Assignment 06: On Phase Trajectories

PH1050 Computational Physics

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Problem Statement

Given the Potential Energy of the particle as a function of x(t) (Position), plot the potential energy and phase trajectories of the particle. Also write the equations of motion of the particle (as a differential equation of order 2 and once more as a system of two first order coupled differential equations) and solve them assuming suitable initial conditions to obtain once again the phase trajectory of the particle.

Aim

To identify the critical points of the particle , obtain the equations of motion of the particle and plot the Phase trajectories of the particle.

Introduction

Given potential energy of a particle as a function of x[t] (position) of the form u(x)=a x^2 Exp[-b*x^2]
I have:
1)Plotted Potential Energy and Kinetic energy of the particle as a function of x[t] by assuming a suitable value of Total Energy.
I also had to assume suitable values for the variables ‘a’ and ‘b’.
2)Then had to find the critical points (Points of maxima and minima) of the potential energy function.
3)Then had to write the Total energy as a function of ‘p’ (momentum) and ‘x’ (position) and Taylor expand the same about the critical points for upto 3 terms.
4)Then drew the phase portraits for the particle for ‘x’ tending to the value of ‘x’ at the critical points.
5)Then plotted the phase portrait of the particle for any ‘x’ , not necessarily x-> critical points. In this part , we are not supposed to taylor expand the Potential energy function.
6)Then had to differentiate the potential energy function to get Force[x] and solve the equation assuming certain initial conditions and corresponding phase trajectory.
7) Then we were to write down the equation of motion as two first order coupled differential equations, solve them for a specific initial condition and plot the corresponding phase trajectory.

8) Then we had to use Show[] to display all the plots obtained in 5,6 and 7 in a single image.

Code Organization

1) Declared U[x], assigned suitable values of ‘a’ and ‘b’.
2)Plotted U[x] and KE[x] by assuming suitable value of total energy.
3)obtained the Critical Points of U[x] and Taylor Expanded TE[p,x] (total energy as a function of ‘p’ and ‘x’) about x=Critical points.
4)Used what we obtained in 3) to get the phase trajectory of the particle for x-> Critical points.
5)Then obtained the phase trajectory, using StreamPlot and ContourPlot, of the particle for any ‘x’, not necessarily near the critical point and without any Taylor expansion.
6)Then wrote the equation of motion of the particle as a differential equation of order 2 , and then by assuming suitable initial conditions, plotted the phase trajectories of the particle.
7)Then wrote the equation of motion as a system of 2 coupled first order differential equations and once again plotted the phase trajectory by assuming suitable initial conditions.
8)Displayed what we obtained in 5-7 using the function Show[].

Code for computation

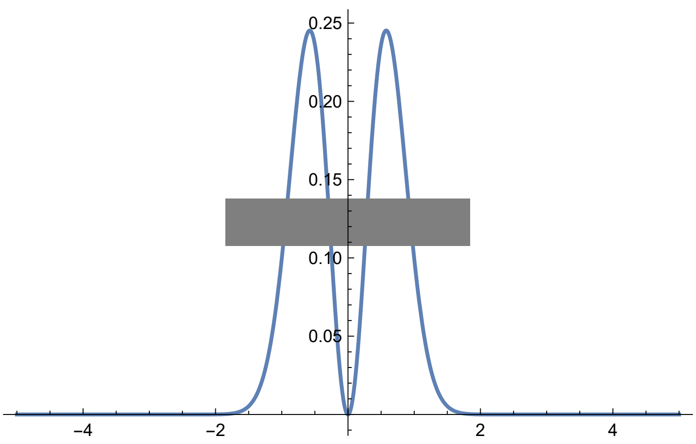
In[1]:=
```
U[x_]= a*x^2*Exp[-b*x^2]  (*Declaring U[x]*)
a=2;
b=3;   (*Giving 'a' and 'b' suitable values*)
U[x]
```

Out[1]= $a\, e^{-b x^2} x^2$

Out[4]= $2\, e^{-3 x^2} x^2$

In[8]:=
```
Plot[U[x], {x, -5, 5}](*Plotting U[x]*)
```

Out[8]=

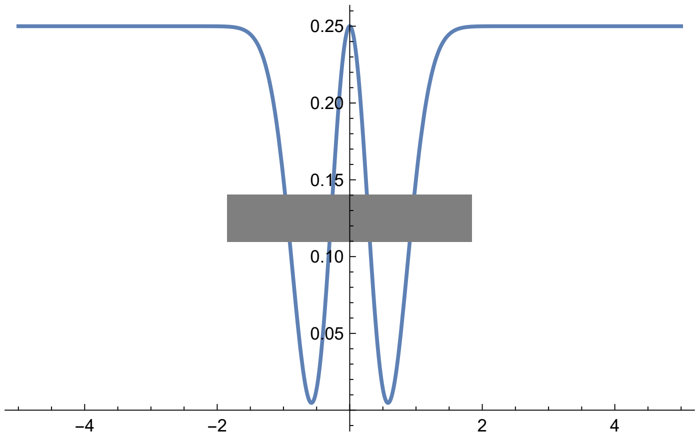


```
totenergy = 0.25;
KE[x_] = totenergy - U[x]
(*Plotting the Kinetic Energy by assuming a suitable value of total energy = 0.25*)
Plot[KE[x], {x, -5, 5}]
```

Out[10]=

$$0.25 - 2\, e^{-3 x^2} x^2$$

Out[11]=




```
Ud[x_] := D[U[x], x]   (*Finding the Critical points of U[x]*)
CriticalPts = SolveValues[Evaluate[Ud[x] == 0], x, Reals]
```

Out[13]=

$$\left\{0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$$

```
TotalEnergy[p_, x_] =
 p^2 / (2 * m) + U[x] (*Total energy as a function of momentum and x[t]*)
```

Out[14]=

$$\frac{p^2}{2\, m} + 2\, e^{-3 x^2} x^2$$

In[15]:=

```
TEexp[p_, x_] := Series[TotalEnergy[p, x], {x, CriticalPts[[1]], 3}]
TEexp[p, x]    (*Taylor expansion of U[x] about Critical points*)
```

Out[16]=

$$\frac{p^2}{2\, m} + 2\, x^2 + O[x]^4$$

In[17]:=

```
(*Normalising what we got*)
TEexpN[p_, x_] := Normal[TEexp[p, x]]
TEexpN[p, x]
```

Out[18]=

$$\frac{p^2}{2\, m} + 2\, x^2$$

In[19]:=

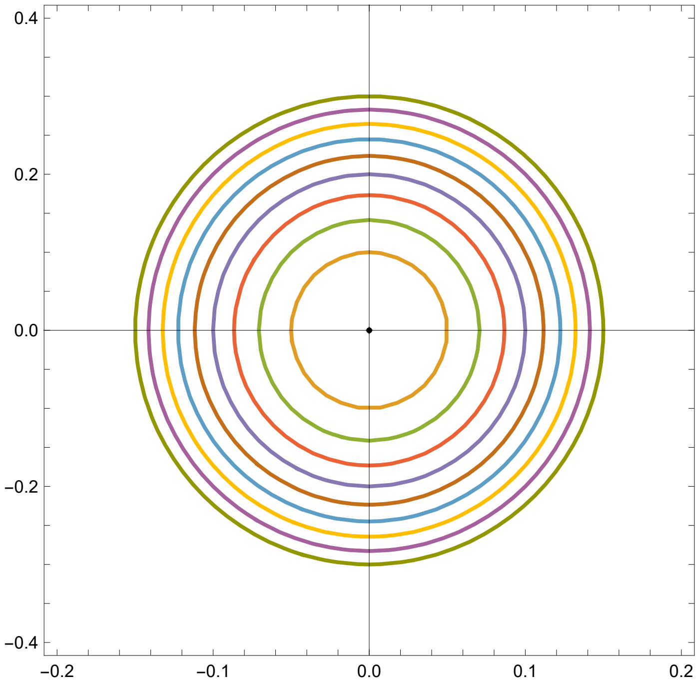
```
TEexpNf[p_, x_] := TEexpN[p, x] /. {m → 1}
TEexpNf[p, x]  (*Putting mass of particle=1*)
```

Out[20]=

$$\frac{p^2}{2} + 2\, x^2$$

In[21]:=
```
data = Table[TEexpNf[p, x] == j, {j, 0.00001, 0.05, 0.005}];
cplt = ContourPlot[Evaluate[data], {x, -0.2, +0.2}, {p, -0.4, 0.4}];
axes = {Line[{{0, -0.5}, {0, 0.5}}], Line[{{-0.5, 0}, {0.5, 0}}]};
ax = Graphics[{Thin, Black, axes}];
point = Graphics[{Point[{0, 0}]}];
Show[cplt, ax, point]
(*Plotting Phase Trajectory of the particle about one of the Critical points*)
```

Out[26]=



In[27]:=
```
(*plotting the phase trajectories for the particle about the other critical
 points by taylor expanding the Total energy about those points also*)
TEexp2[p_, x_] := Series[TotalEnergy[p, x], {x, CriticalPts⟦2⟧, 3}]
TEexp2[p, x]
```

Out[28]=

$$\left(\frac{2}{3\,e} + \frac{p^2}{2\,m}\right) - \frac{4\left(x + \frac{1}{\sqrt{3}}\right)^2}{e} - \frac{4\left(x + \frac{1}{\sqrt{3}}\right)^3}{\sqrt{3}\,e} + O\left[x + \frac{1}{\sqrt{3}}\right]^4$$

In[29]:=
```
TEexpN2[p_, x_] := Normal[TEexp2[p, x]]
TEexpN2[p, x]
```

Out[30]=

$$\frac{2}{3\,e} + \frac{p^2}{2\,m} - \frac{4\left(\frac{1}{\sqrt{3}} + x\right)^2}{e} - \frac{4\left(\frac{1}{\sqrt{3}} + x\right)^3}{\sqrt{3}\,e}$$

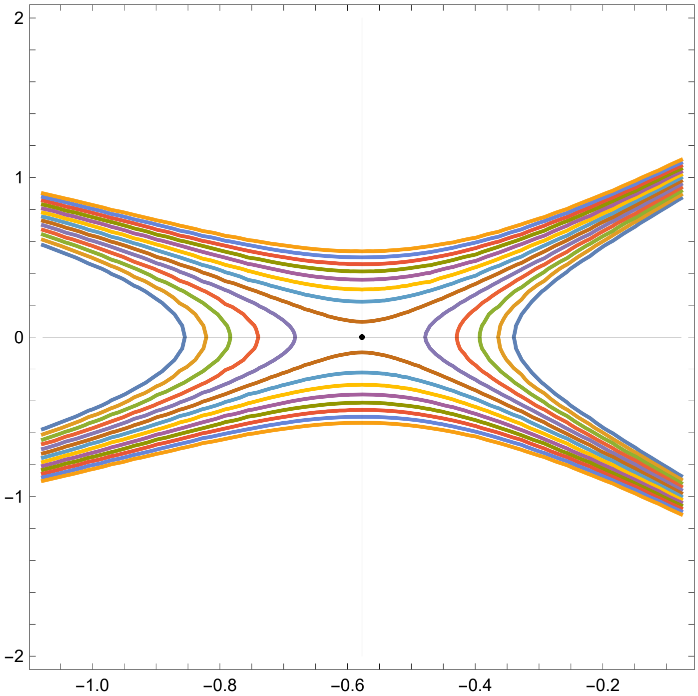
In[31]:=
```
TEexpNf2[p_, x_] := TEexpN2[p, x] /. {m → 1}
TEexpNf2[p, x]
```

Out[32]=

$$\frac{2}{3\,e} + \frac{p^2}{2} - \frac{4\left(\frac{1}{\sqrt{3}} + x\right)^2}{e} - \frac{4\left(\frac{1}{\sqrt{3}} + x\right)^3}{\sqrt{3}\,e}$$

In[33]:=
```
data2 = Table[TEexpNf2[p, x] == j, {j, 0.15, 0.4, 0.02}];
cplt2 =
  ContourPlot[Evaluate[data2], {x, -1 / Sqrt[3] - 0.5, -1 / Sqrt[3] + 0.5}, {p, -2, 2}];
axes2 = {Line[{{-1 / Sqrt[3], -2}, {-1 / Sqrt[3], 2}}],
   Line[{{-1 / Sqrt[3] - 0.5, 0}, {-1 / Sqrt[3] + 0.5, 0}}]};
ax2 = Graphics[{Thin, Black, axes2}];
point2 = Graphics[{Point[{-1 / Sqrt[3], 0}]}];
Show[cplt2, ax2, point2]
```

Out[38]=



In[39]:=
```
(*For the 3rd critical point*)
TEexp3[p_, x_] := Series[TotalEnergy[p, x], {x, CriticalPts[[3]], 3}]
TEexp3[p, x]
```

Out[40]=

$$\left(\frac{2}{3\,e}+\frac{p^2}{2\,m}\right)-\frac{4\left(x-\frac{1}{\sqrt{3}}\right)^2}{e}+\frac{4\left(x-\frac{1}{\sqrt{3}}\right)^3}{\sqrt{3}\,e}+O\left[x-\frac{1}{\sqrt{3}}\right]^4$$

In[41]:=
```
TEexpN3[p_, x_] := Normal[TEexp3[p, x]]
TEexpN3[p, x]
```

Out[42]=

$$\frac{2}{3\,e}+\frac{p^2}{2\,m}-\frac{4\left(-\frac{1}{\sqrt{3}}+x\right)^2}{e}+\frac{4\left(-\frac{1}{\sqrt{3}}+x\right)^3}{\sqrt{3}\,e}$$

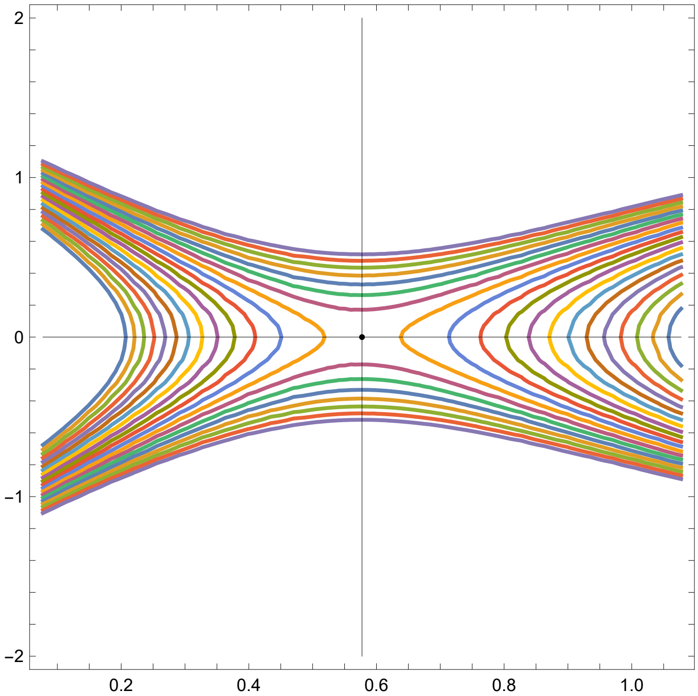
In[43]:=
```
TEexpNf3[p_, x_] := TEexpN3[p, x] /. {m → 1}
TEexpNf3[p, x]
```

Out[44]=

$$\frac{2}{3\,e}+\frac{p^2}{2}-\frac{4\left(-\frac{1}{\sqrt{3}}+x\right)^2}{e}+\frac{4\left(-\frac{1}{\sqrt{3}}+x\right)^3}{\sqrt{3}\,e}$$

In[45]:=
```
data3 = Table[TEexpNf3[p, x] == j, {j, 0.00001, 0.4, 0.02}];
cplt3 =
  ContourPlot[Evaluate[data3], {x, 1 / Sqrt[3] - 0.5, 1 / Sqrt[3] + 0.5}, {p, -2, 2}];
axes3 = {Line[{{1 / Sqrt[3], -2}, {1 / Sqrt[3], 2}}],
   Line[{{1 / Sqrt[3] - 0.5, 0}, {1 / Sqrt[3] + 0.5, 0}}]};
ax3 = Graphics[{Thin, Black, axes3}];
point3 = Graphics[{Point[{1 / Sqrt[3], 0}]}];
Show[cplt3, ax3, point3]
```

Out[50]=



In[51]:=
```
(*Generating the general trajectory of the particle without any taylor expansion*)
TotalEnergy[p, x]
```

Out[51]=

$$\frac{p^2}{2\,m} + 2\,e^{-3\,x^2}\,x^2$$

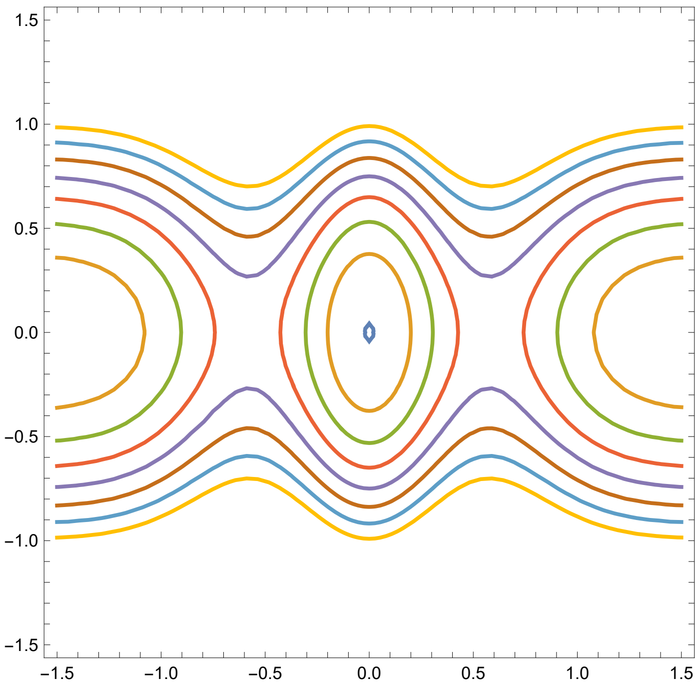
In[52]:=
```
GenTraj[p_, x_] = TotalEnergy[p, x] /. {m → 1}
```

Out[52]=

$$\frac{p^2}{2} + 2\,e^{-3\,x^2}\,x^2$$

In[53]:=
```
dataGenTraj = Evaluate[Table[GenTraj[p, x] == j, {j, 0.001, 0.5, 0.07}]] ;

plot1 = ContourPlot[Evaluate[dataGenTraj], {x, -3 / 2 , 3 / 2 }, {p, -3 / 2, 3 / 2}]
```

Out[54]=



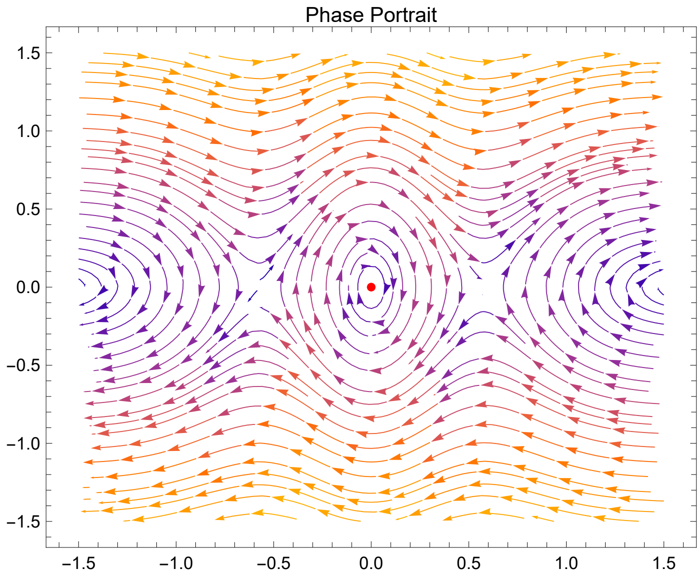
In[55]:=
```
(*Plotting using StreamPlot*)
TotalEnergy[p, x]
```

Out[55]=

$$\frac{p^2}{2\,m} + 2\,e^{-3\,x^2}\,x^2$$

```
field = {v, -U'[x]};
(*Took this as the field as the derivative of its components are {x,p}*)
plot2 = StreamPlot[field, {x, -3 / 2, 3 / 2},
  {v, -3 / 2, 3 / 2}, Frame → True, StreamPoints → Fine,
AspectRatio → 0.8,
  Epilog → {Red, PointSize → Medium, Point[{{0, 0}}]}, PlotLabel → "Phase Portrait"]
```

Out[57]=



```
(*Solving the equation of motion and giving
 suitable initial conditons to obtain phase trajectory*)
U[x_] = a * x^2 * Exp[-b * x^2]
a = 2;
b = 3;
F[x_] = -D[U[x], x]
eqn1 := {x''[t] == F[x[t]], x[0] == 4 / 10, x'[0] == 0.0};
soln = NDSolve[eqn1, x[t], {t, 0, 100}]
solnX[t_] = x[t] /. Flatten[soln]
soln2 = NDSolve[eqn1, x'[t], {t, 0, 100}]
solnVel[t_] = x'[t] /. Flatten[soln2]
d1 = Table[{solnX[t], solnVel[t]}, {t, 0, 30, 0.01}];
plot3 = ListLinePlot[d1]
```

Out[58]=

$2\, e^{-3 x^2} x^2$

Out[61]=

$-4\, e^{-3 x^2} x + 12\, e^{-3 x^2} x^3$

Out[63]=

{{x[t] → InterpolatingFunction[Domain: {{0., 100.}} Output: scalar][t]}}

Out[64]=

InterpolatingFunction[Domain: {{0., 100.}} Output: scalar][t]

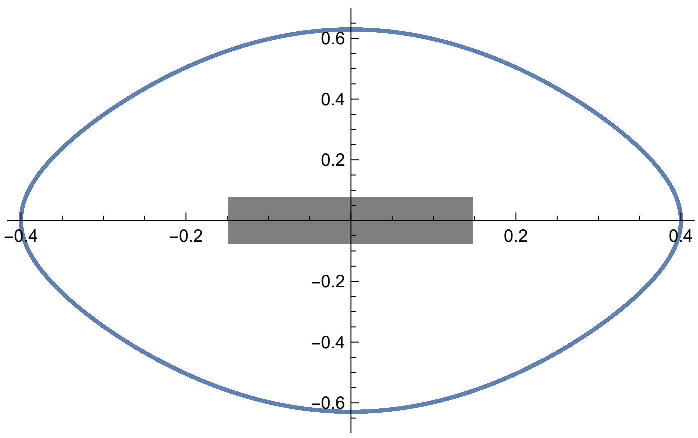
Out[65]=

{{x′[t] → InterpolatingFunction[Domain: {{0., 100.}} Output: scalar][t]}}

Out[66]=

InterpolatingFunction[Domain: {{0., 100.}} Output: scalar][t]

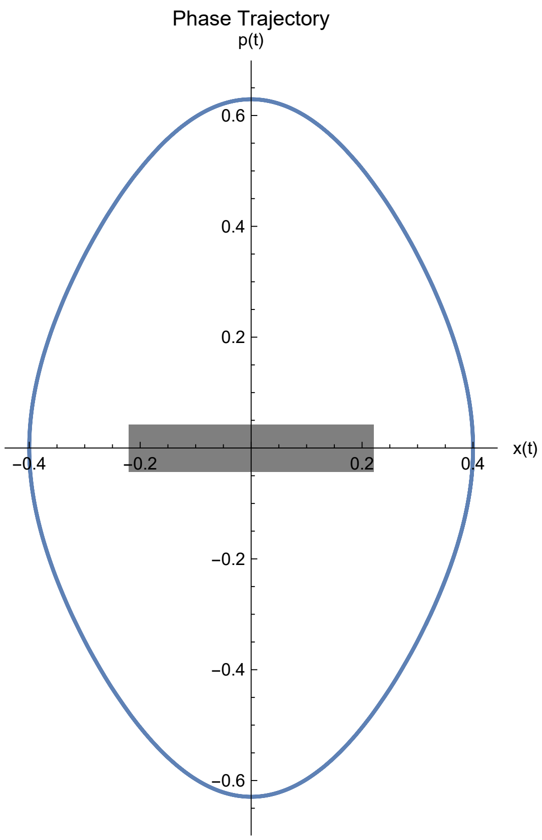
Out[68]=



```
(*Obtained an ellipse indicating bounded motion as expected
 (as total energy is set lesser than the maximum potential energy and
    i have set the initial conditions such that at x==0.4 velocity==0*)
```

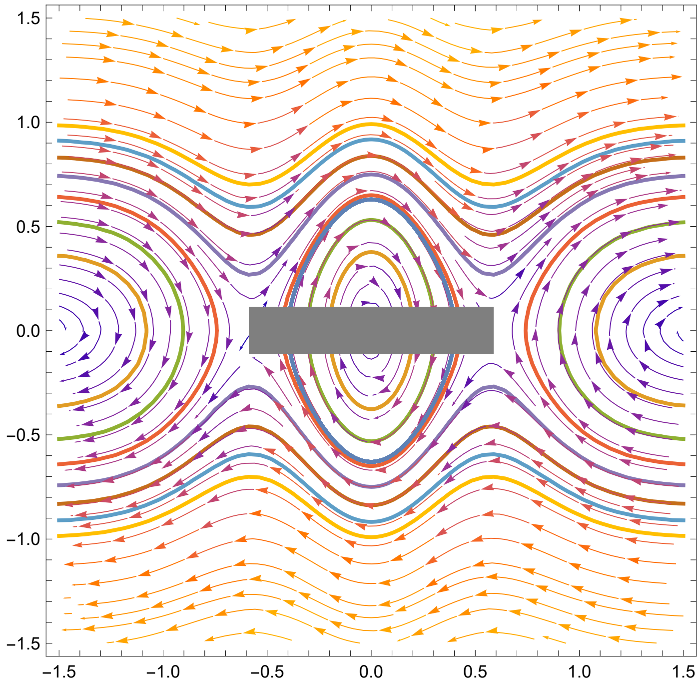
In[69]:=
```
(*Writing the equation of motion as a system of 2 coupled first order differential
 equations and giving suitable initial conditions to obtain the Phase Trajectory*)
U[x_] = a * x^2 * Exp[-b * x^2];
a = 2;
b = 3;
F[x_] = -D[U[x], x];
x0 = 0.4;
p0 = 0;
eqns = {
   1 * x'[t] == p[t],
   p'[t] == F[x[t]]};
sol = NDSolve[{eqns, x[0] == x0, p[0] == p0}, {x, p}, {t, 0, 100}];
plot4 = ParametricPlot[Evaluate[{x[t], p[t]} /. sol], {t, 0, 10},
  PlotLabel → "Phase Trajectory", AxesLabel → {"x(t)", "p(t)"}]
```

Out[77]=



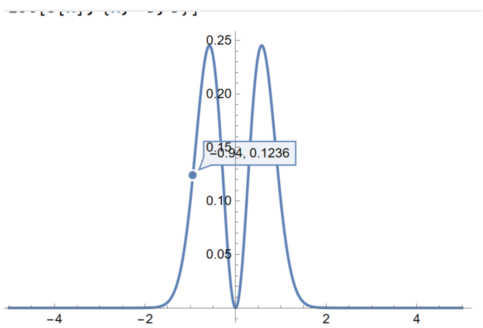
In[78]:=
```
(*Using Show[] to show all the plots we got in 5-7 in a single plot*)
Show[{plot1, plot2, plot3, plot4}]
```

Out[78]=

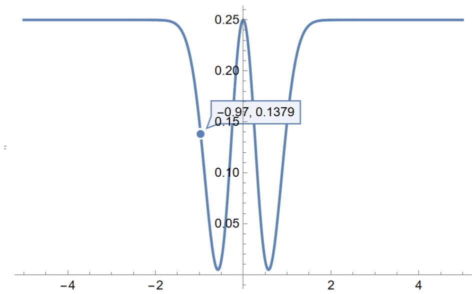


Results

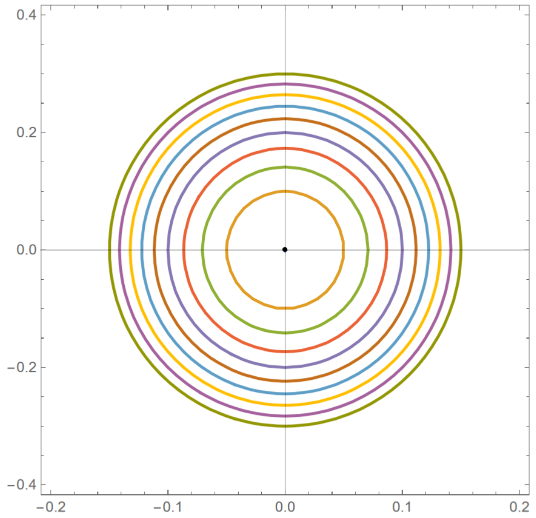
```
(*Plot of U[x]*)
```



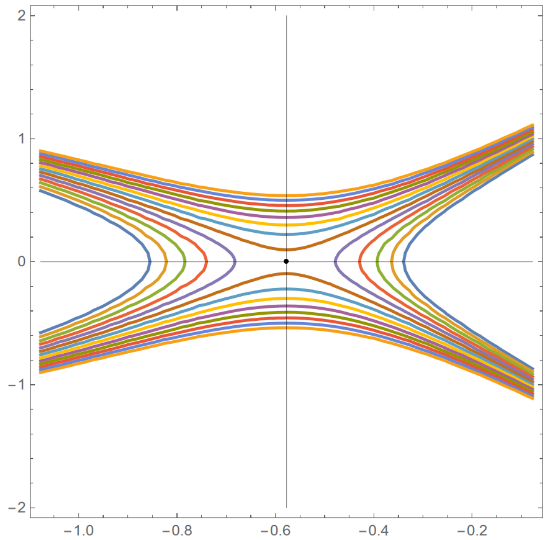

```
(*Plot of KE[x]*)
```

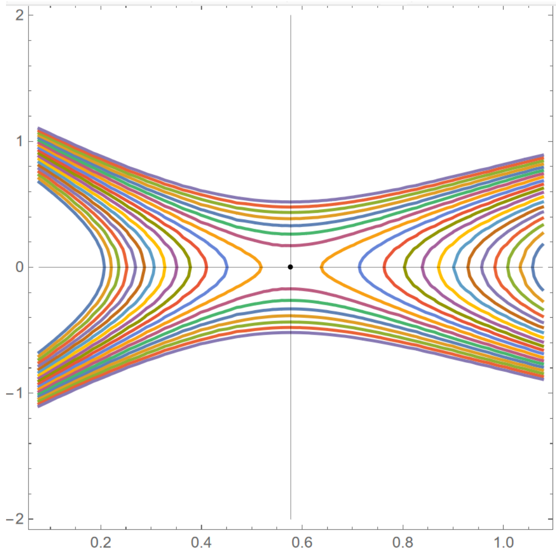



```
(*Phase trajectory about the point of stable eq*)
```

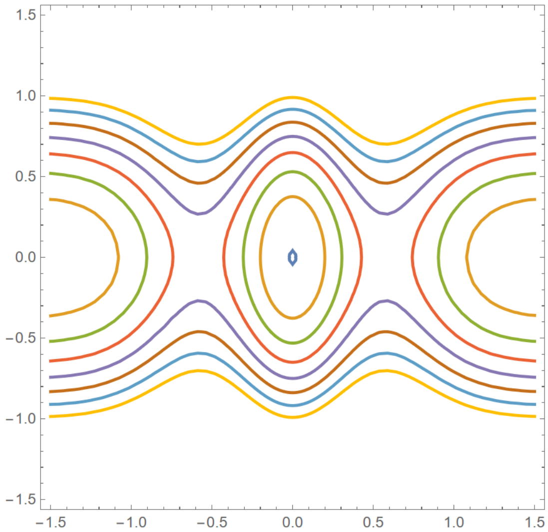


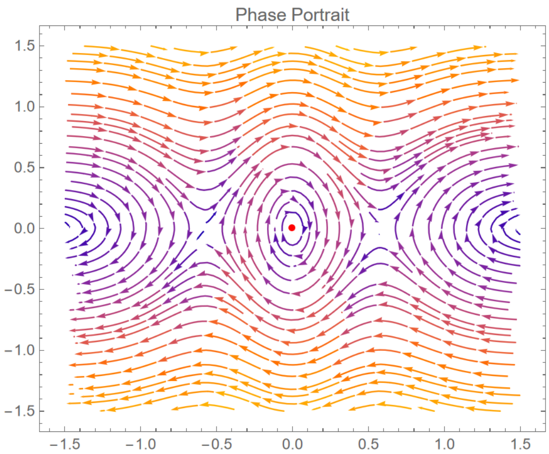

```
(*Phase trajectory about the 2 points of unstable eq*)
```





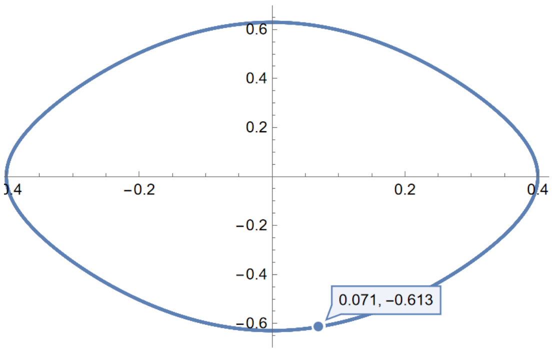
```
(*The 2 plots of the phase trajectory without
 any assumptions using contour plot and streamplot*)
```



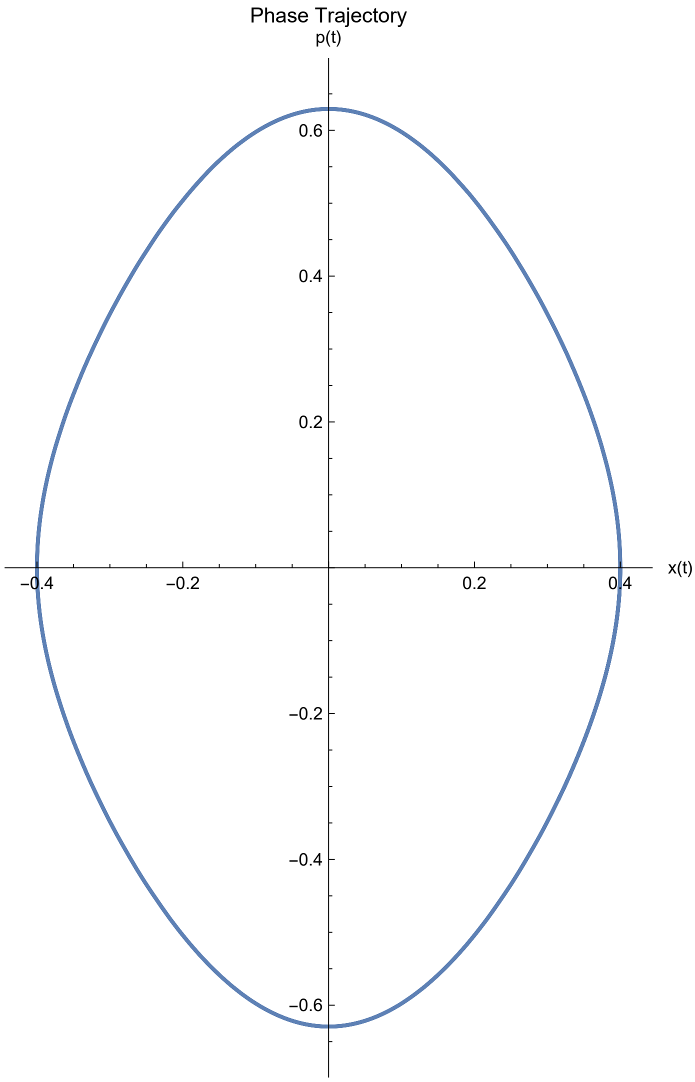



```
(*Solving equation of motion to get phase trajectory*)
```

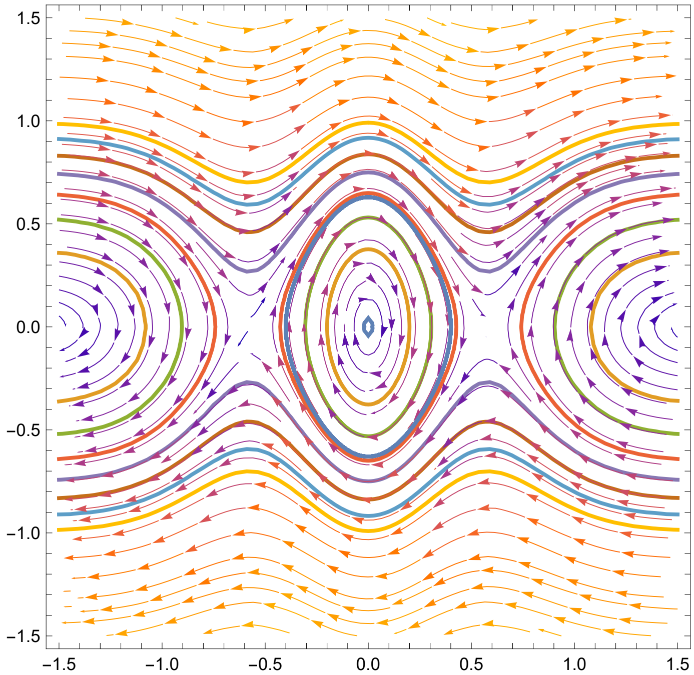
```
{{x[t] → InterpolatingFunction[ Domain: {{0., 100.}} Output: scalar ][t]}}

InterpolatingFunction[ Domain: {{0., 100.}} Output: scalar ][t]

{{x′[t] → InterpolatingFunction[ Domain: {{0., 100.}} Output: scalar ][t]}}

InterpolatingFunction[ Domain: {{0., 100.}} Output: scalar ][t]
```




```
(*Phase trajectory by writing the equation of motion
 as a system of coupled first order differential equations*)
```




```
(*Obtaining all the plots together*)
```



Comments

This assignment was time consuming. It involved lot of Physics (analysing the motion and dynamics of the particle).

References

https://chat.openai.com/