**Topic**: Law of sines

**Question**: Find each side length.

If the measures of two of the interior angles of a triangle are $70^\circ$ and $43^\circ$, and the length of the side which is opposite the $70^\circ$ angle is $12$, what is the length $b$ of the side of the triangle which is opposite the angle of measure $43^\circ$, and what is the length $c$ of the third side of the triangle?

**Answer choices**:

A $b \approx 11.8$ and $c \approx 16.5$

B $b \approx 16.5$ and $c \approx 12.2$

C $b \approx 12.2$ and $c \approx 8.71$

D $b \approx 8.71$ and $c \approx 11.8$

**Solution**: D

By the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where angle $A$ is the interior angle of measure $70^\circ$, angle $B$ is the interior angle of measure $43^\circ$, angle $C$ is the third interior angle of the triangle (i.e., the interior angle which is opposite the side of length $c$), and $a$ is the length of the side which is opposite angle $A$ (i.e., $a = 12$).

Substituting the known values, we have

$$\frac{12}{\sin 70^\circ} = \frac{b}{\sin 43^\circ} = \frac{c}{\sin C}$$

Since the sum of the measures of angles $A$, $B$, and $C$ must be $180^\circ$, we see that the measure of angle $C$ is

$$180^\circ - (70^\circ + 43^\circ) = 180^\circ - 113^\circ = 67^\circ$$

Therefore,

$$\frac{12}{\sin 70^\circ} = \frac{b}{\sin 43^\circ} = \frac{c}{\sin 67^\circ}$$

To determine $b$, we'll solve the equation

$$\frac{12}{\sin 70^\circ} = \frac{b}{\sin 43^\circ}$$

Turning this equation around, we have

$$\frac{b}{\sin 43^\circ} = \frac{12}{\sin 70^\circ}$$

Thus

$$b = \frac{12(\sin 43^\circ)}{\sin 70^\circ}$$

With a calculator, we get $\sin 43^\circ \approx 0.682$ and $\sin 70^\circ \approx 0.940$, so

$$b = \frac{12(\sin 43^\circ)}{\sin 70^\circ} \approx \frac{12(0.682)}{0.940} \approx 8.71$$

To determine $c$, we'll solve the equation

$$\frac{12}{\sin 70^\circ} = \frac{c}{\sin 67^\circ}$$

Turning this equation around, we have

$$\frac{c}{\sin 67^\circ} = \frac{12}{\sin 70^\circ}$$

Therefore,

$$c = \frac{12(\sin 67^\circ)}{\sin 70^\circ}$$

With a calculator, we get $\sin 67^\circ \approx 0.921$, hence

$$c = \frac{12(\sin 67^\circ)}{\sin 70^\circ} \approx \frac{12(0.921)}{0.940} \approx 11.8$$

**Topic**: Law of sines

**Question**: Find the other measures.

If the lengths of two sides of a triangle are $20$ and $30$, and the measure of the interior angle which is opposite the side of length $30$ (call it angle $B$) is $95^\circ$, find the measures of the other two interior angles ($A$ and $C$) of that triangle (where angle $A$ is opposite the side of length $20$) and the length $c$ of its third side.

**Answer choices**:

A The measures of angles $A$ and $C$ are approximately $48.9^\circ$ and $38.6^\circ$, respectively, and $c$ is approximately $16.3$.

B The measures of angles $A$ and $C$ are approximately $138.4^\circ$ and $22.9^\circ$, respectively, and $c$ is approximately $45.2$.

C The measures of angles $A$ and $C$ are approximately $37.6^\circ$ and $47.4^\circ$, respectively, and $c$ is approximately $31.8$.

D The measures of angles $A$ and $C$ are approximately $41.6^\circ$ and $43.4^\circ$, respectively, and $c$ is approximately $20.7$.

**Solution**: D

Let $a = 20$ and $b = 30$, and let angle $B$ be the interior angle opposite the side of length $30$ (i.e., angle $B$ has measure $95^\circ$). By the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where angle $A$ is the interior angle opposite the side of length $20$, $c$ is the length of the third side of the triangle, and angle $C$ is the interior angle opposite the side of length $c$.

Substituting the known values, we have

$$\frac{20}{\sin A} = \frac{30}{\sin 95^\circ} = \frac{c}{\sin C}$$

Multiplying both sides of the equation

$$\frac{20}{\sin A} = \frac{30}{\sin 95^\circ}$$

by

$$(\sin A)\left(\frac{\sin 95^\circ}{30}\right)$$

we get

$$\frac{20(\sin 95^\circ)}{30} = \sin A$$

Turning this equation around, we have

$$\sin A = \frac{20(\sin 95^\circ)}{30} = \left(\frac{2}{3}\right)(\sin 95^\circ)$$

Using a calculator, we find that $\sin 95^\circ \approx 0.996$, so

$$\sin A \approx \left(\frac{2}{3}\right)(0.996) \approx 0.664$$

What this tells us is that if angle $A$ is acute, then its measure is approximately $41.6^\circ$; and if angle $A$ is obtuse, then its measure is (approximately)

$$180^\circ - 41.6^\circ = 138.4^\circ$$

It's impossible for angle $A$ to have measure $138.4^\circ$, because that would mean that the sum of the measures of angles $A$ and $B$ is (approximately)

$$138.4^\circ + 95^\circ = 233.4^\circ$$

which is greater than $180^\circ$. Thus the measure of angle $A$ is $41.6^\circ$.

Now we know that the measures of two of the three interior angles of this triangle are $95^\circ$ and (approximately) $41.6^\circ$. Since the sum of the interior angles of any triangle is $180^\circ$, the measure of the third interior angle of this triangle (angle $C$) is approximately

$$180^\circ - (95^\circ + 41.6^\circ) = 180^\circ - 136.6^\circ = 43.4^\circ$$

To find $c$ (the length of the third side of this triangle), we'll solve the equation

$$\frac{30}{\sin 95^\circ} = \frac{c}{\sin C}$$

Substituting the value we found for the measure of angle $C$, we get

$$\frac{30}{\sin 95^\circ} \approx \frac{c}{\sin 43.4^\circ}$$

Using a calculator, we find that $\sin 43.4^\circ \approx 0.687$, so

$$c \approx \left(\frac{30}{\sin 95^\circ}\right)(\sin 43.4^\circ) \approx \left(\frac{30}{0.996}\right)(0.687) \approx 20.7$$

**Topic**: Law of sines

**Question**: How many triangles with these properties are there?

Consider a triangle that has one side of length $a = 20$ and another side of length $c = 16$, and where the interior angle opposite the side of length $16$ (call it angle $C$) has measure $35^\circ$.

**Answer choices**:

A There are two triangles with the stated properties.

B There is only one triangle with the stated properties.

C There is no triangle with the stated properties.

D The number of triangles with the stated properties cannot be determined.

**Solution**: A

By the law of sines,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

where angle $A$ is the interior angle which is opposite the side of length $20$, $b$ is the length of the third side of the triangle, and angle $B$ is the interior angle which is opposite the side of length $b$.

Substituting the known values, we have

$$\frac{20}{\sin A}=\frac{b}{\sin B}=\frac{16}{\sin 35^\circ}$$

Multiplying both sides of the equation

$$\frac{20}{\sin A}=\frac{16}{\sin 35^\circ}$$

by

$$(\sin A)\left(\frac{\sin 35^\circ}{16}\right)$$

we get

$$\frac{20(\sin 35^\circ)}{16}=\sin A$$

Turning this equation around, we have

$$\sin A=\frac{20(\sin 35^\circ)}{16}=\left(\frac{5}{4}\right)(\sin 35^\circ)$$

Using a calculator, we find that $\sin 35^\circ \approx 0.574$, so

$$\sin A \approx \left(\frac{5}{4}\right)(0.574) \approx 0.718$$

What this tells us is that if angle $A$ is acute, then its measure is approximately $45.9^\circ$; and if angle $A$ is obtuse, then its measure is (approximately)

$$180^\circ - 45.9^\circ = 134.1^\circ$$

Suppose the measure of angle $A$ is (approximately) $134.1^\circ$. Then the sum of the measures of angles $A$ and $C$ is

$$134.1^\circ + 35^\circ = 169.1^\circ < 180$$

Therefore, there are two triangles with the stated properties.

Next, let's set angle $A$ to $45.9^\circ$, compute the measure of angle $B$, and then use the law of sines to determine $b$.

In this case, the measure of angle $B$ is

$$180^\circ - (45.9^\circ + 35^\circ) = 180^\circ - 80.9^\circ = 99.1^\circ$$

To get $\sin B$, we'll solve the equation

$$\frac{b}{\sin B} = \frac{16}{\sin 35^\circ}$$

Substituting $99.1^\circ$ for angle $B$ gives

$$b = \frac{16(\sin 99.1^\circ)}{\sin 35^\circ}$$

Using a calculator, we get that $\sin 99.1^\circ \approx 0.987$, so

$$b = \frac{16(\sin 99.1^\circ)}{\sin 35^\circ} \approx \frac{16(0.987)}{0.574} \approx 27.5$$

Finally, let's set angle $A$ to $134.1^\circ$. Then the measure of angle $B$ is

$$180^\circ - (134.1^\circ + 35^\circ) = 180^\circ - 169.1^\circ = 10.9^\circ$$

To get $\sin B$, we'll solve the equation

$$\frac{b}{\sin B} = \frac{16}{\sin 35^\circ}$$

Substituting $10.9^\circ$ for angle $B$ gives

$$b = \frac{16(\sin 10.9^\circ)}{\sin 35^\circ}$$

Using a calculator, we get that $\sin 10.9^\circ \approx 0.189$, so

$$b = \frac{16(\sin 10.9^\circ)}{\sin 35^\circ} \approx \frac{16(0.189)}{0.574} \approx 5.27$$

Thus there are two triangles with the indicated properties:

- a triangle with interior angles of measure $45.9^\circ$, $99.1^\circ$, and $35^\circ$, and with sides opposite those angles which are of length $20$, $27.5$, and $16$, respectively

- a triangle with interior angles of measure $134.1^\circ$, $10.9^\circ$, and $35^\circ$, and with sides opposite those angles which are of length $20$, $5.27$, and $16$, respectively