Q1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x_1, x_2) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2$$

Determine all points at which gradient of f is zero, and whether any of those points are minimizers.

Q2) Is the set $S = \{x \in \mathbb{R}^2 \mid x_1^2 - x_2^2 + x_1 + x_2 \leq 4\}$ convex?

Q3) Let f be a convex function over $\mathbb{R}^n$ and $x, y \in \mathbb{R}^n$. Let $\alpha > 0$

Let $z := x + \frac{1}{\alpha}(x - y)$. Show that

$$f(y) \geq f(x) + \alpha\left[f(x) - f(z)\right]$$

Q4) Determine if the following are convex optimization problems.

(a)
$$\min_{x \in \mathbb{R}^3} \ \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 = 3$$

(b)
$$\min_{x \in \mathbb{R}^2} \ x_1^2 + 2x_2^2 + 4x_1 x_2$$
$$\text{s.t.} \quad x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$

Q5) Show that projection of $y \in \mathbb{R}^n$ on the set $C = \{x \in \mathbb{R}^n \mid Ax = b\}$ is given by:

$$\Pi_C(y) = y - A^T(AA^T)^{-1}(Ay - b).$$

Note that projection is the optimal sol^n of $\boxed{\begin{array}{ll} \min\limits_{x} & \|x - y\|^2 \\ \text{s.t.} & Ax = b \end{array}}$.

Use Lagrangian to obtain expression of λ & then solve for the projection.

Discussion, 18/02/2024

Q1.7 C: compact and convex set

$$C(\alpha) = \{ x \in C \mid |x_1 - \alpha| \leq |y_1 - \alpha| \quad \forall y \in C \}$$

Let $x, z \in C(\alpha) \Rightarrow |x_1 - \alpha| \leq |y_1 - \alpha| \quad \forall y \in C$

$$|z_1 - \alpha| \leq |y_1 - \alpha| \quad \forall y \in C$$

we want to show

$$\boxed{\lambda x + (1-\lambda) z} \in C(\alpha) \quad \forall \lambda \in [0,1]$$

$$\underbrace{|\lambda x_1 + (1-\lambda) z_1 - \alpha|} \leq |y_1 - \alpha| \quad \forall y \in C$$

$$= |\lambda (x_1 - \alpha) + (1-\lambda)(z_1 - \alpha)|$$

$$\leq \lambda |x_1 - \alpha| + (1-\lambda) |z_1 - \alpha|$$

$$\leq \lambda |y_1 - \alpha| + (1-\lambda) |y_1 - \alpha| \quad \forall y \in C$$

$$= |y_1 - \alpha| \quad \forall y \in C.$$

Q2.3 x^* & λ^* satisfy KKT conditions

$$\Rightarrow \quad f_i(x^*) \leq 0 \quad \forall i \in [m]$$

$$\lambda_i^* \geq 0 \quad \forall i \in [m]$$

$$\lambda_i^* f_i(x^*) = 0 \quad \forall i \in [m]$$

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0$$

Let $\bar{x}$ be feasible $\Rightarrow f_i(\bar{x}) \leq 0 \quad \forall i \in [m]$

we want to show $\nabla f_0(x^*)^T (\bar{x} - x^*) \geq 0$

Since f_i's are convex, $0 \geq f_i(\bar{x}) \geq f_i(x^*) + \nabla f_i(x^*)^T(\bar{x} - x^*)$

$$\Rightarrow \quad 0 \geq \sum_{i=1}^{m} \left[f_i(x^*) + \nabla f_i(x^*)^T (\bar{x} - x^*) \right] \lambda_i^*$$

$$= \underbrace{\sum_{i=1}^{m} \lambda_i^* f_i(x^*)}_{=0} + \sum_{i=1}^{m} \underline{\underline{\lambda_i^* \nabla f_i(x^*)^T}} (\bar{x} - x^*)$$

$$= \quad - \underbrace{\nabla f_0(x^*)^T (\bar{x} - x^*)}$$

$$\Rightarrow \quad \underline{\nabla f_0(x^*)^T (\bar{x} - x^*) \geq 0} .$$

2.10

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

$$\text{s.t.} \quad \boxed{\underline{\underline{a^T x \leq b}} \qquad \forall \; a : \underline{\underline{Ca \leq d}}}$$

$$\Updownarrow$$

$$\boxed{\max_{a \in \mathbb{R}^n, Ca \leq d} a^T x \leq b}$$

$$\underbrace{\begin{aligned} &\max_{a \in \mathbb{R}^n} \; a^T x \\ &\text{s.t. } Ca \leq d \end{aligned}}_{\text{is a LP}} \quad (P)$$

$\Updownarrow$ Dual of (P): $\begin{aligned} &\min \; d^T y \\ &\text{s.t. } C^T y = x \\ &\qquad y \geq 0 \end{aligned} \quad (*)$

$$\min_{y,\, C^T y = x,\, y \geq 0} d^T y \leq b.$$

$$\boxed{\begin{array}{ll} \min_{x,y} & c^T x \\ \text{s.t.} & d^T y \le b \checkmark \\ & C^T y = x \checkmark \\ & y \ge 0 \checkmark \end{array}}$$

Q2.8 $\mathcal{L}(x,\lambda) = -x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3(x_1 - x_2 - 3) \quad \lambda \in \mathbb{R}^3$

$$d(\lambda) = \inf_{x \in \mathbb{R}^2} \mathcal{L}(x,\lambda)$$

$$= \inf_{x \in \mathbb{R}^2} \left[x_1(\lambda_3 - \lambda_1) + x_2(-1 - \lambda_2 - \lambda_3) - 3\lambda_3 \right]$$

$$= \begin{cases} -3\lambda_3 & \text{when } \lambda_3 - \lambda_1 = 0 \\ & \quad -1 - \lambda_2 - \lambda_3 = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Dual: $\max_{\lambda \ge 0} d(\lambda) \equiv$ $\boxed{\begin{array}{ll} \max_{\lambda} & -3\lambda_3 \\ \text{s.t.} & \lambda \ge 0 \\ & \lambda_3 - \lambda_1 = 0 \\ & \lambda_3 + \lambda_2 = -1 \end{array}}$ (*).

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \ge 0 \end{array} \longrightarrow \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \le c \end{array}$$

$$c = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

let $\bar{x}_2 = -x_2$ $\left\{ \begin{array}{ll} \max & -x_2 \\ \text{s.t.} & -x_1 \le 0 \\ & x_2 \le 0 \\ & x_1 + x_2 \le 3 \end{array} \right.$

which is the original problem
$$\begin{cases} \min & -\bar{x}_2 \\ \text{s.t.} & x_1 \geq 0,\ \bar{x}_2 \geq 0 \\ & x_1 - \bar{x}_2 \leq 3 \end{cases}$$

Q1.12 (d) $f_4(x_1, x_2) = \underbrace{x_1 \log(x_1 + \beta x_2)} - \underbrace{x_1 \log x_1}_{\text{convex } f^n.}$

(e) $f_5(x) = g(h(x))$

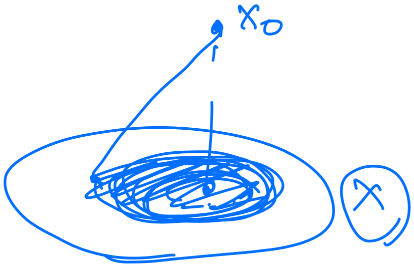
$g(y) = -y^{1/a}$, $\quad g'(y) = -\frac{1}{a} y^{1/a - 1} : -ve$

$h(x) = \underbrace{\sum_{i=1}^{n} \underbrace{x_i^a}_{\text{concave}}}_{= \text{concave.}}$ $\quad g''(y) = \underbrace{-\frac{1}{a}\left(\frac{1}{a} - 1\right) y^{1/a - 2}}_{-ve}$

$$\begin{cases} g : \text{concave \& decreasing.} \\ h : \text{concave.} \end{cases}$$

$\{x_0\}$

X : closed,



$\alpha = \|\bar{x} - x_0\|_2^2$

$S_\alpha(f) = \{x \mid \|x - x_0\|_2^2 \leq \alpha\}$

$x_1, x_2 \in S_\alpha(f)$

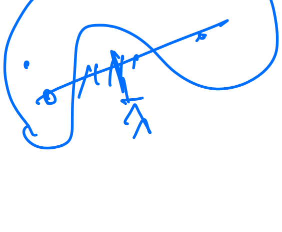
$\|x_1 - x_2\|_2 \leq \|x_1 - x_0\| + (\,)$

$\leq 2\alpha$

Epigraph: Suppose epigraph is not a convex set.

$\exists\ x_1, x_2 \in epi(f)$ & $\bar{\lambda} \in [0,1]$ s.t.

t_1, t_2



$(\bar{\lambda} x_1 + (1-\bar{\lambda}) x_2,\ \bar{\lambda} t_1 + (1-\bar{\lambda}) t_2) \notin epi(f)$

$\Rightarrow f(\bar{\lambda} x_1 + (1-\bar{\lambda}) x_2) > \bar{\lambda} t_1 + (1-\bar{\lambda}) t_2$

$\geq \bar{\lambda} f(x_1) + (1-\bar{\lambda}) f(x_2)$

$\Rightarrow$ f is not a convex function.

$A \Leftrightarrow B$, $A \Rightarrow B$

$B \Rightarrow A \Leftrightarrow A^c \Rightarrow B^c$

If epigraph is a convex set, then function is convex.

to show:

$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall \lambda \in [0,1]$

$(x_1, f(x_1)) \in \text{epi} f$ $\quad$ $(z,t) \in \text{epi}$

$\Leftrightarrow f(z) \leq t$

$(x_2, f(x_2)) \in \text{epi} f$

$\lambda \begin{bmatrix} x_1 \\ f(x_1) \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2 \\ f(x_2) \end{bmatrix} \in \text{epi} f \quad \forall \lambda \in [0,1]$

$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ ✓

$g(x) = \inf_y f(x,y)$ is concave when f is concave in x for every y.

$= -\sup_y -f(x,y)$

$= -h(x)$, where $h(x) = \sup_y \ell(x,y)$, where $\ell(x,y) = -f(x,y)$

concave. $\quad$ $\Downarrow$ convex $\quad$ $\downarrow$ convex in x $\forall y$.

$$\begin{aligned} \min_{x,y} \quad & \|x-y\|_2^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 1 \\ & y_1 - y_2 = 2 \end{aligned}$$

$x \in X_1$

$y \in X_2$

2.9

$$\min_{x \in \mathbb{R}^n} \|y - x\|_2^2$$

$$\text{s.t.} \quad x \geq 0, \quad \sum_{i=1}^{n} x_i \leq 1$$

$$\underbrace{f(x) + \lambda g(x)}$$

$$x^* = f(y)$$