Random Numbers

Pranav B

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
cc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.2/uni_cdf.
    py
```

Run the following command in the terminal to run the code.

```
python3 uni_cdf.py
```

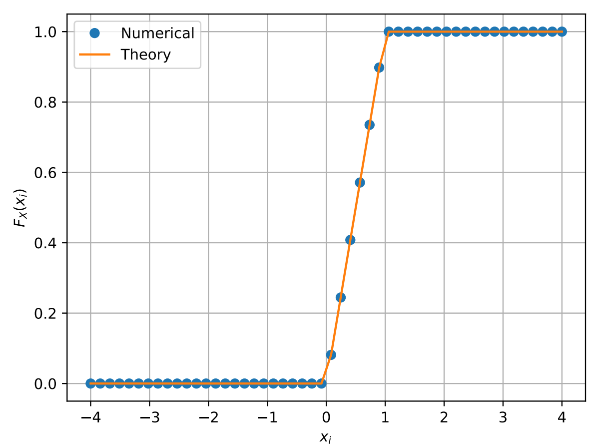

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$
CDF of $P_U(x)$=$F_U(x)$

$$= \int P_U(x)dx \tag{1.2}$$

$$= \int kdx \tag{1.3}$$

$$\text{wkt} \int_0^1 kdx = 1 \tag{1.4}$$

$$\therefore k = 1 \tag{1.5}$$

$$\therefore F_U(x) = x \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.4/
    mean_var.c
```

Use below command to run file,

```
cc mean_var.c -lm
./a.out
```

running the code gives us Mean =0.500137 , Variance =0.083251

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

$$dF_U(x) = dx \tag{1.10}$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.11}$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{1.12}$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.13}$$

$$\because P_X(x) = 0, \forall x \in (1,\infty) \cap (-\infty, 0) \tag{1.14}$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.15}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc exrand.c -lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in 2.2,Properties of the CDF:

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

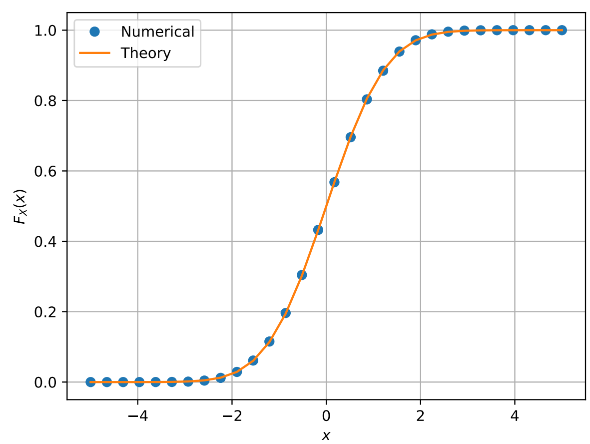

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in 2.3 using the code below

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/2.3/pdf.py
```

Use the below command to run the code:

```
python3 pdf.py
```

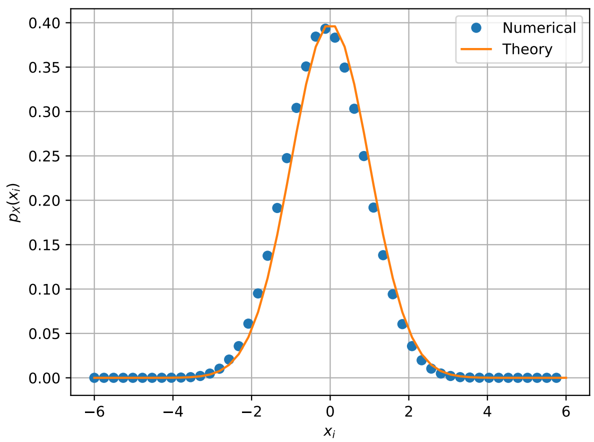

Fig. 2.3: The PDF of X

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives Mean = -0.000417 Variance= 0.999902

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/2.4/
    mean_var(gau).c
```

Command used:

```
cc mean_var(gau).c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Given ,$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$
E[x]=$\int_{-\infty}^{\infty} x p_X(x) dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}}$
$\because x e^{-\frac{-x^2}{2}}$ is a odd function,
$E[x] = 0$

$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (x e^{-\frac{-x^2}{2}}) dx$

Using integration by parts:

$$= x \int x e^{-\frac{-x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{-x^2}{2}} dx \quad (2.4)$$

$$I = \int x e^{-\frac{-x^2}{2}} \quad (2.5)$$

$$\text{Let} \frac{x^2}{2} = t \quad (2.6)$$

$$\implies x dx = dt \quad (2.7)$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \quad (2.8)$$

$$\therefore \int x e^{-\frac{-x^2}{2}} = -e^{-\frac{-x^2}{2}} + c \quad (2.9)$$

Using (2.9) in (2.4)

$$= -x e^{-\frac{-x^2}{2}} + \int e^{-\frac{-x^2}{2}} dx \quad (2.10)$$

$$\text{Also,} \int_{-\infty}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi} \quad (2.11)$$

$$\therefore \text{substituting limits we get,} E[x^2] = 1 \quad (2.12)$$

$$Var(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.13)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/3.1/V.py
```

Use the below command in the terminal to run the code:

```
python3 V.py
```

Now these samples are used to plot (3.1) by running the below code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/3.1/V_cdf.
    py
```

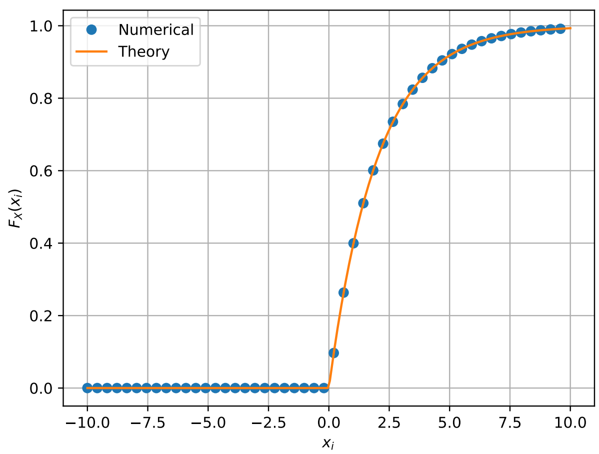

Fig. 3.1: CDF for (3)

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \tag{3.2}$$
$$= P(-2ln(1-U) \leq x) \tag{3.3}$$
$$= P(1 - e^{\frac{-x}{2}} \geq U) \tag{3.4}$$
$$P(U < x) = \int_0^x dx = x \tag{3.5}$$
$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \tag{3.6}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Run the below code to generate T.dat

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.1/
    T_gen_dat.c
```

Run the command below in the terminal

```
cc T_gen_dat.c -lm
./a.out
```

4.2 Find the CDF of T.

$$F_T(t) = P(T < t) \tag{4.2}$$
$$= P(U_1 + U_2 < t) \tag{4.3}$$

we know that $0 \leq U_1 \leq 1$ and $0 \leq U_2 \leq 1$
$\therefore 0 \leq U_1 + U_2 \leq 2$, so
$\forall t > 2, P(U_1 + U_2 < t) = 1$
$\forall t < 0, P(U_1 + U_2 < t) = 0$
for $0 \leq t \leq 2$ let us split it into 2 cases, for $0 \leq t \leq 1$ and $1 < t \leq 2$

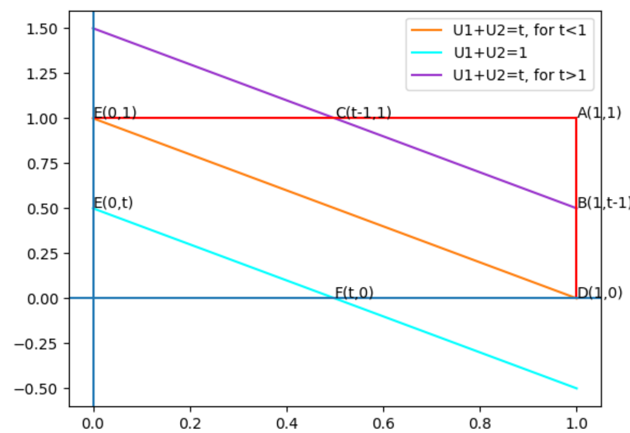

Fig. 4.2: Plot

The above figure is produced by the following code

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.2/T_plot.
    py
```

Run the following command in the terminal to run the code

```
python3 T_plot.py
```

From Fig (4.2)

$$P(U_1 + U_2 < t, 0 \leq t \leq 1) = \frac{\Delta(EOF)}{\Delta(AEOD)} \tag{4.4}$$
$$= \frac{t^2}{2} \tag{4.5}$$
$$P(U_1 + U_2 < t, 1 \leq t \leq 2) = \frac{\Delta(ABC)}{\Delta(AEOD)} \tag{4.6}$$
$$= 1 - \frac{(2-t)^2}{2} \tag{4.7}$$
$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \tag{4.8}$$

4.3 Find the PDF of T.
Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \tag{4.9}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{4.10}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{4.11}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases} \tag{4.12}$$

4.5 Verify your results through a plot.
Solution: Run the below code to get the cdf

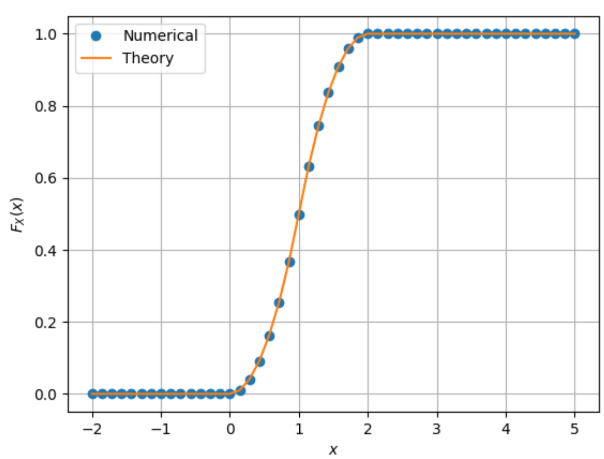

Fig. 4.5: CDF for (4)

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.5/T_cdf.
    py
```

Use the following command in the terminal to run the code

```
python3 T_cdf.py
```

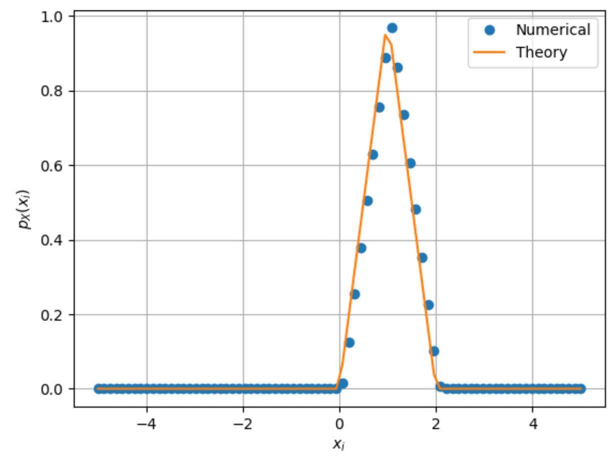

Fig. 4.5: PDF for (4)

Run the below code to get the pdf

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.5/T_pdf.
    py
```

Use the following command in the terminal to run the code

```
python3 T_pdf.py
```

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X1\{1, -1\}$, is Bernoulli and $N \sim 01$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: The sum of squares of k independent standard random normal variables is nothing

but a χ^2 distribution with k degrees of freedom.
$\chi^2(k) = \frac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{\frac{-x}{2}}, \forall x \geq 0$
Here k=2,

$$\therefore \chi^2(2) = P_V(v) = \frac{e^{\frac{-x}{2}}}{2} \tag{6.2}$$

$$\implies F_V(v) = \int_0^v \frac{e^{\frac{-x}{2}}}{2} dx \tag{6.3}$$

$$= 1 - e^{\frac{-x}{2}} \tag{6.4}$$

To generate data for V , run the following code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.1/V_gen.
    c
```

Run the below command in terminal,

```
cc V_gen.c −lm
./a.out
```

The PDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.1/chi_pdf.
    py
```

Use the following command in the terminal to run the code

```
python3 chi_pdf.py
```

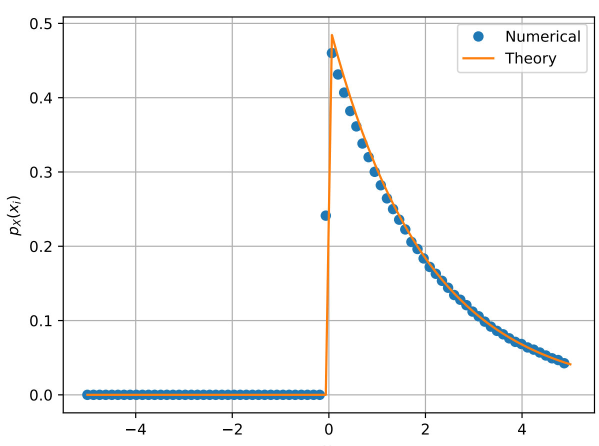

Fig. 6.1: PDF for (6.1)

The CDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.1/chi_cdf.
    py
```

Use the following command in the terminal to run the code

```
python3 chi_cdf.py
```

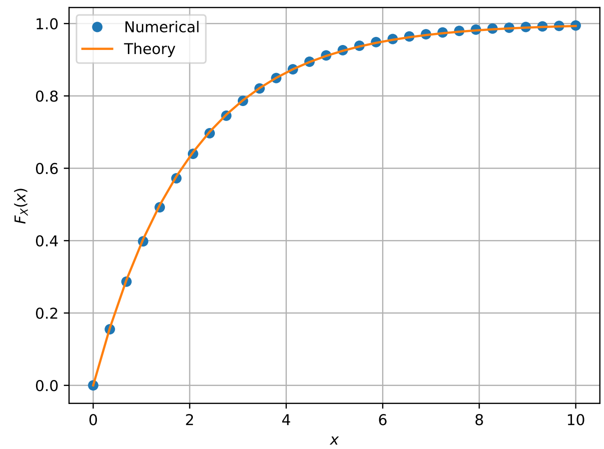

Fig. 6.1: PDF for (6.1)

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.5}$$

find α.
Solution: From (6.4) $\alpha = 0.5$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.6}$$

Solution:

$$F_A(a) = P(A < a) = P(V < a^2) \tag{6.7}$$

$$\text{from(6.4)}, = \begin{cases} 1 - e^{\frac{-a^2}{2}} & a > 0 \\ 0 & a <= 0 \end{cases} \tag{6.8}$$

$$\implies P_A(a) = \frac{d(F_A(a))}{da} \tag{6.9}$$

$$= \begin{cases} ae^{\frac{-a^2}{2}} & a > 0 \\ 0 & a <= 0 \end{cases} \tag{6.10}$$

To generate data for A , run the following code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.3/A_gen.
    c
```

Run the below command in terminal,

```
cc A_gen.c −lm
./a.out
```

The PDF plot of A can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.3/A_pdf.
    py
```

Use the following command in the terminal to run the code

```
python3 A_pdf.py
```

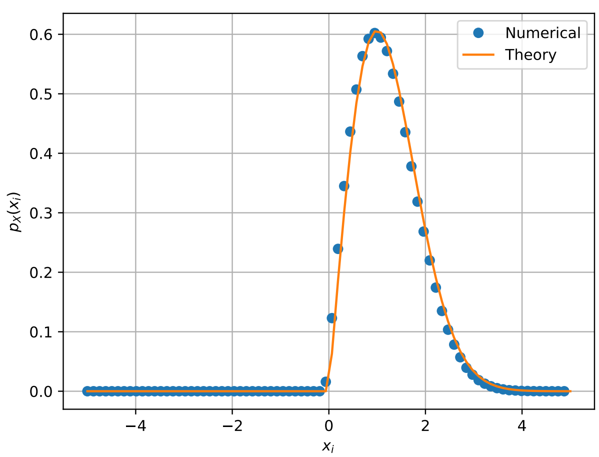

Fig. 6.3: PDF for (6.3)

The CDF plot of the A can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.3/A_cdf.
    py
```

Use the following command in the terminal to run the code

```
python3 A_cdf.py
```

7 Conditional Probability

7.1

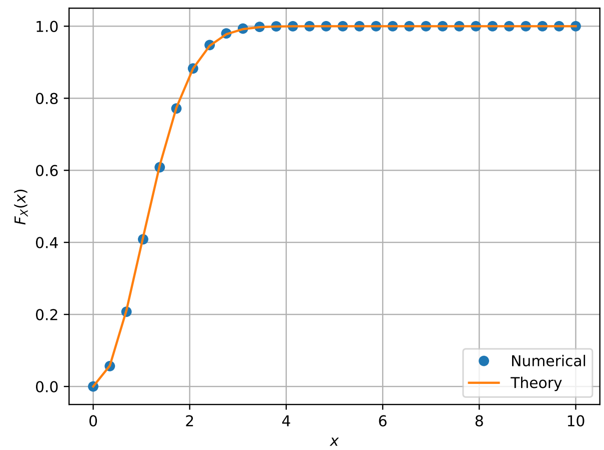

Fig. 6.3: CDF for (6.3)

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.