Functions of negative angles

We already know that the value of a trig function is the same for any pair of coterminal angles. In other words, given the coterminal angles 60° and 420°, we know that

$$\sin 60^\circ = \sin 420^\circ \qquad \csc 60^\circ = \csc 420^\circ$$

$$\cos 60^\circ = \cos 420^\circ \qquad \sec 60^\circ = \sec 420^\circ$$

$$\tan 60^\circ = \tan 420^\circ \qquad \cot 60^\circ = \cot 420^\circ$$

These are true simply because 60° and 420° are coterminal. What we want to remember is that this applies to negative angles as well. For example, the angle -300° is coterminal with 60°, which means that all six trig functions will have the same value at those angles as well.

$$\sin 60^\circ = \sin 420^\circ = \sin(-300^\circ) \qquad \csc 60^\circ = \csc 420^\circ = \csc(-300^\circ)$$

$$\cos 60^\circ = \cos 420^\circ = \cos(-300^\circ) \qquad \sec 60^\circ = \sec 420^\circ = \sec(-300^\circ)$$

$$\tan 60^\circ = \tan 420^\circ = \tan(-300^\circ) \qquad \cot 60^\circ = \cot 420^\circ = \cot(-300^\circ)$$

Example

Find the angle in the interval $[0^\circ,360^\circ)$ that's coterminal with -539°.

Let $\theta = -539^\circ$, and let α be the angle that lies in the interval $[0^\circ, 360^\circ)$ and is coterminal with θ. To find α, let's add 360° to $\theta = -539^\circ$ until we get to an angle that lies in the interval $[0^\circ, 360^\circ)$.

$$-539^\circ + 360^\circ = -179^\circ$$

$$-179^\circ + 360^\circ = 181^\circ$$

This is the angle that's coterminal with $\theta = -539^\circ$ that also lies in the interval $[0^\circ, 360^\circ)$.

We weren't very many full rotations away from 360°, but this might not always be the case. If we're further away from $[0^\circ, 360^\circ)$, we could instead set up the inequality

$$0 \le -539^\circ + n(360^\circ) < 360^\circ$$

where n is the number of full rotations that we are away from the interval $[0^\circ, 360^\circ)$. The first goal would be to solve for n. We'll add 539° to all three quantities, then divide through by 360°,

$$539^\circ \le n(360^\circ) < 899^\circ$$

$$\frac{539^\circ}{360^\circ} \le n < \frac{899^\circ}{360^\circ}$$

about $1.5 \le n <$ about 2.5

Because n has to be an integer, we know that n can only be 2. To find α, we'll substitute $n = 2$ into $\theta + n(360^\circ)$.

$$\alpha = \theta + 2(360^\circ)$$

$$\alpha = -539^\circ + 2(360^\circ)$$

$$\alpha = -539^\circ + 720^\circ$$

$$\alpha = 181^\circ$$

This is the more technical way to find the coterminal angle inside $[0^\circ, 360^\circ)$.

We can follow the same process for radians, except that we'll use $\alpha = \theta + n(2\pi)$ instead of $\alpha = \theta + n(360^\circ)$.

Example

Find the angle in the interval $[0,2\pi)$ that's coterminal with $-(61/4)\pi$.

Let $\theta = -(61/4)\pi$, and let α be the angle that lies in the interval $[0,2\pi)$ and is coterminal with θ. Then

$$0 \le \alpha < 2\pi$$

$$0 \le \theta + n(2\pi) < 2\pi$$

Substituting $\theta = -(61/4)\pi$ gives

$$0 \le -\frac{61}{4}\pi + n(2\pi) < 2\pi$$

$$\frac{61}{4}\pi \le n(2\pi) < 2\pi + \frac{61}{4}\pi$$

$$\frac{61}{4}\pi \le n(2\pi) < \left(2+\frac{61}{4}\right)\pi$$

$$\frac{61}{4}\pi \le n(2\pi) < \left[\frac{4(2)+61}{4}\right]\pi$$

$$\frac{61}{4}\pi \le n(2\pi) < \left(\frac{8+61}{4}\right)\pi$$

$$\frac{61}{4}\pi \le n(2\pi) < \frac{69}{4}\pi$$

Dividing through by 2π,

$$\frac{\left(\frac{61}{4}\pi\right)}{2\pi} \le n < \frac{\left(\frac{69}{4}\pi\right)}{2\pi}$$

$$\frac{61}{4(2)} \le n < \frac{69}{4(2)}$$

$$\frac{61}{8} \le n < \frac{69}{8}$$

about $7.63 \le n <$ about 8.63

So $n = 8$, and we can say

$$\alpha = \theta + n(2\pi)$$

$$\alpha = \theta + 8(2\pi)$$

$$\alpha = -\frac{61}{4}\pi + 16\pi$$

$$\alpha = \left(-\frac{61}{4} + 16\right)\pi$$

$$\alpha = \left[\frac{-61 + 4(16)}{4}\right]\pi$$

$$\alpha = \left(\frac{-61 + 64}{4}\right)\pi$$

$$\alpha = \frac{3\pi}{4}$$

We've been looking at finding the coterminal angle within $[0^\circ, 360^\circ)$ or $[0, 2\pi)$. But we can actually look for coterminal angles in any interval.

Example

Find the angle in the interval $(-900^\circ, -540^\circ]$ that's coterminal with 247°.

The interval $(-900^\circ, -540^\circ]$ is a full 360° rotation. Let $\theta = 247^\circ$, and let α be the angle that's coterminal with θ and lies in the interval $(-900^\circ, -540^\circ]$. In this case, we'll use the general relation $\alpha = \theta + n(360^\circ)$ and solve for the unique integer n such that α lies in that interval.

$$-900^\circ < \alpha \leq -540^\circ$$

$$-900^\circ < \theta + n(360^\circ) \leq -540^\circ$$

Substituting $\theta = 247^\circ$ gives

$$-900^\circ < 247^\circ + n(360^\circ) \le -540^\circ$$

Subtracting 247° and then dividing through by 360°, we get

$$-900^\circ - 247^\circ < n(360^\circ) \le -540^\circ - 247^\circ$$

$$-1{,}147^\circ < n(360^\circ) \le -787^\circ$$

$$-\frac{1{,}147^\circ}{360^\circ} < n \le -\frac{787^\circ}{360^\circ}$$

$$\text{about } -3.12 < n \le \text{about } -2.19$$

Therefore, $n = -3$. To find α, we'll substitute $n = -3$ into the expression $\theta + n(360^\circ)$.

$$\alpha = \theta + n(360^\circ)$$

$$\alpha = \theta + (-3)(360^\circ)$$

$$\alpha = 247^\circ + (-3)(360^\circ)$$

$$\alpha = 247^\circ + (-1{,}080^\circ)$$

$$\alpha = -833^\circ$$

Let's do an example in radians.

Example

Find the angle in the interval $[-\pi, \pi)$ that's coterminal with $(56/3)\pi$.

The interval $[-\pi, \pi)$ is a full 2π rotation. Let $\theta = (56/3)\pi$, and let α be the angle that's coterminal with θ and lies in the interval $[-\pi, \pi)$. We'll use $\alpha = \theta + n(2\pi)$ and solve for the value of n that makes α lie in that interval.

$$-\pi \leq \alpha < \pi$$

$$-\pi \leq \theta + n(2\pi) < \pi$$

Substituting $\theta = (56/3)\pi$ gives

$$-\pi \leq \frac{56}{3}\pi + n(2\pi) < \pi$$

Subtracting $(56/3)\pi$ and dividing through by 2π, we get

$$-\pi - \frac{56}{3}\pi \leq n(2\pi) < \pi - \frac{56}{3}\pi$$

$$-\left(1 + \frac{56}{3}\right)\pi \leq n(2\pi) < \left(1 - \frac{56}{3}\right)\pi$$

$$-\left[\frac{3(1) + 56}{3}\right]\pi \leq n(2\pi) < \left[\frac{3(1) - 56}{3}\right]\pi$$

$$-\left(\frac{3 + 56}{3}\right)\pi \leq n(2\pi) < \left(\frac{3 - 56}{3}\right)\pi$$

$$-\frac{59}{3}\pi \leq n(2\pi) < -\frac{53}{3}\pi$$

$$\frac{\left(-\frac{59}{3}\pi\right)}{2\pi} \leq n < \frac{\left(-\frac{53}{3}\pi\right)}{2\pi}$$

$$-\frac{59}{3(2)} \le n < -\frac{53}{3(2)}$$

$$-\frac{59}{6} \le n < -\frac{53}{6}$$

about $-9.83 \le n <$ about -8.83

So $n = -9$. To find α, we'll substitute $n = -9$ into $\theta + n(2\pi)$.

$$\alpha = \theta + n(2\pi)$$

$$\alpha = \theta + (-9)(2\pi)$$

$$\alpha = \frac{56}{3}\pi - 18\pi$$

$$\alpha = \left(\frac{56}{3} - 18\right)\pi$$

$$\alpha = \left[\frac{56 + 3(-18)}{3}\right]\pi$$

$$\alpha = \left(\frac{56 - 54}{3}\right)\pi$$

$$\alpha = \frac{2\pi}{3}$$