Using Schur complement lemma

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \quad \Leftrightarrow \quad \begin{bmatrix} P & PA-PLC \\ (PA-PLC)^T & P \end{bmatrix} \succ 0$$

Let $PL = Z$ $\quad \Leftrightarrow \quad \begin{bmatrix} P & PA-ZC \\ (PA-ZC)^T & P \end{bmatrix} \succ 0$

Once the above is solved to obtain P^*, Z^*, we find L by choosing $\underline{L = (P^*)^{-1} Z^*}$

Robust Stability and Control

Suppose the system matrices A, B, C are not known with certainty. Rather

$$A(t) = \underline{A_{nom}} + \underbrace{\Delta(t)}$$

possibly time-varying perturbation.

$\dot{x}(t) = A(t)x(t)$
or $x_{t+1} = A_t x_t$

Requiring $A(t)$ to have eigenvalues in the left-half plane for CT system (or unit circle for DT system) for all t is not enough for stability of origin.

Quadratic stability.

Let $V(x) = x^T P x$, $P = P^T \succ 0$

$$\frac{dV(x)}{dt} = x^T \left[A(t)^T P + P A(t) \right] x < 0$$

$$\Leftrightarrow \quad A(t)^T P + P A(t) \prec 0, \quad P \succ 0$$

$$\Leftrightarrow \quad (A_{nom} + \Delta(t))^T P + P(A_{nom} + \Delta(t)) \prec 0, \quad P \succ 0, \quad \forall\, \Delta(t).$$

We will focus on two types of allowed perturbations:

a) $\|\Delta(t)\| \leq \gamma$ for all t. $\gamma > 0$: scalar

(norm - bounded)

b) parametric polytopic:

$$\Delta(t) \in \bar{\Delta} = \{\Delta \mid \Delta = A_1\delta_1 + A_2\delta_2 + \cdots + A_k\delta_k,\ \delta_i \in [-1, 1]\}$$

(A) Perturbations with bounded norm

Let $A(t) = A_{nom} + F\,\Delta(t)\,H$, where $A \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times p}$, $\Delta(t) \in \mathbb{R}^{p\times q}$, $H \in \mathbb{R}^{q\times n}$

$\|\Delta(t)\|_2 \leq \gamma \quad \forall t$

$\sigma_{max}(\Delta) = \sqrt{\lambda_{max}(\Delta^T\Delta)}$

The condition for $A(t)$ to be quadratically stable is

$$(A_{nom} + F\Delta(t)H)^T P + P(A_{nom} + F\Delta(t)H) \prec 0, \quad P \succ 0, \quad \forall \|\Delta(t)\|_2 \leq \gamma$$

Multiplying P^{-1} from both left and right, we obtain:

$$P^{-1}(A_{nom} + F\Delta(t)H)^T PP^{-1} + P^{-1}P(A_{nom} + F\Delta(t)H)P^{-1} \prec 0$$

$$\Leftrightarrow P^{-1}A_{nom}^T + P^{-1}H^T\Delta(t)^T F^T + A_{nom}P^{-1} + F\Delta(t)HP^{-1} \prec 0$$

$$\Leftrightarrow \underbrace{(P^{-1}A_{nom}^T + A_{nom}P^{-1})}_{G} + (HP^{-1})^T\Delta(t)^T F^T + \underset{M}{F}\,\Delta(t)\,\underset{N}{HP^{-1}} \prec 0 \qquad (*)$$

The above inequality needs to hold for all $\|\Delta(t)\|_2 \leq \gamma$ which gives rise to infinite number of LMIs.

In ordere to tackle the above challenge, we apply Petersen's Lemma.

Lemma: Let $G = G^T \in \mathbb{R}^{n\times n}$ and M, N be two other matrices.

Then, $G + M\Delta N + N^T\Delta^T M^T \preceq 0 \quad \forall \ \|\Delta\|_2 \leq 1$

(if and only if)

$\exists \ \varepsilon \in \mathbb{R}$ s.t.

$$\begin{bmatrix} G + \varepsilon M M^T & N^T \\ N & -\varepsilon I \end{bmatrix} \preceq 0,$$

with the above being a LMI in ε.

Applying the above lemma to $(*)$, we see that

$$G \equiv P^{-1}A_{nom}^T + A_{nom}P^{-1}, \quad M = F, \quad N = HP^{-1}.$$

Hence $(*)$ is equivalent to

$$\begin{bmatrix} P^{-1}A_{nom}^T + A_{nom}P^{-1} + \varepsilon FF^T & P^{-1}H^T \\ HP^{-1} & -\varepsilon I \end{bmatrix} \prec 0,$$

which is a LMI in ε and P.

Q1) Can we allow for larger perturbations, i.e..

find $\gamma_{max} = \sup_{\gamma \in \mathbb{R}} \left\{ \gamma \ \middle| \ (A_{nom} + F\Delta H)P + P(A + F\Delta H)^T \prec 0 \text{ for some } P \succ 0 \text{ and all } \|\Delta\|_2 \leq \gamma \right\}.$

Sol^n: $\max \ \gamma$

s.t. $P \succ 0$, $\begin{bmatrix} A_{nom}P^{-1} + P^{-1}A_{nom}^T + \gamma FF^T & P^{-1}H^T \\ HP^{-1} & -I \end{bmatrix} \preceq 0.$

Q2 Robust Stabilization using state feedback

The goal is to find matrix K s.t. if $u(t) = Kx(t)$, then the closed-loop system $\dot{x}(t) = \underline{(A(t)+BK)}\,x(t)$ is quadratically stable.

$$A_{cl}(t) = A_{nom} + F\Delta(t)H + BK$$
$$= \underbrace{(A_{nom}+BK)}_{A_{nom,cl}} + F\Delta(t)H$$

The corresponding matrix inequality is given by:

$$\begin{bmatrix} P^{-1}(A_{nom}+BK)^T + (A_{nom}+BK)P^{-1} + \varepsilon FF^T & P^{-1}H^T \\ HP^{-1} & -\varepsilon I \end{bmatrix} \prec 0,$$

$$P^{-1}A_{nom}^T + \underline{P^{-1}K^T}B^T + A_{nom}P^{-1} + B\underline{KP^{-1}}$$

define new variable $Z = KP^{-1}$.

$$\left(\begin{matrix} P \succ 0,\ \varepsilon \in \mathbb{R}, \\ Z \end{matrix}\right) \begin{bmatrix} P^{-1}A_{nom}^T + A_{nom}P^{-1} + BZ + Z^TB^T + \varepsilon FF^T & P^{-1}H^T \\ HP^{-1} & -\varepsilon I \end{bmatrix} \prec 0$$

once the above LMI is solved, we find Z^*, P^*, and obtain $K = Z^*P^*$

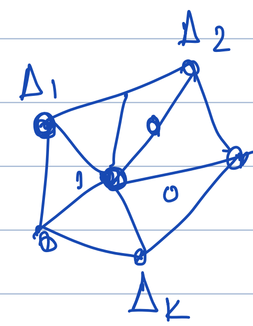
Case 2: $\Delta(t)$ belongs to a polytope

Theorem For any H, $(L_i)_{i=1}^{n}$, $(R_i)_{i=1}^{n}$ matrices of suitable dimension,

$$H + \sum_{i=1}^{n} L_i \Delta R_i \succ 0 \quad \forall\ \Delta \in \mathrm{conv}(\Delta_1, \Delta_2 \ldots \Delta_K)$$

$$\Leftrightarrow \left\{ H + \sum_{i=1}^{n} L_i \Delta_j R_i \succ 0 \quad \text{for } j = 1, 2, \ldots K. \right.$$

$$\equiv \left\{ \begin{array}{l} H + \sum_{i=1}^{n} L_i \Delta_1 R_i \succ 0 \\ H + \sum_{i=1}^{n} L_i \Delta_2 R_i \succ 0 \\ \vdots \\ H + \sum_{i=1}^{n} L_i \Delta_K R_i \succ 0. \end{array} \right.$$



If the uncertain dynamical system $\dot{x} = A(t)x$ with

$$A(t) = A_{nom} + \Delta(t), \qquad \Delta(t) \in \mathrm{conv}(A_1, A_2 \ldots A_K),$$

then quadratic stability requires

$$\exists\ P \succ 0 \text{ s.t. } (A_{nom} + \Delta)^T P + P(A_{nom} + \Delta) \prec 0 \quad \underline{\forall\ \Delta \in \mathrm{conv}(A_1 \ldots A_L)}$$

$$\Leftrightarrow P \succ 0 \text{ s.t. } \left. \begin{array}{l} (A_{nom} + A_1)^T P + P(A_{nom} + A_1) \prec 0 \\ (A_{nom} + A_2)^T P + P(A_{nom} + A_2) \prec 0 \\ \vdots \\ (A_{nom} + A_K)^T P + P(A_{nom} + A_K) \prec 0 \end{array} \right\}$$

Another variation:

Let $\Delta(t) = A_1 \delta_1(t) + A_2 \delta_2(t) + \cdots + A_K \delta_K(t)$,

$\delta_i(t) \in [\delta_i^{min}, \delta_i^{max}]$

then $\Delta(t) \in \text{conv}\left(\sum_{i=1}^{K} A_i \delta_i \,,\; \delta_i \in [\delta_i^{min}, \delta_i^{max}] \right)$

$\rightarrow 2^K$ corner points.

For state-feedback stabilization,

$$\dot{x} = (A + \Delta_A(t))x + (B + \Delta_B(t))u,$$

we wish to find K s.t. $u = Kx$ renders the closed-loop system to be quadratically stable.

$$(\Delta_A, \Delta_B) \in \text{conv}((A_1, B_1), \cdots (A_K, B_K)).$$

This requirement can be stated as matrix inequalities:

$$P \succ 0, \quad ((A + A_i) + (B + B_i)K)^T P + P((A + A_i) + (B + B_i)K) \prec 0,$$

$$\forall i = 1, 2 \ldots K$$

$$(A + A_i)^T P + P(A + A_i) + K^T (B + B_i)^T P + P(B + B_i)K \prec 0$$

$$\Leftrightarrow \quad P^{-1}(A + A_i)^T + (A + A_i)P^{-1} + \underbrace{(KP^{-1})^T}_{Z}(B + B_i)^T + (B + B_i)\underbrace{KP^{-1}}_{Z} \prec 0$$

$$\Leftrightarrow \quad P^{-1}(A + A_i)^T + (A + A_i)P^{-1} + Z^T (B + B_i)^T + (B + B_i)Z \prec 0 \quad \forall i$$

Once we solve it, set $K = Z^* P^*$