Topic: Proving the trig equation

Question: Which of the following is a trigonometric identity?

Answer choices:

A $\tan^2\theta = 0.5$

B $\sec\theta + \tan\theta = \sqrt{2}$

C $2\cos^2\theta + 3\sin^2\theta = 2 + \sin^2\theta$

D $\csc\theta\sin^2\theta = 1$

Solution: C

We claim that the equation in answer choice C is a trigonometric identity. We'll prove that identity, but first we'll show that the equations in A, B, and D aren't trigonometric identities.

To see that the equation in A, $\tan^2\theta = 0.5$, isn't an identity, consider $\theta = 0^\circ$ and recall that $\sin 0^\circ = 0$ and $\cos 0^\circ = 1$. Therefore,

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

From this it follows that

$$\tan^2 0^\circ = 0 \neq 0.5$$

For the equation in B, $\sec\theta + \tan\theta = \sqrt{2}$, let $\theta = \pi/4$ (radians). Recall that

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$$

Therefore,

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

and

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)} = 1$$

Thus we obtain

$$\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)=\sqrt{2}+1\neq\sqrt{2}$$

For the equation in D, $\csc\theta\sin^2\theta=1$, let $\theta=\pi/3$. Recall that

$$\csc\theta=\frac{1}{\sin\theta}$$

is a reciprocal identity and that

$$\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Therefore,

$$\csc\left(\frac{\pi}{3}\right)=\frac{1}{\sin\left(\frac{\pi}{3}\right)}=\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{2}{\sqrt{3}}$$

Substituting these results:

$$\csc\left(\frac{\pi}{3}\right)\sin^2\left(\frac{\pi}{3}\right)=\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)^2=\left(\frac{2}{\sqrt{3}}\right)\left(\frac{3}{4}\right)=\frac{\sqrt{3}}{2}\neq 1$$

Now we'll prove that the equation in C is an identity. We'll start from the left-hand side of the equation:

$2\cos^2\theta+3\sin^2\theta \qquad =2\cos^2\theta+(2\sin^2\theta+\sin^2\theta)$ Write $3\sin^2\theta$ as $2\sin^2\theta+\sin^2\theta$.

$= (2\cos^2\theta + 2\sin^2\theta) + \sin^2\theta$	Regroup.
$= 2(\cos^2\theta + \sin^2\theta) + \sin^2\theta$	Common factor of 2.
$= 2(1) + \sin^2\theta$	Use $\cos^2\theta + \sin^2\theta = 1$.
$= 2 + \sin^2\theta$	Simplify.

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Question: Which of the following is a trigonometric identity?

Answer choices:

A $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

B $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = \sin\theta$

C $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = \cos\theta$

D $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = \csc\theta$

Solution: A

The left-hand side of all four equations is the same. We will prove that the equation in A is an identity. Then we will exhibit an angle θ at which the expressions on both sides of the other three equations are defined and the right-hand side is unequal to 1.

$(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = \csc^2\theta - \cot^2\theta$ Carry out indicated multiplication.

$= (1 + \cot^2\theta) - \cot^2\theta$ Use $\csc^2\theta = 1 + \cot^2\theta$.

$= 1$ Simplify.

To show that the other three equations are not identities, consider $\theta = \pi/6$ (radians) and recall the following:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Therefore,

$$\csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{6}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

and

$$\cot\left(\frac{\pi}{6}\right) = \frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$$

We have shown that $\csc(\pi/6)$ and $\cot(\pi/6)$ are defined, hence

$$\left[\csc\left(\frac{\pi}{6}\right) + \cot\left(\frac{\pi}{6}\right)\right]\left[\csc\left(\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{6}\right)\right]$$

is also defined. Moreover, at $\theta = \pi/6$, the values of the expressions on the right-hand sides of the equations in B, C, and D ($\sin(\pi/6)$, $\cos(\pi/6)$, and $\csc(\pi/6)$, respectively) are all unequal to 1. Therefore, those three equations are not identities, because the equation in A is an identity (and its right-hand side is equal to 1).

Topic: Proving the trig equation

Question: Which of the following is not a trigonometric identity?

Answer choices:

A $\sec\theta\sin\theta = \tan\theta$

B $\sin\theta(\cot\theta + \tan\theta) = \cos\theta$

C $\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$

D $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1$

Solution: B

First, we'll show that the equation in B, $\sin\theta(\cot\theta+\tan\theta)=\cos\theta$, is not a trigonometric identity. Let $\theta=\pi/6$, and recall that

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

and

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

By the definitions of $\tan\theta$ and $\cot\theta$,

$$\tan\left(\frac{\pi}{6}\right)=\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{1}{\sqrt{3}}$$

and

$$\cot\left(\frac{\pi}{6}\right)=\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=\sqrt{3}$$

Therefore,

$$\sin\left(\frac{\pi}{6}\right)\left[\cot\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{6}\right)\right]$$

$$= \left(\frac{1}{2}\right)\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{2}\right)\left[\frac{\sqrt{3}(\sqrt{3}) + 1}{\sqrt{3}}\right]$$

$$= \left(\frac{1}{2}\right)\frac{4}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \neq \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$$

Now we'll prove that the other three equations are indeed trigonometric identities. First, the equation in A: $\sec\theta\sin\theta = \tan\theta$.

$\sec\theta\sin\theta$	$= \frac{1}{\cos\theta}(\sin\theta)$	Use $\sec\theta = \frac{1}{\cos\theta}$
	$= \tan\theta$	Use $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Next, the equation in C: $\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$.

$\sec^4\theta - \sec^2\theta$	$= (\sec^2\theta)^2 - \sec^2\theta$	Write $\sec^4\theta$ as $(\sec^2\theta)^2$
	$= (1 + \tan^2\theta)^2 - (1 + \tan^2\theta)$	Use $\sec^2\theta = 1 + \tan^2\theta$
	$= (1 + 2\tan^2\theta + \tan^4\theta) - (1 + \tan^2\theta)$	Expand $(1 + \tan^2\theta)^2$
	$= (1 - 1) + (2\tan^2\theta - \tan^2\theta) + \tan^4\theta$	Regroup
	$= \tan^2\theta + \tan^4\theta$	Simplify

Finally, the equation in D: $\cos^2\theta\tan^2\theta+\sin^2\theta\cot^2\theta=1$.

$$\cos^2\theta\tan^2\theta+\sin^2\theta\cot^2\theta = \cos^2\theta\left(\frac{\sin\theta}{\cos\theta}\right)^2+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)^2$$

Use $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

$$=\cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right)+\sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

Carry out indicated squaring.

$$=\sin^2\theta+\cos^2\theta$$

Cancel $\cos^2\theta$ in first term, and $\sin^2\theta$ in second term.

$$=1$$

Use $\sin^2\theta+\cos^2\theta=1$