Law of cosines

We talked about the law of sines, but there's also a law of cosines. For any triangle with vertices A, B and C, where side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C, the law of cosines comes in three parts:

$$c^2 = a^2 + b^2 - 2ab\ \cos \text{C}$$

$$b^2 = a^2 + c^2 - 2ac\ \cos \text{B}$$

$$a^2 = b^2 + c^2 - 2bc\ \cos \text{A}$$

Notice that each one of these parts takes the cosine of a different angle, and is solved for a different variable. The part we'll want to use therefore depends on the information we have.

Let's look at an example where we use the law of cosines to complete a triangle when we know all three side lengths, but none of the interior angles.

Example

Solve the triangle that has side lengths 11, 6, and 9.

Because we know all three side lengths, we can simply plug those values into each of the three parts of the law of cosines. In each part, we'll be left only with the cosine of one angle, so we'll be able to use each part to solve for one of the angles of the triangle.

Remember that once we have the first two angles, we can simply find the third angle by subtracting the first two from 180°, instead of using the third part of the law of cosines to find it.

If we let $a = 11$, $b = 6$, and $c = 9$, then for angle A we'll use

$$a^2 = b^2 + c^2 - 2bc\cos \mathsf{A}$$

Solving for $\cos \mathsf{A}$ we get

$$\cos \mathsf{A} = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plugging in the side lengths gives

$$\cos \mathsf{A} = \frac{11^2 - (6^2 + 9^2)}{-2(6)(9)}$$

$$\cos \mathsf{A} = \frac{121 - (36 + 81)}{-12(9)}$$

$$\cos \mathsf{A} = \frac{121 - 117}{-108}$$

$$\cos \mathsf{A} = -\frac{4}{108}$$

$$\cos \mathsf{A} \approx -0.0370$$

$$\mathsf{A} \approx \arccos(-0.0370)$$

$$\mathsf{A} \approx 92.1^\circ$$

For angle C we'll use

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Solving for $\cos C$ we get

$$\cos C = \frac{c^2 - (a^2 + b^2)}{-2ab}$$

Plugging in the side lengths gives

$$\cos C = \frac{9^2 - (11^2 + 6^2)}{-2(11)(6)}$$

$$\cos C = \frac{81 - (121 + 36)}{-22(6)}$$

$$\cos C = \frac{81 - 157}{-132}$$

$$\cos C = \frac{76}{132}$$

$$\cos C \approx 0.576$$

$$C \approx \arccos(0.5760)$$

$$C \approx 54.8^\circ$$

To get angle B, we'll subtract the other two angles from 180°.

$$B = 180^\circ - A - C$$

$$B \approx 180^\circ - 92.1^\circ - 54.8^\circ$$

$$B \approx 33.1^\circ$$

Let's look at how to use the law of cosines to prove that a triangle with given measurements exists.

Example

Determine whether there exists a triangle with sides of length 18, 15, and 2.

Let $a=18$, $b=15$, and $c=2$. If we were to apply the law of cosines and solve for $\cos C$ using

$$\cos C = \frac{c^2-(a^2+b^2)}{-2ab}$$

we'd get

$$\cos C = \frac{2^2-(18^2+15^2)}{-2(18)(15)}$$

$$\cos C = \frac{4-(324+225)}{-36(15)}$$

$$\cos C = \frac{4-549}{-540}$$

$$\cos C = \frac{545}{540}$$

$$\cos C \approx 1.01$$

But remember, the cosine function as a range of $[-1,1]$. So it's impossible to find $\cos \text{C} \approx 1.01$. In other words, if you ever find that $\cos \text{A}$, $\cos \text{B}$, or $\cos \text{C}$ is equal to some value outside the interval $[-1,1]$, then you know right way that the triangle can't exist.

The other scenario in which you can complete a triangle by applying the law of cosines is when you know the lengths of any two sides of the triangle and the measure of the "included angle" which is the angle between those two sides. This would be like knowing side a, side b, and angle C.

Example

Solve the triangle where two of the sides are 25 and 21 and the measure of the included angle is 70°.

Let $a = 25$ and $b = 21$. The included angle will be $C = 70^\circ$.

$$c^2 = a^2 + b^2 - 2ab \cos \text{C}$$

We'll first solve this equation for c. Plugging in what we know, we get

$$c^2 = 25^2 + 21^2 - \left[2(25)(21)(\cos 70^\circ)\right]$$

$$c^2 = 625 + 441 - \left[50(21)(\cos 70^\circ)\right]$$

$$c^2 = 1{,}066 - (1{,}050 \cos 70^\circ)$$

$$c^2 \approx 1{,}066 - (1{,}050)(0.342)$$

$$c^2 \approx 1{,}066 - 359 = 707$$

$$c \approx \sqrt{707} \approx 26.6$$

Now that we have the lengths of all three sides, we can check to be sure that such a triangle exists. As we saw in the last example, we could find the cosine of each angle to ensure that each is inside the interval $[-1,1]$. But there's another test we can use.

If we find the absolute value of the difference of each combination of side lengths. The difference has to be less than the length of the other, third side, AND the sum of those same two sides has to be greater than the length of the other, third side.

In other words, the side lengths of this triangle are $a = 25$, $b = 21$, and $c \approx 26.6$. So $|25 - 21|$ must be less than $c \approx 26.6$, AND $25 + 21$ has to be greater than $c \approx 26.6$. If we check this for each combination of sides (a and b, a and c, and b and c), and all these relationships check out, then we know the triangle exists.

$$|a - b| = |25 - 21| = 4 < 26.6 \approx c \approx 26.6 < 46 = 25 + 21 = a + b$$

$$|a - c| \approx |25 - 26.6| = 1.6 < 21 = b = 21 < 51.6 = 25 + 26.6 \approx a + c$$

$$|b - c| \approx |21 - 26.6| = 5.6 < 25 = a = 25 < 47.6 = 21 + 26.6 \approx b + c$$

We see that it passes all three tests, so such a triangle does exist.

To get the measure of angle B, we'll apply

$$b^2 = a^2 + c^2 - 2ac\cos \mathsf{B}$$

$$\cos \mathsf{B} = \frac{b^2 - (a^2 + c^2)}{-2ac}$$

Plugging in the side lengths gives

$$\cos \mathsf{B} \approx \frac{21^2 - (25^2 + 26.6^2)}{-2(25)(26.6)}$$

$$\cos \mathsf{B} \approx \frac{441 - (625 + 708)}{-50(26.6)}$$

$$\cos \mathsf{B} \approx \frac{441 - 1{,}333}{-1{,}330}$$

$$\cos \mathsf{B} \approx \frac{892}{1{,}330}$$

$$\cos \mathsf{B} \approx 0.671$$

$$\mathsf{B} \approx \arccos(0.671)$$

$$\mathsf{B} \approx 47.9^\circ$$

Which means angle A is

$$180^\circ - 70^\circ - 47.9^\circ$$

$$180^\circ - 117.9^\circ = 62.1^\circ$$

$$62.1^\circ$$