* Finite Groups and Subgroups

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* Solve following examples for Practice.

① Prove that $2Z = \{2n \mid n \in Z\}$ is a subgroup of Z w.r.t. addition.

② Prove that $H = \{0, 4, 8\}$ is a subgroup of Z_{12} under addition modulo 12.

③ Determine whether the given subsets are subgroups of a given group. Justify.

i] $H = \{0, 1, 2\}$ of Z_4

ii] Q^+ the set of +ve rational no., a subset of R^+ under multiplication.

④ Prove that every cyclic group is abelian.

⑤ Is following group cyclic group? If so find its generaters.

i] $(Z_6, +_6)$ ii] $(Z_{10}, \times_{10})$

⑤ Prove that a subgroup of a cyclic group is cyclic.

⑥ ~~Deter,~~ Denote the permutation in the array notation

i] $\sigma = (2, 3, 5, 4) \in S_5$

ii] $\sigma = (1, 2, 3, 4, 5) \in S_5$

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7. Write the order of following groups. Also compute the order of each element in the following groups.

i] $G = \{1, -1, i, -i\}$ under multiplication.

ii] $(Z_{12}, +_{12})$

iii] $U(15)$ under multiplication modulo 15.

8. Determine whether the given subsets are subgroups of a given group. Justify.

i] H = set of odd integers of additive group Z.

ii] $M_{2\times2}(Z)$ the set of non singular matrices with integer entries of $M_{2\times2}(R)$, the additive group of 2×2 matrices with real entries.

9. Consider $G = \{1, -1, i, -i, j, -j, k, -k\}$ under multiplication $i \cdot i = j \cdot j = k \cdot k = 1$, $i \cdot j = k$, $j \cdot i = -k$ .....
write order of each element. Also write subgroups of G.

10. Give example of

1] Finite group 2] Infinite group.
3] finite cyclic group

6. Let a, b be elements in a group G. Let $|a| = m$ and $|b| = n$ such that $(m, n) = 1$. Prove that $|ab| = mn$.

7. Find subgroups of $\mathbb{Z}_{20}$ of order 5 and order 4.

8. Find all subgroups of $G = \{e, a, a^2, \cdots a^7\}$

9. Write all generators for a) $\mathbb{Z}_{18}$ b)U(7) c) $G = \{e, a, a^2 \cdots a^{21}\}$

10. If $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{bmatrix}$ and $\psi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$,
Find $\sigma\psi$, $\psi\sigma$, σ^2 order of σ.

11. Express the following permutation as product of disjoint cycles. Find its order. Also determine whether it is even or odd.

(a) $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 3 & 2 & 4 & 5 & 6 & 7 & 1 \end{bmatrix}$

(b) (5,1,3,6)(6,7,5)(1,2,3)

(c) (1,2)(1,4,3)(2,3)

12. Express each of the following permutation as a product of disjoint cycles and Hence, find its order. $\psi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 6 & 5 & 1 & 7 & 3 & 2 & 9 \end{bmatrix}$

13. Find inverse of the following permutations in S_7. Also find order.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 & 9 \end{bmatrix}$

(c) $(6, 2, 5)(4, 3, 2, 7)$

14. If $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{bmatrix}$,then compute σ^2, $\sigma\tau$, $\sigma^{-1}\tau$.

15. List the even permutations in S_4.

16. State True or false with justification.

(a) Every cyclic group is abelian

(b) If order of the group is 10, then there exists an element of order 6.

(c) Product of two odd permutations is again a odd permutation.

(d) The element $\overline{7}$ is a generator of $\mathbb{Z}_{10}$.

(e) Every permutation is cycle.

(f) The additive group $\mathbb{Z}_{20}$ has a subgroup of order 12.

17. Consider a subgroup $H = \{\overline{0}, \overline{4}, \overline{8}\}$ of $\mathbb{Z}_{12}$. Find all left cosets of H in $\mathbb{Z}_{12}$.

18. Write all right cosets of $H = \{\overline{0},\ \overline{3},\ \overline{6},\ \overline{9}\}$ in $G = \mathbb{Z}_{12}$.

19. G be a cyclic group of order 15 generated by a. Consider a subgroup of G generated by a^5. Find all left and right cosets of H in G.

20. Find all cosets of $3\mathbb{Z}$ in $\mathbb{Z}$.

21. Let H be a subgroup of G such that $|H| = 5$ and $|G| = 20$. How many elements are there in the left coset aH of H? Justify.

22. Let H be a subgroup of G and $a, b \in G$. Then prove that

   (a) $a \in Ha$

   (b) $Ha = H$ if and only if $a \in H$.

   (c) $Ha = Hb$ if and only if $a \in Hb$

   (d) $Ha = Hb$ or $Ha \cap Hb = \emptyset$