

Random Numbers

Pranav B

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
cc -lm exrand.c
.\exrand.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The graph 1.2 is obtained by running the below code

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/cdf_plot.py
```

The above codes plot the figure below

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$

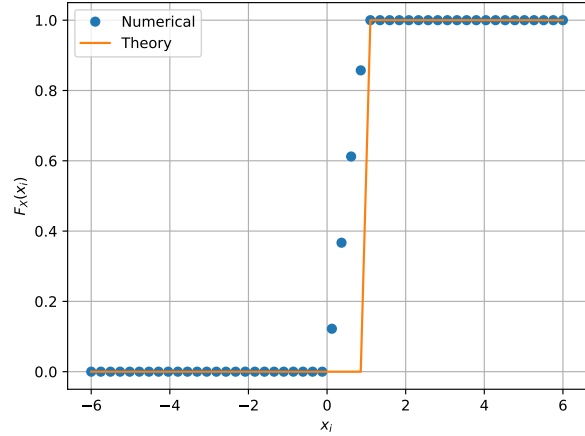


Fig. 1.2: The CDF of U

CDF of $P_U(x)=F_U(x)$

$$= \int P_U(x) dx \quad (1.2)$$

$$= \int k dx \quad (1.3)$$

$$\text{wkt } \int_0^1 k dx = 1 \quad (1.4)$$

$$\therefore k = 1 \quad (1.5)$$

$$\therefore F_U(x) = x \quad (1.6)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.7)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

Write a C program to find the mean and variance of U .

Solution:

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.4/
mean_var.c
```

Use below command to run file,

```
cc -lm mean_var.c
.\mean_var.out
```

running the code gives us Mean =0.500008 ,
Variance =0.083302

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

$$dF_U(x) = dx \quad (1.10)$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \quad (1.11)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.13)$$

$$\therefore P_X(x) = 0, \forall x \in (1, \infty) \cap (-\infty, 0) \quad (1.14)$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.15)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc -lm exrand.c
.\exrand.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in 2.2, Properties of the CDF:

- $\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du$

- $\lim_{x \rightarrow \infty} \Phi(x) = 1, \lim_{x \rightarrow -\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

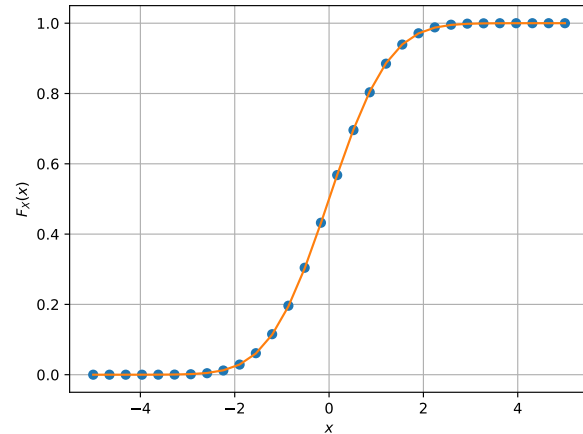


Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

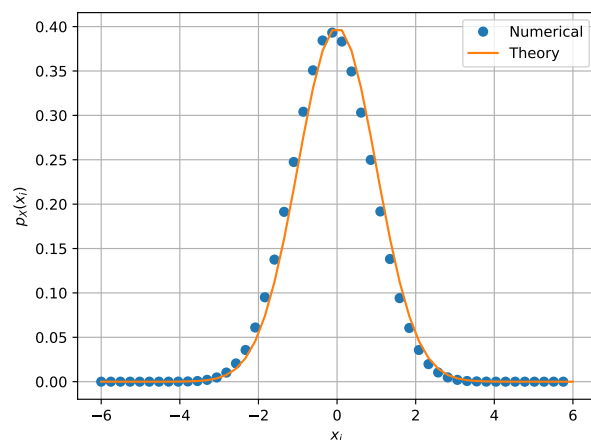


Fig. 2.3: The PDF of X

Solution: The PDF of X is plotted in 2.3 using the code below

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/2.3/pdf.py
```

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.

Solution: Running the below code gives Mean = 0.000337 Variance= 1.000803

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/2.4/
mean_var(gau).c
```

Command used:

```
cc -lm mean_var(gau).c
.\mean_var(gau).out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Given, $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$F_X(x) = \int_{-\infty}^x p_X(x) dx$$

$$= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (2.4)$$

$$= \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \quad (2.5)$$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx \quad (2.7)$$

$$\therefore x e^{-\frac{x^2}{2}} \text{ is a odd function,} \quad (2.8)$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.9)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (x e^{-\frac{x^2}{2}}) dx \quad (2.10)$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.11)$$

$$I = \int x e^{-\frac{x^2}{2}} dx \quad (2.12)$$

$$\text{Let } \frac{x^2}{2} = t \quad (2.13)$$

$$\Rightarrow x dx = dt \quad (2.14)$$

$$\Rightarrow \int e^{-t} dt = -e^{-t} + c \quad (2.15)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.16)$$

Using (2.16) in (2.11)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.17)$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.18)$$

$$\therefore \text{ substituting limits we get, } E[x^2] = 1 \quad (2.19)$$

$$\operatorname{Var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.20)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

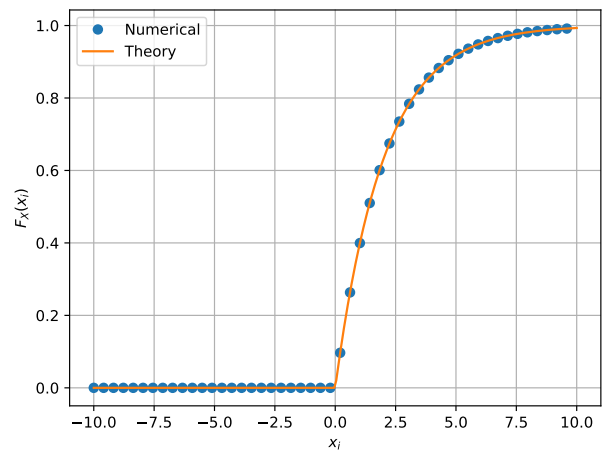


Fig. 3.1: CDF for (3)

Solution: Running the below code generates samples of V from file uni.dat(U).

[https://github.com/Pranavb060504/
Random_numbers/blob/main/3.1/V.py](https://github.com/Pranavb060504/Random_numbers/blob/main/3.1/V.py)

Now these samples are used to plot (3.1) by running the below code,

[https://github.com/Pranavb060504/
Random_numbers/blob/main/3.1/V_cdf.
py](https://github.com/Pranavb060504/Random_numbers/blob/main/3.1/V_cdf.py)

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2\ln(1 - U) \leq x) \quad (3.3)$$

$$= P(1 - e^{\frac{-x}{2}} \geq U) \quad (3.4)$$

$$P(U < x) = \int_0^x dx = x \quad (3.5)$$

$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \quad (3.6)$$