Find the value or say where it's undefined

First, let's remember that there are six trig functions, two of which are $\sin\theta$ and $\cos\theta$, and that the other four can all be defined in terms of $\sin\theta$ and $\cos\theta$.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

The only angles θ where one or more of these six functions is undefined are the angles where either $\sin\theta = 0$ or $\cos\theta = 0$. The only points where $\sin\theta = 0$ are along the x-axis, and the only points where $\cos\theta = 0$ are along the y-axis.

So if we have any point that isn't along one of the major axes, we automatically know that all six of the trig functions are defined.

For these kinds of problems, we'll often use the reciprocal identities, which are

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Example

Determine the values of the trig functions that are defined at $3\pi/2$ radians, and state which of the six circular functions (if any) are undefined there.

An angle of $3\pi/2$ radians is on the negative y-axis, so it intersects the unit circle at $(0, -1)$. Therefore,

$$\sin\frac{3\pi}{2} = y = -1$$

$$\cos\frac{3\pi}{2} = x = 0$$

Using reciprocal identities, we get

$$\sec\frac{3\pi}{2} = \frac{1}{\cos\frac{3\pi}{2}} = \frac{1}{0} \Longrightarrow \sec\frac{3\pi}{2} \text{ is undefined}$$

$$\csc\frac{3\pi}{2} = \frac{1}{\sin\frac{3\pi}{2}} = \frac{1}{-1} = -1$$

The tangent and cotangent functions at $3\pi/2$ are

$$\tan\frac{3\pi}{2} = \frac{\left(\sin\frac{3\pi}{2}\right)}{\left(\cos\frac{3\pi}{2}\right)} = \frac{-1}{0} \Longrightarrow \tan\frac{3\pi}{2} \text{ is undefined}$$

$$\cot\frac{3\pi}{2}=\frac{\cos\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2}\right)}=\frac{0}{-1}=0$$

Now we'll consider an angle that's in one of the quadrants, so all six of the circular functions are defined there.

Example

Determine the values of the six circular functions at $5\pi/4$ radians.

Looking at the unit circle, we know that

$$\sin\frac{5\pi}{4}=y=-\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4}=x=-\frac{\sqrt{2}}{2}$$

Which means that the other four trig functions are

$$\sec\frac{5\pi}{4}=\frac{1}{\cos\frac{5\pi}{4}}=\frac{1}{\left(-\frac{\sqrt{2}}{2}\right)}=-\sqrt{2}$$

$$\csc\frac{5\pi}{4}=\frac{1}{\sin\frac{5\pi}{4}}=\frac{1}{\left(-\frac{\sqrt{2}}{2}\right)}=-\sqrt{2}$$

$$\tan\frac{5\pi}{4}=\frac{\left(\sin\frac{5\pi}{4}\right)}{\left(\cos\frac{5\pi}{4}\right)}=\frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)}=1$$

$$\cot\frac{5\pi}{4}=\frac{1}{\tan\frac{5\pi}{4}}=\frac{1}{1}=1$$

We haven't yet dealt with any angle that's an integer multiple of $\pi/6$ radians but isn't on any of the coordinate axes, so we'll do that now.

Example

Find the values of the circular functions at $\pi/3$ radians, and state which of the six circular functions (if any) are undefined there.

From the unit circle, we know that

$$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3}=\frac{1}{2}$$

Therefore, the values of the other four circular functions at $\pi/3$ are

$$\sec\frac{\pi}{3}=\frac{1}{\cos\frac{\pi}{3}}=\frac{1}{\left(\frac{1}{2}\right)}=2$$

$$\csc\frac{\pi}{3}=\frac{1}{\sin\frac{\pi}{3}}=\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{2\sqrt{3}}{3}$$

$$\tan\frac{\pi}{3}=\frac{\left(\sin\frac{\pi}{3}\right)}{\left(\cos\frac{\pi}{3}\right)}=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=\sqrt{3}$$

$$\cot\frac{\pi}{3}=\frac{1}{\tan\frac{\pi}{3}}=\frac{\sqrt{3}}{3}$$

Now let's look at the values of the trig functions at an angle θ that isn't in the interval $[0,2\pi]$.

Example

Find the values of the circular functions that are defined for an angle of $-11\pi/3$ radians, and state which of the six circular functions (if any) are undefined there.

A full 2π rotation, if we express it in thirds is $6\pi/3$, so we know that $-11\pi/3$ is more than one full rotation, but less than two full rotations since two full rotations would be $12\pi/3$. Since the angle is negative, we're rotating clockwise. We rotate a full rotation to get to $-6\pi/3$, and then continue on an additional $-5\pi/3$ to get to $-11\pi/3$. So we're looking for an angle that's coterminal with $-5\pi/3$. To find it, we'll add this to 2π.

$$2\pi - \frac{5\pi}{3}$$

$$\frac{6\pi}{3} - \frac{5\pi}{3}$$

$$\frac{\pi}{3}$$

So $\pi/3$ is coterminal with $-11\pi/3$, and we can therefore evaluate all six trig functions at $\pi/3$.

$$\sin\left(-\frac{11\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{11\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\csc\left(-\frac{11\pi}{3}\right) = \csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$$

$$\sec\left(-\frac{11\pi}{3}\right) = \sec\frac{\pi}{3} = 2$$

$$\tan\left(-\frac{11\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\cot\left(-\frac{11\pi}{3}\right) = \cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$$