The One-Cut Linear Programming Approach (Model II) for the Cutting Stock Problem

Based on the work of Harald Dyckhoff (1981)

December 1, 2025

Abstract

The classical Gilmore-Gomory approach to the One-Dimensional Cutting Stock Problem relies on column generation to handle an exponential number of cutting patterns. Dyckhoff (1981) proposed an alternative formulation, "Model II", based on a recursive "One-Cut" principle. This document details the mathematical formulation of Model II, which uses a polynomial number of variables and constraints for many practical instances, allowing it to be solved using standard Linear Programming software without column generation.

1 Problem Definition

We consider the **Standard Problem** of one-dimensional cutting stock optimization. Given:

- A set of standard stock lengths $S = \{s_1, \ldots, s_K\}$.
- A set of required order lengths (demand) $D = \{d_1, \ldots, d_I\}$.
- Demand requirements N_l for each order length $l \in D$.
- Costs c_l associated with consuming a standard length $l \in S$.

The objective is to satisfy all demands N_l while minimizing the total cost of stock used.

2 The One-Cut Concept

Unlike the classical approach (Model I), which defines a variable for every possible complex cutting pattern (e.g., "one bin contains 2 items of size A and 3 of size B"), Model II is based on a recursive cutting process.

Assumption 1 (The One-Cut Principle)**.** *The cutting process is modeled as an unlimited sequence of cutting operations. In each operation, a piece of length k is divided into exactly two new pieces:*

1. *A section of an order length $l \in D$ (where $l < k$).*
2. *A residual section of length $k - l$.*

This simple structure $[k; l]$ allows complex patterns to be built successively. For example, cutting a length of 9 into $\{4, 2, 2, 1\}$ is modeled as:

$$[9; 4] \to \text{Residue } 5 \to [5; 2] \to \text{Residue } 3 \to [3; 2] \to \text{Residue } 1$$

3 Mathematical Formulation (Model II)

3.1 Sets and Parameters

Let R be the set of all relevant residual lengths (lengths that can be produced by cutting order lengths from stock lengths). The model considers all lengths in the set $L = S \cup D \cup R$.

3.2 Decision Variables

The fundamental decision variables represent the number of times a specific "one-cut" is performed:

$$y_{k,l} \geq 0 \quad \text{for } k \in S \cup R,\ l \in D,\ l < k \tag{1}$$

$y_{k,l}$ represents the number of pieces of length k that are cut to produce one item of order length l and a remainder of $k - l$.

3.3 Constraints

The model relies on **flow conservation (balance) constraints** for every length l that is not a standard stock length (i.e., for all $l \in (D \cup R) \setminus S$).

The logic is: *Total Input of length l $\geq$ Total Output of length l.*

$$\underbrace{\sum_{k \in A_l} y_{k,l}}_{\text{Production from larger cuts}} + \underbrace{\sum_{k \in B_l} y_{k+l,k}}_{\text{Production as residue}} \geq \underbrace{\sum_{k \in C_l} y_{l,k}}_{\text{Consumption for smaller cuts}} + \underbrace{N_l}_{\text{Final Demand}} \tag{2}$$

Where the sets are defined as:

- $A_l = \{k \in S \cup R \mid k > l\}$: Lengths k that can be cut to produce l as the primary order piece.
- $B_l = \{k \in D \mid k + l \in S \cup R\}$: Lengths $k + l$ that, when cut into order size k, leave l as the residue.
- $C_l = \{k \in D \mid k < l\}$: Order lengths k that can be cut *from* length l.

3.4 Objective Function

The objective is to minimize the net cost of standard lengths consumed.

$$\text{Minimize } Z = \sum_{l \in S} c_l \left(\sum_{k \in C_l} y_{l,k} - \sum_{k \in B_l} y_{k+l,k} \right) \tag{3}$$

This calculates the net consumption of standard length l as: (Total pieces of l cut) minus (Total pieces of l produced as residue from larger stock).

4 Comparison with Classical Model (Model I)

4.1 Model Size Analysis

Model II generally has more constraints than Model I but drastically fewer variables.

- **Model I:** Constraints $\approx I$. Variables $\approx$ Millions.
- **Model II:** Constraints $\approx S_{max}$ (max standard length). Variables $\approx I \cdot (K + S_{max})$.

For problems with many stock lengths or a high ratio of demand sizes to stock size ($|D|/S_{max} \approx 1$), Model II can be significantly more efficient and easier to implement.

Feature	Model I (Gilmore-Gomory)	Model II (Dyckhoff)
Variables	Cutting Patterns (Exponential)	One-Cuts (Polynomial)
Constraints	$\|D\|$ (Number of order lengths)	$\|D\| + \|R\|$ (Orders + Residues)
Solution Method	Column Generation	Standard Simplex
Structure	Static Patterns	Dynamic Flow

Table 1: Comparison of Approaches

5 Conclusion

Dyckhoff's Model II offers a distinct advantage for cutting stock problems where the variety of stock lengths is high or where "trim loss" has value (reusable residue). By treating the cutting process as a flow of materials through "one-cut" transformations, it avoids the complexity of generating all combinatorial patterns, providing an exact solution using standard LP solvers.