Chapter 9 Notes

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Contents

1 Day 1 | 9.1 Sequences & Factorials

1.1 Sequences

A sequence is a discrete function that maps a non-negative integer, typically denoted n, to a corresponding value in a set of numbers. Consider the sequences:

$$a_n = 2n - 5 \text{ whose first five terms are } \{-3, -1, 1, 3, 5\}.$$

$$\text{Since } \lim_{n\to\infty} a_n = \infty,\ a_n \text{ diverges.}$$

$$a_n = \frac{1}{2^n} \text{ whose first five terms are } \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\right\}.$$

$$\text{Since } \lim_{n\to\infty} a_n = 0,\ a_n \text{ converges to } 0.$$

$$a_n = \frac{(-1)^{n+1}}{n} \text{ whose first five terms are } \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}\right\}.$$

$$\text{Since } \lim_{n\to\infty} a_n = 0,\ a_n \text{ converges to } 0.$$

$$a_n = \frac{5n+1}{2n} \text{ whose first five terms are } \left\{3, \frac{11}{4}, \frac{8}{3}, \frac{21}{8}, \frac{13}{5}\right\}.$$

$$\text{Since } \lim_{n\to\infty} a_n = \frac{5}{2},\ a_n \text{ converges to } \tfrac{5}{2}.$$

$$a_n = 3 + (-1)^n \text{ whose first five terms are } \{2, 4, 2, 4, 2\}.$$

$$\text{Since } \lim_{n\to\infty} a_n \text{ does not exist, } a_n \text{ diverges.}$$

Definition 1.1: Convergence & Divergence of Sequences

If $\lim_{n\to\infty} a_n = c$ where c is finite and positive, then a_n converges to c.

If $\lim_{n\to\infty} a_n$ does not exist or equals $\pm\infty$, then a_n diverges.

A series is the sum of the terms in a sequence. The sum of the first k terms of a sequence a_n is symbolically represented with sigma notation as

$$\sum_{n=1}^{k} a_n.$$

1.2 Factorials

The factorial is a function defined on the input set $x \in \mathbb{N}$ that multiplies an input by every consecutive decreasing integer value, until 1.

Definition 1.2: Factorial

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

$$\frac{10!}{7!} \text{ can be simplified to } \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720.$$

$0! = 1$ is a special factorial case.

2 Day 2 | 9.2 Series & Convergence

2.1 Geometric Series

A geometric sequence has a first term and a common ratio by which each subsequent term is multiplied. A geometric sequence where a is the first term and r is the common ratio has terms $\{a, ar, ar^2, ar^3, ar^4, \cdots\}$. An infinite geometric series **converges** if $|r| < 1$ and **diverges** otherwise.

$$\text{Consider: the infinite geometric series } \sum_{n=0}^{\infty} ar^n \text{ where } \lim_{n\to\infty} a_n = 0.$$

Let S represent the value of the series.

$$S = ar^0 + ar^1 + ar^2 + ar^3 + \cdots$$

$$S = r(ar^{-1} + \underbrace{ar^0 + ar^1 + ar^2 + ar^3 + \cdots}_{S})$$

For fun, try factoring r^2 instead of r and continue with the proof.

$$S = r(ar^{-1} + S)$$

$$S = a + rS$$

$$S - rS = a$$

$$(1-r)S = a$$

$$S = \frac{a}{1-r}.$$

Theorem 2.1: Value of an Infinite Geometric Series

If an infinite geometric series converges ($|r| < 1$), then it does so to $\dfrac{a}{1-r}$.

2.2 n^{th} Term Test

If a sequence does not converge to 0, its series **cannot** converge because any nonzero number, no matter how small, diverges to $\pm\infty$ when added infinitely. However, the inverse statement is not necessarily true as a sequence that converges to 0 can still have a series that diverges. (consider $a_n = \frac{1}{x}$)

$$\text{Consider: } \sum_{n=1}^{\infty} \frac{n^2}{3n^2+1} = \frac{1}{4} + \frac{4}{13} + \frac{9}{28} + \frac{16}{49} + \cdots + \underbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots}_{\infty \text{ times}}.$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \frac{n^2}{3n^2+1} \text{ diverges.}$$

Theorem 2.2: n^{th} Term Test

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum^{\infty} a_n \textbf{ diverges.}$$

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then no definitive conclusion can be drawn.}$$

2.3 Telescoping Series

Telescoping series are referred to as such because they expand infinitely and cancel to a finite value.

$$\text{Consider: } S_\infty = \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}.$$

$$S_\infty = \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+2}}_{\text{by partial fraction decomposition}}$$

Generating terms:

$$
\begin{aligned}
S_1 &= \frac{1}{2} - \frac{1}{3} \\
S_2 &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \\
S_3 &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} \\
S_4 &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6}
\end{aligned}
$$

Following $\frac{1}{2}$, all terms cancel with subsequent terms—the defining characteristic of a telescoping series. Therefore, S_∞ converges to $\frac{1}{2}$.

3 Day 3 | 9.3 Integral Test

The integral test works by using an integral to approximate the value of a monotone decreasing series. If an integral converges, the infinite series it represents converges as well. If an integral diverges, the infinite series it represents must also diverge. The integral test is requisite to determine that harmonic series diverge.

Theorem 3.1: The Integral Test

If f is a function that is positive, continuous, and **monotone decreasing** over $[c, \infty]$ such that $f(n) = a_n$, then

$$\int_c^\infty f(x)dx \text{ and } \sum_{n=c}^{\infty} a_n \text{ behave the same way.}$$

$$\text{Consider: } S_\infty = \sum_{n=1}^{\infty} \frac{n}{n^2+3}. \text{ To evaluate, compare to } \int_1^\infty \frac{n}{n^2+3}.$$

$$
\begin{aligned}
\text{Using } u &= n^2 + 3 \\
\text{and } du &= 2ndn, \\
\int_1^\infty \frac{n}{n^2+3}dn &= \frac{1}{2}\int_4^\infty \frac{1}{u}du \\
&= \frac{1}{2}[\ln|u|]_4^\infty \\
&= \infty.
\end{aligned}
$$

Therefore, S_∞ diverges.

4 Day 3 | 9.4 Comparison Tests

4.1 Direct Comparison Test

The direct comparison test works by comparing an unknown series to a known series to determine the qualities of the unknown series. It relies on the principle that a series that is larger[1] than a series that diverges must also diverge. The contrapositive statement is also true.

Theorem 4.1: The Direct Comparison Test

For $0 \leq a_n \leq b_n$:

$$\text{If } \underbrace{\sum^{\infty} b_n}_{\text{larger}} \text{ converges, then } \underbrace{\sum^{\infty} a_n}_{\text{smaller}} \text{ converges as well.}$$

$$\text{If } \underbrace{\sum^{\infty} a_n}_{\text{smaller}} \text{ diverges, then } \underbrace{\sum^{\infty} b_n}_{\text{larger}} \text{ diverges as well.}$$

Consider: $S_\infty = \sum_{n=1}^{\infty} \frac{1}{n^3+1}$. To evaluate, compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which is known to converge.

Since $\sum_{n=1}^{\infty} \frac{1}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges as well.

4.2 Limit Comparison Test

The limit comparison test is used more often than the direct comparison test and provides a more concrete result. It compares the ratio of two similar discrete functions to determine whether they behave similarly.

Theorem 4.2: The Limit Comparison Test

For $a_n \geq 0$ and $b_n \geq 0$:

If $\frac{a_n}{b_n} = L$ where L is finite and **strictly positive**, then

$$\sum^{\infty} a_n \text{ and } \sum^{\infty} b_n \text{ behave the same way.}$$

[1] Both series must be greater than zero, so it is not possible for a series to be *more negative than* (less than) another with this test.

Consider: $\sum_{n=5}^{\infty} \frac{n^2 - 10}{4n^5 - n^3 + 7}$. To evaluate, compare to $\sum_{n=5}^{\infty} \frac{1}{n^3}$.

$$\lim_{n \to \infty} \frac{\frac{n^2-10}{4n^5-n^3+7}}{\frac{1}{n^3}}$$

$$= \lim_{n \to \infty} \frac{n^5 - 10n^3}{4n^5 - n^3 + 7} = \frac{1}{4}.$$

5 Day 4 | 9.5 Alternating Series Test

Alternating series have both positive and negative terms. In *strictly* alternating series, positive and negative terms must be immediately adjacent.

Theorem 5.1: Alternating Series Test

$$\sum^{\infty}(-1)^n a_n \text{ converges } \textbf{only} \text{ if } a_{n+1} \le a_n \textbf{ and } \lim_{n \to \infty} a_n = 0.$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is known to converge. Now consider: $S_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

When expanded, $S_\infty = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$

First, isolate the alternator by rewriting S_∞ as $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

$\frac{1}{n+1} \le \frac{1}{n}$ is always true.

$\lim_{n \to \infty} \frac{1}{n} = 0$ is always true.

Therefore, S_∞ converges by the AST.

This alternating variant of the simple harmonic series converges, but the standard harmonic series does not. This is known as *conditional convergence*.

Definition 5.1: Conditional & Absolute Convergence

Conditional convergence: $\sum^{\infty}(-1)^n a_n$ converges, but $\sum^{\infty} a_n$ diverges.

Absolute convergence: $\sum^{\infty}(-1)^n a_n$ converges, and $\sum^{\infty} a_n$ converges.

6 Day 5 | 9.5 Alternating Series Error Bound

$$\text{Consider: } S_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

The partial sums are:

$$S_1 = 1$$
$$S_2 = \frac{1}{2}$$
$$S_3 = \frac{5}{6}$$
$$S_4 = \frac{7}{12}$$

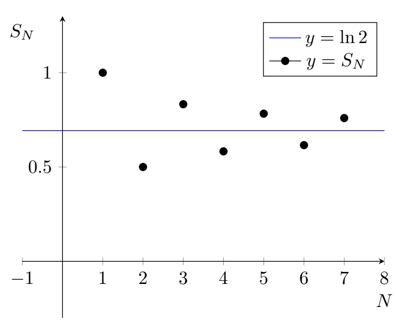


Figure 1: Graph of S_N with respect to N

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges to } \ln 2.$$

Theorem 6.1: Alternating Series Error Bound

The sum of a convergent, *strictly*[a] alternating series of infinite terms S_∞ will differ from the partial sum of N terms S_N by no more than the value of the **first omitted term** (a_{N+1}).

$$|S_\infty - S_N| < a_{N+1}$$

[a] If $S_4 = 1 + 2 - 3 - 4$, S is **not** strictly alternating.

Problems you could be given include:

- Approximate the infinite sum using N terms.
- Determine the error bound for a partial sum of N terms.
- Determine the number of terms required to approximate the infinite sum within (some given error) (ex. error less than $\frac{1}{10}$). The first term less than $\frac{1}{10}$ is $\frac{1}{11}$. Therefore, ten terms are required for the error bound to be less than $\frac{1}{10}$.
- Which term represents (some given error bound)?

Consider: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$.

First, perform a simple AST test. Is $a_{n+1} < a_n$?

$$\frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} \text{ is true.}$$

Then, generate terms.

$$\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

Compute partial sums.

$$S_2 = \frac{5}{6} \text{ whose error bound is } \frac{1}{5!} = \frac{1}{120}.$$

$$S_3 = \frac{101}{120} \text{ whose error bound is } \frac{1}{7!} = \frac{1}{5040}.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \text{ converges to } \sin 1.$$

7 Day 6 | 9.6 Ratio Test & n^{th} Root Test

7.1 Ratio Test

If every term in a sequence is less than its preceding term (the sequence is decreasing quickly), then its series must converge. The inverse statement is also true.

Theorem 7.1: Ratio Test

$$\sum^{\infty} a_n \textbf{ converges} \text{ absolutely if } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\sum^{\infty} a_n \textbf{ diverges} \text{ absolutely if } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$$

$$\sum^{\infty} a_n \text{ is } \textbf{inconclusive} \text{ if } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

$$\textbf{Determine if } \sum_{n=0}^{\infty}\frac{2^n}{n!} \text{ converges.}$$

$$\text{Ratio test: } \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1$$

$$\text{Therefore, } \sum_{n=0}^{\infty}\frac{2^n}{n!} \text{ converges.}$$

$$\textbf{Determine if } \sum_{n=0}^{\infty}\frac{(-1)^n\cdot n^3\cdot 2^{n+2}}{3^n} \text{ converges.}$$

$$\textbf{Let } S_\infty = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot n^3\cdot 2^{n+2}}{3^n}$$

$$\text{Ratio test: } \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot(n+1)^3\cdot 2^{n+3}}{3^{n+1}}\cdot\frac{3^n}{(-1)^n\cdot n^3\cdot 2^{n+1}}\right| = \frac{2}{3} < 1$$

$$\text{Therefore, } \sum_{n=0}^{\infty}\frac{(-1)^n\cdot n^3\cdot 2^{n+2}}{3^n} \text{ converges.}$$

$$\textbf{Determine if } \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ converges.}$$

$$\textbf{Let } S_\infty = \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\text{Ratio test: } \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right| = \lim_{n\to\infty} \left| (1+\frac{1}{n})^n \right| = e > 1$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ converges.}$$

7.2 n^{th} Root Test

$$\text{Let } \sqrt[n]{|a_n|} \leq k < 1.$$

$$\text{Then, } |a_n| \leq k^n < 1.$$

By the direct comparison test, since $\sum^{\infty} k^n$ converges when $k < 1$, $\sqrt[n]{|a_n|}$ also converges.

Theorem 7.2: n^{th} Root Test

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1, \text{ then } a_n \text{ converges absolutely.}$$

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} > 1, \text{ then } a_n \text{ diverges absolutely.}$$

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1, \text{ then the test is inconclusive.}$$

$$\text{Determine if } \sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} \text{converges.}$$

$$\text{Using the } n^{\text{th}} \text{ root test, } \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0 < 1$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} \text{ converges.}$$

8 Day 1 | 9.7 Taylor Polynomials

Polynomial functions are often more cooperative and nicer to deal with than non-polynomial functions. Taylor Polynomials[2] allow for approximating non-polynomial functions with polynomial functions. In fact, a tangent line is just a 1^{st}-degree Taylor Polynomial. Extending the concept of tangent lines yields tangent quadratics, cubics, quartics, and more!

8.1 Introduction to Polynomial Approximations

Consider the generic 5^{th}-degree polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ where a_n represents the coefficient of the n^{th} power of x.

$$\begin{aligned}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 \\
f''(x) &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\
f'''(x) &= 6a_3 + 24a_4x + 60a_5x^2 \\
f^{(4)}(x) &= 24a_4 + 120a_5x \\
f^{(5)}(x) &= 120a_5
\end{aligned}$$

To construct a polynomial function P such that $P(0) = 10$, $P'(0) = 9$, $P''(0) = 8$, $P'''(0) = 7$, $P^{(4)}(0) = 6$, and $P^{(5)}(0) = 5$:

$$\begin{array}{ll}
n = 0 & a_0 = \frac{10}{0!} \\
n = 1 & a_1 = \frac{9}{1!} \\
n = 2 & a_2 = \frac{8}{2!} \\
n = 3 & a_3 = \frac{7}{3!} \\
n = 4 & a_4 = \frac{6}{4!} \\
n = 5 & a_5 = \frac{5}{5!}
\end{array}$$

$$P(x) = 10 + 9x + 4x^2 + \frac{7}{6}x^3 + \frac{1}{4}x^4 + \frac{1}{24}x^5$$

These conditions were arbitrary and therefore generate a function with no significant properties. However, this method can be used to generate polynomial approximations to more useful functions.

[2]Nomenclature: Taylor **Polynomials** are partial sums of Taylor **Series**.

To create a polynomial P_5 that matches the behavior of $f(x) = \ln(x+1)$ through its first five derivatives, first compute coefficients:

$$\begin{aligned}
n &= 0 & a_0 &= \ln 1 & &= 0 \\
n &= 1 & a_1 &= \tfrac{1}{0+1} & &= 1 \\
n &= 2 & a_2 &= -\tfrac{1}{(0+1)^2} & &= -1 \\
n &= 3 & a_3 &= \tfrac{2}{(0+1)^3} & &= 2 \\
n &= 4 & a_4 &= -\tfrac{6}{(0+1)^4} & &= -6 \\
n &= 5 & a_5 &= \tfrac{24}{(0+1)^5} & &= 24
\end{aligned}$$

$$\begin{aligned}
P_5(x) &= \frac{0}{0!} + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + \frac{24}{5!}x^5 \\
&= 1 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5
\end{aligned}$$

Recall that, when $x = 1$, this is the series shown in 9.5.

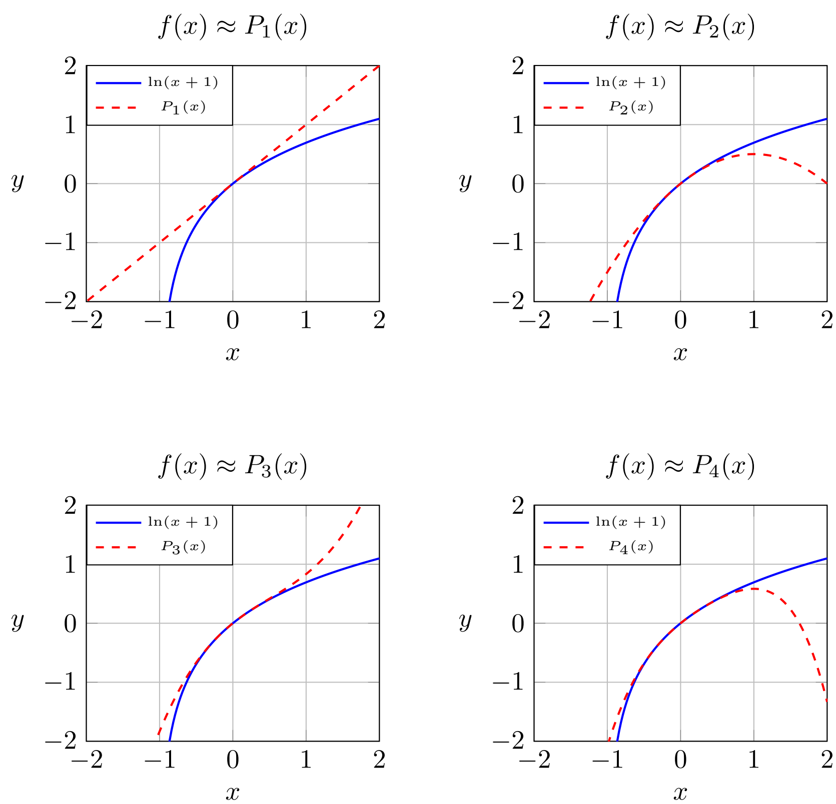


Figure 2: The first four Taylor Polynomials of $f(x) = \ln(x + 1)$

Theorem 8.1: Taylor & Maclaurin Polynomials

Any function f that has derivatives up to the n^{th} order can be approximated by an n^{th}-degree polynomial $P_n(x)$ centered at $x = c$ as such:

$$f(x) \approx P_n(x) = \underbrace{f(c) + f'(c)(x-c)}_{\text{tangent line}} + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

When $c = 0$, this polynomial is referred to as a *Maclaurin Polynomial.*

The same process can be used to find the 5^{th}-degree Maclaurin Polynomial for $\sin x$.

$$\begin{aligned}
n &= 0 \quad & a_0 &= \sin 0 & &= 0 \\
n &= 1 \quad & a_1 &= \cos 0 & &= 1 \\
n &= 2 \quad & a_2 &= -\sin 0 & &= 0 \\
n &= 3 \quad & a_3 &= -\cos 0 & &= -1 \\
n &= 4 \quad & a_4 &= \sin 0 & &= 0 \\
n &= 5 \quad & a_5 &= \cos 0 & &= 1
\end{aligned}$$

$$\text{Therefore, } \sin x \approx P_5(x) = 0 + x^1 + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 = \frac{1}{5!}x^5$$
$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

Since $\sin x$'s Taylor Polynomial (and, similarly, $\cos x$'s) yields a zero every other term, it can be optimized by re-indexing as such:

Theorem 8.2: Maclaurin Series for $\sin x$, $\cos x$, and e^x

$$\text{The } n^{\text{th}}\text{-degree Maclaurin Polynomial for } \sin x \approx \sum_{k=0}^{n} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

$$\text{The } n^{\text{th}}\text{-degree Maclaurin Polynomial for } \cos x \approx \sum_{k=0}^{n} \frac{(-1)^n x^{2n}}{(2n)!}.$$

$$\text{The } n^{\text{th}}\text{-degree Maclaurin Polynomial for } e^x \approx \sum_{k=0}^{n} \frac{x^n}{n!}.$$

To create a 4^{th}-degree Maclaurin Polynomial P_4 to $f(x) = \sin(2x + \frac{\pi}{4})$, first generate coefficients:

$$\begin{array}{lll} n=0 & a_0 = \sin(2\cdot 0 + \frac{\pi}{4}) & = \frac{\sqrt{2}}{2} \\ n=1 & a_1 = \cos 0 & = 1 \\ n=2 & a_2 = -\sin 0 & = 0 \\ n=3 & a_3 = -\cos 0 & = -1 \\ n=4 & a_4 = \sin 0 & = 0 \\ n=5 & a_5 = \cos 0 & = 1 \end{array}$$

These coefficients can then be used to generate P_4.

$$\sin\left(2x + \frac{\pi}{4}\right) \approx P_4(x) = \frac{1}{\sqrt{2}}\left(1 + 2x - \frac{4}{2!}x^2 - \frac{8}{3!}x^3 + \frac{16}{4!}x^4\right) \tag{1}$$

8.2 Taylor Alternating Polynomial Error Bound

To approximate $\sin 0.3$ with a 5^{th}-degree Maclaurin Polynomial, it is essential to first recall the 5^{th}-degree Maclaurin Polynomial for $\sin x$: $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$. Evaluating this polynomial at 0.3 is much simpler than evaluating $\sin 0.3$.

$$\sin 0.3 \approx 0.3 - \frac{1}{3!}(0.3)^3 + \frac{1}{5!}(0.3)^5 = 0.29552025$$

$$\sin x = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\text{alternating series}}$$

Since $\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!} < \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\lim_{n\to\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 0$, Theorem 6.1 states that this approximation will be off by the actual value of $\sin 0.3$ by no more than the absolute value of the first term omitted: $\frac{1}{7!}(0.3)^7$ or 0.000000043392857... The **actual** error of this approximation is $|\sin 0.3 - P_5(0.3)| = 0.00000004333...$, which is indeed less than the error bound. To approximate $\sin 1.5$ with a 3^{rd}-degree Maclaurin Polynomial, one might compute:

$$P_3(1.5) = 1.5 - \frac{1.5^3}{3!} = 0.9375.$$

This approximation's error bound is $\frac{1.5^5}{5!} = 0.06328...$ and its actual error, $|\sin 1.5 - P_3(1.5)|$, is 0.05999489, which is significantly larger than the error when approximating $\sin 0.3$ with a 5^{th}-degree Maclaurin Polynomial.

There are, therefore, two factors that affect the accuracy of a polynomial approximation:

- more terms in the Taylor Polynomial will yield a better approximation
- less distance from the center of the Taylor Polynomial will yield a better approximation

The 4^{th}-degree Maclaurin Polynomial P_4 for $f(x) = \sin\left(2x + \frac{\pi}{4}\right)$ generated in Equation 1 can be used to approximate the value of $\sin\left(3 + \frac{\pi}{4}\right)$ by evaluating $P_4(1.5) = -1.149$. This approximation is very far from the true value because the sin function has range $[-1, 1]$. More importantly, however, this approximation's error bound cannot be determined conventionally because its Maclaurin Series is **not** strictly alternating.

8.3 Lagrange Error Bound

The Lagrange Error Bound theorem gives an upper bound on the error of an approximation of a function using its Taylor Polynomial. Specifically, if $f(x)$ has derivatives up to order $n+1$, then the Lagrange form of the remainder term of the n^{th} Taylor Series centered at $x = c$ is given by:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1.}$$

In other words, the error bound is defined by the **first omitted term**, similarly to the Alternating Series Error Bound in Theorem 6.1.

Theorem 8.3: Lagrange Error Bound

The absolute error $|f(x) - P_n(x)|$ of approximating $f(x)$ by its n^{th}-degree Taylor Polynomial centered at $x = c$ can be given by:

$$\underbrace{|f(x) - P_n(x)|}_{\text{actual error}} \leq \underbrace{\frac{M}{(n+1)!}|x-c|^{n+1}}_{\text{error bound}}$$

where M is an upper bound for $|f^{(n+1)}(x)|$ on the interval $[a, b]$.

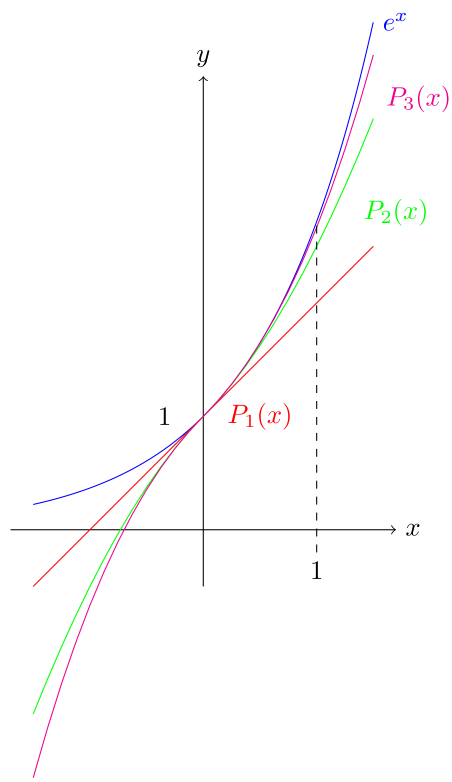
The Lagrange error bound can be useful for estimating the accuracy of an approximation based on the degree of the Taylor Polynomial used. For example, to approximate $\cos(0.5)$ using a Taylor Polynomial centered at $c = 0$ with degree $n = 3$, the 4^{th} derivative of $\cos(x)$, $-\cos(x)$, which has an absolute value of 1 on the interval $[-0.5, 0.5]$, must be used.

Therefore, the Lagrange error bound states that:

$$|\cos(0.5) - P_3(0.5)| \leq \frac{1}{4!}|0.5 - 0|^4 = \frac{1}{384}$$

where $P_3(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ is the third degree Taylor Polynomial for $\cos(x)$. Thus, this approximation is accurate to within $\frac{1}{384}$ of the true value of $\cos(0.5)$.

The following graph illustrates the Lagrange error bound for approximating e^x with its Taylor Polynomial centered at $c = 0$ with degrees $n = 1$, $n = 2$, and $n = 3$. The function e^x is shown in blue, and the Taylor Polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ are shown in red, green, and magenta, respectively. The vertical line at $x = 1$ indicates the point at which e^x is being approximated.



The Taylor Polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ provide increasingly accurate approximations of e^x near $x = 1$, as expected. The Lagrange Error Bound states that the error of each approximation is proportional to the coefficient of the first omitted term. This can be seen visually on the graph as the vertical distance between e^x and $P_n(x)$ appears to decrease with the degree of each Taylor Polynomial.

9 Day 4 | 9.8 Power Series

Maclaurin Series are a type of Taylor series when $c = 0$ and Taylor Series are a type of **power series** that are used to approximate functions.

9.1 Radius of Convergence

A power series' *radius of convergence* is how far x can go from the center while maintaining the series' convergence.

Definition 9.1: Power Series

A **power series** centered at $x = c$ is an infinite series in the form:

$$\sum_{n=0}^{\infty} a_n + (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

A power series given in terms of x can be described as a function of x,

$$f(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$$

where the *domain* of f is the set of all x for which the series converges as determined by the tests of convergence. When a power series is used to mimic the behavior of a given function, the center of the power series is analogous to the point of tangency on the function.

When $x = c$, then $\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} a_n(c-c)^n = a_0 + \underbrace{0 + 0 + 0 + \cdots}_{\infty \text{ times}} = a_0$

Theorem 9.1: Convergence of Power Series at Center

A power series will always converge at its center.

A power series' radius of convergence (RoC) can behave in three ways.

1. The power series converges only at its center: $x = c$. (RoC $= 0$)
2. The power series converges absolutely for all x (RoC $= \infty$)
3. There exists an RoC > 0 such that the power series converges when $|x-c| <$ RoC and diverges when $|x-c| >$ RoC.

The **Ratio Test** is used to find the radius of convergence of a power series.

10 Day 6 | 9.9 Representation of Functions by Power Series

Theorem 10.1: IoC for Common Maclaurins

The Maclaurin series for $\sin x$, $\cos x$, and e^x all have the interval of convergence $(-\infty, \infty)$.

Knowing the Maclaurin series for $\sin x$, $\cos x$, and e^x, it is possible to compose functions with algebraic operations

Operations on series include:

1. $f(kx) = \sum_{n=0}^{\infty} a_n (kx)^n$
2. $f(x^p) = \sum_{n=0}^{\infty} a_n (x^p)^n$
3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n + b) x^n$
4. $f(x)g(x) = \sum_{n=0}^{\infty} f(x) \cdot \sum_{n=0}^{\infty} g(x)$