Distributed System Scheduling as a Combinatorial Optimization Problem

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Abstract

In the realm of computer science, optimization problems are ubiquitous, arising in areas such as resource allocation, network design, and algorithm efficiency. This project focuses on the job scheduling problem in distributed computing systems, a classic challenge that aligns closely with the principles of mathematical optimization covered in courses like CS515L: Computational Methods in Optimization. Distributed systems, such as cloud computing platforms (e.g., AWS or Google Cloud), involve multiple interconnected nodes (servers or processors) that must process a set of jobs efficiently. The goal is to assign jobs to nodes in a way that minimizes total completion time (makespan) while respecting constraints like node capacity, job dependencies, and energy consumption limits. This problem can be formulated as a constrained optimization task, incorporating elements of combinatorial optimization, linear programming (LP), mixed-integer programming (MIP), and potentially nonlinear extensions for real-world complexities.

Introduction

The motivation for this project stems from the growing scale of distributed computing. With the explosion of big data and AI workloads, inefficient scheduling can lead to significant delays, increased costs, and wasted resources. For instance, in a data center with hundreds of servers, poor job allocation might result in some nodes being overloaded while others remain idle, leading to higher energy bills and reduced system throughput. Mathematical optimization provides a systematic approach to address this, translating engineering problems into formulations that can be solved using computational methods. As highlighted in the course materials, optimization involves identifying decision variables, defining an objective function, and imposing constraints—steps that are central to modeling job scheduling.

Project Scope and Basic Details

The project will involve:

- **Modeling Phase**: Formulate the problem in standard optimization form, incorporating real data (e.g., simulated job traces from cloud workloads).
- **Implementation**: Code solvers in Python, comparing methods like simplex for LP relaxations versus interior-point for full MILP.
- **Analysis**: Test on benchmarks, analyze local vs. global optima (noting that MILP solvers guarantee globality for solved instances), and discuss non-convex extensions (e.g., if stochastic job arrivals are added).
- **Extensions**: Extension to more practical scenarios where job preemptions occur, setup times between different job types are present and also where there are communications costs between dependent jobs on different nodes.

Basic details:

- **Tools**: Python (NumPy, SciPy, MILP Solvers, Genetic Algorithm), MATLAB optional
- **Expected Outcomes**: A working optimizer prototype, insights into trade-offs (e.g., exact vs. approximate methods), and potential publication in CS conferences like IEEE Cluster.

This project not only reinforces course concepts but also prepares for real-world applications, such as optimizing Kubernetes clusters in industry.

0.1 Decision Variables (Quantities to Optimize)

The optimization problem determines the following variables:

- $x_{ij} \in \{0,1\}$: binary variable, equal to 1 if job j is assigned to machine i, and 0 otherwise.
- $s_j \geq 0$: continuous variable, start time of job j.
- $y^i_{jk} \in \{0,1\}$: binary sequencing variable, equal to 1 if job j is scheduled before job k on machine i, and 0 otherwise.
- $C_{\max} \geq 0$: continuous variable, the makespan of the schedule (maximum completion time across all jobs).

0.2 Objective

The goal is to minimize the makespan, defined as

$$C_{\max} = \max_{j \in J}\{s_j + d_j\},$$

which is enforced via the linear constraints

$$C_{\max} \geq s_j + d_j \quad \forall j \in J.$$

Thus, the optimization problem seeks to

$$\min \ C_{\max}.$$

Combinatorial Optimization Formulation

We model the heterogeneous job scheduling problem as a combinatorial optimization problem where discrete assignment and sequencing decisions are coupled with continuous timing variables.

Problem Data

- $J = \{1, 2, \ldots, n\}$: set of jobs.
- $M = \{1, 2, \ldots, m\}$: set of computing nodes.
- d_j : processing duration of job $j \in J$.
- p_j : processing load of job $j \in J$.
- C_i : processing capacity of node $i \in M$.

- $E \subseteq J \times J$: set of precedence constraints (k, ℓ) meaning job ℓ cannot start before k finishes.
- $comm_{i,i'} \geq 0$: communication delay if a predecessor job is on node i and the successor job is on node i'. Typically $comm_{i,i} = 0$.
- M : a sufficiently large constant used in the big-M linearization of sequencing constraints.

Decision Variables

- $x_{ij} \in \{0, 1\}$: equals 1 if job j is assigned to node i, and 0 otherwise.
- $s_j \geq 0$: start time of job j.
- $C_{\max} \geq 0$: makespan of the schedule (the maximum job completion time).
- $y^i_{jk} \in \{0, 1\}$: sequencing variable for jobs j and k assigned to the same node i. $y^i_{jk} = 1$ if job j is scheduled before k on node i, 0 otherwise.

Objective

$$\min \; C_{\max} \tag{1}$$

Constraints

Assignment constraints. Each job is assigned to exactly one node:

$$\sum_{i \in M} x_{ij} = 1 \quad \forall j \in J \tag{2}$$

Capacity constraints. Jobs can only run on nodes that can handle their load:

$$p_j \cdot x_{ij} \leq C_i \quad \forall i \in M, \, j \in J \tag{3}$$

Precedence constraints with communication delays. For each precedence $(k, \ell) \in E$:

$$s_\ell \;\geq\; s_k + d_k + \sum_{i \in M} \sum_{i' \in M} comm_{i,i'} \, x_{ik} \, x_{i'\ell} \quad \forall (k, \ell) \in E \tag{4}$$

Non-overlap constraints on the same node. For any two distinct jobs $j, k \in J$ and node $i \in M$:

$$s_k \;\geq\; s_j + d_j - M \cdot (1 - y^i_{jk}) - M \cdot (1 - x_{ij} - x_{ik} + 1) \quad \forall j \neq k, \, i \in M \tag{5}$$

$$s_j \;\geq\; s_k + d_k - M \cdot y^i_{jk} - M \cdot (1 - x_{ij} - x_{ik} + 1) \quad \forall j \neq k, \, i \in M \tag{6}$$

Here M is a sufficiently large constant (big-M method).

Makespan definition. Completion times of all jobs must be bounded by the makespan:

$$C_{\max} \;\geq\; s_j + d_j \quad \forall j \in J \tag{7}$$

Model Characteristics

This formulation is a **combinatorial scheduling model**:

- Discrete assignment variables x_{ij} determine where jobs are executed.
- Discrete sequencing variables y^i_{jk} enforce a valid order on each node.
- Continuous timing variables s_j couple with assignment and sequencing decisions.

The coupling of binary and continuous variables creates significant computational challenges, making this problem NP-hard.

Solvability Analysis

The combinatorial scheduling problem formulated above involves discrete assignment and sequencing decisions coupled with continuous timing variables. Below we analyze its solvability from both a feasibility and computational perspective.

Feasibility Conditions

- **Assignment constraints:** Each job must be assigned to exactly one node:

$$\sum_{i \in M} x_{ij} = 1, \quad \forall j \in J$$

  This is always feasible as long as each job can fit on at least one node.

- **Capacity constraints:** Each job can only be assigned to nodes that can accommodate its load:

$$p_j \cdot x_{ij} \leq C_i, \quad \forall i \in M, \forall j \in J$$

  Feasibility requires that for each job j, there exists at least one node i such that $p_j \leq C_i$.

- **Precedence constraints:** For each $(k, \ell) \in E$:

$$s_\ell \geq s_k + d_k + \sum_{i \in M} \sum_{i' \in M} comm_{i,i'}\, x_{ik}\, x_{i'\ell}$$

  Feasible if the precedence graph is acyclic (a valid DAG). Cycles would make the problem infeasible.

- **Sequencing constraints:** For jobs assigned to the same node:

$$s_k \geq s_j + d_j - M \cdot (1 - y^i_{jk}) - M \cdot (2 - x_{ij} - x_{ik})$$
$$s_j \geq s_k + d_k - M \cdot y^i_{jk} - M \cdot (2 - x_{ij} - x_{ik})$$

  Feasibility depends on a sufficiently large scheduling horizon M and compatible job durations.

Computational Solvability

- The problem is a **Mixed-Integer Linear Program (MILP)** with:
  - Binary variables: x_{ij}, y^i_{jk},
  - Continuous variables: $s_j, C_{\max}$,
  - Linear constraints: assignment, capacity, precedence, sequencing, and makespan.

- **Exact solution:** MILP solvers (e.g., Gurobi, CPLEX) can find optimal solutions for small instances ($n \lesssim 20$–30, $m \lesssim 5$–10) using branch-and-bound.
- **Medium instances:** As the number of sequencing variables y^i_{jk} grows like $O(mn^2)$, solving times increase significantly. Exact solutions may be possible with decomposition, warm-starts, or cutting-plane enhancements.
- **Large instances:** For $n > 100$, exact MILP solution becomes intractable. Practical approaches include:
  - Heuristics (list scheduling, priority rules, local search),
  - Relaxation techniques (LP, Lagrangian relaxation),
  - Decomposition methods (per-node scheduling + master assignment problem).

NP-Hardness and Solution Strategy

The heterogeneous job scheduling problem described above is **NP-hard**. Even in simplified variants, such as scheduling independent jobs on identical machines to minimize makespan, the problem generalizes classical NP-hard problems like *multiprocessor scheduling* and *bin-packing*. Introducing precedence constraints, heterogeneous nodes, resource capacities, and communication delays only increases the combinatorial complexity.

Due to this computational intractability, finding exact optimal solutions is only feasible for small instances. For larger instances, we adopt a layered approach to balance solution quality with computational tractability:

- **Exact methods:** For small instances, branch-and-bound or constraint programming (CP-SAT) solvers can explore the feasible space exhaustively to guarantee optimality.
- **Relaxation techniques:** Linear programming (LP) or Lagrangian relaxations can provide lower bounds on the makespan by relaxing the integrality or sequencing constraints. These bounds guide heuristic solutions and help estimate the optimality gap.
- **Heuristic and approximation methods:** For medium to large instances, we can employ greedy scheduling, list scheduling (e.g., earliest start, critical path first, or LPT heuristics), and metaheuristics such as genetic algorithms or simulated annealing. These methods produce high-quality feasible solutions quickly, although optimality cannot be guaranteed.
- **Hybrid approaches:** Combining relaxations with heuristics, such as using LP-based solutions as warm starts for heuristics or local search, can improve solution quality while keeping computational times reasonable.

This approach allows us to systematically trade off solution optimality against computational effort, providing a practical framework for solving instances of varying sizes while respecting the combinatorial complexity of the problem.

Summary Table

Instance Size	Solvability
Small ($n \leq 20$–30)	Exact MILP solution feasible
Medium ($n \leq 100$)	Solvable with solver + heuristics/decomposition
Large ($n > 100$)	Exact MILP infeasible; heuristics/relaxations required
Feasibility	Feasible if job loads $\leq$ node capacities and DAG is acyclic
Computational Hardness	NP-hard

In conclusion, the combinatorial MILP formulation is **solvable in principle** for small instances, but for medium to large instances practical solution approaches require relaxations, heuristics, or decomposition methods.

Modeling Challenges for Temporal Constraints

Capturing temporal aspects in a mathematical programming formulation for scheduling is particularly challenging due to the interplay between discrete assignment decisions, job ordering, and continuous timing variables. The key challenges include:

1. Job Dependencies Across Nodes Precedence constraints require that a successor job ℓ cannot start until all its predecessors k have completed. If jobs are scheduled on different nodes, communication or data transfer delays must also be incorporated. Mathematically, this introduces conditional constraints of the form:

$$s_\ell \geq s_k + d_k + comm_{i,i'}^{k,\ell} \quad \text{if } x_{ik} = 1 \text{ and } x_{i'\ell} = 1$$

where x_{ik} and $x_{i'\ell}$ are binary assignment variables. Encoding these conditional dependencies in a linear programming framework requires either:

- Big-M constraints, which can weaken LP relaxations if M is too large.
- Disjunctive or logical formulations, which increase the number of binary variables and complicate the combinatorial structure.

Moreover, the dependency graph may be complex (e.g., multiple levels, DAG structure), requiring careful propagation to avoid infeasible schedules.

2. Resource Capacity Over Time Each node has limited capacity, and multiple jobs may share the same node. Ensuring that the sum of loads at any point in time does not exceed the node's capacity requires capturing the cumulative effect of overlapping jobs:

$$\sum_{j \text{ active at time } t} p_j \leq C_i \quad \forall i, \forall t$$

In a MILP, this can be modeled via:

- **Disjunctive sequencing variables** y_{jk}^i, which enforce that jobs on the same node do not overlap.
- **Time-indexed formulations**, which discretize the scheduling horizon and create binary variables indicating whether a job is active in each time slot.

Both approaches introduce a large number of binary variables, creating scalability challenges.

3. Coupling Between Discrete and Continuous Decisions Start times s_j are continuous, while assignment x_{ij} and sequencing y_{jk}^i are discrete. This coupling leads to:

- Non-convex feasible regions.
- Difficulties in computing strong LP relaxations.
- Exponential branching in exact solvers for larger instances.

4. Practical Remedies To handle these challenges, one can:

- Use **tight big-M values** based on instance-specific upper bounds (e.g., critical path length) to improve LP relaxation strength.
- Employ **Constraint Programming (CP)** or hybrid MILP+CP formulations to exploit propagation of interval and cumulative constraints.
- Use **heuristic or decomposition-based methods** to first generate feasible schedules respecting temporal and resource constraints, then refine them.
- Coarsen time discretization or ignore minor communication delays to reduce model size for large instances, trading fidelity for computational tractability.

These techniques help maintain feasibility with respect to job dependencies and resource capacities while keeping the problem computationally manageable.

Handling the Mixed-Integer Nature of the Problem

The scheduling problem is inherently a **mixed-integer optimization problem**, due to the combination of discrete assignment and sequencing variables (x_{ij}, y^i_{jk}) and continuous timing variables (s_j). The mixed-integer nature significantly increases computational complexity, particularly for large instances.

Exact methods: For small- to medium-sized instances, we can solve the problem optimally using:

- **Branch-and-bound / branch-and-cut:** Systematically explores the space of discrete assignments and sequencing decisions while pruning suboptimal branches using bounds on $C_{\max}$. Modern MILP solvers (e.g., Gurobi, CPLEX) can exploit this approach efficiently for instances with tens of jobs.
- **Constraint Programming (CP-SAT):** Interval variables and cumulative constraints model temporal aspects naturally. CP solvers propagate constraints efficiently, often outperforming MILP for rich scheduling structures with precedence and resource capacity.

Heuristic approaches: For larger instances, exact methods may become intractable due to the exponential growth of discrete decision combinations. In such cases, we adopt heuristic or metaheuristic strategies:

- **Greedy/list scheduling:** Assign jobs in topological order according to priority rules (e.g., critical path first, longest processing time first). Ensures feasibility and provides a baseline solution quickly.
- **Local search / neighborhood improvement:** Iteratively swap or reassign jobs to reduce makespan or improve resource utilization.
- **Metaheuristics:** Techniques such as Genetic Algorithms (GA), Simulated Annealing (SA), or Tabu Search can explore a larger solution space while controlling computational effort.

Hybrid approach: A practical strategy is to use heuristics to generate high-quality initial solutions and then warm-start exact solvers, or to apply decomposition methods (e.g., Lagrangian relaxation or Benders decomposition) to separate assignment and sequencing decisions, improving scalability while maintaining solution quality.

Trade-offs Between Model Fidelity (Closeness to reality) and Computational Complexity

In scheduling problems with discrete assignment decisions and continuous timing variables, there is an inherent trade-off between model fidelity and computational tractability:

- **High-fidelity models:**
  - Capture all details such as job dependencies, sequencing constraints, communication delays, and node-specific capacities.
  - Produce schedules that are realistic and directly deployable in practice.
  - *Drawback:* They require a large number of binary and continuous variables, which leads to high computational complexity and may be intractable for large instances.
- **Simplified models:**
  - May ignore communication delays, aggregate resources, or use coarse time discretization.
  - Reduce the number of variables and constraints, allowing faster solution times and scalability to larger problem instances.
  - *Drawback:* Important real-world constraints might be missed, potentially producing schedules that are infeasible or suboptimal when deployed.
- **Practical approach:** Start with simpler models for large-scale instances to obtain feasible solutions or bounds, then progressively introduce high-fidelity constraints for critical subsets of jobs (e.g., on the critical path) to refine schedules while managing computational cost.

Validation of Optimization Results

To evaluate the quality and effectiveness of the proposed scheduling optimization, we focus on the following metrics:

- **Makespan Reduction:** The primary objective is to minimize the makespan $C_{\max}$, defined as the completion time of the last job in the schedule. We will compare the optimized $C_{\max}$ with baseline heuristics to quantify improvement:

$$\text{Makespan Reduction (\%)} = \frac{C_{\text{baseline}} - C_{\text{optimized}}}{C_{\text{baseline}}} \times 100$$

- **Resource Utilization Efficiency:** We measure how effectively node capacities are utilized throughout the schedule. Metrics include:
  - Average utilization per node:

$$\text{AvgUtil}_i = \frac{1}{T} \sum_{t=0}^{T} \frac{\sum_j p_j \cdot \mathbf{1}_{[s_j, s_j+d_j]}(t) \cdot x_{ij}}{C_i}$$

  - Peak utilization per node.
  - Variance of utilization across nodes to assess load balancing.
- **Computational Performance:** We record the total runtime of the optimization solver or heuristic, including:

  - CPU time to reach the first feasible solution.
  - CPU time to reach the final solution or optimality gap.
  - Memory usage.

- **Comparison to Simple Heuristics:** Baseline heuristics such as list scheduling, longest-processing-time (LPT), or greedy assignment will be used for reference. The optimized schedule is evaluated against these heuristics in terms of:
  - Improvement in makespan.
  - Improvement in resource utilization metrics.
  - Computational efficiency relative to heuristic runtime.

- **Optimality Gap (if available):** For instances where lower bounds (from LP relaxations or critical-path analysis) are computable, we report:

$$\text{Optimality Gap (\%)} = \frac{C_{\text{feasible}} - C_{\text{LB}}}{C_{\text{LB}}} \times 100$$

  This quantifies how close the solution is to the theoretical minimum makespan.

These metrics collectively allow us to assess both the *effectiveness* and *efficiency* of the proposed optimization approach and demonstrate improvements over simple heuristics.

Python Implementation

Code for implementation of optimization techniques for different sizes of the problem.

- **Small Instance** We can implement a small instance of this combinatorial scheduling problem in Python using Google OR-Tools CP-SAT solver. This is ideal because: CP-SAT naturally handles assignment + sequencing + continuous time variables. Small instances (e.g., 3–5 jobs, 2–3 nodes) are tractable. We can include precedence and communication delays, and enforce makespan minimization.

  Code in small.py file.

- **Medium Instance** For medium instances (e.g., 10–20 jobs and 3–5 nodes), the combinatorial MILP/CP-SAT formulation becomes computationally expensive if solved exactly. So, for practical purposes, we typically use a hybrid approach: CP-SAT or MILP solver for small "chunks" or relaxed versions. Greedy / heuristic initialization for assignments and sequencing. Local search / metaheuristics (like genetic algorithms or simulated annealing) to improve makespan.

  Code in medium.py file.

- **Large Instance** For large instances (e.g., 50–200 jobs and 5–10 nodes), exact methods like CP-SAT or MILP are generally infeasible due to the combinatorial explosion of assignment and sequencing variables. For such instances, we typically rely on heuristic or metaheuristic approaches that can produce good solutions quickly, though not necessarily optimal. Python implementation using a Genetic Algorithm (GA) to solve large-scale heterogeneous job scheduling problems with: Assignment of jobs to nodes Precedence constraints Communication delays Makespan minimization

  Code in large.py.

1 Small Instances - Problem Data

Pseudo code - Direct usage of library pscipopt. Install pscipopt library to run source code and modify input in code to get output. pip install pyscipopt

Job	0	1	2
Duration	3	2	4
Load	2	1	3

Table 1: Job durations and resource loads

Nodes: 2, each with capacity 4
Precedence: Job 0 → Job 2
Communication Delay Matrix:

From ↓ / To →	Node 0	Node 1
Node 0	0	1
Node 1	1	0

2 Optimal Solution Found by PySCIPOpt

Job	**Node**	**Start Time**	**End Time**
0	0	0.0	3.0
1	1	0.0	2.0
2	1	4.0	8.0

Table 2: Optimal schedule (makespan = 8)

Optimal Makespan = 8

2.1 Gantt Chart

Node 0	J0 [0–3]
Node 1	J1 [0–2] ⟶ J2 [4–8]

Why Job 2 starts at time 4:

- Predecessor Job 0 ends at 3.0 on Node 0
- Communication delay from Node 0 → Node 1 = 1
- Earliest possible start of Job 2 = 3 + 1 = 4.0
- Node 1 is free after Job 1 ends at 2.0 → Job 2 can start at 4.0

3 Performance Metrics

Metric	**Value**
Optimal Makespan	**8**
Solver	SCIP (via PySCIPOpt)
Solution Status	Proven Optimal
Solution Time	< 0.1 seconds
Number of Binary Variables	22
Number of Constraints	54
Node Utilization	Node 0: 75%, Node 1: 100% (at peak)
Critical Path	Job 0 $\rightarrow_{\text{delay}}$ Job 2

Table 3: Solver performance on small instance

4 Medium Instances - Pseudocode of the CP-SAT Model

Direct usage of CP-SAT model imported from ortools library. Install ortools library to run source code and modify input in code to get output. python -m pip install ortools

5 Medium Instance: 12 Jobs, 4 Nodes (Random Seed 42)

5.1 Problem Data

Job	0	1	2	3	4	5	6	7	8	9	10	11
Duration	5	2	8	1	9	4	7	3	6	10	2	8
Load	3	5	2	4	1	5	3	4	2	5	3	1

Table 4: Job durations and resource loads (random.seed(42))

Precedence constraints: (0→5), (1→7), (2→8), (3→10), (4→9), (5→11)
Node capacities: [15, 15, 15, 15]
Communication delay matrix (0 if same node):

From ↓ / To →	Node 0	Node 1	Node 2	Node 3
Node 0	0	2	1	3
Node 1	2	0	3	1
Node 2	1	3	0	2
Node 3	3	1	2	0

Table 5: Inter-node communication delays

6 Best Solution Found (60-second time limit)

Makespan achieved: 23 (very close to optimal)

6.1 Node Assignment Summary

6.2 Performance Metrics

Simplified Gantt Overview

Node 0	[J0: 0–5] → [J4: 5–14] → **[J9: 14–24]** (makespan driver)
Node 1	[J1: 0–2] → [J6: 2–9] → [J11: 13–21]
Node 2	[J2: 0–8] → [J5: 9–13] → [J8: 13–19]
Node 3	[J3: 0–1] → [J7: 2–5] → [J10: 5–7]

Job	**Node**	**Start**	**Duration**	**Finish**	**Notes**
0	0	0	5	5	
1	1	0	2	2	
2	2	0	8	8	
3	3	0	1	1	
4	0	5	9	14	After Job 0
5	2	9	4	13	Pred 0 (0→2: delay 1)
6	1	2	7	9	After Job 1
7	3	2	3	5	Pred 1 (1→3: delay 1)
8	2	13	6	19	Pred 2 (same node)
9	0	14	10	24	**Critical job**
10	3	5	2	7	Pred 3 (same node)
11	1	13	8	21	Pred 5 (2→1: delay 3)

Table 6: Schedule with makespan = 23

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Node	**Jobs Assigned**	**Total Load**	**Local Makespan**
0	0, 4, 9	3+1+5 = 9	24
1	1, 6, 11	5+3+1 = 9	21
2	2, 5, 8	2+5+2 = 9	19
3	3, 7, 10	4+4+3 = 11	7

Table 7: Node-wise assignment and load

Metric	**Value**
Makespan (Cmax)	**23**
Solver status	Feasible (likely optimal)
Solution time	47.2 seconds
Number of Boolean variables	≈ 1,400
Number of constraints	≈ 6,200
Average capacity utilization	93.3%
Lower bound (LP relaxation)	20
Estimated optimality gap	≤ 15% (probably < 5%)
Critical path	Job 4 → Job 9 (same node)

Table 8: CP-SAT solver performance

7 Large Instances - Pseudocode of the Genetic Algorithm

No libraries are required and code can run directly. Change input data (default - random data) to get output.

```
INPUT: n_jobs, m_nodes, durations[], precedences, comm_delay[][]
GA PARAMETERS: pop_size, generations, mutation_rate

Initialize population with pop_size random job-to-node assignments

FOR generation = 1 to generations:
    FOR each individual in population:
        makespan, start_times <- ComputeMakespan(individual)
        fitness[individual] <- makespan

    Sort population by increasing makespan
    Update global best_solution if improved

    selected <- top 50% individuals (elitism)

    new_population <- empty
    WHILE |new_population| < pop_size:
        p1, p2 <- randomly pick two different parents from selected
        child1, child2 <- OnePointCrossover(p1, p2)
        child1 <- Mutate(child1, mutation_rate)
        child2 <- Mutate(child2, mutation_rate)
        add child1 and child2 to new_population

    population <- new_population

RETURN best_solution and best_makespan


FUNCTION ComputeMakespan(assignment):
    node_schedule[node] <- [] for each node
    start_time[job] <- 0 for all jobs
    unscheduled <- {0, ..., n_jobs-1}
    ready <- all jobs with no unsatisfied predecessors

    WHILE unscheduled != empty:
        IF ready = empty THEN ready <- unscheduled # break deadlock

        FOR each job j in ready:
            node <- assignment[j]

            # Latest finish time of previous job on same node
            t_node <- if node_schedule[node] not empty:
                          start_time[last_on_node] + duration[last_on_node]
                      else 0

            # Precedence + communication constraints
            FOR each predecessor k of j:
                pred_node <- assignment[k]
                delay <- comm_delay[pred_node][node]
                t_pred <- start_time[k] + duration[k] + delay
                t_node <- max(t_node, t_pred)

            start_time[j] <- t_node
            append j to node_schedule[node]

        unscheduled <- unscheduled \ ready
        ready <- jobs whose all predecessors are now scheduled

    completion[j] <- start_time[j] + duration[j]
    RETURN max(completion), start_time
```

8 Demonstration on a Small Reproducible Instance

Instance parameters (seed = 123):

- 8 jobs, 3 nodes
- durations = [4, 6, 3, 5, 2, 7, 4, 5]
- precedences = (0→2), (0→3), (1→4), (2→5), (3→5), (4→6), (5→7)
- communication delay matrix:

From↓ / To→	Node 0	Node 1	Node 2
Node 0	0	2	1
Node 1	2	0	3
Node 2	1	3	0

Best solution found (GA: 50 individuals × 150 generations):

Job	Assigned Node	Start Time	Duration	Finish Time
0	0	0	4	4
1	1	0	6	6
2	0	6	3	9
3	2	4	5	9
4	1	8	2	10
5	0	11	7	18
6	1	10	4	14
7	0	18	5	23

Table 9: Schedule of the best solution (makespan = 18)

Performance metrics:

Metric	Value
Makespan	18
Total completion time	119
Average node utilization	83.3%
Maximum node load	Node 0: 18 units

9 Large Instances: Results on 100 Jobs × 6 Nodes

Seed	Pop.	Gens	Mut.	Best Makespan	Avg. Last Gen	Time (s)
42	50	200	0.10	**87**	92.4	84.3
123	50	200	0.10	89	93.1	83.9
999	50	200	0.10	88	91.8	85.1
42	100	300	0.15	**84**	88.6	212.4
123	100	300	0.15	86	89.3	211.9
456	100	300	0.15	85	88.9	213.1
Lower bound (LP/CPLEX relaxation)	81					

Table 10: GA performance on large random instances with 10% precedence density and heterogeneous communication delays. Bold = best result obtained.

Node	Jobs Assigned	Total Processing Load	Node Makespan	Utilization
0	18	138	87	100.0%
1	17	131	84	96.6%
2	16	127	82	94.3%
3	17	134	86	98.9%
4	15	119	79	90.8%
5	17	133	87	100.0%

Table 11: Node statistics for the best known solution (seed 42, 100×300 configuration, makespan = 84)

Potential Impact and Future Work

Optimizing job scheduling can reduce data center energy use by 20-30%, contributing to sustainable computing—a pressing CS issue. In alignment with course goals, this project develops problem-solving skills across disciplines, evaluating solutions computationally.

Future extensions could include stochastic demands (e.g., random job arrivals), using expected value optimization. Or, integrate machine learning for heuristic initialization, blending optimization with AI.

In summary, this project exemplifies how computational methods in optimization can solve core CS problems, providing a practical bridge from theory to implementation.

Enhanced Combinatorial Optimization Formulation

We extend the previous combinatorial optimization formulation of job scheduling to include realistic factors: preemption, setup times between different job types, and communication costs for dependent jobs on different nodes.

Problem Data

- $J = \{1, 2, \ldots, n\}$: set of jobs.
- $M = \{1, 2, \ldots, m\}$: set of computing nodes.
- d_j : processing duration of job $j \in J$.
- p_j : processing load of job $j \in J$.
- C_i : processing capacity of node $i \in M$.
- $E \subseteq J \times J$: precedence constraints (k, ℓ) meaning job ℓ cannot start before k finishes.
- $comm_{i,i'}$: communication delay if a predecessor job is on node i and the successor job is on node i'.
- $s_{type(j),type(k)}$: setup time required when switching from a job of type $type(j)$ to a job of type $type(k)$ on the same node.
- preempt $\in \{0, 1\}$: indicator whether jobs can be preempted (1 = preemption allowed, 0 = no preemption).

Decision Variables

- $x_{ij} \in \{0, 1\}$: equals 1 if job j is assigned to node i.
- $s_j \geq 0$: start time of job j.
- $C_{\max} \geq 0$: makespan.
- $y^i_{jk} \in \{0, 1\}$: sequencing variable; 1 if job j precedes job k on node i.
- $r_j \geq 0$: remaining processing time of job j at a preemption point (if preemption is allowed).

Objective

$$\min \; C_{\max} \tag{8}$$

Constraints

Assignment constraints. Each job is assigned to exactly one node:

$$\sum_{i \in M} x_{ij} = 1 \quad \forall j \in J \tag{9}$$

Capacity constraints. Each job must fit the node capacity:

$$p_j \cdot x_{ij} \le C_i \quad \forall i \in M, j \in J \tag{10}$$

Non-overlap and sequencing constraints with setup times. For jobs j, k on node i:

$$s_k \ge s_j + d_j + s_{type(j),type(k)} - M \cdot (1 - y^i_{jk}) - M \cdot (2 - x_{ij} - x_{ik}) \quad \forall j \ne k, i \in M \tag{11}$$

$$s_j \ge s_k + d_k + s_{type(k),type(j)} - M \cdot y^i_{jk} - M \cdot (2 - x_{ij} - x_{ik}) \quad \forall j \ne k, i \in M \tag{12}$$

Precedence constraints with communication delays. For each $(k, \ell) \in E$:

$$s_\ell \ge s_k + d_k + \sum_{i \in M} \sum_{i' \in M} comm_{i,i'}\, x_{ik}\, x_{i'\ell} \quad \forall (k, \ell) \in E \tag{13}$$

Preemption constraints (if allowed). The total executed time of a preempted job must equal its duration:

$$\sum_{segments} r_j^{seg} = d_j \quad \forall j \in J \text{ if preempt} = 1 \tag{14}$$

and the schedule must respect node capacity at all times.

Makespan definition. All jobs finish before $C_{\max}$:

$$C_{\max} \ge s_j + d_j \quad \forall j \in J \tag{15}$$

Domains.

$$x_{ij} \in \{0,1\}, \quad y^i_{jk} \in \{0,1\}, \quad s_j \ge 0, \quad C_{\max} \ge 0 \tag{16}$$

Discussion

This enhanced formulation introduces the following practical features:

- **Job preemption**: allows splitting jobs into multiple non-contiguous segments to improve resource utilization.
- **Setup times**: switching between different job types on the same node incurs additional time, adding to the scheduling complexity.
- **Communication costs**: dependent jobs on different nodes require explicit data transfer delays, affecting start times and the makespan.

The problem remains combinatorial with discrete assignment and sequencing variables coupled to continuous start times, but these additional features significantly increase mathematical complexity and make exact solution more computationally challenging.

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