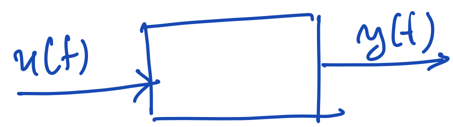
Module C: Convex Optimization in Control

In a nutshell, control theory is the study of influencing trajectories of a dynamical system to satisfy desired properties.

- Static vs. Dynamic System:

  output $y(t)$ depends on $u(\tau)$ for $\tau \le t$. in other words, $u(t)$ has impact on $y(\tau)$ for $\tau \ge t$.

  $y(\tau)$ is only a function of $u(\tau)$ and no other $u(t), t \neq \tau$.

- Example:

  $$\boxed{\frac{d^2 p(t)}{dt} = \frac{F(t)}{m}},$$

  m: mass of the object

  F: force applied to it (input)

  p(t): position of the object

  $x_1(t) = p(t)$, $x_2(t) = v(t)$, $\underbrace{x(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2}$: state variables

- State-space representation:

  - State variable, $x(t)$, is a variable such that knowledge of $x(t)$ and $\{u(\tau)\}_{t \le \tau \le t+T}$ is sufficient to predict $x(T+t)$, and output $y(t+T)$.

  - $$\left.\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t) \end{aligned}\right] \implies$$ Standard form of state-space representation of a dynamical system.

  $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$.

$$\dot{p}(t) = v(t)$$
$$\dot{v}(t) = \frac{F}{m}$$

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ \frac{F}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Questions of Interest

- Stability: $\dot{x} = f(x(t))$, if $f(x^*) = 0$ for some x^*, then x^* is called an equilibrium point.
  → how to check whether an equilibrium point x^* is stable or not.

- Identification: if some parameters governing the dynamics is not known, can we learn their values from input-output data?

- State Estimation : if $y(t) \neq x(t)$, then can we estimate $x(t)$ from the knowledge of $u(t)$ and $y(t)$.

- Optimal Control: initial state x_0, desired state $x^{desired}$. design a control input which achieves this task at smallest cost.

- Robust Control: if the dynamics/some parameters are not perfectly known, how to design a control input which will work well for all possible/likely variation of the parameters.

All the above problems can be answered in a principled manner using convex optimization techniques.

A. Discrete-time Optimal Control

- Discrete-time State-Space Model: $\begin{cases} \boxed{\begin{aligned} z_{k+1} &= f(z_k, u_k) \\ y_k &= h(z_k, u_k), \end{aligned}} \end{cases}$ $k = 0, 1, 2 \ldots$, z_0: specified

– some systems are inherently discrete, while in other cases a CT systems is discretized.

Euler discretization

$$\dot{x} = f(x, u) \implies x_{k+1} = x_k + h f(x_k, u_k),$$

h: small const / sampling time

- Goal: Starting from an initial state z_0, compute a sequence of control inputs $(u_0, u_1, \ldots, u_{T-1})$ such that the state at time T, denoted $\boxed{z_T = z^{\text{des}}}$ which is the desired state.

- Let us formulate an optimization problem to achieve this goal.

- Decision Variables: $x = \begin{bmatrix} (u_0, u_1, u_2, \ldots u_{T-1}) \in \mathbb{R}^{T \times n_u} \\ (z_1, z_2, \ldots, z_T) \in \mathbb{R}^{T \times n_x} \end{bmatrix}$

- Cost Function: $f(x) = \begin{cases} \| z_T - z^{des} \|_2^2 \\ \sum_{k=1}^{T} \| z_k - z^{des} \|_2^2 \\ \sum_{k=1}^{T} \left[\| z_k - z^{des} \|_2^2 + \lambda \| u_{k-1} \|_2^2 \right] \end{cases}$

$z_0 \rightarrow z^{des}$

- Constraints:

$$z_{k+1} = f(z_k, u_k), \quad k = 0, 1, \ldots, T-1$$

$$\begin{bmatrix} z_1 = f(z_0, u_0) \\ z_2 = f(z_1, u_1) \\ z_3 = f(z_2, u_2) \\ \vdots \\ z_T = f(z_{T-1}, u_{T-1}) \end{bmatrix}$$

$$z_T = z^{des}$$

$$\begin{cases} u^{min} \le u_k \le u^{max}, \quad k = 0, 1, \ldots T-1 \\ z^{min} \le z_k \le z^{max} \end{cases}$$

$$g(z_k, u_k) \le 0$$

Finite-Horizon Optimal Control Problem

$$\min_{\substack{u_0, u_1 \ldots u_{T-1} \\ z_1, z_2 \ldots z_T}} \sum_{k=1}^{T} \left[\underbrace{\| z_k - z^{des} \|_2^2} + \lambda \underbrace{\| u_k \|_2^2} \right]$$

$$\text{s.t.} \quad z_{k+1} = f(z_k, u_k), \quad k = 0, 1, 2 \ldots T-1$$

$$u^{min} \le u_k \le u^{max}, \quad ''$$

$$\underline{g(z_k, u_k)} \le 0, \quad ''$$

When is the above problem a convex optimization problem?

→ $\lambda \ge 0$ which will lead to the cost function being convex.

→ Function f should be a linear/affine function of z_k & u_k.

$$\boxed{\text{e.g.: } z_{k+1} = A z_k + B u_k}$$

→ Function g should be a convex function of z_k & u_k.

Discrete-time Linear Quadratic Regulation Problem

$$\min_{\substack{u_0, \dots u_{T-1} \\ z_1, \dots z_T}} \sum_{k=1}^{T} \left[z_k^T Q z_k + u_{k-1}^T R u_{k-1} \right] \quad (z^{des} = 0)$$

s.t. $\quad z_{k+1} = A z_k + B u_k \quad , k = 0, 1, 2 \dots T-1$

Q, R are positive (semi)-definite matrices, under which the problem is convex (QP)

designed by user to achieve a good trajectory / solution.

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1(t), x_2(t) \ldots) \\ f_2(x_1(t), x_2(t) \ldots \\ \vdots \\ f_n(x_1(t) \ldots) \end{bmatrix}$$

B. Stability

- Consider a continuous-time (autonomous) dynamical system: $\dot{x} = f(x)$ with initial state x_0.

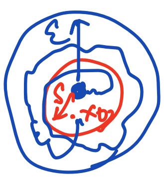
$$\frac{dx(t)}{dt} = f(x), \quad f: \mathbb{R}^n \to \mathbb{R}^n$$

- Equilibrium point: any point $x^* \in \mathbb{R}^n$ is an eq^m point if $f(x^*) = 0$

$$\Rightarrow x(t) = x^* \Rightarrow x(\tau) = x^* \quad \forall \tau \geq t.$$

- Stability of an equilibrium point:

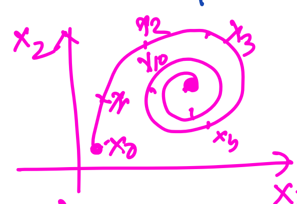
An eq^m point $x^* \in \mathbb{R}^n$ is stable if $\forall \varepsilon > 0$, there exists δ_ε such that whenever $\|x_0 - x^*\| \leq \delta_\varepsilon$, we have $\|x(t) - x^*\| \leq \varepsilon \quad \forall t \geq 0$.



Conversely, if x^* is not stable, then $\exists\, \varepsilon > 0$ s.t. $\|x(t) - x^*\| \geq \varepsilon$ for any x_0 arbitrarily close to x^*. for some $t > 0$.

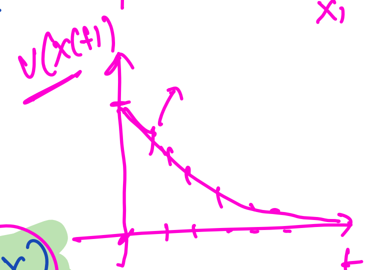
x^* is Globally Asymptotically stable (GAS) if $\lim_{t \to \infty} x(t) = x^* \quad \forall x_0 \in \mathbb{R}^n$

- Lyapunov Stability Theorem: Consider a function $V: \mathbb{R}^n \to \mathbb{R}_+$ satisfying:



Candidate Lyapunov functions.

$$\left\{ \begin{array}{ll} \text{i)} & V(x) \geq 0 \quad \forall x \in \mathbb{R}^n \\ \text{ii)} & V(x) = 0 \text{ if \& only if } x = x^* \end{array} \right.$$



iii) $\underbrace{\dfrac{dV(x)}{dt}}$ < 0

$$\frac{d}{dt} V(x) = \nabla_x V(x)^T \boxed{\frac{d}{dt} x(t)} = \nabla_x V(x)^T f(x)$$

$$= \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x) < 0$$

If $\exists\, V$ satisfying the above, then x^* is a GAS equilibrium point.

$x^* = 0$ is a GAS eqm of $\dot{x} = Ax \iff \underline{\underline{Re[\lambda_i(A)] < 0}}$

Stability of a Continuous-time LTI System

$A \in \mathbb{R}^{n \times n}$

- An autonomous LTI System is stated as $\boxed{\dot{x} = Ax}$ with initial state x_0.

$x^* = 0$ is always an equilibrium point.

Let $V(x) = x^T P x$, $P = P^T$, and positive definite

Easy to see that

i) $V(x) \geq 0 \quad \forall x$

ii) $V(x) = 0 \iff x = 0$

$$\frac{d}{dt} V(x) = (\dot{x})^T P x + x^T P \dot{x}$$

$$= (Ax)^T P x + x^T P A x$$

$$= x^T \left[\underbrace{A^T P + P A} \right] x < 0 \qquad \forall x \in \mathbb{R}^n, \; x \neq 0$$

$$\Rightarrow \boxed{\begin{array}{c} A^T P + P A \prec 0 \\ P \succ 0 \end{array}} \rightarrow \underline{P \in \mathbb{S}^n}$$

If we can find P satisfying the above, then $x^* = 0$ is a globally asymptotically stable eqm point.

$\rightarrow$ Linear matrix inequalities.

$\underline{n = 3}$: $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$

number of unknowns $\frac{n(n+1)}{2}$.

Linear Matrix Inequalities

- Definition: $F_0 + x_1 F_1 + x_2 F_2 + \dots + x_n F_n \preceq 0$,

where $F_0, F_1, \dots F_n$ are symmetric matrices.

$$P \succ 0 \iff \underline{p_{11}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_{11}} + p_{12} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_{12}} + p_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{+ p_{31} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + p_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + p_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \succ 0$$

$$P = \sum_j p_j E_j \qquad \sum_j p_j E_j \succ 0$$

$$\iff \underline{\sum_j p_j (-E_j) \prec 0}$$

$$A^T P + PA$$

$$= A^T \Big(\sum_j p_j E_j \Big) + \Big(\sum_j p_j E_j \Big) A$$

$$= \underbrace{\sum_{j=1}^{6} p_j \left[A^T E_j + E_j A \right] \prec 0}$$

Primal and Dual forms of Optimization with LMI Constraints

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & \underbrace{F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0}, \end{aligned}$$

$c, F_0, F_1, F_2, \ldots F_n$ are given

- the above is a convex optimization problem as the feasibility set is a convex set.
- In order to find a lower bound on the optimal value, we multiply a positive definite matrix $Z \succ 0$ with the constraint to obtain:

$$\langle Z, F_0 \rangle + x_1 \langle Z, F_1 \rangle + x_2 \langle Z, F_2 \rangle + \cdots + x_n \langle Z, F_n \rangle \le 0$$

$$\Rightarrow \langle Z, F_0 \rangle \le -x_1 \langle Z, F_1 \rangle - x_2 \langle Z, F_2 \rangle - \cdots - x_n \langle Z, F_n \rangle$$

$$\Rightarrow \langle Z, F_0 \rangle \le c^T x \quad \text{when} \quad \left. \begin{aligned} \langle Z, F_1 \rangle &= -c_1 \\ \langle Z, F_2 \rangle &= -c_2 \\ &\vdots \\ \langle Z, F_n \rangle &= -c_n \end{aligned} \right\}.$$

Dual problem:

$$\begin{aligned} \max_{Z \in S^n} \quad & \langle Z, F_0 \rangle \\ \text{s.t.} \quad & \langle Z, F_i \rangle + c_i = 0 \quad \forall i = 1, 2 \ldots n \\ & Z \succ 0. \end{aligned}$$

$$\boxed{\langle A, B \rangle = \operatorname{tr}(A^T B)}$$

$$\begin{aligned} \min_{Z} \quad & \|Z_0 - Z\|_F^2 \\ \text{s.t.} \quad & Z \succ 0 \end{aligned}$$

$x^* = 0$ is GAS $\Leftrightarrow$ $\underline{|\lambda_i(A)| < 1}$ $\forall\, i \in \{1, 2 \ldots n\}$.

Stability of a Discrete-time LTI System

- An discrete-time autonomous LTI System is stated as $\boxed{x_{k+1} = Ax_k}$ with initial state x_0.

we once again consider Lyapunov function $V(x) = x^T P x$, $P > 0$

$$\underbrace{V(x_{k+1}) - V(x_k)} = x_{k+1}^T P x_{k+1} - x_k^T P x_k$$

$$= (Ax_k)^T P (Ax_k) - x_k^T P x_k$$

$$= x_k^T \underbrace{[A^T P A - P]} x_k$$

$V(x_{k+1}) - V(x_k) < 0 \quad \forall\, x_k \in \mathbb{R}^n, \underline{x_k \neq 0} \iff \underline{A^T P A - P < 0}$

<u>LMI conditions</u>

$$\boxed{\begin{aligned} &P > 0 \\ &A^T P A - P < 0 \end{aligned}} \rightarrow \text{linear in } P$$

though it is nonlinear in A, A is not a variable.

Good programming practice is to define the above constraint as

$$\begin{aligned} &P \succeq \varepsilon_1 I \\ &A^T P A - P \preceq -\varepsilon_2 I \end{aligned},$$

where $\varepsilon_1, \varepsilon_2 > 0$ are sufficiently small constants.

$\min\ e^x$

s.t. $-x \geq 0 \rightarrow$ open.

Linear State Feedback Control Design.

Consider LTI system with input: $\dot{x} = Ax + Bu$, x_0 is given.

The simplest feedback control scheme is $u = Kx$, where K is a matrix of suitable dimension.

Dynamics of the closed-loop system is given by: $\dot{x} = Ax + BKx$

$$\dot{x} = (A+BK)x$$

It is often possible to stabilize $x(t)$ around $x^* = 0$ by choosing K in a suitable manner.

$x^* = 0$ is stable for $\dot{x} = (A+BK)x$ if

$$\boxed{\begin{aligned} &P \succ 0 \\ &(A+BK)^T P + P(A+BK) \prec 0 \end{aligned}}$$

→ both P & K are decision variables / unknowns.

not an LMI as it contains nonlinear product terms P and K.

In order to tackle the above:

$$(A+BK)^T P + P(A+BK) \prec 0$$

$$\Leftrightarrow \quad PP^{-1}(A+BK)^T P + P(A+BK)P^{-1}P \prec 0$$

$$\Leftrightarrow \quad P\left[P^{-1}(A+BK)^T + (A+BK)P^{-1}\right]P \prec 0$$

$$\Leftrightarrow \quad P^{-1}(A+BK)^T + (A+BK)P^{-1} \prec 0$$

$$\Leftrightarrow \quad P^{-1}A^T + \underbrace{P^{-1}K^T}_{X^T}B^T + AP^{-1} + B\underbrace{KP^{-1}}_{X} \prec 0$$

$$X^T = (P^{-1})^T K^T = P^{-1}K^T$$

If $P \succ 0$, $P^{-1} \succ 0$.
one can show
$(P^{-1})^T Z P^{-1} \succ 0$
$\Leftrightarrow Z \succ 0$

$$\Leftrightarrow \boxed{\begin{array}{r} P^{-1}A^T + X^TB^T + AP^{-1} + BX < 0 \\ P > 0 \end{array}} \Rightarrow \text{LMI in } P \text{ and } X.$$

once we solve the above to obtain P^*, X^* we know

$$X^* = K(P^*)^{-1} \Rightarrow \boxed{K = X^* P^*}$$

6th April

State Feedback Control design for DT System

Recall that $\quad x_{k+1} = Ax_k + Bu_k, \quad A, B$: known.

State feedback controller: $\quad u_k = K x_k$

Closed-loop dynamics: $\quad x_{k+1} = Ax_k + BKx_k = (A+BK)x_k.$

For CL dynamics to be stable:

$$(*) \begin{cases} P > 0 \\ (A+BK)^T P (A+BK) - P < 0 \end{cases}, \quad \text{variables } P, K$$

$\rightarrow$ not a linear matrix inequality due to nonlinear terms $\quad K^TPK$, etc.

Schur Complement lemma:

The following are equivalent:

i) $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0, \quad A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times m}, B^T \in \mathbb{R}^{m\times n}$

$\rightarrow$ dimension: $(n+m)\times(n+m)$

(ii) $\underbrace{A - BC^{-1}B^T}_{\in \mathbb{R}^{n\times n}} > 0, \quad \underbrace{C}_{\in \mathbb{R}^{m\times m}} \succ 0$

(iii) $A \succ 0, \quad C - B^T A^{-1} B \succ 0.$

$C - B^T A^{-1} B$ is called schur complement of A

$A - BC^{-1}B^T$ is called schur complement of C.

We will now apply schur complement lemma to transform $(*)$ into a LMI.

The requirement is $P \succ 0$

$$(A+BK)^T P (A+BK) - P \prec 0$$

we multiply P^{-1} from both right & left.

$$P^{-1}\left[(A+BK)^T P (A+BK) - P\right] P^{-1} \prec 0$$

$$\iff P^{-1}(A+BK)^T P (A+BK) P^{-1} - P^{-1} P P^{-1} \prec 0$$

$$\iff (P^{-1})^T (A+BK)^T P (A+BK) P^{-1} - P^{-1} \prec 0$$

$$\iff \left((A+BK)P^{-1}\right)^T P \left(A+BK\right)P^{-1} - P^{-1} \prec 0.$$

$C \equiv P^{-1}, \quad B \equiv (A+BK)P^{-1}, \quad A = P^{-1}$

$$\begin{bmatrix} P^{-1} & (A+BK)P^{-1} \\ ((A+BK)P^{-1})^T & P^{-1} \end{bmatrix} > 0$$

define $Q = P^{-1}$, $Z = KP^{-1}$, then $(*)$ is equivalent to

$$\begin{bmatrix} Q & AQ+BZ \\ (AQ+BZ)^T & Q \end{bmatrix} > 0$$ which is a LMI, in variables Q and Z

Once we solve the above LMI, we obtain Q^*, Z^*.

Since $Z^* = KQ^*$

$$\Rightarrow K = Z^*(Q^*)^{-1}$$

State Estimation

Consider a discrete-time LTI system:

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

We observe $(u_0, u_1, \ldots, y_0, y_1 \ldots)$

the goal is to estimate the states from our observations.

Observer design: We create an auxiliary system.

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L[y_k - C\hat{x}_k]$$

define $e_k = x_k - \hat{x}_k$

$$\underline{e_{k+1}} = x_{k+1} - \hat{x}_{k+1} = (Ax_k + Bu_k) - \big(A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k)\big)$$

$$= A(x_k - \hat{x}_k) - L(Cx_k - C\hat{x}_k)$$

$$= (A-LC)(x_k - \hat{x}_k)$$

$$= (A-LC)\,\underline{e_k}$$

We need to ensure that eigenvalues of $(A-LC)$ lie within unit circle. Equivalently:

$$(*)\quad \begin{cases} P \succ 0 \\ (A-LC)^T P (A-LC) - P \prec 0 \end{cases} \qquad (L, P \text{ are variables})$$

↳ not a LMI due to nonlinear product terms involving L & P

To transform $(*)$ to a LMI,

$$(A-LC)^T \underset{\uparrow\, P^{-1}P}{P} (A-LC) - P \prec 0$$

$$\Leftrightarrow \quad (A-LC)^T P\, P^{-1}\, P(A-LC) - P \prec 0$$

$$\Leftrightarrow \quad (PA-PLC)^T P^{-1} (PA-PLC) - P \prec 0$$

$$\Leftrightarrow \quad \underbrace{P}_{C} - \underbrace{(PA-PLC)^T}_{B^T} \underbrace{P^{-1}}_{A^{-1}} \underbrace{(PA-PLC)}_{B} \succ 0, \quad \underbrace{P}_{A} \succ 0$$

Using Schur complement lemma

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \iff \begin{bmatrix} P & PA-PLC \\ (PA-PLC)^T & P \end{bmatrix} \succ 0$$

Let $PL = Z$ $\iff \begin{bmatrix} P & PA-ZC \\ (PA-ZC)^T & P \end{bmatrix} \succ 0$

Once the above is solved to obtain P^*, Z^*, we find L by choosing $\boxed{L = (P^*)^{-1} Z^*}$

Robust Stability and Control

Suppose the system matrices A, B, C are not known with certainty. Rather

$$A(t) = A_{nom} + \underbrace{\Delta(t)}_{\text{possibly time-varying perturbation.}}$$

$\dot{x}(t) = A(t)x(t)$
or $x_{t+1} = A_t x_t$

Requiring $A(t)$ to have eigenvalues in the left-half plane for CT system (or unit circle for DT system) for all t is not enough for stability of origin.

Quadratic stability.

Let $V(x) = x^T P x$, $P = P^T \succ 0$

$$\frac{dV(x)}{dt} = x^T\left[A(t)^T P + P A(t)\right] x < 0$$

$$\iff A(t)^T P + P A(t) \prec 0, \quad P \succ 0$$

$$\Leftrightarrow \quad \underline{(A_{nom} + \Delta(t))^T P + P(A_{nom} + \Delta(t)) \prec 0 \ , \ P \succ 0,}$$

$$\underline{\underline{\forall \ \Delta(t).}}$$

we will focus on two types of allowed perturbations:

a) $\underline{\|\Delta(t)\| \leq \gamma}$ for all t. $\gamma > 0$: scalar

(norm - bounded)

b) <u>parametric polytopic</u>:

$$\underline{\Delta(t)} \in \bar{\Delta} = \left\{ \Delta \ \middle| \ \Delta = A_1\delta_1 + A_2\delta_2 + \cdots + A_K\delta_K, \ \delta_i \in [-1, 1] \right\}$$