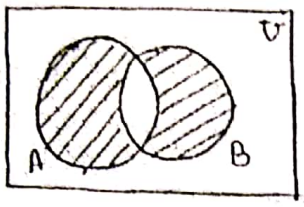
The venn-diagram for $A \Delta B$ is shown below



Shaded region represent $A \Delta B$.

Ex: If $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 7, 8\}$, then

$A \Delta B = \cancel{\{1, 2, 3, 7, 8\}}$ $\{1, 2, 3, 7, 8\}$

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(1) If $U = \{1,2,3,4,5,6,7,8,9\}$, $A = \{1,2,4,6,8\}$ and $B = \{2,4,5,9\}$, find the following:

(i) $\bar{A}$ (ii) $\bar{B}$ (iii) $\bar{A} \cup \bar{B}$ (iv) $\overline{A \cup B}$ (v) $\bar{A} \cap \bar{B}$ (vi) $\overline{A \cap B}$

(vii) $B - A$ (viii) $A - B$ (ix) $A \Delta B$

Sol: (i) $\bar{A} = U - A = \{3,5,7,9\}$

(ii) $\bar{B} = U - B = \{1,3,6,7,8\}$

(iii) $\bar{A} \cup \bar{B} = \{1,3,5,6,7,8,9\}$

(iv) $A \cup B = \{1,2,4,5,6,8,9\}$; $\overline{A \cup B} = \{3,7\}$ $= U - (A \cup B)$

(v) $\bar{A} \cap \bar{B} = \{3,7\}$

$\therefore$ From (iv) & (v) $\Rightarrow$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(vi) $A \cap B = \{2,4\}$: $\overline{A \cap B} = U - (A \cap B) = \{1,3,5,6,7,8,9\}$

From (iii) and (vi) $\Rightarrow$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$

(vii) $B - A = \{5,9\}$ (viii) $A - B = \{1,6,8\}$

(ix) $A \Delta B = (A \cup B) - (A \cap B) = \{1,5,6,8,9\}$

(2) Determine the sets A and B, given that $A - B = \{1,2,4\}$, $B - A = \{7,8\}$ and $A \cup B = \{1,2,4,5,7,8,9\}$

Sol: $A = (A \cup B) - (B - A) = \{1,2,4,5,9\}$

$B = (A \cup B) - (A - B) = \{5,7,8,9\}$

(3) Determine the sets A and B, given that $A - B = \{1,3,7,11\}$, $B - A = \{2,6,8\}$ and $A \cap B = \{4,9\}$

Sol: W.K.T, $A = (A \cap B) \cup (A - B) = \{1,3,4,7,9,11\}$

$B = (A \cap B) \cup (B - A) = \{2,4,6,8,9\}$

(4) Determine the sets A & B, given that $A \cup B = \{1,2,4,5,7,8,9,10\}$, $A \cap B = \{2,4,7\}$, $A - B = \{1,8\}$

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Sol: W.K.T. $A = (A \cap B) \cup (A - B) = \{1, 2, 4, 7, 8\}$

$B = (A \cup B) - (A - B) = \{2, 4, 5, 7, 9, 10\}$

(5) For any two sets A and B, prove that

$P(A \cap B) = P(A) \cap P(B)$

Give an example to show that $P(A \cup B) \neq P(A) \cup P(B)$.

Sol: Let $X \in P(A \cap B)$. Then $X \subseteq (A \cap B)$. Therefore, $X \subseteq A$ and $X \subseteq B$. Hence $X \in P(A)$ and $X \in P(B)$. Thus, $X \in P(A) \cap P(B)$. This implies that

$P(A \cap B) \subseteq P(A) \cap P(B)$ ---- (i)

Next, let $Y \in P(A) \cap P(B)$. Then $Y \in P(A)$ & $Y \in P(B)$. This means that $Y \subseteq A$ and $Y \subseteq B$.

Therefore, $Y \subseteq A \cap B$. Hence $Y \in P(A \cap B)$. This implies that $P(A) \cap P(B) \subseteq P(A \cap B)$ --(ii)

From (i) & (ii) $\Rightarrow$ $P(A \cap B) = P(A) \cap P(B)$.

Now, consider the sets $A = \{1, 2\}$, $B = \{3, 4\}$.

Then $A \cup B = \{1, 2, 3, 4\}$.

Clearly, $\{1, 2, 3\} \subseteq A \cup B$, so that $\{1, 2, 3\} \in P(A \cup B)$.

But $\{1, 2, 3\} \notin A$ and $\{1, 2, 3\} \notin B$.

$\therefore \{1, 2, 3\} \notin P(A) \cup P(B)$.

Hence $P(A \cup B) \neq P(A) \cup P(B)$

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The Laws of Set theory (Set identities)

The operations on set satisfy certain laws. The following are a few of these laws wherein A, B, C are subsets of a universal set U.

(1) Commutative Laws.

(i) $A \cup B = B \cup A$. (ii) $A \cap B = B \cap A$.

Proof: (i) From the defn, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$= \{x \mid x \in B \text{ or } x \in A\} = B \cup A$.
Hence proved

(ii) From the defn, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
$= \{x \mid x \in B \text{ and } x \in A\}$
$= B \cap A$. Hence proved.

(2) Associative laws:

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Proof: Let $x \in A \cup (B \cup C) \Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$. $\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow x \in ((A \cup B) \cup C) \Rightarrow A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Similarly, let $x \in ((A \cup B) \cup C)$ then $x \in (A \cup (B \cup C))$
$\Rightarrow (A \cup B) \cup C \subseteq A \cup (B \cup C)$

Thus $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) Let $x \in (A \cap (B \cap C)) \Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C \Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow x \in (A \cap B) \cap C$. $\Rightarrow A \cap (B \cap C) \subseteq (A \cap B) \cap C$

Similarly, let $x \in ((A \cap B) \cap C)$ then we get $x \in (A \cap (B \cap C))$
$\Rightarrow (A \cap B) \cap C \subseteq A \cap (B \cap C)$

Thus $A \cap (B \cap C) = (A \cap B) \cap C$.

(3) Distributive laws

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

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Proof: (i) Let $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C) \Rightarrow (x \in A$ & $x \in A)$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow (x \in A \cap B)$ or $(x \in (A \cap C))$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ --- (*)

Again, let $y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

$\Rightarrow (y \in A$ or $y \in A)$ and $(y \in B)$ or $y \in C)$

$\Rightarrow y \in A$ and $y \in B \cup C$

$\Rightarrow y \in A \cap (B \cup C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ --- (**)

Combining (*) & (**), we get

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Hence proved.

(ii) Let $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ ~~x∈A and x~~ $(x \in A$ or $x \in A)$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ --- (*)

Again, let $y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B)$ & $y \in (A \cup C)$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow (y \in A$ and $y \in A)$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ or $y \in (B \cap C) \Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ --- (**)

Combining (*) & (**), we get

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Hence proved.

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(4) Idempotent laws:

(i) $A \cup A = A$ (ii) $A \cap A = A$.

proof: i) Let $x \in A \cup A \Rightarrow x \in A$ or $x \in A \Rightarrow x \in A$

$\Rightarrow A \cup A \subseteq A$ --- (*)

Illy, let $x \in A \Rightarrow x \in A$ or $x \in A \Rightarrow x \in A \cup A$

$\Rightarrow A \subseteq A \cup A$ --- (**)

Combining (*) & (**), we get $A \cup A = A$

(ii) let $x \in A \cap A \Rightarrow x \in A$ and $x \in A \Rightarrow x \in A$

$\Rightarrow A \cap A \subseteq A$ --- (*)

Let $y \in A \Rightarrow y \in A$ and $y \in A \Rightarrow y \in A \cap A$

$\Rightarrow A \subseteq A \cap A$ --- (**)

Combining (*) & (**), we get $A \cap A = A$.

(5) Identity Laws:

(i) $A \cup \phi = A$ (ii) $A \cap U = A$.

proof: (i) Let $x \in A \cup \phi \Rightarrow x \in A$ or $x \in \phi$

$\Rightarrow x \in A$ ($\because x \notin \phi$, as ϕ is the null set)

$\Rightarrow A \cup \phi \subseteq A$ --- (*)

We know that $A \subseteq A \cup B$ for any set B.

$\Rightarrow A \subseteq A \cup \phi$ --- (**)

Combining (*) & (**), we get $A \cup \phi = A$.

(ii) Let $x \in A \cap U \Rightarrow x \in A$ and $x \in U \Rightarrow x \in A$

$\Rightarrow A \cap U \subseteq A$ -- (*) ($\because A \subseteq U \Rightarrow x \in A \Rightarrow x \in U$)

Again, let $y \in A \Rightarrow y \in A$ and $y \in U \Rightarrow y \in A \cap U$

$\Rightarrow A \subseteq A \cap U$. --- (**)

Combining (*) & (**), we get $A \cap U = A$.

(6) Law of Double complement.

$\bar{\bar{A}} = A$ or $(A^c)^c = A$.

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Proof

Let $x \in \overline{\overline{A}} \Rightarrow x \notin \overline{A} \Rightarrow x \in A$

$\Rightarrow \overline{\overline{A}} \subseteq A$ ---(*)

Again, let $y \in A \Rightarrow x \notin \overline{A} \Rightarrow x \in \overline{\overline{A}}$

$\Rightarrow A \subseteq \overline{\overline{A}}$ ---(**)

combining (*) & (**), we get

$\overline{\overline{A}} = A$

7. Inverse Laws :

(i) $A \cup \overline{A} = U$ (ii) $A \cap \overline{A} = \phi$.

proof: (i) Let $x \in A \cup \overline{A} \Rightarrow x \in A$ or $x \in \overline{A} \Rightarrow x \in U$

$\therefore A \cup \overline{A} \subseteq U$ ---(*)

Again, let $x \in U \Rightarrow x \in A$ or $x \notin A \Rightarrow x \in A$ or $x \in \overline{A}$

$\Rightarrow x \in A \cup \overline{A} \Rightarrow U \subseteq A \cup \overline{A}$ --(**)

Combining (*) & (**), we get $A \cup \overline{A} = U$

(ii) W.K.T. empty set ie ϕ is subset of every set $\Rightarrow \phi \subseteq A \cap \overline{A}$ ---(*)

Let $x \in A \cap \overline{A} \Rightarrow x \in A$ and $x \in \overline{A} \Rightarrow x \in A$ and $x \notin A$ (a contradictory statement)

$\Rightarrow x \in \phi \Rightarrow A \cap \overline{A} \subseteq \phi$ ---(**)

combining (*) and (**), we get $A \cap \overline{A} = \phi$.

8) De Morgan Laws

(i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

proof: (i) Let $x \in \overline{A \cup B} \Rightarrow x \notin A \cup B$ ($\because x \in A \Leftrightarrow x \notin \overline{A}$)

$\Rightarrow x \notin A$ and $x \notin B \Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cap \overline{B} \Rightarrow \overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Again, let $x \in \overline{A} \cap \overline{B} \Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \notin A$ and $x \notin B$

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$\Rightarrow x \notin A \cup B \Rightarrow x \in (\overline{A \cup B})$

$\Rightarrow \bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$ ----(**)

combining (*) & (**), we get

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

(ii) Let $x \in (\overline{A \cap B}) \Rightarrow x \notin A \cap B$ $(\because x \in A \Leftrightarrow x \notin \bar{A})$

$\Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cup \bar{B}$

$\therefore \overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ ---(*)

Again, let $x \in \bar{A} \cup \bar{B} \Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

$\Rightarrow x \notin A$ or $x \notin B \Rightarrow x \notin A \cap B \Rightarrow x \in (\overline{A \cap B})$

$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$ ---(**)

combining (*) & (**), we get $\overline{A \cap B} = \bar{A} \cup \bar{B}$

(9) <u>Domination Laws</u>

(i) $A \cup U = U$ (ii) $A \cap \phi = \phi$

proof: (i) W.K.T. every set is a subset of universal set $\therefore A \cup U \subseteq U$ ---(*)

let $x \in U \Rightarrow x \in A$ or $x \in U \Rightarrow x \in A \cup U$

$\Rightarrow U \subseteq A \cup U$ --(**)

combining (*) & (**), we get $A \cup U = U$

(ii) Let $x \in A$ then $x \notin \phi$

$\therefore x \in A, x \notin \phi \Rightarrow A \cap \phi = \phi$.

(10) <u>Absorption Laws</u>

(i) $A \cup (A \cap B) = A$ (b) $A \cap (A \cup B) = A$

<u>proof</u>: (i) let $x \in A \cup (A \cap B) \Rightarrow x \in A$ or $x \in A \cap B$

$\Rightarrow x \in A$ or ($x \in A$ and $x \in B$) $\Rightarrow$ ($x \in A$ or $x \in A$) and ($x \in A$ or $x \in B$)

$\Rightarrow x \in A$ and $x \in A \cup B \Rightarrow x \in A \cap (A \cup B) \Rightarrow x \in A$.

$\therefore$ ~~x~~ $A \cup (A \cap B) \subseteq A$ ---(*)

Again $x \in A \Rightarrow x \in A$ or $x \in A \cap B \Rightarrow x \in A \cup (A \cap B)$

$\therefore A \subset A \cup (A \cap B)$ --(**)

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From (*) & (**) $\Rightarrow A \cup (A \cap B) = A$

(ii) Let $x \in A \cap (A \cup B) \Rightarrow x \in A$ and $x \in A \cup B$

$\Rightarrow x \in A$ and ($x \in A$ or $x \in B$)

$\Rightarrow$ ($x \in A$ and $x \in A$) or ($x \in A$ and $x \in B$)

$\Rightarrow x \in A$ or $x \in A \cap B \Rightarrow x \in A \cup (A \cap B) \Rightarrow x \in A$

$\therefore A \cap (A \cup B) \subseteq A$ ...(*)

Again, let $x \in A \Rightarrow x \in A$ and $x \in A \cup B$.

$\Rightarrow x \in A \cap (A \cup B)$

$\therefore A \subseteq A \cap (A \cup B)$ ...(**)

From (*) & (**) $\Rightarrow$ $\underline{A \cap (A \cup B) = A}$.

<u>Duality</u>: Consider a set A formed by the unions and/or intersections of sets which may include the null set ϕ and a universal set U. With these, we form another subset of U by replacing

(i) each union operation by the intersection operation,

(ii) each intersection operation by the union operation,

and by replacing U by ϕ and ϕ by U.

The new set formed is called the <u>dual</u> of the original set A and is denoted by A^d.

<u>Ex</u>: If $A = (A \cup B) \cap (A \cup \bar{B}) \cap (\bar{A} \cup B)$ then

$A^d = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$.

<u>Note</u> (1) If A and B are sets such that $A = B$, then $A^d = B^d$. This is known as the <u>principle of Duality</u> for the equality of sets.

(2) Except the law of double complement, all the laws of set theory contain two part each. An

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∴ examination of these parts show that each part is the dual of the other. As such, if the first part is proved, then, in view of the principle duality, the second part follows.

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Prob (1) P.T., for any three sets A, B, C,

$$\overline{(A\cap B)\cup C} = (\bar{A}\cup\bar{B})\cap\bar{C}$$

Sol: Let $D = A\cap B$. Then

$$\overline{(A\cap B)\cup C} = \overline{D\cup C} = \bar{D}\cap\bar{C} \quad \text{by De Morgan's law}$$

$$= (\overline{A\cap B})\cap\bar{C} = (\bar{A}\cup\bar{B})\cap\bar{C} \quad \text{by De Morgan's law}$$

(2) For any two sets A and B, prove the following:

(i) $A-B = A\cap\bar{B}$ (ii) $\overline{A-B} = \bar{A}\cup B$

(iii) $A-B = A-(A\cap B)$ (iv) $\overline{A-B} = \bar{A}\cup(A\cap B)$

(v) $A-(A-B) = A\cap B$

Sol: (i) W.K.T., $A-B = \{x\in U \mid x\in A \text{ and } x\notin B\}$

$$= \{x\in U \mid x\in A \text{ and } x\in\bar{B}\}$$

$$A-B = A\cap\bar{B}$$

(ii) From the above result, we have

$$A-B = A\cap\bar{B}$$

$$\Rightarrow \overline{A-B} = \overline{(A\cap\bar{B})} = \bar{A}\cup\bar{\bar{B}}, \text{ by De Morgan's law}$$

$$\Rightarrow \overline{A-B} = \bar{A}\cup B, \quad \text{because } \bar{\bar{B}} = B.$$

(iii) From the result (i), we have

~~$\overline{A-B} = \bar{A}\cup B$~~. $A-B = A\cap\bar{B}$

$$\Rightarrow A-(A\cap B) = A\cap(\overline{A\cap B})$$

$$= A\cap(\bar{A}\cup\bar{B}), \text{ by De Morgan's law}$$

$$= (A\cap\bar{A})\cup(A\cap\bar{B}), \text{ by distributive law}$$

$$= \phi\cup(A\cap\bar{B}), \quad \because A\cap\bar{A} = \phi \text{ by domination law}$$

$$= A\cap\bar{B} = A-B$$

$$\Rightarrow A-B = A-(A\cap B)$$

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~~Set Theory~~

(iv) ~~From the Result (iii), we have~~

~~$A-B = A-(A\cap B)$~~

~~$\Rightarrow \overline{A-B} =$~~

(iv) From the result (ii), we have

$\overline{A-B} = \overline{A-(A\cap B)}$

$\Rightarrow \overline{A-B} = \bar{A} \cup (A\cap B)$, using result (ii)

(v) Consider, $A-(A-B) = A-(A\cap\bar{B})$, using (i)

$= A\cap\overline{(A\cap\bar{B})}$, again using (i)

$= A\cap(\bar{A}\cup B)$, using De Morgan's law & $\bar{\bar{B}} = B$

$= (A\cap\bar{A})\cup(A\cap B)$, by distributive law

$= \phi \cup (A\cap B)$, by Domination law

$= A\cap B$, by identity law.

(3) For any sets A, B, C, prove the following.

(i) $A\cap(B-C) = (A\cap B)-C$ (ii) $(A-B)\cap(A-C) = A-(B\cup C)$

Sol: W.K.T. $A-B = A\cap\bar{B}$ ---(*)

(i) Consider, $A\cap(B-C) = A\cap(B\cap\bar{C})$, by (*)

$= (A\cap B)\cap\bar{C}$, by ~~distributive~~ associative law

$= (A\cap B)-C$, using (*).

(ii) consider, $(A-B)\cap(A-C) = (A\cap\bar{B})\cap(A\cap\bar{C})$, using (*)

$= A\cap(\bar{B}\cap A)\cap\bar{C}$, by associative law

$= A\cap(A\cap\bar{B})\cap\bar{C}$, by commutative law

$= (A\cap A)\cap(\bar{B}\cap\bar{C})$, by associative law

$= A\cap(\overline{B\cup C})$, by De Morgan's law

$= A-(B\cup C)$, by (*).

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4) For any three sets A, B, C prove that

$$(A-B)-C = A-(B\cup C) = (A-C)-(B-C)$$

Sol: W.K.T. $A-B = A\cap\bar{B}$ --- (*)

Consider, $(A-B)-C = (A-B)\cap\bar{C}$, by (*)

$= (A\cap\bar{B})\cap\bar{C}$, by (*)

$= A\cap(\bar{B}\cap\bar{C})$, by associative law

$= A\cap(\overline{B\cup C})$, by De Morgan's law

$= A-(B\cup C)$, by (*) --- (i)

Also, $(A-C)-(B-C) = (A-C)\cap(\overline{B-C})$, by (*)

$= (A\cap\bar{C})\cap(\overline{B\cap\bar{C}})$, by (*)

$= (A\cap\bar{C})\cap(\bar{B}\cup C)$, by De Morgan's law

$= \{(A\cap\bar{C})\cap\bar{B}\}\cup\{(A\cap\bar{C})\cap C\}$, by distributive law

$= \{A\cap(\bar{C}\cap\bar{B})\}\cup\{A\cap(\bar{C}\cap C)\}$, by associative law

$= \{A\cap(\overline{B\cup C})\}\cup\{A\cap\phi\}$, by De Morgan's law

$= \{A\cap(\overline{B\cup C})\}\cup\phi$, by Domination law

$= \{A\cap(\overline{B\cup C})\}$, by identity law

$= A-(B\cup C)$, by (*) --- (ii)

From (i) & (ii) $\Rightarrow$

$\therefore (A-B)-C = A-(B\cup C) = (A-C)-(B-C)$

5) If $S, T \subseteq U$, prove that S and T are disjoint if and only if $S\cup T = S\Delta T$

Sol: $S\Delta T = (S\cup T)-(S\cap T)$, by defn

W.K.T. $A-B = A\cap\bar{B}$

$\therefore S\Delta T = (S\cup T)\cap(\overline{S\cap T})$ --- (i)

Now, suppose that S & T are disjoint so that $S\cap T = \phi$. Then the expression (i) gives

$S\Delta T = (S\cup T)\cap\bar{\phi} = (S\cup T)\cap U = S\cup T$

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Conversely, suppose $S \Delta T = S \cup T$. Then the expression (i) becomes $S \cup T = (S \cup T) \cap (\overline{S \cap T})$.

This implies that $\overline{S \cap T} = U$ so that $S \cap T = \overline{U} = \phi$.

$\therefore$ S and T are disjoint.

(6) ~~pro~~ For any sets, prove the following:

(i) $A \Delta B = (B \cap \bar{A}) \cup (A \cap \bar{B}) = (B - A) \cup (A - B)$

(ii) $\overline{A \Delta B} = \bar{A} \Delta B = A \Delta \bar{B}$

Sol: W.K.T. $A - B = A \cap \bar{B}$ ... (*)

(i) Consider, $A \Delta B = (A \cup B) - (A \cap B)$, by defn.

$= (A \cup B) \cap (\overline{A \cap B})$, by (*)

$= (A \cup B) \cap (\bar{A} \cup \bar{B})$, by De Morgan's law

$= \{(A \cup B) \cap \bar{A}\} \cup \{(A \cup B) \cap \bar{B}\}$, by distributive law

$= \{(A \cap \bar{A}) \cup (B \cap \bar{A})\} \cup \{(A \cap \bar{B}) \cup (B \cap \bar{B})\}$ Again by D.L.

$= \{\phi \cup (B \cap \bar{A})\} \cup \{(A \cap \bar{B}) \cup \phi\}$, by inverse law

$= (B \cap \bar{A}) \cup (A \cap \bar{B})$, by identity law ... (i)

$= (B - A) \cup (A - B)$, by (*)

$\overline{(A \cup B) \cap (\overline{A \cap B})}$

$(\overline{A \cup B}) \cup (A \cap B)$

$(\bar{A} \cap \bar{B}) \cup (A \cap B)$

$(\bar{A} \cap \bar{B})$, $\therefore$ $A \Delta B = (B \cap \bar{A}) \cup (A \cap \bar{B}) = (B - A) \cup (A - B)$ ... (i)

(ii) Consider, $\bar{A} \Delta B = (\bar{A} \cup B) - (\bar{A} \cap B)$, by defn

$= (\bar{A} \cup B) \cap (\overline{\bar{A} \cap B})$, by (*)

$= \overline{\overline{(\bar{A} \cup B)}} \cap (\overline{\bar{A} \cap B})$, $\because \bar{\bar{A}} = A$

$= \overline{(\overline{\bar{A} \cup B}) \cup (\bar{A} \cap B)}$, by De Morgan's law

$= \cancel{\overline{(\overline{A \Delta \bar{B}}) \cup (\overline{A \cap B})}}$

$= \overline{(A \Delta \bar{B}) \cup (\bar{A} \cap B)}$, by De Morgan's law

$= \overline{A \Delta B}$, by using (i)

Also, $A \Delta \bar{B} = (A \cup \bar{B}) - (A \cap \bar{B})$, by defn

$= (A \cup \bar{B}) \cap (\overline{A \cap \bar{B}})$, by (*)

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$= \overline{\overline{(A \cup \bar{B})}} \cap \overline{(A \cap \bar{B})}$, $= \{\cancel{\overline{(A \cup B)} \cup}$ $\because \bar{\bar{A}} = A$

$= \{\overline{(A \cup \bar{B})} \cup \overline{(A \cap \bar{B})}\}$, by Demorgan's law

$= \{\overline{(\bar{A} \cap B)} \cup \overline{(A \cap \bar{B})}\}$, by Demorgan's law

$= \overline{A \Delta B}$, using (i)

Thus $\bar{A} \Delta B = A \Delta \bar{B} = \overline{A \Delta B}$.

(7) Using the laws of set theory, simplify the following:

(i) $A \cap (B - A)$ (ii) $(A - B) \cup \overline{(A \cap B)}$

(iii) $\bar{A} \cup \bar{B} \cup (A \cap B \cap \bar{C})$ (iv) $\overline{(A \cup B) \cap C} \cup \bar{B}$

Sol: W.K.T. $A - B = A \cap \bar{B}$ ---(*)

(i) $A \cap (B - A) = A \cap (B \cap \bar{A})$, by (*)

$= A \cap (\bar{A} \cap B)$, by commutative law

$= (A \cap \bar{A}) \cap B$, by associative law

$= \phi \cap B$, by inverse law

$= \phi$, by Domination law

(ii) $(A - B) \cup (A \cap B) = (A \cap \bar{B}) \cup (A \cap B)$, by (*)

$= \{(A \cap \bar{B}) \cup A\} \cap \{(A \cap \bar{B}) \cup B\}$, by distributive law

$= \{(\bar{B} \cap A) \cup A\} \cap \{(A \cap \bar{B}) \cup B\}$, by commutative law

$= \{\bar{B} \cap (A \cup A)\} \cap \{A \cap (\bar{B} \cup B)\}$,

$= \{\bar{B} \cap A\} \cap \{A \cap U\}$. $= A \cap (\bar{B} \cup B)$

$= A \cap U = \underline{\underline{A}}$

(iii) $(\bar{A} \cup \bar{B}) \cup (A \cap B \cap \bar{C}) = (\overline{A \cap B}) \cup \{(A \cap B) \cap \bar{C}\}$, by Demorgan's & associative laws

$= \{(\overline{A \cap B}) \cup (A \cap B)\} \cap \{(\overline{A \cap B}) \cup \bar{C}\}$, by distributive law

$= U \cap (\bar{A} \cup \bar{B} \cup \bar{C})$, by Demorgan's law

$= \cancel{(\bar{U} \cap \bar{A}) \cup}$ $= \bar{A} \cup \bar{B} \cup \bar{C}$, by identity law

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(iv) $\overline{(A \cup B) \cap C \cup \bar{B}} = \overline{(A \cup B) \cap C} \cap \bar{\bar{B}}$, by Demorgan's law

$= (A \cup B) \cap (C \cap B)$, by associative law & $\bar{\bar{A}} = A$

$= (A \cup B) \cap (B \cap C)$, by commutative law

$= \{(A \cup B) \cap B\} \cap C$, by associative law

$= \{(A \cap B) \cup (B \cap B)\} \cap C$, by distributive law

$= \{(A \cap B) \cup B\} \cap C = \underline{\underline{B \cap C}}$. idopotent law

(8) For any sets A, B, C, D, prove, by using set theory, that

$(A \cap B) \cup (A \cap B \cap \bar{C} \cap D) \cup (\bar{A} \cap B) = B$

Sol: $(A \cap B) \cup (A \cap B \cap \bar{C} \cap D) = (A \cap B) \cup \{(A \cap B) \cap (\bar{C} \cap D)\}$ by associative law

$= A \cap B$, by absorption law

$\therefore (A \cap B) \cup (A \cap B \cap \bar{C} \cap D) \cup (\bar{A} \cap B) = (A \cap B) \cup (\bar{A} \cap B)$

$= \cancel{(B \cap A) \cup (\bar{A} \cap}$ $= (B \cap A) \cup (B \cap \bar{A})$, by commutative law

$= \{(B \cap A) \cup B\} \cap \{(B \cap A) \cup \bar{A}\}$, by distributive law

$= B \cap \{B \cap (A \cup \bar{A})\}$, by ~~associative~~ absorption law

$= B \cap \{B \cap U\}$, by inverse law

$= B \cap B$, by identity law

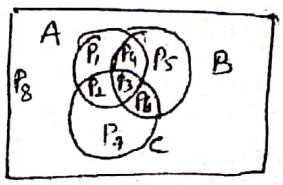
$= B$.

(9) using Venn diagram, P.T., for any three sets A, B, C

a) $\overline{(A \cup B) \cap C} = (\bar{A} \cap \bar{B}) \cup \bar{C}$ (b) $\overline{(A \cap B) \cup C} = (\bar{A} \cup \bar{B}) \cap \bar{C}$

(c) $\{\overline{(A \cup B) \cap C}\} \cup \bar{B} = \overline{B \cap C}$

Sol: Consider the following Venn diagram:



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We observe that

$A = P_1 \cup P_2 \cup P_3 \cup P_4$, $B = P_3 \cup P_4 \cup P_5 \cup P_6$, $C = P_2 \cup P_3 \cup P_6 \cup P_7$

(a) $A \cup B = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$

$\Rightarrow (A \cup B) \cap C = P_2 \cup P_3 \cup P_6$.

$\Rightarrow \overline{(A \cup B) \cap C} = U - (P_2 \cup P_3 \cup P_6) = P_1 \cup P_4 \cup P_5 \cup P_7 \cup P_8$ --(i)

Illy $\bar{A} = U - A = P_5 \cup P_6 \cup P_7 \cup P_8$, $\bar{B} = U - B = P_1 \cup P_2 \cup P_7 \cup P_8$

& $\bar{C} = U - C = P_1 \cup P_4 \cup P_5 \cup P_8$.

$\Rightarrow \bar{A} \cap \bar{B} = P_7 \cup P_8$.

$\Rightarrow (\bar{A} \cap \bar{B}) \cup \bar{C} = P_1 \cup P_4 \cup P_5 \cup P_7 \cup P_8$ ---(ii)

From (i) & (ii) $\Rightarrow \overline{(A \cup B) \cap C} = (\bar{A} \cap \bar{B}) \cup \bar{C}$

(b) $A \cap B = P_3 \cup P_4 \Rightarrow (A \cap B) \cup C = P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7$

$\Rightarrow \overline{(A \cap B) \cup C} = P_1 \cup P_5 \cup P_8$ ---(i)

$\bar{A} \cup \bar{B} = P_1 \cup P_2 \cup P_5 \cup P_6 \cup P_7 \cup P_8$

$(\bar{A} \cup \bar{B}) \cap \bar{C} = P_1 \cup P_5 \cup P_8$ ----(ii)

From (i) & (ii) $\Rightarrow \overline{(A \cap B) \cup C} = (\bar{A} \cup \bar{B}) \cap \bar{C}$.

(c) W.K.T. $(A \cup B) \cap C = P_2 \cup P_3 \cup P_6$

$\Rightarrow \overline{(A \cup B) \cap C} = P_1 \cup P_4 \cup P_5 \cup P_7 \cup P_8$

$\Rightarrow \{\overline{(A \cup B) \cap C}\} \cup \bar{B} = P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_7 \cup P_8$ --(i)

$B \cap C = P_3 \cup P_6 \Rightarrow \overline{B \cap C} = U - (B \cap C) = P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_7 \cup P_8$ --(ii)

From (i) & (ii) $\Rightarrow$

$\{\overline{(A \cup B) \cap C}\} \cup \bar{B} = \overline{B \cap C}$.

(10) Using venn diagram, prove that, for any three sets A, B, C, $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

Sol: consider the following venn diagram:

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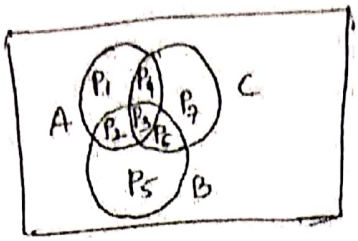

From the Fig, $A = P_1 \cup P_2 \cup P_3 \cup P_4$, $B = P_2 \cup P_3 \cup P_5 \cup P_6$

$C = P_3 \cup P_4 \cup P_6 \cup P_7$.

$\Rightarrow B \cup C = P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7$ and $B \cap C = P_3 \cup P_6$

$\therefore B \Delta C = (B \cup C) - (B \cap C) = P_2 \cup P_4 \cup P_5 \cup P_7$.

$\Rightarrow A \cup (B \Delta C) = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_7$

and $A \cap (B \Delta C) = P_2 \cup P_4$

$\therefore A \Delta (B \Delta C) = \{A \cup (B \Delta C)\} - \{A \cap (B \Delta C)\} = P_1 \cup P_3 \cup P_5 \cup P_7$ --(i)

Illy, $A \cup B = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$ and $A \cap B = P_2 \cup P_3$

$\therefore A \Delta B = (A \cup B) - (A \cap B) = P_1 \cup P_4 \cup P_5 \cup P_6$

$(A \Delta B) \cup C = P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7$

& $(A \Delta B) \cap C = P_4 \cup P_6$

$\therefore (A \Delta B) \Delta C = \{(A \Delta B) \cup C\} - \{(A \Delta B) \cap C\} = P_1 \cup P_3 \cup P_5 \cup P_7$ --(ii)

From (i) and (ii) $\Rightarrow$

$\cancel{(A \Delta B)}$ $\underline{A \Delta (B \Delta C) = (A \Delta B) \Delta C}$.

(11) Write down the dual statement for each of the following statements.

(a) $A = A \cap (A \cup B)$ (b) $A = (A \cup B) \cap (A \cup \phi)$

(c) $A \cup B = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$

(d) $U = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B})$

Sol: The duals of the given statements (a) – (b) are as follows.

(a) $A = A \cup (A \cap B$

(b) $A = (A \cap B) \cup (A \cap U)$

(c) $A \cap B = (A \cup B) \cap (A \cup \bar{B}) \cap (\bar{A} \cup B)$

(d) $\phi = (A \cup B) \cap (A \cup \bar{B}) \cap (\bar{A} \cup B) \cap (\bar{A} \cup \bar{B})$.

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Cardinality of a sets : The total no of unique elements in the set is called the cardinality of the set. The cardinality of the countable infinite set is countably infinite.

Ex: (1) Let $P = \{k, l, m, n\}$

The cardinality of the set P is 4.

(2) Let A is the set of all non-negative integers i.e $A = \{0, 1, 2, 3, \dots\}$

As A is countably infinite set hence the cardinality.

Countable set: A set A is said to be countable if (i) A is a finite set or (ii) $A \sim Z^+$, where Z^+ is the set of all +ve integers.

A set which is not countable is called an uncountable set.

~~Ex: A set which is not countable~~

Ex: The set $A = \{1, 3, 6, 7\}$ is countable because the set is finite. Also, the set E^+ of all +ve even integers is countable because $E^+ \sim Z^+$.

Note: (1) Every subset of a countable set is a countable.

(2) The union of two countable sets is countable.

(3) The intersection of two countable sets need not be uncountable.

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(1) S.T. the set of all integer is countable

<u>Sol:</u> The set of all integers is

$$Z = \{0, \pm 1, \pm 2, \ldots\}$$

The elements of Z can be listed as follows:

$$0, 1, -1, 2, -2, 3, -3 \ldots$$

In the listing 0 is the first element, 1 is the second element, -1 is the third element and so on. Thus, the elements of Z can be arranged as a sequence $\{Z_n\}$, where

$$Z_n = \frac{-n(n-1)}{2} \text{ if } n \text{ is odd}$$

$$Z_n = n/2 \text{ if } n \text{ is even.}$$

$\therefore$ Z is countable.

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<u>Addition principle</u> : Consider a finite set S containing n number of elements. Here, the n number is called the order, size or the cardinality of the set S and is denoted by $O(S)$, or $n(S)$, or $|S|$.

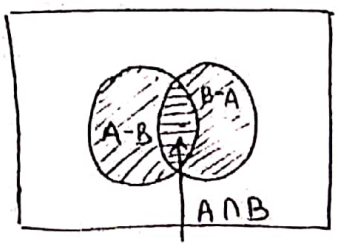
<u>Ex</u>: If $A = \{1, 2, 6\}$ and $B = \{a, b, c, d\}$, then $O(A) = |A| = 3$ and $O(B) = |B| = 4$. Clearly, $|\phi| = 0$ and $|S| \geq 1$ for every non-empty finite set S. Also, for any two finite sets A and B, if $A \subseteq B$ then $|A| \leq |B|$ and if $A \subset B$ then $|A| < |B|$.

<u>Theorem</u>: Let A and B be any two non-disjoint sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

<u>Proof</u>: Obtaining this proof is through the use of a Venn diagram.

From figure, we see that $A \cup B$ can be seen to be the union of three disjoint sets $A - B$, $B - A$ and $A \cap B$.



$\therefore |A \cup B| =$ No of elements in $(A - B)$ + No of elements in $(B - A)$ + No of elements in $(A \cap B)$

$$|A \cup B| = |A - B| + |B - A| + |A \cap B| \quad \text{---(i)}$$

Also $|A| = |A - B| + |A \cap B|$ ---(ii)

and $|B| = |B - A| + |A \cap B|$ ---(iii)

Combining (ii) and (iii), we get

$$|A| + |B| = |A - B| + |B - A| + 2|A \cap B|$$

$$|A| + |B| = |A \cup B| + |A \cap B| \quad (\text{from (i)})$$

$$\Rightarrow |A \cup B| = |A| + |B| - |A \cap B|$$

Hence proved.

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This result through which we can count the number of elements in the union of two sets is known as the principle of inclusion-exclusion for two sets. This means that, for counting the number of elements present in $A \cup B$, we conclude all elements in A and B, but exclude all elements common to A and B. The principle of inclusion-exclusion is also called as the addition principle (Rule).

~~If A and B~~

Suppose, A and B are disjoint sets then the addition principle becomes

$$|A \cup B| = |A| + |B|, \quad \because |A \cap B| = |\phi| = 0$$

This is known as the principle of disjunctive counting for two sets.

The addition principle for two sets can be extended to three or more finite sets.

<u>Result</u> Let A, B and C are ~~three~~ finites. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

<u>Sol:</u> Consider, $|A \cup B \cup C| = |A \cup D|$, where $D = B \cup C$

From addition principle,

$$= |A| + |D| - |A \cap D|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

Again by addition principle & distributive law,

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

$$= |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|]$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|,$$

(by addition principle)

by addition principle.

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$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

This proves the required result.

Result (2): If U is a finite universal set and $A \subseteq U$, $C \subseteq U$ and $B \subseteq U$, prove the following:

(i) $|\bar{A}| = |U| - |A|$ (ii) $|A - B| = |A| - |A \cap B|$

(iii) $|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B|$

(iv) $|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A| - |B| - |C| + |B \cap C| + |A \cap B| + |A \cap C| - |A \cap B \cap C|$

Sol: (i) Since $A \cap \bar{A} = \phi$ and $A \cup \bar{A} = U$

$|U| = |A \cup \bar{A}| = |A| + |\bar{A}|$, by the principle of disjunctive counting

$\Rightarrow |\bar{A}| = |U| - |A|$

(ii) We have $A = (A - B) \cup (A \cap B)$ and $(A - B) \cap (A \cap B) = \phi$

$\therefore |A| = |(A - B) \cup (A \cap B)| = |A - B| + |A \cap B|$, by the principle of disjunctive counting

$\Rightarrow |A - B| = |A| - |A \cap B|$

(iii) Consider, ~~We have,~~ $|\bar{A} \cap \bar{B}| = |\overline{A \cup B}|$, by De Morgan's law

$= |U| - |A \cup B|$, by the result (1)

$= |U| - [|A| + |B| - |A \cap B|]$, by Addition principle

$|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B|$

(iv) Consider, $|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}|$, by De Morgan's law

$= |U| - |A \cup B \cup C|$

$= |U| - [|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|]$, by addition principle

$\therefore |\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |A \cap C| - |A \cap B \cap C|$

Result (3): If A, B, C are finite sets, prove that

$|A - B - C| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

Sol: $(A - B - C)$ means that the set of elements that belong to A, but not to B or C.

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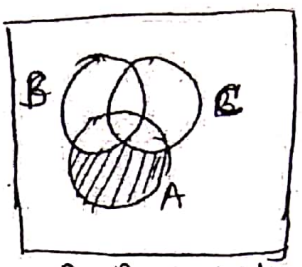
~~This result~~ We prove this result by using venn diagram:

From the Figure,



A - B - C (shaded region)

$$|A-B-C| = |A \cup B \cup C| - |B \cup C|$$

$$= [|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|] - [|B| + |C| - |B \cap C|],$$

by addition principle

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

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Problems: (1) A computer company requires 30 programmers to handle programming jobs and 40 programmers for applications programming. If the company appoints 55 programmers to carry out these jobs, how many of these perform jobs of both types? How many handle only system programming jobs? How many handle only applications programming?

Sol: Let A: the set of programmers who handle systems programming job

B: the set of programmers who handle applications programming.

Then $A \cup B$: the set of programmers appointed to carry out these jobs.

Given, $|A \cup B| = 55$, $|A| = 30$, $|B| = 40$.

By Addition rule, $|A \cup B| = |A| + |B| - |A \cap B|$

$55 = 30 + 40 - |A \cap B|$

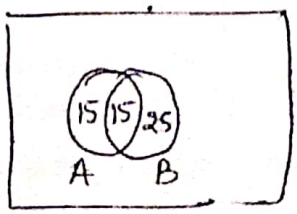
$\therefore |A \cap B| = 70 - 55 = 15$.

This means that 15 programmers perform both types of jobs.

$\therefore$ The no of programmers who handle only systems programming jobs is $|A - B| = |A| - |A \cap B| = 30 - 15 = 15$

Illy the no of programmers who handle only applications programming job is $|B - A| = |B| - |A \cap B| = 40 - 15 = 25$

The results are illustrated in the following Venn diagram:



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(2) In a class of 52 students, 30 are studying C++, 28 are studying pascal and 13 are studying both languages. How many in this class are studying at least one of these languages? How many are studying neither of these languages?

Sol: Let U: the set of all students in the class.

A: The set of students in the class who are studying C++

B: The set of students in the class who are studying pascal.

Then $A \cap B$: The set of students in the class who are studying both languages.

Given, $|U| = 52$, $|A| = 30$, $|B| = 28$, $|A \cap B| = 13$

$\therefore$, by the addition principle,

$$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 28 - 13 = 45$$

$\therefore$ 45 students in the class are studying at least one of these languages.

Also $|\overline{A \cup B}| = |U| - |A \cup B| = 52 - 45 = 7$

This means that 7 students are studying neither of these languages.

(3) Among the integers from 1 to 200, find the number of integers that are

(a) not divisible by 5

(b) divisible by 2 or 5 or 9

(c) not divisible by 2 or 5 or 9.

Sol: Let U: set of all integers from 1 to 200.

A: Set of all even integers from 1 to 200

B: Set of integers from 1 to 200 which are multiples of 5.

C: Set of integers from 1 to 200 which are multiples of 9

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$\therefore |A| = 100$, $|B| = 40$, $|C| = 22$ $|A \cap B| = 20$

$|B \cap C| = 4$, $|A \cap C| = 11$, $|A \cap B \cap C| = 2$.

(a) Number of integers not divisible by 5 is

$$|\bar{B}| = |U| - |B| = 200 - 40 = 160$$

(b) Number of integers divisible by 2 or 5 or 9 is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 100 + 40 + 22 - 20 - 4 - 11 + 2 = 129$$

(c) Number of integers not divisible by 2 or 5 or 9 is

$$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| = 200 - 129 = 71.$$

(4) In a set of 50 students 15 study mathematics, 8 study physics, 6 study chemistry and 3 study all these three subjects. prove that 27 or more students study none of these subjects.

Sol: Let U : set of all students being considered in the problem.

A : set of students studying mathematics

B : set of students studying physics

C : set of students studying chemistry

$\therefore |U| = 50$, $|A| = 15$, $|B| = 8$, $|C| = 6$, $|A \cap B \cap C| = 3$.

$A \cup B \cup C$ denotes the set of students who study atleast one of the three subjects, so that $\overline{A \cup B \cup C}$ denotes the set of students who study none of these subjects. By Addition rule, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C| \quad \cdots (i)$$

Since $A \cap B \cap C$ is a subset of all of $A \cap B$, $B \cap C$, $C \cap A$, we have

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$|A \cap B \cap C| \leq |A \cap B|$, $|A \cap B \cap C| \leq |A \cap C|$, $|A \cap B \cap C| \leq |B \cap C|$

$\therefore$ (i) $\Rightarrow$ $|A \cup B \cup C| \leq |A| + |B| + |C| - |A \cap B \cap C| - |A \cap B \cap C|$
$- |A \cap B \cap C| + |A \cap B \cap C|$

$= |A| + |B| + |C| - 2|A \cap B \cap C|$

$= 15 + 8 + 6 - (2 \times 3) = 23$

$\therefore$ $|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| \geq 50 - 23 = 27$.

This shows that 27 or more students study none of the subjects mentioned.

(5) A survey of 500 television viewers of a sports channel produced the following information: 285 watch cricket, 195 watch ~~hokey~~ hockey, 115 watch football, 45 watch cricket and football, 70 watch cricket & hockey, 50 watch hockey and football and 50 dont watch any of the three kinds of games.

(a) How many viewers in the survey watch all three kinds of games?

(b) How many viewers watch exactly one of the sports?

<u>Sol</u>: Let U: set of all viewers included in the survey

A: set of viewers who watch cricket

B: set of viewers who watch hockey

C: set of viewers who watch football.

$\therefore$ $|U| = 500$, $|A| = 285$, $|B| = 195$, $|C| = 115$

$|A \cap C| = 45$, $|A \cap B| = 70$, $|B \cap C| = 50$, $|\overline{A \cup B \cup C}| = 50$

$\therefore$ $|A \cup B \cup C| = |U| - |\overline{A \cup B \cup C}| = 500 - 50 = 450$

By extended addition rule,

$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |C \cap A|$
$\cancel{- |A \cap B \cap C|}$

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$= 450 - 285 - 195 - 115 + 70 + 50 + 45 = 20$

Thus, the number of viewers who watch all three kinds of games is 20.

Let A_1 denote the set of viewers who watch only cricket, B_1 denote the set of viewers who watch only hockey and C_1 denote the set of viewers who watch only football.

Then $A_1 = A - B - C$

$\Rightarrow |A_1| = |A - B - C| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 285 - 70 - 45 + 20 = 190.$

$\therefore$ The number of viewers who watch only cricket is 190

Illy, the no of viewers who watch only hockey is

$|B_1| = |B - C - A| = |B| - |B \cap A| - |B \cap C| + |B \cap C \cap A|$

$= 195 - 70 - 50 + 20 = 95$

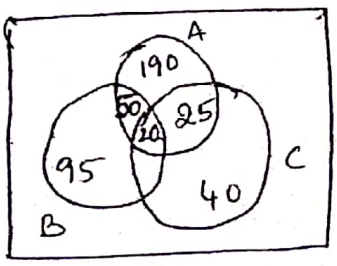
and the no of viewers who watch only football is

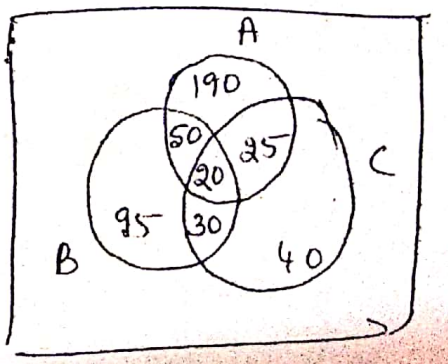
$|C_1| = |C - A - B| = |C| - |C \cap A| - |C \cap B| + |C \cap A \cap B|$

$= 115 - 45 - 50 + 20 = 40.$

Thus, the no of viewers who watch exactly one of the sports is $|A_1| + |B_1| + |C_1| = 190 + 95 + 40 = 325.$

~~Thus~~ The results are illustrated in the following Venn diagram





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(6) A survey of a sample of 25 new cars being sold by an auto-dealer auto-dealer was conducted to see which of the three popular options: air-conditioning, radio and power windows, were already installed. The survey found: 15 had air-conditioning, 12 had radio, 11 had power windows, 5 had ~~air~~ air-conditioning and power windows, 9 had ~~air~~ air-conditioning and radio, 4 had radio and power windows, and 3 had all three options. Find the number of cars that had:

(i) only power windows, (ii) only air-conditioning, (iii) only radio (iv) only one of the options (v) atleast one option (vi) none of the options.

<u>Sol:</u> Let U : set of all cars in the sample.

A : set of cars included in the sample that had air-conditioning

R : set of cars included in the sample that had radio

W : set of cars included in the sample that had power windows.

$\therefore$ $|U| = 25$, $|A| = 15$, $|R| = 12$, $|W| = 11$,

$|A \cap W| = 5$, $|A \cap R| = 9$, $|R \cap W| = 4$, $|A \cap R \cap W| = 3$.

Now, let W_1, A_1 and R_1 respectively denote the sets of cars that had only power windows, only air-conditioning, and only radio. Then

(i) $|W_1| = |W - A - R| = |W| - |W \cap A| - |W \cap R| + |W \cap A \cap R|$

$= 11 - 5 - 4 + 3 = 5$

Thus, the no of cars that had only power windows is 5.

(ii) Similarly, the no of cars which had only air-conditioning is $|A_1| = |A| - |A \cap W| - |A \cap R| + |A \cap W \cap R|$

$= 15 - 5 - 9 + 3 = 4$

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(iii) and the no of cars which had only radio is

$$|R_1| = |R| - |R \cap A| - |R \cap W| + |A \cap R \cap W|$$
$$= 12 - 9 - 4 + 3 = 2.$$

(iv) The no of cars ~~with~~ that had only one of the options, ie $|A_1| + |R_1| + |W_1| = 4 + 2 + 5 = 11$

(v) The no of cars which had at least one option is

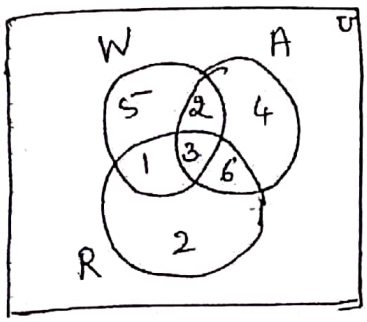
~~|W∩A~~ $|W \cup A \cup R| = |W| + |A| + |R| - |W \cap A| - |W \cap R| - |A \cap R| + |W \cap A \cap R|$

$$= 15 + 12 + 11 - 5 - 9 - 4 + 3 = 23$$

(vi) The no of cars that had none of the options is

$$|\overline{A \cup R \cup W}| = |U| - |A \cup R \cup W| = 25 - 23 = 2.$$

The different cases of this example are depicted in the following Venn diagram:



(7) A student visits a sports club every day from Monday to Friday after school hours and plays one of the three games: Cricket, Tennis, Football. In how many ways can he play each of the three games at least once during a week (from Monday to Friday)?

Sol: On each day, the student has three choices of games. Therefore, the total no of choices of

games in a 5-day period is 3^5. Thus, if U is the set of all choice of games in a 5-day ~~perio~~ period, we have $|U| = 3^5$.

Let A: the set of all choices of games which excludes cricket in a 5-day period

B: The set of all choices of games which excludes Tennis in a 5-day period.

C: The set of all choices of games which excludes Foot ball in a 5-day period.

$\therefore |A| = 2^5, \quad |B| = 2^5, \quad |C| = 2^5$.

So, $A \cap B$ is the set of all choices of games in a 5-day period which excludes cricket and Tennis i.e

$|A \cap B| = 1^5$

Illy, $|B \cap C| = 1^5, \quad |A \cap C| = 1^5$

Also, $A \cap B \cap C$ is the set of all choices of games which excludes all of the three games in the 5-day period, i.e $|A \cap B \cap C| = 0$.

Further, $A \cup B \cup C$ is the set of all choices of games which excludes at least one of the three games in the 5-day period, and $|A \cup B \cup C|$ is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

{by addition principle}

$$= 2^5 + 2^5 + 2^5 - 1^5 - 1^5 - 1^5 + 0$$

$$= 3 \times 2^5 - 3 = 3(2^5 - 1) = 3 \times 31 = 93$$

$\therefore$ the no of choices of games in the 5-day period which does not exclude any game is

$$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| = 3^5 - 93 = 243 - 93 = 150$$

Thus, there are 150 ways for the student to select his

daily games so that he plays every game at least once during a week from Monday to Friday.

8) Let X be the set of all three-digit integers that is $X = \{x \text{ is an integer} \mid 100 \le x \le 999\}$ if A_i is the set of nos in X whose ith digit is i, compute the cardinality of the set the set $A_1 \cup A_2 \cup A_3$.

Sol:

Here, $A_1 = \{100, 101, 102, \ldots, 199\} \Rightarrow |A| = 100$

$A_2 = \{120, 121, \ldots 129, 220, 221, \ldots, 229, 320, 321 \ldots 329, \ldots 920, 921, 922 \ldots 929\}$

$\Rightarrow |A_2| = 90$

$A_3 = \{103, 113, 123, \ldots, 193, 203, 223, 233, \ldots 293, 303, 313, 323, \ldots 393, \ldots 903, 913, \ldots 993\}$

$\Rightarrow |A_3| = 90$

$A_1 \cap A_2 = \{120, 121, \ldots 129\} \Rightarrow |A \cap A_2| = 10$

$A_1 \cap A_3 = \{103, 113, 123, \ldots, 193\} \Rightarrow |A_1 \cap A_3| = 10$

$A_2 \cap A_3 = \{123, 223, 323, \ldots 923\} \Rightarrow |A_2 \cap A_3| = 9$

$A_1 \cap A_2 \cap A_3 = \{123\} \Rightarrow |A_1 \cap A_2 \cap A_3| = 1$

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 A_3|$

$= 100 + 90 + 90 - 10 - 10 - 9 + 1 = \underline{\underline{252}}$.

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