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Bivariate splines on hexagonal lattice for digital image processing

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Abstract

In this paper, the applications of multi-variable spline functions in the digital signal processing was investigated further. Several two-dimensional spline functions were constructed for hexagonal lattice. They are actually variation of bivariate cardinal splines on regular triangular partition that were constructed earlier. Those bivariate cardinal splines have a number of desirable features which make it applicable for image processing and other areas. One feature lies in the fact that the coefficients of the interpolation representations with bivariate cardinal splines are basically the sample data at the interpolation points. The interpolations converge uniformly to the function being interpolated as the sampling increment approaches zero. Compare to the popular tensor product with the same degree of smoothness, our bivariate cardinal splines has the same degree of accuracy with smaller local support, lower degree of the polynomials. Therefore it should lead to the reduction of computation time and storage space. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

This work is the continuation of my paper "Bivariate Cardinal Spline Functions for Digital Signal Processing". In the former paper, two-dimensional cardinal spline functions on the regular triangular partition were constructed with excellent interpolation properties. It came to my attention that the hexagonal partition is frequently used in the digital signal processing. In fact, the hexagonal sampling is the most efficient sampling scheme for circularly band-limited signals [12,18]. Also mentioned in Mersereau's paper, since each sample location in a hexagonal raster has six nearest neighbors, each of which is the same distance away, hexagonal partition can be convenient for certain image processing problems, involving cluster separation, boundary tracing, etc. In this paper, cardinal splines on the hexagonal partition will be discussed. As before, those bivariate cardinal splines have the desirable feature that the function value is one at one grid point and zero at other grid points. Furthermore, as

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a special advantageous feature, the cardinal splines on the hexagonal partition has 12-fold symmetry. The interpolation representation, boundary conditions, and convergence properties will also be given.

2. Bivariate B-splines on a hexagonal partition

Let us first consider the rectangular area $\Omega = \{(x, y) \mid a \leqslant x \leqslant b; c \leqslant y \leqslant d\}$. The hexagonal partition is shown in the Fig. 1.

Let $h = (b - a)/M$, $l = (c - d)/N$; where M and N are both even positive integers, and $x_i = a + ih, y_j = c + jl$, $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$. Notice that partition points are only at (x_{2i}, y_{2j}) and (x_{2i+1}, y_{2j+1}) for the hexagonal lattice. This partition is denoted in this paper by Δ_3. The space of bivariate polynomial spline function with degree k and smoothness degree μ is denoted by $S_k^{\mu}(\Omega, \Delta_3)$. Just like in the case of regular triangular partition, it has been proved [9–11] that when $k > (3\mu+1)/2$, there exist bivariate B-splines in $S_k^{\mu}(\Omega, \Delta_3)$.

Let us start with $\mu = -1$ and the smallest possible $k = 0$. Like the univariate case, there are non-continuous bivariate B-splines on the partition Δ_3: the characteristic functions on each triangle cell of the partition. We can express two linearly independent ones as $B^{\alpha}(x, y) = \chi_{\alpha}(x, y)$ and $B^{\beta}(x, y) = \chi_{\beta}(x, y)$ with supports α and β as triangular cells of different shape.

For $\mu = 0$, $k = 1$, the B-spline $B_1(x, y)$ defined by

$$B_1(x, y) = \int_{-1/2}^{1/2} (\chi_{\alpha} + \chi_{\beta})(x + hv, y + lv)\,\mathrm{d}v \in S_1^0(\Omega, \Delta_3)$$

[9,10] has the minimal support of an hexagon on Δ_3. This B-spline has the value 1 at the center grid point and the value 0 at all other grid points. This is a very important property we will use later.

From those three B-splines, we can construct B-splines in $S_{3\mu+1}^{2\mu}(\Omega, \Delta_3)$ and $S_{3\mu+3}^{2\mu+1}(\Omega, \Delta_3)$ using a integration process [29] and a coordinate transformation.

Let

$$I_1[B_1](x, y) = \int_{-1/2}^{1/2} B_1(x + 2ht, y)\,\mathrm{d}t,$$

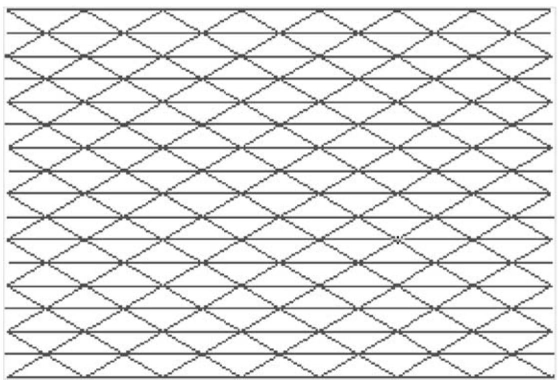

Fig. 1. The hexagonal partition Δ_3 on a rectangular area.

$$I_2[B_1](x,y)=\int_{-1/2}^{+1/2} B_1(x+hv,y+lv)\,\mathrm{d}v,$$

$$I_3[B_1](x,y)=\int_{-1/2}^{+1/2} B_1(x+hu,y-lu)\,\mathrm{d}u.$$

Then $B_{\mu+1}(x,y)=I_3I_2I_1[B_\mu](x,y)\in S^{2\mu}_{3\mu+1}(\Omega,\Delta_3)$, $B^{\alpha}_{\mu}(x,y)=I_3I_2I_1[B^{\alpha}_{\mu-1}](x,y)$, $B^{\beta}_{\mu}(x,y)=I_3I_2I_1$ $[B^{\beta}_{\mu-1}](x,y)\in S^{2\mu-1}_{3\mu}(\Omega,\Delta_3)$, for $\mu=1,2,\ldots$.

As for the approximation orders, we have the following theorem [2,7,8].

Theorem 2.1. *The approximation orders of the bivariate splines spaces* $S^{2\mu-1}_{3\mu}(\Omega,\Delta_3)$ *and* $S^{2\mu}_{3\mu+1}$ (Ω,Δ_3) *are* $2\mu+1,2\mu+2$, *respectively*, *for* $\mu=0,1,2,\ldots$.

3. Bivariate cardinal splines on the hexagonal partition

As discussed in the former paper, the interpolation of a function $f(x,y)\in l^p$ by B-splines in $S^{2\mu-1}_{3\mu}(\Omega,\Delta_3)$ or $S^{2\mu}_{3\mu+1}(\Omega,\Delta_3)$ can be implemented by finding the interpolation coefficients [3–6,8,9, 15–17,21–24]. However, other than $B_1(x,y)\in S^0_1(\Omega,\Delta_3)$, the amount of calculation involved in finding coefficients gets bigger as the partition gets finer. To solve this problem, several bivariate cardinal splines on hexagonal partitions will be given here. Because of the reason stated in the former paper, the emphasis will be given to $H_1(x,y):=B_1(x,y)\in S^0_1(\Omega,\Delta_3)$ and $H_2(x,y)\in S^1_4(\Omega,\Delta_3)$, $H_3(x,y)\in S^2_6(\Omega,\Delta_3)$ such that

$$H_k(0,0)=1,\qquad H_k(i,j)=0;\quad (i,j)\in\mathbf{Z}^2\backslash\{(0,0)\},\ \ k=2,3.$$

Let the supports of $H_1(x,y),H_2(x,y),H_3(x,y)$ be T_1,T_2,T_3, respectively.

If we let the support (which is a hexagon bounded by l_1: $y=(l/h)x-2l$, l_2: $y=-l$, l_3: $y=-(l/h)x-2l$, l_4: $y=(1/h)x+2l$, l_5: $y=l$, l_6: $y=-(l/h)x+2l$) of $H_1(x,y)$ be T_1, and A_i be the triangular cells of T_1 with the outboundary l_i, $i=1,1,\ldots$, then by a coordinate transformation, it is easy to get

$$H_1(x,y)=\begin{cases}1-\dfrac{x}{2h}+\dfrac{y}{2l} & \text{if } (x,y)\in A_1,\\[2mm] 1+\dfrac{y}{l} & \text{if } (x,y)\in A_2,\\[2mm] 1+\dfrac{x}{2h}+\dfrac{y}{2l} & \text{if } (x,y)\in A_3,\\[2mm] 1+\dfrac{x}{2h}-\dfrac{y}{2l} & \text{if } (x,y)\in A_4,\\[2mm] 1-\dfrac{y}{l} & \text{if } (x,y)\in A_5,\\[2mm] 1-\dfrac{x}{2h}-\dfrac{y}{2l} & \text{if } (x,y)\in A_6,\\[2mm] 0 & \text{elsewhere.}\end{cases}$$

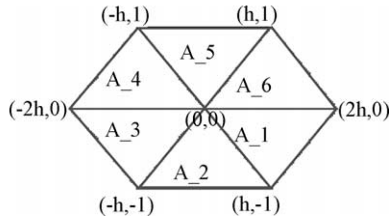

Fig. 2. Support T_1.

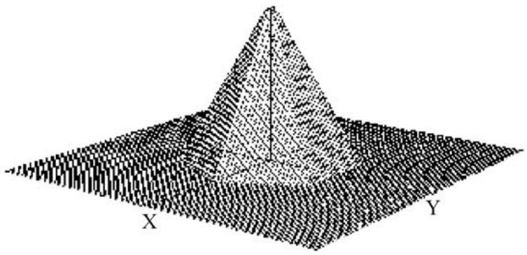

Fig. 3. The graph of $H_1(x,y)$ (Scale: $x:y:z=4:4:0.94$).

To construct a cardinal spline $H_2(x,y)\in S_4^1(\Omega,\Delta_3)$, let $H_0(x,y)$ be the characteristic function of the support T_1 in Fig. 2, i.e.

$$H_0(x,y)=\chi_{T_1}(x,y).$$

Then

$$H_2(x,y)=I_3I_2I_1[2H_1-H_0](x,y).$$

To construct a cardinal spline $H_3(x,y)\in S_6^2(\Omega,\Delta_3)$, the starting point is another spline $H_{13}(x,y)\in S_3^0(\Omega,\Delta_3)$. Let

$$H_{13}(x,y)=\begin{cases} -\dfrac{1}{4h^3l^3}(-2hl+xl-yh)(x^2l^2+3y^2h^2-xhl^2+yh^2l+2h^2l^2) & (x,y)\in A_1,\\ \dfrac{1}{2l^3h^2}(l+y)(3y^2h^2+x^2l^2+2yh^2l+2h^2l^2) & (x,y)\in A_2,\\ \dfrac{1}{4h^3l^3}(2hl+xl+yh)(x^2l^2+3y^2h^2+xhl^2+yh^2l+2h^2l^2) & (x,y)\in A_3,\\ \dfrac{1}{4h^3l^3}(2hl+xl-yh)(x^2l^2+3y^2h^2+xhl^2-yh^2l+2h^2l^2) & (x,y)\in A_4,\\ -\dfrac{1}{2l^3h^2}(-l+y)(3y^2h^2+x^2l^2-2yh^2l+2h^2l^2) & (x,y)\in A_5,\\ -\dfrac{1}{4h^3l^3}(-2hl+xl+yh)(x^2l^2+3y^2h^2-xhl^2-yh^2l+2h^2l^2) & (x,y)\in A_6. \end{cases}$$

Then

$$H_3(x,y) = I_3I_2I_1[16H_{13} - 15H_1](x,y).$$

$H_3(x,y)$ has the same support and symmetry as $H_2(x,y)$ has.

It is obvious that the supports T_1 and T_2 of our cardinal splines have excellent symmetrical properties. In fact, the cardinal splines (Fig. 3) $H_1(x,y)$, $H_2(x,y)$, $H_3(x,y)$ have excellent symmetry, too.

Theorem 3.1. *If we let $h=1/2, l=\sqrt{3}/2$, then the hexagonal partition points draw regular hexagons. In this case, if $H(x,y)$ has 12-fold symmetry, i.e.*

$$\begin{aligned}
H(x,y) = H(-x,y) &= H\left(\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right), \left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)\right) \\
&= H\left(\left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right), \left(-\frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)\right) \\
&= H\left(\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right), \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)\right) \\
&= H\left(\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right), \left(-\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)\right) = H(x,-y),
\end{aligned}$$

then

$$G(x,y) = I_3I_2I_1[H](x,y)$$

has 12-fold symmetry.

Proof. Let $(X,Y) = \left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y + \frac{1}{2}u + \frac{1}{2}v + t, \frac{\sqrt{3}}{2}x + \frac{1}{2}y - \frac{1}{2}\sqrt{3}u + \frac{1}{2}\sqrt{3}v\right)$. Then

$$\begin{aligned}
&\left(-\frac{1}{2}X + \frac{\sqrt{3}}{2}Y, \frac{\sqrt{3}}{2}X + \frac{1}{2}Y\right) \\
&= \left(-\frac{1}{2}\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y + \frac{1}{2}u + \frac{1}{2}v + t\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y - \frac{1}{2}\sqrt{3}u + \frac{1}{2}\sqrt{3}v\right),\right. \\
&\left.\frac{\sqrt{3}}{2}\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y + \frac{1}{2}u + \frac{1}{2}v + t\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y - \frac{1}{2}\sqrt{3}u + \frac{1}{2}\sqrt{3}v\right)\right) \\
&= \left(x - u + \frac{1}{2}v - \frac{1}{2}t, y + \frac{1}{2}\sqrt{3}v + \frac{1}{2}\sqrt{3}t\right).
\end{aligned}$$

Using definition of $G(x,y)$, the symmetry properties of $H(x,y)$ and variable substitution

$$G\left(-\frac{1}{2}x+\frac{\sqrt{3}}{2}y,\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)=I_3I_2I_1[H]\left(-\frac{1}{2}x+\frac{\sqrt{3}}{2}y,\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)$$

$$=\int_{-1/2}^{+1/2}\int_{-1/2}^{+1/2}\int_{-1/2}^{1/2}H(X,Y)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$=\int_{-1/2}^{+1/2}\int_{-1/2}^{+1/2}\int_{-1/2}^{1/2}H\left(-\frac{1}{2}X+\frac{\sqrt{3}}{2}Y,\frac{\sqrt{3}}{2}X+\frac{1}{2}Y\right)\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$=(s=-t,w=-u)=\int_{-1/2}^{+1/2}\int_{-1/2}^{+1/2}\int_{-1/2}^{1/2}H\left(x+\frac{1}{2}s+\frac{1}{2}v+w,y-\frac{\sqrt{3}}{2}s+\frac{\sqrt{3}}{2}v\right)\mathrm{d}s\,\mathrm{d}w\,\mathrm{d}v$$

$$=I_3I_2I_1[H](x,y)=G(x,y).$$

By the similar steps, we can show that

$$G(x,y)=G\left(\left(-\frac{1}{2}x-\frac{\sqrt{3}}{2}y\right),\left(-\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)\right)$$

$$=G\left(\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right),\left(\frac{\sqrt{3}}{2}x-\frac{1}{2}y\right)\right)$$

$$=G\left(\left(\frac{1}{2}x-\frac{\sqrt{3}}{2}y\right),\left(-\frac{\sqrt{3}}{2}x-\frac{1}{2}y\right)\right),$$

$$G(-x,y)=G(x,y)\quad\text{and}\quad G(x,-y)=G(x,y).\qquad\square$$

Corollary. *If we let* $h=\frac{1}{2}, l=\sqrt{3}/2$, *then* $H_i(x,y)$ *has a* 12*-fold symmetry. That means*

$$H_i(x,y)=H_i(-x,y)=H_i\left(\left(-\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right),\left(\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)\right)$$

$$=H_i\left(\left(-\frac{1}{2}x-\frac{\sqrt{3}}{2}y\right),\left(-\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)\right)$$

$$=H_i\left(\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right),\left(\frac{\sqrt{3}}{2}x-\frac{1}{2}y\right)\right)$$

$$=H_i\left(\left(\frac{1}{2}x-\frac{\sqrt{3}}{2}y\right),\left(-\frac{\sqrt{3}}{2}x-\frac{1}{2}y\right)\right)=H_i(x,-y),$$

for $i=1,2,3$.

Proof. It is straight forward to check that

$$
\begin{aligned}
H_1(x,y) = H_1(-x,y) &= H_1\left(\left(-\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right),\left(\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)\right)\\
&= H_1\left(\left(-\frac{1}{2}x-\frac{\sqrt{3}}{2}y\right),\left(-\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right)\right)\\
&= H_1\left(\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right),\left(\frac{\sqrt{3}}{2}x-\frac{1}{2}y\right)\right)\\
&= H_1\left(\left(\frac{1}{2}x-\frac{\sqrt{3}}{2}y\right),\left(-\frac{\sqrt{3}}{2}x-\frac{1}{2}y\right)\right) = H_1(x,-y).
\end{aligned}
$$

Hence by the theorem, so does $H_i(x,y)$, $i=2,3$.

Consequently, if the image we process has the 12-fold symmetry property, we just need one-twelfth information. Furthermore, if the image has 6-fold or 4-fold symmetry, the corresponding symmetry of $H_i(x,y)$ $(i=1,2,3)$ can be used to save the time and space.

4. Boundary conditions for interpolations on the hexagonal partition

First, the consideration will be given to the interpolation in the rectangular area Ω. The hexagonal partition on Ω and regular triangular partition on Ω are different. So here are different formulas from the former paper.

Because that $H_1((x-x_{2i})/h,(y-y_{2j})/l)|_{(x,y)\in\Omega}\neq 0$ for $i=0,1,2,\ldots,M/2$, $j=0,1,2,\ldots,N/2$, and $H_1((x-x_{2i+1})/h,(y-y_{2j+1})/l)|_{(x,y)\in\Omega}\neq 0$ for $i=-1,0,1,2,\ldots,M/2$, $j=0,1,2,\ldots,N/2-1$, the interpolation function can be expressed as

$$
g_1(x,y) := \sum_{i=-1}^{M/2}\sum_{j=0}^{N/2-1} c_{2i+12j+1}H_1\left(\frac{x-x_i}{h},\frac{y-y_j}{l}\right) + \sum_{i=0}^{M/2}\sum_{j=0}^{N/2} c_{2i2j}H_1\left(\frac{x-x_i}{h},\frac{y-y_j}{l}\right),
$$

where

$$
\begin{aligned}
&c_{ij} = f_{ij}, \quad i=0,1,2,\ldots,M,\ j=0,1,2,\ldots,N,\\
&c_{-1,2j+1} = 3f_{1,2j+1}-3f_{3,2j+1}+f_{5,2j+1}, \quad j=0,1,2,\ldots,(N/2)-1,\\
&c_{M+1,2j+1} = 3f_{M-1,2j+1}-3f_{M-3,2j+1}+f_{M-5,2j+1}, \quad j=0,1,2,\ldots,(N/2)-1.
\end{aligned}
$$

Since $H_2(x,y)$, $H_3(x,y)$ have larger supports, $H_k((x-x_i)/h,(y-y_j)/l)|_{(x,y)\in\Omega}\neq 0$ for $i=-3,-2,\ldots,M+3$, $j=-3,-2,\ldots,N+3$, the interpolation function is

$$
\Phi_k[f](x,y) := \sum_{i=-2}^{M/2+1}\sum_{j=-2}^{N/2+1} c_{2i+12j+1}H_k\left(\frac{x-x_i}{h},\frac{y-y_j}{l}\right) + \sum_{i=-1}^{M/2+1}\sum_{j=-1}^{N/2+1} c_{2i2j}H_k\left(\frac{x-x_i}{h},\frac{y-y_j}{l}\right),
$$

$k=2,3,$

where $c_{ij} = f_{ij}$ for $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$, and for boundary coefficients we let $c_{-k,2j+1}$, and $c_{M+k,2j+1}$ for $j = 0, 1, 2, \ldots, N/2$, $k = 1, 2, 3$; $c_{2i+1,-k}$ and $c_{2i+1,N+k}$ for $i = 0, 1, 2, \ldots, M/2$, $k = 1, 2, 3$ be linear combinations of f_{ij}. (For details, please contact the author.) The coefficients are chosen in a way that if $p_2(x, y)$ is a bivariate polynomial of total degree 2, then for $k = 2, 3$, by direct calculation,

$$\Phi_k[p_2](x, y) := \sum_{i=-1}^{M+1} \sum_{j=-1}^{N+1} p_2(x_i, y_j) H_k \left(\frac{x - x_i}{h}, \frac{y - y_j}{l} \right) = p_2(x, y).$$

Hence, we have the following theorem.

Theorem 4.1. *The approximation order of the bivariate cardinal splines spaces* $S_4^1(\Omega, \Delta_3)$ *is* 3.

In the digital signal processing, the hexagonal partition is very useful to the circular band-limited sample functions. The rest of this section will be contributed towards circular area Θ with the hexagonal partition Δ_3.

Let the center of the circle be (x_0, y_0) and the radius be R. Let M, N be positive integers and $h = R/M$, $l = R/N$, $x_i = x_0 + ih$, $y_j = y_0 + jl$, $i = -M, -M+1, \ldots, 0, 1, \ldots, M$, $j = -N, -N+1, \ldots, 0, 1, \ldots, N$. The interpolation expression can be denoted as follows:

$$\Phi_1[f](x, y) := \sum_{i^2+j^2<(R+2)^2} c_{ij} H_1 \left(\frac{x - x_i}{h}, \frac{y - y_j}{l} \right),$$

where

$$\begin{aligned}
&c_{ij} = f_{ij} \quad \text{if } (x_i, y_j) \in \Theta \text{ which means } (ih)^2 + (jl)^2 \leqslant R^2,\ -M \leqslant i \leqslant M,\ -N \leqslant j \leqslant N,\\
&c_{ij} = 2f_{i-1j-1} - f_{i-2j-2} \text{ if } (x_i, y_j) \notin \Theta \quad \text{but} \quad (x_{i-1}, y_{j-1}) \in \Theta,\\
&c_{ij} = 2f_{i-2j} - f_{i-4j} \quad \text{if } (x_{i-1}, y_{j-1}) \notin \Theta \quad \text{but} \quad (x_{i-2}, y_j) \in \Theta,\ 0 \leqslant i \leqslant M,\ 0 \leqslant j \leqslant N,\\
&c_{ij} = 2f_{i+1j-1} - f_{i+2j-2} \text{ if } (x_i, y_j) \notin \Theta \quad \text{but} \quad (x_{i+1}, y_{j-1}) \in \Theta,\\
&c_{ij} = 2f_{i+2j} - f_{i+4j} \quad \text{if } (x_{i+1}, y_{j-1}) \notin \Theta \quad \text{but} \quad (x_{i+2}, y_j) \in \Theta,\ -M \leqslant i \leqslant 0,\ 0 \leqslant j \leqslant N,\\
&c_{ij} = 2f_{i+1j+1} - f_{i+2j+2} \text{ if } (x_i, y_j) \notin \Theta \quad \text{but} \quad (x_{i+1}, y_{j+1}) \in \Theta,\\
&c_{ij} = 2f_{i+2j} - f_{i+4j} \quad \text{if } (x_{i+1}, y_{j+1}) \notin \Theta \quad \text{but} \quad (x_{i+2}, y_j) \in \Theta,\ -M \leqslant i \leqslant 0,\ -N \leqslant j \leqslant 0,\\
&c_{ij} = 2f_{i-1j+1} - f_{i-2j+2} \text{ if } (x_i, y_j) \notin \Theta \quad \text{but} \quad (x_{i-1}, y_{j+1}) \in \Theta,\\
&c_{ij} = 2f_{i-2j} - f_{i-4j} \quad \text{if } (x_{i-1}, y_{j+1}) \notin \Theta \quad \text{but} \quad (x_{i-2}, y_j) \in \Theta,\ 0 \leqslant i \leqslant M,\ -N \leqslant j \leqslant 0.
\end{aligned}$$

Because of the curvature of the boundary of Θ, the choices for the boundary condition are cumbersome to write. However, the comparison is easy to do on the computer. Furthermore, the choices guarantee that for a linear polynomials $p_1(x, y)$,

$$\Phi_1[p_1](x, y) = p_1(x, y).$$

Similarly, the interpolation expressions with $H_k(x, y)$ can be denoted as follows:

$$\Phi_k[f](x, y) := \sum_{i^2+j^2<(R+2)^2} c_{ij} H_k\left(\frac{x - x_i}{h}, \frac{y - y_j}{l}\right),$$

where

$$c_{ij} = f_{ij} \quad \text{if } (x_i, y_j) \in \Theta \text{ which means } (ih)^2 + (jl)^2 \leqslant R^2,\ -M \leqslant i \leqslant M,\ -N \leqslant j \leqslant N.$$

As for the boundary coefficients, they are chosen in a way to guarantee that for a quadratic polynomials $p_2(x, y)$ in two variables (For details, please contact the author),

$$\Phi_k[p_2](x, y) = p_2(x, y).$$

5. Comparison with the tensor product

It is possible to apply tensor product of univariate spline functions to the hexagonal partition in the plane [1,12–14,19,20,25–28]. However, if the univariate B-splines are used, the tensor products have larger support than our bivariate cardinal splines here. Furthermore, the tensor product formed by B-splines will not have the main feature that our bivariate cardinal splines have and the computation involved in the interpolation problems will be a big task. If the univariate cardinal splines are used to form the tensor products, the support will be bigger than ours and the degree of polynomials used will be higher than ours. In conclusion, the two-dimensional cardinal spline function constructed in this paper will reduce computation complexity and time and save storage space when two-dimensional interpolation problems are considered.

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