

## Homework 2: Convex Optimization in Control and Signal Processing

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### Q 2.1: Alternative Form of Farka's Lemma

Following the above theorem, show that exactly one of the following two sets is empty.

$$S_2 = \{x \in \mathbb{R}^n | Ax \leq b\},$$

$$T_2 = \{y \in \mathbb{R}^m | y \geq 0, y^\top A = 0, y^\top b > 0\}.$$

Show that  $S_2$  can be written as  $S_1$ , perhaps by using some ideas discussed in class while looking at equivalence of optimization problems. Then, construct the alternative system for the new  $S_1$  system and show that the alternative system is nothing but  $T_2$ . Hint: any  $x$  can be written as  $x_+ - x_-$  with both  $x_+ \geq 0, x_- \geq 0$ .

### Q 2.2: LP Duality

Consider the following linear optimization problem:

$$\begin{aligned} & \text{maximize}_{x \in \mathbb{R}^2} && 5x_1 + 10x_2 \\ & \text{subject to} && x_1 + 3x_2 \leq 50, \\ & && 4x_1 + 2x_2 \leq 60, \\ & && x_1 \leq 5 \\ & && x_1, x_2 \geq 0. \end{aligned}$$

Find the dual of the above optimization problem. Suppose  $x_1 = 5, x_2 = 15$  is the optimal solution of the above problem. Following complementarity slackness condition, find the optimal dual solution and show that strong duality holds. Verify your answer using CVX/YALMIP.

### Q 2.3: Convex Optimality Conditions

Consider the convex optimization problem:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} && f_0(x) \\ & \text{s.t.} && f_i(x) \leq 0, \quad i \in [m]. \end{aligned}$$

Let there exist  $x^*$  and  $\lambda^*$  that satisfy the KKT conditions for this problem. Then, show that

$$\nabla f(x^*)^\top (x - x^*) \geq 0,$$

for all feasible  $x$ . This shows the equivalence between the general optimality condition derived in class and the KKT conditions derived in the context of duality.

**Q 2.4: Strongly Convex Functions**

Show that if a function is strongly convex, it has a unique minimizer.

**Q 2.5: Linear Support Vector Classification with Soft Margin**

Given labeled dataset  $\{x^i, y^i\}_{i \in [N]}$  where each  $x^i \in \mathbf{R}^n$  is associated with a label  $y^i \in \{1, -1\}$  such that  $y^i = 1$  if  $x^i \in A$  and  $y^i = -1$  if  $x^i \in B$ . Consider the following classification problem with soft margin:

$$\begin{aligned} \min_{w \in \mathbf{R}^n, b \in \mathbf{R}, \epsilon \in \mathbf{R}^N} \quad & \frac{1}{2} \|w\|_2^2 + C \sum_{i=1}^N \epsilon_i \\ \text{s.t.} \quad & 1 - y^i(w^\top x^i + b) \leq \epsilon_i, \quad \forall i \in [N], \\ & \epsilon_i \geq 0, \quad \forall i \in [N]. \end{aligned}$$

Show that the dual of the above problem is given by:

$$\begin{aligned} \min_{\lambda \in \mathbf{R}^N} \quad & -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j y^i y^j (x^i)^\top x^j + \sum_{i=1}^N \lambda_i \\ \text{s.t.} \quad & 0 \leq \lambda_i \leq C, \quad \forall i \in [N], \\ & \sum_{i=1}^N \lambda_i y^i = 0. \end{aligned}$$

Given the optimal dual solution  $\lambda^*$ , find the optimal primal solution  $w^*$  and  $b^*$ .

**Q 2.6: Equivalent formulation of SVM**

Show that the Support Vector Classification problem can be stated equivalently as:

$$\min_{w \in \mathbf{R}^n, b \in \mathbf{R}} \quad \lambda \|w\|_2^2 + \frac{1}{N} \sum_{i=1}^N \max(0, 1 - y^i(w^\top x^i + b)).$$

How does  $\lambda$  relate with  $C$  in the previous formulation?

**Q 2.7: Support vector regression**

Given dataset  $\{x^i, y^i\}_{i \in [N]}$  where each  $x^i \in \mathbf{R}^n$  and  $y^i \in \mathbf{R}$  and linear hypothesis  $f(x) = w^\top x + b$ , the  $\epsilon$ -support vector regression problem is given by

$$\begin{aligned} \min_{w \in \mathbf{R}^n, b \in \mathbf{R}} \quad & \frac{1}{2} \|w\|_2^2 \\ \text{s.t.} \quad & y^i \leq w^\top x^i + b + \epsilon, \quad \forall i \in [N], \\ & y^i \geq w^\top x^i + b - \epsilon, \quad \forall i \in [N]. \end{aligned}$$

Find the dual of the above problem and show how to derive the primal optimal solution from the dual optimal solution.

**Q 2.8: Midsem Spring 2022-23**

Consider the following primal optimization problem:

$$\begin{aligned} & \underset{x \in \mathbb{R}^2}{\text{minimize}} && -x_2 \\ & \text{subject to} && x_2 \geq 0, x_1 \geq 0, \\ & && x_1 - x_2 \leq 3. \end{aligned}$$

Find the dual of the above optimization problem. Determine whether the primal is infeasible, unbounded or has an optimal solution. Determine the dual optimization problem and whether the dual is infeasible, unbounded or has an optimal solution. If either one has an optimal solution, then show that strong duality holds.

**Q 2.9: Midsem Spring 2022-23**

Let  $S_2 = \{x \in \mathbb{R}^n | x \geq 0, \sum_{i=1}^n x_i \leq 1\}$ . Then express that the projection of a point  $y$  on the set  $S_2$  as an optimization problem with constraints, and write the KKT conditions for the problem. Bonus: simplify the KKT solutions as much as possible to derive a closed form expression of the projection.

**Q 2.10: Endsem Spring 2022-23**

Consider the following optimization problem:

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && c^\top x \\ & \text{subject to} && a^\top x \leq b, \quad \forall a \in \{a \in \mathbb{R}^n | Ca \leq d\}, \end{aligned}$$

where  $C \in \mathbb{R}^{m \times n}$  and  $d \in \mathbb{R}^m$ . Reformulate the above problem as a convex optimization problem with finitely many constraints. Hint: You may need to use linear programming duality in a slightly non trivial manner.

**Q 2.11: Endsem Spring 2022-23**

Show that the optimal value of the following optimization problem:

$$\begin{aligned} & \underset{y \in \mathbb{R}^n}{\text{maximize}} && y^\top x \\ & \text{subject to} && \|y\|_1 \leq 1 \end{aligned}$$

is equal to  $\|x\|_\infty$ . Recall that for  $x \in \mathbb{R}^n$ ,  $\|x\|_1 := \sum_{i=1}^n |x_i|$  and  $\|x\|_\infty := \max_{i \in \{1, 2, \dots, n\}} |x_i|$ . Hint: Find the dual of the above optimization problem and compute the optimal dual solution. Explain why strong duality holds for this problem.