Random Numbers

CS21BTECH11044

Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranav−Varma−03/
    Ai1110−Assig/blob/main/Random%20
    Numbers/codes/exrand.c
wget https://github.com/Pranav−Varma−03/
    Ai1110−Assig/blob/main/Random%20
    Numbers/codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. **??**

```
wget https://github.com/Pranav−Varma−03/
    Ai1110−Assig/blob/main/Random%20
    Numbers/codes/cdf_plot_uni.py
```

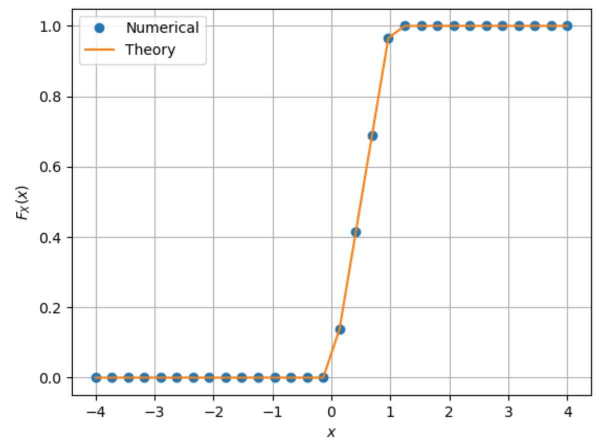

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: As we know
Probability density function of U:

$$P_U(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$\begin{aligned} F_u(x) &= \int_{-\infty}^{x} \Pr(U = a)\, da \\ &= \int_{-\infty}^{0} (0) da + \int_{0}^{x} (1) da \\ &= x \end{aligned}$$

$$F_u(x) = \begin{cases} 0 & \text{for } x < 0 \\ \text{x} & \text{for } x \in (0,1) \\ 1 & \text{for } x > 1 \end{cases} \tag{1.3}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.4}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.5}$$

Write a C program to find the mean and variance of U.
Solution:
The Mean and Variance of X is plotted in Fig. **??**

```
wget https://github.com/Pranav−Varma−03/
    Ai1110−Assig/blob/main/Random%20
    Numbers/codes/q1_4.c
wget https://github.com/Pranav−Varma−03/
    Ai1110−Assig/blob/main/Random%20
    Numbers/codes/coeffs.h
```

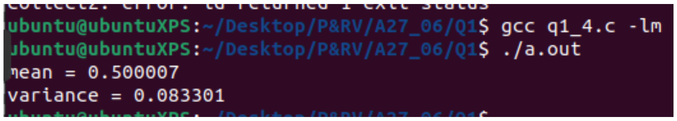

Fig. 1.4: The Mean and Variance of X

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.6}$$

Solution:

1) k = 1 : E[U] is the mean
2) k = 2 : E[U^2]
   where, var[U] = E[U^2] - (E[U])^2

i) Verifying Mean of sample of 10^6 Uniform Random Variable

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x)$$

$$As, F_U(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0,1) \\ 1 & \text{for } x > 1 \end{cases} \tag{1.7}$$

$$E[U] = 0 + \int_0^1 x d(x) + 0$$

$$E[U] = \frac{x^2}{2}\Big|_0^1$$

$$E[U] = 0.5$$

$$E[U] \approx 0.5 \tag{1.8}$$

ii) Verifying Variance of sample of 10^6 Uniform Random Variable

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x)$$

$$As, F_U(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0,1) \\ 1 & \text{for } x > 1 \end{cases}$$

$$E\left[U^2\right] = 0 + \int_0^1 x^2 d(x) + 0$$

$$E\left[U^2\right] = \frac{x^3}{3}\Big|_0^1$$

$$E\left[U^2\right] = \frac{1}{3} = 0.33$$

Thus,

$$Var[U] = E\left[U^2\right] - (E[U])^2$$

$$Var[U] = (\frac{1}{3}) - (\frac{1}{2})^2$$

$$Var[U] = 0.0833$$

$$Var[U] \approx 0.0833 \tag{1.9}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranav-Varma-03/
    Ai1110-Assig/blob/main/Random%20
    Numbers/codes/q1_4.c
wget https://github.com/Pranav-Varma-03/
    Ai1110-Assig/blob/main/Random%20
    Numbers/codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The required python file can be downloaded using

```
wget https://github.com/Pranav-Varma-03/
    Ai1110-Assig/blob/main/Random%20
    Numbers/codes/q2_4.c
```

The CDF of X is plotted in Fig. **??** Properties of CDF:

- cdf is non decreasing fucntion
- $\lim_{x\to-\infty} f(x) = 0$
- $\lim_{x\to+\infty} f(x) = 1$

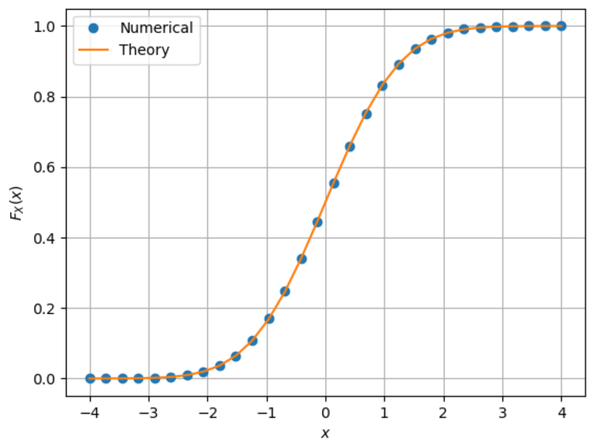

Fig. 2.2: The CDF of X

!ht

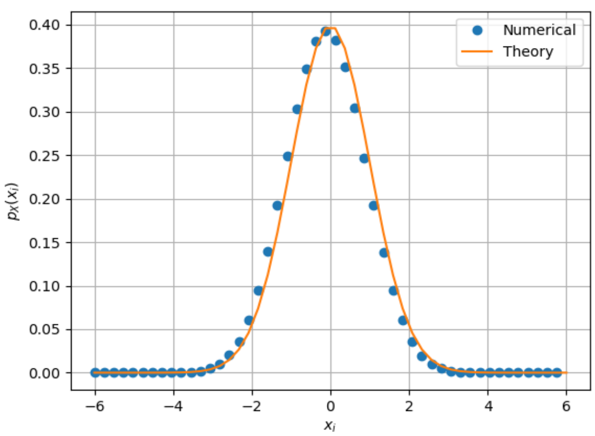

Fig. 2.3: The PDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. **??** using the code below

```
wget https://github.com/Pranav-Varma-03/
    Ai1110-Assig/blob/main/Random%20
    Numbers/codes/pdf_plot_gau.py
```

Properties of PDF:

- It is symmetric about y-axis
- Area under curve is 1 i.e., $\int_{-\infty}^{\infty} P_X(x) = 1$

2.4 Find the mean and variance of X by writing a C program.
Solution: Execute the C program given below:

```
https://github.com/Pranav-Varma-03/Ai1110
    -Assig/blob/main/Random%20Numbers/
    codes/q2_4.c
```

??

Fig. 2.4: The Mean and Variance of X

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution:

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

a) k = 1 : E[U] is the mean
b) k = 2 : E[U^2]

where, var[U] = E[U^2] - (E[U])^2

i) Verifying Mean by using (**??**):

$$E[U] = \int_{-\infty}^{\infty} x p_X(x) dx$$

$$E[U] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$$

$$E[U] = \frac{1}{2}\left[\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\right] + \frac{1}{2}\left[\int_{-\infty}^{\infty} (-x)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\right]$$

$$E[U] = 0$$

$$E[U] \approx 0 \tag{2.4}$$

ii) Verifying Variance by using (**??**):

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x) dx$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$E\left[U^2\right] = 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$E\left[U^2\right] = \sqrt{\frac{2}{\pi}}\left[x \exp\left(-\frac{x^2}{2}\right) - \int_{0}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\right]$$

$$E\left[U^2\right] = \sqrt{\frac{2}{\pi}}\left[0 + \sqrt{\frac{\pi}{2}}\right]$$

$$E\left[U^2\right] = 1$$

$$\begin{aligned} \therefore Var[U] &= E\left[U^2\right] - (E\left[U\right])^2 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$$Var[U] \approx 1 \tag{2.5}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.

Solution: Execute the following python code

```
wget https://github.com/Pranav−Varma−03/
    Ai1110−Assig/blob/main/Random%20
    Numbers/codes/cdf_plot_logg.py
```

The output CDF is plotted in Figure (**??**).

3.2 Find a theoretical expression for $F_V(x)$.
Solution:
Need to find $F_v(x) i.e., \Pr(V \leq x)$:

$$\begin{aligned} &= \Pr(V \leq x) \\ &= \Pr(-2ln(1 - U) \leq x) \\ &= \Pr\left(ln(1 - U) \geq \frac{-x}{2}\right) \\ &= \Pr\left(U \leq 1 - exp\left(\frac{-x}{2}\right)\right) \end{aligned}$$

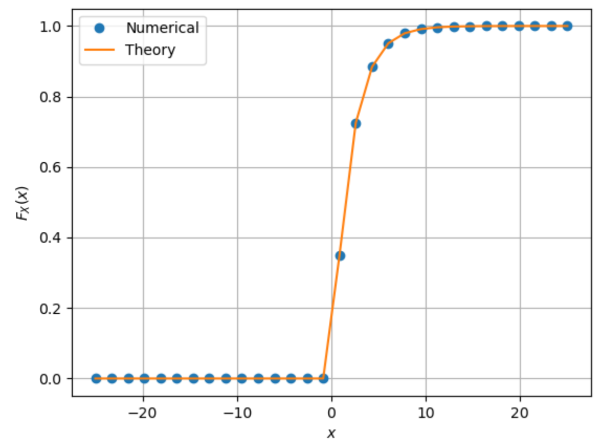

Fig. 3.1: The CDF of X

As we know, $\Pr(U \leq k) = k$:

$$\begin{aligned} \therefore F_V(x) &= \Pr\left(U \leq 1 - exp\left(\frac{-x}{2}\right)\right) \\ &= 1 - exp\left(\frac{-x}{2}\right) \end{aligned}$$

$$As, F_V(x) = \begin{cases} 1 - exp\left(\frac{-x}{2}\right) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{3.2}$$