EE61012 – Spring Semester 2023

Homework 2: Convex Optimization in Control and Signal Processing

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Q 2.1:

Let $C_1 = \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}$, and let $C_2 = \{x \in \mathbb{R}^2 | x_1 \leq 0, x_2 \leq 0\}$. Find a hyperplane that separates C_1 and C_2.

Q 2.2:

Let $x_k \in \mathbb{R}^n, k = 1, 2, \ldots, T$. Let $x = [x_1^\top, x_2^\top, \ldots, x_k^\top]^\top \in \mathbb{R}^{nT}$ where $f : \mathbb{R}^{nT} \to \mathbb{R}$. Let $f(x) = \sum_{k=1}^{T} x_k^\top Q_k x_k$, where each Q_k is positive semi-definite. Then show that f is a convex function.

Q 2.3:

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Show that

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq 0, \quad \forall x, y \in \mathtt{dom} f.$$

Q 2.4:

Suppose that $g(x)$ is convex and $h(x)$ is concave. Suppose the domain of both functions is a closed, convex set C such that both $g(x)$ and $h(x)$ are always positive. Prove that the function $f(x) = \frac{g(x)}{h(x)}$ is quasi-convex, i.e., all sub-level sets of f are convex sets.

Q 2.5:

Determine whether the following functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ are convex, concave or neither.

a. $f_1(x_1, x_2) = x_1 x_2$ with domain $x_1 > 0, x_2 > 0$.
b. $f_2(x_1, x_2) = -\frac{1}{x_1 x_2}$ with domain $x_1 > 0, x_2 > 0$.
c. $f_3(x_1, x_2) = \frac{x_1^2}{x_2}$ with domain $x_1 \in \mathbb{R}, x_2 > 0$.
d. $f_4(x_1, x_2) = x_1 \log\left(1 + \frac{\beta x_2}{x_1}\right)$ with domain $x_1, x_2 > 0$, and $\beta \in \mathbb{R}$ is a constant.

e. $f_5(x) = -(\sum_{i=1}^{n} x_i^a)^{1/a}$ for $a \in (0, 1)$.

f. $f_6(x) = \min(0.5, x, x^2)$.

g. $f_7(x) = \max_{i \in \{1,2,\ldots,n\}} x_i - \min_{i \in \{1,2,\ldots,n\}} x_i$.

Q 2.6:

Let $C \in \mathbb{R}^n$ be a compact (closed and bounded) convex set. For a scalar α, let $C(\alpha)$ be the set of all points in C whose first coordinate is closer to α than any other point. In other words, $C(\alpha) = \{x \in C | |x_1 - \alpha| \leq |y_1 - \alpha|, \quad \forall y \in C\}$. Is $C(\alpha)$ a convex set for any choice of C and α?

Q 2.7:

Formulate an optimization problem to find the largest circle contained in the triangle $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, 3x_1 + 4x_2 \leq 12\}$. Clearly state the decision variables, constraints and cost function. Determine if the problem is a convex optimization problem.

Q 2.8:

If f is a convex function, then $g(x, t) = tf(x/t)$ is also a convex function. You may use the epigraph definition of convex functions to show the above.

Q 2.9:

Formulate an optimization problem to find the minimum distance between a circle $X_1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ and the line $X_2 := \{x \in \mathbb{R}^2 \mid x_1 - x_2 = 2\}$. Clearly state the decision variables, constraints and cost function. Determine if the problem is a convex optimization problem.