Time Evolution of the Quantum Harmonic Oscillator

1 The Quantum Harmonic Oscillator

The Hamiltonian of the one-dimensional quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the lowering and raising operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{2}$$

The energy eigenvalues and eigenstates (number states) are

$$\hat{H}\left|n\right\rangle = E_n\left|n\right\rangle, \qquad n = 0, 1, 2, \ldots \tag{3}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{4}$$

2 Position-Space Wave Functions

The normalized energy eigenfunctions in position representation are

$$\psi_n(x) = \left\langle n\right| x \left|n\right\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n(\xi)e^{-\xi^2/2}, \tag{5}$$

where $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ is the dimensionless coordinate, and $H_n(\xi)$ are the (physicists') Hermite polynomials:

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}. \tag{6}$$

It is common to define the characteristic length

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}, \tag{7}$$

so that $\xi = x/x_0$ and the wave functions can be written as

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\, x_0\sqrt{\pi}}} H_n\left(\frac{x}{x_0}\right)\exp\left(-\frac{x^2}{2x_0^2}\right). \tag{8}$$

3 Time Evolution of Energy Eigenstates

Since $|n\rangle$ are energy eigenstates, their time evolution is simply a phase factor:

$$|n(t)\rangle = e^{-iE_n t/\hbar} |n(0)\rangle = e^{-i\omega(n+1/2)t} |n\rangle . \tag{9}$$

Thus, in position space:

$$\psi_n(x,t) = e^{-i(n+1/2)\omega t}\psi_n(x). \tag{10}$$

4 General State and Time Evolution

Any state can be expanded in the energy basis:

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle , \qquad \sum_{n=0}^{\infty} |c_n|^2 = 1. \tag{11}$$

The time-evolved state is

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-i(n+1/2)\omega t} |n\rangle . \tag{12}$$

The position-space wave function is therefore

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n \, e^{-i(n+1/2)\omega t} \, \psi_n(x). \tag{13}$$

If the expansion is truncated to a finite number of terms (as done numerically), one simply sums over the included n and renormalizes if necessary:

$$\psi(x,t) \approx \frac{\sum_{n=0}^{N} c_n e^{-i(n+1/2)\omega t}\psi_n(x)}{\sqrt{\sum_{n=0}^{N} |c_n|^2}}. \tag{14}$$

5 Summary of Formulas Used in the Python Code

The code you posted implements exactly Eq. (13) with coefficients

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = 3, \quad c_3 = 7 \tag{15}$$

(then normalized inside the function), $x_0 = \sqrt{\hbar/(m\omega)}$, and time evolution factor $e^{-i(n+1/2)\omega t}$ for each term.

The probability density $|\psi(x,t)|^2$ shows a complicated but periodic motion with period $T = 2\pi/\omega$ (revival time, since all phases are multiples of ωt).