Area of a triangle

In this lesson we'll talk about a couple of different formulas to find the area of a triangle. We know from basic geometry that the area of a triangle is

$$\text{Area} = \frac{1}{2}bh$$

where a is area, b is the length of the base, and h is the height. But there are other formulas for the area of a triangle that uses trigonometry.

Law of sines for area of a triangle

In order to find the area of a triangle, we can use the formula

$$\text{Area} = \frac{1}{2}ab\sin \text{C}$$

where a and b are the lengths of any two sides of the triangle and C is their included angle. Keep in mind that it doesn't matter which sides (and included angle we use), so we can actually write this same formula in three ways:

$$\text{Area} = \frac{1}{2}ab\sin \text{C}$$

$$\text{Area} = \frac{1}{2}ac\sin \text{B}$$

$$\text{Area} = \frac{1}{2}bc\sin \text{A}$$

This is sometimes called the law of sines for the area of a triangle. You can apply this law anytime you know the lengths of two sides of a triangle and the measure of the included angle.

Example

Apply the law of sines for the area of a triangle to find the area of the triangle in which two of the sides are lengths 23 and 5 and the measure of the included angle is 38°.

Let $a = 23$ and $b = 5$, and let angle C be the included angle. Then

$$\text{Area} = \frac{1}{2}ab\sin \text{C}$$

Plugging in what we know, we get

$$\text{Area} = \frac{1}{2}(23)(5)(\sin 38^\circ)$$

$$\text{Area} = \frac{23(5)}{2}(\sin 38^\circ)$$

$$\text{Area} = \frac{115}{2}(\sin 38^\circ)$$

$$\text{Area} \approx \frac{115}{2}(0.616) \approx 35.4$$

Heron's formula for area of a triangle

Heron's formula is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the lengths of the sides of the triangle and

$$s = \frac{1}{2}(a+b+c)$$

You can apply Heron's formula whenever you know the lengths of all three sides of a triangle.

Example

Apply Heron's formula to find the area of the triangle that has side lengths 16, 19, and 7.

Let $a = 16$, $b = 19$, and $c = 7$. By Heron's formula,

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$s = \frac{1}{2}(a+b+c)$$

$$s = \frac{1}{2}(16+19+7) = \frac{1}{2}(35+7) = \frac{1}{2}(42) = 21$$

Now we can find $s-a$, $s-b$, and $s-c$.

$$s-a=21-16=5$$

$$s-b=21-19=2$$

$$s-c=21-7=14$$

Therefore,

$$\text{Area}=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Area}=\sqrt{21(5)(2)(14)}$$

$$\text{Area}=\sqrt{105(2)(14)}$$

$$\text{Area}=\sqrt{210(14)}$$

$$\text{Area}=\sqrt{2{,}940}\approx 54.2$$

What we have shown is that if you know the lengths of two sides of a triangle and the measure of the included angle, you can use the law of sines for the area of a triangle to compute the area of that triangle. If you know the lengths of all three sides of a triangle, you can use Heron's formula to compute the area of that triangle.

The question is, How do you get the area of a triangle if you know neither the lengths of two sides of a triangle and the included angle nor the lengths of all three sides?

In that case, you might need to apply the usual law of sines first to find whatever information you're missing, and then go on to apply the law of sines for area of a triangle, or Heron's formula, once you have everything you need.

Example

Find the area of the triangle that has side lengths 10 and 5 and where the angle opposite the side with length 10 is 40°.

Let $a=10$ and $b=5$. Then the angle opposite the side of length 10 is A, the angle opposite the side of length 5 is B, and the included angle is C.

We want to be able to eventually apply the law of sines for the area of a triangle, so we'll first find the measure of B, and then use that to get the measure of C.

By the usual law of sines, we get

$$\frac{a}{\sin \text{A}}=\frac{b}{\sin \text{B}}=\frac{c}{\sin \text{C}}$$

$$\frac{10}{\sin 40^\circ}=\frac{5}{\sin \text{B}}=\frac{c}{\sin\ \text{C}}$$

If we focus on the left side, we can solve for B.

$$\frac{10}{\sin 40^\circ}=\frac{5}{\sin \text{B}}$$

$$\sin \text{B}=5\left(\frac{\sin 40^\circ}{10}\right)=\frac{5}{10}\sin 40^\circ$$

$$\sin B \approx \left(\frac{5}{10}\right)(0.643) \approx 0.322$$

$$B \approx \arccos(0.322)$$

$$B \approx 18.8^\circ$$

So the measure of C is

$$180^\circ - 40^\circ - 18.8^\circ$$

$$180^\circ - 58.8^\circ$$

$$121.2^\circ$$

Now we're ready to compute the area of the triangle. Plugging what we know into the law of sines for the area of a triangle,

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$\text{Area} \approx \frac{1}{2}(10)(5) \sin 121.2^\circ$$

$$\text{Area} \approx \frac{50}{2}(0.855) \approx 21.4$$

Finally, let's consider an example of how to find the area of a triangle when you know the length of one side and the measures of two interior angles.

Example

Find the area of the triangle with angles 59° and 67° and where the side opposite the 67° angle has length 22.

Let $A = 59^\circ$ and $B = 67^\circ$. Then $b = 22$. We want to eventually apply the law of sines for the area of a triangle, so we'll find the measure of angle C and the value of a. Angle C is

$$180^\circ - 59^\circ - 67^\circ$$

$$180^\circ - 126^\circ$$

$$54^\circ$$

To get the value of a, we'll plug what we know into the usual law of sines.

$$\frac{a}{\sin \text{A}} = \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}}$$

$$\frac{a}{\sin 59^\circ} = \frac{22}{\sin 67^\circ} = \frac{c}{\sin 54^\circ}$$

Looking at the left part of this equation, we can solve for a.

$$\frac{a}{\sin 59^\circ} = \frac{22}{\sin 67^\circ}$$

$$a = (\sin 59^\circ)\left(\frac{22}{\sin 67^\circ}\right)$$

$$a \approx (0.857)\left(\frac{22}{0.921}\right) \approx 20.5$$

Plugging what we now know into the usual law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} \approx \frac{1}{2}(20.5)(22)\sin 54^\circ$$

$$\text{Area} \approx \frac{1}{2}(20.5)(22)(0.809)$$

$$\text{Area} \approx \frac{451}{2}(0.809) \approx 182$$