Law of sines

Earlier we learned how to complete a right triangle (find all three side lengths and all three interior angle measures). So at this point, we only know how to complete a triangle when it's a right triangle. But now we'll learn how to complete an oblique triangle, which is a triangle that isn't right. We'll do it using the law of sines.

For any triangle with vertices A, B and C, where side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C, the law of sines says

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

One of the scenarios in which you can complete a triangle by applying the law of sines is when you know the lengths of two sides of the triangle and the measure of the interior angle opposite one of those two sides.

Example

A triangle has one side with length 3 and another with length 5. The angle opposite the side with length 5 is 40°. Complete the triangle.

Let $a = 3$ and $b = 5$, and let $B = 40^\circ$ be the interior angle opposite b. Substituting these values into the law of sines gives

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{3}{\sin\ A} = \frac{5}{\sin 40^\circ} = \frac{c}{\sin\ C}$$

Focusing just on

$$\frac{3}{\sin\ A} = \frac{5}{\sin 40^\circ}$$

we get

$$\sin\ A = \frac{3(\sin 40^\circ)}{5} = \frac{3}{5}(\sin 40^\circ)$$

$$\sin\ A \approx \frac{3}{5}(0.643) \approx 0.386$$

$$A \approx \arcsin(0.386)$$

$$A \approx 22.7^\circ$$

Since the sum of the interior angles of any triangle is 180°, the measure of the third interior angle C is approximately

$$180^\circ - (40^\circ + 22.7^\circ)$$

$$180^\circ - 62.7^\circ$$

$$117.3^\circ$$

To find the length of the third side c, we'll solve

$$\frac{5}{\sin 40^\circ} = \frac{c}{\sin\ C}$$

$$\frac{5}{\sin 40^\circ} \approx \frac{c}{\sin 117.3^\circ}$$

$$c \approx \left(\frac{5}{\sin 40^\circ}\right)(\sin 117.3^\circ) \approx \frac{5(0.889)}{0.643} \approx 6.91$$

The side lengths of the triangle are $a = 3$, $b = 5$, and $c \approx 6.91$, and the angle measures are $A \approx 22.7^\circ$, $B = 40^\circ$, and $C = 117.3^\circ$.

You can also solve a triangle with the law of sines when you know the measures of any two interior angles and the length of one of the sides.

Example

A triangle has one angle that's 38° and another that's 64°. The length of the side opposite the third angle is 55. Solve the triangle and figure out how many different triangles have these properties.

Since the sum of the two known interior angles is $38^\circ + 64^\circ = 102^\circ$, we know that the measure of the third angle is

$$180^\circ - 102^\circ = 78^\circ$$

Since there is only one possible value for the measure of the third interior angle, there can't be more than one triangle with the given properties.

Let angle A be the 38° angle and let B be the 64° angle. That means that $c = 55$. Plugging all of this into the law of sines gives

$$\frac{a}{\sin 38^\circ} = \frac{b}{\sin 64^\circ} = \frac{55}{\sin 78^\circ}$$

We'll focus just on the left part of this equation to solve for a.

$$\frac{a}{\sin 38^\circ} = \frac{55}{\sin 78^\circ}$$

$$a = \frac{(\sin 38^\circ)(55)}{\sin 78^\circ}$$

$$a \approx \frac{0.616(55)}{0.978} \approx 34.6$$

We'll focus on the right part of the equation to solve for b.

$$\frac{b}{\sin 64^\circ} = \frac{55}{\sin 78^\circ}$$

$$b = \frac{(\sin 64^\circ)(55)}{\sin 78^\circ}$$

$$b \approx \frac{0.899(55)}{0.978} \approx 50.6$$

Sometimes, the properties that are given for a triangle are inconsistent, which makes it impossible for the triangle to exist.

Example

Find all the triangles in which there are sides of length 17 and 25, and the interior angle which is opposite the side of length 17 is 80°.

Let $a = 17$ and $b = 25$, and let angle $A = 80^\circ$ be the angle opposite side a. Plugging all of this into the law of sines gives

$$\frac{17}{\sin 80^\circ} = \frac{25}{\sin B} = \frac{c}{\sin C}$$

Focusing on the left part of the equation, we get

$$\frac{17}{\sin 80^\circ} = \frac{25}{\sin B}$$

$$\sin B = \frac{25(\sin 80^\circ)}{17}$$

$$\sin B \approx \frac{25}{17}(0.985) \approx 1.45$$

Since the sine of an angle can't be greater than 1, this "triangle" is impossible.