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Developments in bivariate spline interpolation

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Abstract

The aim of this survey is to describe developments in the field of interpolation by bivariate splines. We summarize results on the dimension and the approximation order of bivariate spline spaces, and describe interpolation methods for these spaces. Moreover, numerical examples are given. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Bivariate spline spaces have been studied intensively in the past 30 years. These spaces consist of piecewise polynomials defined on a triangulation of a polygonal domain. They are of considerable interest in approximation theory and numerical analysis, in particular, in scattered data fitting, the construction and reconstruction of surfaces in fields of application and, classically, in the numerical solution of boundary-value problems by finite-element-type methods.

The aim of this survey is to describe interpolation methods for bivariate splines (including numerical examples) and to summarize related results on the dimension and the approximation order of bivariate spline spaces. In contrast to the univariate case, even standard problems such as the dimension and the approximation order of bivariate spline spaces are difficult to solve. In particular, the construction of explicit interpolation schemes (especially Lagrange interpolation schemes) for spline spaces on given triangulations leads to complex problems.

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The paper is organized as follows. In Section 2, we give some results on the Bézier–Bernstein representation of bivariate polynomials and its relation to bivariate splines. Section 3 deals with the dimension of bivariate spline spaces. First, general lower and upper bounds for the dimension are known in the literature. Moreover, the dimension was determined for arbitrary triangulations if the degree is relatively large compared with the smoothness of the splines. In addition, dimension formulas were derived for general classes of triangulations in the case when the degree is near the smoothness.

In Section 4, we study the approximation order of bivariate spline spaces. For arbitrary triangulations, the approximation order was determined if the degree is sufficiently large compared with the smoothness. In the other case, the approximation order was given for general classes of triangulations. The results were proved by using interpolation and quasi-interpolation methods.

In Section 5, we discuss finite element and macro element methods. Classical finite element methods are based on Hermite interpolation by bivariate polynomials on each triangle of a given triangulation, where the triangles have to be subdivided in the case of low degree polynomials. The polynomials tied together lead to super splines. Macro element methods are generalizations of the finite element methods and also lead to supersplines. Unfortunately, the Hermite interpolation schemes cannot be transformed in a straightforward way into Lagrange interpolation schemes on the whole triangulation. On the other hand, this can be done when spline spaces are used.

Section 6 deals with Hermite and Lagrange interpolation methods for bivariate spline spaces. In contrast to the univariate case, Schoenberg–Whitney-type conditions do not characterize interpolation but the so-called almost interpolation by bivariate spline spaces. The construction of explicit interpolation schemes for bivariate spline spaces leads to complex problems in general. Concerning numerical purposes, it is desirable that the complexity of computing an interpolation spline is linear in the number of triangles. Interpolation methods of this type are known in the literature, where numerical examples with more than 100 000 interpolation points are given.

Such interpolation methods can be used for the construction and reconstruction of surfaces. Concerning scattered data fitting problems, the function values and derivatives which are needed for spline interpolation, respectively, for splines of finite element type can be computed approximately by using local methods. In this context, it is the advantage of Lagrange interpolation that only functional values (and no (orthogonal) derivatives) have to be determined approximately.

In Section 6, we summarize interpolation methods for bivariate spline space (which yield explicit interpolation schemes) of the following type: Interpolation by spline spaces of arbitrary degree and smoothness on uniform-type partitions; the construction of triangulations which are suitable for interpolation by spline spaces; interpolation by spline spaces (of higher degree) on arbitrary triangulations; interpolation by spline spaces (of low degree) on classes of triangulations, respectively quadrangulations; and the construction of local Lagrange interpolation schemes on arbitrary triangulations, respectively quadrangulations. In addition, we discuss the approximation order of these methods.

We finally note that despite the great progress which has been made in the vast literature, several deep problems concerning spline spaces are still unsolved.

We also note that many papers on so-called *multivariate simplex splines* and *multivariate box splines* exist in the literature. Concerning these investigations, we refer to the survey of Dahmen and Michelli [48], the books by Bojanov et al. [20], by de Boor et al. [25] and by Michelli [111].

2. Spline spaces and Bézier–Bernstein techniques

Let Δ be a regular triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$, i.e., a set of closed triangles such that the intersection of any two triangles is empty, a common edge or a vertex.

Following Alfeld, Piper and Schumaker [7], we set

V_{I} = number of interior vertices of Δ,

V_{B} = number of boundary vertices of Δ,

V = total number of vertices of Δ,

E_{I} = number of interior edges of Δ,

E_{B} = number of boundary edges of Δ,

E = total number of edges of Δ,

N = number of triangles of Δ.

It is well known that the following Euler formulas hold:

$$E_{\mathrm{B}} = V_{\mathrm{B}},$$

$$E_{\mathrm{I}} = 3V_{\mathrm{I}} + V_{\mathrm{B}} - 3,$$

$$N = 2V_{\mathrm{I}} + V_{\mathrm{B}} - 2.$$

In the following, we define spline spaces which are natural generalizations of the classical *univariate spline spaces* (cf. the books by de Boor [21], by Nürnberger [116] and by Schumaker [144]) i.e., spaces of splines in one variable.

For given integers $r, q, 0 \leqslant r < q$, the space of *bivariate splines* of degree q and the smoothness r with respect to Δ is defined by

$$S_q^r(\Delta) = \{s \in C^r(\Omega)\colon s|_T \in \Pi_q,\ T \in \Delta\},$$

where

$$\Pi_q = \operatorname{span}\{x^i y^j\colon i, j \geqslant 0, i + j \leqslant q\}$$

is the space of *bivariate polynomials* of total degree q. In addition, suppose ρ_i, $i = 1, \ldots, V$, are integers satisfying $r \leqslant \rho_i < q$, $i = 1, \ldots, V$, and let $\theta = (\rho_1, \ldots, \rho_V)$. The space of *bivariate super splines* with respect to Δ is defined by

$$S_q^{r,\theta}(\Delta) = \{s \in S_q^r(\Delta)\colon s \in C^{\rho_i}(v_i),\ i = 1, \ldots, V\}.$$

Obviously, superspline spaces are subspaces of $S_q^r(\Delta)$.

In this survey, we consider the problem of constructing interpolation sets for bivariate spline spaces $\mathscr{S}$, where $\mathscr{S}$ can be the space $S_q^r(\Delta)$ as well as a superspline space $S_q^{r,\theta}(\Delta)$.

A set $\{z_1, \ldots, z_d\}$ in Ω, where $d = \dim \mathscr{S}$, is called a *Lagrange interpolation set* for $\mathscr{S}$ if for each function $f \in C(\Omega)$, a unique spline $s \in \mathscr{S}$ exists such that

$$s(z_i) = f(z_i), \quad i = 1, \ldots, d.$$

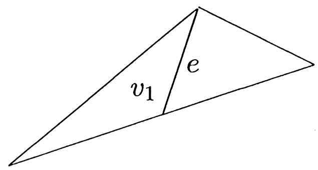

Fig. 1. A degenerate edge e attached to the vertex v_1.

If also partial derivatives of a sufficiently differentiable function f are involved and the total number of Hermite conditions is d, then we speak of a *Hermite interpolation set* for $\mathscr{S}$.

In investigating interpolation by bivariate splines, the following representation of the polynomial pieces of a spline is important. Given a triangle $T \in \Delta$ with vertices v_1, v_2, v_3, the polynomial piece $p^T = s|_T \in \Pi_q$ of a given spline $s \in S_q^r(\Delta)$ can be written as

$$p^T(x,y) = \sum_{i+j+k=q} a_{i,j,k}^T \frac{q!}{i!j!k!} \Phi_1^i(x,y)\Phi_2^j(x,y)\Phi_3^k(x,y), \quad (x,y) \in T, \tag{1}$$

where the so-called *barycentric coordinates* $\Phi_\mu \in \Pi_1$, $\mu = 1,2,3$, are uniquely defined by $\Phi_\mu(v_\nu) = \delta_{\mu,\nu}$, $\nu = 1,2,3$. Representation (1) is called the *Bézier–Bernstein form* of p^T and the real numbers $a_{i,j,k}^T$ are called the *Bézier–Bernstein coefficients* of p^T.

The following result is important for investigating the structure of bivariate spline spaces. This theorem was given by Farin [69] and de Boor [22] (see also [29,47,80]) and characterizes smoothness conditions of polynomial pieces p^{T_l}, $l = 1,2$, in representation (1) on adjacent triangles T_1, T_2 with vertices v_1, v_2, v_3 and v_1, v_2, v_4, respectively.

Theorem 2.1. *Let s be a piecewise polynomial function of degree q defined on $T_1 \cup T_2$. Then $s \in S_q^r(\{T_1, T_2\})$ iff for all $\rho \in \{0,\dots,r\}$:*

$$a_{i,j,\rho}^{T_2} = \sum_{i_1+j_1+k_1=\rho} a_{i+i_1,j+j_1,k_1}^{T_1} \frac{\rho!}{i_1!j_1!k_1!} \Phi_1^{i_1}(v_4)\Phi_2^{j_1}(v_4)\Phi_3^{k_1}(v_4), \quad i+j = q-\rho.$$

It is well known (cf. [29,69]) that for $r = 1$ the smoothness conditions of Theorem 2.1 have the geometric interpretation that the corresponding Bézier–Bernstein points lie in the same plane. Moreover, if the edge $e = [v_1, v_2]$ is *degenerate at* v_1 (i.e., the edges with vertex v_1 adjacent to e lie on a line, see Fig. 1), then for $r = 1$ the geometric interpretation of these smoothness conditions is that this plane degenerates to a line that contains three of the corresponding Bézier–Bernstein points. For $r \geqslant 2$, similar effects appear for degenerate edges. We note that degenerate edges can lead to complex problems in the investigation of bivariate spline spaces.

The following result, given in [22,69] (see also [35,36,125,159]), expresses the relation between the partial derivatives of a polynomial p^T in representation (1) at a vertex and its Bézier–Bernstein coefficients. This lemma plays an important role in the construction of interpolation sets for bivariate splines.

Lemma 2.2. *Let $p^T \in \Pi_q$ be a polynomial on a triangle $T = \Delta(v_1, v_2, v_3)$ in representation* (1) *and d_j, $j = 1, 2$, be unit vectors in direction of the edge $[v_1, v_{j+1}]$, $j = 1, 2$. Then, for all $0 \leqslant \alpha + \beta \leqslant q$,*

$$p^T_{d_1^\alpha d_2^\beta}(v_1) = \frac{q!}{(q-\alpha-\beta)!} \sum_{j=0}^{\alpha} \sum_{k=0}^{\beta} \binom{\alpha}{j} \binom{\beta}{k} (\Phi_1)_{d_1}^{\alpha-j} (\Phi_1)_{d_2}^{\beta-k} (\Phi_2)_{d_1}^{j} (\Phi_3)_{d_2}^{k} a_{q-j-k,j,k}. \quad (2)$$

It easily follows from (2) and induction that if the Bézier–Bernstein coefficients

$$a_{q-j-k,j,k}, \quad j = 0, \dots, \alpha, \;\; k = 0, \dots, \beta$$

are uniquely determined, then all derivatives

$$p^T_{d_1^j d_2^k}(v_1), \quad j = 0, \dots, \alpha, \;\; k = 0, \dots, \beta$$

are given, and conversely. In particular, these relations show that Bézier–Bernstein coefficients can be considered as certain linear functionals (see Section 3).

The Bézier–Bernstein representation (1) of a polynomial $p^T \in \Pi_q$ has important applications in CAGD. We refer the reader to the surveys of Farin [69,70], Boehm et al. [19], the book by Chui [29] and the papers of de Boor [22] and Dahmen [47]. Moreover, concerning the so-called *blossoming approach*, we refer to the tutorial of de Rose et al. [135] and the survey of Seidel [149]. Triangulation methods were described in the survey of Schumaker [147]. For interpolation by bivariate polynomials, we refer to the survey of Gasca and Sauer [74].

Finally, we note that a characterization (different from Theorem 2.1) of the smoothness of polynomial pieces on adjacent triangles, without using Bézier–Bernstein techniques, was proved by Davydov et al. [59].

3. Dimension of spline spaces

In this section, we summarize results on the dimension of bivariate (super) spline spaces for arbitrary triangulations. Results on the dimension play a fundamental role for the construction of interpolation sets.

In contrast to univariate spline theory, it is a nontrivial problem to determine the dimension of bivariate spline spaces when $r \geqslant 1$. (For the case of continuous splines, i.e., $r = 0$, see [143].) In fact, this problem is not yet completely solved.

Historically, Strang [156] was the first in 1973 who posed the problem: What is the dimension of $S_q^r(\Delta)$? In the following we summarize results concerning this problem.

We begin with the following lower bound on the dimension, which was given in 1979 by Schumaker [143].

Theorem 3.1. *Let Δ be an arbitrary triangulation and e_i be the number of edges with different slopes attached to the ith interior vertex of Δ. Set*

$$\sigma_i = \sum_{j=1}^{q-r} (r + j + 1 - je_i)_+, \quad i = 1, \dots, V_I,$$

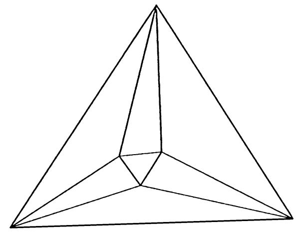

Fig. 2. The Morgan and Scott triangulation Δ_{MS}.

where $(x)_+ = \max\{0, x\}$. *Then*

$$\dim S_q^r(\Delta) \geqslant \binom{q+2}{2} + \binom{q-r+1}{2} E_I - \left(\binom{q+2}{2} - \binom{r+2}{2}\right) V_{\mathrm{I}} + \sum_{i=1}^{V_{\mathrm{I}}} \sigma_i. \tag{3}$$

In [145], it was shown that the lower bound in (3) also holds for spline spaces with respect to partitions more general than triangulations, the so-called *rectilinear partitions*. A lower bound for the dimension of spline spaces with respect to rectilinear partitions with holes was given by Jia [90].

A standard method to determine the exact dimension is to find an upper bound for the dimension of a given spline space that coincides with the lower bound in (3). In order to establish such an upper bound n, it suffices to construct linear functionals λ_i, $i = 1, \ldots, n$, such that

$$\text{if } \lambda_i(s) = 0, \quad s \in S_q^r(\Delta), \;\; i = 1, \ldots, n, \;\; \text{then } s \equiv 0. \tag{4}$$

In [145], an upper bound for the dimension of spline spaces was developed (see also [107]). This upper bound depends on the ordering of the interior vertices. An improved upper bound was given by Ripmeester [134], who used a special ordering of the vertices. We also mention that lower and upper bounds for the dimension hold for superspline spaces and that such bounds for spline spaces in several variables were developed by Alfeld [5]. In general, however, all these upper bounds do not coincide with the lower bound in (3).

The dimension of a bivariate spline space can be larger than the lower bound in (3). Examples concerning this fact were given in [145], where the first example constructed by Morgan and Scott [113] is also discussed (see also [4,76,152]). Morgan and Scott considered triangulations Δ_{MS} as in Fig. 2 and the space $S_2^1(\Delta_{\mathrm{MS}})$. The dimension of this spline space is equal to 7 if Δ_{MS} has certain symmetry properties, and otherwise it is 6, which is equal to the lower bound in (3) for $S_2^1(\Delta_{\mathrm{MS}})$. Hence, in the nonsymmetric case $S_2^1(\Delta_{\mathrm{MS}})$ coincides with Π_2. This example shows that the dimension can depend on the exact geometry of the whole triangulation. In general, such dependencies may appear if the degree q is small compared with the smoothness r. Diener [66] investigated the dimension of the space $S_{2r}^r(\Delta_{\mathrm{MS}})$, $r \geqslant 2$, and found similar dependencies on the exact geometry of Δ_{MS}.

These results show that the structure of $S_q^r(\Delta)$ is becoming more complex when the degree q approaches the smoothness r. This is one of the fundamental phenomena in bivariate spline theory, in contrast to the univariate case.

We proceed by describing cases when the dimension of $S_q^r(\Delta)$ is known. In 1975, Morgan and Scott [112] determined the dimension of $S_q^1(\Delta)$, $q \geqslant 5$. Without using Bézier–Bernstein techniques,

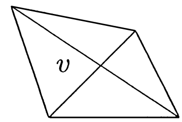

Fig. 3. A singular vertex v.

they showed that the following formula holds for an arbitrary triangulation Δ:

$$\dim S_q^1(\Delta) = \binom{q+2}{2} N - (2q+1)E_{\mathrm{I}} + 3V_{\mathrm{I}} + \sigma, \quad q \geqslant 5. \tag{5}$$

Here and in the following, σ denotes the number of singular vertices of Δ. An interior vertex of Δ that has only two edges with different slopes attached to it is called a *singular vertex* (see Fig. 3). For C^1-spline spaces, $\sigma_i = 1$ in (3) holds if the corresponding vertex is singular, and in all other cases $\sigma_i = 0$. By using this fact and Euler's formulas (see Section 2) it is easy to verify that the dimension formula (5) of Morgan and Scott is equal to the lower bound in (3). Moreover, in [112] a *nodal basis* for C^1-spline spaces of degree at least 5 was constructed. This means that the splines in $S_q^1(\Delta)$, $q \geqslant 5$, are determined by their values and derivatives at d points in Ω, where $d = \dim S_q^1(\Delta)$. Davydov [53] showed that an alternative construction yields a basis for these spline spaces which is *locally linearly independent* (see Section 6).

The results of Morgan and Scott were extended to spline spaces $S_q^r(\Delta)$, $q \geqslant 4r+1$. These extensions are based on the results of Schumaker [146] for spline spaces on *cells* (see Fig. 4) coupled with the methods developed by Alfeld and Schumaker [8] and Alfeld et al. [6] (see also [27]). (For a generalization to trivariate splines of degree at least $8r+1$ on tetrahedral partitions, see [11].) In these papers, Bézier–Bernstein techniques were used and the concept of *minimal determining sets* was introduced. Roughly speaking, the Bézier–Bernstein coefficients $a_{i,j,k}^{\mathrm{T}}$, $i+j+k=q$, $T \in \Delta$, of the polynomial pieces $p^{\mathrm{T}} \in \Pi_q$ in representation (1) of a spline can be considered as linear functionals. A *determining set* is a subset $\{\lambda_i\colon i = 1,\ldots,n\}$ of these functionals such that (4) holds, and such a set is called *minimal* if there exists no determining set with fewer elements.

By using Bézier–Bernstein techniques Hong [80] determined the dimension of $S_q^r(\Delta)$ for arbitrary triangulations Δ in the case when $q \geqslant 3r+2$.

Theorem 3.2. *Let Δ be an arbitrary triangulation. If $q \geqslant 3r+2$, then the dimension of $S_q^r(\Delta)$ is equal to the lower bound in* (3).

This result was generalized by Ibrahim and Schumaker [82] to super spline spaces of degree at least $3r+2$ (see also [36]). The proof of Theorem 3.2 given by Hong [80] (respectively Ibrahim et al. [82]) is based on local arguments by considering vertices, edges and triangles separately. In particular, a basis of *star-supported* splines (i.e. splines whose supports are at most the set of triangles surrounding a vertex, i.e., a cell (for interior vertices)) of $S_q^r(\Delta)$, $q \geqslant 3r+2$, is constructed (see also [59]). Recently, Alfeld and Schumaker [10] showed that such a basis does not exist in general if $q < 3r+2$ and $r \geqslant 1$. Thus, local arguments as in [36,59,80,82] fail in these cases. The problem of finding an explicit formula for the dimension of $S_q^r(\Delta)$, $q < 3r+2$, $r \geqslant 1$, remains open in general.

The only exception known in the literature is the space $S_4^1(\Delta)$. In 1987, Alfeld et al. [7] showed the following result.

Theorem 3.3. *Let Δ be an arbitrary triangulation. Then*

$$\dim S_4^1(\Delta) = 6V + \sigma - 3. \tag{6}$$

Again, the number in (6) is equal to the lower bound in (3). The proof of Alfeld et al. [7] involves arguments from graph theory which are not purely local. As shown in [7], such complex arguments do not have to be used if Δ is a *nondegenerate triangulation* (i.e., a triangulation that contains no degenerate edges, see Section 2). For this class of triangulations Δ, the dimension of $S_{3r+1}^r(\Delta)$, $r \geqslant 2$, has been determined by Alfeld and Schumaker [9].

By Euler's formulas (see Section 2) the lower bound in (3) denoted by lb_3^1 for $S_3^1(\Delta)$ can be written as follows:

$$\mathrm{lb}_3^1 = 3V_\mathrm{B} + 2V_\mathrm{I} + \sigma + 1. \tag{7}$$

Therefore, the dimension of $S_3^1(\Delta)$ is larger than the number of vertices of Δ. This is in contrast to the case of quadratic C^1-splines (where the lower bound in (3) is equal to $V_\mathrm{B} + \sigma + 3$, see also the general comment on these spaces in Section 5) and makes $S_3^1(\Delta)$ interesting for applications. On the other hand, the structure of the space $S_3^1(\Delta)$ is very complex. For instance, it is not known if the following conjecture is true for arbitrary triangulations Δ.

Conjecture. *The dimension of $S_3^1(\Delta)$ is equal to* lb_3^1.

More general, the following problem for $r \geqslant 1$ is unsolved: What is the smallest integer $q \leqslant 3r+2$ (depending on r) such that the dimension of $S_q^r(\Delta)$ coincides with the lower bound in (3) for an arbitrary triangulation Δ?

By using homological methods Billera [17] (see also [18,90,160]) showed that the above conjecture holds *generically*. Roughly speaking, this means that if the dimension of $S_3^1(\Delta)$ is not equal to lb_3^1, then pertubations of the vertices exist such that equality holds.

In addition, the above conjecture holds for general classes of triangulations. In connection with their interpolation method, Davydov et al. [58] (see also [56,57]) proved that lb_3^1 is equal to the dimension of $S_3^1(\Delta)$, where Δ is contained in the general class of *nested polygon triangulations* (see Section 6).

Finally, Gmelig Meyling [75] discussed a numerical algorithm for determining the dimension of $S_3^1(\Delta)$. This is done by computing the rank of a global system associated with the smoothness constraints.

4. Approximation order of spline spaces

In this section, we summarize results on the *approximation order* of bivariate spline spaces. Such results are important for interpolation by these spaces. We say that $S_q^r(\Delta)$ has approximation order

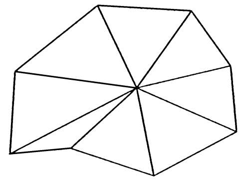

Fig. 4. A cell.

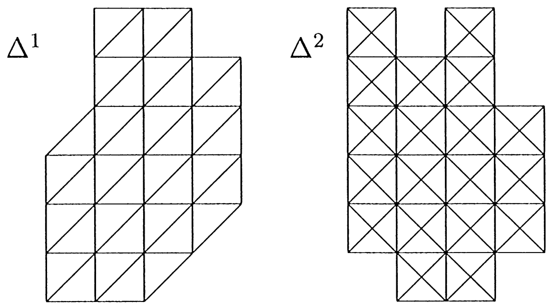

Fig. 5. Δ^i, $i = 1, 2$, triangulations.

ρ, if for any sufficiently differentiable function f

$$\mathrm{dist}(f, S_q^r(\Delta)) \leqslant Kh^\rho, \tag{8}$$

where $h = \max\{\mathrm{diam}(T)\colon T \in \Delta\}$, K is a constant depending on the smallest angle in Δ and f, and the distance is measured in the supremum norm. Moreover, we say that the approximation order is *optimal*, if $\rho = q + 1$.

It is clear that the approximation order of $S_q^r(\Delta)$ cannot be greater than $q + 1$, and that the optimal approximation order of these spaces is obtained for $r = 0$. In the context of the *finite element method*, it follows from a result of Ženišek [162] (see Section 5) that $S_q^r(\Delta)$ has optimal approximation order provided that $q \geqslant 4r + 1$, $r \geqslant 0$.

However, in contrast to the univariate case, the approximation order of a bivariate spline space is not always optimal. This was first proved in 1983 by de Boor and Höllig [23], who considered the space $S_3^1(\Delta^1)$. Here, Δ^1 is a triangulation obtained from a rectangular partition by adding the same diagonal to each rectangle (see Fig. 5). Extensions of this result, were given by Jia [87–89]. In 1993, de Boor and Jia [26] proved the following theorem.

Theorem 4.1. *The approximation order of $S_q^r(\Delta^1)$ is at most q if $q < 3r + 2$, $r \geqslant 1$.*

As shown in [26], this result holds for any L_p, $1 \leqslant p \leqslant \infty$, norm.

In general, for fixed q and r the approximation order of $S_q^r(\Delta)$ depends on the structure of the triangulation Δ. For instance, it was proved by Jia [91] that $S_q^r(\Delta^2)$, $q \geqslant 2r + 1$, $r \geqslant 0$, has optimal approximation order. Here, Δ^2 is a triangulation obtained from a rectangular partition by adding both diagonals to each quadrangle (see Fig. 5).

More general, *triangulated quadrangulations* were considered in the literature. These are triangulations obtained from a quadrangulation $\square$ as follows: for each convex quadrilateral $\mathcal{Q}$ of $\square$ both diagonals are added and for each nonconvex quadrilateral $\mathcal{Q}$ of $\square$ the diagonal is added and the center of this diagonal is connected with the remaining two vertices of $\mathcal{Q}$. A triangulated quadrangulation is called a *triangulated convex quadrangulation*, if every quadrilateral of the corresponding quadrangulation is convex.

Lai and Schumaker [101] proved optimal approximation order for $S_6^2(\Delta)$, where Δ is an arbitrary triangulated quadrangulation. For doing this, a *quasi-interpolation method* for $S_6^2(\Delta)$ was developed, which uses a superspline subspace $S_6^{2,\theta}(\Delta)$, with $\rho_i \in \{2,3\}$, in general. We note that by this method supersplines have only to be used if interior vertices of degree three or certain interior vertices of degree four appear in $\square$. These are the only vertices, where smoothness three is needed.

Generally, given a basis $\{B_i,\ i = 1,\dots,d\}$, of a spline space and a set of linear functionals $\{\lambda_i\colon i = 1,\dots,d\}$, a quasi-interpolant s_f of a sufficiently smooth function f from this spline space can be written in the form

$$s_f = \sum_{i-1}^{d} (\lambda_i f) B_i. \tag{9}$$

These linear functionals λ_i typically consist of linear combinations of certain derivatives at points in Ω. In particular, Bézier–Bernstein coefficients (see Sections 2 and 3) can be considered as linear functionals of this type.

Moreover, Lai and Schumaker [103] proved optimal approximation order for $S_{3r}^r(\Delta)$, $r \geqslant 1$, where Δ is a triangulated convex quadrangulation. Again, this result was shown by using quasi-interpolation for a superspline subspace in general. We note that this quasi-interpolation method lead to an extension of Fraeijs de Veubeke's [71], and Sander's [140] finite element (see Section 5).

Concerning arbitrary triangulations Δ, the following theorem was first stated by de Boor and Höllig [24] in 1988, and later proved completely by Chui et al. [34].

Theorem 4.2. *The approximation order of $S_q^r(\Delta)$ with respect to an arbitrary triangulation Δ is optimal when $q \geqslant 3r + 2$, $r \geqslant 1$.*

While de Boor and Höllig [24] used abstract methods, Chui et al. [34] constructed a quasi-interpolant for a superspline subspace to prove this result.

By using a quasi-interpolation method different from Chui et al. [34], Lai and Schumaker [102] extended Theorem 4.2 to general L_p, $1 \leqslant p \leqslant \infty$, norms. These two quasi-interpolation methods have in common that both use an appropriate superspline subspace and Bézier–Bernstein methods. Recently, for $q \geqslant 3r + 2$, the first interpolation scheme which yields optimal approximation order (in the sense that the constant K only depends on the smallest angle of Δ and is independent of *near-degenerate* edges) such that the fundamental functions have minimal support was developed by Davydov et al. [59]. By using methods which are completely different from those for quasi-interpolation and by applying the concept of *weak interpolation* (introduced by Nürnberger [118]), the authors constructed a Hermite interpolating spline from the super spline space $S_q^{r,\theta^*}(\Delta)$ where $\theta^* = (r + [(r+1)/2],\dots,r + [(r+1)/2])$ and a nodal basis which yield these properties. The following theorem shows that the interpolating spline s_f of Davydov et al. [59] simultaneously

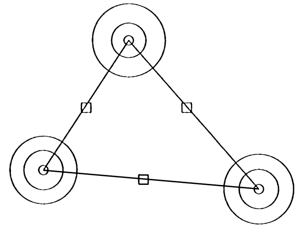

Fig. 6. The Argyis, Fried, and Scharpf element, $S_5^{1,\theta_1}(\Delta)$.

approximates the function f and its derivatives. We note that the quasi-interpolation methods of Lai and Schumaker [102,103] yield similar estimates.

Theorem 4.3. *Let $q \geqslant 3r+2$, $r \geqslant 1$, Δ an arbitrary triangulation, $f \in C^m(\Omega)$, $m \in \{2r, \ldots, q+1\}$ and $s_f \in S_q^{r,\theta^*}(\Delta)$ the interpolating spline of f. Then, for every triangle $T \in \Delta$,*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{L_\infty(T)} \leqslant K h_T^{m-\alpha-\beta} \max\{\|D_x^\mu D_y^{m-\mu} f\|_{C(T)}\colon \mu = 0, \ldots, m\} \tag{10}$$

for all α, $\beta \geqslant 0$, $\alpha + \beta \leqslant m$, where h_T is the diameter of T, and K is a constant which depends only on r, q and the smallest angle in Δ.

We close this section with the following problem.

Problem. Determine the approximation order of $S_q^r(\Delta)$, $q < 3r+2$, $r \geqslant 1$, for general classes of triangulations Δ.

5. Finite elements and macro element methods

In this section, classical *finite elements* and *macro element* methods are considered. Classical finite elements are based either on interpolation by bivariate polynomials on every triangle or on interpolation by C^1 splines with respect to a triangulation that is obtained by applying a splitting procedure to *every* triangle or quadrilateral. Extensions of the latter case lead to macro elements.

We begin by describing finite elements, where each polynomial piece is determined separately. The idea of this classical method is to chose a suitable spline space such that interpolation by bivariate polynomials on every triangle of an arbitrary triangulation Δ automatically leads to a Hermite interpolation set for this spline space. Such spline spaces are superspline spaces of large degree.

As an example, we describe the well-known finite element of Argyis et al. [12], which yields Hermite interpolation by the super spline space $S_5^{1,\theta_1}(\Delta)$, where $\theta_1 = (2, \ldots, 2)$. This Hermite interpolation method is to interpolate function value, first and second-order derivatives at the vertices, and the normal derivative at the midpoint of each edge. (See Fig. 6, where the function value, vertex derivatives and normal derivatives are symbolized by circles and boxes, respectively). The corresponding *condensed scheme* is obtained by replacing the normal derivatives by other conditions, see [14,16]. This method was generalized by Ženišek [162,163] to Hermite interpolation by $S_q^{r,\theta_r}(\Delta)$, $q \geqslant 4r+1$, $r \geqslant 1$, where $\theta_r = (2r, \ldots, 2r)$. As mentioned in Section 4, the corresponding

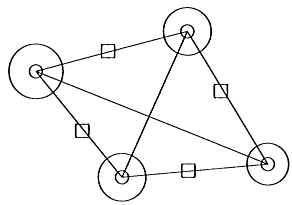

Fig. 7. The Fraeijs de Veubeke, respectively Sander element, $S_3^1(\Delta)$.

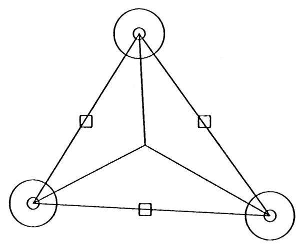

Fig. 8. The Clough–Tocher element, $S_3^1(\Delta_{\mathrm{CT}})$.

Hermite interpolating spline yields optimal approximation order. This follows from a result of Ciarlet and Raviart [43] concerning interpolation by bivariate polynomials. A link between the finite element method, super splines, and Bézier–Bernstein techniques was given by Schumaker [148] (see also [159], for the special case $S_9^{2,\theta_2}(\Delta)$). For further results on finite elements see [109,110] and the references therein.

It is desirable in general, however, to use low degree splines (in relation to the smoothness) to keep computational costs small. The following classical methods have been developed for this purpose. The idea of these methods is to modify the given partition, which can be a triangulation or a convex quadrangulation. This is done by applying a certain splitting procedure to each triangle or quadrilateral. In contrast to the finite element method described above, more than one polynomial piece is needed for each triangle or quadrilateral such that the method is local. These classical approaches lead to Hermite interpolation by cubic and quadratic C^1 splines.

We begin with the classical Hermite interpolation scheme for $S_3^1(\Delta)$ of Fraeijs de Veubeke [71] and Sander [140], where Δ is a triangulated convex quadrangulation (see Section 4). This classical Hermite interpolation set consists of the function and gradient value at the vertices of the underlying convex quadrangulation $\Box$ and the orthogonal derivative at the midpoints of all edges of $\Box$. (See Fig. 7, where the function value, vertex derivatives and normal derivatives are symbolized by circles and boxes, respectively.) The approximation properties of this element were studied by Ciavaldini and Nèdélec [44]. It turns out that the corresponding Hermite interpolating spline yields optimal approximation order. We note that a modification of this Hermite interpolation set involving second-order derivatives (instead of the orthogonal derivatives) was given by Lai [99].

Another well-known element was given by Clough and Tocher [45] in 1966. These authors constructed a Hermite interpolation set for $S_3^1(\Delta_{\mathrm{CT}})$, where Δ_{CT} is a triangulation obtained from an arbitrary triangulation Δ by splitting each triangle $T \in \Delta$ into three subtriangles (see Fig. 8). This classical Hermite interpolation set consists of function and gradient value at the vertices of Δ and

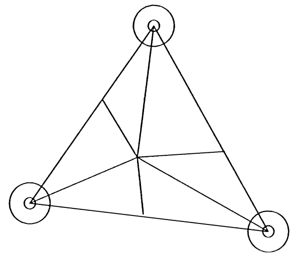

Fig. 9. The Powell-Sabin element, $S_2^1(\Delta_{\mathrm{PS}})$.

the orthogonal derivative at the midpoints of all edges of Δ. (See Fig. 8, where the function value, the vertex derivatives and normal derivatives are symbolized by circles and boxes, respectively.) The approximation properties of this element and its condensed version were studied by Ciarlet [40,42] and Farin [68]. It turns out that the Hermite interpolation set described above yields optimal approximation order. Moreover, Percell [129] considered $S_4^1(\Delta_{\mathrm{CT}})$ and Alfeld [3] developed a trivariate Clough–Tocher-type method.

We turn now to the case of quadratic C^1 splines, $S_2^1(\Delta)$, and begin with a general comment on these spaces. As we mentioned in Section 3, the lower bound in (3) for these spline spaces, is equal to $V_{\mathrm{B}}+\sigma+3$. Since there are many cases, when the dimension of $S_2^1(\Delta)$ is close to this lower bound, it cannot be expected that these spaces have good approximation properties in general. In particular, for $S_2^1(\Delta^1)$. By considering this lower bound and having in mind the example of Morgan and Scott [113] (see Section 3), the only practical way to obtain a quadratic C^1 space in which interpolation yields good approximation order is to increase the number of singular and boundary vertices (cf. the method of Nürnberger and Zeilfelder [125] described in Section 6).

In 1977, Powell and Sabin [131] constructed a Hermite interpolation set for $S_2^1(\Delta_{\mathrm{PS}})$. In that work, the triangulations Δ_{PS} are obtained by splitting every triangle T of a given triangulation Δ into six subtriangles (see Fig. 9). We note that the splitting points are chosen in such a way that each interior edge of Δ leads to a singular vertex of Δ_{PS} and that each boundary edge of Δ leads to an additional boundary vertex of Δ_{PS}. By using Euler's formulas (see Section 2) it can be easily seen that the dimension of the resulting space $S_2^1(\Delta_{\mathrm{PS}})$ is $3V$. The Hermite interpolation set constructed in [131] consists of function and gradient value for all vertices of Δ. (See Fig. 9 where the function values and vertex derivatives are symbolized by circles.) The corresponding Hermite interpolating spline of Powell and Sabin yields optimal approximation order. This was proved by Sablonnière [139] with special attention to the approximation constants. For further results using this space and a modification of this, see [51,67,79,130].

In the following, we describe extensions of these methods. These extensions are called macro element methods. Macro element methods use Bézier–Bernstein techniques and lead to Hermite interpolation by super spline spaces.

We start with the generalization of Fraeijs de Veubeke's and Sander's method, which was developed by Laghchim-Lahlou and Sablonnière [93,94,96]. In [93,94,96], the following cases were considered, where Δ is a triangulated convex quadrangulation: $S_{3r}^{r,\theta}(\Delta)$ if r is odd, $S_{3r+1}^{r,\theta}(\Delta)$ if r is even. Here, the components of θ concerning the vertices of the underlying quadrangulation are $(3r-1)/2$ if r is odd and $3r/2$ if r is even. This Hermite interpolation method is to interpolate function value and derivatives up to order $r+[r/2]$ at the vertices and suitable derivatives at interior points of each edge of the underlying quadrangulation. We note that if r is odd, the super spline spaces considered by Lai and Schumaker [103] (see Section 4) coincide with those of [96]. On the other hand, in [103] different supersplines with lower degree were used for even r.

Generalizations of the Clough–Tocher element were also given in the literature. In 1994, Laghchim-Lahlou et al. [95] (see also [92,94,137]) constructed Hermite interpolation sets for certain superspline space with respect to Δ_{CT}. Again, these sets consist of function value and derivatives up to a certain order at the vertices and suitable derivatives at interior points of each edge of the given (nonsplitted) triangulation. Lai and Schumaker [104] recently constructed a Hermite interpolation set for super spline spaces with respect to Δ_{CT} that additionally contain function value and derivatives up to a certain order at each splitting point (except for the classical case $S_3^1(\Delta)$). These authors show that their construction provides a so-called *stable local basis*, which implies that the associated spline space has optimal approximation order. We mention that such bases have been constructed for $S_q^r(\Delta)$, $q \geqslant 3r+2$, by Chui et al. [34], by Lai and Schumaker [102] and by Davydov and Schumaker [61].

Finally, generalizations of the Powell–Sabin element were given in the literature. In 1996, Laghchim-Lahlou and Sablonnière [97] (see also [94]) considered the triangulation Δ_{PS}^1 that is obtained by applying the Powell–Sabin split to each triangle of a Δ^1 triangulation. There it is shown that the function value and derivatives up to order $r+[r/2]$ at all vertices of Δ^1 yield Hermite interpolation by the superspline spaces $S_{2r}^{r,\theta}(\Delta_{\mathrm{PS}}^1)$ if r is odd, $S_{2r+1}^{r,\theta}(\Delta_{\mathrm{PS}}^1)$ if r is even, where $\theta=(r+[r/2],\dots,r+[r/2])$. Lai and Schumaker [105] recently constructed a Hermite interpolation set for super spline spaces with respect to Δ_{PS} that additionally contains the function values and derivatives of a certain order at points different from the vertices of the underlying given (nonsplitted) triangulation (except for the classical case $S_2^1(\Delta)$). This macro element method uses lower degree splines than earlier methods of this type. Again, in [105] a stable local basis was constructed. We note that the case $S_5^2(\Delta_{\mathrm{PS}})$ has been considered earlier by Sablonnière [138] and Lai [100].

We close this section with two remarks. First, we note that all these Hermite interpolation methods cannot be transformed into Langrange interpolation on the *whole* triangulation straightforwardly. Second, all these methods lead to interpolation by superspline subspaces. With three exceptions: the classical methods of Fraeijs de Veubecke and Sander, Clough–Tocher and Powell–Sabin.

For results on interpolation by splines with constraints (e.g., shape preserving interpolation, convex interpolation) we refer to Dahmen and Micchelli [49,50], Schmidt and Walther [141,142], Lorente-Pardo et al. [106], Constantini and Manni [46], Chalmers et al. [28]. Concerning the finite element method, we refer to the books by Ciarlet [41], Prenter [132] and Strang and Fix [157].

6. Interpolation by spline spaces

If we consider the results discussed in the above sections, the natural problem of constructing interpolation sets for the *spline* space $S_q^r(\Delta)$ appears. In this section, we summarize results on interpolation by these spaces. Here, we do not consider interpolation by subspaces of $S_q^r(\Delta)$ such as super splines.

As mentioned above, results on interpolation by these spaces are strongly connected with the problem of determining the dimension of these spaces. Therefore, interpolation by $S_q^r(\Delta)$ leads to complex problems, in particular, for Lagrange interpolation.

It is well known that for univariate spline spaces, interpolation sets can be characterized through Schoenberg–Whitney-type conditions (see the books by de Boor [21], by Schumaker [144] and by Nürnberger [116]). In contrast to this, the Schoenberg–Whitney theorem cannot be extended to interpolation by bivariate splines, even for the simplest space $S_1^0(\Delta)$ (cf. [29, p. 136]).

It was shown by Sommer and Strauss [154,155] that the natural multivariate analogue of such Schoenberg–Whitney-type conditions characterizes *almost interpolation sets*, i.e., point sets that can be transformed into Lagrange interpolation sets for $S_q^r(\Delta)$ by arbitrary small perturbations of the points (see also [52,53,62–64]). In this context, locally linearly independent systems $\mathscr{S}$ of splines play an important role. This means that each open subset in Ω contains a ball B such that the subsystem consisting of all elements of $\mathscr{S}$ having a support with a nonempty intersection with B is linearly independent on B. In a general topological context, Davydov et al. [64] showed the relations of locally linearly independent systems with almost interpolation. In [64], a Schoenberg–Whitney-type characterization of almost interpolation sets for spline spaces that admit a locally linearly independent basis was developed (see also [53]). A locally linearly independent basis for $S_q^1(\Delta)$, $q \geqslant 5$, was constructed in [53]. For certain superspline spaces of higher smoothness such basis were given by Davydov et al. [59,63]. Recently, Davydov and Schumaker [60] constructed a locally linearly independent basis for $S_q^r(\Delta)$, $q \geqslant 3r+2$. For further results on locally linearly independence, see the references in [60].

Concerning numerical purposes, it is desirable to construct explicit interpolation schemes for $S_q^r(\Delta)$, in particular, such that algorithmical complexity of computing a corresponding interpolating spline is $\mathscr{O}(\operatorname{card} \Delta)$. In the following, we describe explicit Lagrange and Hermite interpolation methods for $S_q^r(\Delta)$.

First, it is obvious that a Lagrange interpolation set for $S_q^0(\Delta)$, $q \geqslant 1$, is obtained by the union of all points $(i/q)v_1 + (j/q)v_2 + (k/q)v_3$, $i+j+k=q$, where v_1, v_2, v_3 are the vertices of a triangle in Δ. In particular, the set of vertices of Δ is a Lagrange interpolation set for $S_1^0(\Delta)$. In 1986, an algorithm for constructing more general Langrange interpolation sets for $S_1^0(\Delta)$ was given by Chui et al. [31]. Davydov et al. [65] recently gave a characterization of Lagrange interpolation sets for $S_1^0(\Delta)$.

The literature shows that for $q \geqslant 2$, it is complex problem to construct interpolation set for $S_q^r(\Delta)$, in particular concerning Lagrange interpolation. This was done for certain classes of triangulations, respectively, for splines of certain degree q and smoothness r. These methods are described in the following.

In the beginning 1990s Nürnberger and Rießinger [120,121] developed a general method for constructing Hermite and Lagrange interpolation sets for $S_q^r(\Delta^i)$, $i=1,2$. We note that the triangulations Δ^i, $i=1,2$ (see Fig. 5), have to be uniform if q,r arbitrary, whereas the triangulations may be nonuniform if $i=1$ and $r \in \{0,1\}$, respectively, $i=2$ and $r \in \{0,1,2\}$. The dimension for such type of spline spaces, and more generally for so-called *crosscut partitions*, was determined by Chui and Wang [37–39] and Schumaker [145] (for *quasi-crosscut partitions*, see [108]). For spline spaces on Δ^i, $i=1,2$, a basis consisting of the polynomials, *truncated power functions* and so-called *cone splines* exists in the above cases.

The method given in [120,121] is to construct line segments in Ω and to place points on these lines which satisfy the *interlacing condition* of Schoenberg and Whitney for certain univariate spline spaces such that the *principle of degree reduction* can be applied.

This construction of Lagrange interpolation sets uses certain basic steps. In this survey paper, we only describe these basic steps for $S_q^r(\Delta^1)$. For an arbitrary triangle $T \in \Delta^1$, one of the following

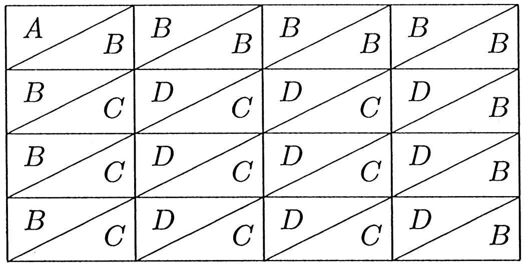

Fig. 10. Interpolation steps for $S_q^r(\Delta^1)$.

steps will be applied. (If the number of lines in Step C or D below is nonpositive, then no points are chosen.)

Step A (*Starting step*): Choose $q+1$ disjoint line segments $a_1,\ldots,a_{q+1}$ in T. For $j=1,\ldots,q+1$, choose $q+2-j$ distinct points on a_j.

Step B: Choose $q-r$ disjoint line segments $b_1,\ldots,b_{q-r}$ in T. For $j=1,\ldots,q-r$, choose $q+1-r-j$ distinct points on b_j.

Step C: Choose $q-2r+[r/2]$ disjoint line segments $c_1,\ldots,c_{q-2r+[r/2]}$ in T. For $j=1,\ldots,q-2r$, choose $q+1-r-j$ distinct points on c_j and for $j=q-2r+1,\ldots,q-2r+[r/2]$ choose $2(q-j)-3r+1$ distinct points on c_j.

Step D: Choose $q-2r-1$ disjoint line segments $d_1,\ldots,d_{q-2r-1}$ in T. For $j=1,\ldots,q-2r-1$, choose $q-2r-j$ distinct points on d_j.

Given a Δ^1 triangulation, the above steps are applied to the triangles of Δ^1 as indicated in Fig. 10.

Hermite interpolation sets for $S_q^r(\Delta^i)$, $i=1,2$, are obtained by using the above Lagrange interpolation sets and by "taking limits". This means that Hermite interpolation sets are constructed by shifting the interpolation points to the vertices. More precisely, the Hermite interpolation conditions are obtained by considering a Lagrange interpolation set and letting certain points and line segments coincide. The corresponding Hermite interpolation conditions are as follows. If certain points on some line segment coincide, then directional derivatives along the line segment are considered, and if certain line segments coincide, then directional derivatives orthogonal to the line segment are considered.

For proving the approximation order of these interpolation methods, Nürnberger [118] introduced the principle of weak interpolation. The following results were proved by Nürnberger [118], respectively, Nürnberger and Walz [122] (see also Nürnberger [117]).

Theorem 6.1. *Let f be a sufficiently differentiable function. The* (*Hermite*) *interpolating spline s_f of f yields* (*nearly*) *optimal approximation order for $S_q^1(\Delta^1)$, $q\geqslant 4$, and $S_q^1(\Delta^2)$, $q\geqslant 2$.*

In particular, the approximation order of the interpolation methods is optimal for $S_q^1(\Delta^1)$, $q\geqslant 5$, and $S_q^1(\Delta^2)$, $q\geqslant 4$. In the remaining cases the interpolating spline yields approximation order q. Later, Davydov et al. [55] showed the following result by extending the weak interpolation principle and by using univariate Hermite–Birkhoff interpolation arguments.

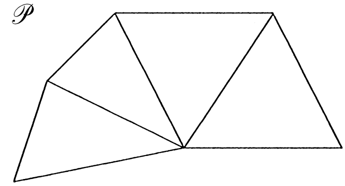

Fig. 11. A polyhedron $\mathscr{P}$.

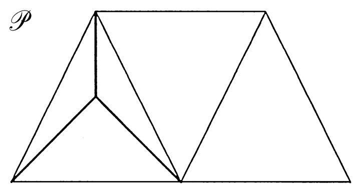

Fig. 12. Split of a triangle.

Theorem 6.2. *Let f be a sufficiently differentiable function and $r \geqslant 1$. The (Hermite) interpolation spline $s_f \in S_q^r(\Delta^1)$, $q \geqslant 3.5r + 1$, of f yields optimal approximation order.*

If we compare Theorem 6.2 with the general results on approximation order of these spaces in Section 4, we see that Theorem 6.2 is very close to what is possible, in general.

In 1996, the method of Nürnberger et al. [120,121] was extended by Adam [1] to cross-cuts partitions (see also [119]). The results in [1,120,121] were obtained by developing methods which were different from Bézier–Bernstein techniques.

In contrast to this, Bézier–Bernstein methods were used for constructing interpolation sets for spline spaces of degree less or equal to three with respect to Δ^i, $i = 1, 2$, and for proving their approximation order.

The following cases were considered between 1981 and 1994: $S_3^1(\Delta^1)$ by Sha [150], Bamberger [13], ter Morsche [114], $S_2^1(\Delta^2)$ by Sibson and Thompson [153], Sha [151], Sablonnière [136], Beatson and Ziegler [15], Chui and He [30], Rießinger [133], Zedek [161], Jeeawock-Zedek [84], Jeeawock–Zedek and Sablonnière [85], $S_3^1(\Delta^2)$ by Jeeawock–Zedek [83] and Lai [98].

We proceed by describing interpolation methods for more general classes than Δ^i, $i = 1, 2$. We start with methods, where triangulations suitable for interpolation by splines are constructed, then we describe interpolation methods for general classes of given triangulations.

In 1999, Nürnberger and Zeilfelder [125] developed an inductive method for constructing triangulations Δ that are suitable for Lagrange and Hermite interpolation by $S_q^r(\Delta)$, $q \geqslant 2r + 1$, $r = 1, 2$. Roughly speaking, the construction of these triangulations works as follows. By starting with one triangle, a polyhedron $\mathscr{P}$ as in Fig. 11 is added in each step to obtain a larger triangulation. The polyhedra, which result from triangulations of locally chosen scattered points, have two common edges with the boundary of the subtriangulation constructed so far. This construction is such that the corresponding splines can be extended in each step. For doing this in the case of C^2 splines, it may be necessary to split *some* of the triangles of Δ (see Fig. 12).

The dimension of the resulting spline spaces was determined by using Bézier–Bernstein methods. Lagrange and Hermite interpolation sets were constructed simultaneously. For doing this, suitable

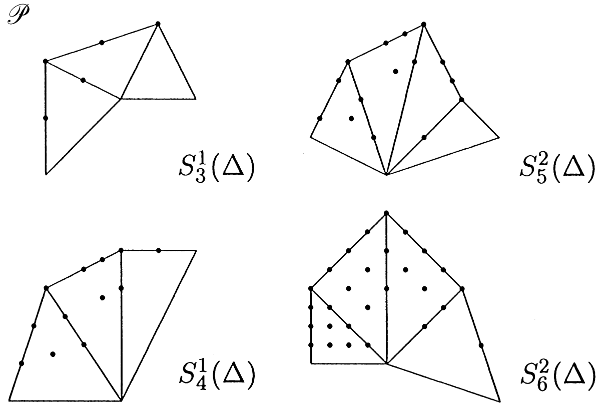

Fig. 13. Lagrange interpolation points in $\mathcal{P}$.

Table 1
Interpolation of Franke's testfunction $f(x, y)$ by splines s_f on a domain Ω containing the unit square

Spline space	Dimension	$\|f - s_f\|_\infty$
$S_3^1(\Delta)$	40 725	$1.62 \cdot 10^{-5}$
$S_4^1(\Delta)$	38 091	$7.80 \cdot 10^{-6}$
$S_6^2(\Delta)$	44 996	$5.83 \cdot 10^{-6}$
$S_7^2(\Delta)$	48 139	$5.30 \cdot 10^{-6}$

extensions of the above interpolation steps A, B, C, D were developed. Examples for Langrange interpolation points in $\mathcal{P}$ are given in Fig. 13. Again, Hermite interpolation sets are obtained by "taking limits". We note that the corresponding interpolating splines can be computed step by step. In each step only small linear systems of equations have to be solved.

A variant of this method can be used to construct triangulations Δ for arbitrary scattered points in the plane which is similar to *Delaunay triangulations* and to construct interpolation points for $S_q^r(\Delta)$, $r=1,2$, as described above. Concerning this variant, in some steps it may be necessary, when small angles appear (in the subtriangulation constructed so far), to add instead of a polyhedron $\mathcal{P}$ with at least two triangles (see Fig. 11) only one triangle which then has to be subdivided by a Clough–Tocher split.

Numerical tests with large numbers of interpolation conditions show that this interpolation method yields good approximations for $S_q^1(\Delta)$, $q \geqslant 4$, and $S_q^2(\Delta)$, $q \geqslant 7$. In order to obtain good approximations in the remaining cases (for nonuniform triangulations Δ) variants of this method were discussed in [125] (for some typical numerical results see Table 1). We note that in contrast to the macro element methods described in Section 5, by this method only some of the triangles have to be subdivided into three subtriangles, and Lagrange interpolation sets for spline spaces were constructed (see also Nürnberger and Zeilfelder [124]). Moreover, this method can be applied to certain classes of given triangulations Δ, in particular the class of triangulated quadrangulations which was investigated by Nürnberger and Zeilfelder [123].

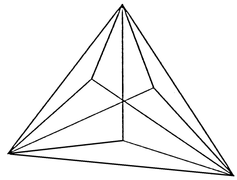

Fig. 14. A double Clough–Tocher split Δ_{DCT}.

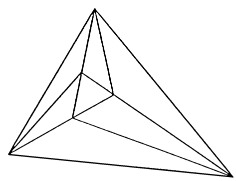

Fig. 15. Wang's special Morgan and Scott triangulation Δ_{MS}.

Earlier, Alfeld [2] and Wang [158] used completely different methods for Hermite interpolation by quintic C^2 splines. As in the finite element case (see Section 5), these approaches are based on a splitting procedure which is applied to *every* triangle of a given triangulation Δ. In [2], a double Clough–Tocher element Δ_{DCT} as in Fig. 14 was considered. Such an element consist of nine triangles. Alfeld [2] proposed to solve a linear system with approximately 100 equations for every triangle of Δ. This system is obtained by interpolating function values and derivatives up to order 2 at the vertices of Δ and certain additional conditions for the space $S_5^2(\Delta_{\mathrm{DCT}})$.

Wang [158] proposed to split *every* triangle of Δ into seven subtriangles. The three corresponding splitting points have to be chosen as in Fig. 15. This is a special case of Morgan and Scott's triangulation Δ_{MS} (see Section 3). In this case, the Hermite interpolation set for $S_5^2(\Delta_{\mathrm{MS}})$ consists of function values and derivatives up to order 2 at the vertices of Δ, of certain first and second order derivatives at interior points of every edge of Δ and function value and of one first derivative at each splitting point. Moreover, we mention that a Hermite interpolation set for $S_6^2(\Delta)$ was constructed by Gao [73], where Δ is a triangulated convex quadrangulation (see Section 4) that has to satisfy additional properties.

We proceed by describing interpolation methods for general classes of given triangulations.

Hermite interpolation sets for $S_q^1(\Delta)$, $q \geqslant 5$, where Δ is an arbitrary triangulation, were defined by Morgan and Scott [112] and Davydov [53]. The Hermite interpolation sets given in [53,112] cannot be transformed to Lagrange interpolation straightforwardly. In 1998, Davydov and Nürnberger [54] gave a different method for constructing explicit Hermite and Lagrange interpolation sets for $S_q^1(\Delta)$, $q \geqslant 5$. As shown in [54], this approach can also be applied to $S_4^1(\Delta)$, where Δ has to be slightly modified if exceptional constellations of triangles occur. Roughly speaking, the inductive method of constructing interpolation sets is as follows: In each step, a vertex of Δ and all triangles with this vertex having a common edge with the subtriangulation considered so far are added. Then the interpolation points are chosen locally on these triangles, where the number of interpolation points depends on so-called *semi-singular* vertices which may result from degenerate edges.

Earlier, Gao [72] defined a Hermite interpolation scheme for $S_4^1(\Delta)$ in the special case when Δ is an *odd degree triangulation*. These are triangulations, where every interior vertex has odd degree.

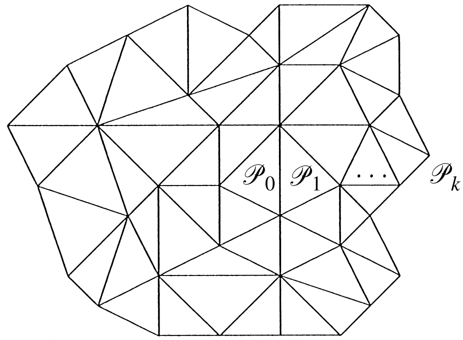

Fig. 16. A nested polygon triangulation.

Moreover, Chui and Hong [33] considered quasi-interpolation by $S_4^1(\Delta)$ (see also [32,81]). There it is shown that in order to achieve optimal approximation order of the quasi-interpolating spline certain edges of a given triangulation Δ have to be swapped.

We mention that interpolation by $S_4^1(\Delta)$ leads to an unsolved problem (see [7]; see also [115]) which we formulate as follows:

Problem. Given an arbitrary triangulation Δ. Does there exist a Hermite interpolation set for $S_4^1(\Delta)$ which includes function and gradient values for *all* vertices of Δ?

The results on Hermite interpolation by $S_4^1(\Delta)$ described above show that for these classes of triangulations, the answer to this question is yes.

The case of cubic C^1 splines, $S_3^1(\Delta)$, is more complex since not even the dimension of these spaces is known for arbitrary triangulations Δ (see Section 3).

In 1987, Gmelig Meyling [75] considered Lagrange interpolation by these spaces. There, a global method for constructing Lagrange interpolation sets involving function values at all vertices of a given triangulation Δ is proposed. This approach requires to solve a large linear system of equations, where it is not guaranteed that this system is solvable. In [75] some numerical experiments were given.

Davydov et al. [58] (see also [56,57]) investigated interpolation by $S_3^1(\Delta)$, where Δ is contained in the general class of the so-called nested polygon triangulations. These are triangulations consisting of nested closed simple polygons $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_k$ whose vertices are connected by line segments (see Fig. 16).

The construction of interpolation sets for $S_3^1(\Delta)$ in [58] is inductive by passing through the points of the nested polygons $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_k$ in clockwise order: In each step, a point of a nested polygon and all triangles with this vertex having a common edge with the subtriangulation considered so far are added. Then the interpolation points are chosen locally on these triangles, where the number of interpolation points is different if semi-singular vertices exist or not. In addition, it was proved in [58] that the number of interpolation points coincides with Schumaker's lower bound lb_3^1 (see Section 3), and therefore the dimension of these spaces is equal to lb_3^1.

The space $S_3^1(\Delta)$ is interesting for applications since the number of interpolation points is relatively small. It was remarked in [58] that numerical examples (with up to 100 000 interpolation conditions)

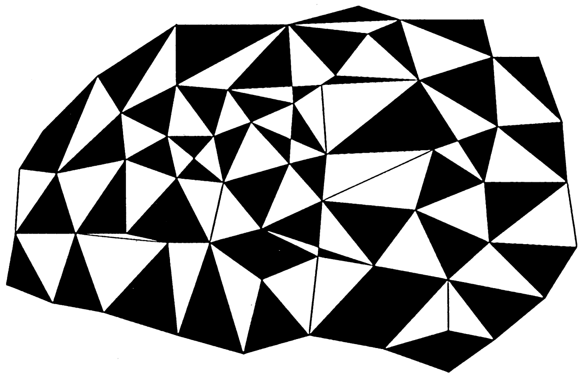

Fig. 17. Coloring of a given triangulation Δ.

show that in order to obtain good approximations, it is desirable to subdivide *some* of the triangles of Δ. The method of constructing interpolation points proposed in [58] also works for these modified triangulations.

We now discuss the problem of *local Langrange interpolation*. Concerning the construction and reconstruction of surfaces, it is sometimes desirable that only function values (and no (orthogonal) derivatives) are involved which for scattered data fitting can be computed approximately by using local methods. The classical method of Clough and Tocher [45] yields a local Hermite interpolation scheme for cubic C^1 splines on triangles which are splitted into three subtriangles (see Section 5). However, this Hermite interpolation scheme cannot be transformed straightforwardly into a local Lagrange interpolation scheme on a given triangulation.

The investigations of Nürnberger and Zeilfelder [127] show that local Lagrange interpolation schemes can be constructed by coloring the triangles of a given triangulation appropriately with two colors (see Fig. 17) and by subdividing the triangles of one color by a Clough–Tocher split. The authors developed an algorithm for constructing local Lagrange interpolation sets for spline spaces $S_3^1(\tilde{\Delta})$. This approach also works for higher degree C^1 spline spaces. Roughly speaking, the triangulations $\tilde{\Delta}$ are obtained by splitting half of the triangles of a given triangulation Δ into three subtriangles. Moreover, in this context "local" means that the corresponding fundamental functions $s_i \in S_3^1(\tilde{\Delta})$, $i = 1, \ldots, d$, determined by

$$s_i(z_j) = \delta_{i,j}, \quad j = 1, \ldots, d$$

have local support (here, $\delta_{i,j}$ denotes Kronecker's symbol).

The algorithm consists of two algorithmical steps. In the first step, Lagrange interpolation points are chosen on the edges of Δ such that interpolating spline is uniquely determined (only) on these edges. In the second step, the triangles are colored black and white (by a fast algorithm) such that at most two consecutive triangles (with a common edge) have the same color (see Fig. 17). Then the black triangles are subdivided by a Clough–Tocher split, and in the interior of the white triangles, additional Lagrange interpolation points are chosen. Then the interpolating spline is uniquely determined on the whole triangulation.

Since recently, Nürnberger et al. [128] are investigating the construction of local Lagrange interpolation schemes by cubic C^1 splines on convex quadrangulations. Since the classical Hermite interpolation scheme of Fraeijs de Veubeke [71] and Sander [140] (see Section 5) cannot be transformed into a local Lagrange interpolation scheme on a given quadrangulation, in [128] a coloring method will be used for local Lagrange interpolation. It turned out that it is a much more complex problem to develop a fast algorithm for coloring quadrangulations in a desired way than for triangulations. The coloring methods known in graph theory (see the book by Jensen and Toft [86]) do not provide such an algorithm. In [128], the authors are investigating this problem for general classes of quadrangulations.

We now discuss an open problem concerning interpolation by cubic C^1 splines.

Problem. Given an arbitrary triangulation Δ. Does there exist an interpolation set for $S_3^1(\Delta)$ which includes function values for *all* vertices of Δ?

Obviously, this question is strongly connected with the conjecture on the dimension of these spaces given in Section 3. We remark that for the interpolation sets constructed by Davydov et al. [58], it may happen that the function values at certain vertices are not included. On the other hand, for subclasses of nested polygon triangulations as described above function values at all vertices are included in the corresponding interpolation sets. This is also true for the local Lagrange interpolation sets constructed in [127].

By using Euler's formula (see Section 2), the lower bound in (3), lb_{2r+1}^r, for $S_{2r+1}^r(\Delta)$, $r \geqslant 1$, is

$$\mathrm{lb}_{2r+1}^r = \binom{r+1}{2} + (r+1)V_{\mathrm{I}} + \binom{r+2}{2} V_{\mathrm{B}} + \sum_{i=1}^{V_{\mathrm{I}}} \sigma_i. \tag{11}$$

Therefore, more general the question arises if an interpolation set for $S_{2r+1}^r(\Delta)$, $r \geqslant 1$, exists which includes function values at all vertices. Gmelig et al. [77] (see also [78]) proposed a method for constructing certain splines from $S_{2r+1}^r(\Delta)$, $r \geqslant 1$, that interpolate the function values at all vertices of a given triangulation Δ. This global method requires to solve a large linear system of equations, where it is not known if this system is solvable.

We turn now to the case $S_2^1(\Delta)$. As a consequence of their general method, Nürnberger and Zeilfelder [125] constructed Lagrange interpolation sets for $S_2^1(\Delta_Q)$, where Δ_Q is a triangulation of the following general type. By starting with one triangle, Δ_Q is described inductively as follows. Given a subtriangulation $\tilde{\Delta}_Q$, a triangle $\tilde{T}$ which has one common edge with $\tilde{\Delta}_Q$ is added. Then in clockwise order, quadrangles (with two diagonals) having one common edge with $\tilde{\Delta}_Q$ and triangles having one common point with $\tilde{\Delta}_Q$ are added, where the last quadrangle also has one common edge with $\tilde{T}$ (see Fig. 18). The resulting subtriangulation is again denoted by $\tilde{\Delta}_Q$. By proceeding with this method Δ_Q is finally obtained. The Lagrange interpolation set for $S_2^1(\Delta_Q)$ given in [125] consists of the vertices of Δ_Q (except the intersection points of the diagonals) together with three additional points in the starting triangle.

If in Δ_Q instead of quadrangles with two diagonals arbitrary quadrangles are considered, then for the quadrangles with only one diagonal no interpolation point can be chosen. In this case, no good approximations can be expected, in general (see the general comment on quadratic C^1 splines in Section 5). It was mentioned in [125] that in order to obtain good approximations, numerical tests (up to 40 000 Lagrange interpolation points) showed that the vertices of Δ_Q should be rather uniformly distributed.

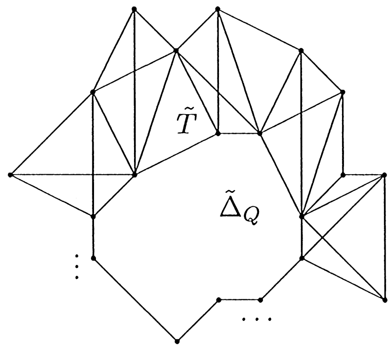

Fig. 18. The triangulation Δ_Q.

We close this section with the following problem (see also Nürnberger and Zeilfelder [126]).

Problem. Construct interpolation sets for $S_q^1(\Delta)$, $q = 2, 3$, for general classes of triangulations Δ.

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