

Random Numbers

CS21BTECH11044*

CONTENTS

1	Uniform Random Numbers	1
2	Central Limit Theorem	2
3	From Uniform to Other	4

Abstract—This manual provides a simple introduction to the generation of random numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/exrand.c
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/coeffs.h
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/cdf_plot_uni.py
```

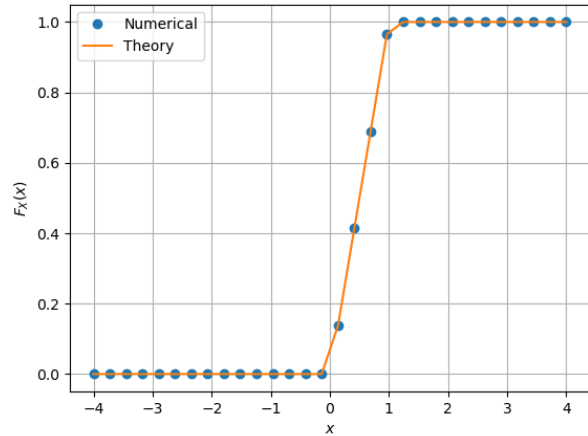


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: As we know

Probability density function of U :

$$P_U(x) = \begin{cases} 1 & \text{for } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

$$\begin{aligned} F_u(x) &= \int_{-\infty}^x \Pr(U = a) da \\ &= \int_{-\infty}^0 (0) da + \int_0^x (1) da \\ &= x \end{aligned}$$

$$F_u(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0, 1) \\ 1 & \text{for } x > 1 \end{cases} \quad (1.3)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.4)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.5)$$

Write a C program to find the mean and

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variance of U .

Solution:

The Mean and Variance of X is plotted in Fig. 2.4

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/q1_4.c
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/coeffs.h
```

```
collect2: error: ld returned 1 exit status
ubuntu@ubuntuXPS:~/Desktop/P&RV/A27_06/Q1$ gcc q1_4.c -lm
ubuntu@ubuntuXPS:~/Desktop/P&RV/A27_06/Q1$ ./a.out
mean = 0.500007
variance = 0.083301
```

Fig. 1.4: The Mean and Variance of X

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.6)$$

Solution:

1) $k = 1$: $E[U]$ is the mean

2) $k = 2$: $E[U^2]$

where, $\text{var}[U] = E[U^2] - (E[U])^2$

i) Verifying Mean of sample of 10^6 Uniform Random Variable

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x)$$

$$\text{As, } F_U(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0, 1) \\ 1 & \text{for } x > 1 \end{cases} \quad (1.7)$$

$$E[U] = 0 + \int_0^1 x dx + 0$$

$$E[U] = \frac{x^2}{2} \Big|_0^1$$

$$E[U] = 0.5$$

$$E[U] \approx 0.5 \quad (1.8)$$

ii) Verifying Variance of sample of 10^6 Uniform Random Variable

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x)$$

$$\text{As, } F_U(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0, 1) \\ 1 & \text{for } x > 1 \end{cases}$$

$$E[U^2] = 0 + \int_0^1 x^2 dx + 0$$

$$E[U^2] = \frac{x^3}{3} \Big|_0^1$$

$$E[U^2] = \frac{1}{3} = 0.33$$

Thus,

$$\text{Var}[U] = E[U^2] - (E[U])^2$$

$$\text{Var}[U] = \left(\frac{1}{3}\right) - \left(\frac{1}{2}\right)^2$$

$$\text{Var}[U] = 0.0833$$

$$\text{Var}[U] \approx 0.0833 \quad (1.9)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

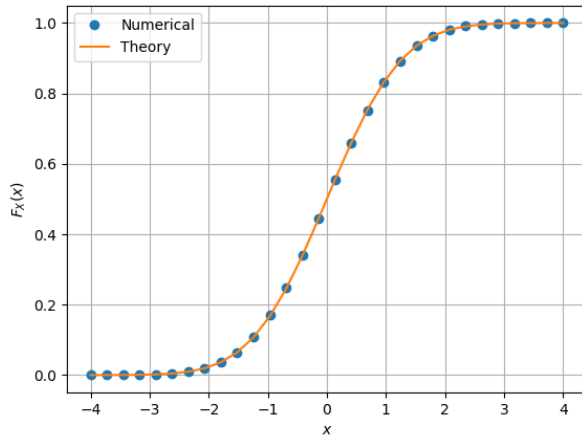
Solution:

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/q1_4.c
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/q2_4.c
```

Fig. 2.2: The CDF of X

The CDF of X is plotted in Fig. 2.2 Properties of CDF:

- cdf is non decreasing function
- $\lim_{x \rightarrow -\infty} f(x) = 0$
- $\lim_{x \rightarrow +\infty} f(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/pdf_plot_gau.py
```

Properties of PDF:

- It is symmetric about y-axis
- Area under curve is 1 i.e., $\int_{-\infty}^{\infty} P_X(x) = 1$

2.4 Find the mean and variance of X by writing a C program. **Solution:** The Mean and Variance of X is plotted in Fig. 2.4

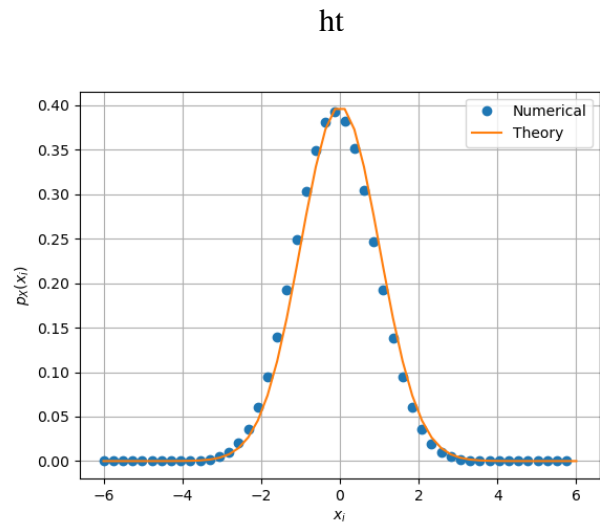
2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution:

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

Fig. 2.3: The PDF of X

```
ubuntu@ubuntuXPS:~/Desktop/P&RV/A27_06/Q1$ gcc q2_4.c -lm
ubuntu@ubuntuXPS:~/Desktop/P&RV/A27_06/Q1$ ./a.out
mean = 0.000326
variance = 1.000907
ubuntu@ubuntuXPS:~/Desktop/P&RV/A27_06/Q1$
```

Fig. 2.4: The Mean and Variance of X

a) $k = 1$: $E[U]$ is the mean

b) $k = 2$: $E[U^2]$

where, $\text{var}[U] = E[U^2] - (E[U])^2$

i) Verifying Mean by using (2.3):

$$E[U] = \int_{-\infty}^{\infty} x p_X(x) dx$$

$$E[U] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$E[U] = \frac{1}{2} \left[\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \right.$$

$$\left. + \int_{-\infty}^{\infty} (-x) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \right] \quad \text{ref}$$

$$E[U] = 0$$

$$E[U] \approx 0 \quad (2.4)$$

ii) Verifying Variance by using (2.3):

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$E[U^2] = 2 \int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$E[U^2] = \sqrt{\frac{2}{\pi}} \left[x \exp\left(-\frac{x^2}{2}\right) - \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \right]$$

$$E[U^2] = \sqrt{\frac{2}{\pi}} \left[0 + \sqrt{\frac{\pi}{2}} \right]$$

$$E[U^2] = 1$$

$$\begin{aligned} \therefore \text{Var}[U] &= E[U^2] - (E[U])^2 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$$\text{Var}[U] \approx 1 \quad (2.5)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

Solution:

```
wget https://github.com/Pranav-Varma-03/
Ai1110-Assig/blob/main/Random%20
Numbers/codes/cdf_plot_logg.py
```

The CDF of X is plotted in Fig. 3.1 and plot its CDF.

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

Need to find $F_V(x)$ i.e., $\Pr(V \leq x)$:

$$\begin{aligned} &= \Pr(V \leq x) \\ &= \Pr(-2 \ln(1 - U) \leq x) \\ &= \Pr\left(\ln(1 - U) \geq \frac{-x}{2}\right) \\ &= \Pr\left(U \leq 1 - \exp\left(\frac{-x}{2}\right)\right) \end{aligned}$$

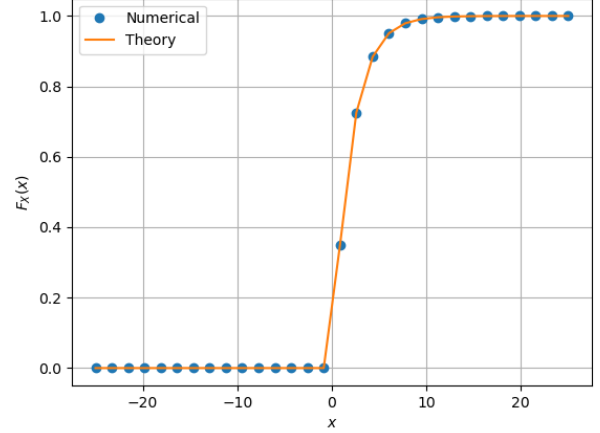


Fig. 3.1: The CDF of X

As we know, $\Pr(U \leq k) = k$:

$$\begin{aligned} \therefore F_V(x) &= \Pr\left(U \leq 1 - \exp\left(\frac{-x}{2}\right)\right) \\ &= 1 - \exp\left(\frac{-x}{2}\right) \end{aligned}$$

$$\text{As, } F_V(x) = \begin{cases} 1 - \exp\left(\frac{-x}{2}\right) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \quad (3.2)$$