Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^T x + d$$

s.t. $\boxed{a_i^T x \le b_i \quad , i = 1, 2 \cdots m} \iff Ax \le b \; , \; A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

$g_j^T x = h_j \quad , j = 1, 2 \cdots p$

$\Downarrow$

$Gx = h \, , \; G \in \mathbb{R}^{p \times n} , \; h \in \mathbb{R}^p$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

In a more compact form,

$$\min_{x \in \mathbb{R}^n} \quad c^T x$$

s.t. $Ax \le b$

$Gx = h$

Quadratic Programming

$$\left.\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}x^T P x + q^T x + r \\ \text{s.t.} \quad & Ax \le b \quad : \quad \lambda \\ & Gx = h \quad : \quad \mu \end{aligned}\right] \text{(QP)}$$

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \,\middle|\, Ax \le b,\ Gx = h \right\} \quad : \text{ convex set}$$

$f(x) = \frac{1}{2}x^T P x + q^T x + r$ is a convex function of x only when P is positive semidefinite.

$$\mathcal{L}(x, \lambda, \mu) = \frac{1}{2}x^T P x + q^T x + r + \lambda^T [Ax - b] + \mu^T [Gx - h]$$

$$= \underbrace{\frac{1}{2}x^T P x + (q^T + \lambda^T A + \mu^T G) x + r - \lambda^T b - \mu^T h}_{\text{convex if } P \text{ is psd}}$$

$$d(\lambda, \mu) = \inf_{x\in\mathbb{R}^n} \mathcal{L}(x, \lambda, \mu)$$

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = Px + (q + A^T\lambda + G^T\mu) = 0$$

$$\Rightarrow x^* = -P^{-1}(q + A^T\lambda + G^T\mu)$$

$d(\lambda, \mu) = \mathcal{L}(x^*, \lambda, \mu)$ we will obtain a function which is quadratic in λ and μ.

Quadratically constrained QPs.

QCQP

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^T P_0 x + q_0^T x + r$$

$$\text{s.t.} \quad x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots m.$$

The above problem is convex if

$P_0, P_1, \ldots P_m$ are positive semi-definite.

Second order cone Programs (SOCP)

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

$$\text{s.t.} \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i \qquad i = 1, 2, \ldots m$$

Note : i) Every LP is a SOCP : if all $A_i = 0$

ii) Every QP is a SOCP :

iii) Every QCQP is a SOCP. :

$$\min_{x \in \mathbb{R}^n} \; x^T P x + q^T x + r \quad \xLeftarrow{\text{Step-1}} \quad \min_{x \in \mathbb{R}^n, t \in \mathbb{R}} \; t$$

$$\text{s.t.} \quad c_i^T x + d_i \geq \alpha_i \qquad\qquad \text{s.t.} \quad c_i^T x + d_i \geq \alpha_i$$

$$\boxed{x^T P x + q^T x + r \leq t}$$

It remains to convert $x^T P x + q^T x + r - t \leq 0$ to $\|A_i \bar{x} + b_i\|_2 \leq c_i^T \bar{x} + d_i$ where $\bar{x} = \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1}$

when P is positive definite.

We need to express $x^T P x + q^T x - t$ as $\|A\bar{x} + b\|_2^2$. HW

$$P = P^{1/2} P^{1/2}$$

Linear Matrix Inequalities (LMIs)

We can even consider decision variables that are matrices $\underline{X \in \mathbb{R}^{m\times n}}$.

We can view the elements of X as part of a vector $\bar{x} \in \mathbb{R}^{mn}$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \quad \bar{x} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} \quad \bar{C} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{bmatrix}, \quad \underline{\bar{c}^T \bar{x}} = \underline{\underline{\sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij} x_{ij}}}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

A linear function of the matrix-valued decision variable is given by $= \text{trace}(C^T X)$

$$\left[\begin{array}{ll} \min\limits_{X \in \mathbb{R}^{n\times m}} & \text{trace}(C^T X) \\ \text{s.t.} & \underbrace{\text{trace}(A_i^T X)}_{\text{scalar}} \le b_i, \quad i = 1, 2, \dots m \end{array}\right] = \text{nothing but a linear program.}$$

scalar. usual "less than equal to" form.

Matrix inequality: For two symmetric matrices A & B, $A \preceq B$ if $A - B$ is negative semidefinite or $B - A$ is positive semidefinite.

LMI: Given $F_0, F_1, \dots F_K \in \mathbb{R}^{n\times n}$,

$$F_0 + x_1 F_1 + x_2 F_2 + \dots + x_K F_K \preceq 0_{n\times n}$$

$\{x \in \mathbb{R}^k \mid F_0 + \sum_i x_i F_i \preceq 0_{n\times n}\}$ is a convex set. (HW).

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^K} & c^T x \\ \text{s.t.}: & F_0 + \sum_{i=1}^{K} x_i F_i \preceq 0_{n\times n} \end{array}$$

Optimization with LMI constraints.

Consider the problem $\min_{x \in \mathbb{R}^K} c^T x$

s.t. $F_0 + \sum_{j=1}^{K} x_j F_i \preceq 0_{n\times n}$.

The multiplier $\Lambda \in \mathbb{R}^{n\times n}$

$$\mathcal{L}(x,\Lambda) = c^T x + \langle \Lambda, F_0 + \sum_{i=1}^{K} x_i F_i \rangle$$

$$= \sum_{i=1}^{K} c_i x_i + \langle \Lambda, F_0 \rangle + \sum_{i=1}^{K} x_i \langle \Lambda, F_i \rangle$$

$$d(\Lambda) = \inf_{x \in \mathbb{R}^K} \mathcal{L}(x,\Lambda) = \begin{cases} \langle \Lambda, F_0 \rangle & \langle \Lambda, F_i \rangle = -c_i \quad \forall i = 1,2..K \\ -\infty, & \text{otherwise} \end{cases}$$

Dual: $\max_{\Lambda \succeq 0_{n\times n}} d(\Lambda)$

$$= \max_{\Lambda} \langle \Lambda, F_0 \rangle$$
$$\text{s.t. } \Lambda \succeq 0_{n\times n}$$
$$\langle \Lambda, F_i \rangle = -c_i$$

dual

linear matrix constraints

$\Lambda \in \mathcal{S}^{n\times n}$ → set of symmetric matrices.

Inner product $\langle \Lambda, F \rangle$
$= \text{trace}(\Lambda^T F)$