

Random Numbers

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
cc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/1.2/uni_cdf.
py
```

Run the following command in the terminal to run the code.

```
python3 uni_cdf.py
```

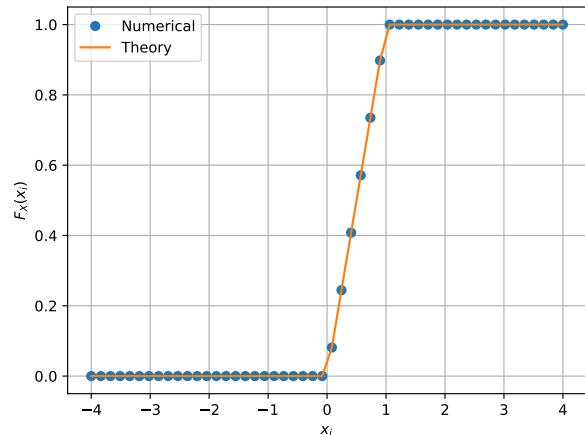


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$
CDF of $P_U(x) = F_U(x)$

$$= \int P_U(x) dx \quad (1.2)$$

$$= \int k dx \quad (1.3)$$

$$\text{wkt } \int_0^1 k dx = 1 \quad (1.4)$$

$$\therefore k = 1 \quad (1.5)$$

$$\therefore F_U(x) = x \quad (1.6)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.7)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

Write a C program to find the mean and variance of U .

Solution:

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.4/
mean_var.c
```

Use below command to run file,

```
cc mean_var.c -lm
./a.out
```

running the code gives us Mean =0.500137 ,
Variance =0.083251

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

$$dF_U(x) = dx \quad (1.10)$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \quad (1.11)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.13)$$

$$\therefore P_X(x) = 0, \forall x \in (1, \infty) \cap (-\infty, 0) \quad (1.14)$$

$$\text{Var}(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.15)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/exrand.c
```

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/1.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in 2.2, Properties of the CDF:

- $\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x \rightarrow \infty} \Phi(x) = 1, \lim_{x \rightarrow -\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

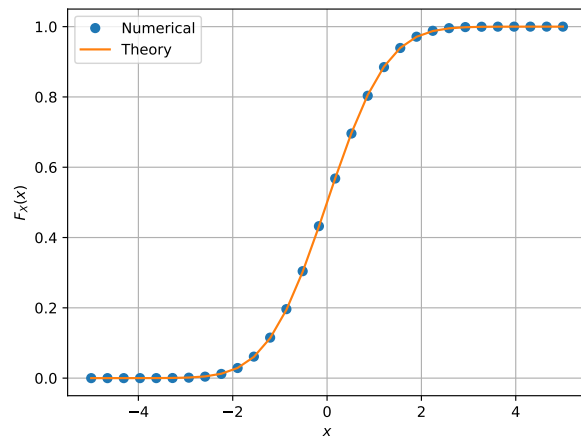


Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in 2.3 using the code below

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/2.3/pdf.py
```

Use the below command to run the code:

```
python3 pdf.py
```

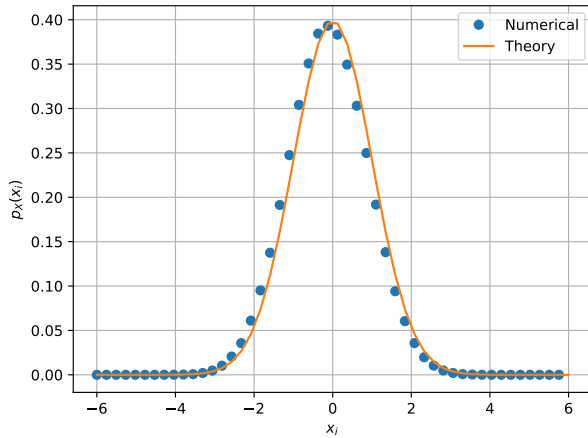


Fig. 2.3: The PDF of X

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.

Solution: Running the below code gives Mean = -0.000417 Variance= 0.999902

```
wget https://github.com/Pranavb060504/
Random_numbers/blob/main/2.4/
mean_var(gau).c
```

Command used:

```
cc mean_var(gau).c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Given, $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx$$

$\therefore x e^{-\frac{x^2}{2}}$ is a odd function,

$$E[x] = 0$$

$$\begin{aligned} E[x^2] &= \int_{-\infty}^{\infty} x^2 p_X(x) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (x e^{-\frac{x^2}{2}}) dx \end{aligned}$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.4)$$

$$I = \int x e^{-\frac{x^2}{2}} dx \quad (2.5)$$

$$\text{Let } \frac{x^2}{2} = t \quad (2.6)$$

$$\Rightarrow x dx = dt \quad (2.7)$$

$$\Rightarrow \int e^{-t} dt = -e^{-t} + c \quad (2.8)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.9)$$

Using (2.9) in (2.4)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.10)$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.11)$$

$$\therefore \text{substituting limits we get, } E[x^2] = 1 \quad (2.12)$$

$$\text{Var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.13)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/3.1/V.py
```

Use the below command in the terminal to run the code:

```
python3 V.py
```

Now these samples are used to plot (3.1) by running the below code,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/3.1/V_cdf.
py
```

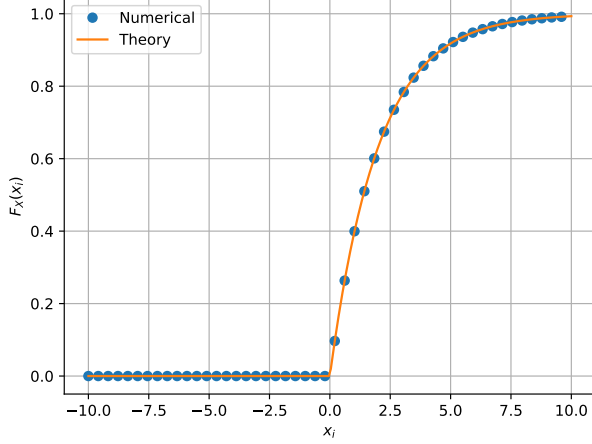


Fig. 3.1: CDF for (3)

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2\ln(1 - U) \leq x) \quad (3.3)$$

$$= P(1 - e^{-\frac{x}{2}} \geq U) \quad (3.4)$$

$$P(U < x) = \int_0^x dx = x \quad (3.5)$$

$$\therefore P(1 - e^{-\frac{x}{2}} \geq U) = 1 - e^{-\frac{x}{2}}, \forall x \geq 0 \quad (3.6)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Run the below code to generate T.dat

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/4.1/
T_gen_dat.c
```

Run the command below in the terminal

```
cc T_gen_dat.c -lm
./a.out
```

4.2 Find the CDF of T .

$$F_T(t) = P(T < t) \quad (4.2)$$

$$= P(U_1 + U_2 < t) \quad (4.3)$$

we know that $0 \leq U_1 \leq 1$ and $0 \leq U_2 \leq 1$

$\therefore 0 \leq U_1 + U_2 \leq 2$, so

$\forall t > 2, P(U_1 + U_2 < t) = 1$

$\forall t < 0, P(U_1 + U_2 < t) = 0$

for $0 \leq t \leq 2$ let us split it into 2 cases, for
 $0 \leq t \leq 1$ and $1 < t \leq 2$

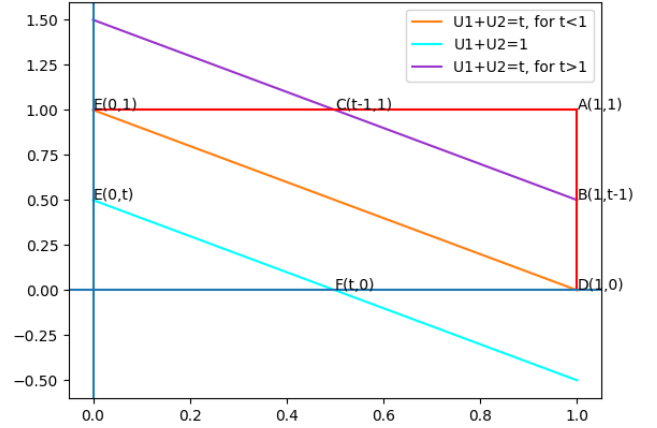


Fig. 4.2: Plot

The above figure is produced by the following code

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/4.2/T_plot.
py
```

Run the following command in the terminal to run the code

```
python3 T_plot.py
```

From Fig (4.2)

$$P(U_1 + U_2 < t, 0 \leq t \leq 1) = \frac{\Delta(EOF)}{\Delta(AEOD)} \quad (4.4)$$

$$= \frac{t^2}{2} \quad (4.5)$$

$$P(U_1 + U_2 < t, 1 \leq t \leq 2) = \frac{\Delta(ABC)}{\Delta(AEOD)} \quad (4.6)$$

$$= 1 - \frac{(2-t)^2}{2} \quad (4.7)$$

$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.8)$$

4.3 Find the PDF of T .

Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \quad (4.9)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.10)$$

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.11)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.12)$$

4.5 Verify your results through a plot.

Solution: Run the below code to get the cdf

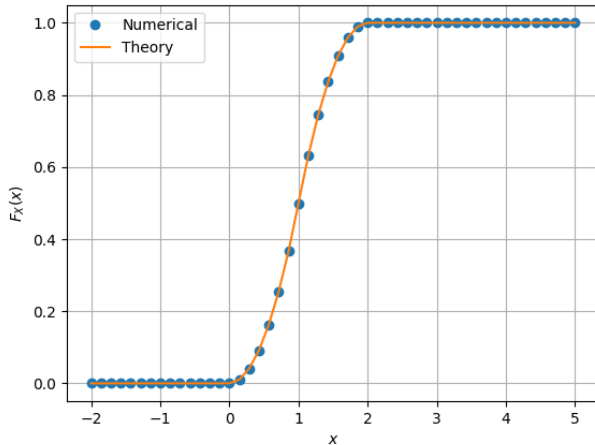


Fig. 4.5: CDF for (4)

https://github.com/Pranavb060504/Random_numbers/blob/main/4.5/T_cdf.py

Use the following command in the terminal to run the code

```
python3 T_cdf.py
```

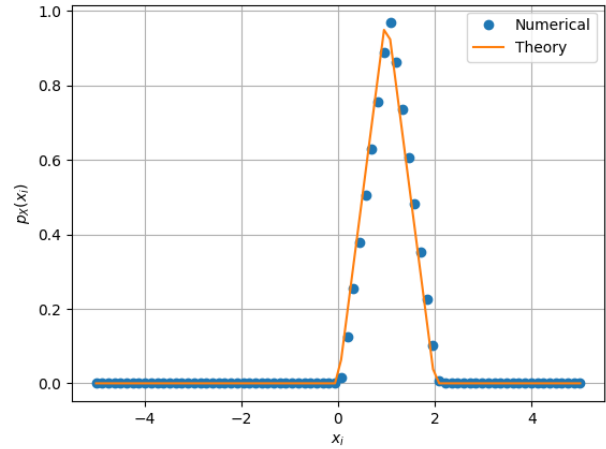


Fig. 4.5: PDF for (4)

Run the below code to get the pdf

https://github.com/Pranavb060504/Random_numbers/blob/main/4.5/T_pdf.py

Use the following command in the terminal to run the code

```
python3 T_pdf.py
```

5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable $X \in \{1, -1\}$.

Solution: Run the below code,

https://github.com/Pranavb060504/Random_numbers/blob/main/5.1/bernoulli.c

Use the below command in the terminal to run the code

```
cc bernoulli.c -lm
./a.out
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

Solution: Run the below code for generating samples of Y ,

https://github.com/Pranavb060504/Random_numbers/blob/main/5.2/Ygen.c

Use the below command in the terminal to run the code

```
cc Ygen.c -lm
./a.out
```

5.3 Plot Y using a scatter plot.

Solution: Run the following code to generate

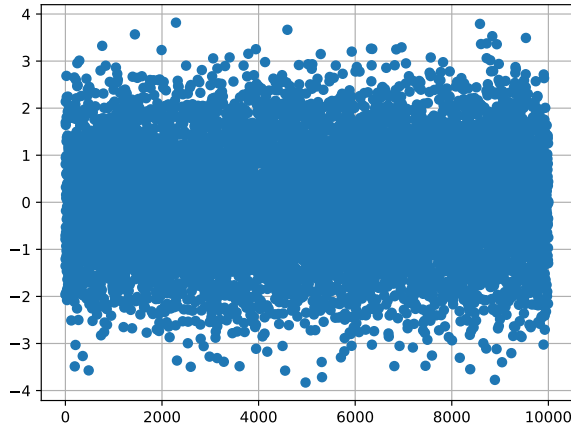


Fig. 5.3: plot for (5.3)

the scatter plot

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/5.3/
Yplot.py
```

Use the below command to run the code,

```
python3 Yplot.py
```

5.4 Guess how to estimate X from Y .

Solution: if the received signal is greater than 0, then the receiver assumes s_1 was transmitted.

if the received signal is less than or equal to 0, then the receiver assumes s_0 was transmitted, where s_0 and s_1 are cases of $X = 1$ and $X = -1$ respectively where threshold 0 is taken to be the decision boundary.

$$y > 0 \implies s_1 \quad (5.2)$$

$$y \leq 0 \implies s_0 \quad (5.3)$$

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.4)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.5)$$

Solution: Here s_1 and s_2 are equally probable ie, $p(s_1) = p(s_0) = \frac{1}{2}$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \quad (5.6)$$

$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{(y-1)^2}{2}} dy \quad (5.7)$$

$$= Q(1)$$

$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{(y+1)^2}{2}} dy \quad (5.8)$$

$$= Q(1) \quad (5.9)$$

5.6 Find P_e assuming that X has equiprobable symbols.

Solution: Total probability of bit error:

$$P_e = p(s_1)p(e|s_1) + p(s_0)p(e|s_0) \quad (5.10)$$

$$= \frac{1}{2}[Q(1) + Q(1)] \quad (5.11)$$

$$\therefore p(s_1) = p(s_0) = \frac{1}{2}, X \text{ has equiprobable symbols} \quad (5.12)$$

$$= Q(1)$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

Solution: Run the following code to generate the scatter plot

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/5.7/
Pplot.py
```

Use the below command to run the code,

```
python3 Pplot.py
```

5.8 Now, consider a threshold δ while estimating X from Y . Find the value of δ that minimizes the theoretical P_e .

Solution: Threshold= δ ,

6 GAUSSIAN TO OTHER

$$y > \delta \implies s_1 \quad (5.13)$$

$$y \leq \delta \implies s_0 \quad (5.14)$$

$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{(y-1)^2}{2}} dy \quad (5.15)$$

$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} e^{-\frac{(y+1)^2}{2}} dy$$

$$P_e = \frac{1}{2\sqrt{2\pi}} \left(\int_{-\infty}^{\delta} e^{-\frac{(y-1)^2}{2}} dy + \int_{\delta}^{\infty} e^{-\frac{(y+1)^2}{2}} dy \right)$$

$$P_e = \frac{Q(\delta+1) + Q(1-\delta)}{2} \quad (5.16)$$

$$P_e = f(\delta) \quad (5.17)$$

$$\text{to minimize } P_e, \frac{d(f(\delta))}{d\delta} = 0 \text{ and } f''(\delta) > 0 \quad (5.18)$$

$$e^{-\frac{(1-\delta)^2}{2}} - e^{-\frac{(1+\delta)^2}{2}} = 0 \quad (5.19)$$

$$\therefore 1 - \delta = 1 + \delta, \implies \delta = 0 \quad (5.20)$$

$$f''(\delta) = k((1-\delta)e^{-\frac{(1-\delta)^2}{2}} + (1+\delta)e^{-\frac{(1+\delta)^2}{2}}) > 0 \quad (5.21)$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.22)$$

Solution: $p_X(0) = p$

$$\implies p_X(1) = 1 - p$$

$$P_e = pP(e|s_0) + (1-p)P(e|s_1) \quad (5.23)$$

$$= pQ(1+\delta) + (1-p)Q(1-\delta) \quad (5.24)$$

$$\frac{d(P_e)}{d(\delta)} = 0 \quad (5.25)$$

$$\implies e^{\frac{(1+\delta)^2 - (1-\delta)^2}{2}} = \frac{p}{1-p} \quad (5.26)$$

$$\therefore \delta = \frac{1}{2} \log\left(\frac{p}{1-p}\right) \quad (5.27)$$

$$\frac{d(P_e)}{d(\delta)} \text{ at } \delta + \epsilon > 0 \quad (5.28)$$

$$\frac{d(P_e)}{d(\delta)} \text{ at } \delta - \epsilon < 0$$

$$\therefore \delta = \frac{1}{2} \log\left(\frac{p}{1-p}\right) \longrightarrow \text{minima}$$

5.10 Repeat the above exercise using the MAP criterion.

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: The sum of squares of k independent standard random normal variables is nothing but a χ^2 distribution with k degrees of freedom.

$$\chi^2(k) = \frac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} e^{-\frac{x}{2}}, \forall x \geq 0$$

Here $k=2$,

$$\therefore \chi^2(2) = P_V(v) = \frac{e^{-\frac{x}{2}}}{2} \quad (6.2)$$

$$\implies F_V(v) = \int_0^v \frac{e^{-\frac{x}{2}}}{2} dx \quad (6.3)$$

$$= 1 - e^{-\frac{x}{2}} \quad (6.4)$$

To generate data for V, run the following code,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/6.1/V_gen.
c
```

Run the below command in terminal,

```
cc V_gen.c -lm
./a.out
```

The PDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/6.1/chi_pdf.
py
```

Use the following command in the terminal to run the code

```
python3 chi_pdf.py
```

The CDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/6.1/chi_cdf.
py
```

Use the following command in the terminal to run the code

```
python3 chi_cdf.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.5)$$

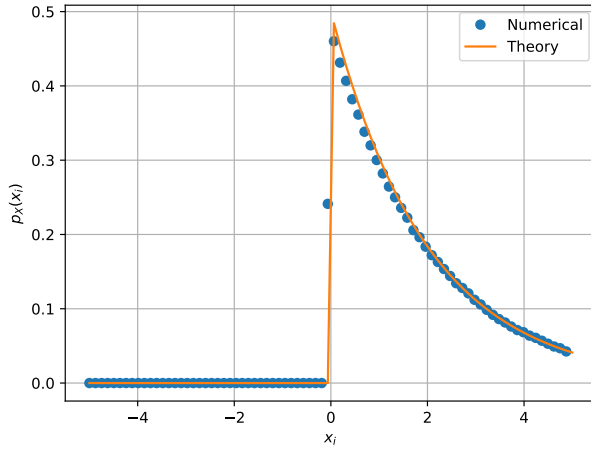


Fig. 6.1: PDF for (6.1)

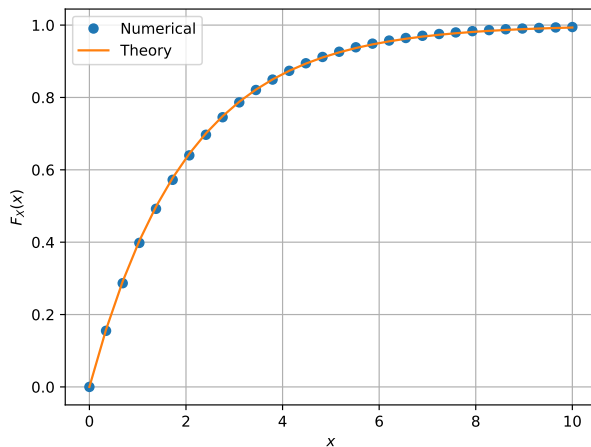


Fig. 6.1: CDF for (6.1)

find α .

Solution: From (6.4) $\alpha = 0.5$

6.3 Plot the CDF and Pdf of

$$A = \sqrt{V} \quad (6.6)$$

Solution:

$$F_A(a) = P(A < a) = P(V < a^2) \quad (6.7)$$

$$\text{from (6.4), } = \begin{cases} 1 - e^{-\frac{a^2}{2}} & a > 0 \\ 0 & a \leq 0 \end{cases} \quad (6.8)$$

$$\Rightarrow P_A(a) = \frac{d(F_A(a))}{da} \quad (6.9)$$

$$= \begin{cases} ae^{-\frac{a^2}{2}} & a > 0 \\ 0 & a \leq 0 \end{cases} \quad (6.10)$$

To generate data for A , run the following code,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/6.3/A_gen.
c
```

Run the below command in terminal,

```
cc A_gen.c -lm
./a.out
```

The PDF plot of A can be obtained by running the code below,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/6.3/A_pdf.
py
```

Use the following command in the terminal to run the code

```
python3 A_pdf.py
```

The CDF plot of the A can be obtained by running the code below,

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/6.3/A_cdf.
py
```

Use the following command in the terminal to run the code

```
python3 A_cdf.py
```

7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

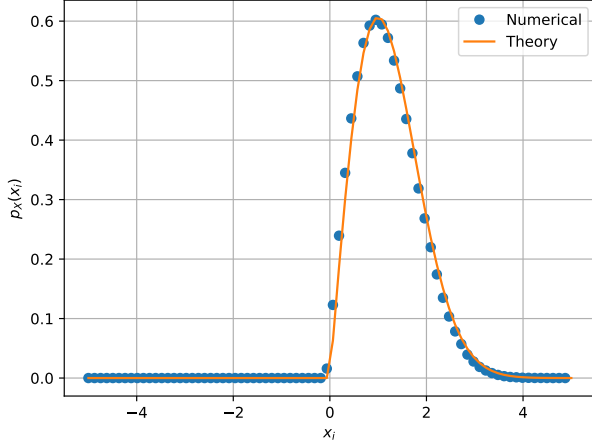


Fig. 6.3: PDF for (6.3)

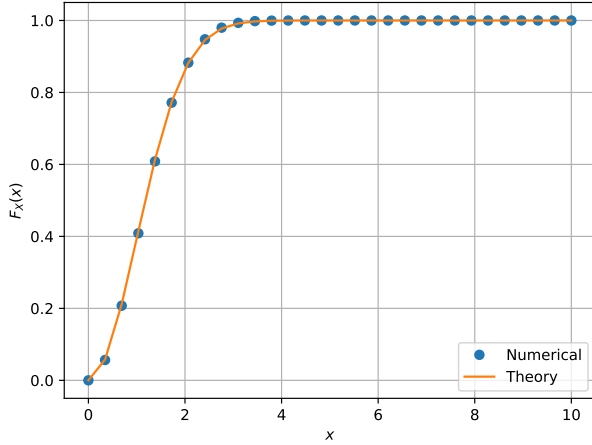


Fig. 6.3: CDF for (6.3)

where A is Raleigh with $E[A^2] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x) dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ . Comment.

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1)$$