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INTERPOLATION (from Lat. *interpolare*, to alter, or insert something fresh, connected with *polire*, a polish), in mathematics, the process of obtaining intermediate terms of a series of which particular terms only are given. The cubes, for instance, shown in the second column of the accompanying table, may be regarded as terms of a series, and the cube of a fractional number, not exceeding the last number in the first column, may be found by interpolation. The process of obtaining the cube of a number exceeding the last number in the first column would be *extrapolation*; the formulae which apply to interpolation apply in theory to extrapolation, but in practice special precautions as to accuracy are necessary. The present article deals only with interpolation.

Number.	Cube of Number.
0	0
1	1
2	8
3	27
4	64
5	125
6	216
.	.
.	.
.	.

The term is usually limited to those cases in which there are two quantities, x and u, which are so related that when x has any arbitrary value, lying perhaps between certain limits, the value of u is determinate. There is a given series of associated values of u and of x, and interpolation consists in determining the value of u for any arbitrary value of x, or the value of x for any arbitrary value of u, lying between two of the values in the series. Either of the two quantities may be regarded as a function of the other; it is convenient to treat one, x, as the "independent variable," the other, u, being treated as the "dependent variable," *i.e.* as a function of x. If, as is usually the case, the successive values of one of the quantities proceed by a constant increment, this quantity is to be regarded as the independent variable. The two series of values may be tabulated, those of *x* being placed in a column (or row), and those of *u* in a parallel column (or row); *u* is then said to be *tabulated in terms of x*. The independent variable *x* is called the *argument*, and the dependent variable *u* is called the *entry*. Interpolation, in the ordinary sense, consists in determining the value of *u* for a value of *x* intermediate between two values appearing in the table. This may be described as *direct interpolation*, to distinguish it from *inverse interpolation*, which consists in determining the value of *x* for a value of *u* intermediate between two in the table. The methods employed can be extended to cases in which the value of *u* depends on the values of two or more independent quantities *x*, *y*,...

In the ordinary case we may regard the values of *x* as measured along a straight line OX from a fixed point O, so that to any value of *x* there corresponds a point on the line. If we represent the corresponding value of *u* by an ordinate drawn from the line, the extremities of all such ordinates will lie on a curve which will be the graph of *u* with regard to *x*. Interpolation therefore consists in determining the length of the ordinate of a curve occupying a particular position, when the lengths of ordinates occupying certain specified positions are known. If *u* is a function of two variables, *x* and *y*, we may similarly represent it by the ordinate of a surface, the position of the ordinate being determined by the values of *x* and of *y* jointly.

The series or tables to which interpolation has to be applied may for convenience be regarded as falling into two main groups. The first group comprises mathematical tables, *i.e.* tables of mathematical functions; in the case of such a table the value of the function u for each tabulated value of *x* is calculated to a known degree of accuracy, and the degree of accuracy of an interpolated value of u can be estimated. The second group comprises tables of values which are found experimentally, *e.g.* values of a physical quantity or of a statistical ratio; these values are usually subject to certain "errors" of observation or of random selection (see PROBABILITY). The methods of interpolation are usually the same in the two groups of cases, but special considerations have to be taken into account in the second group. The line of demarcation of the two groups is not absolutely fixed; the tables used by actuaries, for instance, which are of great importance in practical life, are based on statistical observations, but the tables formed directly from the observations have been "smoothed" so as to obtain series which correspond in form to the series of values of mathematical functions.

It must be assumed, at any rate in the case of a mathematical function, that the "entry" *u* varies continuously with the "argument" *x*, *i.e.* that there are no sudden breaks, changes of direction, &c., in the curve which is the graph of *u*.

Various methods of interpolation are described below. The simplest is that which uses the *principle of proportional parts*; and mathematical tables are usually arranged so as to enable this method to be employed. Where this is not possible, the methods are based either on the use of Taylor's Theorem, which gives a formula involving differential coefficients (see INFINITESIMAL CALCULUS), or on the properties of finite differences (see DIFFERENCES, CALCULUS OF). Taylor's Theorem can only be applied directly to a known mathematical function; but it can be applied indirectly, by means of finite differences, in various cases where the form of the function expressing u in terms of x is unknown; and even where the form of this function is known it is sometimes more convenient to determine the differential coefficients by means of the differences than to calculate them directly from their mathematical expressions. Finally, there are cases where we cannot even employ finite-difference formulae directly. In these cases we must adopt some special method; *e.g.* we may instead of u tabulate some function of u, such as its logarithm, which is found to be amenable to ordinary processes, then determine the value of this function corresponding to the particular value of x, and thence determine the corresponding value of u itself.

In considering methods of interpolation, it will be assumed, unless the contrary is stated, that the values of x proceed by a constant increment, which will be denoted by h.

In order to see what method is to be employed, it is usually necessary to arrange the given series of values of u in the form of a table, as explained above, and then to take the successive *differences* of u. The differences of the successive values of u are called its *first differences*; these form a new series, the first differences of which are the *second differences* of u; and so on. The systems of notation of the differences are explained briefly below. For the fuller discussion, reference should be made to Differences, Calculus of.

I. Interpolation from Mathematical Tables

A. Direct Interpolation.

1. *Interpolation by First Differences.*—The simplest cases are those in which the first difference in u is constant, or nearly so. For example:—

Example 1.—($u = \log_{10}x$).

x.	u.	1st Diff.
		+
4.341	.6375898	
		1000
4.342	.6376898	
		1000
4.343	.6377898	
		1000
4.344	.6378898	
		1000
4.345	.6379898	

Example 2.—($u = \log_{10}x$).

x.	u.	1st Diff.
		+
7.40	.86923	
		59
7.41	.86982	
		58
7.42	.87040	
		59
7.43	.87099	
		58
7.44	.87157	

In Example 1 the first difference of u corresponding to a difference of $h \equiv .001$ in x is .0001000; but, since we are working throughout to seven places of decimals, it is more convenient to write it 1000. This system of ignoring the decimal point in dealing with differences will be adopted throughout this article. To find u for an intermediate value of x we assume the principle of proportional parts, *i.e.* we assume that the difference in u is proportional to the difference in x. Thus for $x = 4.342945$ the difference in u is .945 of 1000 = 945, so that u is .6376898 + .0000945 = .6377843. For $x =$ 4.34294482 the difference in u would be 944.82, so that the value of u would apparently be .6376898 + .000094482 = .637784282. This, however, would be incorrect. It must be remembered that the values of u are only given "correct to seven places of decimals," *i.e.* each tabulated value differs from the corresponding true value by a *tabular error* which may have any value up to $\pm \frac{1}{2}$ of .0000001; and we cannot therefore by interpolation obtain a result which is correct to nine places. If the interpolated value of u has to be used in calculations for which it is important that this value should be as accurate as possible, it may be convenient to retain it temporarily in the form .6376898 + 944 82 = .6377842 82 or $.6376898 + 944^{82} = .6377842^{82}$; but we must ultimately return to the seven-place arrangement and write it as .6377843. The result of interpolation by first difference is thus usually subject to two inaccuracies, the first being the tabular error of u itself, and the second being due to the necessity of adjusting the final figure of the added (proportional) difference. If the tabulated values are correct to seven places of decimals, the interpolated value, with the final figure adjusted, will be within .0000001 of its true value.

In Example 2 the differences do not at first sight appear to run regularly, but this is only due to the fact that the final figure in each value of *u* represents, as explained in the last paragraph, an approximation to the true value. The general principle on which we proceed is the same; but we use the actual difference corresponding to the interval in which the value of *x* lies. Thus for $x = 7.41373$ we should have $u = .86982 + (.373 \text{ of } 58) = .87004$; this result being correct within .00001.

2. *Interpolation by Second Differences.*—If the consecutive first differences of *u* are not approximately equal, we must take account of the next order of differences. For example:—

Example 3.—($u = \log_{10} x$).

x.	*u*.	1st Diff.	2nd Diff.
6.0	.77815		
		+718	
6.1	.78533		−12
		+706	
6.2	.79239		−11
		+695	
6.3	.79934		−11
		+684	
6.4	.80618		−11
		+673	
6.5	.81291		

In such a case the *advancing-difference* formula is generally used. The notation is as follows. The series of values of x and of u are respectively $x_0, x_1, x_2, \ldots$ and $u_0, u_1, u_2, \ldots$; and the successive differences of u are denoted by $\Delta u, \Delta^2 u, \ldots$ Thus Δu_0 denotes $u_1 - u_0$, and $\Delta^2 u_0$ denotes $\Delta u_1 - \Delta u_0 = u_2 - 2u_1 + u_0$. The value of x for which u is sought is supposed to lie between x_0 and x_1. If we write it equal to $x_0 + \theta(x_1 - x_0) = x_0 + \theta h$, so that θ lies between 0 and 1, we may denote it by x_θ, and the corresponding value of u by u_θ. We have then

$$u\theta = u_0 + \theta\Delta u_0 - \frac{\theta\,(1-\theta)}{2!}\Delta^2 u_0 + \frac{\theta\,(1-\theta)\,(2-\theta)}{3!}\Delta^3 u_0 \quad (1) \ldots$$

Tables of the values of the coefficients of $\Delta^2 u_0$ and $\Delta^3 u_0$ to three places of decimals for various values of θ from 0 to 1 are given in the ordinary collections of mathematical tables; but the formula is not really convenient if we have to go beyond $\Delta^2 u_0$, or if $\Delta^2 u_0$ itself contains more than two significant figures.

To apply the formula to Example 3 for $x = 6.277$, we have $\theta = .77$, so that $u_\theta = .79239 + (.77 \text{ of } 695) - (.089 \text{ of } -11) = .79239 + 535\ 15 + 0\ 98 = .79775$.

Here, as elsewhere, we use two extra figures in the intermediate calculations, for the purpose of adjusting the final figure in the ultimate result.

3. *Taylor's Theorem.*—Where differences beyond the second are involved, Taylor's Theorem is useful. This theorem (see INFINITESIMAL CALCULUS) gives the formula

$$u_\theta = u_0 + c_1\theta + c_2\frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \qquad (2),$$

where, c_1, c_2, c_3, . . . are the values for $x = x_0$ of the first, second, third, . . . differential coefficients of u with regard to x. The values of c_1, c_2, . . . can occasionally be calculated from the analytical expressions for the differential coefficients of u; but more generally they have to be calculated from the tabulated differences. For this purpose *central-difference* formulae are the best. If we write

$$\left.\begin{array}{r}\mu\delta u_0 = \tfrac{1}{2}(\Delta u_0 + \Delta u_{-1}) \\ \delta^2 u_0 = \Delta^2 u_{-1} \\ \mu\delta^3 u_0 = \tfrac{1}{2}(\Delta^3 u_{-1} + \Delta^3 u_{-2}) \\ \&c.\end{array}\right\} \qquad (3),$$

so that, if (as in §§ 1 and 2) each difference is placed opposite the space between the two quantities of which it is the difference, the expressions $\delta^2 u_0$, $\delta^4 u_0$, . . . denote the differences of even order in a horizontal line with u_0, and $\mu\delta u_0$, $\mu\delta^3 u_0$, . . . denote the means of the differences of odd order immediately below and above this line, then (see DIFFERENCES, CALCULUS OF) the values of c_1, c_2, . . . are given by

$$\left.\begin{aligned}
c_1 &= \mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \tfrac{1}{140}\mu\delta^7 u_0 + \ldots \\
c_2 &= \delta^2 u_0 - \tfrac{1}{12}\delta^4 u_0 + \tfrac{1}{90}\delta^6 u_0 - \tfrac{1}{560}\delta^8 u_0 + \ldots \\
c_3 &= \mu\delta^3 u_0 - \tfrac{1}{4}\mu\delta^5 u_0 + \tfrac{7}{120}\mu\delta^7 u_0 - \ldots \\
c_4 &= \delta^4 u_0 - \tfrac{1}{6}\delta^6 u_0 + \tfrac{7}{240}\delta^8 u_0 - \ldots \\
c_5 &= \mu\delta^5 u_0 - \tfrac{1}{3}\mu\delta^7 u_0 + \ldots \\
c_6 &= \delta^6 u_0 - \tfrac{1}{4}\delta^8 u_0 + \ldots \\
&\;\;\vdots \qquad \vdots
\end{aligned}\right\} \quad (4).$$

If a calculating machine is used, the formula (2) is most conveniently written

$$\left.\begin{aligned}
u_\theta &= u_0 + P_1\theta \\
P_1 &= c_1 + \tfrac{1}{2}P_2\theta \\
P_2 &= c_2 + \tfrac{1}{3}P_3\theta \\
&\;\;\vdots \qquad \vdots
\end{aligned}\right\} \quad (5).$$

Using θ as the multiplicand in each case, the successive expressions ... P_3, P_2, P_1, u_θ are easily calculated.

As an example, take $u = \tan x$ to five places of decimals, the values of x proceeding by a difference of 1°. It will be found that the following is part of the table:—

Example 4.—($u = \tan x$).

x.	*u*.	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.
		+	+	+	+
			732		16
65°	2.14451				
		10153		96	
66°	2.24604		828		19
		10981		115	
67°	2.35585		943		18

To find u for $x = 66° 23'$, we have $\theta = 23/60 = .3833333$. The following shows the full working: in actual practice it would be abbreviated. The operations commence on the right-hand side. It will be noticed that two extra figures are retained throughout.

u_0.	$\mu\delta u_0$.	$\delta^2 u_0$.	$\mu\delta^3 u_0$.	$\delta^4 u_0$.
2.24604	$+10567^{00}$	$+828^{00}$	$+105^{50}$	$+19^{00}$
	$-\ 17^{58}$	$-\ 1^{58}$		
	$c_1 = +10549^{42}$	$c_2 = +826^{42}$	$c_3 = +105^{50}$	$c_4 = +19^{00}$
$P_1\theta = +4105^{67}$	$\frac{1}{2}P_2\theta = +161^{02}$	$\frac{1}{3}P_3\theta = +13^{71}$	$\frac{1}{8}c_4\theta = +1^{82}$	
$u_\theta = 2.28710$	$P_1 = +10710^{44}$	$P_2 = +840^{13}$	$P_3 = +107^{32}$	

The value 2.2870967, obtained by retaining the extra figures, is correct within .7 of .00001 (§ 8), so that 2.28710 is correct within .00001 1.

In applying this method to mathematical tables, it is desirable, on account of the tabular error, that the differences taken into account in (4) should end with a difference of even order. If, *e.g.* we use $\mu\delta^3 u_0$ in calculating c_1 and c_3, we ought also to use $\delta^4 u_0$ for calculating c_2 and c_4, even though the term due to $\delta^4 u_0$ would be negligible if $\delta^4 u_0$ were known exactly.

4. *Geometrical and Algebraical Interpretation.*—In applying the principle of proportional parts, in such a case as that of Example 1, we in effect treat the graph of u as a straight line. We see that the extremities of a number of consecutive

ordinates lie approximately in a straight line: *i.e.* that, if the values are correct within $\pm\frac{1}{2}\rho$, a straight line passes through points which are within a corresponding distance of the actual extremities of the ordinates; and we assume that this is true for intermediate ordinates. Algebraically we treat u as being of the form A + Bx, where A and B are constants determined by the values of u at the extremities of the interval through which we interpolate. In using first and second differences we treat u as being of the form A + Bx + Cx^2; *i.e.* we pass a parabola (with axis vertical) through the extremities of three consecutive ordinates, and consider that this is the graph of u, to the degree of accuracy given by the data. Similarly in using differences of a higher order we replace the graph by a curve whose equation is of the form $u = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \ldots$ The various forms that interpolation-formulae take are due to the various principles on which ordinates are selected for determining the values of A, B, C . . .

B. *Inverse Interpolation.*

5. To find the value of x when u is given, *i.e.* to find the value of θ when u_θ is given, we use the same formula as for direct interpolation, but proceed (if differences beyond the first are involved) by successive approximation. Taylor's Theorem, for instance, gives

$$\theta = (u_\theta - u_0) \div \left(c_1 + c_2\frac{\theta}{2!} + \ldots\right)$$

$$= (u_\theta - u_0) \div \mathrm{P}_1 \qquad (6).$$

We first find an approximate value for θ: then calculate P_1, and find by (6) a more accurate value of θ; then, if necessary, recalculate P_1, and thence θ, and so on.

II. Construction of Tables by Subdivision of Intervals

6. When the values of u have been tabulated for values of x proceeding by a difference h, it is often desirable to deduce a table in which the differences of x are h/n, where n is an integer.

If n is even it may be advisable to form an intermediate table in which the intervals are $\frac{1}{2}h$. For this purpose we have

$$u_{\frac{1}{2}} = \tfrac{1}{2}(U_0 + U_1) \tag{7}$$

where

$$\begin{aligned} U &= u - \tfrac{1}{8}\delta^2 u + \tfrac{3}{128}\delta^4 u - \tfrac{5}{1024}\delta^6 u + \dots \\ &= u - \tfrac{1}{8}[\delta^2 u - \tfrac{3}{16}\{\delta^4 u - \tfrac{5}{24}(\delta^6 u - \dots)\}] \end{aligned} \tag{8}$$

The following is an example; the data are the values of tan x to five places of decimals, the interval in x being 1°. The differences of odd order are omitted for convenience of printing.

Example 5.

x.	$u \equiv \tan x$.	$\delta^2 u$.	$\delta^4 u$.	$\delta^6 u$.	U.	u = mean of values of U.	x.
		+	+	+			
73°	3.27085	2339	100	5	3.2679495		
						3.37594	73 $\frac{1}{2}$°
74°	3.48741	2808	132	23	3.4839298		
						3.60588	74 $\frac{1}{2}$°
75°	3.73205	3409	187	18	3.7278317		
						3.86671	75 $\frac{1}{2}$°
76°	4.01078	4197	260	51	4.0055922		
						4.16530	76 $\frac{1}{2}$°
77°	4.33148	5245	384	64	4.3250107		

If a new table is formed from these values, the intervals being $\frac{1}{2}$°, it will be found that differences beyond the fourth are negligible.

To subdivide h into smaller intervals than $\frac{1}{2}h$, various methods may be used. One is to calculate the sets of quantities which in the new table will be the successive differences, corresponding to u_0, u_1, . . . and to find the intermediate terms by successive additions. A better method is to use a formula due to J. D. Everett. If we write $\varphi = 1 - \theta$, Everett's formula is, in its most symmetrical form,

$$\left.\begin{aligned} u_\theta &= \theta u_1 + \frac{(\theta+1)\,\theta\,(\theta-1)}{3!}\delta^2 u_1 + \frac{(\theta+2)\,(\theta+1)\,\theta\,(\theta-1)\,(\theta-2)}{5!}\delta^4 u_1 + \ldots \\ &+ \varphi u_0 + \frac{(\varphi+1)\,\varphi\,(\varphi-1)}{3!}\delta^2 u_0 + \frac{(\varphi+2)\,(\varphi+1)\,\varphi\,(\varphi-1)\,(\varphi-2)}{5!}\delta^4 u_0 + \ldots \end{aligned}\right\} \quad (9).$$

For actual calculations a less symmetrical form may be used. Denoting

$$\frac{(\theta+1)\,\theta\,(\theta-1)}{3!}\delta^2 u_1 + \frac{(\theta+2)\,(\theta+1)\,\theta\,(\theta-1)\,(\theta-2)}{5!}\delta^4 u_1 + \ldots \quad (10)$$

by ${}_\theta V_1$, we have, for interpolation between u_0 and u_1,

$$u_\theta = u_0 + \theta\Delta u_0 + {}_\theta V_1 + {}_{1-\theta}V_0 \quad (11),$$

the successive values of θ being $1/n$, $2/n$, . . . $(n-1)/n$. For interpolation between u_1 and u_2 we have, with the same succession of values of θ,

$$u_{1+\theta} = u_1 + {}_{\theta}V_1, \qquad V_2 + {}_{1-\theta}V_1 \qquad (12).$$

The values of ${}_{1-\theta}V_1$ in (12) are exactly the same as those of ${}_{\theta}V_1$ in (11), but in the reverse order. The process is therefore that (i.) we find the successive values of $u_0 + \theta\Delta u_0$, &c., *i.e.* we construct a table, with the required intervals of x, as if we had only to take first differences into account; (ii.) we construct, in a parallel column, a table giving the values of ${}_{\theta}V_1$, &c.; (iii.) we repeat these latter values, placing the set belonging to each interval h in the interval next following it, and writing the values in the reverse order; and (iv.) by adding horizontally we get the final values for the new table.

As an example, take the values of tan x by intervals of $\frac{1}{2}$° in x, as found above (Ex. 5). The first diagram below is a portion of this table, with the differences, and the second shows the calculation of the terms of (11) so as to get a table in which the intervals are 0.1 of 1°. The last column but one in the second diagram is introduced for convenience of calculation.

Example 6.

x.	$u = \tan x$.	δu.	$\delta^2 u$.	$\delta^3 u$.	$\delta^4 u$.
		+	+	+	+
		11147		62	
74°.0	3.48741		700		8
		11847		70	
74°.5	3.60588		770		9
		12617		79	

x	$u_0 + \theta\Delta u_0$.	${}_{\theta}V_1$.	${}_{1-\theta}V_0$.	${}_{\theta}V_1 + {}_{1-\theta}V_0$.	u.
73°.6	·	−22 35	·	·	·
73°.7	·	−39 11	·	·	·
73°.8	·	−44 71	·	·	·
73°.9	·	−33 54	·	·	·
74°.0	3.48741 00				3.48741
74°.1	3.51110 40	−24 58	−33 54	−58 12	3.51052
74°.2	3.53479 80	−43 02	−44 71	−87 73	3.53392
74°.3	3.55849 20	−49 18	−39 11	−88 29	3.55761
74°.4	3.58218 60	−36 89	−22 35	−59 24	3.58159
74°.5	3.60588 00				3.60588

The following are the values of the coefficients of u_1, $\delta^2 u_1$, $\delta^4 u_1$, and $\delta^6 u_1$ in (9) for certain values of n. For calculating the four terms due to $\delta^2 u_1$ in the case of $n = 5$ it should be noticed that the third term is twice the first, the fourth is the mean of the first and the third, and the second is the mean of the third and the fourth. In table 3, and in the last column of table 2, the coefficients are corrected in the last figure.

TABLE 1.—n = 5.

co. u.	co. $\delta^2 u$.	co. $\delta^4 u$.	co. $\delta^6 u$.
+	−	+	−
.2	.032	.006336	.00135168 = 1/740 approx.
.4	.056	.010752	.00226304 = 1/442 ”
.6	.064	.011648	.00239616 = 1/417 ”
.8	.048	.008064	.00160512 = 1/623 ”

TABLE 2.—n = 10.

co. u.	co. $\delta^2 u$.	co. $\delta^4 u$.	co. $\delta^6 u$.
+	−	+	−
.1	.0165	.00329175	.000704591
.2	.0320	.00633600	.001351680
.3	.0455	.00889525	.001887064
.4	.0560	.01075200	.002263040
.5	.0625	.01171875	.002441406
.6	.0640	.01164800	.002396160
.7	.0595	.01044225	.002115799
.8	.0480	.00806400	.001605120
.9	.0285	.00454575	.000886421

TABLE 3.—n = 12.

co. u.	co. $\delta^2 u$.	co. $\delta^4 u$.	co. $\delta^6 u$.
+	−	+	−
1/12	.013792438	.002753699	.000589623
2/12	.027006173	.005363726	.001145822
3/12	.039062500	.007690430	.001636505
4/12	.049382716	.009602195	.002032211
5/12	.057388117	.010979463	.002307357
6/12	.062500000	.011718750	.002441406
7/12	.064139660	.011736667	.002419911
8/12	.061728395	.010973937	.002235432
9/12	.054687500	.009399414	.001888275
10/12	.042438272	.007014103	.001387048
11/12	.024402006	.003855178	.000748981

III. General Observations

7. *Derivation of Formulae.*—The advancing-difference formula (1) may be written, in the symbolical notation of finite differences,

$$u_\theta = (1 + \Delta)^\theta u_0 = \mathrm{E}^\theta u_0 \qquad (13);$$

and it is an extension of the theorem that if n is a positive integer

$$u_n = u_0 + n\Delta u_0 + \frac{n\,(n-1)}{2!}\Delta^2 u_0 + \ldots \qquad (14),$$

the series being continued until the terms vanish. The formula (14) is identically true: the formula (13) or (1) is only formally true, but its applicability to concrete cases is due to the fact that the series in (1), when taken for a definite number of terms, differs from the true value of u_θ by a “remainder” which in most cases is very small when this definite number of terms is properly chosen.

Everett’s formula (9), and the central-difference formula obtained by substituting from (4) in (2), are modifications of a standard formula

$$u_\theta = u_0 + \theta\delta u_{\frac{1}{2}} + \frac{\theta(\theta-1)}{2!}\delta^2 u_0 + \frac{(\theta+1)\theta(\theta-1)}{3!}\delta^3 u_{\frac{1}{2}} + \frac{(\theta+1)\theta(\theta-1)(\theta-2)}{4!}\delta^4 u_0 + \ldots \quad (15),$$

which may similarly be regarded as an extension of the theorem that, if n is a positive integer,

$$u_n = u_0 + \mathrm{n}\delta u_{\frac{1}{2}} + \frac{n(n-1)}{2!}\delta^2 u_0 + \frac{(n+1)n(n-1)}{3!}\delta^3 u_{\frac{1}{2}} + \ldots \quad (16).$$

There are other central-difference formulae besides those mentioned above; the general symbolical expression is

$$u_\theta = (\cosh \theta h\mathrm{D} + \sinh \theta h\mathrm{D})\, u_0 \quad (17).$$

where

$$\cosh \tfrac{1}{2}hD = \mu,\ \sinh \tfrac{1}{2}hD = \tfrac{1}{2}\delta \qquad (18).$$

8. *Comparative Accuracy.*—Central-difference formulae are usually more accurate than advancing-difference formulae, whether we consider the inaccuracy due to omission of the "remainder" mentioned in the last paragraph or the error due to the approximative character of the tabulated values. The latter is the more important. If each tabulated value of u is within $\pm\tfrac{1}{2}\rho$ of the corresponding true value, and if the differences used in the formulae are the *tabular* differences, *i.e.* the actual successive differences of the tabulated values of u, then the ratio of the limit of error of u_θ, as calculated from the first r terms of the series in (1), to $\tfrac{1}{2}\rho$ is the sum of the first r terms of the series

$$1 + 0 + \theta(1-\theta) + \theta(1-\theta)(2-\theta) + \tfrac{7}{12}\theta(1-\theta)(2-\theta)(3-\theta) + \tfrac{1}{4}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta) + \tfrac{31}{360}\theta(1-\theta)\ldots(5-\theta) + \ldots,$$

while the corresponding ratio for the use of differences up to $\delta^{2p}u_0$ inclusive in (4) or up to $\delta^{2p}u_1$ and $o^{2p}u_0$ in (9) (*i.e.* in effect, up to $\delta^{2p+1}u_{\frac{1}{2}}$) is the sum of the first $p + 1$ terms of the series

$$1 + \frac{\theta(1-\theta)}{1.1} + \frac{(1+\theta)\,\theta\,(1-\theta)(2-\theta)}{(2!)^2} + \frac{(2+\theta)(1+\theta)\,\theta\,(1-\theta)(2-\theta)(3-\theta)}{(3!)^2} + \ldots,$$

it being supposed in each case that θ lies between 0 and 1. The following table gives a comparison of the respective limits of error; the lines I. and II. give the errors due to the advancing-difference and the central-difference formulae, and the coefficient ρ is omitted throughout.

TABLE 4.

		Error due to use of Differences up to and including						
		1st.	2nd.	3rd.	4th.	5th.	6th.	7th.
.5	I. . .	.500	.625	.813	1.086	1.497	2.132	3.147
	II. . .	.500	.625	.625	.696	.696	.745	.745
.2	I. . .	.500	.580	.724	.960	1.343	1.976	3.042
	II. . .	.500	.580	.580	.624	.624	.653	.653
.4	I. . .	.500	.620	.812	1.104	1.553	2.265	3.422
	II. . .	.500	.620	.620	.688	.688	.734	.734
.6	I. . .	.500	.620	.788	1.024	1.366	1.886	2.700
	II. . .	.500	.620	.620	.688	.688	.734	.734
.8	I. . .	.500	.580	.676	.800	.969	1.213	1.582
	II. . .	.500	.580	.580	.624	.624	.653	.653

In some cases the differences tabulated are not the tabular differences, but the corrected differences; *i.e.* each difference, like each value of *u*, is correct within $\pm\frac{1}{2}\rho$. It does not follow that these differences should be used for interpolation. Whatever formula is employed, the first difference should always be the tabular first difference, not the corrected first difference; and, further, if a central-difference formula is used, each difference of odd order should be the tabular difference of the corrected differences of the next lower order. (This last result is indirectly achieved if Everett's formula is used.) With these precautions (i.) the central-difference formula is slightly improved by using corrected instead of tabular differences, and (ii.) the advancing-difference formula is greatly improved, being better than the central-difference formula with tabular differences, but still not so good as the latter with corrected differences. For $\theta = .5$, for instance, supposing we have to go to fifth differences, the limits ± 1.497 and $\pm .696$, as given above, become $\pm .627$ and $\pm .575$ respectively.

9. *Completion of Table of Differences.*—If no values of *u* outside the range within which we have to interpolate are given, the series of differences will be incomplete at both ends. It may be continued in each direction by treating as constant the extreme difference of the highest order involved; and central-difference formulae can then be employed uniformly throughout the whole range.

Suppose, for instance, that the values of tan *x* in § 6 extended only from $x = 60°$ to $x = 80°$, we could then complete the table of differences by making the entries shown in italics below.

Example 7.

x.	$u = \tan x$.	δu.	$\delta^2 u$.	$\delta^3 u$.	$\delta^4 u$.	$\delta^5 u$.	$\delta^6 u$.
		+	+	+	+	+	+
		6775		*34*			
60°	1.73205		*425*		*9*		
		7200		*43*			
61°	1.80405		468		*9*		
		7668		52			
62°	1.88073		520		9		
		8188		61			
63°	1.96261		581		10		
		8769		71			
64°	2.05030	.	652	.	9		
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
75°	3.73205	.	3409	.	187	.	18
		27873		788		73	
76°	4.01078		4197		260		51
		32070		1048		124	
77°	4.33148		5245		384		64
		37315		1432		188	
78°	4.70463		6677		572		*64*
		43992		2004		*252*	
79°	5.14455		8681		*824*		*64*
		52673		*2828*		*316*	
80°	5.67128		*11509*		*1140*		*64*

		64182		3968			
						380	

For interpolating between $x = 60°$ and $x = 61°$ we should obtain the same result by applying Everett's formula to this table as by using the advancing-difference formula; and similarly at the other end for the receding differences.

Interpolation by Substituted Tabulation.

10. The relation of *u* to *x* may be such that the successive differences of *u* increase rapidly, so that interpolation-formulae cannot be employed directly. Other methods have then to be used. The best method is to replace *u* by some expression v which is a function of *u* such that (i.) the value of v or of *u* can be determined for any given value of *u* or of v, and (ii.) when v is tabulated in terms of *x* the differences decrease rapidly. We can then calculate v, and thence *u*, for any intermediate value of *x*.

If, for instance, we require tan *x* for a value of *x* which is nearly 90°, it will be found that the table of tangents is not suitable for interpolation. We can, however, convert it into a table of cotangents to about the same number of significant figures; from this we can easily calculate cot *x*, and thence tan *x*.

11. This method is specially suitable for statistical data, where the successive values of *u* represent the area of a figure of frequency up to successive ordinates. We have first to determine, by inspection, a curve which bears a general

similarity to the unknown curve of frequency, and whose area and abscissa are so related that either can be readily calculated when the other is known. This may be called the *auxiliary curve*. Denoting by ξ the abscissa of this curve which corresponds to area u, we find the value of ξ corresponding to each of the given values of u. Then, tabulating ξ in terms of x, we have a table in which, if the auxiliary curve has been well chosen, differences of ξ after the first or second are negligible. We can therefore find ξ, and thence u, for any intermediate value of x.

Extensions.

12. *Construction of Formulae.*—Any difference of u of the rth order involves $r + 1$ consecutive values of u, and it might be expressed by the suffixes which indicate these values. Thus we might write the table of differences

x.	u.	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	(−1, 0)	.	(−2, −1, 0, 1)	.
x_0	u_0		(−1, 0, 1)		(−2, −1, 0, 1, 2)
		(0, 1)		(−1, 0, 1, 2)	
x_1	u_1		(0, 1, 2)		(−1, 0, 1, 2, 3)
		(1, 2)		(0, 1, 2, 3)	
x_2	u_2		(1, 2, 3)		(0, 1, 2, 3, 4)
.	.	(2, 3)	.	(1, 2, 3, 4)	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

The formulae (1) and (15) might then be written

$$u = u_0 + \frac{x - x_0}{h}(0, 1) + \frac{x - x_0}{h} \cdot \frac{x - x_1}{2h}(0, 1, 2) + \frac{x - x_0}{h} \cdot \frac{x - x_1}{2h} \cdot \frac{x - x_2}{3h}(0, 1, 2, 3) + \ldots \quad (19),$$

$$u = u_0 \ \frac{x - x_0}{h}(0, 1) + \frac{x - x_0}{h} \cdot \frac{x - x_1}{2h}(-1, 0, 1) + \frac{x - x_0}{h} \cdot \frac{x - x_1}{2h} \cdot \frac{x - x_{-1}}{3h}(-1, 0, 1, 2) + \ldots \quad (20).$$

The general principle on which these formulae are constructed, and which may be used to construct other formulae, is that (i.) we start with any tabulated value of u, (ii.) we pass to the successive differences by steps, each of which may be either downwards or upwards, and (iii.) the new suffix which is introduced at each step determines the new factor (involving x) for use in the next term. For any particular value of x, however, all formulae which end with the same difference of the rth order give the same result, provided tabular differences are used. If, for instance, we go only to first differences, we have

$$u_0 + \frac{x - x_0}{h}(0, 1) = u_1 + \frac{x - x_1}{h}(0, 1)$$

identically.

13. *Ordinates not Equidistant.*—When the successive ordinates in the graph of u are not equidistant, *i.e.* when the differences of successive values of x are not equal, the above principle still applies, provided the differences are adjusted in a particular way. Let the values of x for which u is tabulated be $a = x_0 + \alpha h$, $b = x_0 + \beta h$, $c = x_0 + \gamma h$, . . . Then the table becomes

<table>
<tr><th rowspan="2">x.</th><th rowspan="2">u.</th><th colspan="3">Adjusted Differences.</th></tr>
<tr><th>1st Diff.</th><th>2nd Diff.</th><th>&c.</th></tr>
<tr><td>.</td><td>.</td><td>.</td><td>.</td><td></td></tr>
<tr><td>.</td><td>.</td><td>.</td><td>.</td><td></td></tr>
<tr><td>.</td><td>.</td><td>.</td><td>.</td><td></td></tr>
<tr><td>$a = x_\alpha$</td><td>u_α</td><td>.</td><td>.</td><td></td></tr>
<tr><td></td><td></td><td>(α, β)</td><td></td><td></td></tr>
<tr><td>$b = x_\beta$</td><td>u_β</td><td></td><td>(α, β, γ)</td><td></td></tr>
<tr><td></td><td></td><td>(β, γ)</td><td></td><td></td></tr>
<tr><td>$c = x_\gamma$</td><td>u_γ</td><td>.</td><td>.</td><td></td></tr>
<tr><td>.</td><td>.</td><td>.</td><td>.</td><td></td></tr>
<tr><td>.</td><td>.</td><td>.</td><td>.</td><td></td></tr>
<tr><td>.</td><td>.</td><td>.</td><td>.</td><td></td></tr>
</table>

In this table, however, (α, β) does not mean $u_\beta - u_\alpha$, but $u_\beta - u_\alpha \div (\beta - \alpha)$; (α, β, γ) means $\{(\beta, \gamma) - (\alpha, \beta)\} \div \frac{1}{2}(\gamma - \alpha)$; and, generally any quantity $(\eta, \ldots \varphi)$ in the column headed "rth diff." is obtained by dividing the difference of the adjoining quantities in the preceding column by $(\varphi - \eta)/r$. If the table is formed in this way, we may apply the principle of § 12 so as to obtain formulae such as

$$u = u_\alpha + \frac{x-a}{h} \cdot (\alpha, \beta) + \frac{x-a}{h} \cdot \frac{x-b}{2h} \cdot (\alpha, \beta, \gamma) + \ldots \quad (21),$$

$$u = u_\gamma + \frac{x-c}{h} \cdot (\beta, \gamma) + \frac{x-c}{h} \cdot \frac{x-b}{2h} \cdot (\alpha, \beta, \gamma) + \ldots \quad (22).$$

The following example illustrates the method, h being taken to be 1°:—

Example 8.

x.	$u = \sin x$.	1st Diff. (adjusted).	2nd Diff. (adjusted).	3rd Diff. (adjusted).
		+	−	−
20°	.3420201			
		162932 50		
22°	.3746066		1125 00	
		161245 00		48 75
23°	.3907311		1222 50	
		158800 00		48 30
26°	.4383711		1303 00	
		156194 00		47 49
27°	.4539905		1445 47	
		151857 60		46 00
32°	.5299193		1583 48	
		145523 67		
35°	.5735764			

To find u for $x = 31°$, we use the values for 26°, 27°, 32° and 35°, and obtain

$$u = .4383711 + \frac{5}{1}(156194\ 00) + \frac{5}{1}\cdot\frac{4}{2}(-1445\ 47) + \frac{5}{1}\cdot\frac{4}{2}\cdot\frac{-1}{3}(-46\ 00) = .5150380,$$

which is only wrong in the last figure.

If the values of u occurring in (21) or (22) are $u_\alpha, u_\beta, u_\gamma, \ldots u_\lambda$, corresponding to values $a, b, c, \ldots l$ of x, the formula may be more symmetrically written

$$u = \frac{(x-b)(x-c)\ldots(x-l)}{(a-b)(a-c)\ldots(a-l)} u_\alpha + \frac{(x-a)(x-c)\ldots(x-l)}{(b-a)(b-c)\ldots(b-l)} u_\beta + \ldots$$

$$\ldots + \frac{(x-a)(x-b)(x-c)\ldots}{(l-a)(l-b)(l-c)\ldots} u_\lambda \qquad (23).$$

This is known as *Lagrange's formula*, but it is said to be due to Euler. It is not convenient for practical use, since it does not show how many terms have to be taken in any particular case.

14. *Interpolation from Tables of Double Entry.*—When u is a function of x and y, and is tabulated in terms of x and of y jointly, its calculation for a pair of values not given in the table may be effected either directly or by first forming a table of values of u in terms of y for the particular value of x and then determining u from this table for the particular value of y. For direct interpolation, consider that Δ represents differencing by changing x into $x + 1$, and Δ' differencing by changing y into $y + 1$. Then the formula is

$$u_{x,y} = (1 + \Delta)^x (1 + \Delta')^y u_{0,0};$$

and the right-hand side can be developed in whatever form is most convenient for the particular case.

References.—For general formulae, with particular applications, see the *Text-book of the Institute of Actuaries*, part ii. (1st ed. 1887, 2nd ed. 1902), p. 434; H. L. Rice, *Theory and Practice of Interpolation* (1899). Some historical references are given by C. W. Merrifield, "On Quadratures and Interpolation," *Brit. Assoc. Report* (1880), p. 321; see also *Encycl. der math. Wiss*. vol. i. pt. 2, pp. 800-819. For J. D. Everett's formula, see *Quar. Jour. Pure and Applied Maths.*, No. 128 (1901), and *Jour. Inst. Actuaries*, vol. xxxv. (1901), p. 452. As to relative accuracy of different formulae, see *Proc. Lon. Math. Soc*. (2) vol. iv. p. 320. Examples of interpolation by means of auxiliary curves will be found in *Jour. Royal Stat. Soc*. vol. lxiii. pp. 433, 637. See also Differences, Calculus of. (W. F. Sh.)

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