EE61012 – Spring Semester 2024

Homework 4: Convex Optimization in Control and Signal Processing

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Q 4.1: Stability and State Feedback Control

Consider the discrete-time dynamical system given by

$$x(k+1) = Ax(k) + Bu(k), \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} -15 \\ 12 \\ -9 \end{bmatrix}.$$

1. Formulate a suitable LMI to determine if 0 is a stable equilibrium point when $u(k) = 0$.
2. Formulate a suitable LMI to determine a static state feedback gain matrix under which 0 is a stable equilibrium point.

Q 4.2: Observer Design and State Reconstruction

Suppose the states are not available to the controller, rather the measured output is given by

$$y(k) = Cx(k), \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Formulate a suitable LMI to an observer gain matrix so that estimation error decays to 0.
2. For the given dataset, estimate the state trajectory.

Q 4.3: Optimal Control

Let $T = 4$ be the time-horizon. Implement the following optimal control problem for the above dynamics and initial state.

$$\begin{aligned} \text{minimize}_{x,u} \quad & \sum_{k=0}^{T-1} \left[||x(k+1)||_2^2 + 0.1u(k)^2 \right] \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k), \\ & -5 \leq u(k) \leq 5, \\ & \begin{bmatrix} -25 \\ -25 \\ -25 \end{bmatrix} \leq x(k+1) \leq \begin{bmatrix} 25 \\ 25 \\ 25 \end{bmatrix}, \quad k = 0, 1, \ldots, T-1. \end{aligned}$$

Determine if the states are at the equilibrium state $T = 4$. Now implement the following *receding horizon control* scheme. At each step k:

1. Treat $x(k)$ as the initial state and solve the above problem.
2. Once you solve it, apply only the first input computed by the optimization solution ($u(0)$ in the above notation, but actually $u(k)$), and let the state evolve to $x(k+1)$ following the dynamics.
3. Then repeat from $k = 0$ to $k = 20$.

Answer the following questions for the *receding horizon control* scheme.

1. Plot the actual trajectory of the state and control input with respect to time in three different plots with proper labels and axis marks.
2. Compare the above trajectory with the trajectory obtained under static state feedback control law. Does the latter respect the bounds on control input and states?

Q 4.4: LMI Problem (Endsem, Spring 2022-23)

Express the problem of finding a matrix A with $||A|| \leq \gamma$ as a linear matrix inequality. Here $||A|| := \sqrt{\lambda_{\max}(A^\top A)}$ where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix.

Q 4.5: Continuous-time Gradient Flow (Endsem, Spring 2022-23)

Consider the continuous time gradient descent scheme $\dot{x} = -\nabla f(x)$ used to minimize a strongly convex function $f(x)$. The optimal solution $x^\star$ is an equilibrium of the above dynamics. For each of the following functions, determine whether it can be used as a Lyapunov function to prove asymptotic stability of $x^\star$.

1. $V_1(x) = \frac{1}{2}||\nabla f(x)||_2^2$.
2. $V_2(x) = \frac{1}{2}||x - x^\star||_2^2$.
3. $V_3(x) = f(x) - f(x^\star)$.