Random Numbers

Pranav B

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
cc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.2/uni_cdf.
    py
```

Run the following command in the terminal to run the code.

```
python3 uni_cdf.py
```

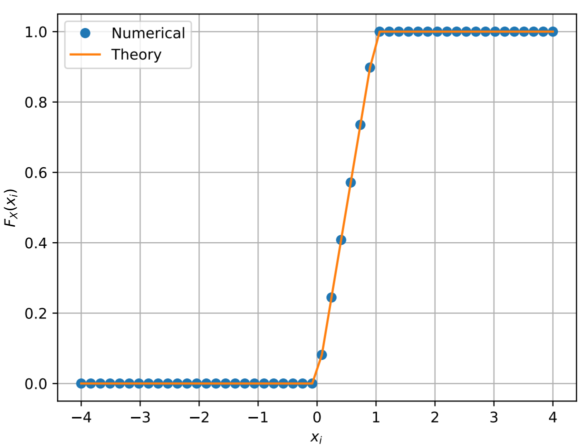

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$
CDF of $P_U(x)$=$F_U(x)$

$$= \int P_U(x)dx \tag{1.2}$$

$$= \int k dx \tag{1.3}$$

$$\text{wkt} \int_0^1 k dx = 1 \tag{1.4}$$

$$\therefore k = 1 \tag{1.5}$$

$$\therefore F_U(x) = x \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.4/
    mean_var.c
```

Use below command to run file,

```
cc mean_var.c -lm
./a.out
```

running the code gives us Mean =0.500137 , Variance =0.083251

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

$$dF_U(x) = dx \tag{1.10}$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.11}$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{1.12}$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.13}$$

$$\because P_X(x) = 0, \forall x \in (1,\infty) \cap (-\infty, 0) \tag{1.14}$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.15}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/1.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc exrand.c -lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in 2.2,Properties of the CDF:

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

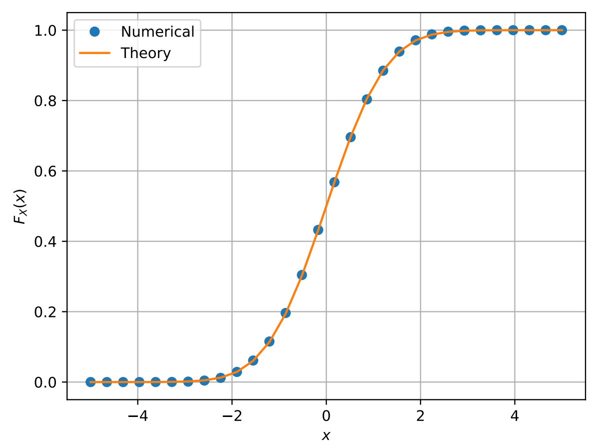

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in 2.3 using the code below

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/2.3/pdf.py
```

Use the below command to run the code:

```
python3 pdf.py
```

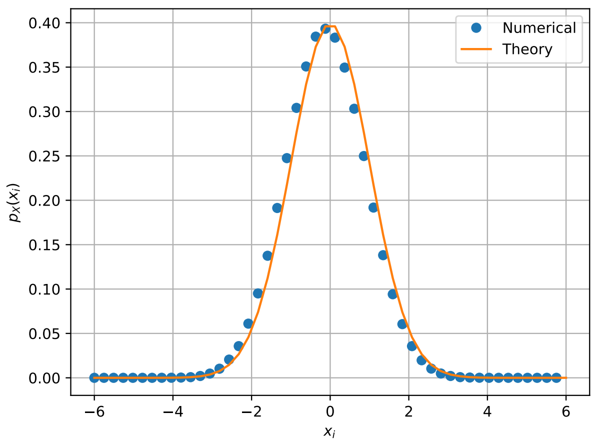

Fig. 2.3: The PDF of X

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives Mean = -0.000417 Variance= 0.999902

```
wget https://github.com/Pranavb060504/
    Random_numbers/blob/main/2.4/
    mean_var(gau).c
```

Command used:

```
cc mean_var(gau).c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Given ,$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.4}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}} \tag{2.5}$$

$$\because x e^{-\frac{-x^2}{2}} \text{is a odd function,} \tag{2.6}$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(x e^{-\frac{-x^2}{2}}) dx \tag{2.8}$$

Using integration by parts:

$$= x \int x e^{-\frac{-x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{-x^2}{2}} dx \tag{2.9}$$

$$I = \int x e^{-\frac{-x^2}{2}} \tag{2.10}$$

$$\text{Let} \frac{x^2}{2} = t \tag{2.11}$$

$$\implies x dx = dt \tag{2.12}$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \tag{2.13}$$

$$\therefore \int x e^{-\frac{-x^2}{2}} = -e^{-\frac{-x^2}{2}} + c \tag{2.14}$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{-x^2}{2}} + \int e^{-\frac{-x^2}{2}} dx \tag{2.15}$$

$$\text{Also,} \int_{-\infty}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi} \tag{2.16}$$

$$\therefore \text{substituting limits we get,} E[x^2] = 1 \tag{2.17}$$

$$Var(X) = E[x^2] - (E[x])^2 = 1 - 0 \tag{2.18}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{3.1}$$

and plot its CDF.

Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/3.1/V.py
```

Use the below command in the terminal to run the code:

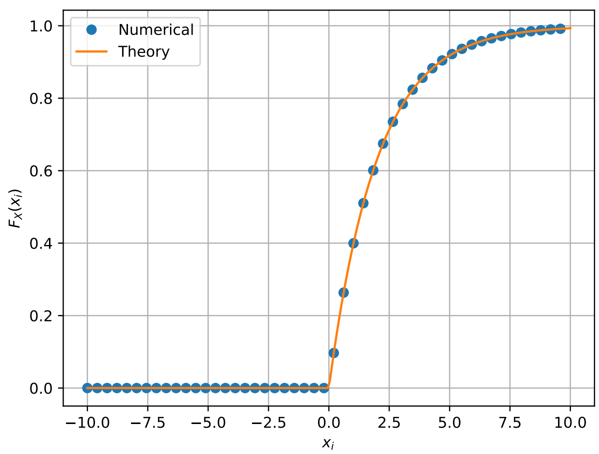

Fig. 3.1: CDF for (3)

```
python3 V.py
```

Now these samples are used to plot (3.1) by running the below code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/3.1/V_cdf.
    py
```

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \tag{3.2}$$
$$= P(-2ln(1-U) \leq x) \tag{3.3}$$
$$= P(1 - e^{\frac{-x}{2}} \geq U) \tag{3.4}$$
$$P(U < x) = \int_0^x dx = x \tag{3.5}$$
$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \tag{3.6}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Run the below code to generate T.dat

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.1/
    T_gen_dat.c
```

Run the command below in the terminal

```
cc T_gen_dat.c −lm
./a.out
```

4.2 Find the CDF of T.

$$F_T(t) = P(T < t) \tag{4.2}$$
$$= P(U_1 + U_2 < t) \tag{4.3}$$

we know that $0 \leq U_1 \leq 1$ and $0 \leq U_2 \leq 1$
$\therefore 0 \leq U_1 + U_2 \leq 2$, so
$\forall t > 2, P(U_1 + U_2 < t) = 1$
$\forall t < 0, P(U_1 + U_2 < t) = 0$
for $0 \leq t \leq 2$ let us split it into 2 cases, for $0 \leq t \leq 1$ and $1 < t \leq 2$

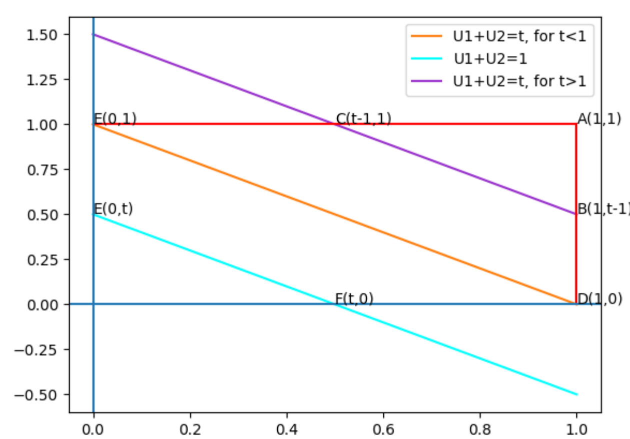

Fig. 4.2: Plot

The above figure is produced by the following code

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.2/T_plot.
    py
```

Run the following command in the terminal to run the code

```
python3 T_plot.py
```

From Fig (4.2)

$$P(U_1 + U_2 < t, 0 \le t \le 1) = \frac{\Delta(EOF)}{\Delta(AEOD)} \quad (4.4)$$

$$= \frac{t^2}{2} \quad (4.5)$$

$$P(U_1 + U_2 < t, 1 \le t \le 2) = \frac{\Delta(ABC)}{\Delta(AEOD)} \quad (4.6)$$

$$= 1 - \frac{(2-t)^2}{2} \quad (4.7)$$

$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \\ 1 - \frac{(2-t)^2}{2} & 1 < t \\ 1 & t > 2 \end{cases} \quad (4.8)$$

4.3 Find the PDF of T.

Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \quad (4.9)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \quad (4.10)$$

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \quad (4.11)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases} \quad (4.12)$$

4.5 Verify your results through a plot.

Solution: Run the below code to get the cdf

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.5/T_cdf.
    py
```

Use the following command in the terminal to run the code

```
python3 T_cdf.py
```

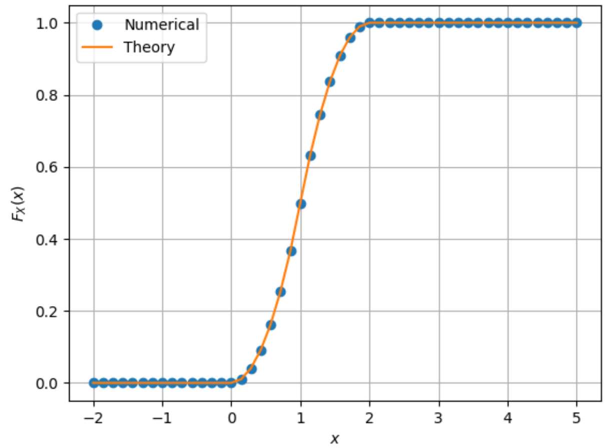

Fig. 4.5: CDF for (4)

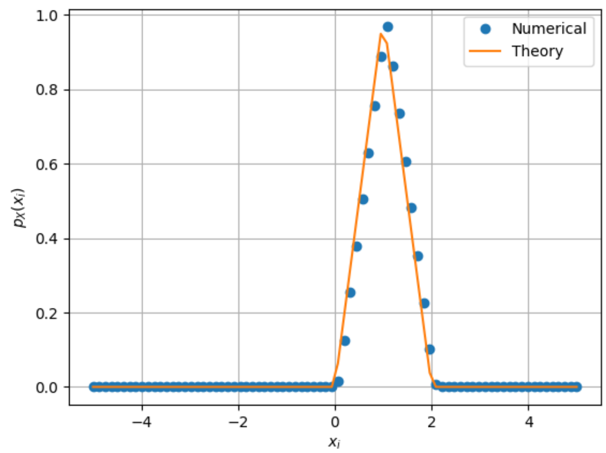

Fig. 4.5: PDF for (4)

Run the below code to get the pdf

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/4.5/T_pdf.
    py
```

Use the following command in the terminal to run the code

```
python3 T_pdf.py
```

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.

Solution: Run the below code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/5.1/
    bernoulli.c
```

Use the below command in the terminal to run the code

```
cc bernoulli.c −lm
./a.out
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Run the below code for generating samples of Y,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/5.2/Ygen.c
```

Use the below command in the terminal to run the code

```
cc Ygen.c −lm
./a.out
```

5.3 Plot Y using a scatter plot.
Solution:

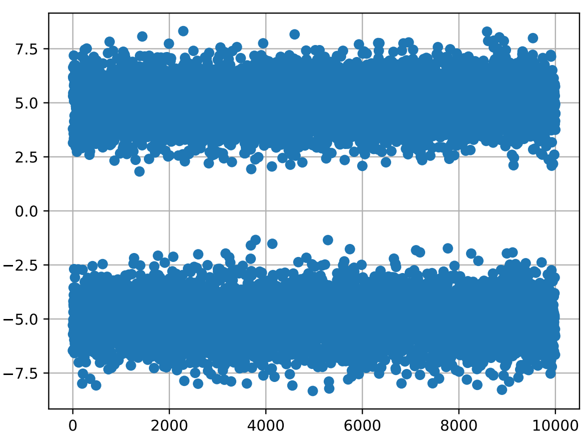

Fig. 5.3: plot for (5.3)

Run the following code to generate the scatter plot

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/5.3/
    Yplot.py
```

Use the below command to run the code,

```
python3 Yplot.py
```

5.4 Guess how to estimate X from Y.
Solution: if the received signal is greater than 0, then the receiver assumes s_1 was transmitted. if the received signal is less than or equal to 0, then the receiver assumes s_0 was transmitted, where s_0 and s_1 are cases of $X = 1$and $X = -1$ respectively where threshold 0 is taken to be the decision boundary.

$$y > 0 \implies s_1 \tag{5.2}$$
$$y \le 0 \implies s_0 \tag{5.3}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.4}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.5}$$

Solution: Here s_1 and s_2 are equally probable ie, $p(s_1) = p(s_0) = \frac{1}{2}$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{\frac{-x^2}{2}} dx \tag{5.6}$$
$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{(y-A)^2}{2}} dy$$
$$= Q(A) \tag{5.7}$$
$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{(y+A)^2}{2}} dy$$
$$= Q(A) \tag{5.8}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Total probability of bit error:

$$P_e = p(s_1)p(e|s_1) + p(s_0)p(e|s_0) \tag{5.9}$$
$$= \frac{1}{2}[Q(A) + Q(A)] \tag{5.10}$$
$$\because p(s_1) = p(s_0) = \frac{1}{2}, \text{X has equiprobable symbols} \tag{5.11}$$
$$= Q(A)$$

= Q(

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:
Run the following code to generate the scatter plot

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/5.7/
    Pplot.py
```

Use the below command to run the code,

```
python3 Pplot.py
```

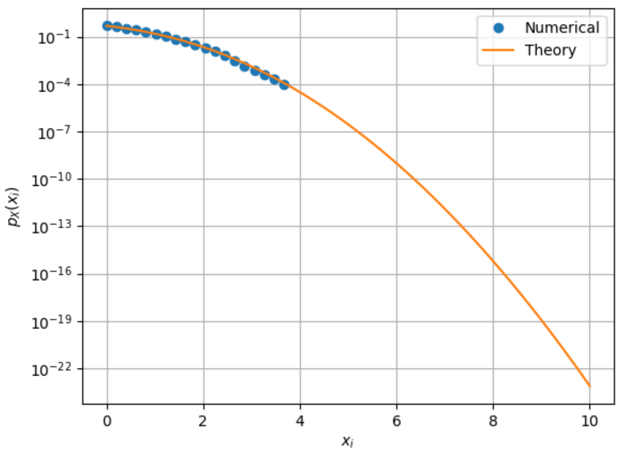

Fig. 5.7: semilogyplot for (5.7)

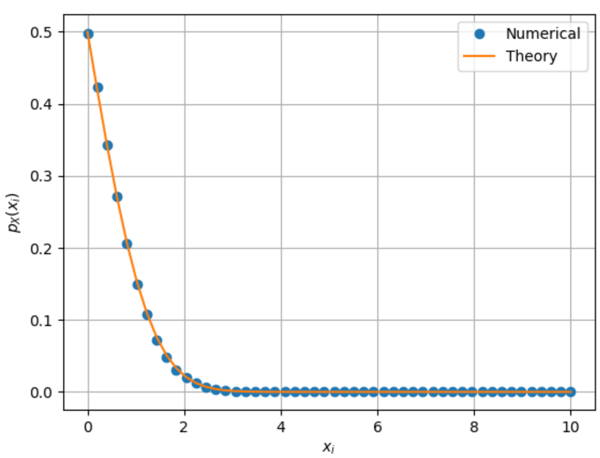

Fig. 5.7: plot for (5.7)

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e.
Solution: Threshold=δ,

$$y > \delta \implies s_1 \tag{5.12}$$
$$y \leq \delta \implies s_0 \tag{5.13}$$
$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy \tag{5.14}$$
$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy$$
$$P_e = \frac{1}{2\sqrt{2\pi}} \left(\int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy + \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy \right)$$
$$P_e = \frac{Q(\delta + A) + Q(A - \delta)}{2} \tag{5.15}$$
$$P_e = f(\delta) \tag{5.16}$$
$$\text{to minimize} P_e, \frac{d(f(\delta))}{d\delta} = 0 \text{ and} f''(\delta) > 0 \tag{5.17}$$
$$e^{\frac{-(A-\delta)^2}{2}} - e^{\frac{-(A+\delta)^2}{2}} = 0 \tag{5.18}$$
$$\therefore A - \delta = A + \delta, \implies \delta = 0 \tag{5.19}$$
$$f''(\delta) = k((A-\delta)e^{\frac{-(A-\delta)^2}{2}} + (A+\delta)e^{\frac{-(A+\delta)^2}{2}}) > 0 \tag{5.20}$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.21}$$

Solution: $p_X(0) = p$
$\implies p_X(1) = 1 - p$

$$P_e = pP(e|s_0) + (1-p)P(e|s_1) \tag{5.22}$$
$$= pQ(A+\delta) + (1-p)Q(A-\delta) \tag{5.23}$$
$$\frac{d(P_e)}{d(\delta)} = 0 \tag{5.24}$$
$$\implies e^{\frac{(A+\delta)^2-(A-\delta)^2}{2}} = \frac{p}{1-p} \tag{5.25}$$
$$\therefore \delta = \frac{1}{2A} log(\frac{p}{1-p}) \tag{5.26}$$
$$\frac{d(P_e)}{d(\delta)} \quad at \quad \delta + \epsilon > 0 \tag{5.27}$$
$$\frac{d(P_e)}{d(\delta)} \quad at \quad \delta - \epsilon < 0$$
$$\therefore \delta = \frac{1}{2A} log\left(\frac{p}{1-p}\right) \longrightarrow minima$$
$$A = 5 \implies \delta = \frac{1}{10} log\left(\frac{p}{1-p}\right) \tag{5.28}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution:

$$P_{X|Y}(x|y)\Big|_{X=1} = \frac{P(Y=y|X=1)P(X=1)}{P(Y=y)} \tag{5.29}$$

$$\begin{aligned} P(Y=y) = P(Y=y|X=1)P(X=1) \\ +P(Y=y|X=-1)P(X=-1) \end{aligned} \tag{5.30}$$

$$P(Y=y|X=1)P(X=1) = pP(Y=A+N) \tag{5.31}$$

$$= p\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y-A)^2}{2}}\right) \tag{5.32}$$

$$\therefore P_{X|Y}(x|y)\Big|_{X=1} = \frac{p\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y-A)^2}{2}}\right)}{P(Y=y)} \tag{5.33}$$

$$P_{X|Y}(x|y)\Big|_{X=-1} = \frac{P(Y=y|X=-1)P(X=-1)}{P(Y=y)} \tag{5.34}$$

$$P(Y=y|X=-1)P(X=-1) = (1-p)P(Y=-A+N)$$

$$= (1-p)\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y+A)^2}{2}}\right) \tag{5.35}$$

$$\therefore P_{X|Y}(x|y)\Big|_{X=-1} = \frac{(1-p)\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y+A)^2}{2}}\right)}{P(Y=y)} \tag{5.36}$$

Now comparing $a = P_{X|Y}(x|y)\Big|_{X=-1}$ and $b = P_{X|Y}(x|y)\Big|_{X=1}$, if $a > b, X = -1$ is more likely ,$a < b, X = 1$ is more likely.

$pe^{\frac{-(y-A)^2}{2}} \gtrless (1-p)e^{\frac{-(y+A)^2}{2}}$

$\implies e^{2Ay} \gtrless \frac{1-p}{p}$

$\implies y \gtrless \frac{1}{2A}log\left(\frac{1-p}{p}\right)$

$\delta = \frac{1}{2A}log\left(\frac{1-p}{p}\right)$

$y > \delta \implies$ X=1 is more likely

$y < \delta \implies$ X=-1 is more likely

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution:

Method1:The sum of squares of k independent standard random normal variables is nothing but a χ^2 distribution with k degrees of freedom.

$\chi^2(k) = \frac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}e^{\frac{-x}{2}}, \forall x \geq 0$

Here k=2,

$$\therefore \chi^2(2) = P_V(v) = \frac{e^{\frac{-x}{2}}}{2} \tag{6.2}$$

$$\implies F_V(v) = \int_0^v \frac{e^{\frac{-x}{2}}}{2}dx \tag{6.3}$$

$$= 1 - e^{\frac{-x}{2}} \tag{6.4}$$

Method2:

$$X_1 = R\cos\theta \tag{6.5}$$

$$X_2 = R\sin\Theta \tag{6.6}$$

$R \in [0,\infty), \Theta \in [0,2\pi)$. Jacobian Matrix is given by

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.7}$$

$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.8}$$

$$\implies |J| = R \tag{6.9}$$

We also know that

$$|J|p_{X_1,X_2}(x_1,x_2) = p_{R,\Theta}(r,\theta) \tag{6.10}$$

$$\implies p_{R,\Theta}(r,\theta) = Rp_{X_1}(x_1)p_{X_2}(x_2) \tag{6.11}$$

$$= \frac{R}{2\pi}e^{\left(-\frac{X_1^2+X_2^2}{2}\right)} \tag{6.12}$$

$$= \frac{R}{2\pi}e^{\left(-\frac{R^2}{2}\right)} \tag{6.13}$$

X_1, X_2 are iid random variables. Thus,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)d\theta \tag{6.14}$$

$$= Re^{\left(-\frac{R^2}{2}\right)} \tag{6.15}$$

However, $V = X_1^2 + X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$ for $x \geq 0$.

$$F_V(x) = F_R(\sqrt{x}) \tag{6.16}$$

$$= \int_0^{\sqrt{x}} re^{\left(-\frac{r^2}{2}\right)}dr \tag{6.17}$$

$$= \int_0^{\frac{x}{2}} e^{-t}dt = 1 - e^{-\frac{x}{2}} \tag{6.18}$$

To generate data for V , run the following code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.1/G_gen.
    c
```

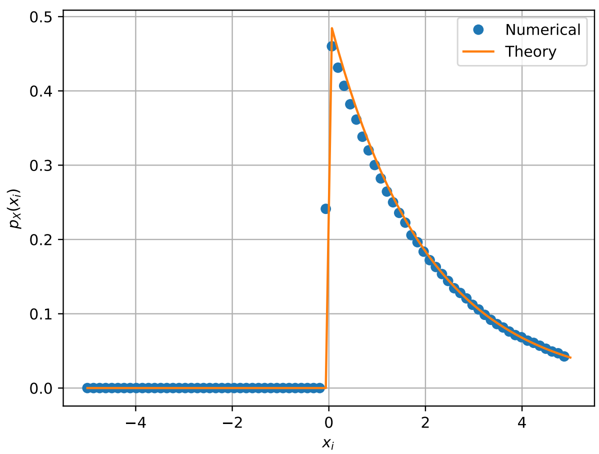

Fig. 6.1: PDF for (6.1)

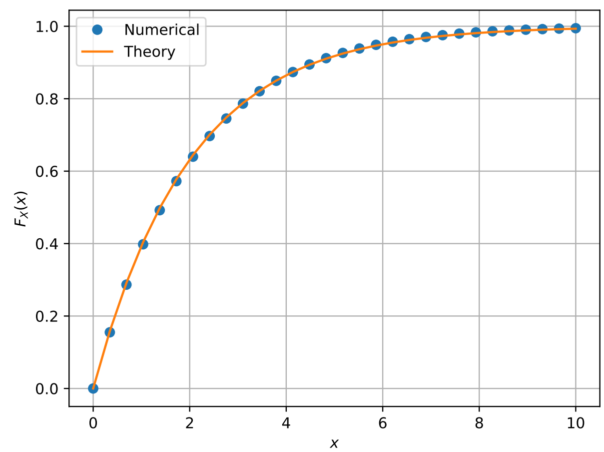

Fig. 6.1: PDF for (6.1)

Run the below command in terminal,

```
cc G_gen.c −lm
./a.out
```

The PDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.1/chi_pdf.
    py
```

Use the following command in the terminal to run the code

```
python3 chi_pdf.py
```

The CDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.1/chi_cdf.
    py
```

Use the following command in the terminal to run the code

```
python3 chi_cdf.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.19}$$

find α.
Solution: From (6.4) $\alpha = 0.5$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.20}$$

Solution:

$$F_A(a) = P(A < a) = P(V < a^2) \tag{6.21}$$

$$\text{from(6.4)}, = \begin{cases} 1 - e^{\frac{-a^2}{2}} & a > 0 \\ 0 & a <= 0 \end{cases} \tag{6.22}$$

$$\implies P_A(a) = \frac{d(F_A(a))}{da} \tag{6.23}$$

$$= \begin{cases} ae^{\frac{-a^2}{2}} & a > 0 \\ 0 & a <= 0 \end{cases} \tag{6.24}$$

To generate data for A , run the following code,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.3/Agen.c
```

Run the below command in terminal,

```
cc Agen.c −lm
./a.out
```

The PDF plot of A can be obtained by running the code below,

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/6.3/A_pdf.
    py
```

Use the following command in the terminal to run the code

```
python3 A_pdf.py
```

The CDF plot of the A can be obtained by running the code below,

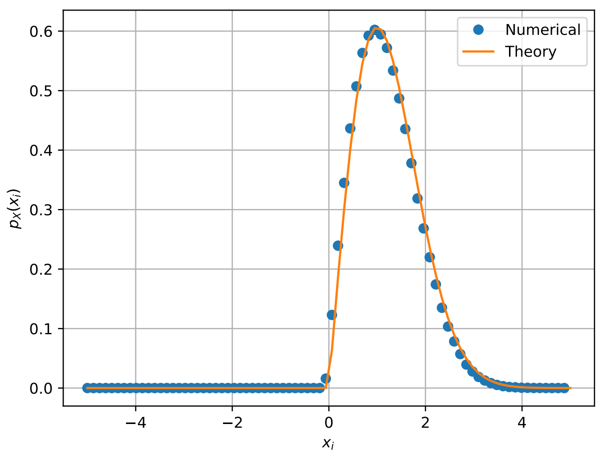

Fig. 6.3: PDF for (6.3)

https://github.com/Pranavb060504/Random_numbers/blob/main/6.3/A_cdf.py

Use the following command in the terminal to run the code

```
python3 A_cdf.py
```

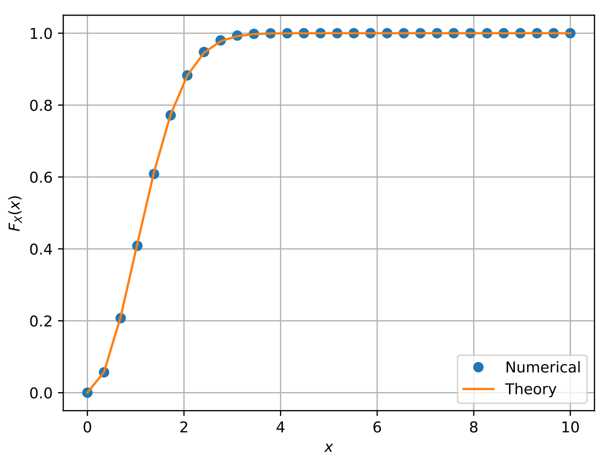

Fig. 6.3: CDF for (6.3)

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.
Solution: Run the following code to generate Y

https://github.com/Pranavb060504/Random_numbers/blob/main/7.1/Ray.c

Use the following command in the terminal to run the code

```
cc Ray.c -lm
```

After generating Y, run the below code to plot points

https://github.com/Pranavb060504/Random_numbers/blob/main/7.1/Raypts.c

Use the following command in the terminal to run the code

```
python3 Raypts.py
```

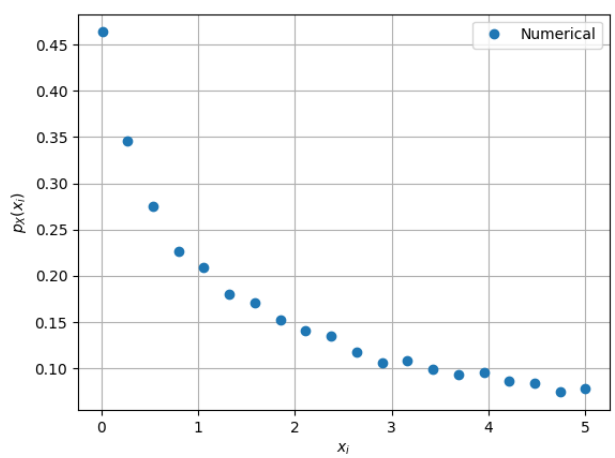

Fig. 7.1: plot for 7.1

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution: We rewrite the previous expression for P_e as

$$P_e(N) = \Pr(A + N < 0) \tag{7.3}$$
$$= \Pr(A < -N) = F_A(-N)$$
$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}} & N \leq 0 \\ 0 & N > 0 \end{cases} \tag{7.4}$$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.5}$$

Find $P_e = E\left[P_e(N)\right]$.
Solution:

$$P_e = \int_0^\infty F_A(x)p_X(x)dx \tag{7.6}$$
$$= \int_0^\infty (1 - e^{-\frac{x^2}{\gamma}})\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \tag{7.7}$$
$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{x^2}{\frac{2\gamma}{\gamma+2}}\right)dx \tag{7.8}$$
$$= \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{\gamma+2}}\right) \tag{7.9}$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.
Solution: Run the below code to plot pdf

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/7.3/Rayplot.
    py
```

Use the following command in the terminal to run the code

```
python3 Rayplot.py
```

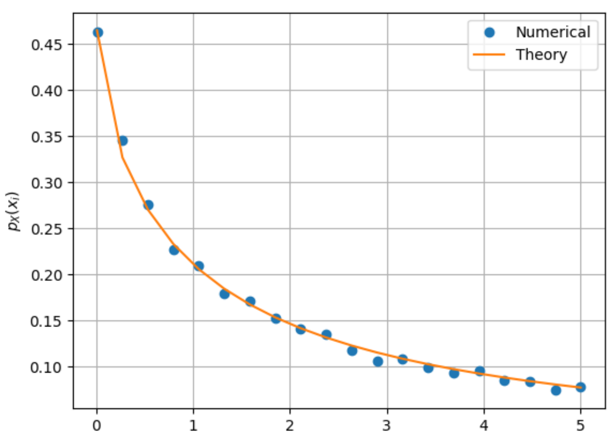

Fig. 7.4: PDF for (7.3)

We observe $P_{e|0} = E[P_e(N)]$.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.2}$$
$$\mathbf{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.
Solution: Run the following code to generate data.

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/8.1/2d.c
```

Use the following command in the terminal to run the code

```
cc 2d.c -lm
```

After generating data run the following python file to get the scatter plot

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/8.1/2d.py
```

Use the following command in the terminal to run the code

```
python3 2d.py
```

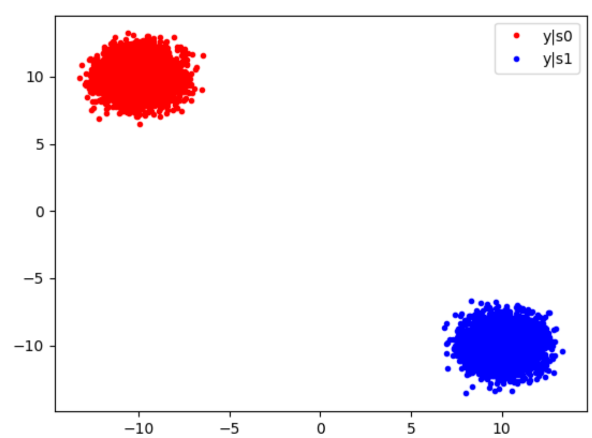

Fig. 8.1: scatter plot for (8.1)

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

Solution:

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{s}_0 & i > j \\ \mathbf{s}_1 & i < j \end{cases} \tag{8.5}$$

where $\mathbf{y} = \begin{pmatrix} i \\ j \end{pmatrix}$

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.6}$$

with respect to the SNR from 0 to 10 dB.
Run the following code to generate data.

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/8.3/2dSNR.
    c
```

Use the following command in the terminal to run the code

```
cc 2dSNR.c −lm
```

After generating data run the following python file to get the scatter plot

```
https://github.com/Pranavb060504/
    Random_numbers/blob/main/8.3/2dSNR.
    py
```

Use the following command in the terminal to run the code

```
python3 2dSNR.py
```

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution: We have,

$$\begin{aligned} P_e &= P(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) && (8.7) \\ &= P(i < j | \mathbf{x} = \mathbf{s}_0) && (8.8) \\ &= P(A + n_1 < -A + n_2) && (8.9) \\ &= P(n_2 - n_1 > 2A) && (8.10) \\ &= P(N > 2A) = Q\left(\sqrt{2}A\right) && (8.11) \end{aligned}$$

where $N \triangleq n_2 - n_1 \sim \mathcal{N}(0, 2)$ and SNR $= \frac{E[A^2]}{\sigma_N^2}$.