Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/2/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Run the below python code

```
https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/2/
    Cancel_noise.py
```

Use the following command in the terminal to run the code

```
python3 Cancel_noise.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: First run the following c code to generate data

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/3/xy.c
```

Use the following command in terminal to run code

```
cc -lm xy.c
```

Then run the below python code which yields Fig. 3.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/3/xnyn.py
```

Use the following command in terminal to run code

```
python3 xnyn.py
```

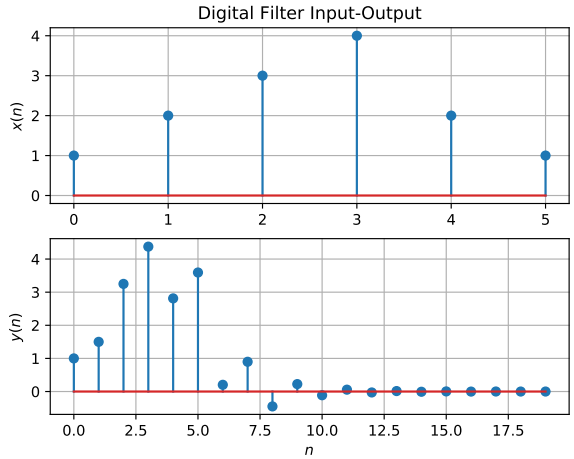

Fig. 3.2

4 Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

$$\mathcal{Z}\{x(n)\} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

4.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.7}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.8}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.9}$$

4.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.12}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.13}$$

and from (4.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.14}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.15}$$

using the formula for the sum of an infinite geometric progression.

4.4 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.16}$$

Solution:

$$a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{4.17}$$

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.18}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.19}$$

Using formula of sum of infinite geometric progression with common ratio<1

$$= \frac{1}{1 - az^{-1}} \left(\because \left| \frac{a}{z} \right| < 1 \right) \tag{4.20}$$

4.5 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.21}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $x(n)$.

Solution: $|H(e^{i\omega})|$ has a fundamental period of 2π. The following code plots Fig. 4.5.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/4/dtft.py
```

Use the following command in the terminal to run the code

```
python3 dtft.py
```

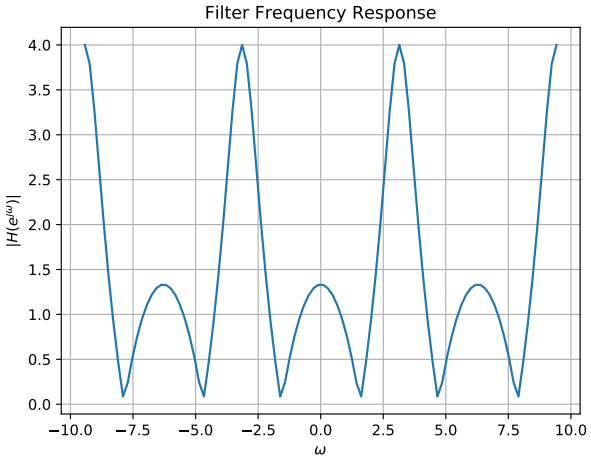

Fig. 4.5: $\left|H\left(e^{\jmath\omega}\right)\right|$

5 Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{z}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.9),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.3}$$

using (4.16) and (4.6).

5.2 Sketch $h(n)$. Is it bounded? Convergent?

Solution: The following code plots Fig. 5.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/5/hn.py
```

Use the following command in the terminal to run the code

```
python3 hn.py
```

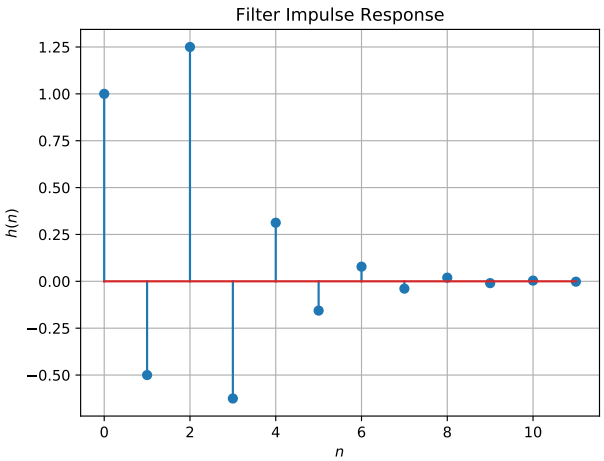

Fig. 5.2: $h(n)$ as the inverse of $H(z)$

5.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.4}$$

Is the system defined by (3.2) stable for the impulse response in (5.1)?

Solution:

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.5}$$

$$u(n-2) = \begin{cases} 1 & n \geq 2 \\ 0 & n < 2 \end{cases} \tag{5.6}$$

$$\therefore h(n) = \begin{cases} 0 & n < 0 \\ \left(\frac{-1}{2}\right)^n & 0 \leq n < 2 \\ \left(\frac{-1}{2}\right)^n + \left(\frac{-1}{2}\right)^{(n-2)} & n \geq 2 \end{cases} \tag{5.7}$$

$$\therefore \sum_{n=-\infty}^{\infty} h(n) = 0 + 1 + \frac{-1}{2} + \sum_{n=2}^{\infty} \left[\left(\frac{-1}{2}\right)^n + \left(\frac{-1}{2}\right)^{(n-} \right] \tag{5.8}$$

$$= \frac{1}{2} + \frac{5}{4}\left(\frac{2}{3}\right) = \frac{4}{3} < \infty \tag{5.9}$$

$$\tag{5.10}$$

$\therefore$ system defined is stable

5.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.11}$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.4. Note that this is the same as Fig. 5.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/5/hndef.py
```

Use the following command in the terminal to run the code

```
python3 hndef.py
```

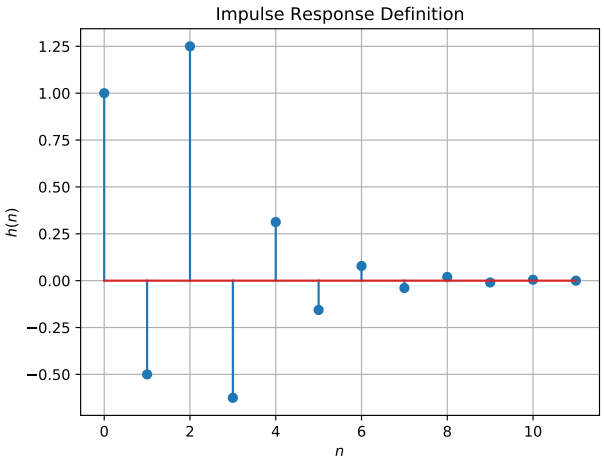

Fig. 5.4: $h(n)$ from the definition

5.5 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.12}$$

Comment. The operation in (5.12) is known as *convolution*.

Solution: $X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$

for some N,

$H = \begin{bmatrix} h(0) & h(1) & h(2) & \cdots & h(N-1) \end{bmatrix}_{(1\times N)}$

Toeplitz matix for

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 3 & 4 & 2 & 1 \end{bmatrix}$$

$\therefore Y^T = X * H$

The following code plots Fig. 5.5. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/ynconv.py
```

Use the following command in the terminal to run the code

```
python3 ynconv.py
```

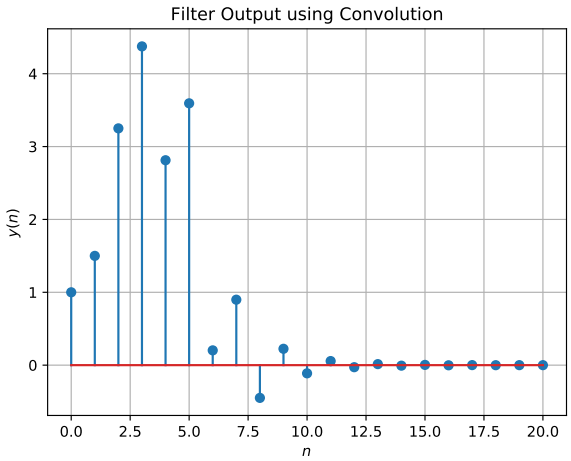

Fig. 5.5: $y(n)$ from the definition of convolution

5.6 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.13}$$

Solution: wkt

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.14}$$

Replacing k with $n-k$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \therefore y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.15}$$

6 DFT AND FFT

6.1 Compute

$$X(k) \stackrel{\triangle}{=} \sum_{n=0}^{N-1} x(n)e^{-\mathrm{j}2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: Run the following code to compute $X(k)$,$H(k)$.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/6_1.py
```

Use the following command in the terminal to run the code

```
python3 6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Run the following code to compute $Y(k)$.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/6_2.py
```

Use the following command in the terminal to run the code

```
python3 6_2.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\mathrm{j}2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.5. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/yndft.py
```

Use the following command in the terminal to run the code

```
python3 yndft.py
```

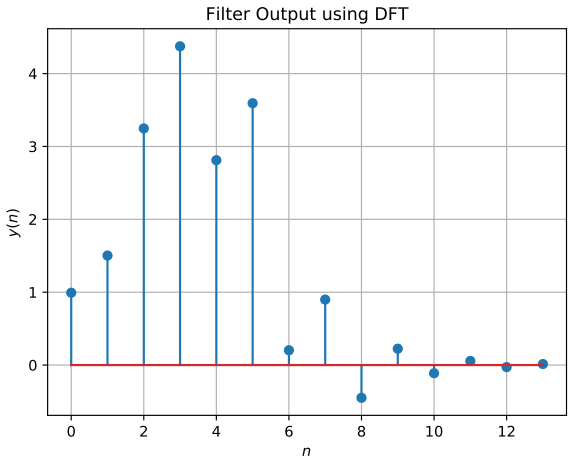

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k)$, $H(k)$ and $y(n)$ through FFT and IFFT.

Solution: Run the following code to generate the plot below

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/xk.py
```

Use the following command in the terminal to run the code

```
python3 xk.py
```

6.5 Wherever possible, express all the above equations as matrix equations.

Solution: Let $X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{(N-1\times 1)}$ and

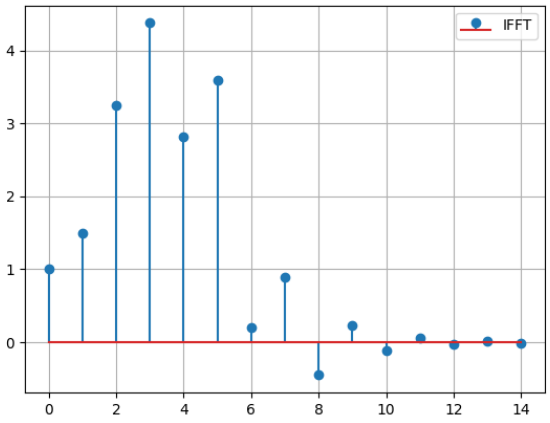

Fig. 6.4: IFFT

$$J = \begin{bmatrix} 0 \\ e^{-2\pi j(1)k/N} \\ e^{-2\pi j(2)k/N} \\ e^{-2\pi j(3)k/N} \\ \vdots \\ e^{-2\pi j(N-1)k/N} \end{bmatrix}_{(N-1\times 1)}$$

$$\therefore X(k) = X^T J$$

$$\text{From (5.7) } H = \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ \vdots \\ h(N-1) \end{bmatrix}_{(N-1\times 1)}$$

$$\therefore H(k) = H^T J$$

$$Y = \begin{bmatrix} H(0)X(0) \\ H(1)X(1) \\ \vdots \\ H(N-1)X(N-1) \end{bmatrix}_{(N-1\times 1)}$$

$$\therefore y(n) = Y^T J$$

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (7.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/7/7_1.py
```

Use the following command in the terminal to run the code

```
python3 7_1.py
```

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.