Digital Signal Processing

Pranav B

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/2/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Run the below python code

```
https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/2/
    Cancel_noise.py
```

Use the following command in the terminal to run the code

```
python3 Cancel_noise.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: First run the following c code to generate data

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/3/xy.c
```

Use the following command in terminal to run code

```
cc -lm xy.c
```

Then run the below python code which yields Fig. 3.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/3/xnyn.py
```

Use the following command in terminal to run code

```
python3 xnyn.py
```

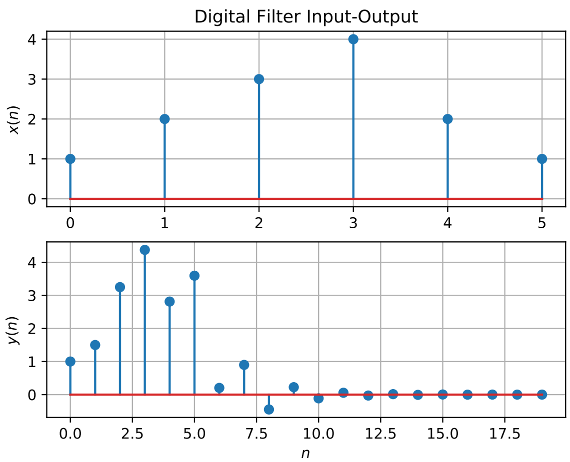

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem $\mathcal{Z}\{x(n)\} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.7}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.8}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.9}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.12}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.13}$$

and from (4.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.14}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.16}$$

Solution:

$$a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{4.17}$$

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.18}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.19}$$

Using formula of sum of infinite geometric progression with common ratio<1

$$= \frac{1}{1 - az^{-1}} \left(\because \left| \frac{a}{z} \right| < 1 \right) \tag{4.20}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.21}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $x(n)$.
Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.22}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.23}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.24}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.25}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.26}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.27}$$

The period of numerator is π and period of denominator is 2π,$\therefore$ the period of $|H(e^{i\omega})|$ is LCM($\pi, 2\pi$)=2π. The following code plots Fig. 4.6.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/4/dtft.py
```

Use the following command in the terminal to run the code

```
python3 dtft.py
```

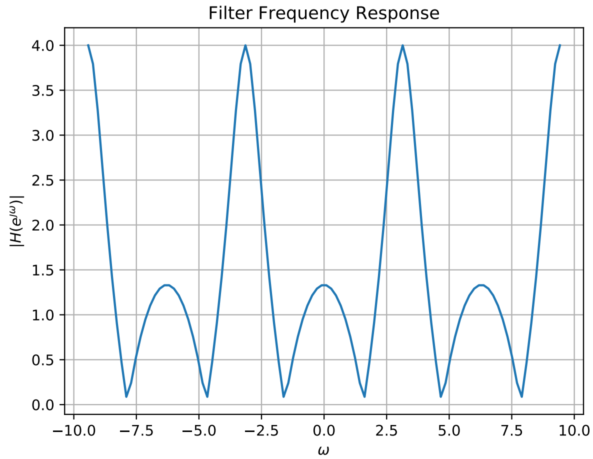

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.
Solution: $h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} e^{j\omega n} d\omega \tag{4.28}$$

$$\tag{4.29}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.9).
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r|l} & 2x - 4 \\ \frac{1}{2}x + 1) & x^2 \quad\quad + 1 \\ & -x^2 - 2x \\ \hline & -2x + 1 \\ & 2x + 4 \\ \hline & 5 \end{array}$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.4}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \tag{5.5}$$

$$\therefore \frac{1}{1 + \frac{z^{-1}}{2}} = 1 - \frac{z^{-1}}{2} + [\frac{z^{-1}}{2}]^2 + \cdots \tag{5.6}$$

$$\therefore = -4 + 2z^{-1} + 5(1 - \frac{z^{-1}}{2} + \left[\frac{z^{-1}}{2}\right]^2 + \cdots) \tag{5.7}$$

$$= 1 - \frac{z^{-1}}{2} + \sum_{n=2}^{\infty} \frac{5}{4}\left(\frac{-1}{2}\right)^{(n-2)} \tag{5.8}$$

$$\tag{5.9}$$

comparing coefficients we get $h(0) = 1$,$h(1) = \frac{-1}{2}$ and $h(n) = \frac{5}{4}\left(\frac{-1}{2}\right)^{(n-2)} \forall n \geq 2$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.10}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.9),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.11}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.12}$$

using (4.16) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/5/hn.py
```

Use the following command in the terminal to run the code

```
python3 hn.py
```

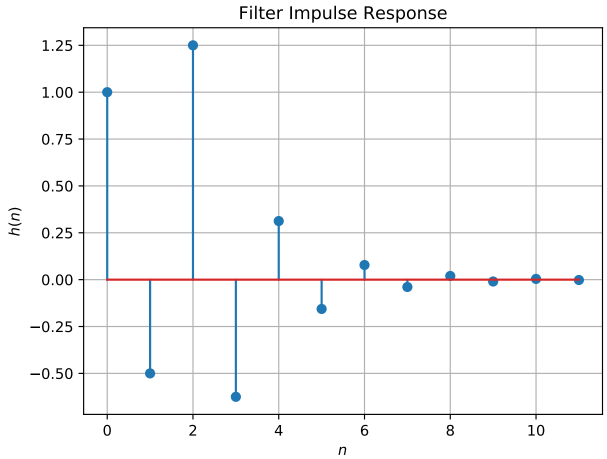

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

$$|u(n)| \leq 1 \tag{5.13}$$

$$\left|\left(-\frac{1}{2}\right)^n\right| \leq 1 \tag{5.14}$$

$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \leq 1 \tag{5.15}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.16}$$

$$\implies h(n) \leq 2 \tag{5.17}$$

Therefore $h(n)$ is bounded. Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \tag{5.18}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.19}$$

$$= \frac{1}{2} < 1 \tag{5.20}$$

Therefore, $h(n)$ is convergent.

5.4 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.21}$$

Is the system defined by (3.2) stable for the impulse response in (5.10)?

Solution:

$$u(n)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases} \tag{5.22}$$

$$u(n-2)=\begin{cases}1 & n\geq 2\\ 0 & n<2\end{cases} \tag{5.23}$$

$$\therefore h(n)=\begin{cases}0 & n<0\\ \left(\frac{-1}{2}\right)^n & 0\leq n\\ \left(\frac{-1}{2}\right)^n+\left(\frac{-1}{2}\right)^{(n-2)}=\left(\frac{-1}{2}\right)^{(n-2)}\left(\frac{5}{4}\right) & n\geq 2\end{cases} \tag{5.24}$$

$$\therefore \sum_{n=-\infty}^{\infty} h(n)=0+1+\frac{-1}{2}+\sum_{n=2}^{\infty}\left[\left(\frac{-1}{2}\right)^n+\left(\frac{-1}{2}\right)^{(n-}\right] \tag{5.25}$$

$$=\frac{1}{2}+\frac{5}{4}\left(\frac{2}{3}\right)=\frac{4}{3}<\infty \tag{5.26}$$

$$\tag{5.27}$$

$\therefore$ system defined is stable

5.5 Compute and sketch $h(n)$ using

$$h(n)+\frac{1}{2}h(n-1)=\delta(n)+\delta(n-2), \tag{5.28}$$

This is the definition of $h(n)$. **Solution:** The following code plots Fig. 5.5. Note that this is the same as Fig. 5.3.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/5/hndef.py
```

Use the following command in the terminal to run the code

```
python3 hndef.py
```

5.6 Compute

$$y(n)=x(n)*h(n)=\sum_{k=-\infty}^{\infty}x(k)h(n-k) \tag{5.29}$$

Comment. The operation in (5.29) is known as *convolution*.

Solution: $X=\begin{bmatrix}1\\2\\3\\4\\2\\1\end{bmatrix}$

for some N,
$H=\begin{bmatrix}h(0) & h(1) & h(2) & \cdots & h(N-1)\end{bmatrix}_{(1\times N)}$
Toeplitz matix for

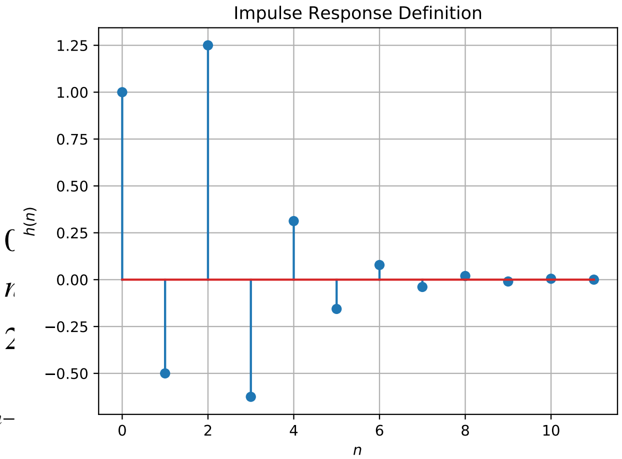

Fig. 5.5: $h(n)$ from the definition

$$X=\begin{bmatrix}1&2&3&4&2&1&\cdots&0&0&0\\0&1&2&3&4&2&1&\cdots&0&0\\0&0&1&2&3&4&2&1&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&0\\0&0&0&\cdots&1&2&3&4&2&1\end{bmatrix}$$

$\therefore Y^T=X*H$

The following code plots Fig. 5.6. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/ynconv.py
```

Use the following command in the terminal to run the code

```
python3 ynconv.py
```

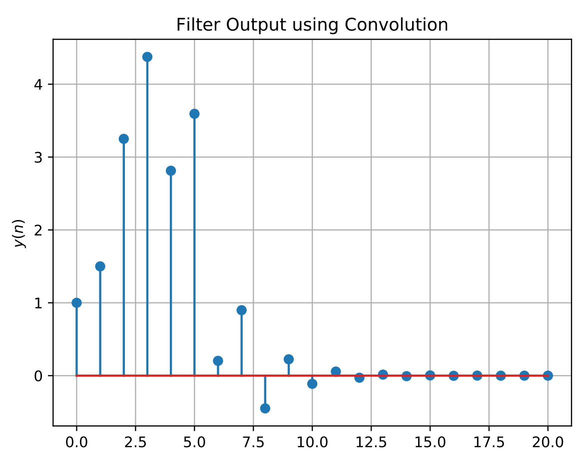

Fig. 5.6: $y(n)$ from the definition of convolution

5.7 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.30}$$

Solution: wkt

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.31}$$

Replacing k with $n-k$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \therefore y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.32}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: Run the following code to compute $X(k)$,$H(k)$.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/6_1.py
```

Use the following command in the terminal to run the code

```
python3 6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Run the following code to compute $Y(k)$.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/6_2.py
```

Use the following command in the terminal to run the code

```
python3 6_2.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.6. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/yndft.py
```

Use the following command in the terminal to run the code

```
python3 yndft.py
```

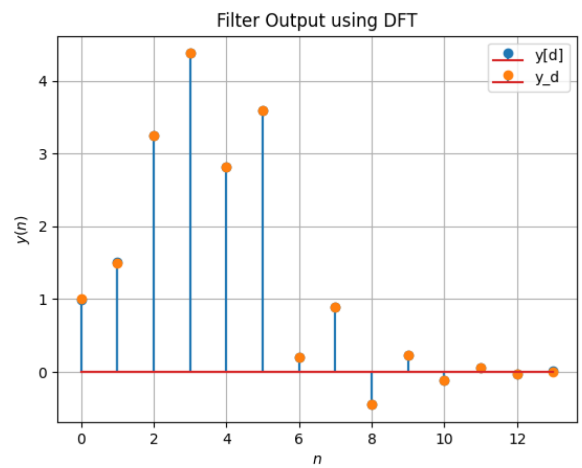

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k)$, $H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Run the following code to generate the plot below

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/6/xk.py
```

Use the following command in the terminal to run the code

```
python3 xk.py
```

6.5 Wherever possible, express all the above equations as matrix equations.

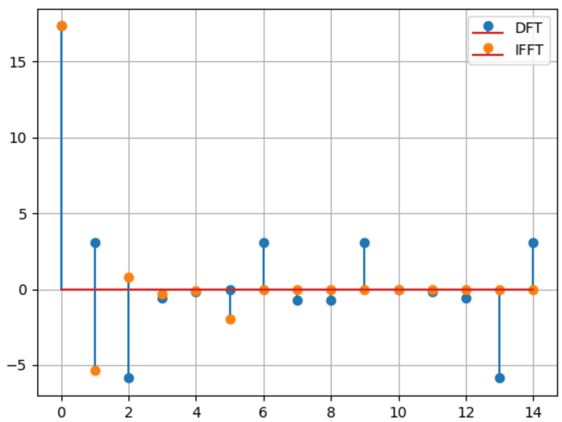

Fig. 6.4: IFFT

Solution:

$$\vec{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.4}$$

$$\vec{h} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.5}$$

$$\vec{y} = \vec{x} \circledast \vec{h} \tag{6.6}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2N-1} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ 0 & 0 & h_{N-1} & \cdots & h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \tag{6.7}$$

The convolution can be written using a Toeplitz matrix.
Consider the DFT matrix

$$\vec{W} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.8}$$

where $\omega = e^{-j2\pi/N}$ is the N^{th} root of unity
Then the discrete Fourier transforms of $\vec{x}$ and $\vec{h}$ are given by

$$\vec{X} = \vec{W}\vec{x} \tag{6.9}$$

$$\vec{H} = \vec{W}\vec{h} \tag{6.10}$$

$\vec{Y}$ is then given by

$$\vec{Y} = \vec{X} \circ \vec{H} \tag{6.11}$$

where $\circ$ denotes the Hadamard product (element-wise multiplication)
But $\vec{Y}$ is the discrete Fourier transform of the filter output $\vec{y}$

$$\vec{Y} = \vec{W}\vec{y} \tag{6.12}$$

Thus,

$$\vec{W}\vec{y} = \vec{X} \circ \vec{H} \tag{6.13}$$

$$\implies \vec{y} = \vec{W}^{-1}\left(\vec{X} \circ \vec{H}\right) \tag{6.14}$$

$$= \vec{W}^{-1}\left(\vec{W}\vec{x} \circ \vec{W}\vec{h}\right) \tag{6.15}$$

This is the inverse discrete Fourier transform of $\vec{Y}$

7 FFT

7.1 Definitions

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\vec{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

3. Let

$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: :
$$W_N^2 = e^{-j4\pi/N} \tag{7.8}$$
$$= e^{-j2\pi/(\frac{N}{2})} = W_{N/2} \tag{7.9}$$
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.10}$$
Solution: :$W_4^{4n} = 1$ and $W_4^{4n+2} = -1 \forall n \in \mathbb{N}$
$$\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.11}$$
$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.12}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.13}$$
$$\implies -\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.14}$$
$$\text{also} \quad \vec{F}_2 = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.15}$$
$$\vec{F}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} \vec{I}_2\vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{I}_2\vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} \tag{7.17}$$
$$= \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.18}$$
Multiplying by $\vec{P}_4$ on both sides we get the desired result.
7. Show that
$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.19}$$
Solution: : $\vec{F}_N = (\vec{f}_N^1 \quad \vec{f}_N^2 \cdots \vec{f}_N^N)$
This can be rewritten as
$$\vec{F}_N = \begin{pmatrix} \vec{f}_{\textbf{odd}} \\ \vec{f}_{\textbf{even}} \end{pmatrix} \vec{P}_N \tag{7.20}$$
Multiplying $\vec{P}_N$ helps in rearranging the terms,$\vec{f}_{\textbf{odd}} = [\vec{f}_i : i(mod)2 = 1]$and vice-versa for even.Also on observation
$$\begin{pmatrix} D_{N/2}F_{N/2} \\ -D_{N/2}F_{N/2} \end{pmatrix} = \vec{f}_{\textbf{even}} \tag{7.21}$$
$$\begin{pmatrix} I_{N/2}F_{N/2} \\ I_{N/2}F_{N/2} \end{pmatrix} = \vec{f}_{\textbf{odd}} \tag{7.22}$$
$$\begin{pmatrix} I_{N/2}F_{N/2} & D_{N/2}F_{N/2} \\ I_{N/2}F_{N/2} & -D_{N/2}F_{N/2} \end{pmatrix} = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix} \tag{7.23}$$
$$= \begin{pmatrix} \vec{f}_{\textbf{odd}} \\ \vec{f}_{\textbf{even}} \end{pmatrix} \tag{7.24}$$
$$\therefore \vec{F}_N = \begin{pmatrix} \vec{f}_{\textbf{odd}} \\ \vec{f}_{\textbf{even}} \end{pmatrix} \vec{P}_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix} P_N \tag{7.25}$$
8. Find
$$\vec{P}_4\vec{x} \tag{7.26}$$
Solution: :
$$\begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} \tag{7.27}$$
$$= \begin{pmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{pmatrix} \tag{7.28}$$
9. Show that
$$\vec{X} = \vec{F}_N\vec{x} \tag{7.29}$$
where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: :
$$\vec{X} = \begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{pmatrix} \vec{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix} \tag{7.30}$$
$$X(k) = \sum_{i=0}^{N-1} x(i)e^{\frac{2\pi jki}{N}} \tag{7.31}$$
Therefore $\vec{X}$ can be represented as :
$$\vec{X} = \vec{F}_N\vec{x} \tag{7.32}$$
where $\vec{F}_N = \left[e^{\frac{2\pi jik}{N}}\right]_{(N-1)\times(N-1)} 0 \le i, k \le N-1$
10. Derive the following Step-by-step visualisation

of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.33}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.34}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.38}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.39}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.40}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.41}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.45}$$

Solution: :

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j2\pi kn/8}, \quad k = 0, \dots, 7 \tag{7.46}$$

$$= \sum_{n=0}^{7} x(n) W_8^{kn} \tag{7.47}$$

$$= \sum_{n \text{ is even}} x(n) W_8^{kn} + \sum_{n \text{ is odd}} x(n) W_8^{kn} \tag{7.48}$$

$$= \sum_{m=0}^{3} x(2m) W_8^{2km} + \sum_{m=0}^{3} x(2m+1) W_8^{2km+k} \tag{7.49}$$

Now substitute $W_8^2 = W_4$

$$X(k) = \sum_{m=0}^{3} x(2m) W_4^{km} + W_8^k \sum_{m=0}^{3} x(2m+1) W_4^{km} \tag{7.50}$$

Consider

$$x_1(n) = \{x(0), x(2), x(4), x(6)\} \tag{7.51}$$

$$x_2(n) = \{x(1), x(3), x(5), x(7)\} \tag{7.52}$$

Thus

$$X(k) = X_1(k) + W_8^k X_2(k) \qquad k = 0, \dots, 7 \tag{7.53}$$

Now, $X_1(k)$ and $X_2(k)$ are 4-point DFTs which means they are periodic with period 4

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \tag{7.54}$$

$$= X_1(k) + e^{-j2\pi(k+4)/8} X_2(k) \tag{7.55}$$

$$= X_1(k) + e^{-j(2\pi k/8+\pi)} X_2(k) \tag{7.56}$$

$$= X_1(k) - e^{-j2\pi k/8} X_2(k) \tag{7.57}$$

$$= X_1(k) - W_8^k X_2(k) \tag{7.58}$$

Therefore, for $k = 0, 1, 2, 3$

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.59}$$

$$X(k+4) = X_1(k) - W_8^k X_2(k) \tag{7.60}$$

which is the same as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.62}$$

Similarly, we can divide $x_1(n)$ into

$$x_3(n) = \{x(0), x(4)\} \tag{7.63}$$

$$x_4(n) = \{x(2), x(6)\} \tag{7.64}$$

i.e.,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \vec{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.65}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \vec{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.66}$$

to get

$$X_1(k) = X_3(k) + W_4^k X_4(k) \tag{7.67}$$

$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \tag{7.68}$$

for $k = 0, 1$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.69}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.70}$$

And on dividing $x_2(n)$ into

$$x_5(n) = \{x(1), x(5)\} \tag{7.71}$$

$$x_6(n) = \{x(3), x(7)\} \tag{7.72}$$

i.e.,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \vec{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.73}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \vec{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.74}$$

to get

$$X_2(k) = X_5(k) + W_4^k X_6(k) \tag{7.75}$$

$$X_2(k+2) = X_5(k) - W_4^k X_6(k) \tag{7.76}$$

for $k = 0, 1$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.77}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.78}$$

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.79}$$

compte the DFT using (7.29)
Solution: :Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/FFT/7_11.c
```

Use the following command in the terminal to run the code

```
cc -lm 7_11.c
./a.out
```

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution: :Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/FFT/7_12.c
```

Use the following command in the terminal to run the code

```
cc -lm 7_12.c
./a.out
```

13. Write a C program to compute the 8-point FFT.
Solution: :Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/FFT/7_13.c
```

Use the following command in the terminal to run the code

```
cc -lm 7_13.c
./a.out
```

14. Represent time complexities of FFT,Convolution and DFT matrix.
Solution: :Run the following code to generate times of convolution and FFT

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/FFT/7_14.c
```

Use the following command in the terminal to run the code

```
cc -lm 7_14.c
./a.out
```

Run the following code to generate time of DFT matrix

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/FFT/dftmatrix
    .py
```

Use the following command in the terminal to run the code

```
python3 dftmatrix.py
```

Then run the following python code to get graph

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/FFT/7_14.py
```

Use the following command in the terminal to run the code

```
python3 7_14.py
```

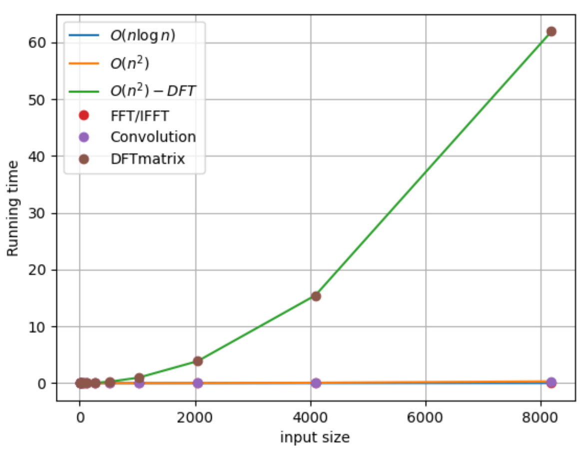

Fig. 7.14: Time complexity

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
        output_signal = signal.lfilter(b, a,
            input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \quad (8.2)$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \quad (8.3)$$

For obtaining the discrete Fourier transform, put $z = \jmath^{\frac{2\pi i}{I}}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$ Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/7/7_1.py
```

Use the following command in the terminal to run the code

```
python3 7_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.

Solution: The polynomial coefficients obtained are

$$\vec{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \quad (8.4)$$

The difference equation is then given by

$$\begin{aligned} &\vec{a}^{\top}\vec{y} = \vec{b}^{\top}\vec{x} \\ &\quad y(n) - (2.519)y(n-1) + (2.561)y(n-2) \\ &-(1.206)y(n-3) + (0.220)y(n-4) = (0.003)x(n) \\ &+(0.014)x(n-1) + (0.021)x(n-2) + (0.014)x(n-3) \\ &\qquad + (0.003)x(n-4) \quad (8.5) \end{aligned}$$

where

$$\vec{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/7/7_2_1.py
```

Use the following command in the terminal to run the code and generate the figure below

```
python3 7_2_1.py
```

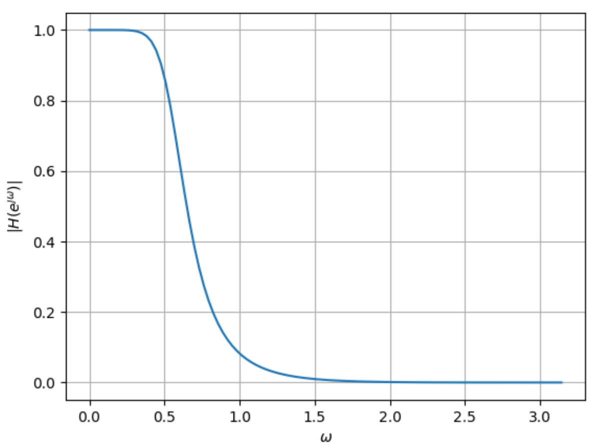

Fig. 8.2: $|H(e^{j\omega})|$

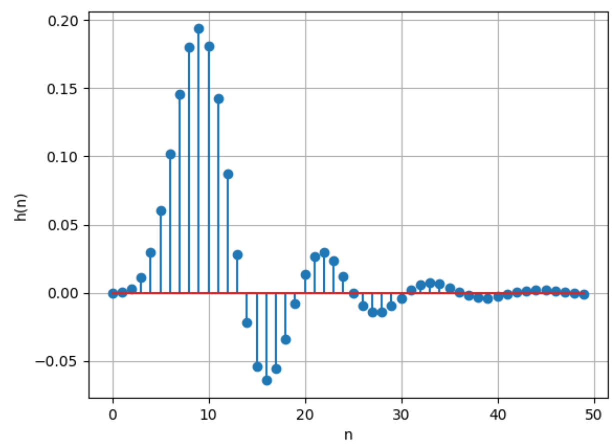

Fig. 8.2: h(n)

Run the following code

```
wget https://github.com/Pranavb060504/
    SIgnalProcessing/blob/main/7/7_2_2.py
```

Use the following command in the terminal to run the code and generate the figure below

```
python3 7_2_2.py
```

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: Order: 10
Cutoff frequency: 3000 Hz