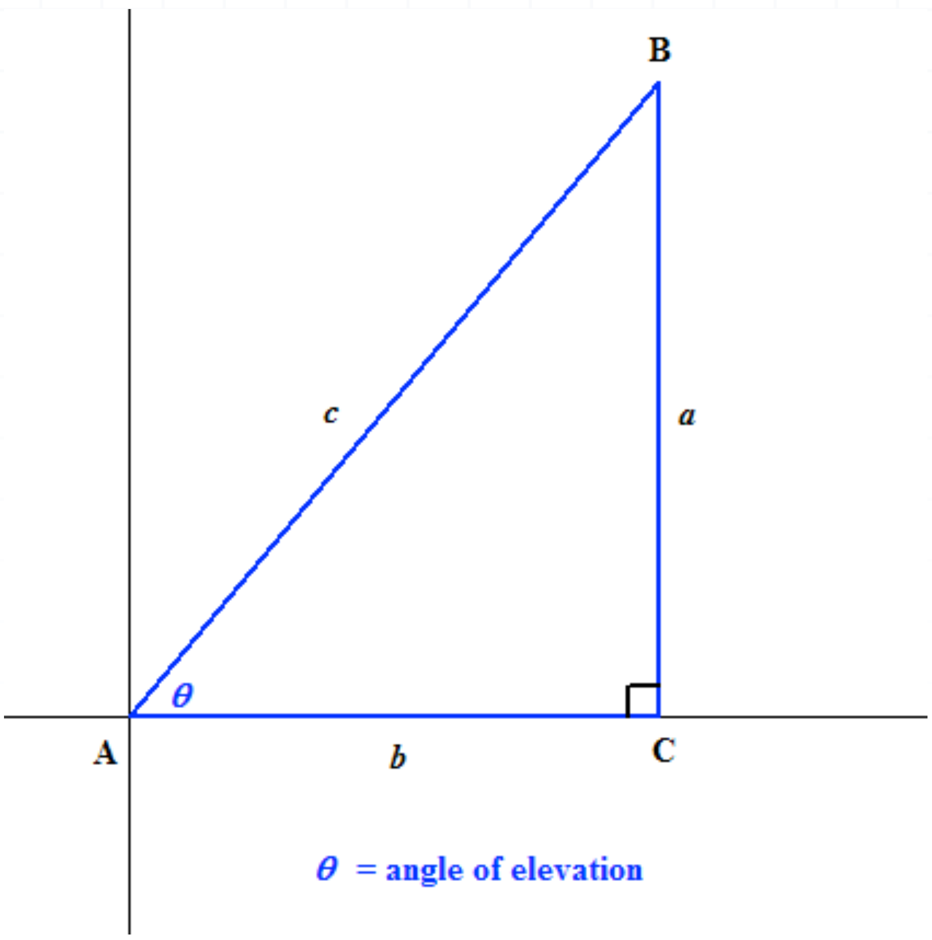
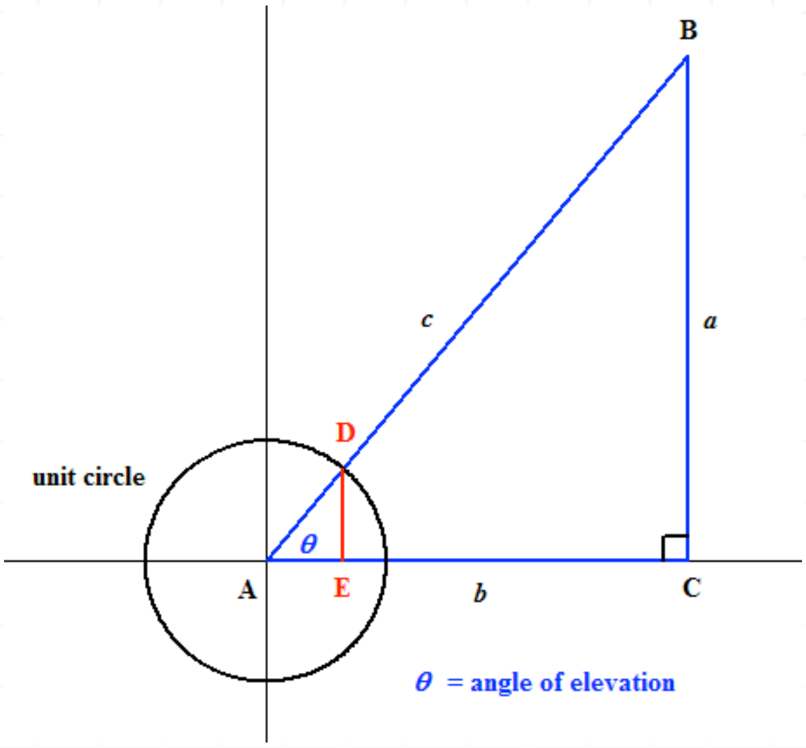
Angle of depression and elevation

Imagine that you're standing on level ground and looking at a bird in the air, and the bird is not directly above your head. If we say that you're located at A, that the bird is at B, and that C is the point on the ground directly below the bird, then a picture of this is



The angle θ at the origin is the "angle of elevation" of the bird with respect to you. An angle of elevation is always acute. If you know the measure of θ and the length of one side of the triangle, you can use trigonometry to find the lengths of the other two sides.

If we draw the unit circle over the top of this triangle, and form a new triangle ADE, then we can say that triangles ADE and ABC are similar.



We can write the proportion

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{1}$$

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

Since D is on the unit circle, its coordinates are $(\cos\theta, \sin\theta)$. The coordinates of E are $(\cos\theta, 0)$ and the coordinates of A are $(0,0)$. Therefore

$$\overline{DE} = \sin\theta - 0$$

$$\overline{DE} = \sin\theta$$

and

$$\overline{AE} = \cos\theta - 0$$

$$\overline{AE} = \cos\theta$$

The proportion becomes

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = c$$

Now if we know c and θ, then we can use

$$\frac{a}{\sin\theta} = c$$

to find a.

$$a = c(\sin\theta)$$

And we can use

$$\frac{b}{\cos\theta} = c$$

to find b.

$$b = c(\cos\theta)$$

Example

If the angle of elevation of a bird with respect to you is 40° and the distance between you and the bird is 15 feet, find the height of the bird above the ground and your distance from the point on the ground that's directly below the bird.

Since the bird is 15 feet away, $c = 15$ feet. The height of the bird above the ground is a, so

$$a = 15(\sin 40^\circ)$$

We can use a calculator (in degree mode) to find $\sin 40^\circ \approx 0.643$, so we get

$$a \approx 15(0.643) \approx 9.65 \text{ feet}$$

Your distance from the point on the ground below the bird is b, so

$$b = 15(\cos 40^\circ)$$

With a calculator we get $\cos 40^\circ \approx 0.766$, so

$$b \approx 15(0.766) \approx 11.5 \text{ feet}$$

Next, we'll consider a situation where we know the angle of elevation of one of the corners of the top of a building and the height of that corner above the ground.

Example

If the angle of elevation of one corner of the top of an 80-foot-tall building with respect to a certain point A on the ground is 65°, find the distance between A and the point on the ground directly below that corner of the building.

In this case, $\theta = 65^\circ$ and $a = 80$ feet. The distance between A and B is c, and the distance from A to the point on the ground directly below B is b.

If we start with

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = c$$

then to find b, we can use

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta}$$

Multiplying both sides by $\cos\theta$, we get

$$\left(\frac{a}{\sin\theta}\right)\cos\theta = b$$

$$b = \left(\frac{a}{\sin\theta}\right)\cos\theta$$

Using a calculator, we know that

$$\sin 65^\circ \approx 0.906$$

$$\cos 65^\circ \approx 0.423$$

Therefore,

$$b \approx \left(\frac{80}{0.906}\right)(0.423) \approx 37.4 \text{ feet}$$

To find c, we can use

$$\frac{a}{\sin\theta} = c$$

Plugging in what we already know, we get

$$c = \frac{a}{\sin\theta} \approx \frac{80}{0.906} \approx 88.3 \text{ feet}$$

Now we'll look at a case where we know the angle of elevation of B with respect to A, and the horizontal distance from A to the point directly below B.

Example

Suppose you measure the horizontal distance of a cell-phone tower from a nearby point on the ground A to be 24 feet. If the angle of elevation of the top of the tower with respect to A is 78°, how far is the top of the tower from A, and what is the height of the tower?

In this case, $\theta = 78^\circ$ and $b = 24$ feet. Also, the distance from A to the top of the tower is c, and the height of the tower is a. You can find c using

$$\frac{b}{\cos\theta} = c$$

Plugging in what we know, we get

$$c = \frac{b}{\cos\theta} = \frac{24}{\cos 78^\circ}$$

$$c \approx \frac{24}{0.208} \approx 115 \text{ feet}$$

To find a, we'll use

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta}$$

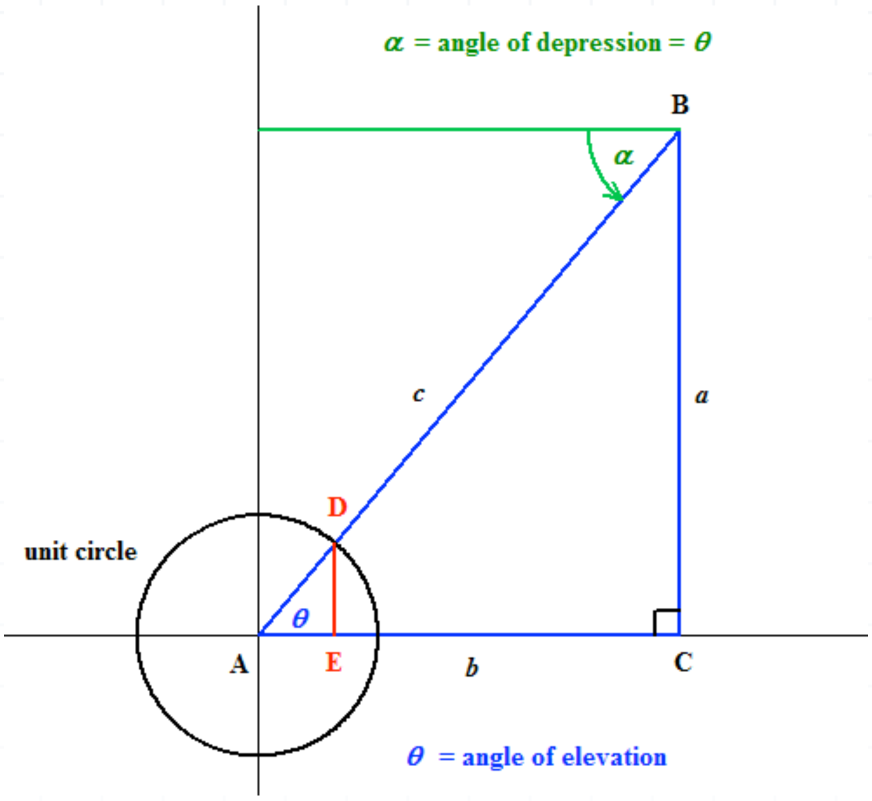
Multiplying both sides of this equation by $\sin\theta$, we have

$$a = \left(\frac{b}{\cos\theta}\right)\sin\theta$$

$$a \approx \left(\frac{24}{0.208}\right)(0.978) \approx 113 \text{ feet}$$

When we talk about the position of a bird in the air compared to you on the ground, you have to look up to see the bird and the bird has to look down to see you.

Now imagine the acute angle α by the bird. The angles θ and α are alternate interior angles, which means they have equal measures. We call α the angle of depression of you with respect to the bird, whereas θ was the angle of elevation of the bird with respect to you.



Example

If your angle of depression with respect to a bird is 57° and your distance from the bird is 19 feet, how far are you, vertically and horizontally, from the bird?

Your angle of depression with respect to the bird is $\alpha = 57^\circ$, so the angle of elevation of the bird with respect to you is $\theta = 57^\circ$. Since the bird is 19 feet away from you, $c = 19$ feet. The height of the bird above the ground is a, so

$$a = 19(\sin 57^\circ)$$

$$a \approx 19(0.839) \approx 15.9 \text{ feet}$$

Your distance from the point on the ground below the bird is b, so

$$b = 19(\cos 57^\circ)$$

$$b \approx 19(0.545) \approx 10.4 \text{ feet}$$

So your vertical distance from the bird is 15.9 feet, and your horizontal distance from the bird is 10.4 feet.