Homogeneous recurrences.

Form: $a_0 t_n + a_1 t_{n-1} + \dots a_k t_{n-k} = 0$.
where t_i are the values we are looking for.

Solution is $t_n = \sum_{i=1}^{k} c_i r_i^n$ k = no of roots

1. Consider the recurrence

$$t_n = \begin{cases} 0 & \text{if } n=0 \\ 5 & \text{if } n=1 \\ 3t_{n-1} + 4t_{n-2} & \text{otherwise.} \end{cases}$$

$$t_n - 3t_{n-1} - 4t_{n-2} = 0$$

The characteristic Polynomial is

$$x^2 - 3x - 4 = (x+1)(x-4) = 0$$

Roots are $r_1 = -1$ $r_2 = 4$

Therefor general solution is

$$t_n = c_1 r_1^n + c_2 r_2^n$$
$$= c_1(-1)^n + c_2(4)^n$$

The initial condition give

$$\left.\begin{aligned} c_1(-1)^0 + c_2 4^0 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}\right\} n=0$$

$$\left.\begin{aligned} c_1(-1)^1 + c_2(4)^1 &= 5 \\ -c_1 + 4c_2 &= 5 \end{aligned}\right\} n=1$$

Solving these equations $\boxed{c_1 = -1}$ $\boxed{c_2 = 1}$

$$t_n = (-1)(-1)^n + 1 \cdot 4^n$$
$$= 4^n - (-1)^n$$

$$\boxed{t_n = \Theta(4^n)}$$

② $t_n = \begin{cases} n & \text{if } n=0 \text{ \& } n=1 \\ 5t_{n-1} - 6t_{n-2} \end{cases}$

Now $t_n - 5t_{n-1} + 6t_{n-2} = 0$.
Characteristic Polynomial is

$x^2 - 5x + 6 = 0$

$(\cancel{x-6})(\cancel{x+1})$

$(x-3)(x-2) = 0 \quad \therefore x = 3 \quad x = 2.$

roots are $\boxed{r_1 = 3}$ $\boxed{r_2 = 2}$

$t_n = c_1 r_1^n + c_2 r_2^n$

$= c_1 3^n + c_2 2^n$

$c_1 3^0 + c_2 2^0 = 0$ when $n = 0$.

$c_1 + c_2 = 0$ — ①

$c_1 \cdot 3 + c_2 \cdot 2 = 1$

ie $3c_1 + 2c_2 = 1$ — ②

Solving these two

$3c_1 - 2c_2 = 1$

$\therefore \boxed{\begin{matrix} c_1 = 1 \\ c_2 = -1 \end{matrix}}$

$\therefore t_n = c_1 r_1^n + c_2 r_2^n$

$= 3^n + (-1) 2^n$

$= O(3^n).$

Inhomogeneous recurrence [illegible]

Inhomogeneous recurrence are of the form [illegible]

[illegible] [illegible] $P(n)$

[illegible] is similar to homogeneous [illegible] [illegible] $P(n)$ where b is a constant [illegible] $P(n)$ is polynomial in n of degree [illegible]

(1) Consider the recurrence

$t_n - 2t_{n-1} = 3^n$

$t_n - 2t_{n-1} = 3^n$ (1)

[illegible] and $P(n) = 1$ of degree 0

Multiply recurrence by 3

$3t_n - 6t_{n-1} = 3^{n+1}$

Now replace n by $n-1$

$3t_{n-1} - 6t_{n-2} = 3^n$ (2)

Subtract eq (2) from (1)

$t_n - 5t_{n-1} + 6t_{n-2} = 0$

which is homogeneous equation

$x^2 - 5x + 6 = (x-2)(x-3) = 0$

$r_1 = 2$ and $r_2 = 3$

$t_n = c_1 2^n + c_2 3^n$ (3)

from eq (1)

$t_1 = 2t_0 + 3$

from (3)

$c_1 + c_2 = t_0$ for $n = 0$

$2c_1 + 3c_2 = 2t_0 + 3$ for $n = 1$

So solving these equations

$c_1 = t_0 - 3$ & $c_2 = 3$.

$t_n = (t_0 - 3)2^n + 3^{n+1}$

$t_n = \Theta(3^n)$ Provided $t_0 \geq 0$.

(4) $t_m = \begin{cases} 0 & \text{if } m = 0 \\ 2t(m-1) + 1 & \text{otherwise.} \end{cases}$

Now $t_m = 2t(m-1) + 1$

$t_{m-1} = 2t(m-2) + 1$.

$t_m - 2t(m-1) = 1$ —— (1)

$t_{m-1} - 2t(m-2) = 1$ —— (2).

subtract (2) from (1).

$t_m - 3t(m-1) + 2t(m-2) = 0$.

$x^2 - 3x + 2 = 0$.

$(x-2)(x-1) = 0$.

$\therefore r_1 = 2 \quad r_2 = 1$.

$t_m = \cancel{t_m} = c_1 2^m + c_2$

$t_0 = c_1 + c_2 = 0$ from $m = 0$. —— (3)

$t_m = 2t(m-1) + 1$ when $m = 1$

$= 2t(0) + 1$

$= 1$.

$\therefore 2c_1 + c_2 = 1$ —— (4)

From (3) and (4)

$c_1 = 1 \quad c_2 = -1$

$t_m = 2^m - 1$

$\Theta = (2^m)$.

Towers of Hanoi (n, x, Y, Z)

Algorithm Towersofhanoi (n, x, Y, Z)
{ // move the top n disks from tower x to tower Y.
if (n ≥ 1) then
{ Towersofhanoi (n-1, x, Z, Y)
write (" move top disk from tower", x, " to top of tower", Y);
Towersofhanoi (n-1, Z, Y, x);
}
}

Alg TOH(3, x, Y, Z)
{ 3≥0
TOH(2, x, Z, Y)
x → Y ①
TOH(2, Z, Y, x)
}

TOH(2, x, Z, Y)
{ 2≥0
TOH(1, x, Y, Z)
x → Z ②
TOH(1, Y, Z, x)
}

TOH(2, Z, Y, x)
{ 2≥0
TOH(1, Z, x, Y)
Z → Y ⑥
TOH(1, x, Y, Z)
}

TOH(1, x, Y, Z)
{ 1≥0
TOH(0, x, Z, Y)
x → Y ①
TOH(0, Z, Y, x)
}

TOH(1, Y, Z, x)
{ TOH(0, Y, x, Z)
Y → Z ③
TOH(0, x, Z, Y)
}

TOH(1, Z, x, Y)
{ 1≥0
TOH(0, Z, Y, x)
Z → x ⑤
TOH(0, Y, x, Z)
}

TOH(1, x, Y, Z)
{ 1≥0
TOH(0, x, Z, Y)
x → Y ⑦
TOH(0, Z, Y, x)
}