Mid-semester Examination: CS60023: Approximation and online algorithms: Time 2 hours: Spring 2020

You may answer questions totalling 100 marks.
Maximum marks one can score is 100.
Answer any four questions from each of the two parts.

February 20, 2020: 2 pm - 4 pm, CSE 120

Part A: Marks 60: Answer any four questions:

(1) State the K-centre problem precisely and the greedy approximation algorithm for the K-centre problem for a complete weighted graph $G(V, E)$, with positive edge weights obeying the triangle inequality.

Once this algorithm identifies K centres, suppose one vertex v has its cluster centre $c(v)$ and nearest neighbour in $G(V, E)$ as w. Let the nearest neighbour of $c(v)$ in $G(V, E)$ be w'. Estimate an upper bound for the distance between w and w' in terms of OPT. [Here, OPT is the optimal radius of the K-centre problem for the given graph.] (6+9 marks)

(2) Show that we can design a polynomial time f-factor algorithm for the *weighted set covering* problem if each element is covered in at most f sets. To show this use an optimal solution for the dual LP of the relaxed primal LP.

You must first clearly state and explain the formulations of the primal and dual LPs. Secondly, you must show that the collection of sets picked up is indeed a set cover. Finally, you have to argue why the set cover computed is an f-factor approximation. (5+5+5 marks)

(3) Construct "tight" examples for the *multiway cut* and k-cut problems for the respective approximation algorithms which have approximation ratios $2 - \frac{2}{k}$. In the k-cut case, show the Gomory-Hu tree as well. (9+6 marks)

(4) Write the primal and dual formulations and the 2-factor primal-dual algorithm for the *min-knapsack* problem and trace the algorithm on the input for three items with sizes 5, 2, 2, and values 3, 5, 6, respectively, with a demand value $D = 8$. Trace the algorithm for sizes 2, 2, 5 and values 3, 5 and 3, respectively, keeping D unchanged. (4+4+7 marks)

(5) Consider the problem of computing the *minimum weighted vertex cover* for connected undirected *planar graphs*. Show that a polynomial time algorithm with approximation ratio $\frac{8}{5}$ is possible. Is it possible to further improve the approximation ratio? If so, state how this can be done. (10+5 marks)

(6) A *wheel* is a graph with a *central* vertex, which is connected to all vertices in a *cycle*. So, an n-vertex wheel has $2n - 2$ edges. Determine $\chi_{FF}(G)$ for wheel graphs G on n vertices. Determine $\chi_{FF}(G)$ for a *star* graph G on n vertices; a star graph has only one node of degree at least one,

connected to all the other $n-1$ vertices of degree one. (9+6 marks)

Part B: : Marks 40: Answer any four questions:

(1) For the greedy weighted set covering algorithm, establish the lower bound on OPT with respect to n, $price(e)$ and k, where, e is the kth element covered in the course of the algorithm and n is the total number of elements. (10 marks)

(2) Show that the minimum weight perfect matching used in the 1.5-factor TSP algorithm has cost at most 0.5 times OPT where OPT is the cost of the optimal TSP tour. (10 marks)

(3) In the case of the K-server problem for one-dimensional geometry, show how the potential function changes in the double coverage step where the online algorithm makes its move subsequent to the move of the offline algorithm. Consider both the cases where the request location is (i) between two servers, and (ii) at the extreme right or left of all the servers. (10 marks)

(4) Describe the notion of *maximality* for a dual feasible solution for the *weighted set cover* problem. Show that the following algorithm results in a maximal dual solution. Here, I is the set of m elements, and w_j is the weight of set S_j. Also, $a_{ij} = 1$ if $i \in S_j$, and 0 otherwise. (4+6 marks)

- Step 0: Set $C = \phi$; $I = \{1, 2, ..., m\}$; $\mathbf{y} = 0$;
- Step 1: Let $i \in I$; Let $w_{j(i)} = min_{(a_{ij}=1)} w_j$; $y_i = w_{j(i)}$; $C \leftarrow C \cup \{j(i)\}$;
- Step 2: For all j such that $a_{ij} = 1$, $w_j \leftarrow w_j - w_{j(i)}$; $I \leftarrow I \setminus S_{j(i)}$;
- Step 3: If $I = \phi$, STOP and OUTPUT C. Else, go to Step 1. ;

(5) Let a dual optimal solution be (v^*, w^*) and an optimal primal solution be (x^*, y^*) for the facilities location problem. State the complementary slackness conditions for the relaxed primal-dual LP formulation. Show that complementary slackness conditions imply $v_j^* \geq c_{ij}$, for some clients j and facilities i.

We wish to select a facility i for every client j, if $v_j^* \geq c_{ij}$. We wish to connect such pairs of clients and facilities. Argue that the connection costs add up to at most OPT. [Here, OPT is the optimal total cost for the facilities location problem with costs for connecting clients to facilities, as well as costs for opening facilities.] In such a process will all clients get connected to some facility? Why? (3+2+2+3 marks)

(6) A *chordal* graph $G(V, E)$ must have a vertex v that is adjacent to some maximal clique of vertices $C \subseteq V$ in $G(V, E)$. The induced subgraph $G(V \setminus H \setminus \{v\}, E')$ has edge set E' with edges of the maximal clique and its connecting edges to v dropped. If we have an independent set I in this induced subgraph then we can add v to I to get an independent set for $G(V, E)$. How does this help us in estimating $\chi_{FF}(G'(V, E''))$, where $G'(V, E'')$ is the complement of the chordal graph $G(V, E)$? Explain. (10 marks)