Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /f.py
```

Use the following command in the terminal to run the code

```
python3 f.py
```

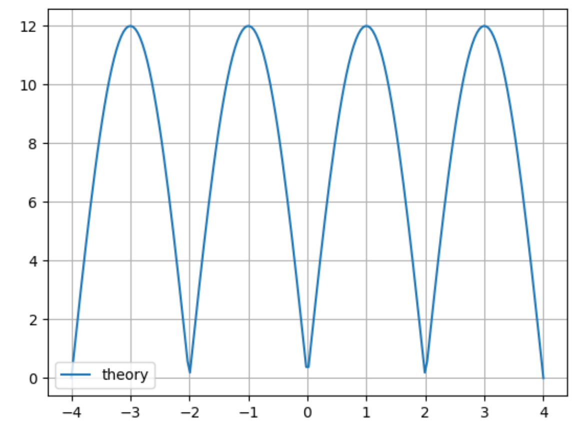

Fig. 1.1

1.2 Show that $x(t)$ is periodic and find its period.
Solution: :

$$x(t + \frac{1}{2f_0}) = A_0 \left|\sin\left(2\pi f_0 (t + \frac{1}{2f_0})\right)\right| \tag{1.2}$$

$$= A_0 \left|\sin\left(2\pi f_0 t + \pi\right)\right| = A_0 \left|\sin\left(2\pi f_0 t\right)\right| = x(t) \tag{1.3}$$

$\therefore$ x(t) is periodic with period $\frac{1}{2f_0}$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: We have for some $n \in \mathbb{Z}$,

$$x(t) e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi (k-n) f_0 t} \tag{2.3}$$

But we know from the periodicity of $e^{\jmath 2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi k f_0 t}\, dt = \frac{1}{f_0}\delta\left(k\right) \tag{2.4}$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt = \frac{c_n}{f_0} \tag{2.5}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.6}$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.7}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos\left(2\pi n f_0 t\right)\, dt + \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \sin\left(2\pi n f_0 t\right)\, dt \tag{2.8}$$

$$= 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \cos\left(2\pi n f_0 t\right)\, dt \tag{2.9}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n+1\right) f_0 t\right)\right)\, dt - f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n-1\right) f_0 t\right)\right)\, dt \tag{2.10}$$

$$= A_0 \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.11}$$

$$= \begin{cases} \frac{2A_0}{\pi\left(1-n^2\right)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.12}$$

2.3 Verify (1.1) using python.
Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /f.py
```

Use the following command in the terminal to run the code

```
python3 f.py
```

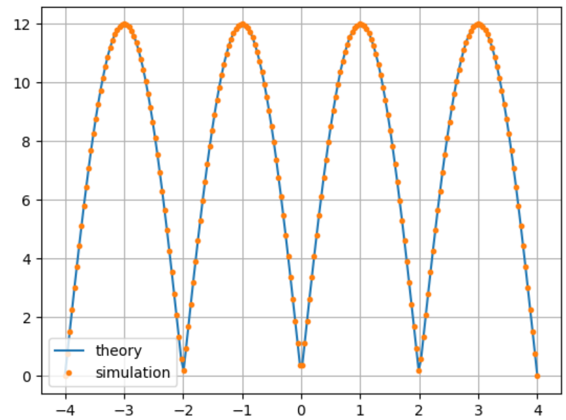

Fig. 2.3

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty}\left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right) \tag{2.13}$$

and obtain the formulae for a_k and b_k.

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.14}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty}\left(c_k + c_{-k}\right)\cos\left(2\pi k f_0 t\right) + \sum_{k=0}^{\infty}\left(c_k - c_{-k}\right)\sin\left(2\pi k f_0 t\right) \tag{2.16}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.17}$$

$$b_k = c_k - c_{-k} \tag{2.18}$$

2.5 Find a_k and b_k for (1.1)
Solution: $\because$ $x(t)$ is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

where we substitute $k \mapsto -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.18) and for

$k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k > 0,\ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

$$b_k = 0 \tag{2.23}$$

2.6 Verify (2.13) using python.
Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /f.py
```

Use the following command in the terminal to run the code

```
python3 f.py
```

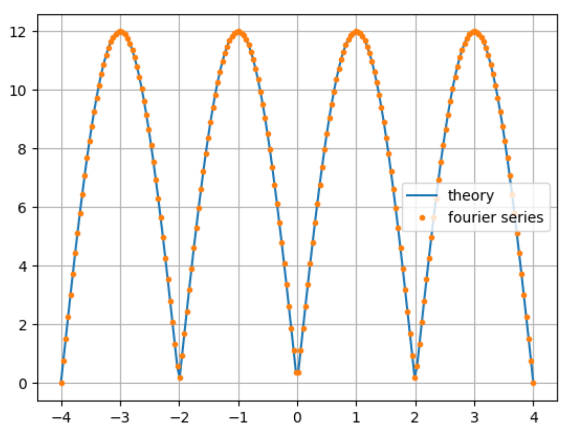

Fig. 2.6

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

$$\tag{3.5}$$

Solution:

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t} dt \tag{3.6}$$

$$\text{Let} \quad t - t_0 = k \tag{3.7}$$

$$\implies g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(k) e^{-j2\pi f (k + t_0)} dk \tag{3.8}$$

$$= G(f) e^{-j2\pi f t_0} \tag{3.9}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.10}$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df \tag{3.11}$$

$t = -f$ and $f = t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t}\, dt \tag{3.12}$$

$$\implies G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.13}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt \tag{3.14}$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f (0)} dt = 1 \tag{3.15}$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\delta(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_0} \tag{3.16}$$

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.17}$$

$$\therefore e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f - f_0) \tag{3.18}$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \tag{3.19}$$

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \mathcal{F}\left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right] \tag{3.20}$$

$$= \frac{\delta(f - f_0) + \delta(f + f_0)}{2} \tag{3.21}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /3_8.py
```

Use the following command in the terminal to run the code

```
python3 3_8.py
```

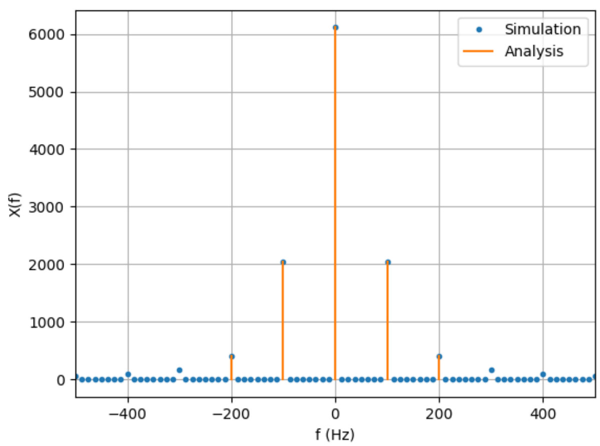

Fig. 3.8

3.9 Show that

$$\operatorname{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{sinc}(t) \tag{3.22}$$

Verify using python.

Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /3_9.py
```

Use the following command in the terminal to run the code

```
python3 3_9.py
```

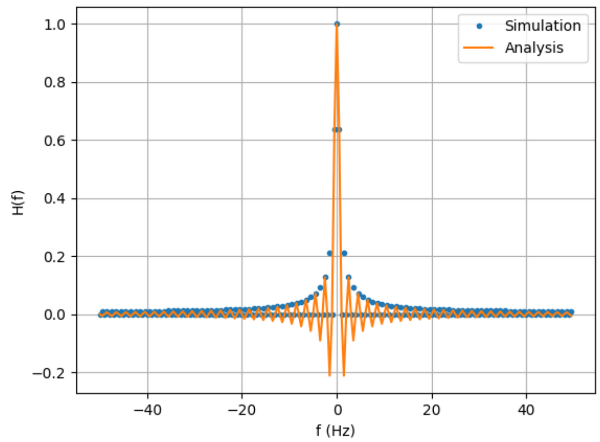

Fig. 3.9

$$\operatorname{rect}(t) = \begin{cases} 1 & \frac{-1}{2} \le t \le \frac{1}{2} \\ 0 & \text{else} \end{cases} \tag{3.23}$$

$$\therefore \operatorname{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \operatorname{rect}(t)\, e^{-j2\pi f t} dt \tag{3.24}$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi f t} dt = \frac{1}{2\pi j f}\left[e^{\pi f j} - e^{-\pi f j}\right] \tag{3.25}$$

$$= \frac{\sin \pi f}{\pi f} = \operatorname{sinc}(f) \tag{3.26}$$

3.10 $\operatorname{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow}$?. Verify using python.
Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /3_10.py
```

Use the following command in the terminal to run the code

```
python3 3_10.py
```

$$\because \operatorname{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{sinc}(f) \tag{3.27}$$

$$\therefore \operatorname{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{rect}(-f) \tag{3.28}$$

$$\operatorname{rect}(-f) = \operatorname{rect}(f) \tag{3.29}$$

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: $H(f)$ is a low pass filter, assume f_0 to be the cutoff frequency,
$H(f) = k\operatorname{rect}\left(\frac{f}{2f_0}\right)$,where k is the scaling factor.

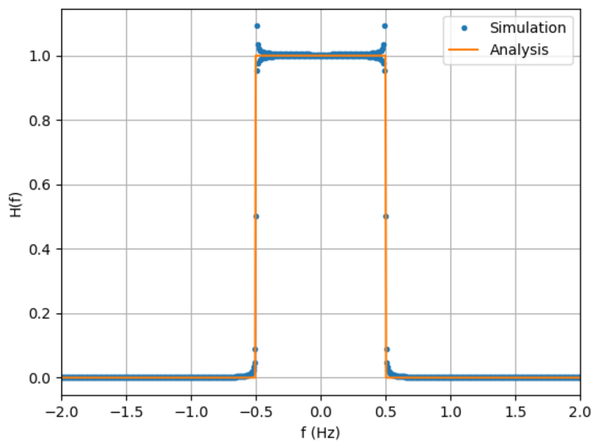

Fig. 3.10

this is a low pass filter $\because$ all frequencies above f_0 are cut-off.
now $k = \frac{5\pi}{24}$ can be obtained, $\because$ output voltage= $5V$.

4.2 Find $h(t)$.
Solution:

$$\mathcal{F}^{-1}[H(f)] = h(t) \tag{4.1}$$

$$g(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{a}G\left(\frac{f}{a}\right) \tag{4.2}$$

$$\therefore h(t) = 2kf_0\text{sinc}\,(2f_0t) \tag{4.3}$$

4.3 Verify your result using through convolution.
Solution: Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/charger/codes
    /4_3.py
```

Use the following command in the terminal to run the code

```
python3 4_3.py
```

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$. **Solution:**

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2n}}} \tag{5.1}$$

n=order of the filter
wkt $A = 10log_{10}|H(f)|^2$

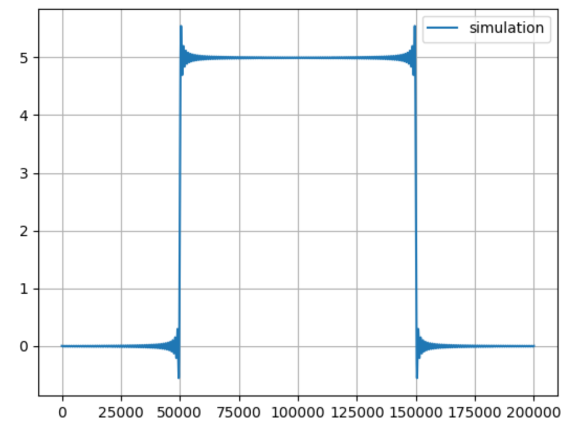

Fig. 4.3

$$A_1 = -10log_{10}\left[1+\left(\frac{f_1}{f_c}\right)^{2n}\right] \tag{5.2}$$

$$A_2 = -10log_{10}\left[1+\left(\frac{f_2}{f_c}\right)^{2n}\right] \tag{5.3}$$

$$\tag{5.4}$$

solving for n,

$$n = \frac{log\left(\frac{10^{-A_1/10}-1}{10^{-A_2/10}-1}\right)}{2log\left(\frac{f_1}{f_2}\right)} \tag{5.5}$$

A_1=-1 db, A_2 =-10 db,f_1=50 Hz and f_2=100 Hz,we get n=2.5596 $\approx$ 3
putting n =3 in above equations we get $f_1^{'}$ and $f_2^{'}$ as 62.628 Hz and 69.336 Hz,so we take $f_c = \sqrt{f_1^{'}f_2^{'}} = 65.897Hz$

5.2 Design a Chebyschev filter for $H(f)$.
Solution:

$$|H_n(f)| = \frac{1}{\sqrt{1+\epsilon^2T_n^2\left(\frac{f}{f_0}\right)}} \tag{5.6}$$

where T_n=nth order Chebyshev polynomial
ϵ=ripple factor which is related to passband ripple in δ as $\sqrt{10^{\delta/10}-1}$

$$A_1 = -10\log_{10}\left[1+\epsilon^2 c_n^2\left(\frac{f}{f_0}\right)\right] \tag{5.7}$$

$$\implies c_n\left(\frac{f}{f_0}\right) = \frac{\sqrt{10^{-\frac{A_1}{10}}-1}}{\epsilon} \tag{5.8}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_1}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f}{f_0}\right)} \tag{5.9}$$

considering $f_0 = 65Hz$,f=120 Hz ,$A_1 = 10dB$,$\delta = 0.2dB$,$\epsilon = 0.217$. on solving we get n=2.71302, $\implies$ n=3

5.3 Design a circuit for your Butterworth filter.
Solution: for this 3rd order filter the $C_1 = 1, C_3 = 1$ and $L_2 = 2$, de-normalizing the values $C_i' = \frac{C_i}{\omega_c}$and similarly $L_i' = \frac{L_i}{\omega_c}$.

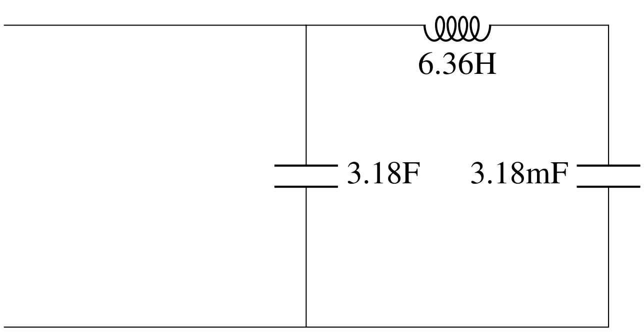

Fig. 5.3

5.4 Design a circuit for your Chebyschev filter.
Solution: the normalised values obtained are $C_1 = 1.2276F, L_2 = 1.1525H, C_3 = 1.2276F$. De-normalizing the values we get:

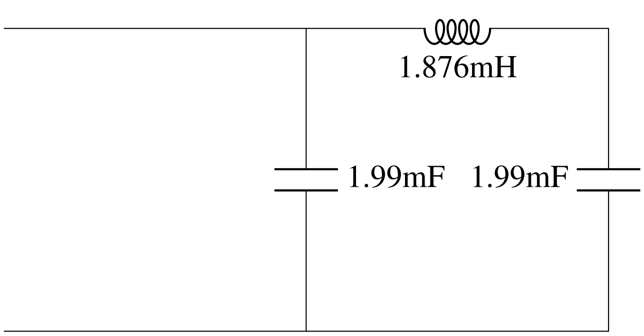

Fig. 5.4