Coterminal angles

When you sketch two angles in standard position, they're coterminal if their terminal sides lie on top of each other. In other words, if both angles finish up at the same place, then they're coterminal.

Angles that differ by 360° or 2π radians will always be coterminal. Which means that, to find a coterminal angle for an angle in degrees, we just add or subtract 360° as many times as we want to. For example, if we want to find angles that are coterminal with 45°, we could add 360° over and over, and say that all of these angles are coterminal with 45°:

$$45^\circ + 360^\circ = 405^\circ$$

$$45^\circ + 2(360^\circ) = 765^\circ$$

$$45^\circ + 3(360^\circ) = 1{,}125^\circ$$

$$\dots$$

And we could subtract 360° over and over, and say that all of these angles are coterminal with 45°:

$$45^\circ - 360^\circ = -315^\circ$$

$$45^\circ - 2(360^\circ) = -675^\circ$$

$$45^\circ - 3(360^\circ) = -1{,}035^\circ$$

$$\dots$$

We can also do this with radians. Instead of adding or subtracting some multiple of 360°, we'd add or subtract any multiple of 2π. For example, all of these angles are coterminal with $\pi/6$:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$

$$\frac{\pi}{6} + 2(2\pi) = \frac{25\pi}{6}$$

$$\frac{\pi}{6} + 3(2\pi) = \frac{37\pi}{6}$$

...

and

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$$

$$\frac{\pi}{6} - 2(2\pi) = -\frac{23\pi}{6}$$

$$\frac{\pi}{6} - 3(2\pi) = -\frac{35\pi}{6}$$

...

Example

Find the three smallest positive angles that are coterminal with 67°.

We can add or subtract 360° to find coterminal angles for 67°. If we were to subtract any multiple of 360°, we'd get a negative angle, but we were asked for only positive angles. Therefore, we need to add multiples of 360° in order to find the angles we need.

$$67^\circ + 1(360^\circ) = 67^\circ + 360^\circ = 427^\circ$$

$$67^\circ + 2(360^\circ) = 67^\circ + 720^\circ = 787^\circ$$

$$67^\circ + 3(360^\circ) = 67^\circ + 1{,}080^\circ = 1{,}147^\circ$$

These are the three smallest positive coterminal angles to 67°.

Example

Find the two smallest positive angles that are coterminal with $-(3/2)\pi$.

We can add or subtract 2π to find coterminal angles for $-(3/2)\pi$. If we were to subtract any multiple of 2π, we'd get another negative angle, but we were asked for only positive angles. Therefore, we need to add multiples of 2π in order to find the angles we need.

$$-\frac{3}{2}\pi + 1(2\pi) = -\frac{3}{2}\pi + \frac{4\pi}{2} = \frac{\pi}{2}$$

$$-\frac{3}{2}\pi + 2(2\pi) = -\frac{3}{2}\pi + \frac{8\pi}{2} = \frac{5\pi}{2}$$

These are the two smallest positive coterminal angles to $-(3/2)\pi$.

Now we'll try an example with negative rotations.

Example

Find four negative angles that are coterminal with $(6/5)\pi$.

In order to find negative angles, we'll need to subtract multiples of 2π.

$$\frac{6}{5}\pi - 1(2\pi) = \frac{6}{5}\pi - \frac{10}{5}\pi = -\frac{4\pi}{5}$$

$$\frac{6}{5}\pi - 2(2\pi) = \frac{6}{5}\pi - \frac{20}{5}\pi = -\frac{14\pi}{5}$$

$$\frac{6}{5}\pi - 3(2\pi) = \frac{6}{5}\pi - \frac{30}{5}\pi = -\frac{24\pi}{5}$$

$$\frac{6}{5}\pi - 4(2\pi) = \frac{6}{5}\pi - \frac{40}{5}\pi = -\frac{34\pi}{5}$$

Let's do two quick examples with angles given in DMS (degrees, minutes, and seconds) so that we know how to handle those.

Example

Find the angle α that's coterminal with $150^\circ 17' 49''$ if you make two full positive rotations around the origin.

Notice that in DMS, we have degrees for the first part, and then the minutes and seconds part. Therefore, to find coterminal angles, we do the same thing we did with angles given in degrees, and we just carry the minutes and seconds along with us.

Since we were asked to make two full positive rotations from $150^\circ 17'49''$ to find α, we can say that alpha is

$$\alpha = 150^\circ 17'49'' + 2(360^\circ)$$

$$\alpha = 150^\circ 17'49'' + 720^\circ$$

$$\alpha = (150 + 720)^\circ 17'49''$$

$$\alpha = 870^\circ 17'49''$$

In the example we just did, the original angle was positive, and we rotated in the positive direction. So the signs of the angle and the rotation matched; they were both positive.

Things are little more complicated when the signs are different (when the angle is positive and you rotate in the negative direction, or when the angle is negative and your rotate in the positive direction in order to find coterminal angles).

Example

Find the angle α that's coterminal with $16^\circ 20'42''$ if you make three full negative rotations around the origin.

Since we were asked to make three full negative rotations from $16^\circ 20' 42''$ to find α, we can say that alpha is

$$\alpha = (16^\circ + 20' + 42'') - 3(360^\circ)$$

$$\alpha = 16^\circ + 20' + 42'' - 1{,}080^\circ$$

$$\alpha = (16^\circ - 1{,}080^\circ) + 20' + 42''$$

$$\alpha = -1{,}064^\circ + 20' + 42''$$

The reason we separated the degrees, minutes, and seconds from each in this example, but kept them together in the last example, is because in DMS, the three parts all must be positive, or all must be negative. We can calculate degrees first like we did here, and find that we have a negative value for degrees, but we now need to make both the minutes and second negative as well.

To make the minutes part negative, we'll borrow -1° from the $-1{,}064^\circ$, and combine that -1° with the $20'$ by using the fact that $1^\circ = 60'$:

$$\alpha = -1{,}064^\circ + 20' + 42''$$

$$\alpha = -1{,}063^\circ + (-1^\circ) + 20' + 42''$$

$$\alpha = -1{,}063^\circ + (-60') + 20' + 42''$$

$$\alpha = -1{,}063^\circ + (-40') + 42''$$

Now to make the seconds part negative, we'll borrow $-1'$ from the $-40'$, and combine that $-1'$ with the $42''$ by using the fact that $1' = 60''$:

$$\alpha = -1{,}063^\circ + (-39') + (-1') + 42''$$

$$\alpha = -1{,}063^\circ + (-39') + (-60'') + 42''$$

$$\alpha = -1{,}063^\circ + (-39') + (-18'')$$

Now that all three parts are negative, we can write the coterminal angle as $\alpha = -1{,}063^\circ 39' 18''$.