

Theory of CHSH Violation for a Rotated Bell State

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Abstract

This document presents the theoretical background for the numerical simulation that evaluates the CHSH quantity for a maximally entangled Bell state under local rotation operations. The code computes the expectation value of the CHSH correlator as a function of rotation angles and illustrates violation of the classical CHSH bound.

1 The Bell State

The simulation uses the well-known Bell state

$$|\psi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad (1)$$

which is a maximally entangled two-qubit state.

In the code this is produced via

$$|0\rangle = \text{basis}(2,0), \quad |1\rangle = \text{basis}(2,1),$$

and

$$|\psi_+\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

2 Local Rotation Operator

Each qubit is independently rotated by a single-qubit unitary of the form

$$R(\theta) = \cos \theta \mathbb{I} + \sin \theta (|1\rangle \langle 0| - |0\rangle \langle 1|). \quad (2)$$

This operator is equivalent (up to phase) to a rotation around the y -axis of the Bloch sphere:

$$R(\theta) = \exp(-i\theta \sigma_y). \quad (3)$$

In the code this is implemented as

$$R(\theta) = \cos \theta I + \sin \theta (|1\rangle \langle 0| - |0\rangle \langle 1|).$$

3 Measurement Observable

Both qubits are measured in the σ_z basis. The joint observable is

$$A \otimes B = \sigma_z \otimes \sigma_z. \quad (4)$$

Expectation values are computed after rotating the Bell state:

$$E(\theta_i, \theta'_j) = \langle \psi_+ | R(\theta_i)^{\otimes 2\dagger} (\sigma_z \otimes \sigma_z) R(\theta'_j)^{\otimes 2} | \psi_+ \rangle.$$

4 CHSH Quantity

The CHSH expression is defined as

$$S = E(a, b) + E(a, b') + E(a', b) - E(a', b'). \quad (5)$$

Classically, any local hidden-variable theory obeys the bound

$$|S| \leq 2.$$

Angle Choices

Your simulation uses

$$a = \frac{t}{2}, \quad a' = -\frac{t}{2}, \quad b = 0, \quad b' = t.$$

For each value of the parameter t , the code computes $S(t)$.

5 Violation of CHSH Inequality

The plot produced by your script shows:

- The CHSH value $S(t)$ as a function of the angle parameter.
- Shaded regions between $|S| = 2$ and $|S| = 5$ emphasizing the quantum-violating region.

Whenever

$$|S(t)| > 2,$$

the Bell inequality is violated, demonstrating that the correlations generated by the Bell state cannot be explained by any local realistic theory.

6 Summary

This simulation evaluates the CHSH correlator for a maximally entangled Bell state subjected to local rotations. The numerical results reproduce the theoretical prediction: the CHSH quantity oscillates and periodically exceeds the classical bound of 2, confirming quantum nonlocality. The method implemented corresponds directly to the theoretical formulation of Bell tests using local unitary rotations and σ_z measurements.