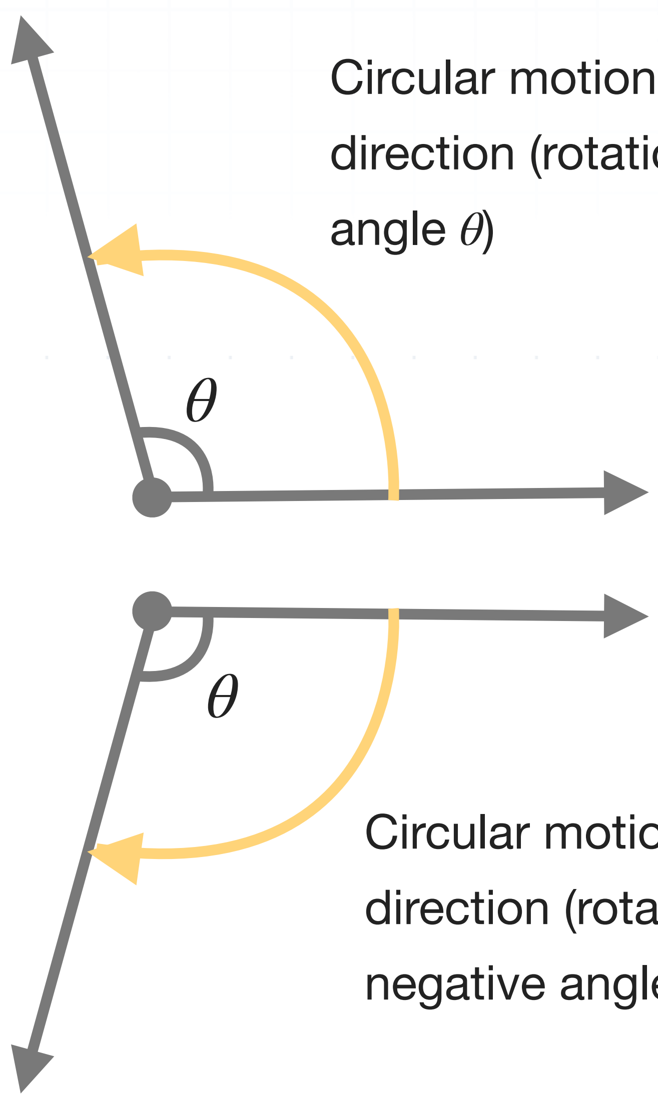
Linear and angular velocity

Before we get into linear and angular velocity, let's first talk about the definition of velocity. It's easy to confuse velocity with speed, but velocity has both magnitude (a "size") and direction, whereas speed has only magnitude. In fact, speed is the magnitude of velocity, so speed is always a non-negative number.

So angular velocity tells us the size and direction of an angle, while linear velocity tells us the size and direction of an arc.

The classic example of angular velocity is when we trace out an angle as we rotate around a stationary point.



Circular motion in counterclockwise direction (rotation through a positive angle θ)

Circular motion in clockwise direction (rotation through a negative angle θ)

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If the magnitude of the angular velocity is constant, the total angle θ that's been swept out at any given time during a rotation is proportional to the amount of time elapsed since the rotation began. Assuming time started at $t=0$, then

$$\omega = \frac{\theta}{t}$$

where ω (omega) is angular speed and t is the time since the rotation started.

Since θ is an angle measure and t is time, ω will be an angle measure per unit time, like radians per second, degrees per minute, etc.

Example

What is the angular speed, in radians per second, of a disc that rotates at a constant rate and sweeps out an angle of 36.4π radians in 8.39 seconds?

To find the angular speed ω, we'll divide the total angle swept out θ, by the total time t.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{(36.4\pi)\text{ rad}}{8.39\text{ sec}}$$

$$\omega = \left(\frac{36.4}{8.39}\pi\right)\text{ radians per second}$$

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$$\omega \approx 4.34\pi \text{ radians per second}$$

You always want to be careful about units, and make sure that you have matching units of time. If you don't, you'll need to do some conversions.

Example

What is the angular speed, in radians per second, of a wheel that rotates at a constant rate and sweeps out an angle of 72.7π radians in 3.2 minutes?

Since we're asked to find angular speed in radians per second, but we're given the rotation in minutes, we'll need to first convert the minutes into seconds.

$$t = (3.2 \text{ min})\left(\frac{60 \text{ sec}}{1 \text{ min}}\right)$$

$$t = 3.2(60) \text{ sec}$$

$$t = 192 \text{ sec}$$

Now we're all set to find the angular speed.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{(72.7\pi) \text{ rad}}{192 \text{ sec}}$$

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$$\omega = \left(\frac{72.7}{192}\pi\right) \text{ radians per second}$$

$$\omega \approx 0.379\pi \text{ radians per second}$$

Let's do an example where we need to convert between radians and degrees.

Example

Find the angular speed, in radians per second, of an object that rotates at a constant rate and sweeps out an angle of $1{,}043^\circ$ in 5.9 seconds.

We'll convert the angle $1{,}043^\circ$ to radians.

$$\theta = 1{,}043^\circ \left(\frac{\pi \text{ rad}}{180^\circ}\right)$$

$$\theta = \left(\frac{1{,}043}{180}\pi\right) \text{ rad}$$

Now we'll find the angular speed.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\left(\frac{1{,}043}{180}\pi\right) \text{ rad}}{5.9 \text{ sec}}$$

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$$\omega = \left[\frac{1{,}043}{180(5.9)}\pi\right] \text{ radians per second}$$

$$\omega \approx 0.982\pi \text{ radians per second}$$

We also often express angular speed as revolutions per second, revolutions per minute, etc. When we want to convert from one set of units to the other, we'll need to remember that there is 1 revolution per 2π radians.

Example

Express angular speed of 31 radians per second in units of revolutions per minute.

We know the angular speed ω, and we just need to convert it to different units.

$$\omega = 31\ \frac{\text{rad}}{\text{sec}}$$

$$\omega = \left(31\ \frac{\text{rad}}{\text{sec}}\right)\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ sec}}{1 \text{ min}}\right)$$

$$\omega = \left[\frac{31(60)}{2\pi}\right] \text{ revolutions per minute}$$

If we say that π is $3.1415...$, we get

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$$\omega \approx 296 \text{ revolutions per minute}$$

Here's an example of a conversion in the opposite direction.

Example

Express angular speed of 86.3 revolutions per minute in units of radians per second.

We already know angular speed, we just need to convert the units.

$$\omega = 86.3 \frac{\text{rev}}{\text{min}}$$

$$\omega = \left(86.3 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$$\omega = \left[\frac{86.3(2)}{60}\pi\right] \text{radians per second}$$

$$\omega \approx 2.88\pi \text{ radians per second}$$

Now we're ready to discuss linear velocity in connection with angular velocity.

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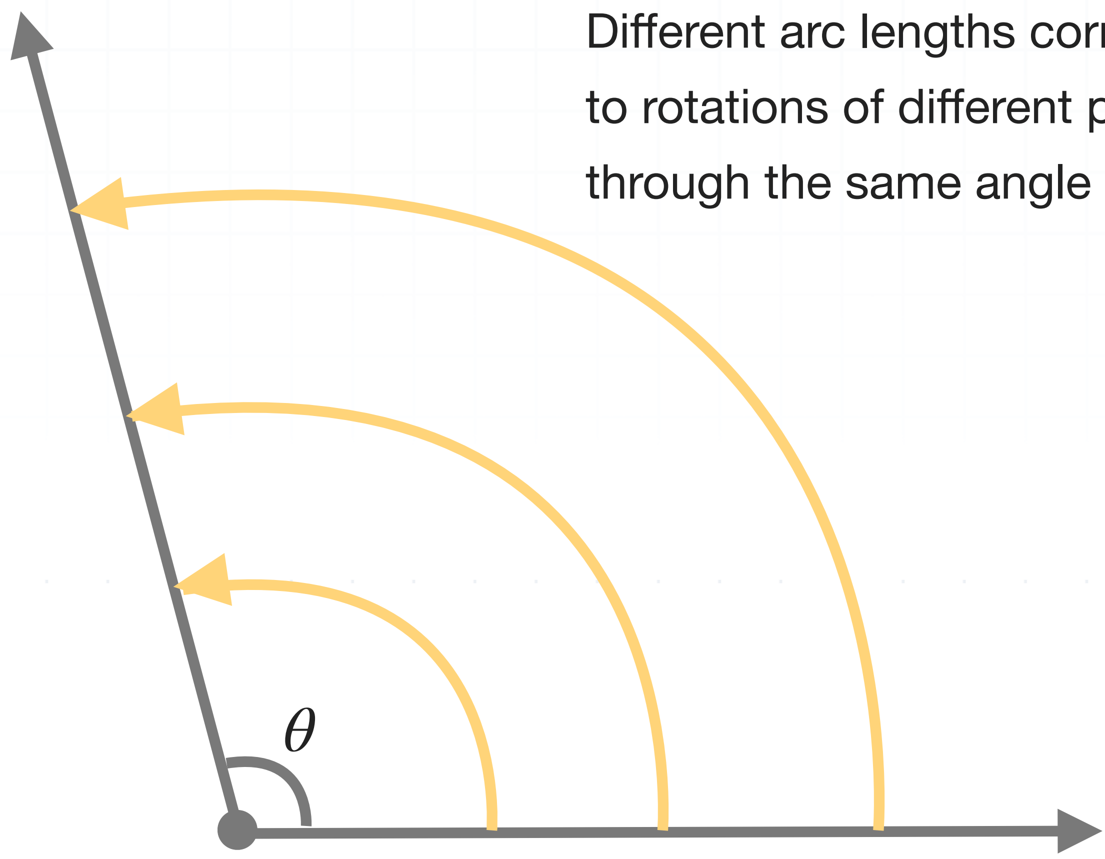
Think of a solid disk that's spinning. Because the entire disk rotates together, the angular speed is the same at every single point on the disk, no matter which point we pick.

If two points on the disk are at different distances from the center, then the further point will trace out a longer arc than the arc that the closer point makes. Therefore, in order for both points to have the same angular speed, the further point would need to move faster than the closer point. Linear velocity takes these differences into account, because it tells us how fast the arc for an individual point is traced out.



Different arc lengths correspond to rotations of different points through the same angle θ.

Remember that angular velocity is always either counterclockwise or clockwise. But linear velocity, is always changing direction.

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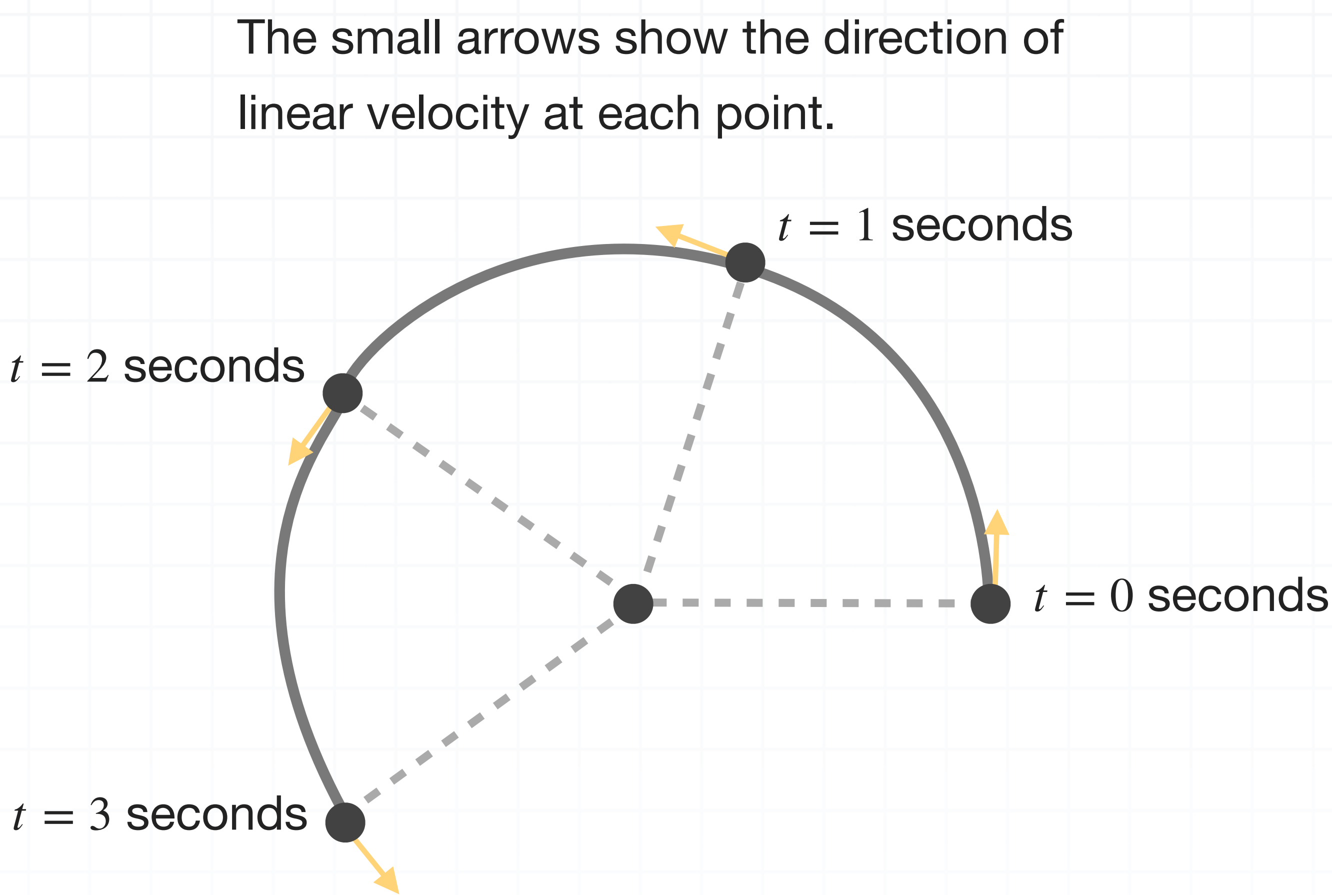

Because arc length s is given by $s = r\theta$, and because we know $\theta = \omega t$, we can replace θ with ωt, and get a new formula for arc length.

$$s = r\theta = r(\omega t) = (r\omega)t$$

If we use v for linear speed, we can say

$$\omega = \frac{v}{r}$$

Example

What are the linear speeds, in inches per second, of points on a rotating disc that has an angular speed of 9.4 radians per second if those points are located 1.3 and 2.6 inches, respectively, from the center of the disc?

For the point that's 1.3 inches from the center of the disc, we'll say $r = 1.3$ and $\omega = 9.4$.

$$v = r\omega$$

$$v = (1.3 \text{ in})\left(\frac{9.4}{\text{sec}}\right)$$

$$v = [1.3(9.4)] \text{ inches per second}$$

$$v = 12.22 \text{ inches per second}$$

For the point that's 2.6 inches from the center of the disc, $r = 2.6$, but ω is still 9.4.

$$v = r\omega$$

$$v = (2.6 \text{ in})\left(\frac{9.4}{\text{sec}}\right)$$

$$v = [2.6(9.4)] \text{ inches per second}$$

$$v = 24.44 \text{ inches per second}$$

Notice that the linear speed of the second point is double the linear speed of the first point. That's because the arc traced out by the second point is twice as long as the arc traced out by the first point.

Sometimes we'll have to convert units before we can solve the problem.

Example

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Consider a disc of radius 7.9 centimeters. What is the linear speed, in inches per second, of a point that's 5.6 centimeters from the center of the disc if the disc is rotating at 42.8 revolutions per minute?

We'll need to convert the lengths from centimeters to inches, and we'll need to convert the angular speed from revolutions per minute to radians per second.

For the first conversion, we'll use the fact that 1 inch is about 2.54 centimeters.

$$r = 5.6 \text{ cm}$$

$$r \approx (5.6 \text{ cm})\left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right)$$

$$r \approx \frac{5.6}{2.54} \text{ in}$$

$$r \approx 2.20 \text{ in}$$

To express the angular speed in radians per second, we'll get

$$\omega = 42.8 \frac{\text{rev}}{\text{min}}$$

$$\omega = \left(42.8 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$$\omega = \left[\frac{42.8(2)}{60}\pi\right] \text{ radians per second}$$

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$$\omega \approx 1.43\pi \text{ radians per second}$$

Now we'll combine those two results to get the linear speed.

$$v = r\omega$$

$$v \approx (2.20\text{in})\left(\frac{1.43\pi}{\text{sec}}\right)$$

$$v \approx [2.20(1.43)\pi] \text{ inches per second}$$

Substituting the numerical value of π, we find that

$$v \approx 9.88$$

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