Oriented arc for a real number

An arc is two points connected by a curved line. When we care about the direction of the arc, and we want to call one end of the arc the starting point, and the other side the ending point, then we have to say that the arc is "oriented", and that it starts at its "initial point" and ends at its "terminal point".

The length s of an arc that traces around the edge of a circle is given by $s = r\theta$, where r is the radius of the circle and θ is the arc's corresponding angle in radians.

The length of an oriented arc can be positive, negative, or 0. It's positive if the terminal point is counterclockwise of the initial point, negative if the terminal point is clockwise of the initial point, and 0 if the terminal and initial points are actually the same point.

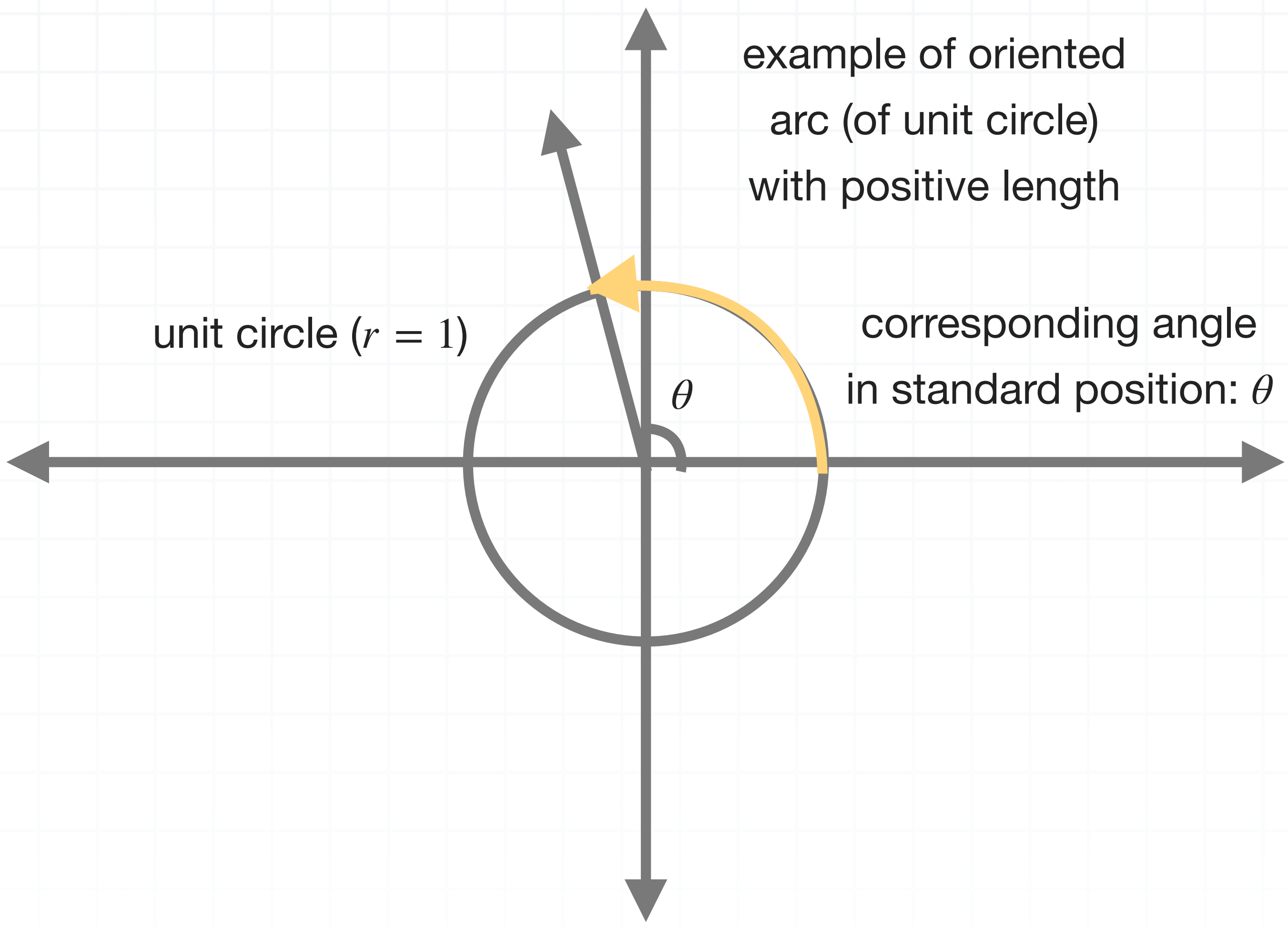
When we talk about an "oriented arc" and that arc is on the unit circle, the radius is $r = 1$, so

$$s = r\theta$$

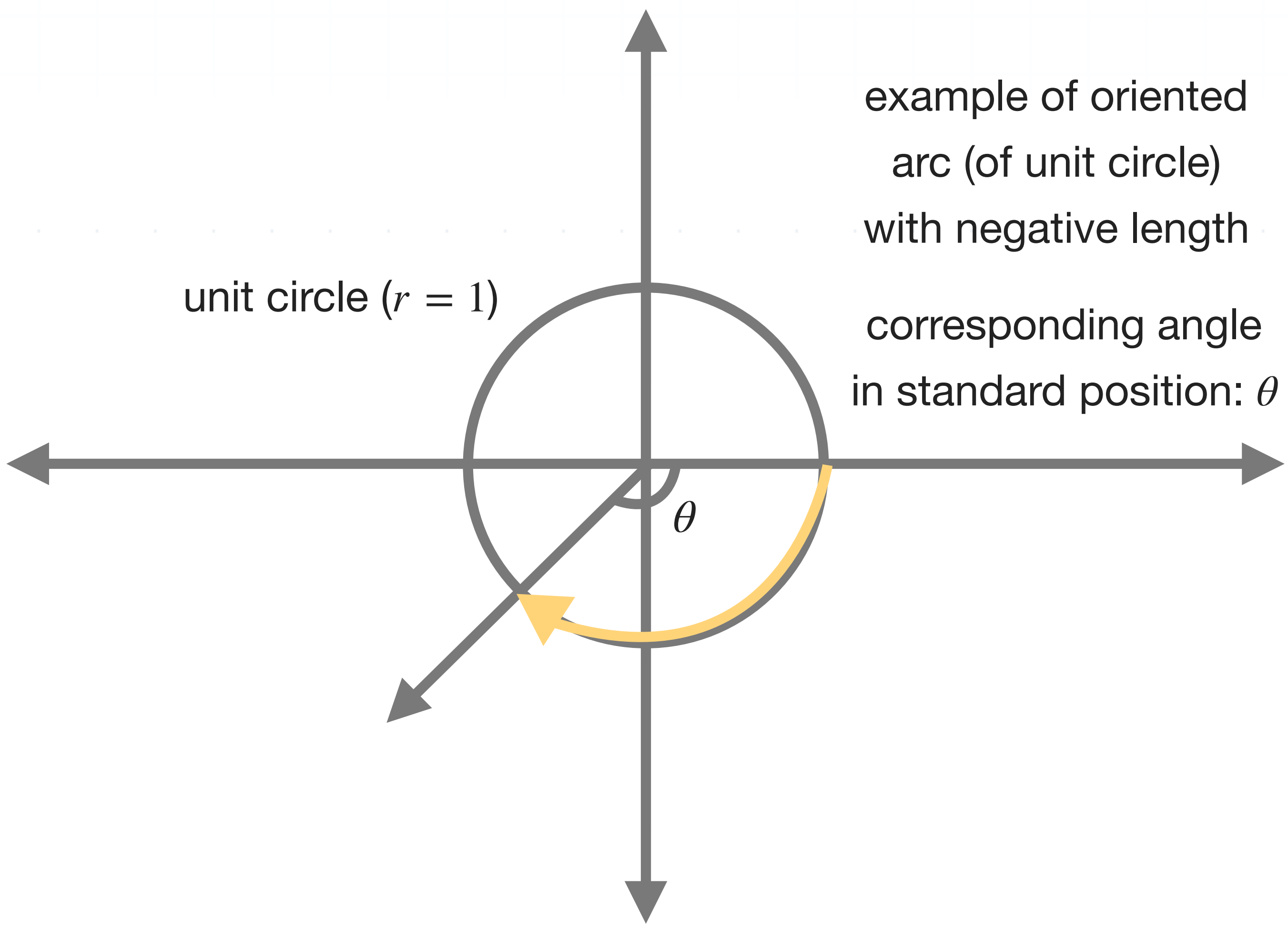
$$s = 1\theta$$

$$s = \theta$$

Which means that the length of the arc is equal to the associated angle. When the length of the arc is positive, it looks like



and when the length of the arc is negative, it looks like



Let's look at an oriented arc in radians.

Example

Find the length of the oriented arc that's traced out along the unit circle and corresponds to $(48/17)\pi$ radians. Find the number of rotations associated with the arc, and say whether the center of the unit circle will be on your right or left as you trace out the arc.

Because we're tracing the arc on the unit circle, the arc length is equal to the angle, so the arc length is $(48/17)\pi$ radians. The number of rotations as the arc is traced out is

$$\frac{\left|\frac{48}{17}\pi\right|}{2\pi} = \frac{48}{17(2)} = \frac{24}{17} \approx 1.412$$

So you'd only make one full turn, and then about 41.2% of an additional turn as you trace out the arc. Since we have a positive length, the center of the unit circle would be to your left as you trace it out.

Let's look at an oriented arc when the angle is a real number.

Example

Find the length of the oriented arc that corresponds to an angle of -15.62 radians. Find the number of rotations associated with the arc, and say whether the center of the unit circle will be on your right or left as you trace out the arc.

Because we're tracing the arc on the unit circle, the arc length is equal to the angle, so the arc length is -15.62 radians. The number of rotations as the arc is traced out is

$$\frac{|-15.62|}{2\pi} = \frac{15.62}{2\pi} \approx 2.486$$

So you'd make two full turns, and then about 48.6% of an additional turn as you trace out the arc. Since we have a negative length, the center of the unit circle would be to your right as you trace it out.

We can also do this backwards, where we start with the arc length and find the measure of the associated angle. If we want the angle in radians, then we can just take the arc length as the angle measure. But if we want the angle in degrees, then we'll have to convert from radians to degrees.

Example

What is the measure in degrees of the angle that corresponds to an oriented arc with length 63.9? Find the number of rotations associated with the arc.

Because the length of the arc is 63.9, the associated angle is $\theta = 63.9$ radians. To convert this to degrees, we multiply 63.9 by the conversion factor $(180^\circ)/\pi$.

$$\theta = 63.9$$

$$\theta = 63.9\left(\frac{180^\circ}{\pi}\right)$$

$$\theta = \left(\frac{11{,}502}{\pi}\right)^\circ$$

$$\theta \approx 3{,}661^\circ$$

To find the number of rotations, we'll divide the length of the arc by 2π.

$$\frac{|63.9|}{2\pi} = \frac{63.9}{2\pi} \approx 10.17$$

So we'd make ten full turns, and then about $17\,\%$ of an additional turn as we trace out the oriented arc.

Let's do one example with an angle in DMS. To find the length of an arc associated with a DMS angle, we have to convert the angle from DMS to degrees, and then from degrees to radians, before getting the length of the arc.

Example

What is the length of the oriented arc that corresponds to $-415^\circ 36' 45''$? Find the number of rotations associated with the arc.

If $\theta = -415^\circ 36' 45''$, then we can say

$$\theta = (-415^\circ) + (-36') + (-45'')$$

We have to convert $(-36') + (-45'')$ to degrees, and then add the result to -415°. We'll use the conversion factors $(1^\circ)/(60')$ and $(1')/(60'')$.

$$(-36') + (-45'')$$

$$(-36')\left(\frac{1^\circ}{60'}\right) + (-45'')\left(\frac{1'}{60''}\right)\left(\frac{1^\circ}{60'}\right)$$

$$-\left(\frac{36}{60}\right)^\circ + \left[-\frac{45}{60(60)}\right]^\circ$$

$$-\left(\frac{36}{60}\right)^\circ - \left(\frac{45}{3{,}600}\right)^\circ$$

$$-\left(\frac{3}{5}\right)^\circ - \left(\frac{1}{80}\right)^\circ$$

$$-\left(\frac{3}{5} + \frac{1}{80}\right)^\circ$$

$$-\left[\frac{16(3) + 1}{80}\right]^\circ$$

$$-\left(\frac{48+1}{80}\right)^\circ$$

$$-\left(\frac{49}{80}\right)^\circ$$

Then θ becomes

$$\theta = (-415^\circ) + (-36') + (-45'')$$

$$\theta = (-415^\circ) + \left[-\left(\frac{49}{80}\right)^\circ\right]$$

$$\theta = -\left[415 + \left(\frac{49}{80}\right)\right]^\circ$$

$$\theta = -\left[\frac{80(415)+49}{80}\right]^\circ$$

$$\theta = -\left(\frac{33{,}200+49}{80}\right)^\circ$$

$$\theta = -\left(\frac{33{,}249}{80}\right)^\circ$$

Convert θ to radians, we'll use the conversion factor $\pi/(180^\circ)$.

$$\theta = -\left(\frac{33{,}249}{80}\right)^\circ\left(\frac{\pi}{180^\circ}\right)$$

$$\theta = -\left[\frac{33{,}249}{80(180)}\right]\pi$$

$$\theta = -\frac{33,249}{14,400}\pi$$

This is the length of the oriented arc associated with $-415^\circ 36' 45''$. The number of rotations is

$$\frac{\left| -\frac{33,249}{14,400}\pi \right|}{2\pi} = \frac{33,249}{14,400(2)} = \frac{33,249}{28,800} \approx 1.154$$

So as we trace out the arc, we'd make one full turn, and then about $15.4\,\%$ of an additional turn.