AI1110
Assignment 7

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Outline

1. Question
2. Solution

Papoulis Chapter 5 problem 5.38

a)Let $x \sim G(\alpha, \beta)$, Show that $E(x) = \alpha\beta$,Var(x)=$\alpha\beta^2$ and $\phi_X(w) = (1 - \beta jw)^{-\alpha}$.
b)Let $x \sim \chi^2(n)$, Show that $E(x) = n$,Var(x)=$2n$ and $\phi_X(w) = (1 - 2e^{jw})^{\frac{n}{2}}$.
c)Let $x \sim B(n, p)$, Show that $E(x) = np$,Var(x)=npq and $\phi_X(w) = (pe^{jw} + q)^n$.
d)Let $x \sim NB(r, p)$, Show that $\phi_X(w) = p'(1 - qe^{jw})^{-r}$.

Solution

a)

$$f_X(x) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} \forall x \in [0, \infty) \tag{1}$$

else $f_X(x) = 0$ Mean(μ)=$\int_{-\infty}^{\infty} x f_X(x) dx$

$$\implies \mu = \int_{-\infty}^{\infty} \frac{x^{\alpha} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} dx \tag{2}$$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \tag{3}$$

$$\text{Let} \frac{x}{\beta} = t \tag{4}$$

$$\implies \frac{dx}{\beta} = dt \tag{5}$$

$$x^{\alpha} = \beta^{\alpha} t^{\alpha} \tag{6}$$

$$\therefore \mu = \beta \int_0^{\infty} \frac{t^{\alpha} e^{-t}}{\Gamma(\alpha)} dt \tag{7}$$

$$\tag{8}$$

Simplification of numerator using integration by parts-

$$\int_0^{\infty} t^{\alpha} e^{-t} dt = \alpha \int_0^{\infty} t^{\alpha-1} e^{-t} dt \tag{9}$$

$$= \alpha \Gamma(\alpha) \tag{10}$$

$$\therefore \mu = \alpha\beta \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \tag{11}$$

$$\implies \mu = \alpha\beta \tag{12}$$

$$E(x^2) = \int_0^\infty \frac{x^{\alpha+1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} dx \text{Let} \frac{x}{\beta} = t \tag{13}$$

$$\implies \frac{dx}{\beta} = dt \tag{14}$$

$$x^{\alpha+1} = \beta^{\alpha+1} t^{\alpha+1} \tag{15}$$

$$\text{Substituting the above we get-} \tag{16}$$

$$E(x^2) = \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty t^{\alpha+1} e^{-t} dt \tag{17}$$

$$\text{Using integration by parts} = \tag{18}$$

$$(\alpha + 1) \int_0^\infty t^\alpha e^{-t} dt \tag{19}$$

$$\text{From (9)–} \tag{20}$$

$$E(x^2) = \frac{\beta^2}{\Gamma(\alpha)} (\alpha + 1) \alpha \Gamma(\alpha) \tag{21}$$

$$\implies E(x^2) = (\alpha + 1)\alpha\beta^2 \tag{22}$$

$$Var(x) = E(x^2) - (E(x))^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 \tag{23}$$

$$\therefore Var(x) = \alpha\beta^2 \tag{24}$$

$$\phi_X(w) = \int_0^\infty \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} e^{iwx} dx \tag{25}$$

$$\implies \phi_X(w) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{\frac{-x(1-i\beta t)}{\beta}} dt \tag{26}$$

$$\text{Let} \frac{\beta}{1 - i\beta t} = \beta^* \tag{27}$$

$$\text{also} \tag{28}$$

$$\int_0^\infty e^{\frac{-x}{\beta^*}} x^{\alpha-1} dx = \Gamma(\alpha)(\beta^*)^\alpha \tag{29}$$

$$\because \int_0^\infty f_X(x) dx = 1 \tag{30}$$

$$\tag{31}$$

$$\therefore \phi_X(w) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\Gamma(\alpha)(\beta^{*})^{\alpha} \tag{32}$$

$$\implies \phi_X(w) = (\frac{\beta^{*}}{\beta})^{\alpha} \tag{33}$$

$$\therefore \phi_X(w) = (1 - i\beta t)^{-\alpha} \tag{34}$$

b)

$$X \sim \chi^2(n) \implies \alpha = \frac{n}{2}, \beta = 2 \tag{35}$$

in Gamma(α, β). This gives

$$\phi_X(\omega) = (1 - j2\omega)^{-n/2} \tag{36}$$

$$E(X) = n \tag{37}$$

$$Var(X) = 2n \tag{38}$$

c)$P_X(x) = {}^nC_x p^x q^{n-x}$, $p+q=1$ for $x \in [0,n]$ else $P_X(x)$=0

$$E(x) = \sum_{-\infty}^{+\infty} x P_X(x) \tag{39}$$

$$\implies E(x) = \sum_{x=0}^{n} x\,{}^nC_x p^x q^{n-x} \tag{40}$$

$$x\,{}^nC_x = n\,{}^{n-1}C_{x-1} \tag{41}$$

$$= \sum_{x=0}^{n} n\,{}^{n-1}C_{x-1} p.p^{x-1} q^{(n-1)-(x-1)} \tag{42}$$

$$= np(p+q)^{n-1} \tag{43}$$

$$= np \tag{44}$$

$$\because p+q=1 \tag{45}$$

$$E(x(x-1)) = \sum_{x=0}^{n} x(x-1)^n C_x p^x q^{n-x} \tag{46}$$

using (41) we can simplify to get

$$E(x(x-1)) = n(n-1)p^2 \tag{47}$$
$$E(x^2 - x) = E(x^2) - E(x) \tag{48}$$
$$\implies E(x^2) = n^2p^2 + npq \tag{49}$$
$$\sigma^2 = E(x^2) - (E(x))^2 \implies \sigma^2 = npq \tag{50}$$

that is $Var(x) = npq$

$$\phi_X(w) = \int_{-\infty}^{\infty} {}^nC_x p^x q^{n-x} e^{iwx} dx \tag{51}$$

$$\implies \phi_X(w) = \int_{-\infty}^{\infty} {}^nC_x e^{iw} p^x q^{n-x} dx \tag{52}$$

$$\implies \phi_X(w) = \sum_{x=0}^{n} {}^nC_x (e^{iw} p)^x q^{n-x} \tag{53}$$

$\because$ x is discrete and $x \in [0, n]$

$$\implies \phi_X(w) = (pe^{iw} + q)^n \tag{54}$$

d)$NB(r,p) = {}^{r+x-1}C_x p^r q^x$ where x=0,1,2.....∞

$$\phi_X(w) = \int_{-\infty}^{\infty} {}^{r+x-1}C_x p^r q^x e^{iwx} dx \tag{55}$$

$$\implies \phi_X(w) = \sum_{x=0}^{\infty} {}^{r+x-1}C_x p^r q^x e^{iwx} \tag{56}$$

$$\implies \phi_X(w) = \sum_{x=0}^{\infty} {}^{r+x-1}C_x p^r (qe^{iw})^x dx \tag{57}$$

$$\therefore \phi_X(w) = p'(1 - qe^{jw})^{-r} \tag{58}$$

$\because$ x is discrete