EE61012 – Spring Semester 2024

Homework 1: Convex Optimization in Control and Signal Processing

Prof. Ashish Ranjan Hota
Department of Electrical Engineering, IIT Kharagpur

Q 1.1: Bounded Set

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence that converges to $x^* \in X$. Show that the set $\{x_n\}_{n\in\mathbb{N}} = \{x_1, x_2, \ldots\} \subseteq X$ is bounded. Recall that a sequence converges if for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $||x^* - x_m|| \leq \epsilon$ for all $m \geq n$.

Q 1.2: Open and Closed Set

Is the set $X = \{x \in \mathbb{R}^2 | x_1 + x_2 = 1\}$ an open set? Is it a closed set?

Q 1.3: Convex Hull

Let $C = \{x \in \mathbb{R} | 1 \leq x \leq 2, 3 \leq x \leq 4\}$. Find the affine hull, conic hull, and convex hull of C.

Q 1.4: Convex Set

Let $x_0, v \in \mathbb{R}^n$. Let $C = \{x \in \mathbb{R}^n | x = x_0 + \alpha v, \alpha \geq 0\}$ be the set of points that lie on the ray originating from x_0 along the direction v. Is C an affine set? Is it a convex set? Is it a cone?

Q 1.5: Convex Set

Show that the set $C = \{x \in \mathbb{R}^2 | x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$ is a convex set. Hint: If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{(1-\theta)} \leq \theta a + (1-\theta) b$.

Q 1.6: Convex Set

Let $x \in \mathbb{R}^n$ and $F_0, F_1, \ldots, F_n \in \mathbb{S}^n$ where $\mathbb{S}^n$ is the set of symmetric matrices of dimension n. A linear matrix inequality (LMI) $F : \mathbb{R}^n \to \mathbb{S}^n$ is an expression of the form:

$$F(x) := F_0 + \sum_{i\in[n]} x_i F_i \preceq 0_{n\times n}.$$

Show that the set $\{x \in \mathbb{R}^n \mid F(x) \preceq 0_{n\times n}\}$ is a convex set.

Q 1.7: Convex Set

Let $C \in \mathbb{R}^n$ be a compact (closed and bounded) convex set. For a scalar α, let $C(\alpha)$ be the set of all points in C whose first coordinate is closer to α than any other point. In other words, $C(\alpha) = \{x \in C | |x_1 - \alpha| \leq |y_1 - \alpha|, \quad \forall y \in C\}$. Is $C(\alpha)$ a convex set for any choice of C and α?

Q 1.8: Separating Hyperplane

Let $C_1 = \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}$, and let $C_2 = \{x \in \mathbb{R}^2 | x_1 \leq 0, x_2 \leq 0\}$. Find a hyperplane that separates C_1 and C_2.

Q 1.9: Convex Function

Let $x_k \in \mathbb{R}^n, k = 1, 2, \ldots, T$. Let $x = [x_1^\top, x_2^\top, \ldots, x_k^\top]^\top \in \mathbb{R}^{nT}$ where $f : \mathbb{R}^{nT} \to \mathbb{R}$. Let $f(x) = \sum_{k=1}^{T} x_k^\top Q_k x_k$, where each Q_k is positive semi-definite. Then show that f is a convex function.

Q 1.10: Convex Function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Show that

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq 0, \quad \forall x, y \in \texttt{dom} f.$$

Q 1.11: Convex Function

Suppose that $g(x)$ is convex and $h(x)$ is concave. Suppose the domain of both functions is a closed, convex set C such that both $g(x)$ and $h(x)$ are always positive. Prove that the function $f(x) = \frac{g(x)}{h(x)}$ is quasi-convex, i.e., all sub-level sets of f are convex sets.

Q 1.12: Convex Function

Determine whether the following functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ are convex, concave or neither.

a. $f_1(x_1, x_2) = x_1 x_2$ with domain $x_1 > 0, x_2 > 0$.

b. $f_2(x_1, x_2) = -\frac{1}{x_1 x_2}$ with domain $x_1 > 0, x_2 > 0$.

c. $f_3(x_1, x_2) = \frac{x_1^2}{x_2}$ with domain $x_1 \in \mathbb{R}, x_2 > 0$.

d. $f_4(x_1, x_2) = x_1 \log\left(1 + \frac{\beta x_2}{x_1}\right)$ with domain $x_1, x_2 > 0$, and $\beta \in \mathbb{R}$ is a constant.

e. $f_5(x) = -(\sum_{i=1}^{n} x_i^a)^{1/a}$ for $a \in (0, 1)$.

f. $f_6(x) = \min(0.5, x, x^2)$.

g. $f_7(x) = \max_{i\in\{1,2,\ldots,n\}} x_i - \min_{i\in\{1,2,\ldots,n\}} x_i$.

Q 1.13: Convex Function

If f is a convex function, then $g(x,t) = tf(x/t)$ is also a convex function. You may use the epigraph definition of convex functions to show the above.

Q 1.14: Convex Optimization

Formulate an optimization problem to find the largest circle contained in the triangle $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, 3x_1 + 4x_2 \leq 12\}$. Clearly state the decision variables, constraints and cost function. Determine if the problem is a convex optimization problem.

Q 1.15: Convex Optimization

Formulate an optimization problem to find the minimum distance between a circle $X_1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ and the line $X_2 := \{x \in \mathbb{R}^2 \mid x_1 - x_2 = 2\}$. Clearly state the decision variables, constraints and cost function. Determine if the problem is a convex optimization problem.

Q 1.16: Midsem Spring 2022-23

Answer the following questions. Each question carries equal number of points.

1. Give an example of two convex sets whose union is not convex.
2. Let $f(x) = \frac{x^2+2}{x+2}$ with $\texttt{dom}(f) = (-\infty, -2)$. Is this function convex, concave or neither?
3. Show that a function is convex if and only if its epigraph is a convex set.
4. Consider an optimization problem with cost function $f(x) = -x_1 + x_2^2$ and constraint set $X = \{x \in \mathbb{R}^2 | -x_1^2 - x_2^2 + 4 \leq 0, x_1 + x_2 \geq -2\}$. Explain with justificiation whether this problem is a convex optimization problem or not.

Q 1.17: Midsem Spring 2022-23

Consider the set $X := \{x \in \mathbb{R}^n | ||x - z_0||_2 \leq ||x - z_i||_2, i = 1, 2, \ldots, k\}$ where $z_0, z_1, \ldots, z_k \in \mathbb{R}^n$. Show that this set is a polyhendron and can be written as $X := \{x \in \mathbb{R}^n | Ax \leq b\}$. Find A and b.

Q 1.18: Endsem Spring 2022-23

Consider the following optimization problem:

$$\begin{aligned} \text{maximize}_{x \in \mathbb{R}^1} \quad & 2x - x^2 \\ \text{subject to} \quad & 0 \leq x \leq 3. \end{aligned}$$

Is the above problem a convex optimization problem? Find a globally optimal solution of the above.

Q 1.19: Endsem Spring 2022-23

Determine if the following statements are true or false. If true, justify. If false, give a counter example.

1. A convex optimization problem can have at most one globally optimal solution.
2. A convex optimization problem must have at least one globally optimal solution.
3. Any optimization problem with an unbounded feasible region does not have an optimal solution.

Q 1.20: Endsem Spring 2022-23

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = (x_1 + x_2^2)^2$ where $x = [x_1 x_2]$.

1. Compute the gradient of f.
2. At $x_0 = [0, 1]$, is the direction $d = [1, -1]$ a descent direction, i.e., with directional derivative negative?
3. Find $\alpha > 0$ that minimizes $f(x_0 + \alpha d)$.